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A Cost-Benefit Analysis of Capital Requirements
Adjusted for Model Risk

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Abstract

Capital adequacy is the key microprudential and macroprudential tool of banking regulation. Financial models of capital adequacy are subject to errors, which may prevent from estimating a sufficient capital base to absorb bank losses during economic downturns. In this paper, we propose a general method to account for model risk in capital requirements calculus related to market risk. We then evaluate and compare our capital requirements values with those obtained under Basel 2.5 and the new Basel 4 regulation. Capital requirements adjusted for model risk perform well in containing losses generates in normal and stressed times. In addition, they are as conservative as Basel 4 capital requirements, but they exhibit less fluctuations over time.

Keywords: Basel framework, Capital requirements, Cost-benefit analysis, Model risk
JEL: D81, G17, G18

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1 Introduction

The historical evidence, and particularly the recent financial crisis, show that banking crises produce huge negative externalities on the financial system, on public finance and on the economy as a whole. Due to distortions such as the tax shield from debt, deposit insurance and implicit government guarantees, banks may want to hold less equity than what it would be socially optimal (Modigliani and Miller, 1963; Gennotte and Pyle, 1991; Admati et al., 2013; Repullo and Suarez, 2013). This provides a rationale for banking regulation. Capital adequacy is the cornerstone of microprudential and macroprudential regulation of banks. Capital requirements foster financial stability by preventing a bank default and its contagion effects in two ways. First, they ensure that banks have a sufficient capital buffer to absorb losses (Diamond and Rajan, 2000; Freixas and Rochet, 2008; Gordy and Heitfield, 2010; Martinez-Miera and Suarez, 2014); second, they limit risk-taking incentives of banks (Furlong and Keeley, 1989; Rochet, 1992; Giammarino et al., 1993; Hellman et al., 2000; Freixas and Rochet, 2008; Gordy and Heitfield, 2010; Allen et al., 2011; Mehran and Thakor, 2011; Admati et al., 2013). While capital requirements allow to internalize the costs of bank failures, if tightened, they can lead to a shrinkage in bank assets e.g. by reducing credit supply to the corporate and household sectors (Aiyar et al., 2014; Kisin and Manela, 2016; Braouezec and Wagalath, 2018; Naceur et al., 2018; Bams et al., 2019; Fraisse et al., 2019; De Jonghe et al., 2020; Nguyen et al., 2020). Overall, this implies that the capital adequacy rules should be set upon a careful cost-benefit analysis.

The current regulation requires banks to hold a sufficient capital base to cover losses in periods of economic stress (European Parliament, 2013a; European Parliament, 2013b; Federal Reserve System, 2013; BCBS, 2017). As a consequence, the calculation of capital requirements crucially relies on risk forecast models. The empirical evidence suggests that such models may deliver inaccurate risk measures. For example, significant errors in the estimation methods used in the finance industry were reported by Berkowitz (2001) and Berkowitz and O’ Brien (2002).\footnote{These errors may involve not only an underestimation of risk, but also an overestimation (Pérignon et al., 2008).} A recent exercise of the Basel Committee on Banking Supervision (BCBS) reveals that internal models used by banks to calculate capital requirements produce inconsistent risk valuations across the financial system for the same hypothetical portfolio (BCBS, 2014a). In general, misspecified models and estimation uncertainty may lead to a significant exposure of banks to model risk (Green and Figlewski, 1999).

The Federal Reserve (Federal Reserve Board of Governors, 2011) mentions explicitly that “model risk should be managed like other types of risk” and that “banks should identify the sources of model risk and assess the magnitude”. Similarly, the European
Banking Authority (European Banking Authority, 2012) states: “Institutions should include the impact of valuation model risk when assessing the prudent value of its balance sheet. [...] Where possible an institution should quantify model risk by comparing the valuations produced from the full spectrum of modeling and calibration approaches”. Nevertheless, how to account for model risk and how to adjust capital requirements accordingly are still very much open to research.

In this paper we propose a methodology to determine the amount of equity capital that is necessary to absorb losses on a given investment with a certain probability. This methodology accounts for potential errors in the calculations (i.e. model risk) and can be applied by both financial and non-financial companies as part of the inputs in their capital decisions. Our approach to handle model risk builds on the derivation of confidence intervals for standard market risk measures and on the development of a model screening tool. According to the “trade-off theory” of capital structure (Kraus and Litzenberger, 1973), firms determine the optimal mix of equity and debt by balancing the interest tax shield on debt with the bankruptcy costs (which increase with the likelihood of a default). Our methodology is helpful to deal with this trade-off by providing a way to map the amount of equity with the default probability. Importantly, we focus on a specific application of this methodology to derive bank capital requirements for market risk that are robust to model risk. We then discuss costs and benefits of our capital requirements adjusted for model risk. This evaluation includes an assessment of the extent to which they absorb losses in different economic scenarios, as well as a comparison with the capital requirements obtained under the latest BCBS regulations. We show that our approach delivers capital buffers that are able to absorb losses in normal and stressed times, while not being excessively conservative nor volatile. Moreover, our capital requirements adjusted for model risk complement and improve the current capital adequacy regulation.

Risk management is an essential element of business for companies operating in any sector. The evaluation and management of risks facilitate the preservation of firm value and help to operate profitably by guiding investment and capital structure decisions (Froot et al., 1993; Froot and Stein, 1998; Lucas and Klaassen, 1998; Basak and Shapiro, 2001; Campbell et al., 2001; Leland, 2002; Valencia, 2016; Borochin and Yang, 2017). Risk measures are commonly used in risk management to quantify the exposure to a given type of risk. This paper focuses on market risk, which can be defined as the potential decrease in value of an investment due to changes in market factors. A popular measure of market risk is Value-at-Risk (VaR). Given an investment in an asset, VaR indicates the maximum loss that can occur with a certain probability. It can be derived from the quantile of the distribution of asset returns and it can be estimated using a variety of statistical models. Estimating VaR carries inevitably model risk due to estimation uncertainty. This means that the pointwise estimate may not correspond to the true value of the risk measure. One way to address model risk is to derive confidence intervals for VaR, which define
a band where the true value of the risk measure lies in with high confidence. Deriving confidence intervals for VaR is challenging, because the true distribution of VaR is usually unknown even under simple distributions of asset returns.

In the first part of this paper we provide a rationale to gauge market risk using quantile risk measures such as VaR and the Median Shortfall (MS). The latter corresponds to the conditional median of losses beyond VaR and it shares the same properties of VaR, being a VaR at a lower critical level. Then, we present a methodology to construct confidence intervals for quantile risk measures. Specifically, we exploit the order statistics calculus to derive model-free confidence intervals and parametric confidence intervals for any given probability distribution of asset returns. The upper bound of the confidence intervals allows risk managers to adjust the point estimate for estimation uncertainty and the adjusted risk measure provides a conservative measure of market risk.\(^2\)

We extend our approach developing a model validation test that builds on the difference between the nonparametric estimates of the MS and VaR. Intuitively, this difference captures the unexpected extra losses incurred if VaR is exceeded, given the confidence level at which VaR is estimated. Our statistical tool allows assessing the goodness of fit of a given model by testing if these extra losses are captured by the envisaged distribution of asset returns. As such, it represents a valid complement to standard backtesting procedures to prevent model risk due to misspecification.

Although we focus on market risk, the scope of our methodology is much broader. Our approach to handle model risk can be applied to quantile risk measures used to estimate market, credit, operational or systemic risk for any practical purpose and both by financial and non-financial companies. In general, our statistical toolkit is useful for many applications in corporate investment and risk management, with the ultimate goal to protect and enhance firm value.

An exhaustive empirical exercise shows how our methodology works in practice. To this end, we consider an investment in an equity index, the S&P 500, over the period 1st January 1980 to 29th July 2016.\(^3\) We focus on the estimation of two risk measures, namely VaR and the MS at 1% level. Whilst the literature offers a myriad of quantitative models to calculate these metrics, in this paper we utilise three of the most widely used classes of one-dimensional distributions of asset returns in the finance industry: the nonparametric, the gaussian and the normal inverse gaussian (NIG).\(^4\) In principle, this

\(^2\)This is in line with the additional valuation adjustment (AVA) recommended by European Banking Authority (2012), which is defined as the “prudent” value accounting for unexpected losses represented by the end point of a confidence interval generated by model risk and the fair value of a financial product.

\(^3\)We replicate the empirical study on a different asset, i.e. a foreign exchange. The results of the analysis on the exchange rate are presented in the Online Supplement.

\(^4\)We limit ourselves to three models since our aim is not to identify the “best” model for risk management, but rather to replicate market approaches to estimate risk measures and to show how to handle model risk associated with these estimates.
set could be extended to any parametric model assuming i.i.d. asset returns. Irrespective of the modeling assumption, we find that our confidence intervals are wider in turbulent times, consistently with the idea that model risk increases during periods of market stress. Moreover, our model validation tool and a suite of standard backtests reveal that the NIG model delivers better estimates of market risk when compared to the gaussian model and the nonparametric approach.

In the second part of this paper we apply our methodology to derive capital requirements for market risk that are robust to model risk due to estimation uncertainty. Accounting for model risk is as crucial as model selection and calibration to ensure that capital buffers are able to absorb losses in stressed periods, as required by the current regulation. In fact, the recent financial crisis revealed that systemic banking crises generate significant social and fiscal costs (Admati et al., 2013). Regulators and policy makers worldwide responded to the crisis enacting new regulations on capital adequacy to ensure that banks are properly capitalized. The new rules are aimed to improve the quality and quantity of regulatory capital to make sure that the banking system has sufficient capacity to absorb losses in times of market and economic stress (BCBS, 2011; Federal Reserve System, 2013; European Parliament, 2013b,a; BCBS, 2017).

With specific regard to market risk, in 2010 the Basel Committee for Banking Supervision (BCBS) introduced the Basel 2.5 capital requirements (BCBS, 2010), which are into force since January 2012. The capital buffers set by Basel 2.5 (BCBS, 2010) remained unchanged under the new regulatory framework named Basel 3 (BCBS, 2011), which was enacted in 2011. Starting in 2016, the BCBS introduced various amendments to Basel 3 (BCBS, 2011), commonly referred as “Basel 4” (PricewaterhouseCoopers, 2016; McKinsey, McKinsey; KPMG, 2018). These include new capital requirements for market risk (BCBS, 2016, 2019b), which will enter into force in January 2022. A common feature of Basel 2.5 (BCBS, 2010) and Basel 4 (BCBS, 2016, 2019b) is that model risk due to estimation uncertainty is captured indirectly by applying a multiplier to the risk measure used to calculate capital requirements. This multiplication factor is determined by backtesting a risk metric (one-day VaR at 1%) that does not correspond to the risk measure prescribed for the computation of capital requirements (ten-day VaR at 1% in Basel 2.5 and ten-day stressed ES at 2.5% in Basel 4).

In this paper we present a straightforward method to calculate capital requirements for market risk which are robust to model risk. Our starting point is the use of the MS at 1% level to estimate the magnitude of extreme losses beyond VaR. In line with the current regulation, we further suggest to calibrate the model to a period of stress of one year. Then, we propose to take the upper bound of a confidence interval for the stressed

\footnote{The i.i.d assumption is common in the risk management literature (Bali, 2007; Kerkhof et al., 2010) and also in the risk management practices of banks (O’Brien and Szerszen, 2017; Alexander and Baptista, 2017).}
MS (our risk measure adjusted for model risk) as the starting point to derive capital buffers for market risk. Our approach has two nice properties when compared to Basel 4 (BCBS, 2019b). First, the same risk measure is used both for model validation and the calculation of capital requirements, ensuring consistency and limiting computational costs. Second, model risk due to estimation uncertainty is incorporated explicitly in the required capital buffers thanks to a robust statistical approach, which builds on confidence intervals. As opposed to the multiplier approach of Basel 4 (BCBS, 2019b), which does not distinguish across different models, these confidence intervals are obtained under the specific modeling assumption that is adopted for asset returns.

In the empirical exercise we focus on market risk capital requirements for an investment in the S&P 500 obtained under an NIG distribution for asset returns, as this is the best model identified in the validation analysis. We evaluate to what extent our capital requirements adjusted for model risk are able to absorb losses in normal and stressed times. In addition, we compare them to the regulatory buffers obtained under Basel 2.5 (BCBS, 2010) and Basel 4 (BCBS, 2019b). We find that our capital requirements adjusted for model risk are able to contain losses generated in periods of economic stress, such as the Lehman Brothers collapse in 2008, once the calibration of the stressed MS is made using historical data that includes an extreme event (in our case the stock market crash of 1987). At the same time, they are not excessively conservative, being close to the extreme losses experienced in times of stress. Taking into account model risk due to estimation uncertainty is crucial to derive capital buffers that are sufficiently conservative throughout the sample period. In terms of performance our capital requirements adjusted for model risk are similar to the Basel 4 capital requirements. In general, these two capital regimes do well in preventing a bank failure and its potential contagion effects. This represents a major benefit, not only for the bank itself, but also for the economy more broadly. In addition, as opposed to the Basel 2.5 capital requirements, our capital requirements adjusted for model risk and the Basel 4 capital requirements are not excessively conservative, ruling out the possibility of an undesirable shrinkage in bank investments. Finally, our capital requirements adjusted for model risk are less volatile and procyclical than the capital requirements computed under Basel 4 (BCBS, 2019b), meaning that they increase less in periods of market stress. As a consequence, our capital requirements adjusted for model risk may generate a weaker contraction in bank assets during economic downturns than Basel 4 capital requirements.

The paper is structured as follows. Section 2 reviews the existing literature on model risk and outlines the contribution of our work. Section 3 provides a discussion of various risk measures supporting why VaR is a very useful risk metric to be used in practice. Section 4 describes our proposed methodology to account for model risk in the calculation of market risk measures. The data and various calibration schemes used in the empirical analysis are discussed in Section 5. Section 6 presents an empirical exercise to show
how our methodology works in practice. In Section 7 we describe how to derive capital requirements that are robust to model risk and we discuss their ability to absorb losses in comparison with capital requirements set by the recent Basel regulations. The last section contains a summary of our results. Some important computational details needed for our methodology are contained in the Appendix. In addition, we also provide an Online Supplement where we present additional theory on order statistics and the output of further empirical analysis.

2 Related Literature and Contribution

This paper contributes to the growing literature on model risk. The seminal work of Hull and Suo (2002) presents a methodology to gauge model risk associated with misspecified models in the evaluation of exotic options. Their focus is on the estimation of the implied volatility surface from current option market data using three different models, Black-Scholes, implied volatility function and a two factor stochastic volatility model. The latter is considered as the benchmark for pricing exotic options and they quantify model risk looking at the level of disagreement of the other two models with the benchmark model. The model performance is measured for pricing and separately for hedging. Cont (2006) starts from an axiomatic perspective, combining elements from Hull and Suo (2002) with the seminal approach of Artzner et al. (1999), and he proposes a general framework to quantify model risk in derivative pricing. Our research departs from the model risk analyses of Hull and Suo (2002) and Cont (2006) in several ways. First, we focus on model risk arising in the calculation of risk measures such as VaR and MS. Secondly, we provide a methodology to gauge model risk due to mainly estimation uncertainty, which builds on the derivation of confidence intervals for pointwise estimates. Differently from option pricing where models are calibrated from market data to capture current conditions, in the computation of risk measures the estimation of model parameters is performed using historical data to identify worst case scenarios. Thus, in calculating risk measures one must deal with substantial estimation uncertainty, which depends on the extent to which a data sample allows identifying the true parameters of the model. In addition, for a large class of models (any probability distribution and iid returns), we also provide a tool to discriminate between various competing models for calculating quantile risk measures, hence, covering model specification risk as well.

More recently, a series of studies investigate how to handle model risk in a risk management context (Kerkhof et al., 2010; Escanciano and Olmo, 2011; Alexander and Sarabia, 2012; Embrechts et al., 2013; Gourieroux and Zakoian, 2013; Boucher et al., 2014; Embrechts and Hofert, 2014; Embrechts et al., 2015; Detering and Packham, 2015). Our paper is closely related to Kerkhof et al. (2010), who develop a method to explicitly account for model risk in the calculation of capital requirements for market risk. Their
approach builds on the asymptotic statistics calculus to derive confidence intervals for VaR and the expected shortfall (ES), but only under a nonparametric approach and a gaussian assumption for asset returns. The upper bound of the confidence intervals for VaR and the ES deliver capital levels adjusted for model risk that are compared with Basel 2.5 capital requirements (BCBS, 2010). Deriving confidence intervals for market risk measures using asymptotic statistics is challenging. For this reason, the approach of Kerkhof et al. (2010) is not viable to get confidence intervals for VaR and the ES considering other modeling assumptions.

Our paper continues the line of research of Kerkhof et al. (2010) providing a wider and more flexible methodology. Our contribution to the literature covers three important points. First, we present a straightforward way to obtain nonparametric and parametric confidence bounds specific to any distribution of asset returns used in finance. Second, we develop a new test for model validation focused on extreme losses. Third, we present a critical comparison of our capital requirements adjusted for model risk with those obtained according to the latest Basel 4 regulation (BCBS, 2019b).

3 A Critical Comparison of VaR and ES

One of the most popular measures of market risk is VaR. Consider an investment of $100 in the S&P500 and suppose that the one-day VaR at 1% is $10. This means that there is a 1% probability of losing more than $10 over the next day. A limitation of VaR is that it does not provide any information about the magnitude of losses that occur with a probability of less than 1%. The Expected Shortfall (ES) has been introduced exactly to capture the entire spectrum of possible tail losses. It is defined as the conditional expectation of the loss given that VaR is exceeded. An alternative measure of tail risk is the MS, which consists in the conditional median of losses beyond the VaR level. It can be shown that the MS at a level \( \alpha \) corresponds to VaR at the level \( \alpha/2 \) (So and Wong, 2012; Kou and Peng, 2016), implying that the MS risk measure shares the same properties of VaR.

During the last decade the ES has grown in popularity, with its usage being advocated for the calculation of capital requirements for market risk from the BCBS since 2012 (BCBS, 2012). There is an extensive body of literature comparing the theoretical advantages and disadvantages of VaR and the ES. The discussion is centered on whether a general risk measure \( \rho \) satisfies the well-known “coherence” conditions over the set of losses and profits (or equivalently returns) \( \Pi = \{ Y : E(Y^2) < \infty \} \) (McNeil et al., 2015). Specifically, Artzner et al. (1999) developed an axiomatic approach to risk measures, identifying a series of properties that a good (coherent) risk measure should have. One of this properties is “sub-additivity”, a condition requiring that \( \rho(Y_1 + Y_2) \leq \rho(Y_1) + \rho(Y_2) \), for any \( Y_1, Y_2 \). Intuitively, this property reflects the idea that portfolio diversification may
reduce risk. While the ES is a “coherent” measure of risk, VaR does not always satisfy the sub-additivity condition.

A strand of recent literature questions the claimed theoretical superiority of the ES over VaR. First, Garcia et al. (2007), Ibragimov (2009), Ibragimov and Walden (2007) and Danielsson et al. (2013) demonstrate that VaR is actually subadditive for various classes of distributions. Dhaene et al. (2006) argue that sub-additivity may not be even ideal in some cases. Specifically, they consider the example of a merger between two portfolios and they prove that the difference between the losses and the risk measure may be higher if the two portfolios are merged rather than separated. Furthermore, Ibragimov (2009) and Ibragimov and Walden (2007) highlight that, for asset distributions which are heavy tailed, diversification may actually increase rather than decrease tail risk. Therefore, the sub-additivity condition is not sufficient to dismiss using a quantile risk measure like VaR.

Secondly, the ES may not satisfy other desirable properties beyond those that characterize a coherent risk measure. Cont et al. (2010) and Kou et al. (2013) formalize the concept of “robust” risk measure. According to their definitions, the “robustness” property requires a risk measure not to be too sensitive to small changes in the data (Cont et al., 2010; Kou et al., 2013) and to the modeling assumptions (Kou et al., 2013). They show that VaR is a robust risk measure, whereas the ES is not, mostly because it is (by definition) highly sensitive to outliers. Kou et al. (2013) argue that the lack of robustness of the ES provides a rationale for using the MS (rather than the ES) to derive capital requirements, as only a robust risk measure ensures that capital regulation is implemented consistently across different institutions.

Third, despite that the ES dominates VaR as a measure of downside risk in portfolio optimization (Alexander and Baptista, 2006), the ES exhibits some limitations in such applications. In particular, Kondor and Varga-Haszonits (2010) identify specific conditions for which portfolio optimization under an ES constraint is unfeasible.

In practice, the estimation of the ES is more challenging than the estimation of VaR. First, in order to reliably estimate the ES at the same level as VaR a much larger sample size is needed (Yamai and Yoshiha, 2005). Moreover, the estimation of the ES is particularly costly in computational terms for fat-tailed distributions (Yamai and Yoshiha, 2005). Second, the ES measures are generally less accurate than the VaR measures, pointing to a higher estimation uncertainty for the ES (Christoffersen and Goncalves, 2005; Danielsson and Zhou, 2016), which cannot be reduced easily as this would require a large dataset which may not be available.

We highlight an additional disadvantage of the ES with respect to VaR, namely the difficulty to get confidence intervals for the ES. Relying on asymptotic statistics calculus, Kerkhof et al. (2010) derive confidence intervals for VaR and the ES, but only under a nonparametric approach and a gaussian distribution of asset returns. The approach of Kerkhof et al. (2010) is not feasible to get confidence intervals for VaR and the ES
considering other modeling assumptions. In this paper we present a methodology that permits the estimation of confidence intervals for quantile risk measures, i.e. VaR and MS, for any distribution of asset returns used in finance, as well as for a nonparametric approach. Our method is straightforward and computationally simple. However, since it builds on order statistic calculus, it cannot be applied to the ES. We believe that the ability to quantify estimation uncertainty by calculating confidence intervals for quantile risk measures under any modeling assumption, as suggested in this paper, provides a further reason to use VaR and the MS in risk management.

Here we consider a practical approach of a risk manager of a corporate institution that must deal with daily challenges of measuring market risk. As highlighted above, when it comes to estimation, the ES shows several shortcomings. On the other hand, for quantile risk measures such as VaR and MS, their theoretical disadvantages are offset but their many practical advantages. In what follows we focus on quantile risk measures and we present our methodology to quantify model risk associated with point estimation.

4 Accounting for Model Risk of Risk Measures Used for Capital Requirements

4.1 Confidence Intervals for Quantile Risk Measures

The estimation of a risk measure carries inevitably model risk. This means that the pointwise estimate is likely not to correspond to the true value of the risk measure. Deriving a confidence interval for the risk measure allows to identify a range of numbers where the true value lies in with a high pre-specified probability level such as 95%. As such, a confidence interval represents a straightforward way to quantify model risk associated with the estimation of a risk measure. Moreover, for a risk manager or regulator concerned with underestimating risk, the upper bound of a confidence interval provides a conservative estimate of the risk measure. This is exactly the approach that we advocate to derive risk measures adjusted for model risk that can be used for capital requirements calculation. In what follows we show two methods to derive confidence intervals for VaR measures.

Let us consider an investment in an asset, such as an equity index or a foreign exchange, whose price at time \( t \) is \( X_t \). We assume that the log-returns of the asset over a fixed horizon \( h \), \( Y_{t+h} = \ln \left( \frac{X_{t+h}}{X_t} \right) \), have an arbitrary cumulative distribution function, \( Y_{t+h} | F_t \sim F \). We characterize \( F \) relying both on parametric models and a nonparametric approach. Specifically, we consider the gaussian model, still widely applied in the finance sector, as well as a model better equipped to deal with fat tails such as the NIG, which is representative for the Lévy models applied in finance (Venter and de Jongh, 2002). De-
spite the variety of statistical models available, many banks seem to prefer a model-free approach\textsuperscript{6}. The reason is twofold: first, it is unlikely that a family of models will perform well across all asset classes, including new emerging ones; second, a nonparametric estimate of $F$ is often less computationally intensive, which is a relevant advantage when it comes to calculate risk measures for a large trading book.

In general, VaR can be conceptualized as a function of the $\alpha$-th quantile of the distribution of asset returns, $q_\alpha$, which is notoriously difficult to estimate. However, when a sample of past returns of the asset is available, it is possible to estimate $q_\alpha$ relying only on the sample quantile $Y_{\lfloor n\alpha \rfloor + 1}$, i.e. a nonparametric approximation. $Y_{\lfloor n \rfloor}$ represents the $i$-th order statistics of the empirical distribution of asset returns in a sample of size $n$ and $\lfloor a \rfloor$ denotes the smallest integer not greater than $a$. Given a sample of returns data $\{Y_1, \ldots, Y_n\}$, because the empirical distribution function generated $F_n(u) = \frac{1}{n} \sum_{i=1}^{n} 1_{\{u \geq Y_i\}}$ is not invertible, Inui and Kijima (2005) suggest to estimate VaR employing the lower empirical distribution value given by the piece-wise constant function equal to the order statistic $Y_{\lfloor k \rfloor}$ with $k - 1/n < \alpha \leq k/n$. This gives a direct estimator for VaR as $-Y_{\lfloor k \rfloor}$, but this estimator is known to carry a positive bias for small critical levels $\alpha$. Alternatively, one can use the upper empirical distribution value $Y_{\lfloor k \rfloor}$ with $k - 1/n \leq \alpha < k/n$ as in Chen (2008), that corresponds to our preferred estimator $Y_{\lfloor n\alpha \rfloor + 1}$.

For risk management purposes, we are interested in the left tail of the distribution of the direct profit and loss $X_{t+h} - X_t$. Then, VaR at $\alpha$ critical level of the profit and loss probability density can be estimated as follows:\textsuperscript{7}

$$\hat{\text{Var}}^\text{nonp}_t (X_{t+h} - X_t) = X_t - X_t e^{q_\alpha} = X_t - X_t e^{Y_{\lfloor \alpha \rfloor + 1}}$$

The same formula can be applied to estimate the MS, which is still a quantile risk measure being VaR at $\alpha/2$ (So and Wong, 2012; Kou and Peng, 2016).

The VaR measures estimated according to formula (1) are subject to sampling estimation error. As discussed earlier, one method to address this issue is to derive, for example, a 95\% confidence interval for VaR. The order statistics calculus allows the construction of a distribution-free confidence interval for quantiles, and hence for the VaR measures. This approach has been pioneered in the risk management literature by Dowd (2010) and it is expanded on here.\textsuperscript{8} Specifically, an important result from order statistics (David and

\textsuperscript{6}O’Brien and Szerszen (2017) describe a sample of U.S. banks for which VaR and ES are calculated. From this sample 60\% of banks employ the nonparametric or historical simulation method.

\textsuperscript{7}Similarly, the ES of the profit and loss probability density is given by

$$\hat{\text{ES}}^\text{nonp}_t (X_{t+h} - X_t) = X_t - \left[ \frac{1}{\lfloor n\alpha \rfloor + 1} X_t \sum_{i=1}^{n} e^{y_i} 1_{\{y_i \in [Y_{\lfloor n\alpha \rfloor}, Y_{\lfloor (n+1)\alpha \rfloor}]\}} \right].$$

\textsuperscript{8}Dowd (2010) shows how to derive distribution-free confidence intervals for the broad class of quantile-based risk measures, which includes both VaR and the Expected Shortfall (ES). The confidence bounds for the ES are obtained by averaging the corresponding confidence bounds derived for VaR under a
Nagaraja, 2003) is that for any $1 \leq i_1 < i_2 \leq n$

$$\text{Prob} \left( Y_{[i_1:n]} \leq q_\alpha \leq Y_{[i_2:n]} \right) = \sum_{j=i_1}^{i_2-1} \binom{n}{j} \alpha^j (1 - \alpha)^{n-j}. \quad (2)$$

This result can be used to directly determine a distribution-free confidence interval for VaR.

However, there may exist several combinations of the order statistics $Y_{[i_1:n]}, Y_{[i_2:n]}$ that make the probability in (2) close to the desired confidence level.

Here, we propose distribution-free confidence intervals at $1 - \beta$ confidence level for VaR using the fractional order statistics $y_{i_1}$ and $y_{i_2}$ satisfying $\text{Prob} \left( y_{i_1} \leq q_\alpha \leq y_{i_2} \right) = 1 - \beta$

$$\text{CI}_{1-\beta}^{\text{VaR nonp}} (X_{t+h} - X_t) = (X_t - X_t e^{y_{i_1}}, X_t - X_t e^{y_{i_2}}). \quad (3)$$

The order statistics calculus provides additional tools when the returns of the asset are assumed to be distributed according to a certain parametric distribution law $F$. If $F_{[i]}(u) = P(Y_{[i:n]} \leq u)$ is the cumulative distribution function of the $i$-th order statistic, then $F_{[1]}(u) = 1 - [1 - F(u)]^n$ and $F_{[n]}(u) = [F(u)]^n$. The distribution of the order statistics of any order $j$ can be derived (David and Nagaraja, 2003) as

$$F_{[j]}(u) = B_{F(u)}(j, n - j + 1) \quad (4)$$

where $B_{U}(a,b) = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)}$ is the incomplete beta function and $B(a,b)$ is the beta function. Thus, the probability density function of the $j$-th order statistics is

$$f_{[j]}(u) = \frac{1}{B(j, n-j+1)} F^{j-1}(u) [1 - F(u)]^{n-j} f(u) \quad (5)$$

where $f$ is the corresponding density to $F$.

This result has a crucial implication: once a parametric distribution model is specified for the returns of the asset, one can derive not only confidence intervals, but also the entire distribution associated with the VaR measure. For example, if $f$ is gaussian or NIG, we can simply replace the expressions for $f$ and $F$ in (5) and obtain the formula of the density of the corresponding VaR. In this way the risk manager can determine how uncertainty series of different critical levels $\alpha$. The discretization of this process and the averaging does not allow to estimate confidence intervals for the ES with an exact probability coverage. Furthermore, for extreme values of $\alpha$, it may not possible to identify the necessary order statistics pertaining to the extreme tail of the distribution, adding a further complication to this approach. In this paper, we focus on deriving exact confidence intervals for risk measures by exploiting the order statistics theory. We are able to accomplish this goal only for quantile risk measures such as VaR, which consist in order statistics.

We rely on fractional order statistics as defined by Hutson (1999) to compute the non-parametric confidence interval with the optimum coverage, meaning that the probability $\text{Prob} \left( y_{i_1} \leq q_\alpha \leq y_{i_2} \right)$ is exactly equal to the desired confidence level. Section 1 of the Online Supplement provides technical self-contained details on this approach.
around VaR is distributed. For example, one can calculate confidence intervals for the quantile-based risk measure, while keeping the estimated vector of parameters of the parametric distribution assumed for asset returns as fixed. Specifically, we derive the following expressions for the parametric confidence intervals for VaR at $1 - \beta$ level:

$$CI_{1-\beta}^{VaR_{param}}(X_{t+h} - X_t) = \left( X_t - X_t e^{B^{-1}(1+n\alpha)}(\lfloor n\alpha \rfloor + 1, n-[n\alpha]) \right) \left( X_t - X_t e^{B^{-1}(1-\beta/2)}(\lfloor n\alpha \rfloor + 1, n-[n\alpha]) \right).$$

(6)

4.2 A Measure of Model Discrimination

If VaR is estimated based on the order statistic $Y_{\lfloor v/n \rfloor}$, there is a direct advantage of estimating the median shortfall (MS) using the truncated sample $Y_{1:n}, \ldots, Y_{v-1:n}$, which is already an ordered sample. In fact, estimating a quantile measure at different orders allows the calculation of the bivariate joint distribution of $(Y_{\lfloor \kappa_1/n \rfloor}, Y_{\lfloor \kappa_2/n \rfloor})$, where $\kappa_1 = n \times \alpha_1$ and $\kappa_2 = n \times \alpha_2$, as:

$$F_{\lfloor \kappa_1, \kappa_2 \rfloor}(u, v) = \sum_{k=\kappa_2}^{n} \sum_{s=\kappa_1}^{\kappa} \frac{n!}{s!(k-s)!(n-k)!} [F(u)]^s [F(v) - F(u)]^{k-s} [1 - F(v)]^{n-k}$$

(7)

for any $u < v$. This formula shows that any two quantile order statistics estimators at different critical levels are not independent. Hence, for risk management purposes, it is worthwhile to look at VaR and MS as a pair of tools for gauging risk. Building on this idea, we consider the difference between the MS estimate and the VaR estimate. Conceptually, this represents an estimate of the unexpected extra losses incurred in case VaR is exceeded, given the confidence level at which VaR is estimated. We exploit this metric to derive a statistical test to verify the goodness-of-fit of each individual parametric model selected by the risk manager to describe the distribution of a quantile risk measure. The model or models which are rejected based on this distance tool should not be employed for computing risk measures.

Let us denote the sample quantile to estimate MS by $Y_{\lfloor m \rfloor}$, with $m < v$. Since $Y_{v}$ and $Y_{m}$ are both order statistics, the difference

$$D = (Y_{v} - Y_{m})$$

(8)

is a natural test statistic for comparing different risk management models. To see why, let us impose that $Y_{m} = z$ and $Y_{v} = y$, so that $D = y - z$. Then, using the order

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10When $v$ is even then the estimate of the median shortfall (of the distribution of asset returns) is $Y_{\lfloor v/2 \rfloor}$, whereas when $v$ is odd then the sample estimate is $\frac{1}{2} [Y_{\lfloor (v-1)/2 \rfloor} + Y_{\lfloor (v+1)/2 \rfloor}]$. Here we consider a simpler estimator $Y_{\lfloor v/2 \rfloor + 1}$.

11$Y_{v}$ and $Y_{m}$ are the quantiles used to estimate VaR and MS obtained from the empirical distribution of financial returns.
statistics calculus, we can show that the probability density function of $D$ is given by

$$q(d) = K \int_{-\infty}^{\infty} F^{n-1}(z) [F(z + d) - F(z)]^{v-m-1} [1 - F(z + d)]^{n-v} f(z) f(z + d) dz$$

(9)

where $K = \frac{n!}{(m-1)(v-m-1)(n-v)!}$ is just a normalizing constant factor. Therefore, the probability density presented in equation (9) can be used to calculate the p-value of the estimated measure $d$. Specifically, for a given model represented by the cdf $F$, the p-value of $d$ is

$$P_F(D \leq d) = \int_0^d q(u) du.$$  

(10)

VaR and MS being monotonic transformations of the order statistics $y$ and $z$, this formula provides also the p-value of the difference between the MS estimate and the VaR estimate. In the Online Supplement we provide more technical details and we show how to simplify equation (10) to get a closed-form expression for the p-value. If this p-value suggests that the observed difference between the two quantiles for VaR and MS is in the extreme tails of the density given in (9), then the risk manager or the regulator can reject the model.

This tool can be very useful to monitor model performance, as well as for model validation.\textsuperscript{12}

To the best of our knowledge, this is the first approach in the literature that allows model validation looking at the distribution of the difference between quantile risk measures estimated at a different level of confidence. This is in the same spirit of Pérignon and Smith (2010) and Colletaz et al. (2013), who suggest a joint backtest of a standard VaR (e.g. at 1%) and an extreme VaR at very low $\alpha$ (e.g. at 0.2%) to control for the quality of risk measures. Our tool can be tailored to this framework by selecting the desired quantiles.

In general, our methodology to handle model risk (based on confidence intervals and a model discrimination test) can be applied to risk measures for market, credit, operational, or systemic risk estimates, and for a single asset or a portfolio held by a bank, an insurance company, a hedge fund or a non-financial company. Being able to derive risk measures adjusted for model risk allows to assess in a reliable way the exposure of a firm to a given type of risk. This, in turn, is crucial in many relevant corporate finance applications.

\textsuperscript{12}A different approach may be taken by the regulator who can ask all relevant market participants, say $m$ companies, to inform about the model used for their capital requirements calculations. Suppose that all market participants rely on models characterized by a vector of parameters $\theta(\mathbf{1})$, denoted by $M_\theta(\mathbf{1})$, and a distribution law given by $F_{M_\theta(\mathbf{1})}$. The set of all models from market participants is then described by the set $\mathcal{M}(\Theta)$ of all $\{M_{\theta(\mathbf{1})}\}_{\mathbf{1}=1,...,m}$. This covers situations where for example, all models are from the same family such as normal distribution with $\theta(\mathbf{1}) = (\mu, \sigma) \in \mathbb{R} \times (0, \infty)$, or from different families of distributions, say normal distribution and the NIG distribution. In this case, the regulator may be interested in testing $H_0 : F \in \mathcal{M}(\Theta)$ against the alternative $H_a : F \notin \mathcal{M}(\Theta)$. Constructing a (good) test for this set of hypotheses requires somehow taking into account possible estimation inaccuracy in $\hat{\theta}(\mathbf{1})$, if testing is carried out on the basis of $F_{M_{\theta(\mathbf{1})}}$. This is a promising avenue for future research.
including but not limited to capital structure decisions (Froot and Stein, 1998; Dangl and Zechner, 2004; Kerkhof et al., 2010; Valencia, 2016; Wilkens and Predescu, 2017), hedging choices (Froot et al., 1993; Leland, 2002), investment decisions (Lucas and Klaassen, 1998; Basak and Shapiro, 2001; Campbell et al., 2001; Agarwal and Naik, 2004), project finance transactions (Gatti et al., 2007), assessment of shareholders value (Hallerbach and Menkveld, 1999). In other words, the scope of our methodology is broad and it encompasses any real application where a firm needs to have a proper assessment of downside risk. In section 7 we will focus on one specific purpose, which is the calculation of capital requirements for banks.

5 Data and Calibration Description

For our empirical application we consider the daily time-series of the S&P500 over a thirty-six year time period that goes from January 1, 1980, to July 29, 2016. The data has been retrieved from Thomson Reuters Datastream. The analysis is focused on three risk measures, VaR, ES and MS, along with the 95% confidence intervals for VaR and MS obtained according to the theoretical approaches explained in the previous section.

The calibration of the models is done using several computational schemes on a daily basis for the entire period of study. Basel 2.5 (BCBS, 2010) requires to calculate capital requirements for market risk starting from VaR at 1% with a horizon of 10 days. Model calibration should be performed using an observation window of at least one year. Ten-day VaR can be calculated by multiplying one-day VaR by \( \sqrt{10} \), as allowed by the Basel 2.5 regulation (BCBS, 2010) and used in practice. So, our first calibration approach employs daily returns and an estimation window of 250 returns to calculate one-day risk measures. These, in turn, are multiplied by \( \sqrt{10} \) to obtain the ten-day counterparts. Basel 2.5 (BCBS, 2010) prescribes also to calculate a stressed VaR by calibrating the models to a period of stress of one year. Thus, as required by the regulation, we derive the stressed VaR, ES and MS measures. To this end, at each date we identify the lowest return occurred up to that point in time and then we select the set of 250 past returns that include this lowest return and exhibit the highest volatility.

Basel 4 (BCBS, 2019b) requires to compute capital requirements based on the ten-day stressed ES at 2.5% level. Differently from Basel 2.5, the new regulation does not allow to derive the ten-day ES starting from the one-day counterpart with a time scaling.

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\(^{13}\)We also collect the daily time series of the USD/GBP exchange rate which is used to replicate the empirical study on a different asset. The results of the analysis on the exchange rate are presented in the Online Supplement.

\(^{14}\)Under the standard assumption that a year includes 250 business days, our observation window corresponds to one year.

\(^{15}\)In the Online Supplement we extend this approach by calibrating the models over a two-year (rather than one-year) observation window.
This means that the model calibration must be performed using ten-day returns. In particular, it is possible to use both non-overlapping and overlapping returns.\(^{16}\) Hence, the second scheme consists in calibrating the models using ten-day overlapping returns and an observation window of 250 returns, as allowed both by Basel 2.5 (BCBS, 2010) and Basel 4 (BCBS, 2019b). In this way, we are able to directly estimate risk measures with an horizon of 10 days.

Last but not least, our third calibration scheme employs ten-day non-overlapping returns and an estimation window of 100 returns. This approach to calculate risk measures is compliant with Basel 4 regulation (BCBS, 2019b).\(^{17}\)

Model calibration under these three schemes is then performed according to the modeling set-up. Under the nonparametric approach, the set of past returns considered represents the empirical distribution of asset returns and the various risk measures are directly estimated starting from a quantile of this empirical distribution. In the case of the gaussian model and the NIG model, we follow a two-step approach: first, we use the samples of past return to estimate the parameters of the distribution function of each model; second, once we have the parameter values, we calculate the required quantiles of the probability distribution and we estimate the risk measures.

Specifically, under the gaussian assumption, the log-returns of the asset are distributed according to a normal distribution with mean \(\mu\) and variance \(\sigma^2\), i.e. \(Y_{t+h}\mid \mathcal{F}_t \sim \mathcal{N}(\mu, \sigma^2)\). We estimate the two model parameters using the sample mean and the sample variance, respectively. Under the NIG model, we assume that \(Y_{t+h}\mid \mathcal{F}_t \sim \text{NIG}(\alpha, \beta, \delta, \mu)\). The NIG distribution is a subclass of the generalized hyperbolic distribution, \(\text{GHyp}(\lambda, \alpha, \beta, \delta, \mu)\), where the shape parameter \(\lambda = -\frac{1}{2}\), the tail parameter \(\alpha > 0\), the skewness parameter \(\beta \in (-\alpha, \alpha)\), the “peakedness” parameter \(\delta > 0\), and the location parameter \(\mu \in \mathbb{R}\) (Paolella, 2007). We estimate \(\alpha, \beta, \delta, \text{ and } \mu\) via maximum-likelihood estimation (MLE).

### 6 Selecting Risk Measures and Models

In this section we show how to apply the methodology presented in section 4 to handle model risk of market risk measures. Our empirical analysis starts by considering a long position on the S&P 500.

In Figure 1 we plot the time series of the one-day nonparametric VaR, gaussian VaR,\(^{16}\)Danielsson and Zhou (2016) highlights that, in principle, the non-overlapping approach is preferred because the overlapping method introduces a dependence structure in the returns which may amplify estimation risk. Specifically, under the overlapping approach returns are auto-correlated and not independent. Nevertheless, the non-overlapping approach might be in some cases unfeasible, as it requires to collect a wide set of historical data.

\(^{17}\)Basel 4 (BCBS, 2019b) requires to calibrate the models using a sample of past returns pertaining to a period of stress of one year. The non-overlapping approach, in this setup, would require to use a sample of 25 returns, which is too small for parameter estimation. For this reason we rely on an estimation window of 100 returns.
and NIG VaR at 1%, 2.5% and 5%. Model calibration is performed using daily returns and a moving estimation window of 250 days. To visualize the evolution of market risk over time, risk measures are expressed as a proportion of the spot price.

As expected, all estimates of market risk follow a similar path and exhibit substantial spikes at marked crisis events, such as the Black Monday in 1987 or the Lehman Brothers collapse in 2008. However, there are important differences across models. The nonparametric and NIG estimates of market risk are roughly the same in most the sample period, but 50% larger than the gaussian estimates. In general, during calm periods the three risk models produce close estimates, whilst in times of distress they tend to diverge substantially, confirming the findings of Danielsson et al. (2016).

In addition, risk measures under the gaussian model are very close at all critical levels even in periods of crisis, perhaps not surprisingly given that the gaussian model cannot generate fat tails. Overall this first line of investigation confirms the well-known fact by risk managers that the gaussian estimates may underestimate market risk by comparison with the nonparametric and NIG estimates of risk.18

6.1 Risk Measures Adjusted for Model Risk

We now implement our methodology to derive confidence intervals for VaR based on order statistics calculus.

The graphs in Figure 2 show the time series of the one-day VaR at 1% along with its 95% confidence intervals. For both distribution-free and parametric approaches, our newly proposed confidence intervals are significantly larger in turbulent than normal times. This points out to an increase in the uncertainty related to VaR estimates (model risk) in periods of market stress, which is again consistent with Danielsson et al. (2016). Focusing on the parametric confidence intervals, we notice that the nonparametric VaR (our order statistics estimate) lies systematically within the range only for confidence intervals produced by the NIG model. Moreover, the various breaches under the gaussian assumption occur mostly in times of crisis. This indicates that the NIG model properly captures the distribution of asset returns and outperforms substantially the gaussian model. This simple exercise highlights also that confidence intervals based on order statistics can be used as a model screening tool in risk management.19

One great advantage of being able to calculate confidence intervals is that risk managers can easily adjust risk estimates for parameter estimation uncertainty. Here we

18Figure D in the Online Supplement shows that this is true also for the USD/GBP exchange rate, but only in times of distress. As we might expect, moving from one-day to ten-day VaR trebles the level of estimated risk under all models, but the patterns are similar. These results are persistent in good and bad times, for both the equity index and the exchange rate (see Figures B-C and F-G in the Online Supplement).

19Similar results apply to calculations under other calibration schemes and for an investment in the USD/GBP exchange rate (see Figures H-N in the Online Supplement).
Figure 1: One-day VaR at 1% estimating using different models

The figure shows the time series of one-day VaR at 1%, 2.5% and 5% for a long position on the S&P 500. VaR is estimated using a nonparametric approach (a), a gaussian model (b) and a NIG model (c) for asset returns. Model calibration is performed using daily returns and an observation window of 250 days. Risk measures are expressed as a proportion of the spot price.

(a) Nonparametric

(b) Gaussian

(c) NIG
The figure shows the time series of one-day VaR at 1% for a long position on the S&P 500 along with its 95% distribution-free and parametric confidence intervals. VaR is estimated using a nonparametric approach (a), a gaussian model (b) and a NIG model (c) for asset returns. Parametric confidence intervals are estimated assuming a gaussian model (b) and a NIG model (c) for asset returns. Model calibration is performed using daily returns and an observation window of 250 days. Risk measures are expressed as a proportion of the spot price.

(a) Distribution-free confidence interval

(b) Parametric confidence interval - Gaussian

(c) Parametric confidence interval - NIG
advocate using the upper bound of our newly proposed confidence intervals as a risk measure adjusted for model risk. Risk measures adjusted for model risk allow a prudent assessment of market risk, which is valuable both for risk managers and regulators, as these are generally more conservative than standard risk measures.

However, prior to adjusting for parameter estimation uncertainty, it is crucial to make sure that the model adopted is appropriate. Ultimately, whether a model works well should be assessed by model validation.

6.2 Model Validation

In section 4 we show that, for any distribution of returns available analytically, we can derive the density function and the cumulative distribution function of the difference between the orders statistics used to estimate VaR and MS. Given the modeling assumption, this allows us to compute the p-value \( P_f(D \leq d) \) under the null hypothesis that the observed difference between two quantiles for VaR and MS belongs to the conditional distribution. In this way one can assess whether the model envisaged for asset returns is appropriate to capture the unexpected losses incurred in case VaR at a certain confidence level is exceeded. To this end, we compute the p-values on a daily basis spanning the entire sample period. Then, we validate the model as being appropriate for estimating risk measures if these p-values are consistently between 0.05% and 99.5%, and reject the model otherwise.

The graphs in Figures 3 show the time series of the p-values obtained for the gaussian model and the NIG model. Model calibration is performed using daily returns and an estimation window of 250 days. Our model validation tool indicates that the NIG distribution is superior. We note that the gaussian model has too many observations in the extreme ranges of the probability distribution of \( D \), especially during the recent financial crisis.20

While our tool validates the NIG model and rejects the gaussian model, it does not allow to say something on whether the NIG model outperforms the nonparametric approach. To address this point and show further evidence on the performance of our models in delivering market risk estimates, we conduct a comprehensive battery of backtests. These are aimed at evaluating if the estimated measures of risk properly contain the realized losses. First, we consider a series of tests that look at the number of risk thresholds breaches and/or the durations between breaches. These include the frequency of excessive losses (FOEL) test, the proportion of failures (POF) test of Kupiec (1995), the independence test of Christoffersen (1998), the independence test of Christoffersen

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20This result holds true also when we adopt different calibration schemes, as well as when we analyze the USD/GBP exchange rate (see Figures O-U in the Online Supplement). Our statistical test enables a straightforward visual assessment of model performance in capturing the left tail of the distribution of asset returns.
Figure 3: P-values of the difference between the quantiles used to estimate VaR and MS at 1%

The figure shows the time series of the p-values for the observed difference between the quantiles used to estimate VaR and MS at 1% for a long position on the S&P 500. P-values are calculated assuming that the true distribution of asset returns is gaussian (a) and NIG (b). Model calibration is performed using daily returns and an observation window of 250 days. Marks highlight rejections of the model, namely the dates in which the p-values are below 0.5% and above 99.5%.

(a) Gaussian

(b) NIG

and Pelletier (2004), and the conditional coverage test of Christoffersen (1998).

A drawback of these tests is that they fail to take into account the magnitude of large losses. Thus, we extend our exercise to the recent backtesting approach of Périsignon and Smith (2010), which tests jointly VaR at standard levels as well as very low critical levels.21 This test accounts for both the number and the magnitude of extreme losses.

Panel A of Table 1 presents the results of the various unconditional, independence, and conditional coverage tests on one-day VaR at 1% and 0.5% (corresponding to MS at 1%). Model calibration is performed using daily returns and a moving estimation window of 250 days. A simple metric as the FOEL indicates that the NIG VaR is the risk estimate that is breached with lower frequency. The significant statistic of the FOEL test and the Kupiec’s POF test reveal that this frequency is also in line with the critical level at which VaR is estimated. When compared to the other competing models, the NIG delivers the lowest test statistics across all backtests, except for the two independence tests. Overall, our findings suggest that the NIG model outperforms not only the gaussian model, but also the nonparametric approach. This result is confirmed, and even reinforced, by the multivariate unconditional coverage test of Périsignon and Smith (2010) reported in panel B. Here we jointly test VaR at five critical levels, that is 0.5%, 1%, 1.25%, 2.5% and 5%, respectively. The NIG model is the only one that passes the test with a highly significant

21 This approach has been further extended by Colletaz et al. (2013).
Table 1: Backtests of one-day VaR


<table>
<thead>
<tr>
<th>Model</th>
<th>Critical level</th>
<th>FOEL</th>
<th>Ku POF</th>
<th>Ch I</th>
<th>Ch-Pe I</th>
<th>Ch CC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nonparametric VaR</td>
<td>0.5%</td>
<td>1.00%</td>
<td>6.84</td>
<td>36.24</td>
<td>5.73*</td>
<td>22.54</td>
</tr>
<tr>
<td>Normal VaR</td>
<td>0.5%</td>
<td>1.45%</td>
<td>13.02</td>
<td>111.78</td>
<td>8.12</td>
<td>52.52</td>
</tr>
<tr>
<td>NIG VaR</td>
<td>0.5%</td>
<td>0.65%</td>
<td>1.99</td>
<td>3.63**</td>
<td>7.29</td>
<td>24.90</td>
</tr>
<tr>
<td>Nonparametric VaR</td>
<td>1%</td>
<td>1.35%</td>
<td>3.34</td>
<td>10.10</td>
<td>4.44*</td>
<td>36.01</td>
</tr>
<tr>
<td>Normal VaR</td>
<td>1%</td>
<td>2.04%</td>
<td>10.12</td>
<td>78.68</td>
<td>14.09</td>
<td>66.93</td>
</tr>
<tr>
<td>NIG VaR</td>
<td>1%</td>
<td>1.28%</td>
<td>2.72</td>
<td>6.79</td>
<td>5.14*</td>
<td>36.08</td>
</tr>
</tbody>
</table>

Panel B

<table>
<thead>
<tr>
<th>Pe-Sm MUC (critical levels 0.5%, 1%, 1.25%, 2.5% and 5%)</th>
<th>T-stat</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nonparametric VaR</td>
<td>52.65</td>
</tr>
<tr>
<td>Normal VaR</td>
<td>138.99</td>
</tr>
<tr>
<td>NIG VaR</td>
<td>5.92**</td>
</tr>
</tbody>
</table>

* , ** and *** denotes the model that passes the test at 10% level, 5% level and 1% level test statistic.22

Overall our model validation exercise emphasizes that the NIG distribution is the most appropriate model for the equity index (also for the exchange rate and across various calibration methods, see Online Supplement). After selecting the NIG as our model of asset returns, we can derive model risk adjusted quantile measures using the upper bound of the parametric confidence interval derived in section 6.1. Then we can proceed to calculate “capital requirements adjusted for model risk” as described in the next section.

7 Capital Requirements

Capital adequacy is the key microprudential and macroprudential tool of banking regulation. The role of capital requirements is multifold. First, they are intended to keep banks solvent providing a safety buffer for senior liability holders (Diamond and Rajan, 2000; Freixas and Rochet, 2008; Gordy and Heitfield, 2010). Second, they reduce systemic risk associated with bank defaults (Freixas and Rochet, 2008; Gordy and Heitfield, 2010; Martinez-Miera and Suarez, 2014). Third, they work as a discipline device limiting moral-hazard incentives due to government guarantees such as deposit insurance and implicit bailout (Furlong and Keeley, 1989; Rochet, 1992; Giammarino et al., 1993; Hellman et al., 2000; Freixas and Rochet, 2008; Gordy and Heitfield, 2010; Allen et al., 2011; Mehran and Thakor, 2011; Admati et al., 2013), as well as preventing the “leverage ratchet effect” described by Admati et al. (2018). However, equity financing is perceived

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by banks as especially costly. This is due to distortions such as taxation (passive interests on debt are tax deductible), deposit insurance schemes (with a premium independent of risk) and implicit bailout guarantee (Modigliani and Miller, 1963; Gennette and Pyle, 1991; Admati et al., 2013). This means that there are private costs for banks associated with capital requirements that stem from missed savings on taxes and government subsidies. As a consequence, banks may want to fund themselves with less capital than what would be optimal from a social welfare perspective (Repullo and Suarez, 2013). This is why the regulation requires banks to comply with minimum capital requirements. However, if on the one hand capital requirements allow to internalize the costs of bank failures, on the other hand, if tightened, they can lead to a contraction of credit supply (Aiyar et al., 2014; Kisin and Manela, 2016; Braouezec and Wagalath, 2018; Naceur et al., 2018; Fraisse et al., 2019; Bams et al., 2019; De Jonghe et al., 2020; Nguyen et al., 2020), a loss in consumption (Van den Heuvel, 2008) and an increase in shadow banking, which is outside the scope of banking regulation (Hanson et al., 2010, 2011; Plantin, 2015). Posner (2015) argues that regulators should set capital requirements standards upon a careful cost-benefit analysis.

Admati et al. (2013) highlight that, even though banks had capital levels well above the regulatory minima between 1990s and 2000s, the recent financial crises reveal that they were in fact undercapitalized vis-à-vis a social safety level. Vallascas and Hagendorff (2013) show that the risk weighted assets (the denominator of the risk-based capital ratio) reported by banks between 2000 and 2010 were only marginally related to the volatility of bank asset returns, suggesting that banks may under-represent their risk exposures to limit regulatory capital buffers.

In the aftermath of the unprecedented cascade of financial crises (subprime, liquidity and sovereign) over the period 2007-2011, new regulations have been developed to improve capital requirements calculations and to ensure that banks are properly capitalized. Specifically, in 2010 the BCBS enacted the Basel 2.5 capital requirements standards for market risk BCBS (2010), which are into force since January 2012. In the same year, the Committee introduced a broad new regulatory framework named Basel 3 (BCBS, 2011), which was partially revised in 2011. Basel 3 (BCBS, 2011) was adopted in the USA by the Federal Reserve System (Federal Reserve System, 2013) and in the European Union by a legislative package including the Directive 2013/36/EU (European Parliament, 2013a) and the Regulation (EU) No. 575/2013 (European Parliament, 2013b). Basel 2.5 capital requirements for market risk were maintained under the Basel 3 framework. Starting in 2016, the BCBS enacted a series of amendments to Basel 3 (BCBS, 2011), commonly referred as “Basel 4” (PricewaterhouseCoopers, 2016; McKinsey, McKinsey; KPMG, 2018). These include the new Basel 4 capital requirements for market risk, which were firstly introduced in 2016 (BCBS, 2016), finalized in 2019 (BCBS, 2019b), and will enter into force in January 2022.
Under Basel 3 (BCBS, 2011) “banks are required to maintain more capital of higher quality to cover unexpected losses” (BCBS, 2017). Along the same line, the US and EU regulations state, respectively, “the recent financial crisis demonstrated that the amount of high-quality capital held by banking organizations was insufficient to absorb the losses generated over that period. [...] the BCBS assessed the international capital framework and, in 2010, published Basel 3, a comprehensive reform package designed to improve the quality and quantity of regulatory capital and build additional capacity into the banking system to absorb losses in times of market and economic stress.” (Federal Reserve System, 2013), “In light of the nature and magnitude of unexpected losses experienced by institutions during the financial and economic crisis, it is necessary to improve further the quality and harmonisation of own funds that institutions are required to hold. This should include the introduction of a new definition of the core elements of capital available to absorb unexpected losses as they arise [...] It is also necessary to raise significantly the level of own funds, including new capital ratios focusing on the core elements of own funds available to absorb losses as they arise.” (European Parliament, 2013b), “It is therefore appropriate to require credit institutions and relevant investment firms to hold, in addition to other own fund requirements, a capital conservation buffer and a countercyclical capital buffer to ensure that they accumulate, during periods of economic growth, a sufficient capital base to absorb losses in stressed periods.” (European Parliament, 2013a).

Consistently with the latest regulations, capital requirements should be set in such a way to cover losses generated in periods of economic stress to ward off a bank default. To this end, capital requirements should be computed relying on models identified as appropriate by a careful model validation. This process would help capital requirements computation to be robust to model risk due to misspecification. In addition, in line with the current regulation, capital requirements should be estimated calibrating the model to a period of stress in the market.

### 7.1 Basel 2.5 and Basel 4 Capital Requirements

The Basel 2.5 regulation (BCBS, 2010) that is still into effect requires banks to calculate minimum capital requirements for market risk on the basis of unstressed and stressed VaR measures. Specifically, banks that rely on the internal models approach must meet, on a daily basis, a capital requirements defined by

\[
CR = \max \left\{ VaR_{t-1}, k \cdot \overline{VaR}_{avg} \right\} + \max \left\{ sVaR_{t-1}, k_s \cdot s\overline{VaR}_{avg} \right\}
\]

(11)

\[23\text{Linking capital requirements calculation to the validation of the adopted internal models should motivate banks to disclose the actual level of risk that they take and to handle this risk with adequate levels of capitalization (Cuoco and Liu, 2006).}\]
where \( \hat{VAR}_{t-1} \) is the previous day’s VaR estimate, computed at 1% level and with a horizon of 10 working days (two weeks), \( \bar{VAR}_{avg} \) is the average of the daily VaR on each of the preceding 60 business days, computed at 1% level and with a horizon of 10 working days, \( k \) is a multiplication factor subject to an absolute minimum of 3 and \( s\hat{VAR}_{t-1} \) is the previous day’s stressed VaR estimate, at the same critical level and horizon. The stressed VaR has to be computed using historical data drawn from a continuous 12-months period of significant financial stress relevant to the bank’s portfolio.

The multiplication factor \( k \) is in part aimed to account implicitly for model risk (Kerkhof et al., 2010). It is defined according to a backtesting procedure of the one-day VaR computed at 1% level. Interestingly, risk measures in the capital requirements formula are based on ten-day VaR, whereas \( k \) is derived by backtesting one-day VaR. The minimum value of 3 seems arbitrary. However, Barrieu and Scandolo (2015) argue this value corresponds to the multiplier that allows to get the upper bound of VaR under the assumption that asset returns are normally distributed.

The new Basel 4 (BCBS, 2019b) regulation has changed substantially the calculation of capital requirements under the internal models approach. The formula to derive capital requirements for market risk in our setup\(^{24}\) is

\[
CR = \max \left\{ s\hat{ES}_{t-1}, m \cdot s\hat{ES}_{avg} \right\}
\]

where \( s\hat{ES}_{t-1} \) is the previous day’s stressed ES estimate, computed at 2.5% level and with a horizon of 10 working days, \( s\hat{ES}_{avg} \) is the average of the daily stressed ES on each of the preceding 60 business days, computed at 2.5% level and with a horizon of 10 working days, and \( m \) is a multiplication factor subject to a floor of 1.5. The stressed ES has to be computed using historical data drawn from a continuous 12-months period of significant financial stress relevant to the bank’s portfolio.

As in Basel 2.5 (BCBS, 2010), the calculation of the multiplication factor \( m \) is based on a backtesting procedure of the daily VaR at 1% level and one-day horizon. In Basel 4 the minimum value of 1.5 is half the corresponding value under Basel 2.5 (BCBS, 2010). The BCBS justifies the switch from VaR at 1% of Basel 2.5 (BCBS, 2010) to the ES at 2.5% of Basel 4 (BCBS, 2019b) as the risk metric to calculate capital requirements highlighting that the ES captures tail risk which is not accounted for in VaR (BCBS, 2019a). However, the Committee acknowledges that, from a quantitative perspective, these two measures have the same order of magnitude: “if the same cut-off point is used for VaR and for ES, the value of ES will be higher than the value of VaR. The difference between ES and VaR outcomes increases in cases of fat-tailed distributions. In the revised market risk framework, the 97.5th percentile ES is roughly equivalent to the

\(^{24}\)See the Online Supplement for a detailed description of the general formula for capital requirements and how we simplified it to derive capital requirements for an investment in a single asset subject to a modellable risk factor with a liquidity horizon of ten days.
7.2 Capital Requirements Adjusted for Model Risk

In this paper, we advocate a simple approach to explicitly account for model risk in capital requirements calculations. First, we propose to use the MS at 1% to capture tail risk, specifically the median of losses occurring in case VaR is exceeded. We keep the same critical level of 1% specified in the original Basel 2.5 regulation (BCBS, 2010) to ensure that we estimate the magnitude of losses above the 99th percentile of VaR, which was the original confidence level adopted in the regulatory framework. This should deliver a more conservative measure of market risk than the ES at 2.5%. We opt for the MS rather than ES because a measure of tail risk as we can calculate confidence intervals for the MS according to our approach. Moreover, consistently with the regulation, we would further ensure that our estimate of risk is sufficiently conservative by calibrating the model to a period of stress of one year.

Then, we compute the capital requirements accounting for model risk as the upper bound of the 95% confidence interval for the stressed MS at 1%. Adjusting capital requirements for model risk safeguards from parameter estimation uncertainty. Combined with a model calibration to a period of stress, this procedure should deliver capital requirements that are able to absorb extreme losses such as those experienced during the recent global financial crisis.

In principle, our approach has two nice properties with respect to capital requirements standards under Basel 4 (BCBS, 2019b). First, the same risk measure, i.e. the MS at 1%, is used both for backtesting and capital requirements calculation, limiting computational time and ensuring consistency throughout the whole risk management process. Second, uncertainty is taken into account relying on a robust statistical approach, which builds on confidence intervals for the MS quantile measure. These confidence intervals are derived under the modeling assumption that is adopted for the specific asset class under analysis. This is quite different from the multiplier approach of Basel 4 (BCBS, 2019b), which does not distinguish across models used to capture the distribution of asset returns and that is derived by backtesting a different metric (VaR at 1%) than the one used for capital requirements calculation (ES at 2.5%).

However, empirical evidence is needed to evaluate the costs and benefits of our capital requirements adjusted for model risk in practice. This analysis includes an assessment of the extent to which capital requirements adjusted for model risk absorb losses in normal and stressed times, as well as a comparison with capital requirements obtained according to Basel 2.5 (BCBS, 2010) and Basel 4 (BCBS, 2019b). In what follows we focus the attention on capital requirements computed under the NIG model for asset returns

Additional results concerning other distributions of asset returns are available upon request from
this is the distributional assumption accepted by our model validation procedure.

We start presenting jointly our capital requirements adjusted for model risk and capital requirements computed according to Basel 2.5 (BCBS, 2010). Figure 4 shows the ten-day realized losses, the nonparametric stressed VaR at 1%, the nonparametric stressed MS at 1%, our capital requirements adjusted for model risk (i.e. the upper bound of the 95% parametric confidence interval for the stressed MS at 1%) and the capital requirements computed using formula (11) for an investment of $100 in the S&P 500. The risk metrics used to derive our capital requirements adjusted for model risk and the Basel 2.5 capital requirements are estimated under the NIG model for asset returns. Panel (a) shows the results when model calibration is performed using daily returns and an estimation window of 250 observations, and ten-day risk measures are calculated scaling the corresponding one-day measures by $\sqrt{10}$ (as allowed under Basel 2.5, see BCBS (2010)). Panel (b) covers the results when model calibration is performed using ten-day overlapping returns and an estimation window of 250 days.\textsuperscript{26}

Irrespective of the calibration scheme adopted, our capital requirements adjusted for model risk are able to systematically contain the realized losses after the 1987 market crash. At the same time, they are not excessively conservative, maintaining a level that from 1987 is slightly higher than the extreme losses experienced during the recent financial crisis. Adjusting the MS for model risk to derive capital requirements that are robust to estimation uncertainty is pivotal to ensure that the capital levels absorb actual losses. In fact, the realized losses breach the MS at 1% (not adjusted for model risk) during the recent financial crisis, indicating that adjusting for model risk is crucial. The graphs highlight a huge difference between the Basel 2.5 capital requirements and our capital requirements adjusted for model risk, with the former being considerably higher. In some cases the Basel 2.5 capital requirements exceed the initial investment of $100, which seems paradoxical.\textsuperscript{27} Our analysis reveals that the Basel 2.5 regulation (BCBS, 2010) is excessively conservative, probably in an effort to respond strongly to the flaws emerged in the banking industry during the global financial crisis. Our finding is not singular, similar conclusions being reported in the literature (Alexander et al., 2013; Alexander and Baptista, 2017; Kerkhof et al., 2010). While it is crucial that capital requirements are able to absorb unexpected losses, such extreme capital levels may lead to an undesirable contraction in banks investments. In the consultative document BCBS (2014b), though, the BCBS recognizes itself that “basing regulatory capital on both

\textsuperscript{26} The reason why the estimated values in Figure 4 start from 1981 (rather than 1980) is due to the combination of (i) the calibration procedure, which employs a sample of 250 past returns, and (ii) the BCBS formula (11) for capital requirements, which includes the average of the daily VaR on each of the preceding 60 business days. This means that a sample of data covering more than one year is needed in order to estimate the first value of the Basel 2.5 capital requirements.

\textsuperscript{27} A similar finding is reported by Kerkhof et al. (2010).
current VaR and stressed VaR calculations may be unnecessarily duplicative”.  

We now analyze our capital requirements adjusted for model risk in comparison with those under Basel 4 regulation (BCBS, 2019b). The graphs in Figure 5 depict the ten-day realized losses, the nonparametric stressed VaR at 1%, “SVaR 1%”, the nonparametric stressed MS at 1%, “SVaR 0.5% (SMS 1%)”, our capital requirements adjusted for model risk (the upper bound of the 95% confidence interval for the stressed MS at 1%), “95% UCI SVaR 0.5% (SMS 1%)”, and the Basel 4 capital requirements computed using formula (12), “BCBS CR”, for an investment of $100 in the S&P 500. Our capital requirements adjusted for model risk and the Basel 4 capital requirements are obtained assuming a NIG distribution for the returns of the equity index. In panel (b) we remove the colored face patch for the upper part of the the 95% parametric confidence interval for the stressed MS at 1% to ease the visualization of Basel 4 capital requirements.  

Comparing panels (a)-(b) of Figure 5 with panel (b) of Figure 4, it is evident that the new capital requirements are substantially lower than those obtained under Basel 2.5 (BCBS, 2010). This is due to two main reasons. First, the Basel 4 quantitative standards (BCBS, 2019b) require banks to calculate capital requirements using a single ES metric instead of a combination of VaR and stressed VaR. This has ultimately erased the duplicative effect characterising Basel 2.5 (BCBS, 2010) discussed above. Secondly, the possible values for the multiplication factor in the capital requirements formula have been dramatically decreased, with the minimum value being exactly half of what it used to be. As such, the new Basel 4 capital requirements are consistently at a level slightly above the realized losses after the 1987 market crash. The same holds for our capital requirements adjusted for model risk, but they are less volatile and sensitive to extreme losses over time. This is due to the fact that the multiplication factor in formula (12) increases as the backtesting performance of VaR worsens, which is likely to happen after a market crash.  

Ideally one may want to use ten-day non-overlapping returns when calculating capital

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28 Our results are confirmed when using an estimation window of 500 observations for model calibration. Similar findings apply also when we consider an investment in the USD/GBP exchange rate. However, we observe a good performance of our capital requirements adjusted for model risk after the Lehman collapse when model calibration is performed using daily returns and after the Black Wednesday of 1992 (when the British government withdraw the pound sterling from the Exchange Rate Mechanism) when model calibration is done using ten-day overlapping returns (see Figures V-W in the Online Supplement).  

29 The reason why the estimated values in Figure 5 start from 1981 (rather than 1980) is due to the combination of (i) the calibration procedure, which employs a sample of 250 past returns, and (ii) the BCBS formula (12) for capital requirements, which includes the average of the daily ES on each of the preceding 60 business days. This means that a sample of data covering more than one year is needed in order to estimate the first value of the Basel 4 capital requirements.  

30 Results are similar when we look at an investment in the USD/GBP exchange rate, but in this case our capital requirements adjusted for model risk and the Basel 4 capital requirements result consistently above the realized losses after the Black Wednesday of 1992 (see Figure X in the Online Supplement).
Figure 4: Our capital requirements adjusted for model risk and Basel 2.5 capital requirements

The figure shows the time series of ten-day realized losses, the nonparametric stressed VaR at 1%, “SVaR 1%”, the nonparametric stressed MS at 1%, “SVaR 0.5% (SMS 1%)”, our capital requirements adjusted for model risk (i.e. the upper bound of the 95% parametric confidence interval for the stressed MS at 1%), “95% UCI SVaR 0.5% (SMS 1%)”, and the Basel 2.5 capital requirements computed using formula (11), “BCBS CR”, for an investment of $100 in the S&P 500. The risk measures used to calculate our capital requirements adjusted for model risk and the Basel 2.5 capital requirements are estimated under the NIG model for asset returns. Panel (a) shows the results when model calibration is done using daily returns and an estimation window of 250 observations, and ten-day risk measures are calculated by multiplying the corresponding one-day measures by $\sqrt{10}$ (as allowed from Basel 2.5 (BCBS, 2010)). Panel (b) shows the results when model calibration is done using ten-day overlapping returns and estimation window of 250 days.

(a) Daily returns and 250 observation window

(b) Ten-day overlapping returns and 250 observation window
Figure 5: Our capital requirements adjusted for model risk and Basel 4 capital requirements using ten-day overlapping returns

The figure shows the time series of ten-day realized losses, the nonparametric stressed VaR at 1%, “SVaR 1%”, the nonparametric stressed MS at 1%, “SVaR 0.5% (SMS 1%)”, our capital requirements adjusted for model risk (i.e., the upper bound of the 95% parametric confidence interval for the stressed MS at 1%), “95% UCI SVaR 0.5% (SMS 1%)”, and the Basel 4 capital requirements computed using formula 12, “BCBS CR”, for an investment of $100 in the S&P 500. The risk measures used to calculate our capital requirements adjusted for model risk and the Basel 4 capital requirements are estimated under the NIG model for asset returns. Model calibration is done using ten-day overlapping returns and estimation window of 250 days. In panel (b) the colored face patch for the upper part of the 95% parametric confidence interval for the stressed MS at 1% is removed to ease the visualization of Basel 4 capital requirements.

(a) Ten-day overlapping returns and 250 observation window

(b) Ten-day overlapping returns and 250 observation window no face patch
requirements to limit estimation risk.\textsuperscript{31} The graph in panel (a) of Figure 6 shows that,\textsuperscript{32} under this computational scheme and using an estimation window of 100 observations\textsuperscript{33}, our capital requirements adjusted for model risk are able to cover any realized loss after the 1987 market crash. This holds true for the Basel 4 capital requirements most of the time. A relevant exception, though, is the failure in absorbing the losses occurred at the outset of the recent financial crisis. In the graph of panel (a) of Figure 6 both our capital requirements adjusted for model risk and the Basel 4 capital requirements are often hidden. This is due to the fact that from the stock market crash of 1987 the stressed nonparametric MS at 1\% is consistently higher than the two types of capital requirements. Thus, to visualize better our capital requirements adjusted for model risk and the Basel 4 capital requirements, we add panel (b) where we remove the colored face patch for the stressed MS at 1\%.\textsuperscript{34}

8 Summary Remarks

In this paper we propose a framework for handling model risk in the calculation of capital requirements for market risk. Our approach builds on the order statistics calculus to derive confidence intervals for quantile risk measures and perform a model validation test. In an extensive empirical exercise focused on an investment in the S&P 500 equity index, we investigate the extent to which our capital requirements adjusted for model risk absorb losses in normal and stressed times and we compare them to the capital requirements set by the Basel Committee on Banking Supervision regulation.

Our study reveals that our capital requirements adjusted for model risk, based on the upper bound of the stressed median shortfall at 1\%, are able to cover losses generated in periods of economic stress, such as the Lehman Brothers collapse in 2008, once the calibration of the stressed median shortfall is made using historical data that include an extreme event (in our example the stock market crash of 1987). Taking into account model risk in capital requirements calculations is crucial. In fact, ignoring estimation uncertainty would lead to capital levels that are not sufficiently conservative. The performance of our

\textsuperscript{31}As discussed above, the overlapping approach introduces auto-correlation in asset returns (Danielsson and Zhou, 2016).

\textsuperscript{32}The reason why the estimated values in Figure 6 start from 1984 (rather than 1980) is due to the combination of (i) the calibration procedure, which employs a sample of 100 non-overlapping past returns, and (ii) the BCBS formula (12) for capital requirements, which includes the average of the daily ES on each of the preceding 60 business days. This means that a sample of data covering almost for years is needed in order to estimate the first value of the Basel 4 capital requirements.

\textsuperscript{33}The regulation states that the stressed ES has to be computed by calibrating the model to the most severe 12-month period of stress occurred in the past. Relying on ten-day non-overlapping returns implies using a sample of only 25 returns, which is too short for a robust inference. Hence, we opt for a window of 100 ten-day non-overlapping returns that corresponds to roughly 4 years of historical data.

\textsuperscript{34}Results are similar when we look at an investment in the USD/GBP exchange rate, but in this case our capital requirements adjusted for model risk and the Basel 4 capital requirements result consistently above the realized losses after the Black Wednesday of 1992 (see Figure X in the Online Supplement).
Figure 6: Our capital requirements adjusted for model risk and Basel 4 capital requirements using ten-day non-overlapping returns

The figure shows the time series of ten-day realized losses, the nonparametric stressed VaR at 1%, “SVaR 1%”, the nonparametric stressed MS at 1%, “SVaR 0.5% (SMS 1%)”, our capital requirements adjusted for model risk (i.e. the upper bound of the 95% parametric confidence interval for the stressed MS at 1%), “95% UCI SVaR 0.5% (SMS 1%)”, and the Basel 4 capital requirements computed using formula 12, “BCBS CR”, for an investment of $100 in the S&P 500. The risk measures used to calculate our capital requirements adjusted for model risk and the Basel 4 capital requirements are estimated under the NIG model for asset returns. Model calibration is done using ten-day non-overlapping returns and estimation window of 100 days. In panel (b) the colored face patch for the stressed MS at 1% is removed to ease the visualization of our capital requirements adjusted for model risk and Basel 4 capital requirements.

(a) Ten-day non-overlapping returns and 100 observation window

(b) Ten-day non-overlapping returns and 100 observation window no face patch
capital requirements adjusted for model risk in absorbing losses is similar to that of Basel 4 capital requirements. The ability to prevent a bank failure is a major benefit of both capital regimes, not only for the bank itself but also from a broader social perspective. As discussed above, capital requirements entail some costs for banks in terms of savings on taxes and government subsidies, but these are private costs.

Furthermore, our capital requirements adjusted for model risk and the Basel 4 capital requirements are superior to the Basel 2.5 capital requirements. In fact, the latter are excessively conservative, exposing to a potential risk of an undesirable contraction in banks investments. Specifically, Basel 2.5 capital requirements calculations (BCBS, 2010) include a multiplicative effect due to the sum of VaR and stressed VaR in the capital requirements’ formula, as acknowledged by the Basel Committee on Banking Supervision, plus an inflating effect due to the implicit model risk multiplicative factors for which the minimum is 3. These rules lead to very high capital requirements. Basel 4 regulation (BCBS, 2019b) has introduced relevant changes. In particular, the old “multiplicative effect” has been eradicated, while the “inflating effect” has been strongly reduced.

Finally, our capital requirements adjusted for model risk look very stable over time. When compared to the Basel 4 capital requirements, they are less volatile and responsive to market crashes or bear markets. For example Figure 5 shows that, after the Lehman’s collapse in 2008, Basel 4 capital levels increase sharply of about $20 (corresponding to 20% of the initial investment) whereas our capital requirements adjusted for model risk rise only marginally. Correcting the procyclicality of capital requirements to prevent contractionary effects in downturns is the rational behind the capital conservation buffer and the countercyclical buffer of Basel 3 (BCBS, 2011; Repullo and Suarez, 2013). In light of that, the stability of our capital requirements adjusted for model risk across market scenarios is a good feature. Moreover, the volatility of capital requirements may be costly for banks as it translates de facto in higher capital buffers (Fuster and Vickery, 2018; Matousek et al., 2020), but this would represent a private cost of banks only. Overall, our analysis highlights various private and social benefits of capital requirements adjusted for model risk.

In our empirical analysis we focus on model risk in the calculation of capital requirements for an investment in an equity index (and a foreign exchange in the Online Supplement). According to the Basel 4 regulation (BCBS, 2019b), our setup falls in the case of a single risk factor and a liquidity horizon of ten days. Basel 4 (BCBS, 2019b) emphasizes the importance to distinguish among risk classes assigning a specific liquidity horizons to each risk factor category. Assessing the impact of model risk across different risk classes and liquidity horizons represents an interesting avenue for future research.
Appendix: Closed-form expression for the p-values of the difference between the quantiles for VaR and MS

Assuming that we have \( n \) order statistics, \( Y_{[1]}, \ldots, Y_{[n]} \), which represent our ordered financial returns, we define VaR and MS, respectively, as a monotonic transformation of the order statistics \( Y_{[v]} \) and \( Y_{[m]} \), with \( v > m \). We want to calculate the probability \( P_{F}(Y_{[v]} - Y_{[m]} \leq d) \) i.e. the probability that the difference between the two quantiles for VaR and MS is lower or equal to a certain value \( d \). We impose \( d \) to be the difference between the estimated quantiles, namely the quantiles obtained from the empirical distribution of financial returns. If \( Y_{[m]} = z \) and \( Y_{[v]} = y \), the density function of \( D = y - z \) is:

\[
q(d) = K \int_{-\infty}^{\infty} F^{m-1}(z) [F(z + d) - F(z)]^{v-m-1} [1 - F(z + d)]^{n-v} f(z) f(z + d) dz \tag{13}
\]

with \( K = \frac{n!}{(m-1)!(v-m-1)!(n-v)!} \). The order statistics \( Y_{[1]}, \ldots, Y_{[n]} \) in a sample from any absolutely continuous distribution with cdf \( F \) can be transformed by the order-preserving probability integral transformation \( \tilde{y} = F(y) \) into order statistics drawn from a uniform distribution on the interval \([0,1]\), \( U_{[1]}, \ldots, U_{[n]} \). Thus we can transform \( z \) and \( y \) into \( \tilde{z} = F(z) \) and \( \tilde{y} = F(y) \) and we can denote by \( \tilde{D} = \tilde{y} - \tilde{z} \). Recalling the expressions for the density function and the distribution function of a uniform random variable in \([0,1]\), we have:

\[
q(\tilde{d}) = K \int_{0}^{\tilde{d}} \tilde{z}^{m-1} \tilde{d}^{v-m-1} (1 - \tilde{z} - \tilde{d})^{n-v} \tilde{d} \tilde{z} \tag{14}
\]

This density function is absolutely equivalent to the density function in formula (13). Setting \( \tilde{z} = \nu (1 - \tilde{d}) \) where \( 0 \leq \tilde{d} \leq 1 \), formula (14) can be rewritten as:

\[
q(\tilde{d}) = K \int_{0}^{1} \nu^{m-1} (1 - \tilde{d})^{m-1} \tilde{d}^{v-m-1} \left[ (1 - \nu) (1 - \tilde{d}) \right]^{n-v} (1 - \tilde{d}) d\nu
\]

\[
= K (1 - \tilde{d})^{m-1} \tilde{d}^{v-m-1} (1 - \tilde{d}) \int_{0}^{1} \nu^{m-1} \left[ (1 - \nu) (1 - \tilde{d}) \right]^{n-v} d\nu
\]

\[
= \frac{n!(m-1)!(n-v+1-1)!}{(m-1)!(v-m-1)!(n-v)!(n-v+m+1-1)!} (1 - \tilde{d})^{n-v+m} \tilde{d}^{v-m-1}
\]

\[
= \frac{1}{B(v-m,n-v+m+1)} (1 - \tilde{d})^{n-v+m} \tilde{d}^{v-m-1}
\]

33
As a last step we compute the probability $P_F(D \leq d) = P_F(\hat{D} \leq \hat{d})$ as follows:

$$P_F(D \leq d) = \frac{1}{B(v - m, n - v + m + 1)} \int_0^{\hat{d}} (1 - u)^{n-v+m} u^{v-m-1} du$$

$$= B(\hat{d}, v - m, n - v + m + 1)$$

$0 \leq \hat{d} \leq 1$.  

Formula (16) is exactly the closed-form expression we use to calculate the p-values.

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