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Diversification with Volatility Products

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ABSTRACT

Recent changes to clearing-house regulations have promoted exchange-traded products offering risk premia previously accessible only over-the-counter. Thus, as correlations increase between equity, bonds and commodities, a new strand of research questions the benefits of home-grown diversification using volatility products. First we ask: “What expected returns will induce equity and bond investors to perceive ex-ante diversification benefits from adding volatility?” We call this the *optimal diversification threshold*. We derive the theoretical thresholds for minimum–variance, mean–variance and Black–Litterman optimization. Empirical analysis of US and European markets shows that volatility diversification is frequently perceived to be optimal, ex-ante, but these apparent benefits are almost never realized, being eroded by high roll and transaction costs. Exchange-traded volatility only proved an effective diversifier during the banking crisis. At other times long equity and bond portfolios diversified with volatility futures have not performed as well as those without diversification, or even those diversified with commodities.

JEL classification: G11, G15, G23

Keywords: Variance risk premium, VIX, VSTOXX, futures, volatility ETN, commodity ETN

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1 Introduction

Equities, bonds and commodities have become more highly correlated globally since the banking crisis.¹ Within home-grown investments alternative markets have developed for real estate, funds of hedge funds – even wine and art – amid vigorous debate on the benefits of international portfolio diversification.² Amongst alternative domestic diversifiers the highly innovative asset class of equity volatility arises as a natural diversification choice because its negative correlation with equity increases exactly when diversification is needed most – a fact that has been well documented since Bekaert and Wu (2000). For example, during 2008 – 2010 the negative correlation between the S&P 500 index and its corresponding volatility index VIX was about -0.85 , measured on daily returns. Consequently if, on 1 April 2010 an S&P 500 investor had put 30% of his capital in the risk-free asset and taken an equivalent long position in the June 2010 VIX futures contract, closing the position a week before expiry, he would have achieved an (equivalent) Sharpe ratio of 3.61. Holding the S&P 500 exchange-traded fund (SPY) alone gave a negative mean excess return over the same period.

These observations motivate the question whether volatility could be an effective diversification tool for pension or mutual funds, public companies and indeed any investor that is long in domestic capital assets, i.e. equities and bonds. Many investment entities are forbidden by law to short equity, because this is generally considered as speculation rather than a position which fits long-term investment. Over-the-counter (OTC) trades such as variance swaps are also disallowed for many investors. However, changes in regulations have recently prompted a huge demand for exchange-listed volatility products. Specifically, the EMIR directive in the European Union and the Dodd-Frank Act in the U.S. now require OTC transactions to be cleared by central counter-parties in much the same way as exchange-traded products; and this has acted as a catalyst for growth in listed products such as volatility futures, notes and funds which attempt to mimic the risk-return characteristics of variance swaps.³

¹See Daskalaki and Skiadopoulos (2011), Cheung and Miu (2010) and others.

²See Kroencke and Schindler (2012), Liu et al. (2014) and many others.

³For instance, from 2009 to 2014, the number of traded VIX futures contracts increased from 4,500 to more than 215,000 contracts with a value of more than \$4bn traded on average each day. Accordingly, the market for exchange-traded products based on volatility futures has exploded in recent years and trading volume on some of these products can reach about \$5bn per day. See Alexander et al. (2015) for further details.

We begin by introducing a new theoretical concept. Given an investor that has a long position on each of the assets or financial instruments X_1, X_2, \dots, X_{k-1} , the *optimal diversification threshold* for the asset/instrument X_k is the lowest expected return q_k on X_k for which an additional long position on X_k is perceived to be optimal, ex-ante. We derive a general expression for the optimal diversification threshold in the context of three standard optimization paradigms: minimum-variance, mean-variance and the Black and Litterman (1992) framework.

Our theoretical results are then used in an empirical study to analyze the perceived benefits of volatility diversification for long equity (or equity-bond) investors in the U.S. and European Union markets. In general, both the threshold and the corresponding optimal diversification frequency will depend on the investor, as characterized by his risk aversion, optimization framework and model parameters (and the covariance matrix of the k assets in particular). Using data from January 2006 to April 2015 we apply the optimal diversification threshold at regular monthly rebalancing points, hence identifying the frequency with which different types of investors would perceive diversification to be ex-ante optimal. Our parameter estimation method is based on historical data, and we take standard equilibrium portfolios for the Black-Litterman extension. Finally, we compare the realized performance of the optimally-diversified portfolios with that of traditional equity-bond portfolios and with the performance of an equity-bond portfolio that is diversified using commodities.⁴

There is a vast literature on volatility diversification which is reviewed in the next section. Our study is the first to apply a proper ex-ante analysis within a rolling framework, i.e. a situation where the investor periodically rebalances his portfolio based on new information. We employ three standard optimization models and we also use a much longer sample period than any previous study, almost all of which have focused on the years surrounding the banking crisis, when the realized performance of volatility futures was unusually good. Our empirical findings may be a timely warning to market players in volatility products, and especially to the investors whose interests we seek to protect.

⁴There are two possible reasons why an investor may choose to adopt a cautious stance on the benefits of volatility as a diversification tool. It may be that, in the past over a long historical period, his expected return was rarely high enough to perceive that volatility diversification was ex-ante optimal. However, it may also be that taking a position in volatility has frequently been viewed as ex-ante optimal, but that taking such position actually deteriorated rather than enhanced performance.

In the following: Section 2 motivates our work by setting it in the context of the relevant literature on diversification; Section 3 presents our theoretical results and applies them to the problem of volatility diversification for (i) a minimum-variance investor, (ii) a mean-variance investor and (iii) an investor using the framework of Black and Litterman (1992); Section 4 presents and discusses our empirical results; Section 5 concludes. All proofs for main theoretical results are in an Appendix.

2 Literature Review

One of the earliest and most influential papers on the benefits of home-made diversification is Errunza et al. (1999). Based on monthly data between 1976 and 1993 for seven developed and nine emerging markets they use return correlations, mean–variance spanning and Sharpe ratio tests as evidence that, once the investor has employed home-grown diversification tools, gains from international diversification are statistically and economically insignificant. An associated finding is that international stock market correlations are particularly high during times of market stress, and this lowers international diversification benefits at a time when it is needed most. To support this, Butler and Joaquin (2002) measure the correlations of US, UK, Japanese, Australian and European stock market indices between January 1970 and December 2000, observing a non-normal behavior of returns correlations with significantly higher correlations in bear markets than in calm or bull markets. Kearney and Lucey (2004) survey the previous literature on international equity market integration and provide further evidence of declining diversification benefits in international equity markets. More recently, Liu et al. (2014) explore ten European equity markets from 2001 to March 2013. They construct optimal portfolios for the PIIGS (Portugal, Italy, Ireland, Greece and Spain) and CORE (Austria, Finland, France, Germany and the Netherlands) countries to demonstrate that there are only limited diversification benefits within the Eurozone. In fact, the optimal portfolio in each group mainly consists of one index. Vermeulen (2013) demonstrates a strong negative relationship between foreign equity holdings and stock market correlations during the recent financial crisis. This stresses the importance of new markets for effective diversification, as advocated by Coerdacier and Guibaud (2011).

Several studies attest to the limited diversification potential of commodities. Cheung and Miu (2010) show that it is only when commodities are bullish that diversifying into their futures is beneficial. Rudolf et al. (1993) investigate the regime-switching behavior of equity–commodity correlation. Using the S&P Goldman Sachs commodity index GSCI from April 1970 to April 1991, they show that correlations rise during stressful periods.⁵ In a similar vein, Daskalaki and Skiadopoulos (2011) consider a portfolio setting where the investor allocates funds between equities, bonds, the risk-free asset and commodities. They consider both an in-sample and an out-of-sample setting and also take the higher moments of the asset returns distribution into account. They also apply various utility functions that describe the preferences of the investor. A rich data set is employed that covers the period between January 1989 and December 2009. Overall, diversification benefits from the inclusion of commodities are only identified during the 2005 – 2008 commodity boom. Their results cover a wide variety of commodities (with the exception of gold) and they are robust to different performance measures and utility functions.

There is also scant evidence that alternative asset classes such as real estate or funds of funds can complement traditional capital asset portfolios – see Mull and Soenen (1997) and Gueyie and Amvella (2006) for further details. In particular, Kroencke and Schindler (2012) show that alternative’s diversification benefits were statistically and economically insignificant during the financial crisis. So the main message from all this research is that international equities, commodities and alternatives all offer reduced diversification potential during stressful markets, i.e. just when it becomes most important. This finding fuels the growing literature on volatility diversification, which we now survey.

Early academic papers which advocate volatility as an effective diversifier studied variance swaps, as in Dash and Moran (2005) and Daigler and Rossi (2006). Hafner and Wallmeier (2008) focus on European variance swaps and Egloff et al. (2010) consider US equity investors – and both find good evidence for the diversification benefits of variance swaps for long equity investors. Unfortunately, trading in variance swaps is not accessible to many investors and

⁵Simple calculations with more recent data still support these findings: for instance, the sample correlation between the daily returns on the S&P 500 stock index and those on the GSCI was only 0.17 when estimated between January 2004 and December 2008 but it rose to 0.55 between January 2009 and June 2013.

demand for volatility diversification has moved to exchange-traded products. As Alexander et al. (2015) observe, the average holding times of volatility futures are so long (e.g. in comparison with equity index futures) that players must include investors and hedgers, not only traders and speculators. However, these products have very different trading and statistical characteristics to variance swaps, as shown by Alexander and Korovilas (2013) and others.

Relatively few previous studies have examined equity diversification using volatility futures and most of these have employed an ex-post analysis based on ad-hoc allocations with samples that focus on the turbulent period covering the credit and banking crises from 2007 to 2009. In fact, we assert that previous studies on diversification using volatility futures offer no adequate demonstration of its benefits. Using only ex-post analysis Hill (2013) confirms the recommendation of Whaley (2000) that VIX mid-term futures are useful diversification instruments for long-term investors. Other studies, like Szado (2009) and Stanescu and Turnaru (2012), simply apply ad-hoc allocations to volatility and other asset classes and examine how such allocations have performed ex-post. Guobuzaitė and Martellini (2012) verify that mid-term futures are more suitable for diversification than short-term futures, this time for European markets. Warren (2012) analyzes a base portfolio which includes US equity, fixed income and real estate exposures, finding that only a *short* position in the prompt VIX futures enhances the Sharpe ratio. Again, the empirical design is limited to an in-sample analysis but the data set covers a wider period than many previous studies.

The first paper to apply any optimization method in this context was Brière et al. (2010). They find diversification benefits for long-equity investors under minimum-variance optimization, but only based on an in-sample analysis with data ending in 2008. Applying a similar methodology, Brière et al. (2012) consider an European equity investor who has the choice of investing in VIX or VSTOXX futures. The optimal portfolio is determined by minimizing the modified conditional Value-at-Risk which takes higher-order moments into account, and volatility-diversified portfolios are found to have significantly lower risk and higher returns than the equity-only portfolio. The authors do not employ an out-of-sample analysis and the sample ends in 2010. Chen et al. (2011) use a mean–variance approach to add VIX futures to four base Fama and French (1992, 1993) US stock portfolios. Again, only an in-sample analysis

is presented and the sample ends in 2008.

More recent work is presented by Hancock (2013), who uses four different hedging methodologies to determine the number of short-term VIX futures contracts to add to an S&P 500 portfolio. However, the results are highly sensitive to the portfolio strategy used and require a careful choice of the appropriate optimization tool. Also, her study focuses on hedging equity risk with volatility futures, rather than portfolio diversification. Although Stanton (2011) describes long-equity investors as being implicitly short in volatility, and thus considers long volatility as a hedge rather than a diversifier, hedging equity with its own futures is more effective and less costly than buying volatility futures.

In summary, our paper fills an important gap in the volatility-diversification literature. It is relevant for mutual funds, pension funds and other long-term investors who cannot trade variance swaps, and who seek new sources of diversification especially during sharp bear markets when other types of diversification fail. We are the first to use a rigorous ex-ante optimization framework and our results are not restricted to the mean-variance approach which has dominated almost all prior research. We also introduce a novel theoretical concept – the optimal diversification threshold – which has potential for other applications and further development.

3 Ex-Ante Optimal Volatility Diversification

The three optimization criteria in increasing order of complexity are: the minimum-variance criterion; its extension to the mean-variance framework introduced by Markowitz (1952); and a further extension to incorporate the effect of personal views on expected returns, as advocated by Black and Litterman (1992).

3.1 Minimum-Variance Optimality

Denote by \mathbf{w} and Σ the portfolio weights and the excess returns covariance matrix. The minimum-variance criterion is to choose \mathbf{w} to minimize $\mathbf{w}'\Sigma\mathbf{w}$ with $\mathbf{1}'\mathbf{w} = 1$. The solution is

$$\mathbf{w}^* = \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}'\Sigma^{-1}\mathbf{1}}. \tag{1}$$

From this it follows immediately that, with only two assets (in our case, equity s and volatility v) a long-position in both is optimal if, and only if their returns correlation ρ is less than both relative volatilities σ_s/σ_v and σ_v/σ_s . Since a relative volatility is always positive, such an investor would always choose to diversify when $\rho < 0$, which it is in our case.

However, the simplicity of this result does not extend to more than two assets. It is well known that minimum-variance weights are positive on all k assets if, and only if all column sums of Σ^{-1} are positive, i.e.

$$\mathbf{1}\Sigma^{-1}\mathbf{e}_i > 0, \tag{2}$$

where \mathbf{e}_i denotes the standard i th basis vector with i th element 1 and zeros elsewhere, $i \in \{1, \dots, k\}$. However, there is no simple criterion for (2) to hold when $k \geq 3$. Indeed deriving general conditions for positively-weighted minimum-variance portfolios is a complex problem that has challenged many researchers since Green (1986).

3.2 Mean-Variance Optimality

The risk-free asset does not affect the minimum-variance problem because it has zero variance. But this is not the case for a mean-variance investor. Here the allocation of funds between risky assets and a risk-free asset may be considered in two stages: (i) find all mean-variance efficient combinations of risky assets; and (ii) find the optimal mix of one of these portfolios with the risk-free asset. The portfolio chosen from the stage (i) analysis is the tangency portfolio that when connected with the risk-free asset yields a linear efficient frontier with slope equal to the maximized Sharpe ratio; the optimal choice along this frontier in stage (ii) is the portfolio which maximizes the investor's expected utility.

In stage (i) the problem is simply one of maximizing a Sharpe ratio and we need not consider specific risk preferences, indeed because we are only concerned with the convex frontier in $\{\text{expected return, standard deviation}\}$ space. But the solution is more complex for stage (ii). As in sub-section 3.1 we first consider the two-dimensional case, i.e. a long equity-only investor who seeks to diversify with volatility. To allocate between equity, volatility and the discount

bond this investor maximizes his certainty equivalent to obtain:

$$\mathbf{w}^{\text{mv}} = \gamma^{-1} \boldsymbol{\Sigma}^{-1} \mathbf{q}, \quad (3)$$

where $\mathbf{w}^{\text{mv}} = (w_s^{\text{mv}}, w_v^{\text{mv}})'$ are allocations to the risky assets,⁶ γ denotes the investor's coefficient of risk aversion; $\mathbf{q} = (q_s, q_v)'$ is the vector of expected excess returns; and $\boldsymbol{\Sigma}$ denotes their covariance matrix with elements σ_s^2 , σ_v^2 and σ_{sv} . The solution may be written as:

$$w_s^{\text{mv}} = \gamma^{-1} |\boldsymbol{\Sigma}|^{-1} (\sigma_v^2 q_s - \sigma_{sv} q_v), \quad w_v^{\text{mv}} = \gamma^{-1} |\boldsymbol{\Sigma}|^{-1} (\sigma_s^2 q_v - \sigma_{sv} q_s), \quad (4)$$

where $|\boldsymbol{\Sigma}| = (\sigma_s^2 \sigma_v^2 - \sigma_{sv}^2)$ is the determinant of $\boldsymbol{\Sigma}$. Since $\gamma > 0$, $|\boldsymbol{\Sigma}| > 0$ and $\sigma_{sv} < 0$, requiring both $w_s^{\text{mv}} > 0$ and $w_v^{\text{mv}} > 0$ simultaneously results in the following condition for the expected return on volatility:

$$q_v > \max \left[\frac{\sigma_v^2}{\sigma_{sv}} q_s, \frac{\sigma_{sv}}{\sigma_s^2} q_s \right]. \quad (5)$$

We call the right-hand side of (5) the *optimal diversification threshold* for a long-equity investor, based on the mean-variance criterion. This is the expected return on volatility that would justify an investor with a long equity position to add a long position in volatility for the purpose of diversification. Notice that, since the covariance $\sigma_{sv} < 0$, this threshold is negative iff $q_s > 0$, which means that an investor might perceive including volatility to be optimal even if he expects negative returns on volatility. Further, the threshold does not depend on the investor's risk aversion γ .

Including additional risky assets such as bonds or commodities in the portfolio increases the dimension of the covariance matrix and makes the derivation of an explicit formula for the corresponding diversification threshold very difficult. It is not possible to generalize (4) for $k \geq 3$, but the general condition for all weights being simultaneously positive still does not

⁶These are not constrained to sum to 1, the allocation being completed with the residual invested in the risk-free asset

depend on γ , because

$$\mathbf{w}^{\text{mv}} = \gamma^{-1} \Sigma^{-1} \mathbf{q} > \mathbf{0} \iff \Sigma^{-1} \mathbf{q} > \mathbf{0}. \quad (6)$$

3.3 Personal Views and Equilibrium Returns

Black and Litterman (1992) argue that investors should not base their decisions entirely on historical data, or more generally on their own personal views about expected returns. Any long-term investment should also take account of equilibrium expected returns, and if investor's personal views are highly uncertain, then the resulting allocations will be more stable than mean-variance allocations, because they will not deviate too far from the equilibrium returns.

In this section we consider the perspective of a long-equity (or equity-bond) investor who uses the classical Black-Litterman model, as interpreted and implemented by He and Litterman (1999). This assumes that asset returns follow a normal distribution and an expression for the posterior distribution of returns is obtained by conjugating two normal distributions, one for the investor's personal views and the other for the equilibrium returns.⁷

In a capital asset pricing model market equilibrium, all investors hold the market portfolio \mathbf{w}^{M} and share the same beliefs about expected returns, encapsulated by a normal prior distribution with mean $\boldsymbol{\mu}^{\text{M}}$ and covariance matrix $\zeta \Sigma$, where Σ is the historical covariance matrix. The parameter ζ is a positive constant representing the uncertainty in the prior distribution for expected returns. Black and Litterman (1992) propose that ζ should be set close to zero, as the investor is more certain about the distribution of expected returns than for returns themselves. Since, under the i.i.d. assumption, the variance of a sample mean is inversely proportional to the sample size $n \in \mathbb{N}$, we follow He and Litterman (1999) and Blamont and Firoozy (2003) and set $\zeta = n^{-1}$.

In addition to prior beliefs, which are based on equilibrium returns, an individual investor holds his own, subjective views about the distribution of expected returns. These views might

⁷Several studies have extended the original Black-Litterman model allowing for other return distributions. See for instance, Martellini and Ziemann (2007), who take preferences about higher moments of asset return distributions into account or Giacometti et al. (2009) who apply t-student and the stable distributions and use alternative risk measures.

be about the distribution of expected returns on individual assets, and/or about certain portfolios of these assets. The views are represented using a matrix $\mathbf{P} \in \mathbb{M}^{l \times k}$, where l denotes the number of personal views about expected returns $\boldsymbol{\mu}$ on the k risky assets. We suppose that views are such that $\mathbf{P}\boldsymbol{\mu}$ follows a normal distribution with mean $\mathbf{q} \in \mathbb{R}^l$ and covariance matrix $\boldsymbol{\Lambda} \in \mathbb{M}^{l \times l}$, which defines the investor's confidence in each view.

Now, blending equilibrium with subjective views yields a posterior normal distribution for expected returns with mean given by the following expression:

$$\boldsymbol{\mu}^{BL} = [(\zeta\boldsymbol{\Sigma})^{-1} + \mathbf{P}'\boldsymbol{\Lambda}^{-1}\mathbf{P}]^{-1} [(\zeta\boldsymbol{\Sigma})^{-1}\boldsymbol{\mu}^M + \mathbf{P}'\boldsymbol{\Lambda}^{-1}\mathbf{q}] \quad (7)$$

and covariance matrix

$$\boldsymbol{\Theta} = [(\zeta\boldsymbol{\Sigma})^{-1} + \mathbf{P}'\boldsymbol{\Lambda}^{-1}\mathbf{P}]^{-1}. \quad (8)$$

Note that the above describes the distribution that a Black-Litterman investor ascribes to *expected* returns. The assets' actual returns are normal with mean $\boldsymbol{\mu}$ and covariance $\bar{\boldsymbol{\Sigma}} = \boldsymbol{\Sigma} + \boldsymbol{\Theta}$.

Let $\mathbf{w}^M = \gamma^{-1}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}^M$ denote the equilibrium portfolio weights. Applying the mean-variance optimizer to the posterior distribution for actual returns, one obtains the following solution for the unconstrained optimal portfolio weights:

$$\mathbf{w}^{BL} = (1 + \zeta)^{-1}(\mathbf{w}^M + \mathbf{P}\boldsymbol{\lambda}), \quad (9)$$

where

$$\boldsymbol{\lambda} = \gamma^{-1}(1 + \zeta)\mathbf{X}^{-1}\mathbf{q} - \mathbf{X}^{-1}\mathbf{P}\boldsymbol{\Sigma}\mathbf{w}^M, \quad \text{with} \quad \mathbf{X} = \mathbf{P}\boldsymbol{\Sigma}\mathbf{P}' + \zeta^{-1}(1 + \zeta)\boldsymbol{\Lambda}. \quad (10)$$

Black and Litterman (1992) and He and Litterman (1999) assume that $\boldsymbol{\Lambda} \in \mathbb{M}^{l \times l}$ is a diagonal matrix. Meucci (2005) relaxed this assumption, suggesting that $\boldsymbol{\Lambda}$ is directly proportional to $\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}'$. We follow Meucci (2005) and set $\boldsymbol{\Lambda} = \eta\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}'$. Thus, the uncertainty in each personal view is proportional to the historical variance, with the same proportionality constant η for each view.⁸ Now the equilibrium returns are obtained via reverse mean-variance optimization, so that $\boldsymbol{\mu}^M = \gamma\boldsymbol{\Sigma}\mathbf{w}^M$.

⁸Note that a more restricting assumption, where η is set equal to ζ , has been used in the implementations of He and Litterman (1999) and Da Silva et al. (2009). We prefer to include η as a free parameter so that we can investigate how the Black-Litterman solution behaves as the investor becomes relatively more or less confident in his own views, i.e. as η decreases or increases, respectively, but ζ remains fixed.

We shall consider two possibilities for the current views of the investor: (a) only one view on the volatility asset, i.e. $l = 1$, and (b) views on all assets in portfolio, i.e. $l = k$. The following theorem gives the optimal diversification threshold for a Black-Litterman investor. That is, we generalize (5) to find the minimum expected return on volatility that will justify a long position when the investor combines personal views with equilibrium expected returns.

Theorem 1. *When a Black-Litterman investor has only one view and it is about the return on volatility, q_v , the optimal portfolio weights in (9) are positive for all assets if, and only if*

$$q_v > (1 + \zeta)^{-1} \mu_v^M. \quad (11)$$

Theorem 1 shows that an investor who has personal views only on volatility asset, not on other risky assets, should allocate positively to volatility whenever his expected return is greater than the equilibrium return (scaled for the uncertainty about the prior that is captured by the parameter ζ).⁹ In this case he may choose to diversify more or less frequently than a mean-variance investor, depending on his views about volatility.

Our next result considers the case where the investor has views on all assets (i.e. $l = k$):

Theorem 2. *When a Black-Litterman investor has asset-specific views on all assets, then positive allocations to all assets are guaranteed if, and only if,*

$$\Sigma^{-1} \mathbf{q} > -\zeta^{-1} \eta \gamma \mathbf{w}^M. \quad (12)$$

For the two-asset equity-volatility portfolio the condition (12) can be simplified to

$$q_v > \max \left[\frac{\sigma_{sv}}{\sigma_s^2} q_s, \frac{\sigma_v^2}{\sigma_{sv}} q_s + a \right], \quad \text{where } a := \zeta^{-1} \eta \gamma \sigma_{sv}^{-1} |\Sigma|. \quad (13)$$

The parameter a must be negative because $\sigma_{sv} < 0$ and $|\Sigma| > 0$. Thus, the diversification threshold for the mean-variance investor in (5) is greater than or equal to (13). It follows that a Black-Litterman long-equity investor with views on *both* equity and volatility will always

⁹However, the diversification threshold is independent of η , and would also be independent of ζ under the modification of the BL model suggested by P ezier (2007), which argues that, since $\Theta = \zeta \Sigma$ in the absence of any personal views, we should set $\mu^M = \gamma(1 + \zeta) \Sigma \mathbf{w}^M$ rather than $\mu^M = \gamma \Sigma \mathbf{w}^M$, so that in (7) μ^M should be replaced by $(1 + \zeta) \mu^M$ and (9) becomes simply $\mathbf{w}^{\text{BL}} = \mathbf{w}^M + \mathbf{P} \lambda$. See P ezier (2007) for further details. The factor $(1 + \zeta)^{-1}$ would not appear in (9) and consequently nor in (5), so that diversification would be optimal simply when the expected return on volatility exceeds its equilibrium return.

diversify into volatility at least as often as a mean-variance investor.

In the case of three risky assets there is no analytic solution such as (13). In the special case $\eta = \zeta$, (12) reduces to $\Sigma^{-1}\mathbf{q} > -\gamma\mathbf{w}^M$, and with only two assets we have $a = \gamma\sigma_{sv}^{-1}|\Sigma|$ in (13). But these expressions cannot be further simplified without assumptions on the signs and sizes of correlations. In practice we only know that the equity-volatility correlation is negative, but other risky assets have correlations which can vary considerably over time. For instance, between January 2006 and April 2015 the 1-year rolling correlation between the bonds fund AGG and the VIX ETN VXX ranged from -0.14 to 0.46 .

3.4 Hypotheses

Given the lack of closed form solutions for the optimal diversification threshold when $k \geq 3$ we end this section by formulating some hypotheses which we shall test in our empirical study.

First we use our insights for the two-asset cases to formulate two hypotheses about the relative diversification frequencies of different investors for a portfolio containing equity, bonds and another risky asset.

H_1 : *A minimum-variance investor holding no short positions in equity or bonds will choose to diversify with volatility more frequently than a mean-variance investor using the same covariance forecasts.*

H_2 : *A Black-Litterman investor holding a long equity position with personal views on both, equity and volatility assets, will diversify into volatility more frequently than a mean-variance investor using the same covariance forecasts.*

Most of the literature advocating volatility diversification through exchange-traded products has only presented ex-post results.¹⁰ The few ex-ante studies based on proper optimization have focused on the period of the banking crisis, and soon after, when the success of volatility diversification is not surprising. Several works omit to perform an out-of-sample performance analysis.¹¹ Our empirical study makes a clear distinction between the ex-ante perceived benefits

¹⁰See Hill (2013), Szado (2009), Warren (2012) and Guobuzaitė and Martellini (2012) for example.

¹¹See Brière et al. (2010), Brière et al. (2012), Chen et al. (2011) for example.

of volatility diversification and its eventual success. We verify previous findings in the literature but our sample period from 2006 to 2015 is much longer than previous studies. This leads to our third hypothesis:

H₃: Volatility diversification for a US (or European) investor, using minimum-variance, mean-variance or Black-Litterman optimization, who is long in equity (or equity and bonds), was only optimal during the few months surrounding the 2008 banking crisis.

This hypothesis has far-reaching economic implications for investors, providers of volatility products and exchanges that list them.

4 Empirical Study

We shall analyze the following portfolios:

- P₁: US equity and bonds only;
- P₂: US equity, bonds and short-term volatility futures;
- P₃: US equity, bonds and mid-term volatility futures;
- P₄: US equity, bonds and commodities;
- P₅: European equity, bonds and short-term volatility futures.

To explore our hypotheses thoroughly we compute ex-ante optimal allocations under a wide variety of conditions, and then we monitor the performance of all ex-ante optimal portfolios using a meticulous out-of-sample methodology. In particular, we ensure that all our data represent *investable* returns and we take actual transaction costs into account by using closing bid and ask prices at each rebalancing point. To avoid complex data-crunching, for replicating constant-maturity investable futures returns we employ equivalent data on an exchange-traded fund, note or i-share when possible. The table below summarizes the somewhat cumbersome acronyms used for each series:

Table 1: Acronyms for Data

	Equity	Bonds	Commodities	Short-Term Volatility	Mid-Term Volatility
US	SPY	AGG	DJP	VXX	VXZ
Euro	SX5EEX	EUAGG	–	VSXX	–

SPY denotes the S&P 500 ETF ‘spider’; SX5EEX is the EURO STOXX 50 ETF; AGG is Barclay’s iShares aggregate US bond funds and EUAGG is the equivalent European fund (ticker: EUN4); DJP is the iPath Bloomberg commodity ETN; VXX and VXZ are the 1-month and 5-month VIX futures tracker ETNs; VSXX is the ETN that tracks 1-month VSTOXX mini-futures.

We employ over 9 years of daily data on the closing, bid and ask prices of SPY, AGG, SX5EEX and EUAGG, i.e. from January 2006 to the end of April 2015. However, volatility ETNs were not issued until 2009 (and 2010 in Europe). So in order to perform an in-sample and out-of-sample empirical analysis over a longer period we use the methodology explained by Galai (1979) to construct synthetic and investable prices of constant-maturity portfolios of volatility futures with 1-month and 5-month maturities.¹² In fact the premiums are very small, so switching to market prices as they become available, instead of using indicative values for the entire period, has a negligible effect on our results. Nevertheless, for convenience, in the following we denote our constant-maturity volatility futures series by these tickers. That is, VXX, VXZ and VSXX refer to the constant-maturity volatility futures portfolios which determine the indicative values of these ETNs. We further assume that each investor has access to a 1-month risk-free asset for financing his investment. Rebalancing is monthly, so for the risk-free rate we use the 1-month US Treasury bill and the 1-month EURIBOR rate.

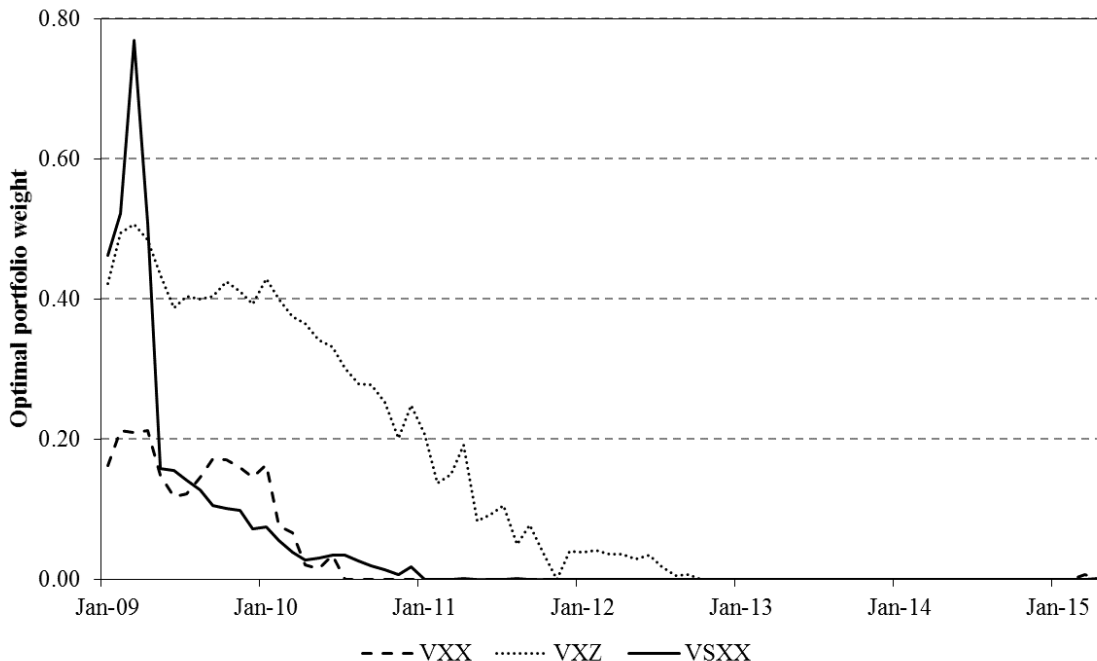
4.1 Ex-Post Results

Similar to the many other studies reviewed in Section 2, our ex-post analysis considers a mean-variance investor who allocates optimally between equity, bonds and volatility, rebalancing his portfolio monthly. We suppose that he uses the last three years of historical data to estimate sample means and covariances at each rebalancing point. Then he uses these estimates for \mathbf{q} and Σ to choose ex-post optimal allocations \mathbf{w} which maximize the Sharpe ratio. Figure

¹²The alternative of rolling over the futures position at or soon before expiry has also been explored but made little difference to our results. In other words, the roll cost which dominates positions on constant-maturity volatility futures trackers can be taken daily, or periodically, without affecting results significantly. The roll cost makes it especially important to produce synthetic volatility futures that are achievable via investment. See Alexander and Korovilas (2013) for instance.

1 displays the resulting optimal allocations to three different volatility assets starting from January 2009.

Figure 1: Ex-post Optimal Allocations to Volatility

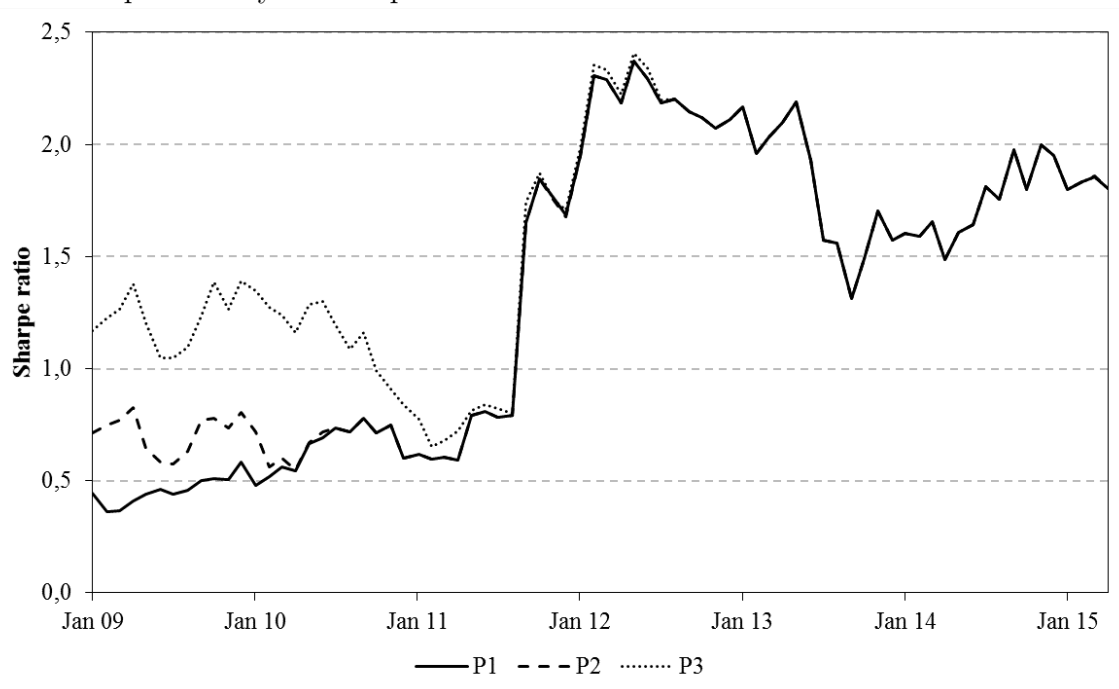


1-month rolling optimal allocations to VXX, VXZ and VSXX, calculated using the mean-variance criterion. Expected returns and covariances are based on the last three years of daily data.

Given that \mathbf{q} and Σ are estimated using the previous three-year period, the optimal allocations from 2009 to 2011 are based on historical data covering the financial crisis. Returns on equity and bonds were negative during this time, and returns on volatility were positive, so it is not surprising that all ex-post optimal portfolios include volatility at the start of our sample. But our data extends beyond the banking crisis. We find that the rapidly-falling prices on volatility derivatives precipitated a reallocation to bonds and equity markets. For instance, the allocation to VXX was positive until June 2010, but since then the optimal portfolio had no exposure to VXX at all. And while the portfolio diversified with the mid-term volatility VXZ has more stable positive allocations, over a longer period (a finding that is consistent with those of Hill (2013), Whaley (2000) and others), using an extended sample demonstrates that the VXZ only remained a significant part of the equity-bond portfolio until the end of 2012. For the European investor it was ex-post optimal to invest in the VSTOXX short-term

volatility futures during the first half of 2009, but from May the weights dropped below 20% and since January 2011 they have been zero. In short, once the data pertaining to the 2008 – 2009 crisis have dropped out of the sample, we find no ex-post justification to add volatility to long equity-bond portfolios. Indeed, our negative results would be even stronger if we used an in-sample period less than three years.

Figure 2: Ex-post Analysis for Optimal Portfolios



1-month rolling Sharpe ratios for the ex-post mean-variance optimal diversified equity-bond portfolio (P1) compared with Sharpe ratios for similarly optimized volatility diversified portfolios based on the VXX (P2) and VXZ (P3) weights depicted in Figure 1. From mid 2012 onward the three lines coincide because the ex-post optimal allocations are to equity and bonds alone.

Turning to the performance that could be achieved via volatility diversification, Figure 2 compares the Sharpe ratios that would have attained for the base, ex-post optimally diversified equity-bond portfolio when diversified using the weights shown in Figure 1. Clearly, adding volatility to an equity-bond portfolio during the banking crisis would have increased the overall portfolio return and decreased its standard deviation. As noted, these results are broadly in line with previous research based on ex-post analysis, although the length of in-sample period does vary between studies.

We conclude that previous research in this area has drawn conclusions that were highly

sample-specific. Indeed, given our extended sample period, our ex-post analysis has shed new light on the apparent benefits of volatility diversification. We have shown that it has never even been ex-post optimal for a US or European long-equity and bond investor to diversify into volatility using exchange-traded products since at least the middle of 2012. It was only ex-post optimal to include volatility during the banking crisis. However, what matters to investors seeking to implement our research is an ex-ante analysis, which we follow for the rest of this study.

4.2 Ex-Ante Methodology and Data Construction

The empirical design of a proper ex-ante analysis is much more complex than the simple ex-post analysis just discussed. Decisions need to be made about: the potential asset classes for investment; the investor's optimization model and his risk aversion; and the way in which he makes forecasts for expected returns and covariances. We shall consider the ex-ante allocations that are made by three types of investors: a minimum-variance investor (I_{minvar}), a mean-variance investor (I_{mv}), and a Black-Litterman investor with specific views on all assets (I_{BL}), setting $\eta = \zeta$.¹³ The two latter investors are assumed to have the widely-used value $\gamma = 4$ for the risk aversion coefficient.¹⁴ Finally, forecasts are based on a historical mean vector for \mathbf{q} and a corresponding covariance matrix Σ , each based on daily data with the same in-sample period of size n ,¹⁵ covering either 1 month, 3 months or 12 months of trading days. When selecting n , there is a trade-off between statistical accuracy (where large n is better) and the ability to reflect current market conditions in markets that have been changing rapidly (where small n is better). We restrict n to be no greater than 12 months so as to generate a very long period for ex-ante results, starting in January 2007.¹⁶

¹³Recall that we also set $\zeta = n^{-1}$. For robustness, later on we shall set $\eta = 1$ so that personal views are held with far greater uncertainty than equilibrium expected returns and optimal allocations tend to be close to the equilibrium portfolio allocation; the opposite is true when $\eta = \zeta$.

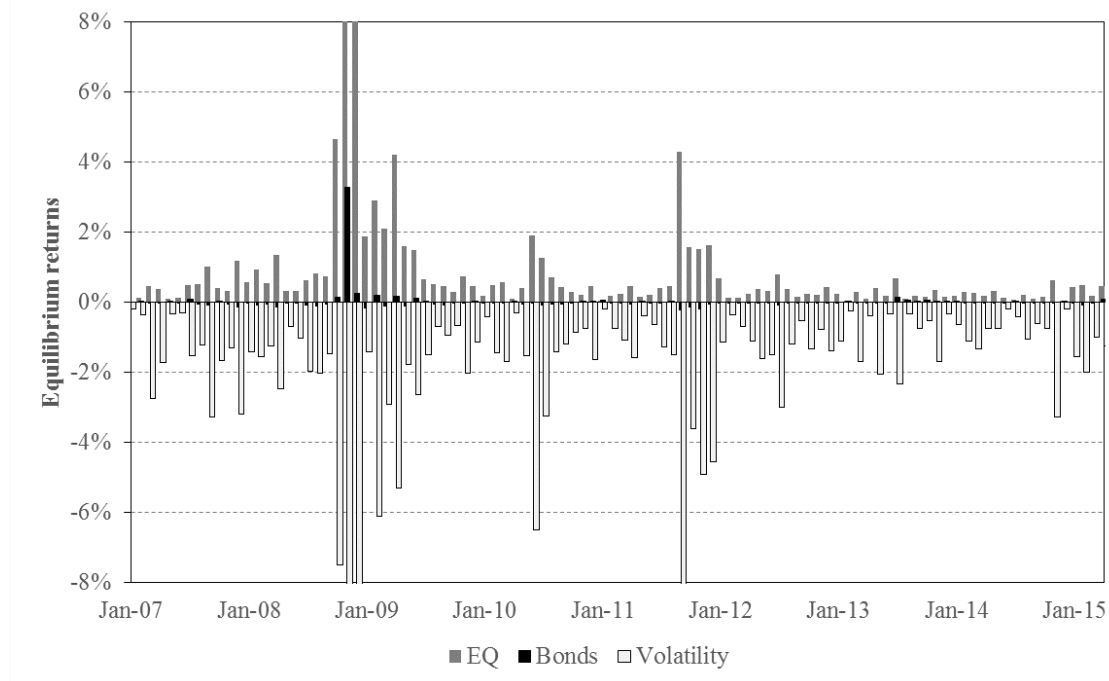
¹⁴An earlier version of this paper considered risk-aversion coefficients $\gamma = 1$ and $\gamma = 4$, to represent more or less risk tolerance in the decision maker, and also reported results when the mean-variance criterion was extended to a more general Sharpe ratio used by a skewness-aware investor. However, this flexibility did not provide much further insight to our results.

¹⁵Practitioners may use forecasts which are entirely subjective, or expected returns are based on a proper asset pricing model, and covariance forecasts based on a conditional volatility model such as GARCH. But this is a vast area of research in its own right and way beyond the scope of this paper.

¹⁶As noted later, using a longer in-sample period adds nothing of any qualitative value to our conclusions. However, these results are also available upon request. So too are results for equilibrium returns on the

To summarize our assumptions: we analyze the optimal allocations for three different investors $\{I_{\text{minivar}}, I_{\text{mv}}, I_{\text{BL}}\}$, with the last two having risk aversion $\gamma = 4$, considering to invest in two different regions (US or Europe) in an ex-ante optimal fashion in five portfolios ($P_1 - P_5$ listed above) when they form their forecasts using historical data with three possible in-sample periods, $n = 1, 3$ or 12 months. In each case re-balancing is monthly and performance is monitored daily. In total, this gives us the opportunity to compare the performance of $3 \times 5 \times 3 = 45$ different daily time series of out-of-sample returns. Finally, the equilibrium weights for the Black-Litterman investor are set to 60% for equity, 40% for bonds according to a widely-used reference portfolio.¹⁷ The equilibrium weights are zero for both volatility and commodities because futures are in zero net supply.

Figure 3: Equilibrium Returns for the US Portfolio



The equilibrium expected returns on equity (SPY), bonds (AGG) and volatility (VXX) between monthly rebalancing points. We assume that the investor has risk aversion $\gamma = 4$ and uses 1 month of daily data for estimations. The equilibrium market portfolio is set to $\mathbf{w}^{\mathbf{M}} = (0.6, 0.4, 0)'$. For clarity we set the range for the vertical scale to $\pm 8\%$. The cut-off values for equity equilibrium returns are 18.63% in November 2008 and 9.06% in December 2008; the cut-off values for equilibrium returns on volatility are -14.20% in November 2008, -11.67% in December 2008 and -11.17% in September 2011.

European portfolio. These have been excluded, for brevity, because they paint a very similar picture to that depicted in Figure 3.

¹⁷Results for alternative weights were explored but made no difference to our conclusions, for any reasonable equilibrium allocation.

Table 2: Statistics for Equilibrium Returns

Correlation	Equity	Bonds	Volatility	Commodities
Equity	1	0.85	-0.91	0.95
Bonds		1	-0.63	0.74
Volatility			1	-0.89
Range				
Min	0.04%	-0.21%	-17.20%	-0.34%
Max	18.63%	3.28%	-0.19%	5.59%

Summary statistics for the equilibrium returns depicted in Figure 3.

Figure 3 depicts the equilibrium returns for the three-asset US portfolio P_2 and Table 2 summarizes their statistics.¹⁸ Equilibrium returns are calculated using a historical covariance matrix based on the last month of daily data.¹⁹ They are positive for equity and negative for volatility, with strong correlations as reported in Table 2. The highest correlation of 0.95 is between equity and commodity equilibrium returns; equity and bonds also have highly correlated equilibrium returns (0.85) and volatility equilibrium returns have a strong negative correlation with all other assets, especially with equity. During the banking crisis equilibrium returns on equity and volatility achieved their largest values in the range, of 18.63% and -17.20% respectively. The equilibrium returns for bonds are very small and slightly negative, typically ranging from -0.21% and 0.26%, except in November 2008 when US interest rates were cut sharply at the onset of the financial crisis resulting in a single outlier of 3.28%.

4.3 How Often is Volatility Diversification Perceived to be Optimal?

This sub-section implements our theoretical results on the ex-ante optimal diversification thresholds and tests our first two hypotheses. We calculate optimal diversification frequencies for portfolios $P_2 - P_5$ as follows: Between 1 January 2007 and 1 April 2015 there are 100 monthly rebalancing points. At each point the investor compares his forecasts with the current

¹⁸The equilibrium returns on commodities, based on the three-asset equity-bond-commodity portfolio P_4 , are typically very small, and are therefore not shown, although their statistical characteristics are included in Table 2. Except for the outlier of 5.59% in November 2008, they typically ranged between -0.34% and 3.88% throughout the sample.

¹⁹Recall that the equilibrium market portfolio has weights vector corresponding to the reference portfolio with 60% equity and 40% bonds and with a zero weight on volatility futures. We report the returns between each rebalancing point only for $\gamma = 4$, since equilibrium returns for other values of γ can easily be scaled up or down from these. The corresponding equilibrium returns for the European portfolio have similar features to US equilibrium returns and are available on request.

optimal diversification threshold. That is, he puts his values for \mathbf{q} and Σ into the relevant diversification condition, i.e. (2) for I_{minivar} , (6) for I_{mv} and (12) for I_{BL} . If the corresponding inequality is fulfilled, the investor holds the ex-ante optimal portfolio with a diversified long position in volatility (or commodities in the case of P_4); otherwise, he holds the equity-bond portfolio, or equity only if preferred under his optimization, until the next rebalancing point. If the portfolio is diversified with volatility (or commodities) at this rebalancing point, then we indicate a 1; otherwise we record 0. Then we sum the indicator over all 100 rebalancing points and divide by 100 to derive the ex-ante optimal diversification frequency.

Table 3 reports the proportion of points when diversification is perceived as optimal out a total of 100 monthly rebalancing points. Results are disaggregated according to the sample size n used to form the forecasts of \mathbf{q} and Σ , i.e. 1 month, 3 months or 12 months. On the left of the table (table 3a) we impose the constraint that optimal weights must be strictly positive on both equity and volatility (or, for P_4 , equity and commodities), but the optimal weight on bonds may be zero. On the right (table 3b) we also allow the weight on equity to be zero, if perceived as optimal.²⁰

The results in Table 3 show that an investor who acts according to objective forecasts and makes decisions in accordance with allocations that are rational, based on a standard portfolio optimization model, would very frequently choose to allocate capital to volatility or commodities. The commodity portfolio P_4 exhibits the lowest diversification frequencies for every optimization model, for all three sample sizes used for parameter estimation. This clearly demonstrates that volatility exposure is perceived as more suitable for equity-bond diversification than commodities, by all three investor types, especially for US investors that apply the minimum-variance criterion. US short-term volatility is included in the optimal portfolio slightly more frequently than European short-term volatility (65% on average, over all investor types and both parts of table 3, versus 60% on average). We find only a minimal difference between diversification into short-term or mid-term volatility futures.

Our results provide very strong evidence in support of hypothesis H_1 , i.e. a minimum-

²⁰So, the optimal diversification frequency may be higher than on the left. To clarify once more, on the right side of Table 3 the investor may select a volatility-bond (or commodities-bond) portfolio or a pure volatility (or commodity) position. On the left, the investor must also regard a long position in equity as ex-ante optimal.

Table 3: Frequency of Optimal Equity-Bond Portfolio Diversification

n	Table 3a			Table 3b		
	1m	3m	12m	1m	3m	12m
<i>P₂: US equity, bonds and short-term volatility</i>						
MinVar	92%	96%	100%	92%	96%	100%
MV	41%	45%	52%	58%	53%	56%
BL	42%	45%	52%	55%	49%	52%
<i>P₃: US equity, bonds and mid-term volatility</i>						
MinVar	91%	98%	100%	91%	98%	100%
MV	44%	43%	57%	57%	51%	62%
BL	44%	43%	57%	55%	48%	59%
<i>P₄: US equity, bonds and commodities</i>						
MinVar	55%	71%	90%	65%	82%	91%
MV	31%	31%	28%	47%	44%	37%
BL	32%	33%	32%	44%	42%	35%
<i>P₅: European equity, bonds and short-term volatility</i>						
MinVar	74%	83%	95%	75%	84%	95%
MV	37%	39%	41%	63%	57%	57%
BL	37%	39%	45%	57%	55%	52%

The proportion of rebalancing periods when the ex-ante optimal weights to volatility resp. commodities and equity assets are greater than 0 and less than one (Table 3a) or when the weights to volatility resp. commodities are allowed to be equal to one (Table 3b). For (\mathbf{q}, Σ) the investor uses the historical estimate based on 1 month, 3 months or 12 months of daily data.

variance investor holding no short positions in equity or bonds does choose to diversify with volatility more frequently than a mean-variance investor using the same covariance forecasts. Moreover, given the identical figures on both sides of the table corresponding to the minimum variance investor in portfolios P_2 , P_3 and (almost) P_5 , there are only few months where I_{minvar} chooses no exposure to equity at all, holding only bonds and volatility.

Our second hypothesis, i.e. that the Black-Litterman investor diversifies more frequently than the mean-variance investor, ceteris paribus, is somewhat supported by the left part of Table 3. Nevertheless, the difference between results for I_{mv} and I_{BL} is relatively small. This is to be expected because, as already noted by Black and Litterman (1992), when $\mathbf{P} = \mathbf{I}$ and $\eta \rightarrow 0$ the Black-Litterman portfolio converges to the views portfolio. So in our setting, when $\eta \rightarrow 0$, (12) approaches (6), whereas when $\eta \rightarrow \infty$ the Black-Litterman (posterior) portfolio converges to the equilibrium (prior) portfolio because the views become less and less

informative. Examining the diversification frequency for a larger value of η is one of the many robustness tests that we report at the end of this Section.

Another observation from Table 3 is that the perceived benefits from diversification may become more frequent when a longer sample period is used for forecasts. Although the difference between the 1-month and 3-month results is quite small, when using the 12-month in-sample period for forecasts, the optimal diversification frequency increases by up to 30% for I_{mv} and I_{BL} investors and by up to 68% for I_{minvar} . Later on we shall test whether the out-of-sample performance improves when forecasts are based on 12 months of historical data, or whether the ex-ante optimally diversified portfolios actually do better when the investor uses a shorter sample for making forecasts, and therefore chooses to diversify less frequently.

4.4 How Much Capital is Allocated (Optimally) to Volatility?

Now we consider the size and the exact timing of the position when volatility diversification is perceived to be optimal. In other words, we report (a) the times when the investor chooses, ex-ante, to take a long position in volatility and (b) how much of his capital is allocated to volatility. Again, there are a myriad of results for different optimizers, look-back periods for forecasting and maturity of the volatility futures, so we must be selective but also representative, so as not to present a biased report. First, having previously found greater differences between I_{minvar} and I_{mv} than between I_{mv} and I_{BL} , from henceforth we shall only compare results for I_{minvar} and I_{mv} . In this sub-section we also focus purely on portfolio P_2 , using only the shortest and longest in-sample periods for computing forecasts, viz. 1 month and 12 months.

Recall from Table 3 that we observe much the highest diversification frequency for I_{minvar} . However, this investor's optimal allocations are much lower than they are for I_{mv} . For instance, I_{mv} allocates, on average over the 100 rebalancing points, 21% of his capital to VXX. By contrast, I_{minvar} 's average allocations to VXX are below 10%. To see this in greater detail, Figures 4 and 5 depict the entire time series of allocations to short-term volatility futures (on the left-hand scale) and the value of VXX (on the right-hand scale) with look-back periods of 1 month and 12 months, respectively, for minimum-variance investors (Figure 4) and mean-

variance investors (Figure 5).²¹

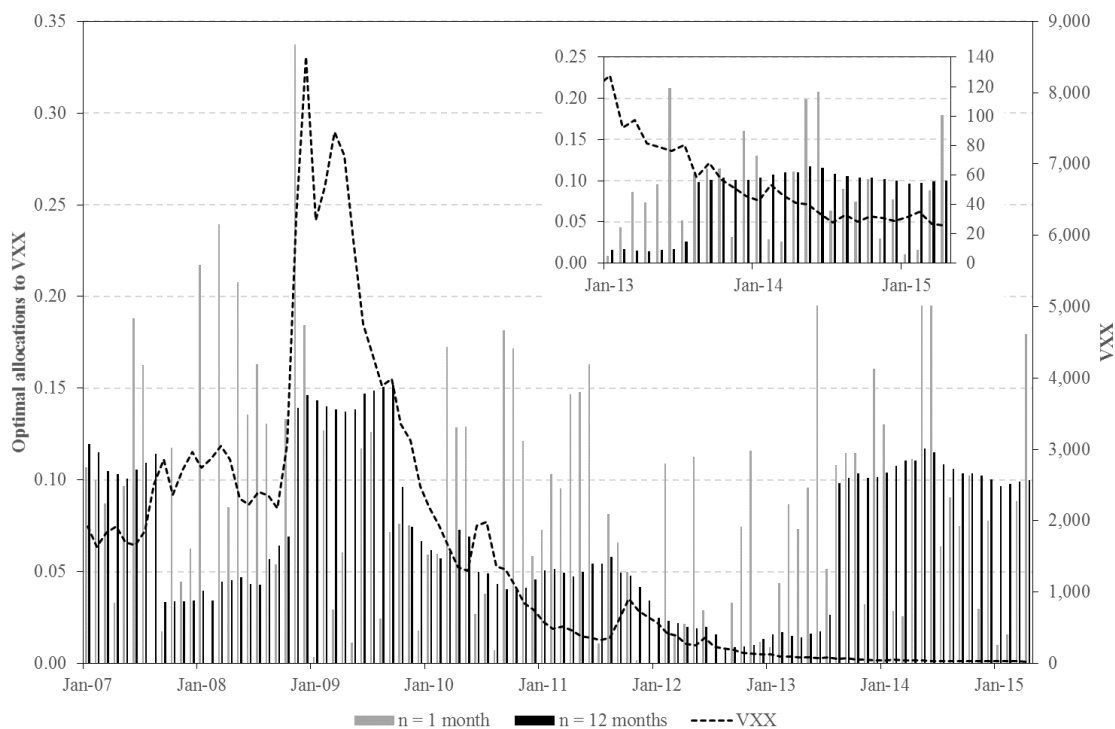
In Figure 4, based on a short look-back period ($n = 1$ month), there are only 8 out of 100 rebalancing months where volatility was not included in a minimum-variance optimal portfolio; and with a longer look-back period ($n = 12$ months) these investors diversify into VXX at every rebalancing point, although their optimal allocations are lower and more stable than they are, when forecasts are based on $n = 1$ month.

To discuss the mean-variance investor's optimal behavior, depicted in Figure 5, it helps to divide the sample into two parts: (a) 1 January 2007 to 31 December 2009, i.e. the three years surrounding the 2008 banking crisis; and (b) 1 January 2010 to 30 April 2015, i.e. the years following the crisis which remained quite turbulent, especially with events associated with the Eurozone debt crisis, starting in 2011. During the first sub-sample investors with a short look-back period ($n = 1$ month) diversify into VXX during only 61% out of 36 periods, with an average allocation of 30%, whereas investors with a longer look-back period ($n = 12$ months) diversify into VXX 81% of the time, however, with a slightly lower average allocation of 24%. This picture changes during the later period, when positive volatility positions are taken less frequently (in 59% periods for $n = 1$ month and 44% periods for $n = 12$ months, out of 64 rebalancing point in total), and much greater allocations tend to be made when $n = 1$ month (average allocation of 15%, compared with 2% when $n = 12$ months).²²

²¹Figure 5 has a different vertical scale to Figure 4. The inset in each figure is just a magnification of the graph for the last part of the sample, which is necessary because the value of the VXX eroded so much over the period, that 3 1-for-4 reverse splits were required. We have adjusted backwards for these splits and hence the magnitude of the right-hand scale in the main figure.

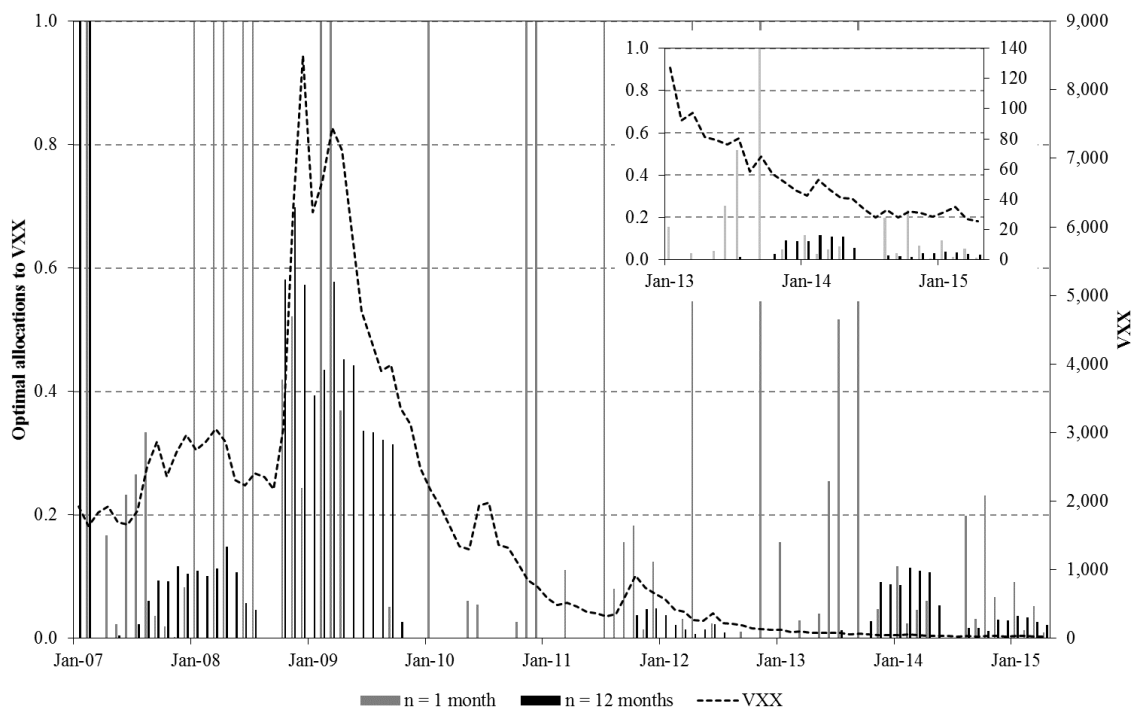
²²From the European point of view, it would be interesting to see the corresponding numbers for the short-term ETN VSXX, in particular during the European crisis in the second part of our sample. For $n = 1$ month, however, the allocation frequency for VSXX is only slightly higher than for VXX (67% vs. 59%) and even lower for $n = 12$ months (39% vs. 44%). The average allocations are, in contrast, significantly higher (24% for $n = 1$ and 14% for $n = 12$). Full results are available upon request.

Figure 4: Ex-Ante Allocation to Short-Term Volatility for Minimum-Variance Investor



Ex-ante allocations to short-term volatility ETN VXX based on minimum-variance optimization for look-back periods $n=1$ month and $n=12$ months vs VXX evolution.

Figure 5: Ex-Ante Allocation to Short-Term Volatility for Mean-Variance Investor



Ex-ante allocations to short-term volatility ETN VXX based on mean-variance optimization for look-back periods $n=1$ month and $n=12$ months vs VXX evolution.

4.5 Out-of-Sample Performance Analysis

The ex-ante diversification results presented above indicate that most rational investors do indeed perceive volatility to be an effective diversifier for a long-equity investor. But is the performance of these diversified portfolios better than the performance of portfolios that are diversified with bonds alone, or diversified with commodities? To answer this question, in this sub-section we compare the out-of-sample performance of the US portfolios: P_1 (US equity and bonds only); P_2 (US equity, bonds and short-term volatility); P_3 (US equity, bonds and mid-term volatility); and P_4 (US equity, bonds and commodities).

Consider first how the investor reallocates between the risky assets in his optimal portfolio. Every month he optimizes his allocations using his forecasts for \mathbf{q} and Σ and compares the optimal weights with the previous month's portfolio. If rebalancing is required, then a cost equal to the product of the bid-ask spread and the absolute change in weights, summed over all assets, is subtracted from the portfolio return. It is important that performance is reported net of transactions costs, because these are relatively high on volatility futures, as noted by Alexander et al. (2015) and others. The optimal portfolio is then held and marked to market each day until the next rebalancing point, when the optimization is repeated. This way we obtain a daily time series representing the out-of-sample performance for each investor, from January 2007 to April 2015.

Again we shall divide the sample into two parts, viz. (a) 1 January 2007 to 31 December 2009; and (b) 1 January 2010 to 30 April 2015; and again we compare results for $n = 1$ month, i.e. investors with a very short look-back period, and $n = 12$ months for investors with a longer look-back period. Figures 6 and 7 depict the evolution of an investment of \$100 invested in each of the US portfolios, based on the minimum-variance criterion (above) and mean-variance criterion (below). On the left we show the evolution of \$100 invested on 1 January 2007, and on the right the growth when \$100 is invested on 1 January 2010. All graphs are plotted on the same vertical scale for ease of comparison. Below each graph we summarize the following performance statistics for each portfolio: the total return (TR) over the sub-sample; the corresponding Sharpe ratio (SR); and the number of rebalancing periods when it

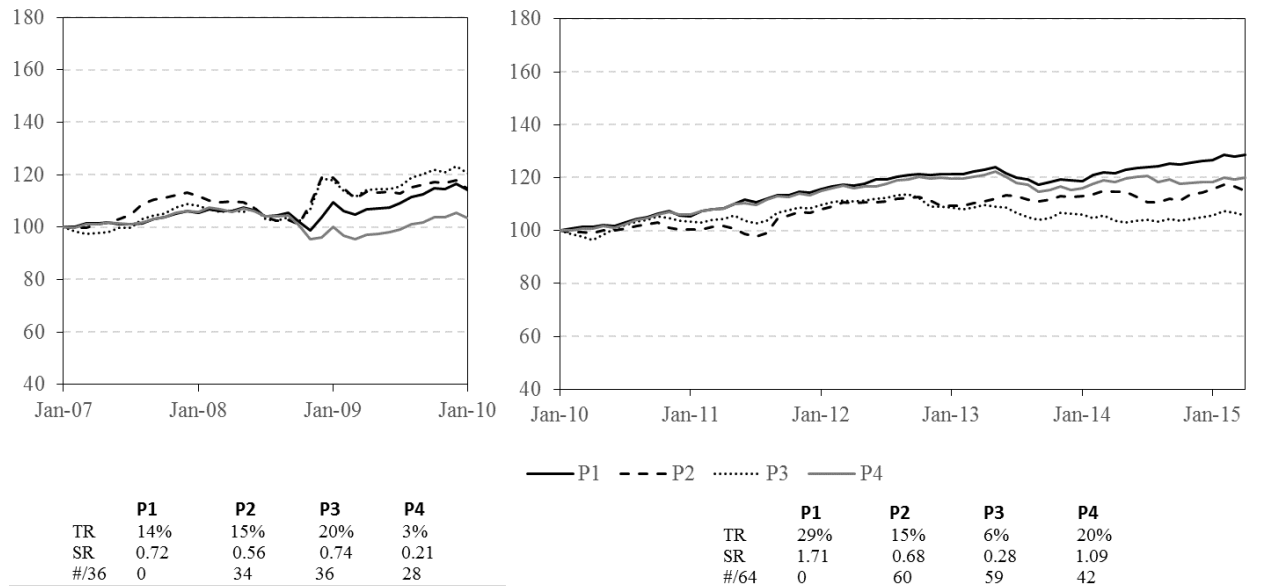
was ex-ante optimal to diversify the long equity-bond position. This is zero by definition for the equity-bond portfolio P_1 .

First consider Figure 6, which sets $n = 1$, and the minimum-variance investors depicted in the top pair of time series plots. By diversifying into mid-term volatility futures during period (a) they could obtain a total return of 20% over three years, with an average annual Sharpe ratio of 0.74, and a volatility position would form part of the diversified portfolio in 36 out of the 36 months. The results are still good for those investing in short-term volatility, having a total return of 15% with a Sharpe ratio of 0.56. By contrast, if investors ignored the possibility for diversification, they would have obtained a lower total return of 14% with a Sharpe ratio of 0.72. The worst performance was that of commodities, with a total return of only 3% over three years. However, during the years since the banking crisis, i.e. period (b) between January 2010 and April 2015, investors will have diversified into short-term volatility in 60 out of 64 months, yet they would only obtain a total return of 15% and a Sharpe ratio of 0.68, compared with a total return of 29% with a Sharpe ratio of 1.71 with no diversification. The addition of commodities (which is optimal in 42 of the 64 months) still damages the performance but not nearly as much as it did during period (a).

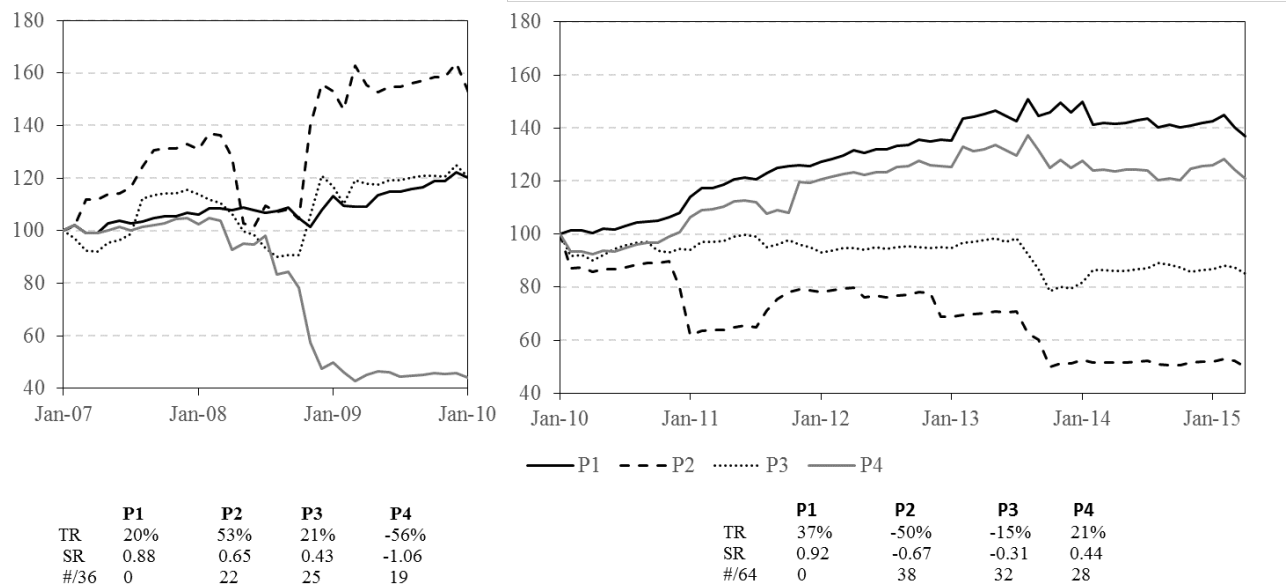
The lower graphs in Figure 6 depict results for the mean-variance investor. During period (a) their highest total return was indeed obtained from a volatility-diversified portfolio. By investing optimally in short-term volatility futures, investors could have achieved a total return of 53% over three years based on the ex-ante optimal mean-variance criterion. The annual Sharpe ratio averages 0.65 over the same period, and the volatility position would have formed part of the diversified portfolio in 22 out of the 36 months. The results are less spectacular but still good for those investing in mid-term volatility, i.e. a total return of 21% with a Sharpe ratio of 0.43. By contrast, if they ignored the possibility for diversification, they would have obtained a lower total return of 20%. However, the volatility of the diversified positions is so high that the Sharpe ratio without diversification is the highest of all, at 0.88. The worst performance was that of P_4 : adding commodities to the equity-bond portfolio would have resulted in losing more than half the portfolio value over the three-year period. During the years since the banking crisis period (right sub-sample) the mean-variance investors with a

Figure 6: Performance of Diversified Portfolios (1-month Look-Back)

Minimum-Variance



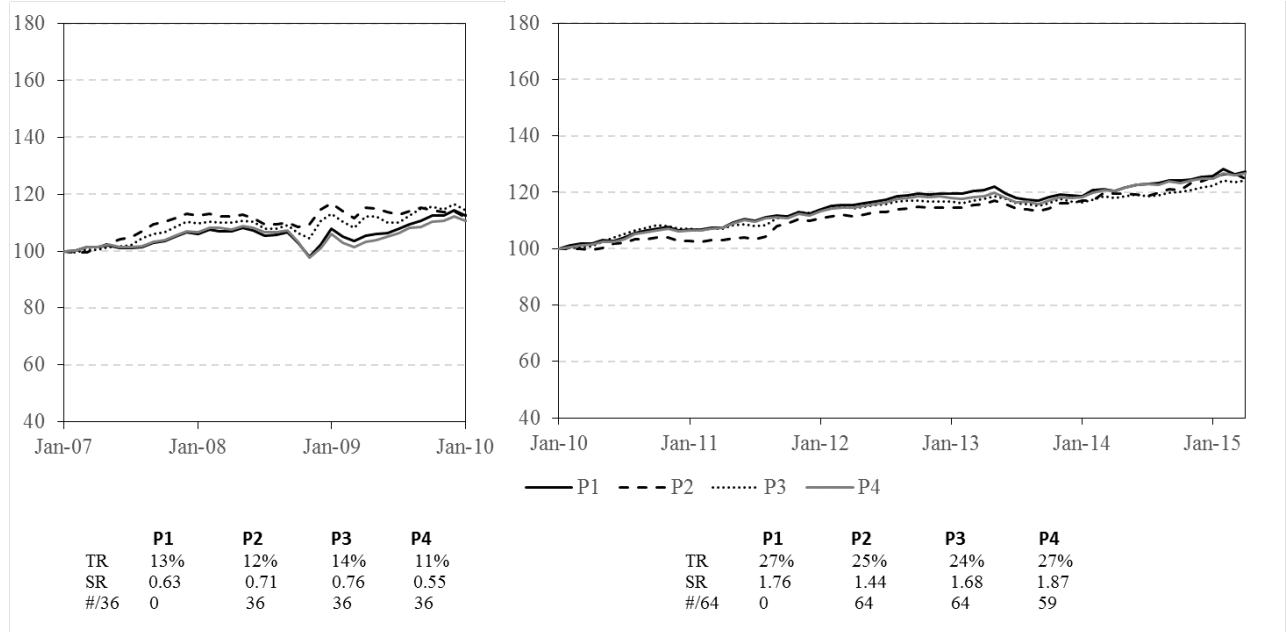
Mean-Variance



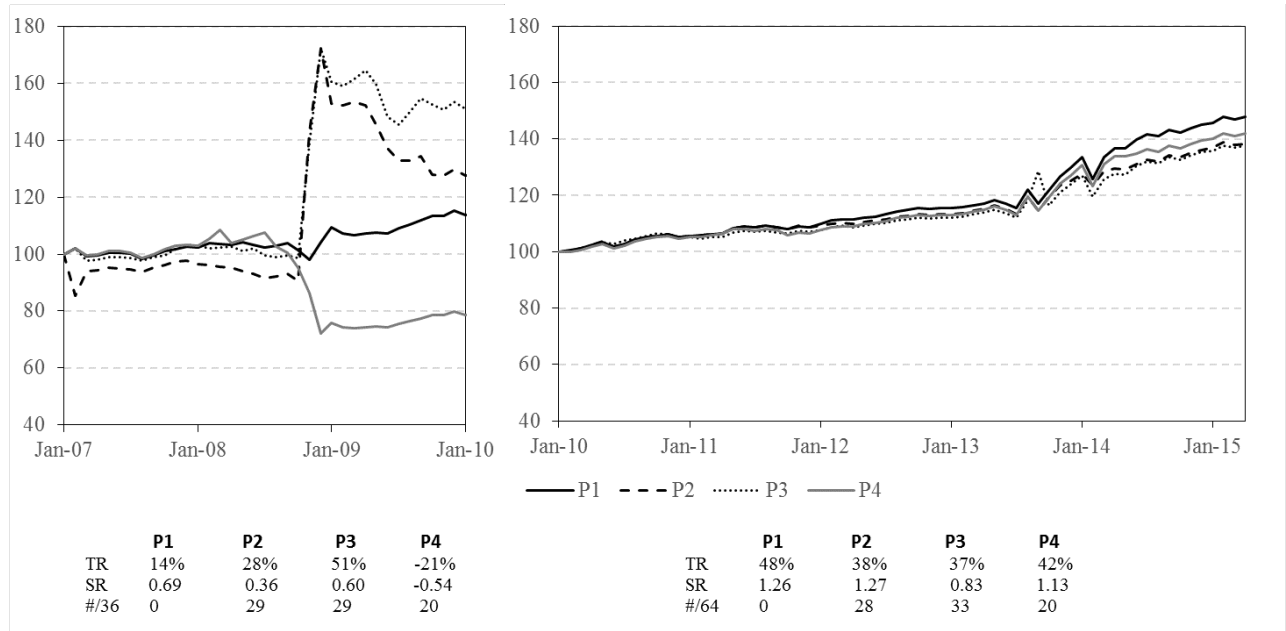
Growth of a theoretical \$100 investment (a) on 1 January 2007 and (b) on 1 January 2010, for a minimum-variance investor (above) and mean-variance investor (below) using a 1-month look-back period to make forecasts. Comparison of the portfolios: P₁ (US equity and bonds only); P₂ (US equity, bonds and short-term volatility); P₃ (US equity, bonds and mid-term volatility); and P₄ (US equity, bonds and commodities). The vertical scale represents the portfolio value in \$. TR is the total return over the sub-sample, SR the annual Sharpe ratio and below this we report the number of rebalancing periods when it was ex-ante optimal to diversify the long equity-bond position.

Figure 7: Performance of Diversified Portfolios (12-month Look-Back)

Minimum-Variance



Mean-Variance



Growth of a theoretical \$100 investment (a) on 1 January 2007 and (b) on 1 January 2010, for a minimum-variance investor (above) and mean-variance investor (below) using a 12-month look-back period to make forecasts. Comparison of the portfolios: P₁ (US equity and bonds only); P₂ (US equity, bonds and short-term volatility); P₃ (US equity, bonds and mid-term volatility); and P₄ (US equity, bonds and commodities). The vertical scale represents the portfolio value in \$. TR is the total return over the sub-sample, SR the annual Sharpe ratio and below this we report the number of rebalancing periods when it was ex-ante optimal to diversify the long equity-bond position.

1-month look-back period would have lost 50% of their investment by diversifying into short-term volatility, and 15% by investing in mid-term volatility. Again, the non-diversified position performed best, with a total return of 37% and an average annual Sharpe ratio of 0.92.

To check robustness of our findings Figure 7 depicts the same two investors, but now using a longer look-back period (i.e. $n = 12$ months). Here the benefits of diversification are even less apparent for the minimum-variance investor (above) except during the crisis period (on the left) when diversifying with volatility does yield marginally higher Sharpe ratios. The period since the crisis is shown on the right. Now the addition of short-term or mid-term volatility makes little difference to the performance of equity-bonds alone because the optimal allocations were very small. Diversification with commodities gives the highest Sharpe ratio (of 1.87) but this is only a marginal improvement on the equity-bond portfolio without any diversification.

The mean-variance investor (below) has a clear advantage by diversifying into volatility, especially mid-term volatility, based on a total return of 51% over the three years. However, the average Sharpe ratio, at 0.69, is highest for the equity-bond portfolio. By contrast, an investor who diversified with commodities would have lost about one-fifth of the portfolio value over three years. During the second sub-sample (right) long equity-bond investors employing mean-variance with a 12-month look-back period for forecasting would have enjoyed spectacular results. On the equity-bond portfolio the total return over the entire period was 48%, net of transactions costs, with monthly rebalancing. The addition of further assets for diversification, whether volatility or commodities, only serves to deteriorate the total return. However, volatility did provide some hedge during the instability surrounding the Eurozone crisis in late 2013 and early 2014 and, as a result, the portfolio with short-term volatility diversification has a Sharpe ratio of 1.27, which is very slightly greater than the non-diversified portfolio's Sharpe ratio (of 1.26), even though the total return was reduced from 48% to 38%.

4.6 Comments on Robustness

Our study has constrained many of the choice variables in order to reduce the quantity of results presented. Despite this, the features and performance of 45 different portfolios have

been discussed. We end this sub-section with comments on some of the results that have been excluded from this paper (more detailed results are available from the authors on request):

1. Risk Aversion : Diversification frequencies for the minimum-variance or mean-variance investors are independent of γ . In the Black-Litterman case the diversification frequency may increase with γ , because the right-hand side of (12) is decreasing in γ . Hence diversification is more likely to occur as the investor becomes more risk-averse. However, empirically, we found little difference when setting γ in a normal range between 1 and 5. Much higher (perhaps unrealistic) levels of risk aversion would need to be assumed to see a significant effect on the diversification frequency.

2. Data and Rebalancing Frequency: We present results based only on daily data with monthly rebalancing. However, the qualitative conclusions are unchanged when we use weekly data, or when we rebalance at a higher frequency (e.g. weekly).

3. Equilibrium Returns: For the Black-Litterman results presented above we have set the equilibrium three-asset portfolio weights to $\mathbf{w}^{\mathbf{M}} = (0.6, 0.4, 0)'$. However, the conclusions are very similar when we use different equilibrium weights. This is not surprising because \mathbf{I}_{BL} is assumed to hold views on all assets. That is, $\mathbf{P} = \mathbf{I}$, and in this case $w_v^{BL} = (1 + \zeta^{-1}(1 + \zeta)\eta)^{-1}\gamma^{-1}\boldsymbol{\Sigma}^{-1}q_v$. Hence, the optimal volatility weight does not depend on the equilibrium weights for equity or bonds.

4. Confidence in Views: Only the results for the case $\eta = \zeta$ are reported in detail, this being the case most commonly assumed in the literature. Setting $\eta = 1$ tends to lower the optimal diversification frequency for the Black-Litterman investor, but often only marginally. This is expected, because then optimal allocation should move closer to the equilibrium portfolio, which only has positions in equities and bonds.

5 Conclusions

The exchange-traded market for volatility is booming. For instance, during 2014 an average of nearly \$4bn was traded every day on VIX futures alone. When we include related exchange-traded notes and funds the total daily trading volume can easily exceed \$10bn. But volatility futures have characteristics quite unlike those of traditional financial markets such as the S&P 500 or commodity futures. First, instead of minutes, the average holding period for a volatility futures is several days, or even weeks for mid-term and longer-term contracts. Given that many futures traders are very short-term speculators, this statistic alone implies that many investors are holding positions in volatility over a fairly long horizon.

Several types of large institutions are interested in promoting volatility trading, including the exchanges that list the products, the data providers of the indices, and the banks that issue the exchange-traded notes as an effective diversifier. Regulators, on the other hand, are wary of allowing purely speculative products to be listed at a time when the spotlight is on the exploitation of the public by financial institutions. Given the high demand, institutions have a vested interest in marketing exchange-traded volatility as an effective diversifier for investors, and not just an instrument for speculators. Whether this claim is true is therefore a very interesting topic for research.

However, to our knowledge, ours is the first paper which tackles this problem in the context of a rigorous and comprehensive, ex-ante academic study. Hardly any studies model the decisions that would be made by asset managers that optimize, ex-ante, their holdings in risky assets using portfolio theory. Most papers perform only an empirical analysis that allocates to volatility using an ad-hoc rule, and all lack a thorough out-of-sample analysis based on ex-ante optimized portfolios. Moreover, much of the published work is confined to a relatively small sample which ends shortly after the banking crisis.

We provide new theoretical results on diversification of equity-bond exposure within three standard optimization frameworks: minimum-variance, mean-variance and Black-Litterman. In each case we derive a general formula for the ex-ante optimal diversification threshold, i.e. the expected return on a risky asset that is sufficient to justify diversification by adding a long

position to a portfolio containing long positions on other risky assets. The condition depends on the parameters of the optimization model that is employed by the investor, including: the covariances between asset returns (all three models); the expected asset returns (the mean-variance and Black-Litterman models); and also on the investor's risk aversion (only the Black-Litterman model).

Our empirical study employs data from January 2006 to April 2015, a longer period than any previous study on the benefits of volatility diversification. We find that diversification of equity-bond portfolios by adding long positions in volatility (or commodity) futures is frequently perceived as optimal, ex-ante, by both US and European investors. Diversification is much more common for a minimum-variance investor than it is for an investor using the mean-variance approach, or its extension to Black-Litterman; but the minimum-variance investor tends to allocate a smaller proportion of capital to volatility (or commodities).

However, none of the optimally-diversified portfolios has out-performed a traditional equity-bond portfolio except during the few months surrounding the collapse of Lehman Brothers in 2008. At other times volatility mean-reverts so quickly that its return becomes negative just at the point when an investor takes his position. Diversification with commodities has been better in recent years, but the traditional long equity-bond mean-variance optimal portfolio still performs best for investments that are longer than a few months.

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Appendix

Proof of Theorem 1.

We prove (11) only for the portfolio with three assets as the proof is similar for the two-asset portfolio. For an investor with only one view on volatility, i.e. $l = 1$, the vectors $\boldsymbol{\lambda}$ and \mathbf{q} each have only one element, i.e. $\boldsymbol{\lambda} = \lambda_v$ and $\mathbf{q} = q_v$ and the matrix of views becomes $\mathbf{P} = [0, 0, 1]$. We also have

$$\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}' = \sigma_v^2, \quad \mathbf{P}\boldsymbol{\mu}^M = \mu_v^M, \quad \boldsymbol{\Lambda} = \eta\sigma_v^2, \quad \mathbf{X} = (\mathbf{1} + \eta + \eta\zeta^{-1})\sigma_v^2 > \mathbf{0}. \quad (14)$$

Using (9), the optimal allocation becomes $\mathbf{w}^{BL} = (1 + \zeta)^{-1}[w_s^M, w_b^M, \lambda_v]'$. Consequently, $w_s^{BL} > 0$, $w_b^{BL} > 0$ for $w_s^M > 0$, $w_b^M > 0$. Requiring $w_v^{BL} > 0$ yields the condition: $\lambda_v > 0$, or equivalently $\gamma\lambda_v > 0$. Substituting (14) in (10) yields

$$\gamma\lambda_v = [(1 + \zeta)q_v - \mu_v^M] (1 + \eta + \eta\zeta^{-1})^{-1}\sigma_v^{-2},$$

so $w_v^{BL} > 0$ if, and only if, $q_v > (1 + \zeta)^{-1}\gamma[w_s^M\sigma_{sv} + w_b^M\sigma_{bv}] = (1 + \zeta)^{-1}\mu_v^M$.

Proof of Theorem 2.

With $\boldsymbol{\Lambda} = \eta\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}'$ and $\mathbf{P} = \mathbf{I}$, we have $\mathbf{X} = x\boldsymbol{\Sigma}$, where $x = 1 + \zeta^{-1}(1 + \zeta)\eta$. Then

$$\begin{aligned} \mathbf{w}^{BL} > \mathbf{0} &\iff \gamma(1 + \zeta)\mathbf{w}^{BL} = [\gamma\mathbf{w}^M + \gamma\boldsymbol{\lambda}] > \mathbf{0}, \\ &\iff \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}^M + (1 + \zeta)x^{-1}\boldsymbol{\Sigma}^{-1}\mathbf{q} - x^{-1}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}^M > \mathbf{0}, \\ &\iff \boldsymbol{\Sigma}^{-1}\mathbf{q} > (1 - x)(1 + \zeta)^{-1}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}^M. \end{aligned}$$

The condition (12) then follows with $(1 - x) = -\zeta^{-1}\eta(1 + \zeta)$ and $\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}^M = \gamma\mathbf{w}^M$.

For the two-asset portfolio with $\mathbf{q} = (q_s, q_v)'$ equation (12) yields

$$\boldsymbol{\Sigma}^{-1}\mathbf{q} = \frac{1}{|\boldsymbol{\Sigma}|} \begin{pmatrix} \sigma_v^2 q_s - \sigma_{sv} q_v \\ \sigma_s^2 q_v - \sigma_{sv} q_s \end{pmatrix} > \begin{pmatrix} -\zeta^{-1}\eta\gamma \\ 0 \end{pmatrix}.$$

After some tedious algebra, we have two inequalities for q_v :

$$q_v > \sigma_{sv}^{-1} (\zeta^{-1} \eta \gamma |\Sigma| + \sigma_v^2 q_s), \quad (15)$$

$$q_v > \frac{\sigma_{sv}}{\sigma_s^2} q_s, \quad (16)$$

so the threshold for a positive allocation to volatility is the maximum of (15) and (16).