Sharp asymptotics for Fredholm Pfaffians related to interacting particle systems and random matrices

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Dedicated to the memory of Marc Kac

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Abstract

It has been known since the pioneering paper of Mark Kac [20], that the asymptotics of Fredholm determinants can be studied using probabilistic methods. We demonstrate the efficacy of Kac' approach by studying the Fredholm Pfaffian describing the statistics of both non-Hermitian random matrices and annihilating Brownian motions. Namely, we establish the following two results. Firstly, let $\sqrt{N} + \lambda_{max}$ be the largest real eigenvalue of a random $N \times N$ matrix with independent $N(0,1)$ entries (the 'real Ginibre matrix'). Consider the limiting $N \to \infty$ distribution $\mathbb{P}[\lambda_{max} < -L]$ of the shifted maximal real eigenvalue λ_{max}. Then

$$\lim_{L\to\infty} e^{\frac{1}{2\sqrt{2\pi}}\zeta\left(\frac{3}{2}\right)L}\mathbb{P}\left(\lambda_{max} < -L\right) = e^{C_e},$$

where ζ is the Riemann zeta-function and

$$C_e = \frac{1}{2}\log 2 + \frac{1}{4\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(-\pi + \sum_{m=1}^{n-1}\frac{1}{\sqrt{m(n-m)}}\right).$$

Secondly, let $X_t^{(max)}$ be the position of the rightmost particle at time t for a system of annihilating Brownian motions (ABM's) started from every point of $\mathbb{R}_-$. Then

$$\lim_{L\to\infty} e^{\frac{1}{2\sqrt{2\pi}}\zeta\left(\frac{3}{2}\right)L}\mathbb{P}\left(\frac{X_t^{(max)}}{\sqrt{4t}} < -L\right) = e^{C_e}.$$

These statements are a sharp counterpart of the results of [22], improved by computing the $O(L^0)$ term in the asymptotic $L \to \infty$ expansion of the corresponding Fredholm Pfaffian.

1 Introduction and the main result

The present paper continues the investigation of the statistics of the real eigenvalues for random matrices with independent normal matrix elements (the so-called real Ginibre ensemble) and particles for the system of annihilating Brownian motions started in [22].

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A mathematical way of describing random arrangements of points representing eigenvalues or particle positions is the theory of point processes, see [9] for a review. Important subclasses of point processes are determinantal and Pfaffian point processes, whose correlation functions are given by determinants or Pfaffians of kernels of certain integral operators. Well known examples of determinantal point processes are the laws of eigenvalues for random Hermitian, unitary and complex Gaussian matrix models; the eigenvalue statistics for symmetric, symplectic and real random Gaussian matrices are described by Pfaffian point processes, see [1], [21] for reviews. Moreover, determinantal and Pfaffian point processes describe the distribution of particles for a number of interacting particle systems such as the totally asymmetric simple exclusion process [19] and reaction-diffusion systems for certain combinations of annihilation, coalescence, branching and immigration of particles, [17], [18], [26].

Of a particular importance for the current investigation is the fact that the law of the real eigenvalues for the real Ginibre ensemble is a Pfaffian point process, [5], [14], [24]. Moreover, its bulk scaling limit coincides (up to a diffusive rescaling) with the fixed time law of annihilating Brownian motions started at every point of the real line [26], its edge scaling limit coincides with the fixed time law of annihilating Brownian motions started at every point of the negative part of the real line [17], [6].

The probabilities of 'gaps' (regions of space void of any particles) are a fundamental object for point processes, which in fact characterise the law of a simple one-dimensional process uniquely. For determinantal and Pfaffian point processes gap probabilities are given by Fredholm determinants and Pfaffians of integral operators determined by the kernels of the corresponding processes. A particular instance of gap probability is the distribution of the rightmost or leftmost particle for the process, for example the statistics of the largest eigenvalue of a random Hermitian matrix (the Tracy-Widom distribution, [21]), or the largest real eigenvalue of a real random matrix (the Rider-Sinclair distribution, [23]).

An exact calculation of a Fredholm determinant or a Pfaffian is impossible in all but a few special cases (for example the celebrated link between Painlevé functions and the GOE and GSE ensembles, see [1] Section 3.1). Fortunately, the asymptotics of gap probabilities in the limit of large empty intervals can be studied in many important cases. If, for example, the operator is translationally invariant, the asymptotics of the corresponding Fredholm determinant can be studied using Szegő's theorem and its modifications, see [8] for a review. In particular, Szegő's theorem was used by Derrida and Zeitak to calculate the asymptotics of a single gap probability in a coalescence-annihilation model started at every point of the real line [11]. Note that for the purely annihilating case, Derrida-Zeitak's calculation is non-rigorous due to the presence of Fisher-Hartwig singularities in the kernel, but the final answer is believed to be correct and can be rigorised using an appropriate modification of Szegő's formula, see [8], Chapter 6. In [15] Forrester used the Derrida-Zeitak formula and the connection between the real Ginibre random matrix model and annihilating Brownian motions stated above to calculate the asymptotics of gap probabilities for the distribution of real eigenvalues in the bulk. For us [15] was a crucial paper which inspired our current research.

In the absence of translational invariance, the situation is more complicated. If the operator is integrable (as is the case for all point processes describing eigenvalues of complex Hermitian random matrices), the asymptotics of the distribution of extreme eigenvalues can be studied by reducing the problem to a matrix Riemann-Hilbert problem and analyzing the latter, see e.g. [10]. As was discovered recently in [2], the operator K which defines the Pfaffian point process for the annihilating Brownian motions and the edge scaling limit of the real Ginibre ensemble is Fourier-conjugated to an integrable operator. This is a significant development, placing the real Ginibre ensemble firmly in the realm of integrable systems. Unfortunately, the associated Riemann-Hilbert problem turned out to be somewhat complicated allowing the calculation of the asymptotics of the Fredholm Pfaffian only for the operator γK, where $\gamma < 1$, thus making it difficult to relate the answer to the distribution of the largest real eigenvalue ($\gamma = 1$). Moreover, the constant factor C_e is not directly accessible by these methods for any γ. Notice however, that in the context of particle system, γK for $\frac{1}{2} \leqslant \gamma \leqslant 1$ has a clear probabilistic meaning - it describes the statistics of mixed annihilating-coalescing Brownian motions, [17].

An alternative approach to the asymptotic analysis of Fredholm determinants was pioneered by Mark Kac [20] who was the first to state and prove a continuous version of Szegő's theorem, which originally was formulated for Toeplitz matrices rather than translationally invariant trace class integral operators. The main idea due to Kac is to interpret the log-det expansion of the Fredholm determinant of the trace-class integral operator T acting on L^2 functions on $I \subset \mathbb{R}$,

$$\log\det(I-T) = -\sum_{n=1}^{\infty}\frac{1}{n}\int_{I^n} T(x_1-x_2)T(x_2-x_3)\dots T(x_n-x_1)\,dx_1\dots dx_n,$$

as a certain expectation with respect to the measure of a random walk whose increments have a (pseudo) distribution $T(x)dx$. Of course, the result of [20] can be derived directly by taking the continuous limit of Szegő's theorem for Toepltiz matrices.

However, as we showed in [22], the probabilistic approach can be used for the derivation of new results, namely the tails of the distribution of the rightmost real eigenvalue (the rightmost particle) for the edge scaling limit of the real Ginibre ensemble (annihilating Brownian motions). This is the most difficult $\gamma = 1$ case in the terminology of [2], see the discussion above. In [22] we already calculated the asymptotic of the relevant Fredholm Pfaffian up to $O(1)$ errors. In the current paper we will show that by sticking closer to the original Kac argument we can calculate the constant term as well as characterise the size of the correction. The calculation turns out to be rather short and intuitive. It is based on some classical properties of random walks with general increments, as discussed in [13].

The main result of the paper is the following statement. Let $\sqrt{N}+\lambda_{max}$ be the largest real eigenvalue for the $N\times N$ real Ginibre ensemble. Let $\mathbb{E}_N$ denote the ensemble expectation. Let

$$\mathbb{P}(\lambda_{max} < -L) = \lim_{N\to\infty}\mathbb{E}_N(\mathbb{1}(\lambda_{max} < -L)) \tag{1}$$

be the edge scaling limit of the distribution of the largest real eigenvalue.

Theorem 1.1. *For $L > 0$,*

$$\lim_{L\to\infty} e^{\frac{\zeta(3/2)}{2\sqrt{2\pi}}L}\,\mathbb{P}\left(\lambda_{max} < -L\right) = e^{C_e}, \tag{2}$$

where

$$C_e = \frac{1}{2}\log 2 + \frac{1}{4\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(-\pi + \sum_{m=1}^{n-1}\frac{1}{\sqrt{m(n-m)}}\right). \tag{3}$$

More precisely,

$$\log\mathbb{P}\left(\lambda_{max} < -L\right) = -\frac{\zeta(3/2)}{2\sqrt{2\pi}}L + C_e + o(L^{-1+}), \tag{4}$$

where for $o(L^{-1+})$: for any $\mu > 0$, $\lim_{L\to\infty} L^{1-\mu}o(L^{-1+}) = 0$.

The problem of computing $\exp C_e$, the constant factor in the asymptotic expansion of Fredholm determinants, is a known difficult problem in statistical physics, see for example [3] for the calculation of the constant factor for the partition function of the six-vertex model and a review of similar cases.

Let us analyse the presented asymptotic formula for the gap probability at the edge of the spectrum in more detail. Numerically, $\exp(C_e) \approx 0.75$, which is consistent with its numerical value obtained in [2]. Let us also compare (4) with the bulk scaling limit of the probabilty $\mathbb{P}(N(-L,0)=0)$ that the interval $(-L,0)$ contains no real (unshifted) eigenvalues. As predicted by the Derrida-Zeitak formula [11] applied to the real Ginibre ensemble in [15],

$$\log \mathbb{P}\left(N(-L,0)=0\right) = -\frac{\zeta(3/2)}{2\sqrt{2\pi}}L + C_b + o(L^0), \tag{5}$$

where

$$C_b = \log 2 + \frac{1}{4\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(-\pi + \sum_{m=1}^{n-1}\frac{1}{\sqrt{m(n-m)}}\right). \tag{6}$$

Comparing (5) and (4) we see that the leading terms coincide. This is not very surprising, see [22] for a heuristic explanation. However, there is no reason why the $O(1)$ terms should be the same. In fact, we see from the above formulae that

$$\lim_{L\to\infty}\frac{\mathbb{P}\left(\lambda_{max}<-L\right)}{\mathbb{P}\left(N(-L,0)=0\right)} = e^{C_e-C_b} = \frac{1}{\sqrt{2}}. \tag{7}$$

It would be interesting to see if it were possible to derive relation (7) without computing $\mathbb{P}(\lambda_{max}<-L)$ and $\mathbb{P}(N(-L,0)=0)$ separately.

Interpreted in terms of particle systems, our result reads as follows:

Corollary 1.2. *Consider the system of instantaneously annihilating Brownian motions on the real line started from every point of $\mathbb{R}_-$ (half-space maximal entrance law). Let $X_t^{(max)}$ be the position of the rightmost particle at a fixed time $t>0$. Then*

$$\lim_{X\to\infty} e^{\frac{1}{2\sqrt{2\pi}}\zeta\left(\frac{3}{2}\right)\frac{X}{\sqrt{4t}}}\mathbb{P}\left(X_t^{(max)}<-X\right) = e^{C_e}. \tag{8}$$

The Corollary is a direct consequence of the observation that

$$X_t^{(max)} \overset{(d)}{\sim} \sqrt{4t}\lambda_{max}. \tag{9}$$

This in turn follows from the fact that the edge scaling limit of the law of real eigenvalues for the real Ginibre ensemble and the single time distribution of ABM's with half-space maximal entrance law rescaled by $1/\sqrt{4t}$ can be characterised by the same Pfaffian point process, see [5], [6] and [17] for a proof of this fact.

The rest of the paper is organised as follows. In Section 2 we collect the probabilistic tools necessary to establish our main result, explain the main idea for the argument and finally prove Theorem 1.1. In Section 3 we prove the probabilistic lemmas used to derive the statement of the Theorem. For the sake of completeness we also present a streamlined proof of the key identity due to Kac [20], which underpins our argument.

2 The proof of Theorem 1.1

The starting point for the proof is the Rider-Sinclair formula [23], which gives a Fredholm Pfaffian expression for $\mathbb{P}(\lambda_{max}<-L)$. More specifically, we will use a probabilistic re-statement of Rider-Sinclair's result proved in [22], which can be explained as follows. Let $(B_n, n\geqslant 0)$ be the discrete time random walk with Gaussian $N(0,1/2)$ increments started at zero. Let

$$\begin{cases} \tau_L &= \inf_{n>0}\{2n-1: \ B_{2n-1}\geqslant L\}, \\ \tau_0 &= \inf_{n>0}\{2n: \ B_{2n}\leqslant 0\}. \end{cases} \tag{10}$$

In words: τ_0 is the smallest *even* time such that $B_{\tau_0}\leqslant 0$, τ_L is the smallest *odd* time such that $B_{\tau_L}\geqslant L$. Also, let

$$M_{2n} = \sup\{B_k : k \textit{ is odd}, \ k<2n\} \tag{11}$$

be the supremum of the random walk $(B_k)_{k\geqslant 0}$ taken over all odd times not exceeding the time $2n$. Then

Theorem 2.1.

$$\mathbb{P}(\lambda_{max} < -L) = \sqrt{\mathbb{P}(\tau_L < \tau_0)}\, e^{-\frac{L}{2}\,\mathbb{E}\left(\delta_0\left(B_{\tau_0}\right)\right)} e^{\frac{1}{2}\mathbb{E}\left(\min(L, M_{\tau_0})\delta_0\left(B_{\tau_0}\right)\right)}. \tag{12}$$

Remark 2.1. *We use the expression $\mathbb{E}(X\delta_y(Y))$ to mean a continuous Lebesgue density for the measure $\mathbb{E}(X\mathbb{1}(Y \in dy))$ evaluated at y. If $y = 0$, we sometimes write $\mathbb{E}(X\delta_0(Y))$ as $\mathbb{E}(X\mathbb{1}(Y \in d0))$.*

Remark 2.2. *Notice a slight change of notations in (11), (12) in comparison with formulae (1.7), (1.9) of [22].*

We will also need the following two facts from [22]:

Lemma 2.2. *As $L \to \infty$,*

$$\mathbb{P}[\tau_L < \tau_0] = \frac{1}{\sqrt{2}L}(1 + o(L^{-1+})). \tag{13}$$

Also,

$$\mathbb{E}\left(\delta_0\left(B_{\tau_0}\right)\right) = \frac{\zeta(3/2)}{\sqrt{2\pi}}. \tag{14}$$

Formula (13) is a slight improvement on Lemma 3.2 of [22], which only claims the error bound of magnitude $O(L^{-1/2})$. The improved bound is obtained simply by using Hölder rather than the Cauchy-Schwarz inequality in the proof, without changing the rest of the argument. Equation (14) is equally straightforward to check and we will do it below to illustrate the utility of a probabilistic approach, see Remark 3.1 below. In addition, we need the following key statement due to Mark Kac [20]:

Lemma 2.3 (Mark Kac, 1954)**.** *Let $(X_i)_{i\geqslant 1}$ be independent identically distributed random variables having continuous even density function ρ on $\mathbb{R}$, and $S_k = X_1+X_2+\ldots+X_k, k \geqslant 1$. Then*

$$\rho^{(n)}(0)\mathbb{E}\left(\max(0, S_1, S_2, \ldots, S_{n-1})|S_n = 0\right) = \frac{n}{2}\int_0^\infty x\sum_{k=1}^{n-1}\frac{\rho^{(k)}(x)\rho^{(n-k)}(x)}{k(n-k)}dx, \tag{15}$$

where $\rho^{(k)}$ denotes the k-fold convolution of ρ with itself or, in other words, the density function of S_k.

Substituting (13, 14) into (12), we find

$$\log\mathbb{P}(\lambda_{max} < -L) + \frac{\zeta(3/2)}{2\sqrt{2\pi}}L = R(L), \tag{16}$$

where

$$R(L) = \frac{1}{2}\mathbb{E}\left(\min(L, M_{\tau_0})\delta_0\left(B_{\tau_0}\right)\right) - \frac{1}{2}\log L - \frac{1}{4}\log 2 + o(L^{-1+}). \tag{17}$$

It remains to calculate the leading asymptotic of $R(L)$, as $L \to \infty$, which turns out to be $O(L^0)$. Decomposing over the values of $\tau_0 = 2n$,

$$\mathbb{E}\left(\min(L, M_{\tau_0})\delta_0\left(B_{\tau_0}\right)\right) = \sum_{n=1}^{\infty} p_n(L), \tag{18}$$

where

$$p_n(L) = \mathbb{E}\left(\min(L, M_{2n})\mathbb{1}(\tau_0 = 2n)\delta_0\left(B_{2n}\right)\right). \tag{19}$$

The summands $p_n(L),\ n \geqslant 1$, can be simplified using the cyclic invariance of the increments of the random walk B. Namely, we have the following result proved in Section 3.

Lemma 2.4. *Let $(S_n)_{n\geqslant 1}$ be a discrete time random walk such that the distribution of increments has a continuous density.. Then*

$$p_n(L) = \frac{1}{n}\mathbb{E}\left(\min\left(L, M_{2n} - m_{2n}\right)\delta_0(S_{2n})\right), \tag{20}$$

where

$$m_{2n} = \inf\{S_k : k \text{ is even}, \ k \leqslant 2n\} \tag{21}$$

is the infimum of the random walk taken over even times not exceeding $2n$ and M_n is the supremum of the walk over odd times defined in (11).

Remark 2.3. *Notice that the above statement does not rely on the Gaussianity of increments. It is a particular instance of a family of results for random walks conditioned to finish at zero found in [13].*

We conclude that p_n's are fully determined by a joint distribution of the maximum, the minimum and the final position of the random walk. Let us fix $\epsilon \in (0, 2)$. Then

$$\mathbb{E}\left(\min(L, M_{\tau_0})\delta_0\left(B_{\tau_0}\right)\right) = \sum_{n=1}^{\lfloor L^{2-\epsilon}\rfloor} p_n(L) + \sum_{\lfloor L^{2-\epsilon}\rfloor+1}^{\infty} p_n(L). \tag{22}$$

For $n \leqslant L^{2-\epsilon}$, $M_{2n} - m_{2n} < L$ with probability close to 1. Therefore (20) is well approximated by $p_n(L) \approx \frac{1}{n}\mathbb{E}\left((M_{2n} - m_{2n})\,\delta_0(B_{2n})\right)$, which can be computed adapting the original Kac' argument [20]. This approximation decays as $1/2n$ at large n's, leading to the logarithmic divergence of $\sum_{n=0}^{\infty} p_n(L)$ and thus the necessity for a separate analysis for large n. Fortunately, for $n > L^{2-\epsilon}$, the random walk can be well approximated by a Brownian motion. Then $p_n(L)$ can be computed using the classical Levy's result for the trivariate distribution of the supremum, the infimum and the final value of the Brownian motion on an interval, see e.g. [4], [13] for review. Rigorising the argument, we arrive at the following Lemma proved in Section 3.

Lemma 2.5. *For $n \leqslant L^{2-\epsilon}$,*

$$p_n(L) = \frac{1}{2\pi n}\sum_{k=1}^{n-1}\frac{1}{\sqrt{k(n-k)}} + E_n^{(1)}(L), \tag{23}$$

where

$$|E_n^{(1)}(L)| \leqslant \sqrt{\frac{2}{\pi n^3}}L\frac{e^{-\frac{2L^2}{n}}}{1-e^{-2L^{\epsilon}}}. \tag{24}$$

For $n \geqslant L^{2-\epsilon}$,

$$p_n(L) = \frac{1}{2n} - \sqrt{\frac{2}{\pi n^3}}L\sum_{k=1}^{\infty} e^{-\frac{2k^2L^2}{n}} + E_n^{(2)}(L), \tag{25}$$

where, for any fixed $\gamma \in (0, \frac{3}{2})$, there exists an n-independent constant $C_\gamma > 0$ such that

$$|E_n^{(2)}(L)| \leqslant C_\gamma n^{-3/2+\gamma}. \tag{26}$$

Substituting (23, 25) into (22) and then into (17), we find

$$\begin{aligned} R(L) \quad = \quad & \frac{1}{4\pi}\sum_{n=1}^{\lfloor L^{2-\epsilon}\rfloor}\frac{1}{n}\sum_{k=1}^{n-1}\frac{1}{\sqrt{k(n-k)}} + \frac{1}{2}\sum_{n=\lfloor L^{2-\epsilon}\rfloor+1}^{\infty}\left(\frac{1}{2n} - \sqrt{\frac{2}{\pi n^3}}L\Omega\left(\frac{2L^2}{\pi n}\right)\right) \\ & -\frac{1}{4}\log\left(2L^2\right) + \frac{1}{2}\sum_{n=1}^{\lfloor L^{2-\epsilon}\rfloor} E_n^{(1)}(L) + \frac{1}{2}\sum_{n=\lfloor L^{2-\epsilon}\rfloor+1}^{\infty} E_n^{(2)}(L) + o\left(L^{-1+}\right), \end{aligned} \tag{27}$$

where

$$\Omega(t) := \sum_{k=1}^{\infty} e^{-\pi k^2 t} \tag{28}$$

is a function on $\mathbb{R}$ closely related to Jacobi's θ-function, $\Omega(t) = \frac{\theta(0,it)-1}{2}$, see [27] for a review. It satisfies $\Omega(t) = O(e^{-\pi t})$ for $t \to \infty$ and $\Omega(t) = \frac{1}{2\sqrt{t}} - \frac{1}{2} + o(1)$ for $t \downarrow 0$, which we use to verify the convergence of integral bounds derived below.

First, let us estimate the error terms in (27) using (24, 26). Notice that the function $f(x) = x^{-3/2+\gamma}$ is decreasing on $\mathbb{R}_+$, if $\gamma \in (0, \frac{3}{2})$, and the function $g(x) = x^{-3/2}\exp(-2k^2L^2/x)$ is increasing for $0 < x < L^{2-\epsilon}$ provided $k \geqslant 1$. Therefore, $\sum_{n=a}^{b} f(n)$ and $\sum_{n=a}^{b} g(n)$ can be bounded above using integrals:

$$\begin{aligned}\frac{1}{2}\left|\sum_{n=1}^{\lfloor L^{2-\epsilon}\rfloor} E_n^{(1)}(L) + \sum_{n=\lfloor L^{2-\epsilon}\rfloor+1}^{\infty} E_n^{(2)}(L)\right| &\leqslant \frac{C_\gamma}{2}\int_{L^{2-\epsilon}}^{\infty}\frac{dx}{x^{3/2-\gamma}} + \sqrt{\frac{2L^2}{\pi}}\frac{1}{1-e^{-2L^\epsilon}}\int_0^{1+L^{2-\epsilon}}\frac{e^{-\frac{2L^2}{x}}}{x^{3/2}}dx \\ &\leqslant \frac{C_\gamma}{1-2\gamma}L^{-(2-\epsilon)\frac{1-2\gamma}{2}} + \sqrt{\frac{2L^2}{\pi}}\frac{1}{1-e^{-2L^\epsilon}}\frac{1+L^{2-\epsilon}}{L^{3-3\epsilon/2}}e^{-\frac{2L^2}{1+L^{2-\epsilon}}}.\end{aligned}$$

Since $\gamma > 0$ can be chosen to be arbitrarily small, we conclude from the above that

$$\frac{1}{2}\left|\sum_{n=1}^{\lfloor L^{2-\epsilon}\rfloor} E_n^{(1)}(L) + \sum_{n=\lfloor L^{2-\epsilon}\rfloor+1}^{\infty} E_n^{(2)}(L)\right| = o\left(L^{-1+\epsilon/2+}\right). \tag{29}$$

Therefore,

$$\begin{aligned}R(L) &= \frac{1}{4\pi}\sum_{n=1}^{\lfloor L^{2-\epsilon}\rfloor}\frac{1}{n}\sum_{k=1}^{n-1}\frac{1}{\sqrt{k(n-k)}} + \frac{1}{2}\sum_{n=\lfloor L^{2-\epsilon}\rfloor+1}^{\infty}\left(\frac{1}{2n} - \sqrt{\frac{2}{\pi n^3}}L\Omega\left(\frac{2L^2}{\pi n}\right)\right) \\ &\quad -\frac{1}{4}\log\left(2L^2\right) + o\left(L^{-1+\epsilon/2+}\right).\end{aligned} \tag{30}$$

Using integral bounds it is elementary to establish an estimate, as $n \to \infty$,

$$\sum_{k=1}^{n-1}\frac{1}{\sqrt{k(n-k)}} = \pi + O(n^{-1/2}), \tag{31}$$

which can be used to re-write $R(L)$ as follows:

$$\begin{aligned}R(L) &= \frac{1}{4\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\sum_{k=1}^{n-1}\frac{1}{\sqrt{k(n-k)}} - \pi\right) + \frac{1}{2}\sum_{n=\lfloor L^{2-\epsilon}\rfloor+1}^{\infty}\left(\frac{1}{2n} - \sqrt{\frac{2}{\pi n^3}}L\Omega\left(\frac{2L^2}{\pi n}\right)\right) \\ &\quad + \frac{1}{4}\sum_{n=1}^{\lfloor L^{2-\epsilon}\rfloor}\frac{1}{n} - \frac{1}{4}\log\left(2L^2\right) + o\left(L^{-1+\epsilon/2+}\right).\end{aligned} \tag{32}$$

Recall a classical result for the sum of harmonic series [27], as $N \to \infty$,

$$\sum_{n=1}^{N}\frac{1}{n} = \log N + \gamma + O(N^{-1}), \tag{33}$$

where γ is the Euler-Mascheroni constant. Using (33) in (32) we find

$$\begin{aligned}R(L) &= \frac{1}{4\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\sum_{k=1}^{n-1}\frac{1}{\sqrt{k(n-k)}} - \pi\right) + \frac{1}{2}\sum_{n=\lfloor L^{2-\epsilon}\rfloor+1}^{\infty}\left(\frac{1}{2n} - \sqrt{\frac{2}{\pi n^3}}L\Omega\left(\frac{2L^2}{\pi n}\right)\right) \\ &\quad - \frac{\epsilon}{4}\log L + \frac{\gamma}{4} - \frac{1}{4}\log 2 + o\left(L^{-1+\epsilon/2+}\right).\end{aligned} \tag{34}$$

An application of the mean value theorem to the terms of the second sum on the right hand side of (34) leads to

$$\sum_{n=\lfloor L^{2-\epsilon}\rfloor+1}^{\infty}\left(\frac{1}{2n}-\sqrt{\frac{2}{\pi n^3}}L\Omega\left(\frac{2L^2}{\pi n}\right)\right)=\int_{L^{-\epsilon}}^{\infty}\left(\frac{1}{2x}-\sqrt{\frac{2}{\pi x^3}}\Omega\left(\frac{2}{\pi x}\right)\right)dx+o(L^{-2+\epsilon})$$

$$=\frac{\epsilon}{2}\log L-\int_0^1 dx\sqrt{\frac{2}{\pi x^3}}\Omega\left(\frac{2}{\pi x}\right)+\int_1^{\infty}\left(\frac{1}{2x}-\sqrt{\frac{2}{\pi x^3}}\Omega\left(\frac{2}{\pi x}\right)\right)dx+o(L^{-2+\epsilon}). \quad (35)$$

Substituting (35) into the right hand side of (34) we discover that

$$\begin{aligned} R(L) &= \frac{\gamma}{4}-\frac{1}{4}\log 2+\frac{1}{4\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\sum_{k=1}^{n-1}\frac{1}{\sqrt{k(n-k)}}-\pi\right)-\frac{1}{2}\int_0^1\sqrt{\frac{2}{\pi x^3}}\Omega\left(\frac{2}{\pi x}\right)dx \\ &+\frac{1}{2}\int_1^{\infty}\left(\frac{1}{2x}-\sqrt{\frac{2}{\pi x^3}}\Omega\left(\frac{2}{\pi x}\right)\right)dx+o\left(L^{-1+\epsilon/2+}\right). \end{aligned} \quad (36)$$

As $\epsilon>0$ is arbitrary, we conclude that the magnitude of the error term is $o(L^{-1+})$.

In principle, (36) gives an answer for the $O(1)$ term in the expansion of $\mathbb{P}(\lambda_{max}<-L)$. It can however be considerably simplified, which probably means that the calculation detailed above can also be significantly streamlined. The rest of the proof is an exact calculation based on the relation between the Euler-Mascheroni constant and Jacobi's theta functions.

The calculation is based on the following two remarks: firstly,

$$\gamma=\log(4\pi)-2+2\int_1^{\infty}(1+\sqrt{t})\frac{\Omega(t)}{t}dt, \quad (37)$$

see [7] containing this as well as a large collection of other expressions for the Euler-Mascheroni constant. Formula (37) follows from combining a more standard expression for γ in terms of Riemann's ζ-function,

$$\gamma=\lim_{s\to 1}\left[\zeta(s)-\frac{1}{s-1}\right],$$

see [27] for the derivation, and Riemann's integral representation of ζ,

$$\zeta(s)=\frac{\pi^{s/2}}{s(s-1)\Gamma(s/2)}+\frac{\pi^{s/2}}{\Gamma(s/2)}\int_1^{\infty}\left(t^{(1-s)/2}+t^{s/2}\right)\frac{\Omega(t)}{t}dt,$$

see e.g. [12]. Secondly,

$$1+2\Omega\left(t^{-1}\right)=\sqrt{t}\left(1+2\Omega(t)\right), \quad (38)$$

which follows from the standard transformation properties of the theta function and can be proved directly using Poisson summation formula, see [27] for review. Now let us modify the right hand side of (36) as follows: express γ using (37), change variables in the penultimate integral according to $t=\frac{2}{\pi x}$, apply (38) to the integrand of the last integral. The result is

$$\begin{aligned} &R(L)-\frac{1}{4\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\sum_{k=1}^{n-1}\frac{1}{\sqrt{k(n-k)}}-\pi\right)+o(L^{-1+}) \\ &= \frac{1}{4}\log 2\pi-\frac{1}{2}+\sqrt{\frac{1}{2\pi}}-\frac{1}{2}\int_{2/\pi}^{1}\frac{\Omega(t)}{\sqrt{t}}dt+\frac{1}{2}\int_1^{\pi/2}\frac{\Omega(t)}{t}dt \\ &= \frac{1}{4}\log 2\pi-\frac{1}{2}+\sqrt{\frac{1}{2\pi}}+\frac{1}{2}\int_1^{\pi/2}\frac{\sqrt{t}\Omega(t)-\Omega(t^{-1})}{t^{3/2}}dt \\ &= \frac{1}{2}\log 2 \end{aligned} \quad (39)$$

where the last equality follows from another application of (38) to the integral in the previous expression. Theorem 1.1 is proved. □

3 The proof of probabilistic lemmas

3.1 Lemma 2.4

Let $(X_k)_{1\leqslant k\leqslant 2n}$ for $n = 1, 2, \ldots$, be a sequence of independent indentically distributed random variables with a continuous density. Let $S = (S_k)_{0\leqslant k\leqslant 2n}$ be the associated random walk started at zero,

$$S_k = \sum_{m=1}^{k} X_m. \tag{40}$$

We will frequently consider the walk conditioned to be at 0 at time $2n$, so that $S_{2n} = 0$. Let $S^{(p)}$ be the random walk associated with the cyclic shift of the increments X's by p steps to the right,

$$S_k^{(p)} = \sum_{m=1}^{k} X_{m+p}, \tag{41}$$

where the addition in the time indices is performed modulo $2n$. As it is easy to check, under the conditioning that $S_{2n} = 0$,

$$S_k^{(p)} = S_{k+p} - S_p,\ S_{2n}^{(p)} = 0. \tag{42}$$

Under the conditioning $S_{2n} = 0$, for any p, $(S_k^{(p)})_{0\leqslant k\leqslant 2n}$ is a bridge, whose law is p-independent,

$$(S_k^{(p)})_{0\leqslant k\leqslant 2n} \overset{(d)}{\sim} (S_k^{(q)})_{0\leqslant k\leqslant 2n},\ 0 \leqslant p, q \leqslant 2n-1. \tag{43}$$

Recall that τ_0 is the first even time the value of the bridge S becomes negative. We write $\tau_0^{(p)}$ for the corresponding exit time for the bridge $S^{(p)}$. Similarly, $M_{2n}^{(p)}$ is the odd time maximum of the walk $S^{(p)}$. Using the fact that the even time global minimum of the bridge is unique almost surely, one can easily show that

$$\sum_{p=0}^{n-1} \mathbb{1}\left(\tau_0^{(2p)} = 2n\right) = 1\ a.\ s. \tag{44}$$

Notice that the event $\tau_0^{(2p)} = 2n$ corresponds to the shift to the time at which the even time global minimum of the random walk has been achieved. Recall that

$$m_{2n} = \min_{0\leqslant k\leqslant n} S_{2k}. \tag{45}$$

Therefore,

$$\begin{aligned}
p_n(L) \quad &:= \quad \mathbb{E}\left(\min(L, M_{2n})\mathbb{1}(\tau_0 = 2n)\delta_0(S_{2n})\right) \\
&\overset{(43)}{=} \quad \frac{1}{n}\sum_{p=0}^{n-1} \mathbb{E}\left(\min(L, M_{2n}^{(2p)})\mathbb{1}(\tau_0^{(2p)} = 2n) \mid S_{2n}^{(2p)} = 0\right) \Pr(S_{2n}^{(2p)} \in d0) \\
&\overset{(42)}{=} \quad \frac{1}{n}\sum_{p=0}^{n-1} \mathbb{E}\left(\min(L, M_{2n} - S_{2p})\mathbb{1}(\tau_0^{(2p)} = 2n) \mid S_{2n} = 0\right) \Pr(S_{2n} \in d0) \\
&\overset{(45)}{=} \quad \frac{1}{n}\sum_{p=0}^{n-1} \mathbb{E}\left(\min(L, M_{2n} - m_{2n})\mathbb{1}(\tau_0^{(2p)} = 2n) \mid S_{2n} = 0\right) \Pr(S_{2n} \in d0) \\
&= \quad \frac{1}{n}\mathbb{E}\left(\min(L, M_{2n} - m_{2n})\sum_{p=0}^{n-1}\mathbb{1}(\tau_0^{(2p)} = 2n) \mid S_{2n} = 0\right) \Pr(S_{2n} \in d0) \\
&\overset{(44)}{=} \quad \frac{1}{n}\mathbb{E}\left(\min(L, M_{2n} - m_{2n}) \mid S_{2n} = 0\right) \Pr(S_{2n} \in d0) \\
&= \quad \frac{1}{n}\mathbb{E}\left(\min(L, M_{2n} - m_{2n})\delta_0(S_{2n})\right).
\end{aligned}$$

□

Remark 3.1. *Using the notations developed for the proof, it is very easy to rederive (14), even though the computation presented below uses no new ideas in comparison with [22]:*

$$\begin{aligned}
\mathbb{E}\left(\delta_0(S_{\tau_0})\right) &= \sum_{n=1}^{\infty} \mathbb{E}\left(\mathbb{1}(\tau_0 = 2n)\delta_0(S_{2n})\right) = \sum_{n=1}^{\infty} \mathbb{E}\left(\mathbb{1}(\tau_0 = 2n) \mid S_{2n} = 0\right) \Pr(S_{2n} \in d0) \\
&= \sum_{n=1}^{\infty} \frac{1}{n} \sum_{p=0}^{n-1} \mathbb{E}\left(\mathbb{1}(\tau_0^{(2p)} = 2n) \mid S_{2n}^{(2p)} = 0\right) \Pr(S_{2n} \in d0) \\
&= \sum_{n=1}^{\infty} \frac{1}{n} \sum_{p=0}^{n-1} \mathbb{E}\left(\mathbb{1}(\tau_0^{(2p)} = 2n \mid S_{2n} = 0\right) \Pr(S_{2n} \in d0) \\
&= \sum_{n=1}^{\infty} \frac{1}{n} \mathbb{E}\left(\sum_{p=0}^{n-1} \mathbb{1}(\tau_0^{(2p)} = 2n) \mid S_{2n} = 0\right) \Pr(S_{2n} \in d0) \\
&\overset{(44)}{=} \sum_{n=1}^{\infty} \frac{1}{n} \mathbb{E}\left(1 \mid S_{2n} = 0\right) \Pr(S_{2n} \in d0) = \sum_{n=1}^{\infty} \frac{1}{n} \Pr(S_{2n} \in d0) \\
&= \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{\sqrt{2\pi n}} = \frac{\zeta(3/2)}{\sqrt{2\pi}}.
\end{aligned}$$

3.2 Lemma 2.5

3.2.1 $n \leqslant L^{2-\epsilon}$

Starting from formula (20) of Lemma 2.4,

$$p_n(L) = \frac{1}{n}\mathbb{E}\left((M_{2n} - m_{2n})\,\delta_0(B_{2n})\right) + E_n^{(1)}(L), \tag{46}$$

where

$$E_n^{(1)}(L) = -\frac{1}{n}\mathbb{E}\left((M_{2n} - m_{2n} - L)_+\,\delta_0(B_{2n})\right), \tag{47}$$

and $x_+ := x\mathbb{1}(x \geqslant 0)$.

We start with estimating the error term $E_n^{(1)}(L)$. Let $(B_t)_{t\geqslant 0}$ be the rate-1/2 Brownian motion. It follows from the definition of M_{2n}, m_{2n} that

$$M_{2n} \leqslant \sup_{0\leqslant t\leqslant 2n} B_t,\ m_{2n} \geqslant \inf_{0\leqslant t\leqslant 2n} B_t. \tag{48}$$

As the function $x \mapsto x_+$ is increasing,

$$\begin{aligned}
|E_n^{(1)}(L)| &= \frac{1}{n}\mathbb{E}\left((M_{2n} - m_{2n} - L)_+\,\delta_0(B_{2n})\right) \\
&\leqslant \frac{1}{n}\mathbb{E}\left(\left(\sup_{0\leqslant t\leqslant 2n} B_t - \inf_{0\leqslant t\leqslant 2n} B_t - L\right)_+ \delta_0(B_{2n})\right) \\
&\leqslant \frac{1}{n}\mathbb{E}\left(\left(\sup_{0\leqslant t\leqslant n} W_t - \inf_{0\leqslant t\leqslant n} W_t - L\right)_+ \delta_0(W_n)\right),
\end{aligned} \tag{49}$$

where $(W_t)_{t\geqslant 0}$ is the standard Brownian motion. Thus the error term $\mathbb{E}_n^{(1)}(L)$ is bounded by an expectation with respect to Wiener measure, which can be computed using the known joint distribution of the supremum, infimum and the final position of the Brownian motion,

$$\begin{aligned}&\mathbb{P}\left(\inf_{0\leqslant t\leqslant n} W_t \geqslant a, \sup_{0\leqslant t\leqslant n} W_t \leqslant b, W_t \in d0\right)\\ =\ & \Psi_n(a,b) := \sum_{k\in\mathbb{Z}} \frac{1}{\sqrt{2\pi n}}\left(e^{-\frac{2k^2}{n}(b-a)^2} - e^{-\frac{2}{n}(b-k(b-a))^2}\right),\end{aligned} \tag{50}$$

where $a \leqslant 0, b \geqslant 0$, (see for example [4] (9.1)). Applying (50) leads, after some work very similar to the upcoming calculation (62), to the equality

$$\begin{aligned}&\mathbb{E}\left(\left(\sup_{0\leqslant t\leqslant n} W_t - \inf_{0\leqslant t\leqslant n} W_t - L\right)_+ \delta_0(W_n)\right)\\ =\ & \sqrt{\frac{2}{\pi n}} L \sum_{k=1}^{\infty} e^{-\frac{2k^2L^2}{n}} - \int_{[-L,L]^c} \frac{1}{\sqrt{2\pi n}} e^{-\frac{2b^2}{t}}\, db.\end{aligned} \tag{51}$$

Using this for the right hand side of (49), approximating $e^{-\frac{2k^2L^2}{n}} \leqslant e^{-\frac{2kL^2}{n}}$ for $k \geqslant 1$ and then summing the resulting geometric series when $n \leqslant L^{2-\epsilon}$ leads to the desired bound (24) of Lemma 2.5.

To finish the proof of (23), we need to calculate $\frac{1}{n}\mathbb{E}\left(\left(M_{2n} - m_{2n}\right)\delta_0(B_{2n})\right)$. Using reflection symmetry,

$$\mathbb{E}\left(m_{2n}\delta_0(B_{2n})\right) = -\mathbb{E}\left(\max_{0\leqslant k\leqslant n}(B_{2k})\delta_0(B_{2n})\right) = -\mathbb{E}\left(\max_{0\leqslant k\leqslant n}(W_k)\delta_0(W_n)\right), \tag{52}$$

where $(W_k)_{k\geqslant 0}$ is the random walk with $N(0,1)$ increments. Let us denote the increments of random walk B by $X_1, X_2, \ldots \sim N(0,1/2)$, let $Y_1, Y_2, , \ldots \sim N(0,1)$ be the increments of the walk W. Then

$$\begin{aligned}&\mathbb{E}\left(\left(M_{2n}\right)\delta_0(B_{2n})\right) = \mathbb{E}\left(\max_{0\leqslant k\leqslant n-1}(B_{2k+1})\delta_0(B_{2n})\right)\\ =\ & \mathbb{E}\left(\max(X_1, X_1+X_2+X_3, \ldots, X_1+X_2+\ldots+X_{2n-1})\delta_0(B_{2n})\right)\\ =\ & \mathbb{E}\left(X_1 + \max(0, X_2+X_3, X_2+X_3+X_4+X_5, \ldots, X_3+X_5+\ldots+X_{2n-1})\delta_0(B_{2n})\right)\\ =\ & \mathbb{E}\left(\max(0, Y_1, Y_1+Y_2, \ldots, Y_1+Y_2+\ldots+Y_{n-1})\delta_0(W_n)\right) = \mathbb{E}\left(\max_{0\leqslant k\leqslant n}(W_k)\delta_0(W_n)\right).\end{aligned} \tag{53}$$

Substituting (52) and (53) into $\frac{1}{n}\mathbb{E}\left(\left(M_{2n} - m_{2n}\right)\delta_0(B_{2n})\right)$ we find

$$\frac{1}{n}\mathbb{E}\left(\left(M_{2n} - m_{2n}\right)\delta_0(B_{2n})\right) = \frac{2}{n}\mathbb{E}(\max_{0\leqslant k\leqslant n} W_k\delta_0(W_n)) = \frac{1}{2\pi n}\sum_{k=1}^{n-1}\frac{1}{\sqrt{k(n-k)}}, \tag{54}$$

where the last step used formula (1.11a) from [20] (Lemma 2.3 of the present paper). Formula (23) of Lemma 2.5 is proved.

3.2.2 $n \geqslant L^{2-\epsilon}$

For large values of n it is natural to approximate the Gaussian random walk with Brownian motion to re-write (20) as follows:

$$p_n(L) = \frac{1}{n}\mathbb{E}\left(\min\left(L, \sup_{0\leqslant t\leqslant 2n} B_t - \inf_{0\leqslant t\leqslant 2n} B_t\right)\delta_0\left(B_{2n}\right)\right) + E_n^{(2)}(L), \tag{55}$$

where the correction term is

$$\begin{aligned}&E_n^{(2)}(L)\\ =\ & \frac{1}{n}\mathbb{E}\left(\left(\min\left(L, \sup_{0\leqslant 2k+1\leqslant 2n} B_{2k+1} - \inf_{0\leqslant 2k\leqslant 2n} B_{2k}\right) - \min\left(L, \sup_{0\leqslant t\leqslant 2n} B_t - \inf_{0\leqslant t\leqslant 2n} B_t\right)\right)\delta_0\left(B_{2n}\right)\right).\end{aligned} \tag{56}$$

For any $L, x, y \in \mathbb{R}$,

$$|\min(L,x) - \min(L,y)| \leqslant |x-y|,$$

which allows us to bound (56) as below:

$$\begin{aligned} E_n^{(2)}(L) &\leqslant \frac{1}{n}\mathbb{E}\left(\left|\sup_{0\leqslant t\leqslant 2n} B_t - \sup_{0\leqslant k<n} B_{2k+1}\right| \delta_0\left(B_{2n}\right)\right) \\ &+ \frac{1}{n}\mathbb{E}\left(\left|\inf_{0\leqslant t\leqslant 2n} B_t - \inf_{0\leqslant k\leqslant n} B_{2k}\right| \delta_0\left(B_{2n}\right)\right). \end{aligned} \tag{57}$$

The two terms on the right hand side are very similar and can be bounded by the same function of the index n. We will present the derivation of the bound for the first term only. In what follows, $(W_t)_{t\geqslant 0}$ is the standard Brownian motion, $(WB_t)_{0\leqslant t\leqslant 1}$ is the Brownian bridge.

Rescaling time,

$$\begin{aligned} &\frac{1}{n}\mathbb{E}\left(\left|\sup_{0\leqslant t\leqslant 2n} B_t - \sup_{0\leqslant 2k+1\leqslant 2n} B_{2k+1}\right| \delta_0\left(B_{2n}\right)\right) \\ &= \frac{1}{n}\mathbb{E}\left(\left|\sup_{t\in[0,1]} W_t - \sup_{t\in\{\frac{k}{n}+\frac{1}{2n}\}_{k=0}^{n-1}} W_t\right| \delta_0\left(W_1\right)\right) \\ &= \sqrt{\frac{1}{2\pi n^2}}\mathbb{E}\left(\left|\sup_{t\in[0,1]} W_t - \sup_{t\in\{\frac{k}{n}+\frac{1}{2n}\}_{k=0}^{n-1}} W_t\right| \Big| W_1 = 0\right) \\ &= \sqrt{\frac{1}{2\pi n^2}}\mathbb{E}\left(\left|\sup_{t\in[0,1]} WB_t - \sup_{t\in\{\frac{k}{n}+\frac{1}{2n}\}_{k=0}^{n-1}} WB_t\right|\right). \end{aligned} \tag{58}$$

Recall the following fact about Brownian motions hence the Brownian bridges: for any fixed $\gamma > 0$ there is a non-negative random variable H_γ defined on the same probability space as the bridge itself such that,

$$|WB_t - WB_\tau| \leqslant H_\gamma |t-\tau|^{\frac{1}{2}-\gamma}, \text{ for all } t,\tau \in [0,1]. \tag{59}$$

Moreover, $\mathbb{E}(H_\gamma) < \infty$, see e.g. [4]. Exploiting (59) to bound the right hand side of (58) one finds that

$$\frac{1}{n}\mathbb{E}\left(\left|\sup_{0\leqslant t\leqslant 2n} B_t - \sup_{0\leqslant 2k+1\leqslant 2n} B_{2k+1}\right| \delta_0\left(B_{2n}\right)\right) \leqslant \frac{\mathbb{E}(H_\gamma)}{\sqrt{2\pi n^2}}\left(\frac{1}{2n}\right)^{\frac{1}{2}-\gamma}. \tag{60}$$

The second second term on the right hand side of (57) obeys the same bound. Thus combining (57) with (60) we conclude that

$$E_n^{(2)}(L) \leqslant C_\gamma n^{-3/2+\gamma}, \tag{61}$$

where $C_\gamma = 2\frac{\mathbb{E}(H_\gamma)}{\sqrt{2\pi}}\left(\frac{1}{2}\right)^{\frac{1}{2}-\gamma}$ is an n-independent constant. The estimate (26) of Lemma 2.5 is proved.

In order to calculate the leading term in the expression (55) for $p_n(L)$ we just need to evaluate the expectation of $\min\left(L, \sup_{0\leqslant t\leqslant 2n} B_t - \inf_{0\leqslant t\leqslant 2n} B_t\right)$ using the distribution $\Psi_n(a,b)$ given in (50). Recall that B is a rate $\frac{1}{2}$ Brownian motion. Then, integrating by parts,

$$\begin{aligned}
&\mathbb{E}\left(\min\left(L, \sup_{0\leqslant t\leqslant 2n} B_t - \inf_{0\leqslant t\leqslant 2n} B_t\right)\delta_0\left(B_{2n}\right)\right) \\
&= -\int_{-\infty}^{0}\int_{0}^{\infty} \min(L, b-a)\frac{\partial^2}{\partial a\partial b}\Psi_n(a,b)\,db\,da \\
&= \int_{-\infty}^{0}\int_{0}^{\infty} \mathbb{1}(b-a\leqslant L)\frac{\partial}{\partial a}\Psi_n(a,b)\,db\,da + L\Psi_n(-\infty,\infty) \\
&= -\int_{0}^{L}\Psi_n(b-L,b)\,db + \frac{L}{\sqrt{2\pi n}}.
\end{aligned}$$

To evaluate the boundary term above we have used the facts that $\Psi_n(0,b) = 0$ and $\Psi_n(a,0) = 0$, and that $\Psi_n(a,b)$ extends to the region $a \in (-\infty, 0], b \in [0,\infty)$ where $\Psi_n(-\infty,\infty) = 1/\sqrt{2\pi n}$. Using the value of Ψ_n from (50) gives

$$\Psi_n(b-L,b) = \sum_{k\in\mathbb{Z}} \frac{1}{\sqrt{2\pi n}}\left(e^{-\frac{2k^2L^2}{n}} - e^{-\frac{2}{n}(b-kL)^2}\right)$$

and this leads to

$$\mathbb{E}\left(\min\left(L, \sup_{0\leqslant t\leqslant 2n} B_t - \inf_{0\leqslant t\leqslant 2n} B_t\right)\delta_0\left(B_{2n}\right)\right) = \frac{1}{2} - \frac{2L}{\sqrt{2\pi n}}\sum_{k=1}^{\infty} e^{-\frac{2k^2}{n}L^2}. \tag{62}$$

Formula (25) of Lemma 2.5 is proved. □

3.3 Lemma 2.3

Fix $k < n$ and define

$$E_k := \mathbb{E}\left(\max(0, S_1, \dots, S_k) \mid S_n = 0\right). \tag{63}$$

Decomposing the expectation according to the events $S_k < 0$ and $S_k > 0$ we find

$$\begin{aligned}
E_k &= \mathbb{E}\left(\mathbb{1}(S_k > 0)\max(S_1,\dots,S_k) \mid S_n = 0\right) + \mathbb{E}\left(\mathbb{1}(S_k < 0)\max(0,S_1,\dots,S_k) \mid S_n = 0\right) \\
&= \mathbb{E}\left(\mathbb{1}(S_k > 0)X_1 \mid S_n = 0\right) + \left(\mathbb{E}(\mathbb{1}(S_k > 0)\max(0, X_2, X_2 + X_3 \dots, S_k - X_1) \mid S_n = 0\right) \\
&+ \mathbb{E}(\mathbb{1}(S_k < 0)\max(0, S_1, \dots, S_{k-1}) \mid S_n = 0)) = \frac{1}{k}\mathbb{E}(\mathbb{1}(S_k > 0)S_k \mid S_n = 0) + E_{k-1},
\end{aligned}$$

where the last equality is due to the invariance of the law of the increments $X_1, X_2, \dots X_n$ with respect to a permutation of the first k increments and the $X \to -X$ symmetry of the distribution of increments. Solving the resulting difference equations for E_k's with the initial condition $E_0 = 0$, we find

$$\mathbb{E}\left(\max(0, S_1, \dots, S_{n-1}) \mid S_n = 0\right) = \sum_{k=1}^{n-1}\frac{1}{k}\mathbb{E}(\mathbb{1}(S_k > 0)S_k \mid S_n = 0). \tag{64}$$

(Formula (64) is attributed in [20] to Freeman Dyson.) Therefore,

$$\begin{aligned}
\rho^{(n)}(0)\mathbb{E}(\max(0, S_1, \quad \dots \quad, S_{n-1}) \mid S_n = 0) &\overset{(64)}{=} \sum_{k=1}^{n-1}\frac{1}{k}\mathbb{E}(\mathbb{1}(S_k > 0)S_k \mid S_n = 0)\rho^{(n)}(0) \\
&= \sum_{k=1}^{n-1}\frac{1}{k}\mathbb{E}((S_k)_+\delta_0(S_n)) = \sum_{k=1}^{n-1}\frac{1}{k}\int_0^\infty x\rho^{(k)}(x)\rho^{(n-k)}(x)dx \\
&= \frac{n}{2}\sum_{k=1}^{n-1}\frac{1}{k(n-k)}\int_0^\infty x\rho^{(k)}(x)\rho^{(n-k)}(x)dx.
\end{aligned} \tag{65}$$

Lemma 2.3 is proved. □

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