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GEOMETRIC ESTIMATES ON WEIGHTED p-FUNDAMENTAL TONE AND APPLICATIONS TO THE FIRST EIGENVALUE OF SUBMANIFOLDS WITH BOUNDED MEAN CURVATURE

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Abstract. This paper generalises to the context of smooth metric measure spaces and submanifolds with negative sectional curvatures some well-known geometric estimates on the p-fundamental tone by using vector fields satisfying a positive divergence condition. Choosing the vector field to be the gradient of an appropriately chosen distance function yields generalised McKean estimates whilst other choices of vector fields yield new geometric estimates generalising certain results of Lima, Montenegro, Santos (Nonlin. Anal., 72 (2010) 771-781). We also obtain a lower bound on the spectrum of the weighted p-Laplacian on a complete noncompact smooth metric space with the underlying space being a submanifold with bounded mean curvature in the hyperbolic space form of constant negative sectional curvature generalising results of Du and Mao (J. Math. Anal. Appl., 456 (2017), 787-795).

1. Introduction

The goal of this paper is to provide some geometric lower estimates on weighted p-fundamental tones *viz* a *viz* the first eigenvalues of the weighted p-Laplacian on smooth metric measure spaces with certain conditions on the potential function and sectional or mean curvatures of the underlying space. The estimates obtained in this paper generalise some established results in recent literature. The intricate connections between the spectra and geometry of Riemannian manifolds has been a fascinating area of research with still a vast number of open and challenging problems at its core.

Let (M, g) be an m-dimensional Riemannian manifold with metric g and volume measure $dv = dv_g$ and let ϕ be a smooth function on M (called the potential). The triple (M, g, ϕ) is called a smooth metric measure space and is endowed with the measure $d\mu = e^{-\phi}dv$ that is conformally related to the volume measure dv. Associated with this smooth metric measure space there is a symmetric self-adjoint second order elliptic differential operator, called the weighted Laplacian (also known as the Witten or drifting Laplacian), defined for any smooth function u on M by

$$L^{\phi}u := \mathrm{div}_{\phi}(e^{-\phi}\nabla u) = \Delta_g u - \langle \nabla\phi, \nabla u\rangle_g. \tag{1.1}$$

Here $\mathrm{div}_{\phi} = e^{\phi}\mathrm{div}$ where div is the usual divergence operator while Δ_g, ∇ and $\langle\cdot,\cdot\rangle_g$ respectively denote Laplace Beltrami operator, gradient and inner product

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all with respect to the metric g. It is easily seen that L^ϕ satisfies

$$\int_M h(L^\phi f)e^{-\phi}\,dv = -\int_M \langle\nabla f,\nabla h\rangle_g\, e^{-\phi}\,dv = \int_M f(L^\phi h)e^{-\phi}\,dv, \tag{1.2}$$

for all $f, h \in C_0^\infty(M)$. A smooth metric measure space is a natural extension of a Riemannian manifold and the weighted Laplacian is the natural counterpart for the Laplace-Beltrami operator (in fact setting ϕ a constant gives the Laplacian). Smooth metric measure spaces play a central role throughout geometric analysis, probability theory, quantum field theory and statistical mechanics and have close links with Markov diffusion operators, generalised curvature and geometry ([4, 5]). A striking application of this class of spaces is already seen in Perelman's approach to resolving the Poincaré and geometrisation conjectures [29]. See also [19, 27, 28].

Let (M, g, ϕ) be a complete noncompact smooth metric space and let $\Omega \subset M$ be a relatively compact domain (an open connected subset with compact closure in M). The Dirichlet eigenvalue problem on Ω consists of finding eigenvalues λ and (non-zero) eigen-functions f so that:

$$\begin{cases} -L^\phi f = \lambda f & \text{in } \Omega, \\ f = 0 & \text{on } \Omega. \end{cases} \tag{1.3}$$

It is well known that the weighted Laplacian has a discrete spectrum here and all eigenvalues can be listed increasingly according to their multiplicities. Moreover, by Min-Max principle, the first eigenvalue $\lambda_1(\Omega)$ of $-L^\phi$ can be characterised by the Rayleigh quotient

$$\lambda_1(\Omega) = \inf_f \left\{ \frac{\int_\Omega \|\nabla f\|^2 e^{-\phi}\,dv}{\int_\Omega |f|^2 e^{-\phi}\,dv} : f \in W_0^{1,2}(\Omega), f \neq 0 \right\}. \tag{1.4}$$

Evidently $\Omega \mapsto \lambda_1(\Omega)$ has the domain monotonicity property in that $\Omega_1 \subset \Omega_2 \implies \lambda_1(\Omega_1) \geq \lambda_1(\Omega_2)$. Therefore if $(\Omega_j : j \geq 1)$ is an expanding sequence of relatively compact domains exhausting M the sequence $[\lambda_1(\Omega_j) : j \geq 1]$ is decreasing and so has a well-defined non-negative limit. As the limit is easily seen to be independent of the choice of the sequence $(\Omega_j : j \geq 1)$ this leads to the notion of the bottom of the spectrum of the weighted Laplacian $-L^\phi$ on (M, g, ϕ) as being

$$\inf \Sigma(M, g, \phi) = \lim_{j\to\infty} \lambda_1(\Omega_j). \tag{1.5}$$

Finding conditions under which $\inf \Sigma(M, g, \phi) > 0$ (strict inequality) has been the subject of various studies with many specific geometric properties formulated to imply this (see [8, 19, 26, 30, 34]). One aim of this paper is to consider nonlinear generalisations of the latter, namely, to the Dirichlet eigenvalue problem for the weighted p-Laplacian. This is the operator defined for $p \in (1, \infty)$ by

$$L_p^\phi f := \mathrm{div}_\phi(e^{-\phi}\|\nabla f\|^{p-2}\nabla f) = L_p f - \|\nabla f\|^{p-2}\langle\nabla\phi, \nabla f\rangle, \tag{1.6}$$

where $L_p f = \mathrm{div}(\|\nabla f\|^{p-2}\nabla f)$ is the usual p-Laplacian of f. Note that when $p = 2$, L_p^ϕ reduces to the weighted Laplacian and when ϕ is a constant, $L_p^\phi = L_p$. Here for a given relatively compact domain $\Omega \subset M$ we have the nonlinear eigenvalue problem of finding λ_p and non-zero f so that

$$\begin{cases} -L_p^\phi f = \lambda_p |f|^{p-2} f & \text{in } \Omega \\ f = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

The above equation is interpreted in a weak sense, meaning that, for $f \in W_0^{1,p}(\Omega)$ and every test function $\psi \in C_0^\infty(\Omega)$ it must be that

$$\int_\Omega \|\nabla f\|^{p-2} \langle \nabla f, \nabla \psi \rangle e^{-\phi} \, dv = \lambda_p \int_\Omega |f|^{p-2} f \psi e^{-\phi} \, dv.$$

Here by $W_0^{1,p}(\Omega)$ we understand the completion of $C_0^\infty(\Omega)$ – the space of smooth compactly supported functions on Ω – with respect to the Sobolev norm [19, 31]

$$\|f\|_{1,p,\Omega} = \left(\int_\Omega [|f|^p + \|\nabla f\|^p] e^{-\phi} \, dv \right)^{\frac{1}{p}}. \tag{1.8}$$

Again the first Dirichlet eigenvalue $\lambda_{1,p}(\Omega)$ of the weighted p-Laplacian on Ω can be characterised by

$$\lambda_{1,p}(\Omega) = \inf_f \left\{ \frac{\int_\Omega \|\nabla f\|^p e^{-\phi} \, dv}{\int_\Omega |f|^p e^{-\phi} \, dv} : f \in W_0^{1,p}(\Omega), f \neq 0 \right\}. \tag{1.9}$$

Moreover as is easily seen $\Omega \mapsto \lambda_{1,p}(\Omega)$ has the domain monotonicity property and so as before by considering a sequence of relatively compact domains exhausting M and noting the independence of the limit from the particular choice of sequence one is led to the following.

Definition 1.1. The bottom of the spectrum of the weighted p-Laplacian $-L_p^\phi$ on (M, g, ϕ) is the non-negative quantity defined as $\inf \Sigma_p(M, g, \phi) = \lim_{j \to \infty} \lambda_{1,p}(\Omega_j)$.

Definition 1.2. (Weighted p-fundamental tone) Let $\Omega \subset M$ be a domain in a smooth metric measure space (M, g, ϕ). The weighted p-fundamental tone of Ω, denoted $\Lambda_p^\star(\Omega, g, \phi)$, or $\Lambda_p^\star(\Omega)$ for short when the choices of g, ϕ are clear from the context, is the non-negative quantity

$$\Lambda_p^\star(\Omega, g, \phi) = \inf_f \left\{ \frac{\int_\Omega \|\nabla f\|^p e^{-\phi} \, dv}{\int_\Omega |f|^p e^{-\phi} \, dv} : f \in W_0^{1,p}(\Omega), f \neq 0 \right\}. \tag{1.10}$$

As for relatively compact domains $\Omega \subset M$ we have $\Lambda_p^\star(\Omega) = \lambda_{1,p}(\Omega)$, by (1.9), and as $\Omega \mapsto \Lambda_p^\star(\Omega)$ has the domain monotonicity property it follows at once that

$$\Lambda_p^\star(M, g, \phi) = \inf \Sigma_p(M, g, \phi). \tag{1.11}$$

Geometric estimates on fundamental tones and the bottom of the spectrum of the Laplacian on Riemannian manifolds have been obtained in various contexts (see, e.g., [6, 7, 8, 13, 22] for $p = 2$ and ϕ constant, [16, 23] for $p \in (1, \infty)$ and ϕ constant) and for other results [1, 2, 17, 18, 21, 25, 33] and [3, 14, 19, 22, 24, 32, 34]. In this paper we present generalisations of some of these results to the setting of smooth metric measure spaces. The main statements and related discussions appear in Section 2 while the proofs are deferred to Section 3. In Section 4 we use the results in Sections 2 and 3 to derive lower bounds on the weighted p-fundamental tones of geodesic balls whose radius does not exceed the injectivity radius.

2. Main results

Before stating the first result we introduce a quantity associated to any domain in a smooth metric measure space which is then used to give a lower bound on the weighted p-fundamental tone of the domain. For motivation and background see [6, 19, 23, 32, 34] and the references therein.

Definition 2.1. Let (M, g, ϕ) be a smooth metric measure space. For a domain $\Omega \subset M$ we denote by $\mathcal{X}(\Omega)$ the set of all smooth vector fields V on Ω satisfying the two properties: $\|V\|_\infty = \sup_\Omega \|V\| < \infty$ and $\inf_\Omega[e^\phi \text{div}(e^{-\phi} V)] > 0$. We then set:

$$h(\Omega) := \sup_V \left\{ \frac{\inf_\Omega[e^\phi \text{div}(e^{-\phi} V)]}{\sup_\Omega \|V\|} : \ V \in \mathcal{X}(\Omega) \right\}. \tag{2.1}$$

Evidently h is non-negative and the assignment $\Omega \mapsto h(\Omega)$ satisfies the domain monotonicity property: $\Omega_1 \subset \Omega_2 \implies h(\Omega_1) \geq h(\Omega_2)$. [Note that if $V \in \mathcal{X}(\Omega_2)$ then by restriction $V|_{\Omega_1} \in \mathcal{X}(\Omega_1)$.] Moreover h depends only on the data (Ω, g, ϕ) (and no p) which is particularly interesting in view of the way it appears in the lower bound formula (2.6) for weighted p-fundamental tones. Finally if $\mathcal{X}(\Omega) = \varnothing$ we set $h(\Omega) = 0$ and so clearly $h(\Omega) > 0$ iff $\mathcal{X}(\Omega) \neq \varnothing$ by (2.1). Let us now briefly discuss some upper and lower bounds on h. Firstly, by taking any sufficiently regular $E \subset\subset \Omega$, we can write

$$\begin{aligned} \inf_\Omega[e^\phi \text{div}(e^{-\phi} V)] \int_E e^{-\phi}\, dv &\leq \int_E [e^\phi \text{div}(e^{-\phi} V)] e^{-\phi}\, dv = \int_{\partial E} \langle V, \nu \rangle e^{-\phi}\, d\sigma \\ &\leq \int_{\partial E} |\langle V, \nu \rangle| e^{-\phi}\, d\sigma \leq \sup_\Omega ||V|| \int_{\partial E} e^{-\phi} d\sigma, \end{aligned} \tag{2.2}$$

giving

$$\frac{\inf_\Omega[e^\phi \text{div}(e^{-\phi} V)]}{\sup_\Omega ||V||} \leq \frac{\text{Per}_\phi(\partial E)}{\text{Vol}_\phi(E)}. \tag{2.3}$$

Here Vol_ϕ and Per_ϕ stand for the weighted volume and perimeter with respect to the volume and surface measures $e^{-\phi} dv$ and $e^{-\phi} d\sigma$ on M respectively. Now upon taking supremum on the left over $V \in \mathcal{X}(\Omega)$ and infimum on the right over E we arrive at an upper bound on h in terms of the generalised or weighted Cheeger constant $I_\phi(\Omega) = \inf_E[\text{Per}_\phi(\partial E)/\text{Vol}_\phi(E)]$, specifically,

$$h(\Omega) \leq I_\phi(\Omega). \tag{2.4}$$

As for a lower bound note that when $\|\nabla \phi\| \leq c$ for some constant $c \geq 0$ then for any vector field $V \in \mathcal{X}(\Omega)$ we have the bound

$$\inf_\Omega[e^\phi \text{div}(e^{-\phi} V)] = \inf_\Omega[\text{div}(V) - \langle \nabla \phi, V \rangle] \geq [\inf_\Omega \text{div}(V) - c \sup_\Omega \|V\|]_+. \tag{2.5}$$

Hence a resulting lower bound on $h(\Omega)$ can be deduced by quotienting over $||V||_\infty$ and then taking supremum over all $V \in \mathcal{X}(\Omega)$. We will use this way of bounding h from below quite frequently in sequel. Note that here and elsewhere we write $X_+ = \max(X, 0)$ and $X_- = \max(-X, 0)$ for $X \in \mathbb{R}$. The first result below gives a lower bound for the weighted p-fundamental tone of a domain in a smooth metric measure space. This is an extension of a nice result in [23, Theorem 1.1].

Theorem 2.2. *Let (M, g, ϕ) be a smooth metric measure space and $\Omega \subset M$ a domain with $\partial\Omega \neq \varnothing$. Then referring to (1.10) and (2.1) we have*

$$\Lambda_p^\star(\Omega) \geq \frac{1}{p^p} h(\Omega)^p. \tag{2.6}$$

In particular $\inf \Sigma_p(M) \geq h(M)^p/p^p$.

Remark 2.3. Taking a gradient field $V = \nabla f$ gives $e^\phi \mathrm{div}(e^{-\phi}V) = L^\phi f$ and more generally $V = |\nabla f|^{q-2}\nabla f$ gives $e^\phi \mathrm{div}(e^{-\phi}V) = L_q^\phi f$. Referring to (2.1) and (2.6) we can deduce the following: Assume that there exists a smooth function f on Ω with $\|\nabla f\| \leq 1$ such that $L^\phi f \geq \alpha > 0$ (or more generally $L_q^\phi f \geq \alpha > 0$). Then $h(\Omega) \geq \alpha$ and for any $1 < p < \infty$ we have the lower bound $\Lambda_p^\star(\Omega) \geq (\alpha/p)^p$. For a related upper bound see Remark 3.1. By slightly refining the argument in the theorem one can relax the regularity of f to being only Lipschitz and satisfying $L^\phi f \geq \alpha$ in a distributional sense or being smooth away from a null set [6, 19, 34].

McKean's result gives a sharp lower bound on the (non-weighted) 2-fundamental tone of a Cartan-Hadamard manifold (a complete noncompact simply-connected Riemannian manifold with non-positive sectional curvatures). In this setting [26] gives $\Lambda_2^\star(M) \geq (m-1)^2a^2/4$ ($p = 2$, $\phi \equiv c$ and $K_M \leq -a^2$). As an application of Theorem 2.2 a generalised form of McKean's estimate is obtained below.

Theorem 2.4. *Let (M, g, ϕ) be an m-dimensional complete noncompact simply-connected smooth metric measure space with sectional curvature $K_M \leq -a^2$ for some $a > 0$ and potential ϕ satisfying $\|\nabla\phi\| \leq c$ for some $c \geq 0$. Then*

$$\Lambda_p^\star(M) \geq \frac{[(m-1)a - c]_+^p}{p^p}. \tag{2.7}$$

One important class of such manifolds are the Riemannian symmetric spaces of non-compact type of rank-one. These are the real, the complex, the quaternionic hyperbolic spaces $\mathbb{H}^n = \mathbf{R}\mathbb{H}^n$, $\mathbf{C}\mathbb{H}^n$, $\mathbf{H}\mathbb{H}^n$ along with the Cayley plane $\mathbf{O}\mathbb{H}^2$ with $m = n, 2n, 4n, 16$ respectively. These spaces are all simply-connected with their sectional curvatures K_M pinched between -4 and -1 while the bottom of the L^2-spectrum $\inf \Sigma_2(M) = \Lambda_2^\star(M)$ can be explicitly described in terms of the half-sum of their positive roots [20]. For related results in this direction and extensions to higher ranks and locally symmetric spaces see [3, 14, 20, 22]. See also [9, 10] and the references therein.

In order to present the next result we need to introduce the space $\mathscr{W}^1(M)$. To this end let V be a locally integrable vector field on M in the sense that $||V|| \in L^1_{loc}(M)$. (Note that the choice of measures $d\mu = e^{-\phi}dv$ or dv are equivalent here due to the local integrability condition and the local boundedness of ϕ, resulting from continuity.) Then $V^\phi = e^{-\phi}V$ is a locally integrable vector field on M and hence a vector distribution.

Definition 2.5. Let V be a locally integrable vector field on M. The function $h_V \in L^1_{loc}(M)$ is referred to as the weak ϕ-divergence of V^ϕ iff $e^{-\phi}h_V = \mathrm{div}(V^\phi)$ in the sense of distributions, that is,

$$\int_M \psi h_V e^{-\phi}\, dv = -\int_M \langle\nabla\psi, V\rangle e^{-\phi}\, dv \qquad \forall \psi \in C_0^\infty(M). \tag{2.8}$$

We denote by $\mathscr{W}^1(M)$ the space of all locally integrable vector fields V on M for which V^ϕ has a weak ϕ-divergence. We also write $\operatorname{div}_\phi(V^\phi) = h_V$.

When V is of class C^1 then the above implies that $\operatorname{div}_\phi(V^\phi) = \operatorname{div}(V) - \langle \nabla\phi, V\rangle$ where div is the divergence in the classical sense. Moreover when $V \in \mathscr{W}^1(M)$ and $f \in C^\infty(M)$ it follows that $|f|^p V \in \mathscr{W}^1(M)$ and from (2.8) we get

$$\operatorname{div}_\phi([|f|^p V]^\phi) = \operatorname{div}_\phi(e^{-\phi}|f|^p V) = \langle \nabla|f|^p, V\rangle + |f|^p \operatorname{div}_\phi(V^\phi). \tag{2.9}$$

In particular if $f \in C_0^\infty(M)$ then $\operatorname{supp}(e^{-\phi}|f|^p V) \subset\subset M$ and so choosing ψ in (2.8) so that $\psi \equiv 1$ in a neighbourhood of $\operatorname{supp}(f)$, basic considerations lead to

$$\int_M \operatorname{div}_\phi(e^{-\phi}|f|^p V) e^{-\phi}\, dv = \int_M [\langle \nabla|f|^p, V\rangle + |f|^p \operatorname{div}_\phi(V^\phi)] e^{-\phi}\, dv = 0. \tag{2.10}$$

Theorem 2.6. *Let (M, g, ϕ) be an m-dimensional smooth metric measure space. Then the weighted p-fundamental tone of (M, g, ϕ) satisfies the following estimate*

$$\Lambda_p^\star(M, g, \phi) \geq \sup_{V \in \mathscr{W}^1} \left\{ \inf_M \left[(1-p)\|V\|^{\frac{p}{p-1}} + \operatorname{div}_\phi(V^\phi) \right] \right\}. \tag{2.11}$$

Finally, the last result of this paper specialises to the weighted p-fundamental tone of submanifolds of the hyperbolic space $\mathbb{H}^m$.

Theorem 2.7. *Let (M, g, ϕ) be an m-dimensional complete noncompact smooth metric measure space where M is a submanifold of the hyperbolic space $\mathbb{H}^m$ with sectional curvature $K_{\mathbb{H}} = -1$ and ϕ satisfying $\|\nabla\phi\| \leq c$ for some $c \geq 0$. Assume the mean curvature vector H of M in $\mathbb{H}^m$ verifies the bound $||H|| \leq b < m-1-c$. Then*

$$\Lambda_p^\star(M) \geq \frac{(m-1-b-c)^p}{p^p}. \tag{2.12}$$

Moreover, if M is a minimal submanifold of $\mathbb{H}^m$, then $\Lambda_p^\star(M) \geq (m-1-c)^p/p^p$.

Remark 2.8. If the potential ϕ is a constant we can set $c = 0$ and then (2.12) gives $\Lambda_p^\star(M) \geq (m-1-b)^p/p^p$ which is exactly [15, Theorem 1.3]. If additionally $p = 2$, then the weighted p-Laplacian becomes the Laplacian and the above result reduces to [13, Theorem 2]. If further M is a complete minimal submanifold of $\mathbb{H}^m$; then $\Lambda_2^\star(M) \geq (m-1)^2/4$. McKean's result [26] here asserts that the bound is sharp when $M = \mathbb{H}^m$ as indeed $\Lambda_2^\star(\mathbb{H}^m) = (m-1)^2/4$.

3. Proof of the main results

Proof of Theorem 2.2. Pick $V \in \mathcal{X}(\Omega)$, $f \in C_0^\infty(\Omega)$. The vector field $|f|^p e^{-\phi} V$ has compact support in Ω. Computing its divergence we have:

$$\begin{aligned} \operatorname{div}(|f|^p e^{-\phi} V) &= \langle \nabla|f|^p, V\rangle e^{-\phi} + |f|^p \operatorname{div}(e^{-\phi} V) \\ &= p|f|^{p-2} f \langle \nabla f, V\rangle e^{-\phi} + |f|^p \operatorname{div}(e^{-\phi} V) \\ &\geq (-p|f|^{p-1}\|\nabla f\|\|V\| + e^\phi \operatorname{div}(e^{-\phi} V)|f|^p) e^{-\phi}. \end{aligned} \tag{3.1}$$

Integrating both sides over Ω and applying the divergence theorem then gives

$$0 = \int_\Omega \operatorname{div}(|f|^p e^{-\phi} V)\, dv \geq \int_\Omega \left(-p|f|^{p-1}\|\nabla f\|\|V\| + e^\phi \operatorname{div}(e^{-\phi} V)|f|^p \right) e^{-\phi}\, dv. \tag{3.2}$$

Recalling next Young's inequality, we have for $\Upsilon \geq 0$, $\Phi \geq 0$ and $\epsilon > 0$ arbitrary: $\Upsilon\Phi \leq p^{-1}(\Upsilon/\epsilon)^p + q^{-1}(\epsilon\Phi)^q$, where $1 < p, q < \infty$ are conjugate exponents. Now applying this inequality with the choices $\Upsilon = p\|\nabla f\|$ and $\Phi = |f|^{p-1}\|V\|$ gives

$$p|f|^{p-1}\|\nabla f\|\|V\| \leq \frac{p^{p-1}}{\epsilon^p}\|\nabla f\|^p + \frac{(p-1)}{p}\epsilon^{\frac{p}{p-1}}|f|^p\|V\|^{\frac{p}{p-1}}, \tag{3.3}$$

where $\epsilon > 0$ is a positive constant to be determined later. Substituting (3.3) into (3.2) yields

$$\begin{aligned}
0 \geq& \int_\Omega \left[-\frac{p^{p-1}}{\epsilon^p}\|\nabla f\|^p + \left(e^\phi \mathrm{div}(e^{-\phi}V) - \frac{(p-1)}{p}\epsilon^{\frac{p}{p-1}}\|V\|^{\frac{p}{p-1}}\right)|f|^p\right] e^{-\phi}\,dv \\
\geq& \frac{p^{p-1}}{\epsilon^p}\int_\Omega -\|\nabla f\|^p e^{-\phi}\,dv + \inf_\Omega\left(e^\phi \mathrm{div}(e^{-\phi}V) - \frac{(p-1)}{p}\|\epsilon V\|^{\frac{p}{p-1}}\right)\int_\Omega |f|^p e^{-\phi}\,dv.
\end{aligned}$$

A rearrangement of terms and basic considerations leads to the integral inequality

$$\int_\Omega \|\nabla f\|^p e^{-\phi}\,dv \geq \frac{\epsilon^p}{p^{p-1}}\left(\inf_\Omega[e^\phi \mathrm{div}(e^{-\phi}V)] - \frac{(p-1)}{p}\sup_\Omega\|\epsilon V\|^{\frac{p}{p-1}}\right)\int_\Omega |f|^p e^{-\phi}\,dv.$$

Now prompted by the above inequality we consider the task of maximising for $\epsilon > 0$ the scalar function Ψ given by

$$\epsilon \longmapsto \Psi(\epsilon) = \epsilon^p\left(\inf_\Omega[e^\phi \mathrm{div}(e^{-\phi}V)] - \frac{(p-1)}{p}\epsilon^{\frac{p}{p-1}}\sup_\Omega\|V\|^{\frac{p}{p-1}}\right), \tag{3.4}$$

or more conveniently and written in a shorter form $\Psi(\epsilon) = \epsilon^p(A - \epsilon^{\frac{p}{p-1}}B)$ where $A = \inf_\Omega[e^\phi \mathrm{div}(e^{-\phi}V)] > 0$ and $B = [(p-1)/p]\sup_\Omega\|V\|^{p/(p-1)} > 0$. In order to do so we compute the first and second derivatives of Ψ respectively which are seen to be

$$\begin{aligned}
\Psi'(\epsilon) &= p\epsilon^{p-1}\left[A - \frac{\epsilon^{\frac{p}{p-1}}pB}{p-1}\right], \\
\Psi''(\epsilon) &= p\epsilon^{p-2}\left\{(p-1)A - \epsilon^{\frac{p}{p-1}}\left[p + \frac{p^2}{(p-1)^2}\right]B\right\}.
\end{aligned}$$

A straightforward calculation shows that the critical point of $\Psi(\epsilon)$ for $\epsilon > 0$ occurs at

$$\epsilon^\star = \left(\frac{p-1}{p}\frac{A}{B}\right)^{\frac{p-1}{p}} \implies \Psi''(\epsilon^\star) = -\frac{Ap^2}{p-1}[\epsilon^\star]^{p-2} \leq 0.$$

Consequently, by basic considerations, $\epsilon^\star$ is a maximum point and the maximum achieved is given by $\max\Psi(\epsilon) = \Psi(\epsilon^\star) = [(p-1)^{p-1}A^p]/[p^pB^{p-1}]$. Substituting for A and B in the above it is seen that the maximum of the factor on the right-hand side of the above integral inequality is precisely the value

$$\frac{1}{p^{p-1}}\max_{\epsilon>0}\Psi(\epsilon) = \frac{1}{p^p}\left(\frac{\inf_\Omega[e^\phi \mathrm{div}(e^{-\phi}V)]}{\sup_\Omega\|V\|}\right)^p. \tag{3.5}$$

Thus substituting (3.5) back into the integral inequality we arrive at

$$\int_\Omega \|\nabla f\|^p e^{-\phi}\,dv \geq \frac{1}{p^p}\left(\frac{\inf_\Omega[e^\phi \mathrm{div}(e^{-\phi}V)]}{\sup_\Omega\|V\|}\right)^p\int_\Omega |f|^p e^{-\phi}\,dv,$$

and so taking the supremum on the right over all vector fields $V \in \mathcal{X}(\Omega)$ gives

$$\int_\Omega \|\nabla f\|^p e^{-\phi}\, dv \geq \frac{1}{p^p}\left(\sup_{V\in\mathcal{X}(\Omega)} \frac{\inf_\Omega[e^\phi \mathrm{div}(e^{-\phi}V)]}{\sup_\Omega \|V\|}\right)^p \int_\Omega |f|^p e^{-\phi}\, dv.$$

We have thus justified the inequality

$$\int_\Omega \|\nabla f\|^p e^{-\phi}\, dv \geq \frac{1}{p^p} h(\Omega)^p \int_\Omega |f|^p e^{-\phi}\, dv. \tag{3.6}$$

Referring to the definition of the p-fundamental tone (1.10), the required estimate (2.6) now follows at once from the above by writing,

$$\Lambda_p^\star(\Omega) = \inf_f \left\{ \frac{\int_\Omega \|\nabla f\|^p e^{-\phi}\, dv}{\int_\Omega |f|^p e^{-\phi}\, dv} : f \in W_0^{1,p}(\Omega), f \neq 0 \right\} \geq \frac{1}{p^p} h(\Omega)^p.$$

Considering an exhaustion of M by a sequence of expanding relatively compact $(\Omega_j : j \geq 1)$ we have $\Lambda_p^\star(\Omega_j) \geq h(\Omega_j)^p/p^p$. Noting $\inf \Sigma_p(M) = \lim_{j\to\infty} \Lambda_p^\star(\Omega_j)$ and $h(\Omega_j) \geq h(M)$ gives the second inequality $\inf \Sigma_p(M) \geq h(M)^p/p^p$. □

Remark 3.1. Whilst Theorem 2.2 gives a lower bound on $\Lambda_p^\star(M)$ here is a context where one can obtain an (often matching) upper bound. Towards this end suppose there exists a Lipschitz function u with $\|\nabla u\| \leq 1$ such that (i) $e^{-\sigma u} \in L^1(M; d\mu)$ for some $\sigma > 0$ and (ii) $u \to \infty$ as $x \to \infty$. Put $f = e^{-\sigma u/p}$. Then $f \in L^p(M; d\mu)$, f vanishes near the boundary and $\nabla f = -\sigma/p f \nabla u$. Hence (with $d\mu = e^{-\phi}\, dv$)

$$\int_M |\nabla f|^p e^{-\phi}\, dv = \frac{\sigma^p}{p^p}\int_M |f|^p|\nabla u|^p e^{-\phi}\, dv \leq \frac{\sigma^p}{p^p}\int_M |f|^p e^{-\phi}\, dv,$$

giving the bound $\Lambda_p^\star(M) \leq (\sigma/p)^p$. For related results and more see [8, 19, 30, 34].

As an application of Theorem 2.2, a generalised form of McKean's estimate is derived on noncompact smooth metric measure space whose potential satisfies $\|\nabla\phi\| \leq c$ for some $c \geq 0$.

Proof of Theorem 2.4. Pick a proper subdomain $\Omega \subset M$ and let $\rho = \rho(x)$ be the distance function measured from a fixed base point in M outside Ω. Now since $K_M \leq -a^2$ by the Laplacian comparison theorem $\Delta\rho \geq (m-1)a$ and as $\|\nabla\rho\| = 1$ a straightforward computation gives $L^\phi\rho = \Delta\rho - \langle\nabla\phi, \nabla\rho\rangle \geq (m-1)a - c$. Let us hence set $V = \nabla\rho$. Then $e^\phi\mathrm{div}(e^{-\phi}V) = L^\phi\rho \geq (m-1)a - c$ whilst $\|V\| = \|\nabla\rho\|$. Thus $V \in \mathcal{X}(\Omega)$ if $c < (m-1)a$. Therefore by using (2.6) in Theorem 2.2,

$$\begin{aligned}\Lambda_p^\star(\Omega) &\geq \frac{1}{p^p}h(\Omega)^p = \frac{1}{p^p}\left(\sup_V \frac{\inf_\Omega e^\phi\mathrm{div}(e^{-\phi}V)}{\sup_\Omega\|V\|}\right)^p \geq \frac{1}{p^p}\left(\frac{\inf_\Omega e^\phi\mathrm{div}(e^{-\phi}\nabla\rho)}{\sup_\Omega\|\nabla\rho\|}\right)^p\\ &\geq \frac{1}{p^p}\left(\frac{\inf_\Omega[\Delta\rho - \langle\nabla\phi,\nabla\rho\rangle]}{\sup_\Omega\|\nabla\rho\|}\right)^p \geq \frac{1}{p^p}[(m-1)a - c]^p.\end{aligned}$$

As the bound on the right does not dependent on Ω the conclusion follows by taking an increasing sequence of domains $(\Omega_j : j \geq 1)$ exhausting M and passing to the limit noting $\Lambda_p^\star(M) = \lim \Lambda_p^\star(\Omega_j)$ as $j \to \infty$. □

Proof of Theorem 2.6. Let $V \in \mathscr{W}^1(M)$ and pick $f \in C_0^\infty(M)$. Then by the discussion preceding the statement of the theorem [*cf.* (2.10)] we have

$$\begin{aligned} 0 &= \int_M p|f|^{p-2} f \langle \nabla f, V\rangle e^{-\phi}\, dv + \int_M \mathrm{div}_\phi(V^\phi)|f|^p e^{-\phi}\, dv \\ &\geq \int_M -p|f|^{p-1}\|\nabla f\|\|V\| e^{-\phi}\, dv + \int_M \mathrm{div}_\phi(V^\phi)|f|^p e^{-\phi}\, dv. \end{aligned} \tag{3.7}$$

Now applying Young's inequality with evident choice of exponents on the first integrand on the right gives $p|f|^{p-1}\|\nabla f\|\|V\| \leq \|\nabla f\|^p + (p-1)|f|^p\|V\|^{p/(p-1)}$ and therefore substituting back into (3.7) results in

$$0 \geq \int_M -\|\nabla f\|^p e^{-\phi}\, dv + \int_M \left[(1-p)\|V\|^{\frac{p}{p-1}} + \mathrm{div}_\phi(V^\phi)\right] |f|^p e^{-\phi}\, dv.$$

Rearranging terms and basic considerations yields

$$\int_M \|\nabla f\|^p e^{-\phi}\, dv \geq \inf_M \left\{(1-p)\|V\|^{\frac{p}{p-1}} + \mathrm{div}_\phi(V^\phi)\right\} \int_M |f|^p e^{-\phi}\, dv.$$

As this holds true for every $V \in \mathscr{W}^1(M)$ it thus follows that subject to $f \neq 0$,

$$\frac{\int_M \|\nabla f\|^p e^{-\phi}\, dv}{\int_M |f|^p e^{-\phi}\, dv} \geq \sup_{V\in\mathscr{W}^1(M)} \inf_M \left\{(1-p)\|V\|^{\frac{p}{p-1}} + \mathrm{div}_\phi(V^\phi)\right\}.$$

The required estimate (2.11) now follows from this by recalling the definition of the weighted p-fundamental tone,

$$\Lambda_p^\star(M) = \inf_{\substack{f\in W_0^{1,p}(M)\\ f\neq 0}} \frac{\int_M \|\nabla f\|^p e^{-\phi}\, dv}{\int_M |f|^p e^{-\phi}\, dv} \geq \sup_{V\in\mathscr{W}^1(M)} \inf_M \left\{(1-p)\|V\|^{\frac{p}{p-1}} + \mathrm{div}_\phi(V^\phi)\right\}$$

which is the required inequality. □

To show the significance of the quantity $(1-p)\|V\|^{\frac{p}{p-1}} + \mathrm{div}_\phi(V^\phi)$ in the theorem and the sharpness of the bound, let $\Omega \subset M$ be a relatively compact domain with first eigenvalue $\lambda_{1,p}(\Omega)$ and first (positive) eigenfunction f_1. Then by (1.6)-(1.7):

$$-L_p^\phi f_1 = -e^\phi \mathrm{div}(e^{-\phi}\|\nabla f_1\|^{p-2}\nabla f_1) = \lambda_{1,p}(\Omega)|f_1|^{p-2} f_1.$$

Now choosing $V = -(\|\nabla f_1\|^{p-2}\nabla f_1)/(|f_1|^{p-2} f_1)$ and proceeding formally we have

$$\begin{aligned} \mathrm{div}_\phi(V^\phi) = -\,\mathrm{div}_\phi\left(\frac{e^{-\phi}\|\nabla f_1\|^{p-2}\nabla f_1}{|f_1|^{p-2} f_1}\right) &= \frac{-\mathrm{div}_\phi(e^{-\phi}\|\nabla f_1\|^{p-2}\nabla f_1)}{|f_1|^{p-2} f_1} \\ + (p-1)\frac{\|\nabla f_1\|^p}{|f_1|^p} &= \lambda_{1,p}(\Omega) + (p-1)\frac{\|\nabla f_1\|^p}{|f_1|^p}. \end{aligned}$$

Also $\|V\| = \|\nabla f_1\|^{p-1}/|f_1|^{p-1}$. Hence putting together the above ingredients gives

$$(1-p)\|V\|^{\frac{p}{p-1}} + \mathrm{div}_\phi(V^\phi) = (1-p)\frac{\|\nabla f_1\|^p}{|f_1|^p} + \lambda_{1,p}(\Omega) + (p-1)\frac{\|\nabla f_1\|^p}{|f_1|^p}$$

$$\Lambda_p^\star(\Omega) = \lambda_{1,p}(\Omega) = \inf_\Omega[(1-p)\|V\|^{\frac{p}{p-1}} + \mathrm{div}_\phi(V^\phi)] \leq \Lambda_p^\star(\Omega).$$

Remark 3.2. Note that in the special case where ϕ is constant, $\mathrm{div}_\phi(V^\phi) = \mathrm{div}(V)$ and the conclusion of Theorem 2.6 coincides with [23, Theorem 1.3].

An estimate will be required on the weighted Laplacian of a distance function on M in the proof of Theorem 2.7. To obtain this we require an identity from [15, Lemma 4]. Let (M, g, ϕ) be an m-dimensional smooth metric measure space with M being a submanifold of the hyperbolic space $\mathbb{H}^m$ with $K_{\mathbb{H}^m} \equiv -1$. Then

$$L^\phi \rho = (m - \|\nabla\rho\|^2)\coth\rho + \langle H, \overline{\nabla}\rho\rangle|_M - \langle\nabla\rho, \nabla\phi\rangle, \tag{3.8}$$

where H is the mean curvature vector of M in $\mathbb{H}^m$, ρ denotes the distance function measured from a base point in $\mathbb{H}^m \setminus M$ and $\overline{\nabla}$ is the connection on $\mathbb{H}^m$. Noting that $\|\overline{\nabla}\rho\| = 1$, along with $\|\nabla\rho\| \leq 1$, by applying the stated conditions $\|\nabla\phi\| \leq c$ and $\|H\| \leq b < m - 1 - c$ in the theorem we have

$$\begin{aligned} L^\phi \rho &= (m - \|\nabla\rho\|^2)\coth\rho + \langle H, \overline{\nabla}\rho\rangle|_M - \langle\nabla\rho, \nabla\phi\rangle \\ &\geq m - 1 - \|H\|\|\overline{\nabla}\rho\| - \|\nabla\phi\|\|\nabla\rho\| \geq m - 1 - b - c. \end{aligned} \tag{3.9}$$

Proof of Theorem 2.7. Let ρ denote the distance function on M with respect to a fixed base point in $\mathbb{H}^m$ outside M. Set $V = \nabla\rho$ and pick $f \in C_0^\infty(M)$. Then $\mathrm{supp}(|f|^p e^{-\phi}\nabla\rho) \subset\subset M$ and an application of the divergence theorem gives

$$\int_M e^\phi \mathrm{div}(|f|^p e^{-\phi}\nabla\rho) e^{-\phi}\, dv = 0. \tag{3.10}$$

Considering the integrand we have

$$e^\phi \mathrm{div}(|f|^p e^{-\phi}\nabla\rho) = \langle\nabla|f|^p, \nabla\rho\rangle + |f|^p L^\phi\rho = p|f|^{p-2} f\langle\nabla f, \nabla\rho\rangle + |f|^p L^\phi\rho.$$

Applying the inequality $\|\nabla\rho\| \leq 1$ and the bound $L^\phi\rho \geq m-1-b-c$ from (3.9) yields

$$e^\phi \mathrm{div}(|f|^p e^{-\phi}\nabla\rho) \geq -p|f|^{p-1} f\|\nabla f\| + (m-1-b-c)|f|^p.$$

Using the inequality $p|f|^{p-1}f\|\nabla f\| \leq \epsilon^p\|\nabla f\|^p + (p-1)\epsilon^{p/(p-1)}|f|^p$ on the first term on the right, with $\epsilon > 0$ arbitrary, gives

$$\mathrm{div}_\phi(|f|^p e^{-\phi} V) \geq -\epsilon^p\|\nabla f\|^p - (p-1)\epsilon^{p/(p-1)}|f|^p + (m-1-b-c)|f|^p,$$

and so substituting in (3.10) and rearranging terms leads to

$$\int_M \|\nabla f\|^p e^{-\phi}\, dv \geq \epsilon^{-p}\left[m - 1 - b - c - (p-1)\epsilon^{p/(p-1)}\right]\int_M |f|^p e^{-\phi}\, dv.$$

Optimising the coefficient $\epsilon^{-p}(m-1-b-c-(p-1)\epsilon^{p/(p-1)})$ on the right for $\epsilon > 0$ and noting that the maximum occurs at $\epsilon^\star = [p/(m-1-b-c)]^{(p-1)/p}$ with the maximum value being $[(m-1-b-c)/p]^p$ it follows that

$$\int_M \|\nabla f\|^p e^{-\phi}\, dv \geq \left(\frac{m-1-b-c}{p}\right)^p \int_M |f|^p e^{-\phi}\, dv. \tag{3.11}$$

This immediately leads to the desired conclusion. If, moreover, M is a minimal submanifold in $\mathbb{H}^m$, then $b = 0$, and so the second result follows. $\square$

4. Weighted p-fundamental tones of geodesic balls

In this final section we utilise Theorem 2.2 to give lower bounds on the first eigenvalue of the weighted p-Laplacian on geodesic balls whose radius does not exceed the injectivity radius. To this end let us first recall the following Hessian comparison theorem (see [12] or [6, 16, 23]).

Theorem 4.1. *Let (M,g) be a complete Riemannian manifold and let ζ, $\xi \in M$. Assume $\gamma : [0,\rho(\xi)] \to M$ is a minimising geodesic joining ζ, ξ where $\rho = \rho(x)$ is the distance function measured from the base point ζ. Let K_M denote the sectional curvature of M and $\mu = \mu(\rho)$ the function*

$$\mu(\rho) = \begin{cases} k\coth(k\rho) & \text{if } \sup_\gamma K_M = -k^2, \\ 1/\rho & \text{if } \sup_\gamma K_M = 0, \\ k\cot(k\rho) & \text{if } \sup_\gamma K_M = +k^2 \text{ and } \rho < \pi/(2k). \end{cases} \tag{4.1}$$

Then the Hessians of ρ and ρ^2 at the point x satisfy the lower bounds

$$\begin{cases} \nabla^2\rho(x)(X,X) \ge \mu(\rho(x))\|X\|^2, & \text{and} \quad \nabla^2\rho(x)(\gamma',\gamma') = 0, \\ \nabla^2\rho^2(x)(X,X) \ge 2\rho(x)\mu(\rho(x))\|X\|^2, & \text{and} \quad \nabla^2\rho(x)(\gamma',\gamma') = 2, \end{cases} \tag{4.2}$$

respectively where X is any vector in T_xM perpendicular to $\gamma'(\rho(x))$.

In particular it follows from (4.2) that the Laplacian of the distance and squared distance functions ρ and ρ^2 respectively satisfy the lower bounds

$$\begin{cases} \Delta\rho(x) \ge (m-1)\mu(\rho(x)), \\ \Delta\rho^2(x) \ge 2(m-1)\rho(x)\mu(\rho(x)) + 2. \end{cases} \tag{4.3}$$

Theorem 4.2. *Let (M,g,ϕ) be a smooth metric measure space with potential function satisfying $\|\nabla\phi\| \le c$ for some $c \ge 0$. Let $\mathcal{B} = \mathcal{B}_r(a)$ denote the geodesic ball centred at a with radius $r < \mathrm{Inj}(a)$ and let $k_r(a) = \sup\{K_M(x) : x \in \mathcal{B}_r(a)\}$ where $K_M(x)$ are the sectional curvatures of M at x. Then*

$$\Lambda_p^\star(\mathcal{B}_r(a)) \ge \frac{1}{p^p}\begin{cases} \max([m-cr]_+^p/r^p, [(m-1)k\coth(kr)-c]_+^p) & k_r(a) = -k^2, \\ [m-cr]_+^p/r^p & k_r(a) = 0, \\ [(m-1)kr\cot(kr)+1-cr]_+^p/r^p & k_r(a) = +k^2, \\ & r < \pi/(2k). \end{cases}$$

where k is a non-zero constant. Note that since a geodesic ball is relatively compact we have $\Lambda_p^\star(\mathcal{B}) = \lambda_{1,p}(\mathcal{B})$.

Proof. In view of the smoothness of ρ^2 in $\mathcal{B}$ we set $V = \nabla\rho^2$. Then $\sup_{\mathcal{B}}\|V\| = 2r$ and $e^\phi\mathrm{div}(e^{-\phi}V) = L^\phi\rho^2 = \Delta\rho^2 - \langle\nabla\phi,\nabla\rho^2\rangle$. Now irrespective of the sign of the infimum of the latter ϕ-divergence on $\mathcal{B}$, by referring to (2.1) we have,

$$\begin{aligned} h(\mathcal{B}) &= \sup_V\left\{\frac{\inf_{\mathcal{B}}[e^\phi\mathrm{div}(e^{-\phi}V)]}{\sup_{\mathcal{B}}\|V\|} : V \in \mathcal{X}(\mathcal{B})\right\} \ge \frac{\inf_{\mathcal{B}} L^\phi\rho^2}{\sup_{\mathcal{B}}||\nabla\rho^2||} \\ &\ge \frac{\inf_{\mathcal{B}}\Delta\rho^2}{2r} - c \ge \frac{1}{r}\inf_{\mathcal{B}}[(m-1)\rho\mu(\rho)+1] - c, \end{aligned} \tag{4.4}$$

where we have made use of the lower bound on the second line of (4.3). Hence by Theorem 2.2 this gives

$$\Lambda_p^\star(\mathcal{B}) \geq \frac{1}{p^p}\left[\frac{1}{r}\inf_{\mathcal{B}}[(m-1)\rho\mu(\rho)+1]-c\right]_+^p. \tag{4.5}$$

In the second and third cases where $k_r(a) \geq 0$ this leads to the desired conclusion. Moreover in the first case where $k_r(a) = -k^2$ this gives $\Lambda_p^\star(\mathcal{B}) \geq [m-cr]_+^p/(rp)^p$. However if we take instead $X = \nabla\rho^s$ with $1 < s < 2$ (here X is smooth in $\mathcal{B}\backslash\{a\}$ and continuous in $\mathcal{B}$ and the argument of Theorem 2.2 still works with minor modifications) we have upon utilising the first inequality in (4.3) the bound

$$\Lambda_p^\star(\mathcal{B}) \geq \frac{1}{p^p}\left[\frac{\inf[s(s-1)\rho^{s-2}+(m-1)sk\rho^{s-1}\coth(k\rho)]}{sr^{s-1}}-c\right]_+^p. \tag{4.6}$$

Letting now $s \searrow 1$ gives $\Lambda_p^\star(\mathcal{B}) \geq [(m-1)k\coth(kr)-c]_+^p/p^p$ and so the desired conclusion in the first case follows too. □

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