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ELLIPTIC GRADIENT ESTIMATES FOR A NONLINEAR f -HEAT EQUATION ON WEIGHTED MANIFOLDS WITH EVOLVING METRICS AND POTENTIALS

ABIMBOLA ABOLARINWA AND ALI TAHERI

ABSTRACT. We develop local elliptic gradient estimates for a basic nonlinear f -heat equation with a logarithmic power nonlinearity and establish pointwise upper bounds for the weighted heat kernel in the context of weighted manifolds where the metric and potential evolve under a Perelman-Ricci type flow. For the heat bounds use is made of entropy monotonicity arguments and ultracontractivity estimates with the bounds directly expressed in terms of the best constant in the logarithmic Sobolev inequality. Some interesting consequences of these estimates are presented and discussed.

1. INTRODUCTION AND MAIN RESULTS

1.1. Introduction. Suppose $(M, g, e^{-f} dv)$ is a weighted manifold meaning that (M, g) is a complete Riemannian manifold of dimension n (here $n \geq 2$), f is a smooth potential on M and $d\mu = e^{-f} dv$ is a weighted measure on M conformally equivalent to the Riemannian volume measure $dv = dv_g$. This paper considers positive smooth solutions to the basic logarithmically nonlinear f -heat equation: ¹

$$\left(\mathcal{L}_f - \frac{\partial}{\partial t} \right) u(x, t) + Lu(x, t) |\log u(x, t)|^\alpha = 0, \quad t > 0, \quad (1.1)$$

where $\mathcal{L}_f$ denotes the f -Laplacian (see (1.8) below), L is a real constant and $\alpha > 1$. In the context studied here both the metric $g = g(t)$ and the potential $f = f(t)$ are taken time dependent; in fact, a case of particular interest is when the pair evolve under an unconstrained or constrained (k, m) Perelman-Ricci flow of the form:

$$\frac{1}{2} \frac{\partial g}{\partial t} + Ric_f^m(g) = -kg, \quad (1.2)$$

$$\frac{\partial f}{\partial t} - \frac{1}{2} \text{Tr} \left(\frac{\partial g}{\partial t} \right) = 0. \quad (1.3)$$

Here k, m are fixed constants with k real and m extended real with $m \geq n$. In (1.2), $Ric_f^m(g)$ denotes the generalised Bakry-Emery m -Ricci curvature tensor defined by

$$Ric_f^m(g(t)) = Ric(g(t)) + \nabla_{g(t)} \nabla_{g(t)} f - \frac{\nabla_{g(t)} f \otimes \nabla_{g(t)} f}{m - n}, \quad (1.4)$$

where $Ric(g)$ is the usual Ricci tensor, ∇_g denotes the gradient operator with respect to g and Tr stands for the metric trace on $(0, 2)$ -tensors. Note additionally that in (1.4), when $m = n$, one admits only constant functions f in which case $Ric_f^n(g) \equiv Ric(g)$, whilst one extends the formulation to $m = \infty$ via $Ric_f(g) = Ric_f^\infty(g) \equiv Ric(g) + \nabla \nabla f$.

¹We also consider the nonlinearity $Lu(\log u)^\alpha$ for a more restricted choice of α (see Remark 2.3).

All these different Ricci type tensors carry significant geometric information and are of great importance in the theory. The first goal of this article is to derive local gradient estimates of elliptic type for positive solutions u to (1.1) under a (k, m) Perelman-Ricci flow. The statement of the result is contained in Theorem 1.1. Such elliptic gradient estimates were first obtained by Hamilton [10] for the linear heat equation in the static case and later extended by various authors to different settings including to evolving geometries and nonlinear equations (see the discussion and references below). Notice that when $L = 0$, then (1.1) reduces to the linear weighted heat or f -heat equation

$$\left(\mathcal{L}_f - \frac{\partial}{\partial t}\right)u(x, t) = 0, \quad (1.5)$$

(where if additionally f is constant this is the ordinary heat equation). Here by analogy with the heat equation case the fundamental solution to (1.5) is called the weighted heat kernel. The second goal of this article (Theorem 1.2) is to derive pointwise bounds on the weighted heat kernel of $(M, g, e^{-f} dv)$ verifying a constrained volume $(0, m)$ -super Perelman-Ricci flow

$$\frac{1}{2} \frac{\partial g}{\partial t} + Ric_f^m(g) \geq 0, \quad (1.6)$$

$$\frac{\partial f}{\partial t} - \frac{1}{2} \text{Tr} \left(\frac{\partial g}{\partial t} \right) = 0. \quad (1.7)$$

The importance of gradient estimates in the fields of geometric analysis and PDEs cannot be overlooked (see [3, 6, 10, 14, 15, 16, 25, 27, 31, 33, 34] and [9, 13, 29]). In case of evolution equations on weighted manifolds involving the f -Laplacian there has been a surge of recent research interest due to the intimate connections here with differential geometry, probability theory, quantum field theory and statistical mechanics. Here the remarkable theory of "carré du champ" for Markov diffusion operators and the curvature dimension condition $CD(k, m)$ as developed by Bakry-Émery (see, e.g., [4, 5]) naturally lead to the generalised m -Ricci curvature tensor Ric_f^m [cf. (1.4)] and therefore the interplay between these objects and the various local and global PDE estimates is of huge research interest. As a matter of fact some closely related work here include: extensions of Li-Yau and Hamilton Harnack estimates to the weighted manifold setting [16, 18], derivation of the monotonicity formula and rigidity results for the W -entropy functional and the curvature condition [17, 19] and the proof of the optimal logarithmic Sobolev inequalities for time dependent weighted manifolds [15, 19] (see [20, 26, 32, 36] for other related results and discussion).

Motivated by the above this article makes a contribution to theory by establishing local elliptic type gradient estimates for positive smooth solutions to the nonlinear f -heat equation (1.1) when the metric and potential evolve under a (k, m) Perelman-Ricci flow. Next, by invoking entropy monotonicity arguments, it establishes ultracontractive estimates and pointwise upper bounds on the weighted heat kernel under a $(0, m)$ -super Perelman-Ricci flow. The latter bounds and estimates are directly expressed in terms of the sharp constants of the associated family of logarithmic Sobolev inequalities. The article ends by presenting some useful implications and consequences of these bounds and estimates, in particular, some global estimates and Liouville type results.

1.2. Background. Let (M, g) be a complete n -dimensional Riemannian manifold. For $f \in C^\infty(M)$ we put $d\mu = e^{-f} dv$, where dv is the Riemannian volume measure on (M, g) . The triple $(M, g, d\mu)$ is called a weighted manifold (see [4] for an introduction to the subject). The weighted manifold is endowed with the f -Laplacian, which is a symmetric diffusion operator with respect to $d\mu$, acting on smooth functions $U \in C^\infty(M)$ by,

$$\mathcal{L}_f U := \Delta U - \nabla f \nabla U. \quad (1.8)$$

The f -Laplacian (also known as the Witten Laplacian) is a natural extension of the standard Laplace-Beltrami operator and has close links to many areas in geometry and physics. The integration by parts formula in this setting gives for $U, V \in C_0^\infty(M)$:

$$\int_M V \mathcal{L}_f U \, d\mu = - \int_M \nabla U \nabla V \, d\mu = - \int_M \nabla V \nabla U \, d\mu = \int_M U \mathcal{L}_f V \, d\mu. \quad (1.9)$$

Regarding the curvature properties of $(M, g, d\mu)$ one introduces the generalised Bakry-Émery Ricci tensor linked with the f -Laplacian $\mathcal{L}_f$ by ([4, 5])

$$Ric_f^m(g) = Ric(g) + \nabla \nabla f - \frac{\nabla f \otimes \nabla f}{m - n}, \quad m \in [n, \infty], \quad (1.10)$$

which is a natural extension of the Ricci tensor $Ric = Ric(g)$ to the weighted manifold context. Here $m \geq n$ is a constant. In the event $m = n$, to make sense of (1.10), one admits only constant functions f in which case $Ric_f^n(g) \equiv Ric(g)$ whilst in the event $m = \infty$ one sets $Ric_f(g) = Ric_\infty(g) = Ric(g) + \nabla \nabla f$.

By a (k, m) -super Perelman-Ricci flow on a complete manifold M we understand the evolution of a family of time dependent Riemannian metrics $g(t)$ and smooth potentials $f(t)$ on M , denoted hereafter $(M, g(t), f(t))_{t \in [0, T]}$, under

$$\frac{1}{2} \frac{\partial g}{\partial t} + Ric_f^m(g) \geq -kg. \quad (1.11)$$

Taking $k = 0$ gives the $(0, m)$ -super Perelman-Ricci flow [cf. (1.6)] and replacing the inequality in (1.11) with equality gives the (k, m) Perelman-Ricci flow (1.2). Under the assumption that the weighted measure $d\mu = e^{-f} dv$, remains static in time, it follows from (1.2) and (1.3) that the potential $f(t)$ evolves by

$$\frac{\partial f}{\partial t} = -R - \Delta f + \frac{|\nabla f|^2}{m - n} - nk. \quad (1.12)$$

Interestingly, the specific case $(k, m) = (0, \infty)$ in (1.2)-(1.3) is precisely what was considered by Perelman in [23] as a gradient flow of the energy functional $\mathcal{F}(g, f) = \int_M (R + |\nabla f|^2) e^{-f} dv$, subject to a stationary volume measure constraint as seen. In this case $Ric_f^m(g) = Ric_f(g) = Ric(g) + \nabla \nabla f$, with f satisfying the heat type equation,

$$\frac{\partial f}{\partial t} = -R - \Delta f. \quad (1.13)$$

Recall that the Hamilton-Ricci flow [11] can be recovered from (1.2)-(1.3) by pulling back $(g(t), f(t))$ through a family of diffeomorphisms generated by the gradient of f (see [23] for more or [8, 12, 36]). For other but related problems with volume measure constraints in PDEs see [21, 22, 29] and the references therein.

1.3. Main results. Fix a reference point x_0 on M and let $d = d_{g(t)}(x, x_0)$ denote the Riemannian distance from x_0 to x with respect to the metric $g = g(t)$. We sometimes write $d(x, x_0; t)$ for simplicity, and when the choice of the reference point is clear from the context, we write $d(x, t)$. For $R, T > 0$ we define the compact set $\mathcal{Q}_{R,T}$ by

$$\mathcal{Q}_{R,T} \equiv \{(x, t) : d(x, x_0; t) \leq R, 0 \leq t \leq T\} \subset M \times [0, T]. \quad (1.14)$$

When the weighted measure $d\mu$ is required to be *stationary*, i.e., time independent, we additionally assume (1.3) to hold by virtue of

$$\frac{\partial d\mu_t}{\partial t} = \frac{\partial}{\partial t} (e^{-f(t)} dv) = \left[-\frac{\partial f}{\partial t} + \frac{1}{2} \text{Tr} \left(\frac{\partial g}{\partial t} \right) \right] d\mu_t = 0. \quad (1.15)$$

The first result is a local elliptic gradient estimate for positive smooth solutions to the nonlinear f -heat equation (1.1). Note that here $\mathcal{L}_f = \Delta_{g(t)} - \nabla_{g(t)} f \nabla_{g(t)}$ [cf. (1.8)]. Moreover $x_0 \in M$ and $R \geq 2$ are chosen and will be fixed whilst $T > 0$ and $\mathcal{Q}_{R,T}$ is as just defined. The estimate makes use of the bounds on u and $\text{Ric}_f^m(g)$ on the compact set $\mathcal{Q}_{R,T}$. Here a bound on the modulus of the latter symmetric tensor means a simultaneous upper and lower bound on the tensor by the positive and negative multiples of the same constant bounding the modulus times the metric (see below and the proof). For extensions, refinements and other discussion see the comments following the proof, in particular see Remarks 2.2 and 2.3. Throughout we write $X_+ = \max(X, 0)$ and $X_- = \max(-X, 0)$. Note finally that due to the local nature of the estimate here all one needs is for u to be a positive solution in any open set containing $\mathcal{Q}_{R,T}$.

Theorem 1.1. *Let $(M, g(t), f(t))_{t \in [0, T]}$ be a complete solution to the (k, m) Perelman-Ricci flow (1.2) and let u be a solution to (1.1). Assume $u > 0$ and $|\text{Ric}_f^m(g)| \leq (m-1)\mathbf{k}$ (or $|\text{Ric}_f^\infty(g)| \leq (n-1)\mathbf{k}$ respectively) for some $\mathbf{k} \geq 0$ in $\mathcal{Q}_{R,T}$. Then there exists $C > 0$ depending on n, m such that, for all $(x, t) \in \mathcal{Q}_{R/2, T}$ with $t > 0$,*

$$\frac{|\nabla u|}{u + \sqrt{u}} \leq C \left(\frac{1}{R} + \sqrt{\frac{\mu_+}{R}} + \sqrt{\mathbf{K}} + \frac{1}{\sqrt{t}} + \sqrt{L_+(\mathbf{M}^\alpha + \alpha \mathbf{M}_1^{\alpha-1}) + \alpha L_- \mathbf{M}_0^{\alpha-1}} \right) \left(\sup_{\mathcal{Q}_{R,T}} \sqrt{u} \right). \quad (1.16)$$

Here $\mu_+ = [\max_{\{(x,t): d(x, x_0; t)=1, 0 \leq t \leq T\}} \mathcal{L}_f r]_+$, $\mathbf{M} = \sup_{\mathcal{Q}_{R,T}} |\log u|$, $\mathbf{M}_1 = \sup_{\mathcal{Q}_{R,T}} [\log u]_+$ and $\mathbf{M}_0 = \sup_{\mathcal{Q}_{R,T}} [\log u]_-$ whilst $\mathbf{K} = \sqrt{k_+^2 + \mathbf{k}^2}$.

Recall that the weighted heat kernel, denoted $H(x, y; t)$, $t > 0$, is, the integral kernel to the solution operator for the weighted heat equation. Alternatively, for any fixed, say $y \in M$, $H(x, y; t)$ is the fundamental (minimal positive) solution to (1.5) with initial data $\lim_{t \searrow 0} H(x, y; t) = \delta_y^f(x)$ where δ_y^f is the weighted or f -Dirac delta concentrated at y . For background on heat kernels, in particular, existence, uniqueness and non-uniqueness and other qualitative properties the interested reader is referred to [5, 7, 9] (see also [13, 20, 35, 36] and the references therein).

The next result gives pointwise upper bounds on the heat kernel by exploiting its relationship with the logarithmic Sobolev inequality. To this end let $(M, g(t), f(t))_{t \in [0, T]}$ be a complete solution to (1.6)-(1.7) and $W_m = W_m(g, v, \tau)$ denote the weighted heat entropy functional associated with $(M, g, e^{-f} dv)$ and the flow as defined in Section 3. We assume here that M is closed and write $H = H(x, y; t; T)$ to emphasise the role of $0 < t < T$. Here $n \leq m < \infty$.

Theorem 1.2. *Let $(M, g(t), f(t))_{t \in [0, T]}$ satisfy the $(0, m)$ -super Perelman-Ricci flow (1.6)-(1.7) and let $H(x, y; t; T)$ be the fundamental solution to the weighted heat equation (1.5). Then with $\mathcal{B}_m = \inf\{\mu_m(t) : [0, T]\}$ and for $0 < t < T$ it holds*

$$H(x, y; t; T) \leq (4\pi t)^{-m/2} e^{-\mathcal{B}_m}. \quad (1.17)$$

Here $\mu_m(t) = \inf\{W_m(g, v, t) : \|(4\pi t)^{-m/2} e^{-v}\|_{L^1(M; d\mu)} = 1\}$ is the best constant in the logarithmic Sobolev inequality associated with $\mathcal{L}_f$ on $(M, g(t), f(t))_{t \in [0, T]}$.

To illustrate some consequences and applications of the local gradient estimate (1.16), in the last section we present global parabolic estimates on positive solutions to (1.1) under a (k, m) Perelman-Ricci flow as well as Liouville type results on the elliptic (non-evolutionary) counterpart of (1.1). The final result here is a curvature free estimate on positive solutions to (1.1) and the closely related equation $(\partial/\partial t - \mathcal{L}_f)u - Lu(\log u)^\alpha = 0$ with $\alpha \geq 1$ integer. Here M is taken to be closed.

Theorem 1.3. *Assume $(M, g(t), f(t))_{t \in [0, T]}$ is a solution to the $(0, m)$ -super Perelman-Ricci flow (1.6). Let u be a positive solution to $(\partial/\partial t - \mathcal{L}_f)u - Lu|\log u|^\alpha = 0$ with $\alpha > 1$. Then if $L \leq 0$ and $u \geq 1$ we have the global estimate:*

$$0 \leq |u(x, t)|^2 + |\sqrt{2t} \nabla u(x, t)|^2 \leq \sup_M |u(x, 0)|^2. \quad (1.18)$$

Assume now that u is a positive solution to $(\partial/\partial t - \mathcal{L}_f)u - Lu(\log u)^\alpha = 0$ with $\alpha \geq 1$ integer. For α even we have exactly as before. For α odd, if $L \geq 0$ and $0 < u \leq e^{-\alpha}$ or if $L \leq 0$ and $u \geq 1$, then the curvature free estimate (1.18) holds.

2. PROOF OF THEOREM 1.1

2.1. Important lemma. Before we attend to the proof of Theorem 1.1, we will discuss some tools that would be used. One of these is the following lemma and the other is a cut-off function technique that is discussed afterwards.

Lemma 2.1. *Let $(M, g(t), f(t))_{t \in [0, T]}$ be a solution to the (k, m) -super Perelman-Ricci flow (1.11) and u a positive solution to the nonlinear f -heat equation (1.1). Set $W = |\nabla \log(1 + h)|^2 = |\nabla h|^2/(1 + h)^2$ where $h = u^\beta$ with $\beta \in (0, 1)$. Then*

$$\begin{aligned} \left(\mathcal{L}_f - \frac{\partial}{\partial t}\right) W &\geq -2 \left(\frac{1}{1+h} - \frac{\beta-1}{\beta h}\right) \nabla h \nabla W - \frac{2(\beta-1)}{\beta h^2} (1+h) W^2 \\ &\quad - 2\beta^{1-\alpha} L \left(\frac{|\log h|^\alpha}{1+h} + \alpha |\log h|^{\alpha-2} \log h\right) W - 2kW. \end{aligned} \quad (2.1)$$

Proof. A straightforward calculation with $h = u^\beta$ gives $h_t = \beta u^{\beta-1} u_t$, $\nabla h = \beta u^{\beta-1} \nabla u$ and $\Delta h = \beta(u^{\beta-1} \Delta u + (\beta-1)u^{\beta-2} |\nabla u|^2)$. Moreover it is easily seen that

$$\frac{|\nabla u|^2}{u^2} = |\nabla \log u|^2 = \frac{1}{\beta^2} |\nabla \log h|^2 = \frac{|\nabla h|^2}{\beta^2 h^2}.$$

Hence referring to (1.1) and (1.8) we have

$$\begin{aligned}
\mathcal{L}_f h &= \beta u^{\beta-1} \Delta u + \beta(\beta-1)u^{\beta-2} |\nabla u|^2 - \beta u^{\beta-1} \nabla f \nabla u \\
&= \beta u^{\beta-1} \mathcal{L}_f u + \beta(\beta-1)u^{\beta-2} |\nabla u|^2 \\
&= \beta u^{\beta-1} (u_t - Lu |\log u|^\alpha) + \beta(\beta-1)u^{\beta-2} |\nabla u|^2 \\
&= h_t - \beta^{1-\alpha} Lh |\log h|^\alpha + (\beta-1) |\nabla h|^2 / (\beta h),
\end{aligned}$$

and subsequently this gives

$$\left(\mathcal{L}_f - \frac{\partial}{\partial t} \right) h = (\beta-1) \frac{|\nabla h|^2}{\beta h} - L_\beta h |\log h|^\alpha, \quad (2.2)$$

where $L_\beta = \beta^{1-\alpha} L$. Let us now derive an evolution for the function $W = |\nabla h|^2 / (1+h)^2$. First we note that

$$W_t = \frac{(|\nabla h|^2)_t}{(1+h)^2} - \frac{2|\nabla h|^2 h_t}{(1+h)^3}. \quad (2.3)$$

Following standard computations in the theory of geometric flows, the evolution of the metric under the (k, m) -super Perelman-Ricci flow (1.11) gives

$$\frac{\partial}{\partial t} |\nabla h|^2 = -\frac{\partial g_{ij}}{\partial t} \nabla_i h \nabla_j h + 2 \nabla h \nabla h_t \leq 2 Ric_f^m(\nabla h, \nabla h) + 2 \nabla h \nabla h_t + 2k |\nabla h|^2. \quad (2.4)$$

Thus referring to (2.3) and substituting from (2.2) and (2.4) results in

$$\begin{aligned}
W_t &\leq \frac{2 Ric_f^m(\nabla h, \nabla h)}{(1+h)^2} + \frac{2 \nabla h \nabla h_t}{(1+h)^2} + \frac{2k |\nabla h|^2}{(1+h)^2} - \frac{2 |\nabla h|^2 h_t}{(1+h)^3} \\
&\leq \frac{2 Ric_f^m(\nabla h, \nabla h)}{(1+h)^2} + \frac{2 \nabla h}{(1+h)^2} \nabla \left(\mathcal{L}_f h - \frac{\beta-1}{\beta} \frac{|\nabla h|^2}{h} + L_\beta h |\log h|^\alpha \right) \\
&\quad - \frac{2 |\nabla h|^2}{(1+h)^3} \left(\mathcal{L}_f h - \frac{\beta-1}{\beta} \frac{|\nabla h|^2}{h} + L_\beta h |\log h|^\alpha \right) + \frac{2k |\nabla h|^2}{(1+h)^2}. \quad (2.5)
\end{aligned}$$

Next taking the gradient inside the second term on the right-hand side gives

$$\begin{aligned}
W_t &\leq \frac{2 Ric_f^m(\nabla h, \nabla h)}{(1+h)^2} + \frac{2 \nabla h}{(1+h)^2} \left\{ \nabla \mathcal{L}_f h - \frac{\beta-1}{\beta} \frac{\nabla |\nabla h|^2}{h} \right. \\
&\quad \left. + \frac{(\beta-1)}{\beta} \frac{|\nabla h|^2}{h^2} \nabla h + L_\beta |\log h|^\alpha \nabla h + \alpha L_\beta \log h |\log h|^{\alpha-2} \nabla h \right\} \\
&\quad - \frac{2 |\nabla h|^2}{(1+h)^3} \left(\mathcal{L}_f h - \frac{\beta-1}{\beta} \frac{|\nabla h|^2}{h} + L_\beta h |\log h|^\alpha \right) + \frac{2k |\nabla h|^2}{(1+h)^2}, \quad (2.6)
\end{aligned}$$

and so it is plain that

$$\begin{aligned}
W_t &\leq \frac{2 Ric_f^m(\nabla h, \nabla h)}{(1+h)^2} + \frac{2 \nabla h \nabla \mathcal{L}_f h}{(1+h)^2} - \frac{4(\beta-1)}{\beta} \frac{\nabla^2 h(\nabla h, \nabla h)}{h(1+h)^2} \\
&\quad + \frac{1}{(1+h)^2} \left\{ \frac{2(\beta-1)}{\beta} \frac{|\nabla h|^4}{h^2} + 2L_\beta |\nabla h|^2 (|\log h|^\alpha + \alpha |\log h|^{\alpha-2} \log h) \right\} \\
&\quad - \frac{2 |\nabla h|^2}{(1+h)^3} \left(\mathcal{L}_f h - \frac{\beta-1}{\beta} \frac{|\nabla h|^2}{h} + L_\beta h |\log h|^\alpha \right) + \frac{2k |\nabla h|^2}{(1+h)^2}. \quad (2.7)
\end{aligned}$$

Now using the Ricci identity $Ric(\nabla h, \nabla h) = \nabla h \Delta \nabla h - \nabla h \nabla \Delta h$ we first note that

$$\begin{aligned} \nabla h \Delta \nabla h - \nabla^2 h(\nabla f, \nabla h) &= \nabla h(\nabla \Delta h + Ric(\nabla h, \cdot)) - \nabla^2 h(\nabla f, \nabla h) \\ &= \nabla h \nabla(\Delta h - \nabla f \nabla h) + (Ric + \nabla^2 f)(\nabla h, \nabla h), \end{aligned}$$

and then by adding relevant terms to both sides the following identity on Ric_f^m follows

$$\frac{2\nabla h \Delta \nabla h}{(1+h)^2} - \frac{2\nabla^2 h(\nabla f, \nabla h)}{(1+h)^2} - \frac{2[\nabla f \otimes \nabla f](\nabla h, \nabla h)}{(m-n)(1+h)^2} = \frac{2\nabla h \nabla \mathcal{L}_f h}{(1+h)^2} + \frac{2Ric_f^m(\nabla h, \nabla h)}{(1+h)^2}.$$

Substituting the above identity back into (2.7) cancels out the Ric_f^m term and yields

$$\begin{aligned} W_t &\leq \frac{2\nabla h \Delta \nabla h}{(1+h)^2} - \frac{2\nabla^2 h(\nabla f, \nabla h)}{(1+h)^2} - \frac{2[\nabla f \otimes \nabla f](\nabla h, \nabla h)}{(m-n)(1+h)^2} \\ &\quad + \frac{1}{(1+h)^2} \left(\frac{2(\beta-1)}{\beta} \frac{|\nabla h|^4}{h^2} - \frac{4(\beta-1)}{\beta} \frac{\nabla^2 h(\nabla h, \nabla h)}{h} \right) \\ &\quad + 2L_\beta \frac{|\nabla h|^2}{(1+h)^2} (|\log h|^\alpha + \alpha |\log h|^{\alpha-2} \log h) + \frac{2k|\nabla h|^2}{(1+h)^2} \\ &\quad - \frac{2|\nabla h|^2}{(1+h)^3} \left(\mathcal{L}_f h - \frac{\beta-1}{\beta} \frac{|\nabla h|^2}{h} + L_\beta h |\log h|^\alpha \right). \end{aligned} \quad (2.8)$$

Similarly, for the gradient and Laplacian of W we have,

$$\begin{aligned} \nabla W &= \frac{\nabla |\nabla h|^2}{(1+h)^2} - \frac{2|\nabla h|^2 \nabla h}{(1+h)^3}, \\ \Delta W &= \frac{2|\nabla^2 h|^2}{(1+h)^2} + \frac{2\nabla h \Delta \nabla h}{(1+h)^2} - \frac{8\nabla^2 h(\nabla h, \nabla h)}{(1+h)^3} - \frac{2|\nabla h|^2 \Delta h}{(1+h)^3} + \frac{6|\nabla h|^4}{(1+h)^4}, \end{aligned}$$

which then gives

$$\begin{aligned} \mathcal{L}_f W &= \Delta W - \nabla f \nabla W = \frac{2|\nabla^2 h|^2}{(1+h)^2} + \frac{2\nabla h \Delta \nabla h}{(1+h)^2} - \frac{8\nabla^2 h(\nabla h, \nabla h)}{(1+h)^3} \\ &\quad + \frac{6|\nabla h|^4}{(1+h)^4} - \frac{2\nabla^2 h(\nabla f, \nabla h)}{(1+h)^2} - \frac{2|\nabla h|^2 \mathcal{L}_f h}{(1+h)^3}. \end{aligned} \quad (2.9)$$

Now putting together the individual descriptions of W_t and $\mathcal{L}_f W$ from (2.8) and (2.9) respectively and taking into account the relevant cancellations we have

$$\begin{aligned} \left(\mathcal{L}_f - \frac{\partial}{\partial t} \right) W &\geq \frac{2|\nabla^2 h|^2}{(1+h)^2} - \left(\frac{8}{1+h} - \frac{4(\beta-1)}{\beta h} \right) \frac{\nabla^2 h(\nabla h, \nabla h)}{(1+h)^2} \\ &\quad + \left(\frac{6}{1+h} - \frac{2(\beta-1)}{\beta} \frac{1+h}{h^2} - \frac{2(\beta-1)}{\beta h} \right) \frac{|\nabla h|^4}{(1+h)^3} \\ &\quad - 2L_\beta \frac{|\nabla h|^2}{(1+h)^2} (|\log h|^\alpha + \alpha |\log h|^{\alpha-2} \log h) - \frac{2k|\nabla h|^2}{(1+h)^2} \\ &\quad + 2L_\beta \frac{|\log h|^\alpha}{1+h} \frac{h|\nabla h|^2}{(1+h)^2} + \frac{2[\nabla f \otimes \nabla f](\nabla h, \nabla h)}{(m-n)(1+h)^2}, \end{aligned} \quad (2.10)$$

or after a rearrangement of terms

$$\begin{aligned}
\left(\mathcal{L}_f - \frac{\partial}{\partial t}\right) W &\geq \frac{2|\nabla^2 h|^2}{(1+h)^2} - \frac{4\nabla^2 h(\nabla h, \nabla h)}{(1+h)^3} + \frac{2|\nabla h|^4}{(1+h)^4} \\
&\quad - \left(\frac{4}{1+h} - \frac{4(\beta-1)}{\beta h}\right) \frac{\nabla^2 h(\nabla h, \nabla h)}{(1+h)^2} \\
&\quad + \left(\frac{4}{1+h} - \frac{2(\beta-1)}{\beta} \frac{1+h}{h^2} - \frac{2(\beta-1)}{\beta h}\right) \frac{|\nabla h|^4}{(1+h)^3} \\
&\quad - 2L_\beta \frac{|\nabla h|^2}{(1+h)^2} (|\log h|^\alpha + \alpha |\log h|^{\alpha-2} \log h) - \frac{2k|\nabla h|^2}{(1+h)^2} \\
&\quad + 2L_\beta \frac{|\log h|^\alpha}{1+h} \frac{h|\nabla h|^2}{(1+h)^2} + \frac{2[\nabla f \otimes \nabla f](\nabla h, \nabla h)}{(m-n)(1+h)^2}. \tag{2.11}
\end{aligned}$$

A close examining of the first three terms on the right-hand side followed by a routine set of calculations next lead to

$$\begin{aligned}
\left(\mathcal{L}_f - \frac{\partial}{\partial t}\right) W &\geq \frac{2}{(1+h)^2} \left| \nabla^2 h - \frac{\nabla h \otimes \nabla h}{1+h} \right|^2 + \left(\frac{4|\nabla h|^4}{(1+h)^4} - \frac{4\nabla^2 h(\nabla h, \nabla h)}{(1+h)^3} \right) \\
&\quad + \frac{4(\beta-1)}{\beta} \frac{\nabla^2 h(\nabla h, \nabla h)}{h(1+h)^2} - \frac{2(\beta-1)}{\beta h} \left(\frac{1}{h} + \frac{1}{1+h} \right) \frac{|\nabla h|^4}{(1+h)^2} \\
&\quad - \frac{2k|\nabla h|^2}{(1+h)^2} - 2L_\beta \frac{|\nabla h|^2}{(1+h)^2} (|\log h|^\alpha + \alpha |\log h|^{\alpha-2} \log h) \\
&\quad + 2L_\beta \frac{|\log h|^\alpha}{1+h} \frac{h|\nabla h|^2}{(1+h)^2} + \frac{2[\nabla f \otimes \nabla f](\nabla h, \nabla h)}{(m-n)(1+h)^2}. \tag{2.12}
\end{aligned}$$

Notice also that by direct evaluation we have the relations

$$\begin{aligned}
\frac{[\nabla f \otimes \nabla f](\nabla h, \nabla h)}{(m-n)(1+h)^2} &= \frac{|\nabla f \nabla h|^2}{(m-n)(1+h)^2} \geq 0, \\
\frac{2\nabla h \nabla W}{1+h} &= \frac{4\nabla^2 h(\nabla h, \nabla h)}{(1+h)^3} - \frac{4|\nabla h|^4}{(1+h)^4},
\end{aligned}$$

the first inequality being true by virtue of $m \geq n$. Putting all these together we have

$$\begin{aligned}
\left(\mathcal{L}_f - \frac{\partial}{\partial t}\right) W &\geq -2 \left(\frac{1}{1+h} - \frac{\beta-1}{\beta h} \right) \nabla h \nabla W - \frac{2(\beta-1)}{\beta h^2} \frac{|\nabla h|^4}{(1+h)^3} - \frac{2k|\nabla h|^2}{(1+h)^2} \\
&\quad - 2L_\beta \frac{|\nabla h|^2}{(1+h)^2} (|\log h|^\alpha + \alpha |\log h|^{\alpha-2} \log h) + 2L_\beta \frac{|\log h|^\alpha}{1+h} \frac{h|\nabla h|^2}{(1+h)^2},
\end{aligned}$$

or upon invoking the identity $W = |\nabla h|^2/(1+h)^2$ on the right-hand side that

$$\begin{aligned}
\left(\mathcal{L}_f - \frac{\partial}{\partial t}\right) W &\geq -2 \left(\frac{1}{1+h} - \frac{\beta-1}{\beta h} \right) \nabla h \nabla W - \frac{2(\beta-1)}{\beta h^2} (1+h) W^2 - 2kW \\
&\quad - 2L_\beta (|\log h|^\alpha + \alpha |\log h|^{\alpha-2} \log h) W + 2L_\beta \frac{|\log h|^\alpha}{1+h} hW, \tag{2.13}
\end{aligned}$$

from where the required inequality follows at once by setting $L_\beta = \beta^{1-\alpha} L$. $\square$

In the rest of this section, Lemma 2.1 and a localisation in space-time, using suitable cut-off functions (see below) will be combined to establish the estimate in Theorem 1.1. Here apart from the technical details and proof tricks involved, the method essentially goes back to Li and Yau [14] and is by now a folklore. (See, e.g., Souplet and Zhang [25], Băileşteanu et al [3], Brighton [6], Wei and Wylie [32] and Wu [33]. For more discussion on the use of cut-off techniques and localisation in deriving various estimates in PDEs see also [28, 29].) Now to proceed fix $R, T > 0$ and pick $\tau \in (0, T]$. In order to localise the evolution inequality obtained in the previous lemma to a compact space-time region for the sake of our estimate, it is convenient to introduce through composition, the radially symmetric function

$$\phi(x, t) = \bar{\phi}(r(x, t), t), \quad (2.14)$$

that by way of construction (see notes below) will constitute a smooth cut-off function supported in the compact set $\mathcal{Q}_{R,T} \subset M \times [0, T]$. Here $r(x, t) = d_{g(t)}(x, x_0)$ is the geodesic radial variable at time t measuring the distance between the reference point x_0 and the point x on M whilst $0 \leq t \leq T$. Referring to (2.14), $\bar{\phi} : [0, \infty) \times [0, T] \rightarrow \mathbb{R}$ (we write $\bar{\phi} = \bar{\phi}(r, t)$) is a smooth function satisfying the following properties:

- (i) $\text{supp } \bar{\phi}(r, t) \subset [0, R] \times [0, T]$ and $0 \leq \bar{\phi}(r, t) \leq 1$ in $[0, R] \times [0, T]$.
- (ii) $\bar{\phi} = 1$ in $[0, R/2] \times [\tau, T]$ and $\partial \bar{\phi} / \partial r = 0$ in $[0, R/2] \times [0, T]$, respectively.
- (iii) $|\partial \bar{\phi} / \partial t| \leq c \bar{\phi}^{1/2} / \tau$ on $[0, \infty) \times [0, T]$ for some $c > 0$ and $\bar{\phi}(r, 0) = 0 \forall r \in [0, \infty)$.
- (iv) $-c_a \bar{\phi}^a / R \leq \partial \bar{\phi} / \partial r \leq 0$ and $|\partial^2 \bar{\phi} / \partial r^2| \leq c_a \bar{\phi}^a / R^2$ hold on $[0, \infty) \times [0, T]$ for every $0 < a < 1$ and some $c_a > 0$.

2.2. Proof of Theorem 1.1. Fix $\tau \in (0, T]$ and let ϕ be as in (2.14) with $\bar{\phi}$ as above. We will show that the estimate in the theorem holds at all (x, τ) with $d(x, x_0; \tau) \leq R/2$. The arbitrariness of τ will then justify the assertion for all (x, t) in $\mathcal{Q}_{R/2,T}$ with $t \neq 0$. Now moving on, first by a straightforward calculation it is seen that,

$$\left(\mathcal{L}_f - \frac{\partial}{\partial t} \right) (\phi W) = \phi \left(\mathcal{L}_f - \frac{\partial}{\partial t} \right) W + 2 \nabla \phi \nabla W + W \left(\mathcal{L}_f - \frac{\partial}{\partial t} \right) \phi. \quad (2.15)$$

Thus making use of (2.1) in Lemma 2.1 and writing $b = 2[1/(1+h) - (\beta-1)/(\beta h)] \nabla h$, for the sake of convenience and brevity, we have

$$\begin{aligned} \left(\mathcal{L}_f - \frac{\partial}{\partial t} \right) (\phi W) &\geq -b \phi \nabla W - \frac{2(\beta-1)}{\beta h^2} (1+h) \phi W^2 + 2 \nabla \phi \nabla W \\ &\quad - 2L_\beta \left(\frac{|\log h|^\alpha}{1+h} + \alpha |\log h|^{\alpha-2} \log h \right) \phi W \\ &\quad + W \left(\mathcal{L}_f - \frac{\partial}{\partial t} \right) \phi - 2k \phi W. \end{aligned} \quad (2.16)$$

To proceed forward next note that by the product rule $b \phi \nabla W = b \nabla(\phi W) - (b \nabla \phi) W$. Moreover by a direct calculation $\nabla \phi \nabla W = (\nabla \phi / \phi) \nabla(\phi W) - (|\nabla \phi|^2 / \phi) W$. Therefore

upon substituting in (2.16) we have

$$\begin{aligned}
\left(\mathcal{L}_f - \frac{\partial}{\partial t}\right)(\phi W) &\geq (b\nabla\phi)W - b\nabla(\phi W) + \frac{2\nabla\phi}{\phi}\nabla(\phi W) - \frac{2|\nabla\phi|^2}{\phi}W \\
&\quad - \frac{2(\beta-1)}{\beta h^2}(1+h)\phi W^2 - 2L_\beta \left(\frac{|\log h|^\alpha}{1+h} + \alpha |\log h|^{\alpha-2} \log h\right) \phi W \\
&\quad + W\mathcal{L}_f\phi - W\phi_t - 2k\phi W.
\end{aligned} \tag{2.17}$$

Suppose that ϕW reaches its maximum on $\mathcal{Q}_{R,T}$ at the point (x_1, t_1) . We can assume without loss of generality that x_1 is not on the cut-locus of M by Calabi's argument [14]. We also assume that $(\phi W)(x_1, t_1) > 0$ as otherwise the desired estimate becomes trivial with $W(x, \tau) \leq 0$ whenever $d(x, x_0; \tau) \leq R/2$. Thus in particular $t_1 > 0$ by (iii) and at (x_1, t_1) we have $\mathcal{L}_f(\phi W) \leq 0$, $(\phi W)_t \geq 0$ and $\nabla(\phi W) = 0$. Hence (2.17) implies after the necessary cancellations that

$$\begin{aligned}
-\frac{2(\beta-1)}{\beta h^2}(1+h)\phi W^2 &\leq - (b\nabla\phi)W + \frac{2|\nabla\phi|^2}{\phi}W \\
&\quad + 2L_\beta \left(\frac{|\log h|^\alpha}{1+h} + \alpha |\log h|^{\alpha-2} \log h\right) \phi W \\
&\quad - W\mathcal{L}_f\phi + W\phi_t + 2k\phi W,
\end{aligned} \tag{2.18}$$

at (x_1, t_1) . This can also be written as follows: Using the fact that $\beta \in (0, 1)$ and hence $-(\beta-1) > 0$ after multiplying through by $-\beta h^2/[(\beta-1)(1+h)]$ leads at (x_1, t_1) to

$$\begin{aligned}
2\phi W^2 &\leq \frac{\beta}{\beta-1}(b\nabla\phi)\frac{Wh^2}{1+h} - \frac{2\beta}{\beta-1}\frac{|\nabla\phi|^2}{\phi}\frac{Wh^2}{1+h} \\
&\quad - \frac{2L_\beta\beta}{\beta-1}\left(\frac{|\log h|^\alpha}{1+h} + \alpha |\log h|^{\alpha-2} \log h\right)\phi\frac{Wh^2}{1+h} \\
&\quad - \frac{2\beta}{\beta-1}k\phi\frac{Wh^2}{1+h} + \frac{\beta Wh^2\mathcal{L}_f\phi}{(\beta-1)(1+h)} - \frac{\beta Wh^2\phi_t}{(\beta-1)(1+h)}.
\end{aligned} \tag{2.19}$$

The aim is now to use (2.19) to establish the required estimate at (x, τ) . To this end and prompted by the nature of the technicalities as well as the complexity of terms involved, the argument proceeds by considering two separate cases according as to whether the distance between the reference point and the space-time maximum point is larger or smaller than one respectively. Firstly, we consider the case $d(x_1, x_0; t_1) \geq 1$. Below we apply the properties (i)-(iv) of ϕ as listed earlier, the Cauchy-Schwarz and Young inequalities respectively to obtain suitable upper bounds for each of the terms on the right-hand side of (2.19).

(a) For the first term, by substituting for b [see the line after (2.15)] and noting the

bound $0 < h/(1+h) \leq 1$ we can write,

$$\begin{aligned}
\frac{\beta}{\beta-1}(b\nabla\phi)\frac{Wh^2}{1+h} &\leq \frac{\beta}{1-\beta}\left|(b\nabla\phi)\frac{Wh^2}{1+h}\right| \\
&\leq \frac{2\beta}{1-\beta}\left(\frac{h}{1+h} + \frac{1-\beta}{\beta}\right)\frac{h|\nabla h|}{1+h}|\nabla\phi|W \\
&\leq \frac{2}{1-\beta}\left(\sup_{\mathcal{Q}_{R,T}} h\right)|\nabla\phi|W^{3/2} \\
&\leq \frac{1}{4}\phi W^2 + C(\beta)\left(\frac{|\nabla\phi|(\sup_{\mathcal{Q}_{R,T}} h)}{\phi^{3/4}}\right)^4 \\
&\leq \frac{1}{4}\phi W^2 + \frac{C(\beta)}{R^4}\left(\sup_{\mathcal{Q}_{R,T}} h\right)^4.
\end{aligned} \tag{2.20}$$

(b) For the second term, proceeding in a similar way we have

$$\begin{aligned}
-\frac{2\beta}{\beta-1}\frac{|\nabla\phi|^2}{\phi}\frac{Wh^2}{1+h} &= \frac{2\beta}{1-\beta}\frac{|\nabla\phi|^2}{\phi}\frac{Wh^2}{1+h} \\
&\leq \frac{2\beta}{1-\beta}\frac{|\nabla\phi|^2}{\phi^{3/2}}\phi^{1/2}W\left(\sup_{\mathcal{Q}_{R,T}} h\right)^2 \\
&\leq \frac{1}{4}\phi W^2 + C(\beta)\left(\frac{|\nabla\phi|^2}{\phi^{3/2}}\right)^2\left(\sup_{\mathcal{Q}_{R,T}} h\right)^4 \\
&\leq \frac{1}{4}\phi W^2 + \frac{C(\beta)}{R^4}\left(\sup_{\mathcal{Q}_{R,T}} h\right)^4.
\end{aligned} \tag{2.21}$$

(c) For the third term, recalling the definition of $\mathbf{M}_0$, $\mathbf{M}_1$ and $\mathbf{M}$ from the statement of the theorem and $L_\beta = \beta^{1-\alpha}L$ and $\alpha \geq 1$, we can write

$$\begin{aligned}
-\frac{2\beta L_\beta}{\beta-1}\left(\frac{|\log h|^\alpha}{1+h} + \alpha|\log h|^{\alpha-2}\log h\right)\phi\frac{Wh^2}{1+h} \\
&= \frac{2\beta^{2-\alpha}}{1-\beta}L\left(\frac{|\log h|^\alpha}{1+h} + \alpha|\log h|^{\alpha-2}\log h\right)\phi\frac{Wh^2}{1+h} \\
&\leq \frac{2\beta^{2-\alpha}}{1-\beta}\sqrt{\phi}W\left(\left[\frac{L|\log h|^\alpha}{1+h}\right]_+ + \alpha[L|\log h|^{\alpha-2}\log h]_+\right)\frac{\sqrt{\phi}h^2}{1+h} \\
&\leq \frac{2\beta}{1-\beta}\sqrt{\phi}W(L_+\mathbf{M}^\alpha + \alpha(L_+\mathbf{M}_1^{\alpha-1} + L_-\mathbf{M}_0^{\alpha-1})\left(\sup_{\mathcal{Q}_{R,T}} h\right)^2 \\
&\leq \frac{1}{4}\phi W^2 + C(\beta)\mathbf{M}_{L,\alpha}^2\left(\sup_{\mathcal{Q}_{R,T}} h\right)^4,
\end{aligned} \tag{2.22}$$

where $\mathbf{M}_{L,\alpha} = L_+(\mathbf{M}^\alpha + \alpha\mathbf{M}_1^{\alpha-1}) + \alpha L_-\mathbf{M}_0^{\alpha-1}$. Note that in deducing the second inequality we have made use of the bounds $[L|\log h|^{\alpha-2}\log h]_+ = L[\log h]_+^{\alpha-1} \leq L(\beta\mathbf{M}_1)^{\alpha-1}$ for when $L > 0$ and $[L|\log h|^{\alpha-2}\log h]_+ = |L|[\log h]_-^{\alpha-1} \leq |L|(\beta\mathbf{M}_0)^{\alpha-1}$ for when $L < 0$ by basic considerations and use of the identities $[\log h]_+ = \beta[\log u]_+$, $[\log h]_- = \beta[\log u]_-$.

(d) For the fourth term we can write

$$\begin{aligned} -\frac{2\beta}{\beta-1}k\phi\frac{Wh^2}{1+h} &= \frac{2\beta}{1-\beta}k\phi\frac{Wh^2}{1+h} \\ &\leq \frac{2\beta}{1-\beta}\sqrt{\phi}Wk_+\sqrt{\phi}\left(\sup_{\mathcal{Q}_{R,T}}h\right)^2 \leq \frac{1}{4}\phi W^2 + C(\beta)k_+^2\left(\sup_{\mathcal{Q}_{R,T}}h\right)^4. \end{aligned} \quad (2.23)$$

(e) For the fifth term we utilise the Wei-Wylie weighted Laplacian comparison theorem, taking advantage of the fact that it only depends on the lower bound on $Ric_f^m(g)$ ([32]). Indeed recalling $r(x, t) = d(x, x_0; t)$, $1 \leq d(x_1, x_0; t_1) \leq R$ and noting that both f and g depend on t we proceed by separating the two cases: (1) $m = \infty$ and (2) $n \leq m < \infty$. (1) Given $Ric_f(g) = Ric_f^\infty(g) \geq -(n-1)kg$ with $k \geq 0$ and setting $\mu_+ = \max(\mu, 0) \geq 0$ with $\mu = \max_{\{(x,t): d(x,x_0;t)=1, 0 \leq t \leq T\}} \mathcal{L}_f r(x, t)$, a straightforward application of Theorem 3.1 in [32] yields $\mathcal{L}_f r(x, t) \leq \mu + (n-1)k(R-1)$ for $r \geq 1$. From $\bar{\phi}$ being radial and $\bar{\phi}_r \leq 0$ in particular it then follows that

$$\mathcal{L}_f \phi = \bar{\phi}_{rr}|\nabla r|^2 + \bar{\phi}_r \mathcal{L}_f r \geq \bar{\phi}_{rr} + \bar{\phi}_r[\mu_+ + (n-1)k(R-1)].$$

and so

$$-\mathcal{L}_f \phi \leq |\bar{\phi}_{rr}| + [\mu_+ + (n-1)k(R-1)]|\bar{\phi}_r|.$$

Hence proceeding on to bounding the fifth term in this case we can write

$$\begin{aligned} \frac{\beta h^2 W \mathcal{L}_f \phi}{(\beta-1)(1+h)} &= \frac{\beta h^2 W (-\mathcal{L}_f \phi)}{(1-\beta)(1+h)} \\ &\leq \frac{\beta h^2 \sqrt{\bar{\phi}} W}{(1-\beta)(1+h)} \left(\frac{|\bar{\phi}_{rr}|}{\sqrt{\bar{\phi}}} + [\mu_+ + (n-1)k(R-1)] \frac{|\bar{\phi}_r|}{\sqrt{\bar{\phi}}} \right) \\ &\leq \frac{\beta \sqrt{\bar{\phi}} W}{1-\beta} \left(\frac{|\bar{\phi}_{rr}|}{\sqrt{\bar{\phi}}} + [\mu_+ + (n-1)k(R-1)] \frac{|\bar{\phi}_r|}{\sqrt{\bar{\phi}}} \right) \left(\sup_{\mathcal{Q}_{R,T}} h \right)^2 \\ &\leq \frac{1}{4} \bar{\phi} W^2 + C(\beta) \left(\frac{|\bar{\phi}_{rr}|^2}{\bar{\phi}} + \mu_+^2 \frac{|\bar{\phi}_r|^2}{\bar{\phi}} + n^2 k^2 R^2 \frac{|\bar{\phi}_r|^2}{\bar{\phi}} \right) \left(\sup_{\mathcal{Q}_{R,T}} h \right)^4 \\ &\leq \frac{1}{4} \phi W^2 + n^2 C(\beta) \left(\frac{1}{R^4} + \frac{\mu_+^2}{R^2} + k^2 \right) \left(\sup_{\mathcal{Q}_{R,T}} h \right)^4. \end{aligned} \quad (2.24)$$

(2) Given $Ric_f^m(g) \geq -(m-1)kg$, $k \geq 0$ the generalised Laplace comparison theorem (see [32]) gives

$$\mathcal{L}_f r \leq (m-1)\sqrt{k} \coth(\sqrt{k}r).$$

Therefore by virtue of $\bar{\phi}$ being radial and $\bar{\phi}_r \leq 0$ it follows that,

$$\mathcal{L}_f \phi = \bar{\phi}_{rr}|\nabla r|^2 + \bar{\phi}_r \mathcal{L}_f r \geq \bar{\phi}_{rr} + (m-1)\bar{\phi}_r \sqrt{k} \coth(\sqrt{k}r).$$

Using the bound $\sqrt{k} \coth(\sqrt{k}r) \leq \sqrt{k} \coth(\sqrt{k}R/2) \leq (2 + \sqrt{k}R)/R$ for $R/2 \leq r \leq R$ (here we are making use of $x \coth x \leq 1 + x$ and the monotonicity of $\coth x$ for $x > 0$) and noting that $\bar{\phi}_r \equiv 0$ for $0 \leq r \leq R/2$ it then follows that

$$-\mathcal{L}_f \phi \leq -[\bar{\phi}_{rr} + (m-1)\bar{\phi}_r \sqrt{k} \coth(\sqrt{k}R/2)] \leq |\bar{\phi}_{rr}| + (m-1)(2/R + \sqrt{k})|\bar{\phi}_r|.$$

Therefore again proceeding on to bounding the fifth term in this case we can write

$$\begin{aligned}
 \frac{\beta h^2 W \mathcal{L}_f \phi}{(\beta - 1)(1 + h)} &= \frac{\beta h^2 W(-\mathcal{L}_f \phi)}{(1 - \beta)(1 + h)} \\
 &\leq \frac{\beta \sqrt{\bar{\phi}} W}{1 - \beta} \left(\frac{|\bar{\phi}_{rr}|}{\sqrt{\bar{\phi}}} + (m - 1)(2/R + \sqrt{k}) \frac{|\bar{\phi}_r|}{\sqrt{\bar{\phi}}} \right) \left(\sup_{\mathcal{Q}_{R,T}} h \right)^2 \\
 &\leq \frac{1}{4} \bar{\phi} W^2 + C(\beta) \left(\frac{|\bar{\phi}_{rr}|^2}{\bar{\phi}} + m^2(1/R^2 + k) \frac{|\bar{\phi}_r|^2}{\bar{\phi}} \right) \left(\sup_{\mathcal{Q}_{R,T}} h \right)^4 \\
 &\leq \frac{1}{4} \phi W^2 + m^2 C(\beta) \left(\frac{1}{R^4} + \frac{k}{R^2} \right) \left(\sup_{\mathcal{Q}_{R,T}} h \right)^4. \tag{2.25}
 \end{aligned}$$

(f) For the sixth term we first estimate ϕ_t as follows: For $d(x_1, x_0; t) \leq R$ and fixed $t > 0$ let $\gamma = \gamma(s) : [0, d] \rightarrow M$ be a minimal geodesic with respect to $g(t)$ connecting $x_0 = \gamma(0)$ to $x_1 = \gamma(d)$. Then a straightforward calculation utilising the flow equation $\partial g / \partial t = -2(\text{Ric}_f^m(g) + kg)$ and the upper bound $\text{Ric}_f^m(g) \leq \bar{k}g$ in the theorem gives

$$\begin{aligned}
 \frac{\partial}{\partial t} r(x_1, t) &= \frac{\partial}{\partial t} d_{g(t)}(x_1, x_0) = \frac{\partial}{\partial t} \int_0^d |\gamma'(s)|_{g(t)} ds = \int_0^d \frac{(\partial g / \partial t)(\gamma', \gamma')}{2|\gamma'|} ds \\
 &= \int_0^d -\frac{[\text{Ric}_f^m(g) + kg](\gamma', \gamma')}{|\gamma'|_{g(t)}} ds \geq -(\bar{k} + k)r(x_1, t) \geq -(\bar{k} + k_+)R. \tag{2.26}
 \end{aligned}$$

Here for the uniformity of notation $\bar{k}$ stands for either quantity $(m - 1)k$ or $(n - 1)k$ as given in the theorem. Now referring to (2.14) and applying the properties of $\bar{\phi}$ yields

$$\begin{aligned}
 \frac{\partial \phi}{\partial t} &= \frac{\partial \bar{\phi}}{\partial t} + \frac{\partial \bar{\phi}}{\partial r} \frac{\partial r}{\partial t} \leq \frac{\partial \bar{\phi}}{\partial t} - (\bar{k} + k_+) \frac{\partial \bar{\phi}}{\partial r} R \\
 &\leq \left| \frac{\partial \bar{\phi}}{\partial t} \right| + (\bar{k} + k_+) \left| \frac{\partial \bar{\phi}}{\partial r} \right| R \leq C \left(\frac{1}{\tau} + \bar{k} + k_+ \right) \sqrt{\bar{\phi}}. \tag{2.27}
 \end{aligned}$$

Hence we can write

$$\begin{aligned}
 -\frac{\beta h^2 W \phi_t}{(\beta - 1)(1 + h)} &= \frac{\beta h^2 W \phi_t}{(1 - \beta)(1 + h)} \leq \frac{C \beta h^2}{1 - \beta} \sqrt{\bar{\phi}} W \left(\frac{1}{\tau} + \bar{k} + k_+ \right) \\
 &\leq \frac{1}{4} \phi W^2 + C(\beta) \left(\frac{1}{\tau} + \bar{k} + k_+ \right)^2 \left(\sup_{\mathcal{Q}_{R,T}} h \right)^4 \\
 &\leq \frac{1}{4} \phi W^2 + C(\beta) \left(\frac{1}{\tau^2} + (\bar{k} + k_+)^2 \right) \left(\sup_{\mathcal{Q}_{R,T}} h \right)^4. \tag{2.28}
 \end{aligned}$$

Having estimated all the individual terms on the right-hand side of (2.19) by inserting back and adjusting the constant (recall that $R \geq 2$) we obtain at (x_1, t_1) the inequality

$$2\phi W^2 \leq \frac{3}{2} \phi W^2 + C(\beta) \left(\sup_{\mathcal{Q}_{R,T}} h \right)^4 \left(\frac{1}{R^4} + \frac{\mu_+^2}{R^2} + \frac{1}{\tau^2} + K^2 + M_{L,\alpha}^2 \right)$$

where $K = \sqrt{k_+^2 + k^2}$. (Note that in the case resulting from (2.25) we can remove the term μ_+^2/R^2 .) Now invoking the maximal characterisation of the point (x_1, t_1) we then

have for all $(x, t) \in \mathcal{Q}_{R,T}$ the bound

$$\begin{aligned} \phi^2 W^2(x, t) &\leq \phi^2 W^2(x_1, t_1) \leq \phi W^2(x_1, t_1) \\ &\leq C(\beta) \left(\sup_{\mathcal{Q}_{R,T}} h \right)^4 \left(\frac{1}{R^4} + \frac{\mu_+^2}{R^2} + \frac{1}{\tau^2} + K^2 + M_{L,\alpha}^2 \right). \end{aligned}$$

Since $\phi(x, \tau) = 1$ in $d(x, x_0; \tau) \leq R/2$ and $W = |\nabla h|^2/(1+h)^2$, we then obtain

$$\frac{|\nabla h|}{1+h}(x, \tau) \leq C(\beta) \left(\sup_{\mathcal{Q}_{R,T}} h \right) \left(\frac{1}{R} + \sqrt{\frac{\mu_+}{R}} + \frac{1}{\sqrt{\tau}} + \sqrt{K} + \sqrt{M_{L,\alpha}} \right).$$

Taking $\beta = 1/2$ and noting $2|\nabla h| = 2|\nabla \sqrt{u}| = |\nabla u|/\sqrt{u}$ by virtue of $h = \sqrt{u}$ we have

$$\frac{|\nabla \sqrt{u}|}{1+\sqrt{u}} \leq C \left(\sup_{\mathcal{Q}_{R,T}} \sqrt{u} \right) \left(\frac{1}{R} + \sqrt{\frac{\mu_+}{R}} + \frac{1}{\sqrt{\tau}} + \sqrt{K} + \sqrt{M_{L,\alpha}} \right),$$

which is immediately seen to be the desired result.

It remains to consider the case $d(x_1, x_0; t_1) \leq 1$. As by (ii) ϕ is a constant function in the space direction (when $d(x, x_0; t) \leq R/2$, $t \in [0, T]$ with $R \geq 2$), referring to (2.19), the terms involving space derivatives of ϕ vanish or simplify (in particular $\phi_t = \bar{\phi}_t$) and so as before, with $M_{L,\alpha} = L_+(M^\alpha + \alpha M_1^{\alpha-1}) + \alpha L_- M_0^{\alpha-1}$, it follows that at the point (x_1, t_1) ,

$$\begin{aligned} W &\leq \frac{\beta^{2-\alpha}}{1-\beta} \left\{ L \left(\frac{|\log h|^\alpha}{1+h} + \alpha |\log h|^{\alpha-2} \log h \right) + \left(k + \frac{1}{2} \frac{\phi_t}{\phi} \right) \right\} \frac{h^2}{1+h} \\ &\leq \frac{\beta^{2-\alpha}}{1-\beta} \left\{ (L_+ |\log h|^\alpha + \alpha [L |\log h|^{\alpha-2} \log h]_+) + \left(k_+ + \frac{1}{2} \frac{|\bar{\phi}_t|}{\bar{\phi}} \right) \right\} \frac{h^2}{1+h} \\ &\leq C(\beta) [M_{L,\alpha} + k_+ + 1/\tau] h^2. \end{aligned} \tag{2.29}$$

Noting $W(x, \tau) = \phi W(x, \tau) \leq \phi W(x_1, t_1) \leq W(x_1, t_1)$ due to $\phi(x_1, t_1) \leq 1$, $\phi(x, \tau) = 1$ when $d(x, x_0; \tau) \leq R/2$, the above and the arbitrariness of $\tau > 0$ easily lead to a special case of (1.16). We have thus shown that in either case the estimate holds true and this therefore completes the proof. $\square$

Remark 2.2. An easy inspection of the proof shows that the same estimate (1.16) holds for the (k, m) -super Perelman-Ricci flow (1.11) by using the bounds $Ric_f^m \geq -(m-1)kg$ [or $Ric_f^\infty \geq -(n-1)kg$] and $\partial g/\partial t \geq -2hg$ in $\mathcal{Q}_{R,T}$ with $k, h \geq 0$. [Thus $k = (m-1)k+h$ or $k = (n-1)k+h$.] The only change here will be in the second line of (2.26) where one uses the lower bound on $\partial g/\partial t$ in place of the upper bound on Ric_f^m in $\mathcal{Q}_{R,T}$. Therefore (2.26)-(2.28) hold with h replacing $\bar{k} + k_+$ and in (1.16) we have $K = (k^2 + h^2)^{1/2}$. This is particularly useful when considering the static case $\partial g/\partial t \equiv 0$ where we can set $h = 0$. We will return to this point in the final section of the paper. See also [27].

Remark 2.3. The restriction $\alpha > 1$ in (1.1) is prompted by the failure of differentiability of the modulus function at the origin ($\alpha = 1$). As an interesting observation, replacing the nonlinearity in (1.1) with $Lu(\log u)^\alpha$ for integer $\alpha \geq 1$ gives an identical estimate to (1.16) only with the last term in the square root replaced by $L_+(M_+^\alpha + \alpha M_1^{\alpha-1}) + L_- M_-^\alpha$ for when α is odd and obviously no change when α is even. Due to the sign changing nature of the logarithm considering other values of α is in general meaningless here.

Remark 2.4. For the purpose of the Laplace comparison theorem in part (e) of proof, a lower bound on $Ric_f = Ric_f^\infty$ is a *weaker* assumption than one on Ric_f^m for $m \in [n, \infty)$.

3. THE W_m -ENTROPY AND POINTWISE HEAT KERNEL BOUNDS

3.1. The W_m -entropy functional. Assume $(M, g(t), f(t))_{t \in [0, T]}$ is a solution to the volume constrained $(0, m)$ -super Perelman-Ricci flow (1.6)-(1.7) and let $w = w(x, t)$ be a positive solution to the f -heat equation (1.5). Suppose the initial data $w(x, 0)$ is normalised by

$$\int_M w(x, 0) d\mu(g(0)) = 1. \quad (3.1)$$

A reference to $(\mathcal{L}_f - \partial/\partial_t)w = 0$ followed by an application of the divergence theorem results in (3.1) being true for all $0 \leq t \leq T$. In view of the measure $d\mu(g(t)) = e^{-f} dv$ being static by (1.7), hereafter when necessary and without causing any confusions, we write $d\mu = d\mu(g(t))$. We next recall the definition of the W_m -entropy functional

$$W_m(g, v, t) = \int_M (t|\nabla v|^2 + v - m)(4\pi t)^{-m/2} e^{-v} d\mu, \quad \int_M \frac{e^{-v} d\mu}{(4\pi t)^{m/2}} = 1. \quad (3.2)$$

Note that with the relation $w = (4\pi t)^{-m/2} e^{-v}$ it is at times convenient to rewrite the associated $W_m(g, v, t)$ as

$$W_m(g, v, t) = \int_M (t|\nabla \log w|^2 - \log w) w d\mu - \frac{m}{2} [2 + \log(4\pi t)]. \quad (3.3)$$

It is proved in [19, Theorem 1.2] that along a solution $w = (4\pi t)^{-m/2} e^{-v}$ of the f -heat equation on a closed manifold with (1.7), the W_m -entropy satisfies the entropy formula:

$$\begin{aligned} \frac{d}{dt} W_m(g, v, t) = \int_M \left\{ -2t \left| \nabla^2 \log w + \frac{1}{2t} g \right|^2 - \frac{2t}{m-n} \left(\nabla f \nabla \log w - \frac{m-n}{2t} \right)^2 \right. \\ \left. - 2t \left(\frac{1}{2} \frac{\partial g}{\partial t} + Ric_f^m(g) \right) |\nabla \log w|^2 \right\} w d\mu. \end{aligned} \quad (3.4)$$

Thus in particular under (1.6)-(1.7) we have $d/dt[W_m(g, v, t)] \leq 0$ for all $t \in (0, T]$ and so the W_m -entropy is decreasing. As a consequence of this the optimal logarithmic Sobolev constant associated with $\mathcal{L}_f$ on $(M, g(t), f(t))$ is decreasing in time too [cf. (3.5) below]. On passing we point out that the W_m -entropy has intimate connections with the Li-Yau Harnack inequality and a host of other well known geometric and functional inequalities (cf., e.g., [2, 5, 13, 16, 17, 29, 36]).

Definition 3.1. Let $(M, g(t), f(t))_{t \in [0, T]}$ satisfy (1.6)-(1.7). The optimal logarithmic Sobolev constant $\mu = \mu_m(t)$ is defined by setting

$$\mu_m(t) := \inf \left\{ W_m(g, v, t) : \int_M \frac{e^{-v} d\mu}{(4\pi t)^{m/2}} = 1 \right\}. \quad (3.5)$$

Upon writing $w = e^{-v/2}$ with v being the minimiser at fixed t (see Lemma 3.2 below) it then follows that w is a weak solution to the nonlinear Euler-Langrange equation

$$-4t\mathcal{L}_f w - 2w \log w - mw = \mu_m(t)w. \quad (3.6)$$

For the sake of future reference and with a slight reformulation we can rewrite (3.2), (3.3) and (3.5) as

$$\mu_m(t) = \inf \left\{ \int_M 4t |\nabla w|^2 - w^2 \log w^2 - mw^2 (4\pi t)^{-m/2} d\mu : \int_M \frac{w^2 d\mu}{(4\pi t)^{m/2}} = 1 \right\}. \quad (3.7)$$

Using standard variational techniques it is seen that the infimum in (3.7) is achieved by some nonnegative function $w \in H^1(M)$. Moreover w satisfies (3.6) and by elliptic regularity theory is smooth. For the sake of clarity and convenience of the reader the main facts from [19] are summarised in Lemma 3.2 below. (For further discussion see [2, 12, 23, 24, 30, 36] as well as [18]-[15] and the references therein.)

Lemma 3.2. *Let $(M, g(t), f(t))_{t \in [0, T]}$ be as above and let $\mu = \mu(t)$ as defined by (3.7). Then the following are true:*

- (1) $\mu_m(t)$ is finite and attained; hence it is a minimum,
- (2) there is a positive minimiser w satisfying (3.6) in the weak sense,
- (3) by elliptic regularity theory w and hence $v = -2 \log w$ are smooth,
- (4) by the stated monotonicity of the W_m -entropy, the minimum $\mu_m(t)$ in (3.7) is monotone decreasing under (1.6)-(1.7),
- (5) $\mu_m(t)$ corresponds to the best constant in the logarithmic Sobolev inequality

$$\int_M v^2 \log v^2 d\mu \leq \int_M 4t |\nabla v|^2 d\mu - \frac{m}{2} \log(4\pi t) - m - \mu_m(t), \quad (3.8)$$

$$\text{subject to } \int_M v^2 d\mu = 1.$$

3.2. Ultracontractive estimates.

Lemma 3.3. *Let $(M, g(t), f(t))_{t \in [0, T]}$ be a solution to the $(0, m)$ -super Perelman-Ricci flow (1.6)-(1.7) and w be a positive solution to the f -heat equation (1.5). Consider the constant $\mathcal{B}_m := \inf \mu_m(t)$ with $0 \leq t \leq T$. Then*

$$\|w(\cdot, T)\|_{L^\infty(M; d\mu)} \leq (4\pi T)^{-m/2} \exp(-\mathcal{B}_m) \|w(\cdot, 0)\|_{L^1(M; d\mu)}. \quad (3.9)$$

Proof. Let $w = w(x, t)$ be a positive solution to the f -heat equation $(\mathcal{L}_f - \partial/\partial t)w = 0$. Writing $w(t) = w(\cdot, t)$ let $\mathcal{G}(t) = \log \|w(t)\|_{L^{p(t)}(M; d\mu)}$ and

$$\mathcal{A}(t) = \|w(t)\|_{L^{p(t)}(M; d\mu)}^{p(t)} = \int_M |w(t)|^{p(t)} d\mu = \int_M w^{p(t)}(t) d\mu. \quad (3.10)$$

Here $p = p(t) \in C^1([0, T]; [1, \infty))$ is a continuously differentiable monotone increasing function satisfying the end-point conditions $p(0) = 1$ and $p(T) = \lim_{t \rightarrow T} p(t) = \infty$ with $T > 0$ as above. By routine computations we have

$$\begin{aligned} \frac{d}{dt} \mathcal{A}(t) &= \frac{d}{dt} \|w(t)\|_{L^{p(t)}(M; d\mu)}^{p(t)} = \frac{d}{dt} \int_M w^{p(t)}(t) d\mu \\ &= \int_M (\dot{p}(t) w^{p(t)} \log w + p(t) w^{p(t)-1} w_t) d\mu \\ &= \dot{p}(t) \int_M w^{p(t)} \log w d\mu + p(t) \int_M w^{p(t)-1} w_t d\mu. \end{aligned} \quad (3.11)$$

Likewise taking time derivative of $\mathcal{G}$, using the above and applying the integration by parts formula [cf. (1.9)] we have

$$\begin{aligned}
 \frac{d}{dt}\mathcal{G}(t) &= \frac{d}{dt} \log \|w(t)\|_{L^{p(t)}(M;d\mu)} = \frac{d}{dt} [p^{-1}(t) \log \mathcal{A}(t)] \\
 &= -\dot{p}(t)p^{-2}(t) \log \mathcal{A}(t) + [p(t)\mathcal{A}(t)]^{-1} \frac{d}{dt} \mathcal{A}(t) \\
 &= -\dot{p}(t)p^{-2}(t) \log \mathcal{A}(t) + [p(t)\mathcal{A}(t)]^{-1} \times \\
 &\quad \times \left[\dot{p}(t) \int_M w^{p(t)} \log w \, d\mu + p(t) \int_M w^{p(t)-1} \mathcal{L}_f w \, d\mu \right] \\
 &= \dot{p}(t)p^{-2}(t) \left[p(t)\mathcal{A}^{-1}(t) \int_M w^{p(t)} \log w \, d\mu - \log \mathcal{A}(t) \right] \\
 &\quad - (p(t) - 1)\mathcal{A}^{-1}(t) \int_M w^{p(t)-2} |\nabla w|^2 \, d\mu.
 \end{aligned} \tag{3.12}$$

Making the change of variables $v^2 = \mathcal{A}^{-1}(t)w^{p(t)}$ whereby $\|v\|_{L^2(M;d\mu)} = 1$ [cf. (3.10)] a straightforward calculation gives

$$\int_M v^2 \log v^2 \, d\mu = p(t)\mathcal{A}^{-1}(t) \int_M w^{p(t)} \log w \, d\mu - \log \mathcal{A}(t), \tag{3.13}$$

and $4|\nabla v|^2 = p^2(t)\mathcal{A}^{-1}(t)w^{p(t)-2}|\nabla w|^2$. Hence it is seen from (3.12) that

$$\frac{d}{dt}\mathcal{G}(t) = \frac{\dot{p}(t)}{p^2(t)} \left\{ \int_M v^2 \log v^2 \, d\mu - \frac{4(p(t) - 1)}{\dot{p}(t)} \int_M |\nabla v|^2 \, d\mu \right\}. \tag{3.14}$$

Now note that with the choice of $p(t) = T/(T-t)$ we have $p(0) = 1$, $\lim_{t \rightarrow T} p(t) = \infty$, p is continuously differentiable and monotone increasing on $[0, T)$ whilst

$$\frac{\dot{p}(t)}{p^2(t)} = \frac{1}{T} \quad \text{and} \quad \frac{p(t) - 1}{\dot{p}(t)} = \frac{t(T-t)}{T}. \tag{3.15}$$

Hence invoking the optimal logarithmic Sobolev inequality (3.8) with the choice of the "t" variable there prompted from above, that is, $t(T-t)/T$ (note that this quotient takes values in the interval $(0, T/4)$ for $0 < t < T$) and substituting in (3.14) yields

$$\begin{aligned}
 \frac{d}{dt}\mathcal{G}(t) &\leq \frac{\dot{p}(t)}{p^2(t)} \left[-\mu_m \left(\frac{p(t) - 1}{\dot{p}(t)} \right) - m - \frac{m}{2} \log \left(4\pi \frac{p(t) - 1}{\dot{p}(t)} \right) \right] \\
 &\leq -\frac{1}{T} \inf \mu_m - \frac{1}{T} \left[m + \frac{m}{2} \log 4\pi + \frac{m}{2} \log \frac{t(T-t)}{T} \right].
 \end{aligned}$$

Now referring to Lemma 3.2 and the basic inequality $\mu_m(s) \geq \inf \mu_m = \mathcal{B}_m$, it follows upon integrating both sides of the last inequality with respect to t from 0 to T we have

$$\begin{aligned}
 \log \frac{\|w(T)\|_{L^{p(T)}}}{\|w(0)\|_{L^{p(0)}}} &= \int_0^T \frac{d}{dt}\mathcal{G}(t) \leq -\frac{1}{T} \int_0^T \left\{ \inf \mu_m + \left[m + \frac{m}{2} \log 4\pi + \frac{m}{2} \log \frac{t(T-t)}{T} \right] \right\} \\
 &\leq -\mathcal{B}_m - m - \frac{m}{2} \log 4\pi - \frac{m}{2T} \int_0^T \log \frac{t(T-t)}{T} \, dt \\
 &\leq -\mathcal{B}_m - m - \frac{m}{2} (\log 4\pi T - 2) = -\mathcal{B}_m - \log(4\pi T)^{m/2},
 \end{aligned}$$

which upon exponentiating implies the ultracontractivity bound

$$\frac{\|w(T)\|_{L^\infty(M;d\mu)}}{\|w(0)\|_{L^1(M;d\mu)}} \leq \frac{e^{-\mathcal{B}_m}}{(4\pi T)^{m/2}}. \quad (3.16)$$

The desired conclusion now follows at once from this ultracontractivity bound. $\square$

3.3. Proof of Theorem 1.2.

Proof. For nonnegative smooth initial data $w(x, 0) = w_0(x)$, we write, by using the heat kernel H :

$$w(x, t) = \int_M H(x, y; t; T) w_0(y) d\mu(y). \quad (3.17)$$

Then $w(x, t)$ is a positive solution to $(\partial_t - \mathcal{L}_f)w = 0$. Thus by standard considerations

$$\sup_{x, y \in M} H(x, y; t; T) = \sup_{w \neq 0} \frac{\|w(t)\|_{L^\infty(M;d\mu)}}{\|w(0)\|_{L^1(M;d\mu)}}. \quad (3.18)$$

The conclusion in (1.17) now follows by applying inequality (3.9) in Lemma 3.3. $\square$

3.4. Global estimates and Liouville type results. In this section we briefly discuss some consequences and corollaries of the gradient estimate in Theorem 1.1 of a global nature.

Corollary 3.4. *Under the assumptions of Theorem 1.1 on $(M, g(t), f(t))_{t \in [0, T]}$ and the global curvature bounds $|Ric_f^m(g)| \leq (m-1)\mathbf{k}$ (or $|Ric_f^\infty(g)| \leq (n-1)\mathbf{k}$ respectively) on $M \times [0, T]$, for any bounded positive solution u to (1.1) on M and $t > 0$ we have*

$$\frac{|\nabla u|}{\sqrt{u}} \leq CA \left(\sqrt{\mathbf{K}} + \frac{1}{\sqrt{t}} + \sqrt{L_+(\mathbf{M}^\alpha + \alpha \mathbf{M}_1^{\alpha-1}) + \alpha L_- \mathbf{M}_0^{\alpha-1}} \right). \quad (3.19)$$

Here $\mathbf{M}$, $\mathbf{M}_0$ and $\mathbf{M}_1$ have the same meaning as in Theorem 1.1 except for the supremums that are being taken over $M \times [0, T]$ instead of $\mathcal{Q}_{R, T}$ and $A = \max(1, \sup_{M \times [0, T]} u)$.

Proof. By virtue of the estimate being true for every $R \geq 2$, and with the constants being independent of R , passing to the limit $R \rightarrow \infty$ and making note of the inequality $(1 + \sup \sqrt{u}) \sup \sqrt{u} \leq 2 \max(1, \sup u)$ gives the desired conclusion. $\square$

Note that the boundedness of u does not necessarily imply that the right-hand side of (3.19) is always finite, however, this is not an issue for the estimate itself. Moreover, in light of Remark 2.2, in the static case of (1.11), that is, with $\partial g / \partial t \equiv 0$, $\partial f / \partial t \equiv 0$ and $Ric_f^\infty \geq -(n-1)\mathbf{k}g$, upon taking $\mathbf{h} = 0$, we still retain (3.19) with $\mathbf{K} = \sqrt{\mathbf{k}^2 + \mathbf{h}^2} = \mathbf{k}$, for any positive bounded solutions u to (1.1). Interestingly this recovers and improves many of the existing estimates on static manifolds. Furthermore, as a byproduct, for the elliptic PDE associated with (1.1) (time independent solutions u), it results in

$$\frac{|\nabla u|}{\sqrt{u}} \leq CA \left(\sqrt{\mathbf{k}} + \sqrt{L_+(\mathbf{M}^\alpha + \alpha \mathbf{M}_1^{\alpha-1}) + \alpha L_- \mathbf{M}_0^{\alpha-1}} \right). \quad (3.20)$$

The above observation leads to the following Liouville type result on static weighted manifolds. The proof is an easy application of (3.20) upon setting $\mathbf{k} = 0$, $L_+ = 0$ and $\mathbf{M}_0 = 0$, which then makes the right-hand side of the estimate identically zero.

Corollary 3.5. *Let $(M, g, e^{-f} dv)$ be a weighted manifold with $Ric_f^\infty \geq 0$ and consider the equation (with real $\alpha > 1$):*

$$\mathcal{L}_f u + Lu |\log u|^\alpha = 0. \quad (3.21)$$

Then if $L \leq 0$ any solution u to (3.21) with $1 \leq u \leq A$ is a constant. Additionally if $L < 0$ then necessarily $u \equiv 1$.

Remark 3.6. Note that an identical conclusion holds if the nonlinear term in (3.21) is replaced with $Lu(\log u)^\alpha$, that is, for the equation $\mathcal{L}_f u + Lu(\log u)^\alpha = 0$ (with integer $\alpha \geq 1$). Note that for $\alpha \geq 2$ even the two equations coincide whereas when $\alpha \geq 1$ is odd they have interesting differences. See Remark 2.3.

3.5. Proof of Theorem 1.3. Using $\partial u / \partial t = \mathcal{L}_f u + Lu |\log u|^\alpha$ and the flow inequality $\partial g / \partial t \geq -2(Ric_f^m(g) + kg)$ we can write

$$\begin{aligned} (|\nabla u|^2)_t &\leq 2Ric_f^m(\nabla u, \nabla u) + 2k|\nabla u|^2 + 2\nabla u \nabla(\mathcal{L}_f u + Lu |\log u|^\alpha) \\ &\leq \mathcal{L}_f |\nabla u|^2 - 2|\nabla^2 u|^2 + 2k|\nabla u|^2 + 2L\nabla u \nabla[u |\log u|^\alpha] - 2 \frac{[\nabla f \nabla u]^2}{m-n} \end{aligned} \quad (3.22)$$

where in deducing the second line we have used the weighted Bochner-Weitzenböck formula. Likewise

$$(u^2)_t = 2u(\mathcal{L}_f u + Lu |\log u|^\alpha) = \mathcal{L}_f u^2 - 2|\nabla u|^2 + 2Lu^2 |\log u|^\alpha. \quad (3.23)$$

Now for $t \geq 0$ and X to be specified below put $\Phi = t|\nabla u|^2 + Xu^2$. Then differentiating and making use of (3.22) and (3.23) gives

$$\begin{aligned} \Phi_t &\leq t \left(\mathcal{L}_f |\nabla u|^2 - 2|\nabla^2 u|^2 + 2k|\nabla u|^2 + 2L\nabla u \nabla[u |\log u|^\alpha] - 2 \frac{[\nabla f \nabla u]^2}{m-n} \right) \\ &\quad + |\nabla u|^2 + X (\mathcal{L}_f u^2 - 2|\nabla u|^2 + 2Lu^2 |\log u|^\alpha), \end{aligned} \quad (3.24)$$

or upon rearranging terms

$$\begin{aligned} \Phi_t &\leq \mathcal{L}_f [t|\nabla u|^2 + Xu^2] + (1 + 2kt - 2X)|\nabla u|^2 - 2t|\nabla^2 u|^2 + 2LXu^2 |\log u|^\alpha \\ &\quad + 2tL\nabla u \nabla[u |\log u|^\alpha] - 2t \frac{[\nabla f \nabla u]^2}{m-n}. \end{aligned} \quad (3.25)$$

Therefore basic considerations lead to

$$\Phi_t \leq \mathcal{L}_f \Phi + (1 + 2kt - 2X)|\nabla u|^2 + 2LXu^2 |\log u|^\alpha + 2tL\nabla u \nabla[u |\log u|^\alpha]. \quad (3.26)$$

Thus in particular if $X = 1/2$ and $k \leq 0$ we have

$$\Phi_t \leq \mathcal{L}_f \Phi + Lu^2 |\log u|^\alpha + 2tL|\nabla u|^2 (|\log u|^\alpha + \alpha |\log u|^{\alpha-2} \log u). \quad (3.27)$$

The conclusion now follows by an application of the maximum principle upon considering the sign of the "log" terms.

For the equation $\partial u / \partial t = \mathcal{L}_f u + Lu(\log u)^\alpha$ with $\alpha \geq 1$ integer we have the following counterpart to the inequality (3.27)

$$\Phi_t \leq \mathcal{L}_f \Phi + Lu^2 (\log u)^\alpha + 2tL|\nabla u|^2 [(\log u)^\alpha + \alpha (\log u)^{\alpha-1}]. \quad (3.28)$$

Hence if α is even then exactly as before. If α is odd then if $L \geq 0$ and $0 < u \leq e^{-\alpha}$ or if $L \leq 0$ and $u \geq 1$ then in either case the curvature free estimate (1.18) holds. $\square$

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