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ABSTRACT

It is widely accepted that some of the most accurate Value-at-Risk (VaR) estimates are based on an appropriately specified GARCH process. But when the forecast horizon is greater than the frequency of the GARCH model, such predictions have typically required time-consuming simulations of the aggregated returns distributions. This paper shows that fast, quasi-analytic GARCH VaR calculations can be based on new formulae for the first four moments of aggregated GARCH returns. Our extensive empirical study compares the Cornish–Fisher expansion with the Johnson SU distribution for fitting distributions to analytic moments of normal and Student *t*, symmetric and asymmetric (GJR) GARCH processes to returns data on different financial assets, for the purpose of deriving accurate GARCH VaR forecasts over multiple horizons and significance levels.

1. Introduction

Since the 1996 Amendment to the Basel I Accord, Value-at-Risk (VaR) has become the standard metric for financial risk assessment and reporting, not only in the major banks that must now use VaR forecasts as a basis for their assessment of market risk capital reserves, but also in asset management, hedge funds, mutual funds, pension funds, corporate treasury and indeed in virtually every large institution worldwide that has dealings in the financial markets. As a result the academic literature on forecasting VaR is huge.[1]

Given the widely documented characteristics of financial asset returns, quite complex dynamic models are needed for predicting their distributions. A salient feature is their volatility clustering and generalised autoregressive conditional heteroscedastic (GARCH) models, introduced by Engle (1982), Bollerslev (1986) and Taylor (1986), have proved very successful in capturing this behaviour. Such models can also partially explain why asset returns distributions are skewed and leptokurtic. Some of the most influential academic research concerns the use of GARCH processes to forecast VaR at the aggregate ("top–down") level, rather than utilizing standard ("bottom–up") VaR models for assessing a firm's market risk capital. A path-breaking paper by Berkowitz and O'Brien (2002) utilizes aggregate profit and loss data from six of the world's major banks to demonstrate a very clearly superior accuracy in top–down GARCH-based VaR estimates relative to more traditional, bottom–up VaR estimates.[2]

An $\alpha\%$ n-day VaR estimate is the loss that will not be exceeded, with a $(1-\alpha)\%$ level of confidence, if the portfolio is left unmanaged over a period of n days. When VaR is quoted as a percentage the current portfolio value, it may therefore be derived from the α-quantile of the n-period portfolio return distribution, as:

$$\mathrm{VaR}_{n,\alpha,t} = -\hat{F}^{-1}_{t;t+n}(\alpha), \quad \text{or} \quad \text{equivalently} \quad \text{as} \int_{-\infty}^{-VaR_{n,\alpha,t}} \hat{f}_{t;t+n}(x)dx = \alpha \quad (1)$$

where $\hat{F}^{-1}_{t;t+n}$ is the time t forecast of the inverse distribution function (also called quantile function) for the returns aggregated from time t

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[1] Recent reviews may be found in Alexander (2008), Angelidis and Degiannakis (2009) and Christoffersen (2009).

[2] Perignon and Smith (2010) note that historical simulation is the most widely-used approach, based on a survey of major banks around the world. However, Alexander and Sheedy (2008) demonstrate that historical simulation is highly inaccurate without additional 'filtering'—see, for example, Barone-Adesi, Bourgoin, & Giannopoulos, (1998), and Barone-Adesi, Giannopoulos, & Vosper, (1999), where the historical VaR methodology is augmented with a GARCH model. See also Boudoukh, Richardson, and Whitelaw (1998), for an alternative filtering approach.

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to time $t+n$, and $\hat{f}_{t;t+n}$ is the corresponding density function. For the purpose of VaR estimation the GARCH model is usually estimated using daily data. However, for many empirical applications – and especially for computing regulatory capital to cover market risks in banks, which is typically based on a 10-day VaR estimate derived from daily data – we are often interested in longer horizons. The problem is that for $n>1$ neither $\hat{F}_{t;t+n}$ nor $\hat{F}^{-1}_{t;t+n}$ is known in closed form (in a GARCH context based on daily data) so they are obtained using simulations. This is in accordance with Engle (2003), who argued in his Nobel lecture that simulations are required to predict the quantiles of the returns distribution over a time horizon which is longer than the frequency of the model, when aggregated returns are generated by a GARCH process. But simulations are only asymptotically exact, so it can be very time consuming to simulate aggregated GARCH returns distributions that allow VaR to be forecast with a satisfactory degree of accuracy.

This computational burden has been an impediment to the adoption of VaR models based on GARCH processes in practice. Furthermore, from an academic perspective, it has reduced the scope for extensive out-of-sample tests of GARCH-based VaR forecasts.

Hence, the need arises for an alternative method of resolution that is less time-consuming than simulation, while retaining the great advantage of accurate GARCH modelling. Given the frequent turmoil in financial markets and the pervasive use of the VaR metric throughout the industry, the construction of fast, accurate and easily implemented VaR forecasts is of significant practical and regulatory importance.

In this paper we forecast aggregated returns distributions using analytic formulae for the higher-order conditional moments of GARCH aggregated returns. Given these moments we compare two VaR forecasts obtained using two different methods to approximate the future returns distribution. The importance of our paper is that it provides a means of generating VaR forecasts based on generic, asymmetric GARCH processes without recourse to time-consuming simulations, thus making the GARCH VaR methodology more generally accessible for practical applications. We present some extensive empirical results of VaR forecasts over different risk horizons and at different quantiles, and based on an out-of-sample period that spans almost 13-years and includes the banking crisis as well as the current European crisis. Using the coverage tests of Christoffersen (1998) we demonstrate that, even during crisis periods, very accurate VaR forecasts can be generated for three broad market risk factors: an equity index (S&P 500), a cross-currency pair (Euro/USD), and a discount bond (3-month US Treasury bill). We also draw interesting comparisons between our proposed methodology and two benchmark methodologies: the first based on accurate but time consuming GARCH simulations, the second based on the square root of time rule (SRTR) often applied in practice for quantile (VaR) scaling.

The approach we propose is closely related to a stream of academic research on forecasting VaR. Zangari (1996) was the first to introduce a parametric method for estimating VaR based on higher moments, based on the Cornish–Fisher expansion. The Cornish–Fisher expansion has subsequently been applied to quantile estimation by Amenc, Martellini, and Vaissie (2003), Boudt, Peterson, and Croux (2009), Favre and Galeano (2002), Gueyie and Amvella (2006), Mina and Ulmer (1999), Qian (2006) and Simonato (2011). Amenc et al. (2003), Favre and Galeano (2002) and Gueyie and Amvella (2006) all use the resulting VaR in a portfolio optimisation setting, while Qian (2006) employs it in a risk-budgeting application. Mina and Ulmer (1999) compare four alternative methods for constructing an approximate delta-gamma portfolio distribution, namely Johnson distributions, Cornish–Fisher expansion, Fourier transforms (for the moment generating function) and partial Monte Carlo. Boudt et al. (2009) derive conditional VaR (also called expected tail loss, ETL) as an application of the Edgeworth expansion, while Simonato (2011) considers VaR and ETL derived for Cornish–Fisher and Cram–Charlier expansions and for Johnson distribution approximations, in the context of Merton's (1976) model.

We estimate the first four moments of aggregated returns for GARCH processes using formulae derived for a generic GJR-GARCH(1,1) model by Alexander, Lazar, and Stanescu (2011), formulae which are similar but not identical to those derived by Wong and So (2003). We prefer to use moments of the asymmetric GJR-GARCH(1,1) process with a generic conditional distribution, instead of the AGARCH (p, q) model considered by Wong and So (2003), because the former model encompasses the majority of the GARCH models that are favoured in the financial forecasting literature: see Awartani and Corradi (2005), Asai and McAleer (2008) and many others.

The reminder of this paper is organised as follows: Section 2 presents the theoretical methodology that we shall implement for our empirical results and explains how analytic formulae for the first four moments of aggregated GARCH returns can be used to approximate VaR; Section 3 presents the data and empirical results; and Section 4 concludes.

2. Analytic approximations for GARCH VaR

We construct quasi-analytic VaR estimates that capture the important characteristics of financial asset returns (i.e. their volatility clustering and non-normal distributions) by applying established moment-based approximation methods to analytic formulae for the first four conditional moments of GARCH aggregated returns.

Consider the following generic GJR specification, introduced by Glosten et al. (1993), for the generating process of a continuously compounded portfolio return from time $t-1$ to time t, denoted r_t:

$$r_t=\mu+\varepsilon_t,\quad \varepsilon_t=z_t h_t^{1/2},\quad z_t\sim D(0,1),$$

with[3]

$$h_t=\omega+\alpha\varepsilon^2_{t-1}+\lambda\varepsilon^2_{t-1}I^-_{t-1}+\beta h_{t-1},$$

where $h_t=V(r_t|\Omega_{t-1})$ is the variance of the portfolio return, conditional on the information set $\Omega_{t-1}=\{r_{t-j}, j\geq 1\}$. In this specification the conditional mean equation is as simple as possible, containing just a constant and an error on the right hand side.[4] The conditional variance equation falls into the class of asymmetric GARCH models when $\lambda\neq 0$ because it contains the indicator function I^-_{t-1}, which equals 1 if $\varepsilon_{t-}<0$ and zero otherwise. This way, the response of the conditional volatility h_t to errors differs according to whether the error is positive or negative.

The GARCH error ε_t is a disturbance process and z_t is a sequence of i.i.d. zero mean unit variance random variables with distribution $D(0,1)$. From henceforth we shall allow $D(0,1)$ to be either a standard normal or a standardized Student t distribution, with degrees of freedom v estimated by maximum likelihood along with the other GARCH model parameters. The symmetric GARCH(1,1) model is obtained by setting $\lambda=0$. Thus we shall consider four different possibilities for the GARCH processes that are most appropriate for different types of asset returns, namely the normal and Student t GJR and GARCH(1,1) models.[5]

The steady-state variance $\overline{h}$ of the GARCH model corresponds to setting $\varepsilon_t^2=h_t=\overline{h}$ for all t, in which case the conditional variance equation becomes

$$\overline{h}=\omega(1-\varphi)^{-1},$$

[3] We employ the standard notation α for one of the parameters (reaction) of the GARCH models; this should not be confused with the α notation for the VaR significance level.

[4] If there is significant autocorrelation in returns then one or more lagged returns could also be included as explanatory variables. It is not standard to use exogenous explanatory variables in the conditional mean because the focus of the GARCH model is to capture the clustering in volatility that is present in many time series, especially financial time series such as portfolio returns.

[5] While the methodology we present is set in the context of univariate GARCH modelling, it could be extended to a multivariate setting that can be used for portfolio optimization, but this is beyond the scope of the current paper.

with $\varphi = \alpha + \frac{1}{2}\lambda + \beta$.[6] Note that φ captures the convergence of the conditional variance to the steady-state $\bar{h}$ following a shock to the error term ε_t. The higher the value of φ, the greater the variance-mediated dependence between one-period returns, and the slower the convergence of the conditional variance to its steady-state.

The n-period future aggregated returns generated by the model above are:

$$R_{t,n} = \sum_{s=1}^{n} r_{t+s}.$$

It follows immediately from the linearity of the conditional expectation operator that the conditional mean of these n-period returns is $n\mu$. However, expressions for the variance and higher moments of the n-period return are more complex.

Denote the first four central moments of the n-period returns as $M_{R,n}^{(i)} = E_t[(R_{t,n} - n\mu)^i]$, for $i = 1, \ldots, 4$. Alexander et al. (2011) derive exact formulae for these central moments, and the corresponding standardized moments in a more general set-up. The following special case of these formulae applies for the normal and Student t GJR models that we consider:

$$M_{R,n}^{(2)} = n\bar{h} + (1-\varphi)^{-1}(1-\varphi^n)\left(h_{t+1} - \bar{h}\right); \tag{2}$$

$$M_{R,n}^{(3)} = 3\sum_{s=1}^{n}\sum_{u=1}^{n-s} E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}^2\right); \tag{3}$$

$$\begin{aligned} M_{R,n}^{(4)} = {} & \kappa_z \sum_{s=1}^{n} E_t\left(h_{t+s}^2\right) + 6\sum_{s=1}^{n}\sum_{u=1}^{n-s} E_t\left(\varepsilon_{t+s}^2\varepsilon_{t+s+u}^2\right) \\ & + 12\sum_{s=1}^{n}\sum_{u=1}^{n-s}\sum_{v=1}^{n-s-u} E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}\varepsilon_{t+s+u+v}^2\right), \end{aligned} \tag{4}$$

where κ_z denotes the kurtosis of D, i.e. $\kappa_z = 3$ for the standard normal and $\kappa_z = 3\left(\frac{\nu-2}{\nu-4}\right)$ for the Student t.

The expression (2) for the variance consists of two terms: the first term $n\bar{h}$ corresponds to the variance of n-period returns when one-period returns are independent; the second term captures the increasing variance of n-period returns when there is variance-induced dependence in the series of one-period returns, and this increases with both φ and n. In other words, the variance of n-period returns increases both with the dependence between one-period returns and the aggregation period.

The third and fourth moments (3) and (4) depend on complex (but still closed-form) expressions for the conditional co-dependences of the errors and squared and cubed errors.[7] Because we only consider a symmetric error distribution D, the third moment of the aggregated return depends only on the co-dependence between the error at some time $t + s$ and its square at time $t + s + u$. However, the fourth moment contains more complex error co-dependence terms, in addition to a term which depends on the kurtosis of D. The skewness $T_{R,n}$ and kurtosis $K_{R,n}$ follow immediately, on dividing Eq. (3) by $(M_{R,n}^{(2)})^{-3/2}$ and $(M_{R,n}^{(2)})^{-2}$ respectively.

We now present analytic approximations for Eq. (1) based on the first four moments of aggregated GARCH returns and two quite well-established approximation methods—namely the Cornish–Fisher expansion and the Johnson SU distribution.[8] These distributions, which have never before been applied in the GARCH framework, allow analytic approximations for GARCH VaR to be derived purely in terms of the estimated GARCH model parameters.

The expression for the Cornish–Fisher VaR as a function of the first four standardized moments of the n-day aggregated returns is:

$$\mathrm{VaR}_{n,\alpha,t}^{\mathrm{CF}} = -\left[z_\alpha + \frac{\mathrm{T}_{R,n}}{6}\left(z_\alpha^2 - 1\right) + \frac{\left(\mathrm{K}_{R,n} - 3\right)}{24} z_\alpha\left(z_\alpha^2 - 3\right) - \frac{\mathrm{T}_{R,n}^2}{36} z_\alpha\left(2z_\alpha^2 - 5\right)\right] \sqrt{M_{R,n}^{(2)} - \tilde{M}_{R,n}^{(1)}} \tag{5}$$

where $z_\alpha = \Phi^{-1}(\alpha)$ is the lower α-quantile of the standard normal distribution. The Cornish–Fisher approximation is popular in empirical applications mainly due to its speed and relative simplicity. While expected to perform well in the vicinity of the normal, it has a number of well-documented disadvantages: increasing the order of the approximation does not necessarily improve its error, the resulting quantile function is not necessarily monotonic as a function of the tail probability, and the approximation error increases at extreme quantiles.[9]

The other approximation method we use here, the Johnson SU distribution, differs from the Cornish–Fisher approach in that it is a proper distribution rather than an expansion. A random variable x is said to follow a Johnson SU distribution if[10]:

$$x = \xi + \lambda \sinh\left(\frac{z - \gamma}{\delta}\right) \tag{6}$$

where z is a standard normal variable. Tuenter (2001) developed a very fast algorithm for the estimation of the four parameters δ, γ, λ and ξ. Specifically, using Tuenter's (2001) algorithm, we are matching the first four conditional moments of the n-period aggregated GARCH returns (detailed above) to the corresponding moments of a Johnson SU distribution. Although flexible, the main disadvantage of this approach is that a Johnson SU distribution is not guaranteed to exist for any set of mean, variance, skewness and (positive) excess kurtosis. Using Eq. (6), one can immediately write the expression for the Johnson SU VaR as:

$$\mathrm{VaR}_{n,\alpha,t}^{\mathrm{JSU}} = -\lambda \sinh\left(\frac{z_\alpha - \gamma}{\delta}\right) - \xi. \tag{7}$$

3. Empirical methodology and results

The performance of our proposed VaR methodology is tested using equity index (S&P 500), foreign exchange (Euro/USD) and interest rate (3-month Treasury bill) daily data. These three series represent three major market risk types (equity, foreign exchange and interest rate risk, respectively) and within each class they represent the most important risk factors in terms of volumes of exposures. The three data sets used in this application were obtained from Datastream and each comprise over 20 years of daily data, from 1st January 1990 to 31st October 2012.[11] Fig. 1 plots the daily log returns for the equity and exchange rate data and the daily changes in the interest rate.[12]

Table 1 presents the sample statistics of the empirical unconditional distribution of returns. In accordance with stylized facts on daily financial returns, the mean of every series is not statistically different from zero and the unconditional volatility is highest for equity and lowest for interest rates. The skewness is negative and low (in absolute value) but significant for all three series, so that extreme negative returns are more likely than extreme positive returns of the same

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[6] Of course, this expression for the steady-state variance only holds when $\varphi \in (0, 1)$; otherwise it is not defined. Also, for asymmetric distributions D(0, 1) replace $\frac{1}{2}$ by the distribution function evaluated at zero.

[7] Expressions for these are given in Appendix 1 of this paper.

[8] See Cornish and Fisher (1937) and Fisher and Cornish (1960) for pioneering results, Hill and Davis (1968) for a generalization, and Wallace (1958) and Jaschke (2002) for discussions of the properties of the Cornish–Fisher approximation. The leptokurtic SU distribution was proposed by Johnson (1949); see also Bowman and Shenton (1983).

[9] See Jaschke (2002) and Wallace (1958).

[10] Here we follow the notation of Tuenter (2001) for the four parameters of the Johnson SU distribution. However, parameters λ and γ of the distribution should not be confused with the GJR-GARCH parameter λ or the constant γ used in Section 2.

[11] The Euro was introduced only in 1999, so prior to this the ECU/USD exchange rate is used.

[12] First differences in fixed maturity interest rates are the equivalent of log returns on corresponding bonds.

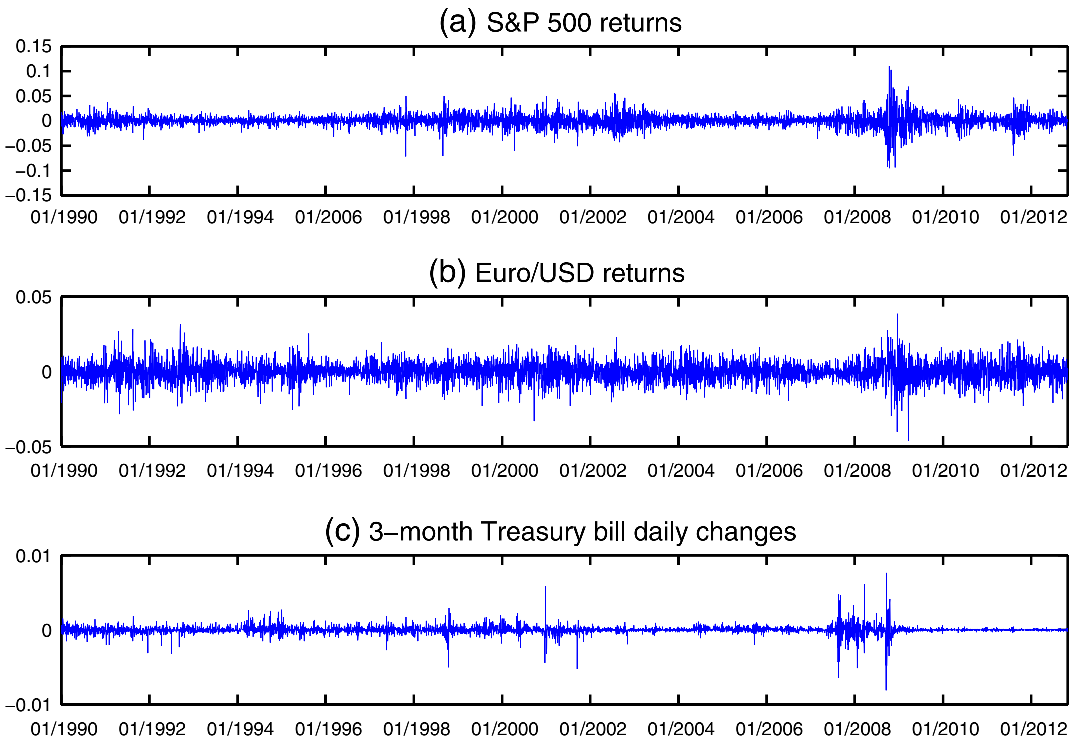

Fig. 1. Returns. The equity and exchange rate daily returns are computed as the first differences in the log of the S&P 500 index values and Euro/USD exchange rates, respectively. The interest rate daily changes are computed as first differences in interest rate values. All returns (daily changes) are computed over the period 1st January 1990 to 31st October 2012.

magnitude, while the excess kurtosis is always positive and highly significant, suggesting that the unconditional distributions of the series have more probability mass in the tails than the normal distribution.

We notice that the interest rate sample exhibits the most significant departures from normality, while the Euro/USD series is the closest to normality amongst the three we analyse.[13]

Four different GARCH models, namely the baseline GARCH(1,1) and the asymmetric GJR, each with normal and Student *t* error distributions, are estimated for each of the three time series.[14] The estimation is conducted in a rolling window format, where a window of ten years of daily data (window size approximately 2500 observations) is rolled daily for almost thirteen years. The resulting time series of model parameters are subsequently used to estimate the first four conditional moments of aggregated returns based on the analytic formulae from Section 2, from 3rd January 2000 to 31st October 2012, for three time horizons: $n = 5, 10, 20$ working days, respectively. For the symmetric models – the normal and Student *t* GARCH(1,1) – the skewness is zero by construction. However, the asymmetric specifications – the normal and Student *t* GJR – lead to non-zero skewness estimates. All four models yield positive excess kurtosis for all horizons and all time series.

We evaluate the accuracy of the proposed VaR estimates over 5, 10 and 20-day risk horizons[15] using the now standard coverage tests of Christoffersen (1998).[16] We combine the four different GARCH models with two approximation methods, the Johnson SU distribution and the Cornish–Fisher expansion, and derive the VaR estimates for each GARCH model, and for each approximation method, and for $\alpha = 10\%$, 5%, 1% and 0.1%.

Tables 2–4 summarize the results of the likelihood ratio (LR) tests for the unconditional coverage, independence and conditional coverage for log returns (or, in the case of Treasury bill rates, absolute changes) aggregated over horizons of $n = 5, 10$ and 20 working days.

3.1. S&P 500

The results in Table 2 show that the model that performs best across all horizons, significance levels and approximation methods is the normal GJR, incurring the smallest number of rejections (and sometimes only marginal rejections) in the coverage tests. The Student *t* GJR also performs very well, especially when coupled with

Table 1
Summary statistics. The summary statistics are of the equity and exchange rate daily log returns, and of the daily changes in interest rates from 1st January 1990 to 31st October 2012. Asterisks denote significance at 10% (*), 5% (**) and 1% (***). The standard error of the sample mean is equal to the (sample) standard deviation, divided by the square root of the sample size. The standard errors are approximately $(6/T)^{1/2}$ and $(24/T)^{1/2}$ for the sample skewness and excess kurtosis, respectively, where T is the sample size. We used 252 risk days per year to annualize the standard deviation into volatility. ADF is the Augmented Dickey Fuller test statistic. ARCH test is Engle's test for ARCH effects.

	S&P 500	Euro/USD	3-Month Treasury bill
Mean	0.0233%	−0.0002%	−0.0013%
Maximum	10.96%	3.84%	0.76%
Minimum	−9.47%	−4.62%	−0.81%
Volatility	18.32%	10.03%	0.84%
Skewness	−0.2295***	−0.0915***	−0.62662***
Excess kurtosis	8.8602***	2.4270***	41.28672***
ADF	−81.9684***	−75.4894***	−15.0652***
ARCH test	262.8835***	35.2867***	282.6141***

[13] We also note that all series are stationary (ADF test statistics are highly significant) and exhibit ARCH effects (highly significant ARCH test statistics).

[14] Based on the BIC and AIK information criteria, an AR(3) model was used to remove the autocorrelation in the data for the 3-month Treasury bill sample, while for the S&P 500 sample an AR(2) suffices to remove all autocorrelation in the returns; in what follows, estimation and testing are based on the residuals from these regressions for the two samples. No autocorrelation was found in the foreign exchange data.

[15] To avoid using overlapping observations, as this would violate the independence assumption for the indicator process in the unconditional coverage test, we use only every *n*-th set of parameter/moments estimates, where *n* is either 5, 10 or 20 working days.

[16] These tests are described in Appendix 2. See also Kupiec (1995) for earlier related results and Sarma, Thomas, and Shah (2003) for a critical discussion of the methodology as well as an empirical implementation.

Table 2
Coverage tests for the S&P 500 index. Christoffersen's (1998) likelihood ratio tests for correct conditional coverage for the S&P 500 returns at horizons $n = 5$, 10 and 20 working days. Rejections of the null – of correct coverage – are marked with (*), (**) and (***) for the 10%, 5% and 1% significance levels, respectively. Empty entries indicate that no exceedances were recorded.

Signif. level	Coverage test	Cornish–Fisher VaR				Johnson SU VaR			
		Normal	Student *t*	Normal	Student *t*	Normal	Student *t*	Normal	Student *t*
		GARCH(1,1)	GARCH(1,1)	GJR	GJR	GARCH(1,1)	GARCH(1,1)	GJR	GJR
$n = 5$									
0.1%	LRuc	7.6708***	1.7251	4.3566**	0.1431	7.6708***	1.7251	4.3566**	4.3566**
	LRind	0.0000	0.0000	0.0000	0.0006	0.0000	0.0000	0.0000	0.0000
	LRcc	7.6708**	1.7251	4.3566	0.1437	7.6708**	1.7251	4.3566	4.3566
1%	LRuc	7.7331***	3.4618*	2.3614	0.2478	7.7331***	6.1596**	2.3614	0.7341
	LRind	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	LRcc	7.7331**	3.4618	2.3614	0.2478	7.7331**	6.1596**	2.3614	0.7341
5%	LRuc	6.7466***	9.3901***	1.7027	2.1626	6.7466***	9.3901***	1.7027	3.2338*
	LRind	0.5809	0.2479	2.0985	1.9634	0.5809	0.2479	2.0985	1.4762
	LRcc	7.3274**	9.6380***	3.8012	4.1261	7.3274**	9.6380***	3.8012	4.7100*
10%	LRuc	3.1629*	8.8539***	0.8362	3.6102*	3.1629*	5.6746**	0.8362	3.6102*
	LRind	1.2284	1.3423	0.8800	0.6485	1.2284	1.3054	0.8800	0.6485
	LRcc	4.3913	10.1963***	1.7161	4.2587	4.3913	6.9800**	1.7161	4.2587
$n = 10$									
0.1%	LRuc	7.8768***	3.8454**	3.8454**	3.8454**	12.5717***	0.8666	3.8454**	3.8454**
	LRind	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	LRcc	7.8768**	3.8454	3.8454	3.8454	12.5717***	0.8666	3.8454	3.8454
1%	LRuc	8.7881***	6.6546***	3.1021*	1.7471	6.6200**	6.6546***	3.1021*	3.1021*
	LRind	1.1116	1.4504	2.3509	0.0000	0.0000	1.4504	2.3509	2.3509
	LRcc	9.8997***	8.1050**	5.4530*	1.7471	6.6200**	8.1050**	5.4530*	5.4530*
5%	LRuc	3.8456**	3.8456**	1.1088	1.6508	2.2499	3.8456**	1.1088	1.6508
	LRind	0.0093	0.0093	0.0978	0.1796	0.1149	0.0093	0.0978	0.1796
	LRcc	3.8549	3.8549	1.2066	1.8304	2.3647	3.8549	1.2066	1.8304
10%	LRuc	1.0334	3.5063*	0.0571	0.7083	0.0840	2.3525	0.0571	0.7083
	LRind	0.5372	0.3846	0.5126	0.0151	0.0343	0.1671	0.5126	0.0151
	LRcc	1.5707	3.8908	0.5697	0.7233	0.1183	2.5195	0.5697	0.7233
$n = 20$									
0.1%	LRuc	1.9277	–	–	–	1.9277	–	–	–
	LRind	0.0022	–	–	–	0.0022	–	–	–
	LRcc	1.9300	–	–	–	1.9300	–	–	–
1%	LRuc	2.3893	0.0660	0.3090	0.3090	2.3893	0.8817	0.3090	0.0660
	LRind	0.0000	0.0000	0.0023	0.0022	0.0000	0.0000	0.0023	0.0000
	LRcc	2.3893	0.0660	0.3113	0.3112	2.3893	0.8817	0.3113	0.0660
5%	LRuc	12.6694***	12.6694***	3.4470*	5.9849**	12.6694***	12.6694***	3.4470*	5.9849**
	LRind	0.1530	0.1530	0.4494	0.0932	0.1530	0.1530	0.4494	0.0932
	LRcc	12.8224***	12.8224***	3.8964	6.0781**	12.8224***	12.8224***	3.8964	6.0781**
10%	LRuc	7.3698***	8.6167***	5.1363**	7.3698***	7.3698***	8.6167***	5.1363**	7.3698***
	LRind	1.1768	1.5816	1.9375	2.6874	1.1768	1.5816	1.9375	2.6874
	LRcc	8.5466**	10.1983***	7.0738**	10.0572***	8.5466**	10.1983***	7.0738**	10.0572***

the Cornish–Fisher approximation. Also, none of the models is rejected in the independence test for this sample, across all horizons and significance levels. For the 5- and 10-day horizons, we notice that while there are inter-model differences in terms of the test results obtained for different GARCH specifications, the results obtained by combining the same GARCH model with different approximation methods are either very similar (for the normal models) or slightly better with the Johnson SU approximation in most cases. Actually, the only GARCH model which yields better results when coupled with the Cornish–Fisher approximation than with the Johnson SU is the Student *t* GJR. At the 20-day horizon, results are similar across all GARCH models and approximation methods, with good performance at the lower significance levels, but rejections in the coverage tests for higher significance levels. Bearing in mind that these are out-of-sample results we can argue that our methodology is indeed accurate.

3.2. Euro/USD

The Euro/USD sample displays the least significant non-normality features, and the results in Table 3 are even better for this sample. Again, none of the normal models are rejected in the independence test, across all horizons and significance levels, while the Student *t* models are only rejected in the independence test for the 10% 10-day VaR. Overall, the normal models perform slightly better than the Student *t* models, and the Johnson SU distribution is the marginally better approximation of the two.

3.3. 3-Month Treasury bill

Interest rates tend to remain stable for a period of time and then move in discrete jumps. Hence, the 3-month Treasury bill sample is the one exhibiting the most pronounced non-normalities. Despite this, the results for this sample in Table 4 indicate that our methodology still performs well; however, out of the three samples we analyse, this is the only one for which the models incur a number of rejections in the independence tests.[17] For the 20-day horizon we find that the normal GARCH(1,1) and normal GJR produce no rejections in the coverage tests across all significance levels and approximation methods; for the 5- and 10-day horizons, no model performs perfectly.

[17] We also note that for this sample the degrees of freedom parameter for the Student *t* models is constrained ($\nu = 6$) in order to ensure that the kurtosis exists and is finite.

Table 3
Coverage tests for the Euro/USD exchange rate. Christoffersen's (1998) likelihood ratio tests for correct conditional coverage for the Euro/dollar returns at horizons $n = 5$, 10 and 20 working days. Rejections of the null – of correct coverage – are marked with (*), (**) and (***) for the 10%, 5% and 1% significance levels, respectively. Empty entries indicate that no exceedances were recorded.

Signif. level	Coverage test	Cornish–Fisher VaR				Johnson SU VaR			
		Normal	Student *t*	Normal	Student *t*	Normal	Student *t*	Normal	Student *t*
		GARCH(1,1)	GARCH(1,1)	GJR	GJR	GARCH(1,1)	GARCH(1,1)	GJR	GJR
n = 5									
0.1%	LRuc	–	–	–	–	–	–	–	–
	LRind	–	–	–	–	–	–	–	–
	LRcc	–	–	–	–	–	–	–	–
1%	LRuc	4.5692**	4.5692**	4.5692**	4.5692**	4.5692**	4.5692**	4.5692**	4.5692**
	LRind	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	LRcc	4.5692	4.5692	4.5692	4.5692	4.5692	4.5692	4.5692	4.5692
5%	LRuc	0.0113	0.0795	0.0051	0.0113	0.0113	0.0795	0.0051	0.0795
	LRind	0.0288	0.6334	0.9535	0.7844	0.0288	0.6334	0.9535	0.6334
	LRcc	0.0401	0.7129	0.9586	0.7957	0.0401	0.7129	0.9586	0.7129
10%	LRuc	0.1680	3.1629*	0.1680	3.6102*	0.1680	1.9909	0.1680	1.9909
	LRind	0.0517	0.1421	0.0517	0.4024	0.0517	0.4195	0.0517	0.4195
	LRcc	0.2196	3.3050	0.2196	4.0126	0.2196	2.4104	0.2196	2.4104
n = 10									
0.1%	LRuc	–	–	–	–	–	–	–	–
	LRind	–	–	–	–	–	–	–	–
	LRcc	–	–	–	–	–	–	–	–
1%	LRuc	0.7331	0.0342	0.7331	0.0342	0.7331	0.1279	0.7331	0.0342
	LRind	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	LRcc	0.7331	0.0342	0.7331	0.0342	0.7331	0.1279	0.7331	0.0342
5%	LRuc	0.9080	0.4683	0.9080	0.4683	0.9080	0.4683	0.9080	0.4683
	LRind	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	LRcc	0.9080	0.4683	0.9080	0.4683	0.9080	0.4683	0.9080	0.4683
10%	LRuc	0.0030	1.0334	0.0030	0.7083	0.0030	0.7083	0.0030	0.7083
	LRind	1.0025	6.7323***	2.3680	5.2553**	1.0025	5.2553**	2.3680	5.2553**
	LRcc	1.0056	7.7657**	2.3710	5.9636*	1.0056	5.9636*	2.3710	5.9636*
n = 20									
0.1%	LRuc	–	–	–	–	–	–	–	–
	LRind	–	–	–	–	–	–	–	–
	LRcc	–	–	–	–	–	–	–	–
1%	LRuc	0.8817	0.8817	0.8817	0.8817	0.8817	2.3893	0.8817	2.3893
	LRind	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	LRcc	0.8817	0.8817	0.8817	0.8817	0.8817	2.3893	0.8817	2.3893
5%	LRuc	4.6426**	4.6426**	4.6426**	4.6426**	4.6426**	4.6426**	4.6426**	4.6426**
	LRind	0.1219	0.1219	0.1219	0.1219	0.1219	0.1219	0.1219	0.1219
	LRcc	4.7645*	4.7645*	4.7645*	4.7645*	4.7645*	4.7645*	4.7645*	4.7645*
10%	LRuc	1.2059	2.4784	1.2059	2.4784	1.2059	2.4784	1.2059	2.4784
	LRind	0.8031	1.2469	0.8031	1.2469	0.8031	1.2469	0.8031	1.2469
	LRcc	2.0090	3.7253	2.0090	3.7253	2.0090	3.7253	2.0090	3.7253

3.4. Comparisons with other GARCH VaR methodologies

3.4.1. Comparisons with the square root of time rule

In a GARCH model, one-step ahead returns have distribution D (i.e. the conditional error distribution), hence, in the GARCH set-up introduced in Section 2, 1-day VaR, for any significance level α, is given by:

$$\mathrm{VaR}_{1;\alpha;t} = -\left(\sqrt{h_{t+1}}D^{-1}(\alpha) + \mu\right)$$

where D^{-1} is the inverse distribution (or quantile) function of the error distribution D and h_{t+1} is the one-step ahead variance forecast. The square root of time rule (SRTR), often used in practice for the scaling of quantiles (or VaR estimates),[18] implies:

$$\mathrm{VaR}_{h;\alpha;t} = \sqrt{h}\,\mathrm{VaR}_{1;\alpha;t}.$$

The results obtained in the Christoffersen tests for the VaR estimates based on the normal GARCH(1,1) model and the square root of time rule are reported in Table 5. The sample used is the same as before, ranging from 1st January 1990 to 31st October 2012, where a window of ten years of daily data (window size approximately 2500 observations) is rolled daily for almost thirteen years. By comparing the results from Table 5 with the corresponding results from Tables 2–4 (columns 3 and 7), we notice that the superiority of our proposed methodology is most apparent for the 3-month Treasury Bill rate sample. Indeed, by comparing the results in Table 4, columns 3 and 7, we notice that our proposed methodology is only once rejected in the tests for unconditional coverage (for the 10-day 0.1% VaR), while for the same sample the VaR estimates based on the SRTR incur six additional rejections in the unconditional coverage tests (Table 5, column 5).

3.4.2. Comparisons with simulated GARCH VaR

Since the true distribution function is not known in analytical form, the GARCH VaR for horizons longer than the frequency of the model is obtained via simulations, which are (asymptotically) accurate. Hence, for a large enough number of simulations we expect the simulated VaR to be a good estimate of the true quantile, and thus a good measure to compare our quasi-analytical, faster VaR estimates. Therefore we selected 150 days (and corresponding estimation windows) from a low volatility period (January to August

[18] For an interesting and detailed examination of the accuracy of the SRTR for quantile scaling, see Wang, Yeh, and Cheng (2011).

Table 4
Coverage tests for the 3-month Treasury bill. Christoffersen's (1998) likelihood ratio tests for correct conditional coverage for the changes in the 3-month Treasury bill rate at horizons $n = 5$, 10 and 20 working days. Rejections of the null – of correct coverage – are marked with (*), (**) and (***) for the 10%, 5% and 1% significance levels, respectively. Empty entries indicate that no exceedances were recorded.

		Cornish–Fisher VaR				Johnson SU VaR			
Signif level	Coverage test	Normal	Student *t*	Normal	Student *t*	Normal	Student *t*	Normal	Student *t*
		GARCH(1,1)	GARCH(1,1)	GJR	GJR	GARCH(1,1)	GARCH(1,1)	GJR	GJR
$n = 5$									
0.1%	LRuc	1.7251	1.7251	1.7251	1.7251	1.7251	4.3566	1.7251	1.7251
	LRind	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	LRcc	1.7251	1.7251	1.7251	1.7251	1.7251	4.3566	1.7251	1.7251
1%	LRuc	0.0152	0.0152	0.4675	0.0724	0.0152	0.2478	0.4675	0.0152
	LRind	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	LRcc	0.0152	0.0152	0.4675	0.0724	0.0152	0.2478	0.4675	0.0152
5%	LRuc	1.2948	1.7027	0.0626	0.6397	1.2948	1.7027	0.0795	1.2948
	LRind	13.1609***	12.3616***	5.7291**	14.8694***	13.1609***	12.3616***	7.0142***	13.1609***
	LRcc	14.4556***	14.0643***	5.7917*	15.5092***	14.4556***	14.0643***	7.0937**	14.4556***
10%	LRuc	2.4494	0.0007	6.4537**	0.0107	2.8970*	0.5746	6.4537**	1.0492
	LRind	13.6205***	17.6564***	7.6601***	15.7522***	11.5822***	20.3874***	7.6601***	19.1786***
	LRcc	16.0699***	17.6570***	14.1138***	15.7629***	14.4791***	20.9620***	14.1138***	20.2278***
$n = 10$									
0.1%	LRuc	3.8454**	3.8454**	0.8666	3.8454**	3.8454**	3.8454**	3.8454**	3.8454**
	LRind	0.0000	0.0000	0.0011	0.0000	0.0000	0.0000	0.0000	0.0000
	LRcc	3.8454	3.8454	0.8677	3.8454	3.8454	3.8454	3.8454	3.8454
1%	LRuc	0.7331	1.7471	0.6261	0.7331	0.7331	1.7471	0.0342	0.7331
	LRind	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	LRcc	0.7331	1.7471	0.6261	0.7331	0.7331	1.7471	0.0342	0.7331
5%	LRuc	1.5079	0.4683	4.4102**	0.4683	1.5079	0.4683	3.2423*	0.4683
	LRind	0.5992	2.3845	0.0000	2.3845	0.5992	2.3845	0.0000	2.3845
	LRcc	2.1071	2.8527	4.4102	2.8527	2.1071	2.8527	3.2423	2.8527
10%	LRuc	0.3746	0.2376	2.4937	0.0030	1.9084	0.3746	4.7821	0.3746
	LRind	9.0732***	19.1907***	10.7328***	19.7387***	6.5143**	20.1835***	6.6066**	12.3606***
	LRcc	9.4478***	19.4284***	13.2265***	19.7418***	8.4227**	20.5581***	11.3887***	12.7352***
$n = 20$									
0.1%	LRuc	1.9277	1.9277	–	1.9277	1.9277	1.9277	1.9277	1.9277
	LRind	0.0022	0.0022	–	0.0022	0.0022	0.0022	0.0022	0.0022
	LRcc	1.9300	1.9300	–	1.9300	1.9300	1.9300	1.9300	1.9300
1%	LRuc	0.3090	2.3893	0.3090	2.3893	2.3893	4.4145**	0.0660	2.3893
	LRind	0.0022	0.0000	0.0022	0.0000	0.0000	0.0000	0.0000	0.0000
	LRcc	0.3112	2.3893	0.3112	2.3893	2.3893	4.4145	0.0660	2.3893
5%	LRuc	0.0606	0.3450	0.7394	0.0115	0.0606	0.3450	0.2259	0.0606
	LRind	0.4756	2.4559	0.0000	0.7818	0.4756	2.4559	1.1876	0.4756
	LRcc	0.5362	2.8009	0.7394	0.7933	0.5362	2.8009	1.4135	0.5362
10%	LRuc	0.1280	0.7312	0.0244	0.7312	0.4753	0.3701	2.3542	0.1280
	LRind	2.2221	5.4431**	0.1550	5.4431**	2.2221	6.5278**	0.1052	7.7423***
	LRcc	2.3502	6.1743**	0.1794	6.1743**	2.6975	6.8979**	2.4593	7.8703**

2006) and 150 days (and corresponding estimation windows) from a high volatility period (August 2008 to March 2009) for which we also computed the simulated GARCH VaR, for a 5-day horizon, using simulations from the normal GARCH(1,1). Fig. 2 plots the percentage differences between the Cornish–Fisher VaR and the corresponding simulated VaR for the normal GARCH(1,1) model. The percentage differences are computed as (Cornish–Fisher VaR − Simulated VaR) / Simulated VaR, where the VaRs are obtained for four different significance levels: $\alpha = 0.1\%$, 1%, 5% and 10%. The corresponding average percentage differences are 0.50%, 0.06%, 0.08% and −0.23%, for the four significance levels, respectively.

To give an example of the speed of our methodology relative to Monte Carlo simulation, on a PC with Intel i5-650 (dual core) and 4Gb RAM using Excel 2010 VBA, the time recorded for computing the Student *t* GJR-GARCH VaR estimates for a 10-day horizon using our quasi-analytic methodology was only 0.254 s. By comparison, to compute the 10-day VaR based on 10,000 Monte Carlo simulation took 13 s. That is, our methodology is at least 50 times faster than the standard simulation-based VaR estimation and the computation time ratio would be greater for VaR computations over longer horizons. Moreover, 10,000 is typically regarded as the minimum number of simulations to be used for a passable degree of accuracy, and the time would be extrapolated linearly as the number of simulations increases. Therefore, the methodology we propose appears to yield results which are very similar to the asymptotically accurate but time consuming method based on GARCH simulations.

4. Conclusions

This paper demonstrates empirically that quasi-analytic GARCH VaR forecasts can be accurately constructed using analytic formulae for higher moments of aggregated GARCH returns. The great accuracy of our results for all time-horizons and significance levels that we considered shows that time-consuming simulations are no longer needed for GARCH VaR forecasting.

Based on their occurrence in the related literature and on the feasibility of obtaining fast, analytic formulae for the associated VaRs, we selected two alternative moment-based approximation methods, namely the Cornish–Fisher expansion and the Johnson SU distribution. A comprehensive testing exercise used very long time series of financial returns representing three major sources of market risk, namely equity (S&P 500), foreign exchange (Euro/USD) and interest

Table 5
Coverage tests for normal GARCH(1,1) VaR measures using the square root of time rule. Christoffersen's (1998) likelihood ratio tests for correct conditional coverage for the returns on the S&P 500 and Euro/USD exchange rates and for the changes in the 3-month Treasury bill rate at horizons $n = 5$, 10 and 20 working days. Rejections of the null – of correct coverage – are marked with (*), (**) and (***) for the 10%, 5% and 1% significance levels, respectively. Empty entries indicate that no exceedances were recorded.

Signif. level	Coverage test	S&P 500	Euro/USD	3-Month Treasury bill
n = 5				
0.1%	LRuc	15.7214***	–	7.6708***
	LRind	0.0000	–	0.0000
	LRcc	15.7214***	–	7.6708**
1%	LRuc	7.7331***	4.5833**	0.7341
	LRind	0.0000	0.0000	0.0000
	LRcc	7.7331**	4.5833	0.7341
5%	LRuc	4.4997**	0.0095	0.0795
	LRind	1.0646	2.6570	13.7492***
	LRcc	5.5643*	2.6664	13.8286***
10%	LRuc	1.0801	0.4226	6.4537**
	LRind	0.3677	0.0195	7.6601***
	LRcc	1.4478	0.4421	14.1138***
n = 10				
0.1%	LRuc	17.8223***	–	12.5938***
	LRind	0.0000	–	0.0000
	LRcc	17.8223***	–	12.5938***
1%	LRuc	8.7881***	0.7331	1.7471
	LRind	1.1116	0.0000	0.0000
	LRcc	9.8997***	0.7331	1.7471
5%	LRuc	3.0230*	0.9080	2.2810
	LRind	0.0470	0.0000	0.8305
	LRcc	3.0700	0.9080	3.1115
10%	LRuc	0.0950	0.2376	5.7346**
	LRind	0.0343	1.2780	1.8584
	LRcc	0.1293	1.5156	7.5929**
n = 20				
0.1%	LRuc	1.9277	–	11.7470***
	LRind	0.0022	–	0.0000
	LRcc	1.9300	–	11.7470***
1%	LRuc	2.3893	2.3893	4.4145**
	LRind	0.0000	0.0000	0.0000
	LRcc	2.3893	2.3893	4.4145
5%	LRuc	7.4656***	4.6426**	0.2259
	LRind	0.0163	0.1219	1.1876
	LRcc	7.4819**	4.7645*	1.4135
10%	LRuc	3.2680*	0.7312	1.5523
	LRind	1.1270	0.0942	1.3238
	LRcc	4.3950	0.8254	2.8761

rate risk (3-month Treasury bill). We applied the Cornish–Fisher expansion and the Johnson SU distribution to four different GARCH specifications (normal and Student *t* GARCH(1,1) and GJR) to test and compare eight alternative VaR models. VaR was estimated at four significance levels (0.1%, 1%, 5%, and 10%) and for time horizons of 5, 10 and 20 days.

Our quasi-analytic GARCH VaR estimation is at least 50 times faster than the standard simulation-based estimation procedure and our estimates are very accurate. We test the accuracy of our methodology for VaR estimation using the likelihood ratio tests for conditional coverage, proposed by Christoffersen (1998). The Johnson SU distribution performs marginally better than the Cornish–Fisher expansion overall. Yet none of the tests are rejected very often. When they are it is generally due to inappropriate unconditional coverage and rarely (and almost exclusively for the interest rate sample only) due to rejections in the independence tests. In fact, especially at higher confidence levels, the models often yield no consecutive violations. Our results are even more remarkable when we consider that the analysis is entirely out-of-sample and that the testing period (2000–2012) contains several years of excessively turbulent financial markets.

Appendix 1

Define the following constants: $\gamma = \varphi^2 + (\kappa_z - 1)\left(\alpha + \frac{1}{2}\lambda\right)^2 + \frac{1}{4}\kappa_z\lambda^2$, $a = \left(\omega^2 + 2\omega\varphi\bar{h}\right)(1-\gamma)^{-1}$, $b = 2\omega\varphi\left(h_{t+1} - \bar{h}\right)(\varphi-\gamma)^{-1}$, and $c = \lambda \int_{z=-\infty}^{0} z^3 f(z)dz$, where f is the density function of $D(0, 1)$, and for the two standard cases for D that we consider:

$$\int_{z=-\infty}^{0} z^3 f(z)dz = \begin{cases} -\sqrt{\dfrac{2}{\pi}} & \text{for } D(0,1) = N(0,1), \text{ standard} \\ -\dfrac{2}{\sqrt{\pi}}\dfrac{(\nu-2)^{3/2}}{(\nu-1)(\nu-3)}\dfrac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} & \text{for } D(0,1) = \text{standardized} \end{cases} \quad (8)$$

Then, for the GARCH model given in Section 2 we have:

1. $E_t(h_{t+s}) = \bar{h} + \varphi^{s-1}\left(h_{t+1} - \bar{h}\right)$;

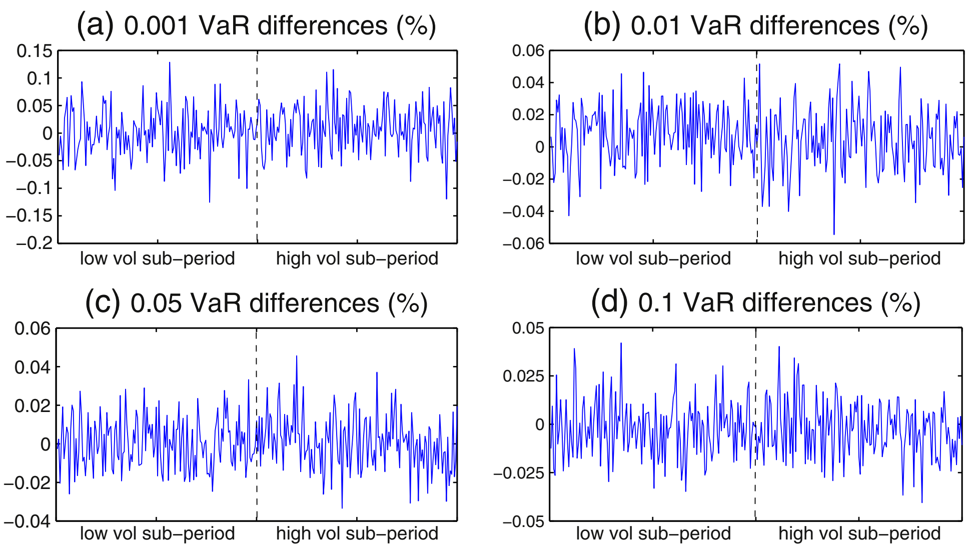

Fig. 2. Percentage differences between the Cornish–Fisher and simulated normal GARCH(1,1) VaRs. Percentage differences are computed as (Cornish–Fisher VaR − Simulated VaR) / Simulated VaR, where the VaRs are computed at four different significance levels: $\alpha = 0.1\%$, 1%, 5% and 10%, for 150 days (and corresponding estimation windows) from a low volatility period (January to August 2006) and 150 days (and corresponding estimation windows) from a high volatility period (August 2008 to March 2009).

2. $E_t\left(h^2_{t+s}\right) = a + \left(h^2_{t+1} - a - b\right)\gamma^{s-1} + b\,\varphi^{s-1}$;

3. $E_t\left(h^{3/2}_{t+s}\right) \simeq \frac{5}{8}\left(E_t(h_{t+s})\right)^{3/2} + \frac{3}{8}E_t\left(h^2_{t+s}\right)\left(E_t(h_{t+s})\right)^{-1/2}$;

4. $E_t\left(\varepsilon_{t+s}\varepsilon^2_{t+s+u}\right) = c\,\varphi^{u-1}E_t\left(h^{3/2}_{t+s}\right)$;

5. $E_t\left(\varepsilon^2_{t+s}\varepsilon^2_{t+s+u}\right) = \overline{h}(1-\varphi^u)E_t(h_{t+s}) + \varphi^{u-1}\kappa_z\left(\alpha + \frac{1}{2}\lambda + \kappa_z^{-1}\beta\right)E_t\left(h^2_{t+s}\right)$;

6. $E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}\varepsilon^2_{t+s+u+v}\right) = c\,\varphi^{v-1}E_t\left(\varepsilon_{t+s}h^{3/2}_{t+s+u}\right)$.

Appendix 2. LR tests

To evaluate the accuracy of our quasi-analytic GARCH VaR we apply the coverage (or 'likelihood ratio' (LR)) tests for VaR accuracy introduced by Kupiec (1995) and extended by Christoffersen (1998) and others. These are statistical tests based on VaR exceedances that have become standard in the applied financial economics literature and are now the most frequently used statistical tool for evaluating the performance of VaR models. A VaR exceedance occurs when we observe a loss that is more severe than predicted by the respective VaR; in other words, the actual loss *exceeds* the corresponding VaR forecast. A good VaR model is one that produces a percentage of exceedances (out of the total number of observations in the backtesting period) that is not statistically significant from the significance level (α) of the VaR. Given a backtesting sample of T non-overlapping observations, Christoffersen's conditional coverage LR test ($LR_{cc;\,\alpha}$) asserts that a good VaR model is one that produces a series of indicator functions.

$$\left\{I_{t,\alpha}\right\}_{t=1}^{T} \equiv \left\{I_{R_{tn}<-\mathrm{VaR}_{n,\alpha,t}}\right\}_{t=1}^{T}$$

which are Bernoulli (α) i.i.d. He also proved that:

$$LR_{cc;\alpha} = LR_{uc;\alpha} + LR_{ind;\alpha}$$

where $LR_{uc;\,\alpha}$ tests for the correct unconditional coverage, given that $\{I_t\}_{t=1}^{T}$ is independent, while $LR_{ind;\,\alpha}$ tests for the independence of this series, against the alternative of first order Markov dependence. He also derives the following test statistics and their respective distributions under the null to make the concepts operational:

$$LR_{uc;\alpha} = -2\ln\left(\left(\frac{1-\alpha}{1-\pi_\alpha}\right)^{n_{0;\alpha}}\left(\frac{\alpha}{\pi_\alpha}\right)^{n_{1;\alpha}}\right)\sim\chi^2(1)$$

$$LR_{ind;\alpha} = -2\ln\left(\frac{\left(1-\pi_{2;\alpha}\right)^{n_{00;\alpha}+n_{10;\alpha}}\pi_{2;\alpha}^{n_{01;\alpha}+n_{11;\alpha}}}{\left(1-\pi_{01;\alpha}\right)^{n_{00;\alpha}}\pi_{01;\alpha}^{n_{01;\alpha}}\left(1-\pi_{11;\alpha}\right)^{n_{10;\alpha}}\pi_{11;\alpha}^{n_{11;\alpha}}}\right)\sim\chi^2(1)$$

$$LR_{cc;\alpha} = LR_{uc;\alpha} + LR_{ind;\alpha}\sim\chi^2(2)$$

where in an empirical implementation $\hat{L}R_{uc;\alpha}$, $\hat{L}R_{ind;\alpha}$, and $\hat{L}R_{cc;\alpha}$ are obtained for:

$$n_{1;\alpha} = \sum_{t=1}^{T}\hat{I}_{t,\alpha};\quad n_{0;\alpha} = T - n_{1;\alpha};\quad \hat{\pi}_\alpha = \frac{n_{1;\alpha}}{T};\quad n_{ij;\alpha} = \sum_{t=1}^{T}\hat{J}_{ij;t;\alpha}$$

$$\hat{\pi}_{01;\alpha} = \frac{n_{01;\alpha}}{n_{00;\alpha}+n_{01;\alpha}};\quad \hat{\pi}_{11;\alpha} = \frac{n_{11;\alpha}}{n_{10;\alpha}+n_{11;\alpha}}+;$$

$$\hat{\pi}_{2;\alpha} = \frac{n_{01;\alpha}+n_{11;\alpha}}{n_{00;\alpha}+n_{10;\alpha}+n_{01;\alpha}+n_{11;\alpha}}\hat{I}_{t,\alpha} = I\left(r_{tn}<-\mathrm{V\hat{a}R}_{n,\alpha,t}\right);$$

$$\hat{J}_{ij;t;\alpha} = I\left(\hat{I}_{t-1,\alpha} = i \cap \hat{I}_{t,\alpha} = j\right);\quad i,j = 0,1$$

where r_{tn} are the sample realizations of the random variable R_{tn}.[19]

$LR_{uc;\alpha}$ is essentially a simple hypothesis test: the null hypothesis is that the difference between the empirical exceedance rate and the desired level α is zero. $LR_{ind;\alpha}$ on the other hand tests the null hypothesis that the exceedances are i.i.d. against the alternative that they exhibit some form of dependence (or clustering), in this case driven by a first order Markov process. By also taking into account the clustering of exceedances, as well as the number of times the VaR is exceeded, $LR_{cc;\alpha}$ is a joint test of correct coverage and independence of exceedances, i.e. correct conditional coverage.

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[19] If there are no consecutive violations, we follow Christoffersen (2011) and compute $\hat{L}R_{ind;\alpha}$ using the formula $\hat{L}R_{ind;\alpha} = (1-\hat{\pi}_{01;\alpha})^{n_{00;\alpha}}\hat{\pi}_{01;\alpha}{}^{n_{01;\alpha}}$.

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