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Article (Accepted Version)

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September 8, 2020

MSC 2010: 34C25; 34D20; 37C27.
Keywords: Periodic orbit, stability, contraction metric, converse theorem, matrix-valued Partial Differential Equation, existence, uniqueness.

Abstract

The stability and the basin of attraction of a periodic orbit can be determined using a contraction metric, i.e., a Riemannian metric with respect to which adjacent solutions contract. A contraction metric does not require knowledge of the position of the periodic orbit and is robust to perturbations.

In this paper we characterize such a Riemannian contraction metric as matrix-valued solution of a linear first-order Partial Differential Equation. This enables the explicit construction of a contraction metric by numerically solving this equation in [7]. In this paper we prove existence and uniqueness of the solution of the PDE and show that it defines a contraction metric.

1 Introduction

Ordinary differential equations arise in many important applications and the determination of periodic orbits, their stability and basins of attraction are important tasks. We consider a general autonomous Ordinary Differential Equation (ODE) of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}),$$

where $\mathbf{f}\colon \mathbb{R}^n \to \mathbb{R}^n$ is sufficiently smooth.

The stability and the basin of attraction of a periodic orbit can be determined using a Lyapunov function, however, its definition requires the exact position of the periodic orbit. A contraction metric can show the existence, uniqueness and stability of a periodic

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orbit without knowledge of its position. Moreover, a contraction metric is robust to small perturbations of the system or the metric, which ensures that even a good approximation to a contraction metric, e.g. using numerical methods, is itself a contraction metric.

A contraction metric is a Riemannian metric such that the distance between adjacent trajectories decreases over time with respect to the Riemannian metric. Such solutions are also called incrementally stable and a contraction metric is a special type of a Finsler-Lyapunov function [4]. The contraction condition can be formulated as a local condition in a point $\mathbf{x} \in \mathbb{R}^n$ and all adjacent solutions through $\mathbf{x}+\mathbf{v}$ for small $\mathbf{v} \in \mathbb{R}^n$. If the contraction condition holds for all points $\mathbf{x}$ in a compact, positively invariant and connected set K, then there exists one and only one attractor in K, it is exponentially stable and K is a subset of its basin of attraction. If the contraction holds for all adjacent directions $\mathbf{v}$, then the attractor is an equilibrium. If the contraction only holds for $\mathbf{v}$ perpendicular to $\mathbf{f}(\mathbf{x})$ and K does not contain any equilibrium, then the attractor is a periodic orbit, see Theorem 2.1.

Contraction metrics for periodic orbits have been studied by Borg [2] with the Euclidean metric and Stenström [18] with a general Riemannian metric. Further results using a contraction metric have been obtained in [11, 10, 12, 14].

Converse theorems, showing the existence of a contraction metric defined in the basin of attraction of an exponentially stable periodic orbit, have been obtained in [5]. [15, Section 3.5] gave a converse theorem, but the Riemannian metric $M(t, \mathbf{x})$ depends on t and, in general, can become unbounded as $t \to \infty$. In [16, Theorem 3], the authors have expressed a transverse contraction condition, i.e. a contraction metric for periodic orbits, using Linear Matrix Inequalities and have used SOS (sum of squares) to construct it.

In the case of contraction metrics for an equilibrium, converse theorems have been established in [6], characterizing the contraction metric as solution of a matrix-valued PDE. Hence, an approximate solution to the PDE, e.g. using numerical methods [9], constructs a contraction metric.

In this paper we seek to establish a similar result for contraction metrics for periodic orbits. The non-trivial challenge is to restrict the space of adjacent solutions in direction $\mathbf{v}$ at the point $\mathbf{x}$ to the $(n-1)$-dimensional hyperplane perpendicular to $\mathbf{f}(\mathbf{x})$ by using a projection onto it.

Let us compare this paper to [7]: the main result of the current paper is that the PDE problem (3.1) and (3.2) has a unique solution (Theorem 3.1 and 4.2 of this paper, cited as Theorem 2.1 in [7]); furthermore, we show that the solution of the PDE problem is a contraction metric (Theorem 2.1, cited as Theorem 1.1 in [7]). In [7], these results are used and extended by showing that an approximate solution to the PDE problem is a contraction metric; furthermore, using error estimates, it is shown that such an approximate solution can be constructed using a numerical method, namely mesh-free collocation.

Let us give an outline of the contents of this paper: in Section 2 we define a contraction metric and show that it provides a sufficient condition for the existence, uniqueness and exponential stability of a periodic orbit and determines its basin of attraction. Furthermore, we show that the solution of a matrix-valued PDE defines such a contraction metric. In Section 3 we prove the existence of a solution of the above matrix-valued PDE and in Section 4 we prove its uniqueness. Section 5 contains an illustrative example, for which we can compute the solution analytically. We conclude in Section 6.

2 Sufficiency

In this section we show that the solution of a certain matrix-valued PDE is a contraction metric and gives information about the existence and uniqueness of a periodic orbit as well as its basin of attraction. Let us consider the ODE

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{2.1}$$

where $\mathbf{f} \in C^\sigma(\mathbb{R}^n, \mathbb{R}^n)$, $n \in \mathbb{N}$ and $\sigma \geq 1$.
psiWe denote the solution $\mathbf{x}(t)$ of (2.1) with initial value $\boldsymbol{\xi}$ by $S_t\boldsymbol{\xi} := \mathbf{x}(t)$ and assume that it exists for all $t \geq 0$. A periodic orbit of (2.1) is a set $\Omega = \{S_t\mathbf{x} \mid t \in [0, T)\}$, where $S_T\mathbf{x} = \mathbf{x}$ with $T > 0$, i.e. $S_t\mathbf{x}$ is a periodic solution. The minimal $T > 0$ with this property is called the period of the periodic orbit. We denote $\text{dist}(\mathbf{x}, \Omega) = \min_{\mathbf{z}\in\Omega} \|\mathbf{x} - \mathbf{z}\|$, where the vector norm here and for the rest of the paper is the Euclidean norm $\|\cdot\| = \|\cdot\|_2$. A periodic orbit Ω is asymptotically stable if it is stable, i.e. for all $\epsilon > 0$ there exists $\delta > 0$ such that $\text{dist}(\mathbf{y}, \Omega) < \delta$ implies $\text{dist}(S_t\mathbf{y}, \Omega) < \epsilon$ for all $t \geq 0$, and there exists $\delta' > 0$ such that $\text{dist}(\mathbf{y}, \Omega) < \delta'$ implies $\lim_{t\to\infty} \text{dist}(S_t\mathbf{y}, \Omega) = 0$. It is exponentially stable, if it is stable and there are $\nu > 0$ and $\delta' > 0$ such that $\text{dist}(\mathbf{y}, \Omega) < \delta'$ implies $\lim_{t\to\infty} \text{dist}(S_t\mathbf{y}, \Omega)e^{\nu t} = 0$. The basin of attraction of an asymptotically stable periodic orbit Ω is defined as $A(\Omega) = \{\mathbf{y} \in \mathbb{R}^n \mid \lim_{t\to\infty} \text{dist}(S_t\mathbf{y}, \Omega) = 0\}$.

To determine the basin of attraction of a periodic orbit, we seek to find a suitable Riemannian metric such that the distance between adjacent solutions decreases with respect to the Riemannian metric. A Riemannian metric is a matrix-valued function $M \in C^1(D, \mathbb{S}^n)$, where $D \subset \mathbb{R}^n$ is a domain and $\mathbb{S}^n$ denotes the symmetric $\mathbb{R}^{n\times n}$ matrices, such that $M(\mathbf{x})$ is positive definite for all $\mathbf{x} \in D$. In particular, $\langle \mathbf{v}, \mathbf{w}\rangle_{\mathbf{x}} = \mathbf{v}^T M(\mathbf{x})\mathbf{w}$ defines a point-dependent scalar product for all $\mathbf{x} \in D$ and $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, which induces the distance $\|\mathbf{v}\|_{\mathbf{x}} = \sqrt{\mathbf{v}^T M(\mathbf{x})\mathbf{v}}$.

Let us give an explanation of Theorem 2.1 below and the meaning of the function L_M in (2.5): consider two adjacent solutions at $\mathbf{x}$ and $\mathbf{x} + \delta\mathbf{v}$ for small $\delta > 0$, where $\mathbf{v}$ is perpendicular on $\mathbf{f}(\mathbf{x})$. We will give a heuristic argument that $L_M(\mathbf{x}; \mathbf{v})$ is negative, if the distance between solutions through $\mathbf{x}$ and $\mathbf{x} + \delta\mathbf{v}$ with respect to the metric $M(\mathbf{x})$ decreases.

To measure the distance between the two solutions $S_t(\mathbf{x})$ and $S_\theta(\mathbf{x}+\delta\mathbf{v})$, we synchronize the times such that the difference vector between the solutions, $S_\theta(\mathbf{x} + \delta\mathbf{v}) - S_t\mathbf{x}$, is perpendicular to $\mathbf{f}(S_t\mathbf{x})$. In particular, we define $\theta(t)$ such that $\theta(0) = 0$ and

$$\left(S_{\theta(t)}(\mathbf{x} + \delta\mathbf{v}) - S_t\mathbf{x}\right)^T \mathbf{f}(S_t\mathbf{x}) = 0 \text{ for all } t \geq 0. \tag{2.2}$$

The implicit function theorem shows that

$$\dot{\theta}(0) = \frac{\|\mathbf{f}(\mathbf{x})\|^2 - \delta\mathbf{v}^T D\mathbf{f}(\mathbf{x})\mathbf{f}(\mathbf{x})}{\mathbf{f}(\mathbf{x} + \delta\mathbf{v})^T\mathbf{f}(\mathbf{x})} \approx 1 - \delta\frac{\mathbf{v}^T(D\mathbf{f}(\mathbf{x})^T + D\mathbf{f}(\mathbf{x}))\mathbf{f}(\mathbf{x})}{\|\mathbf{f}(\mathbf{x})\|^2 + \delta\mathbf{v}^T D\mathbf{f}(\mathbf{x})^T\mathbf{f}(\mathbf{x})} \tag{2.3}$$

for small $\delta > 0$. Now we consider the squared distance between the trajectories with respect to the Riemannian metric

$$d(t) = \left(S_{\theta(t)}(\mathbf{x}+\delta\mathbf{v}) - S_t\mathbf{x}\right)^T M(S_t\mathbf{x}) \left(S_{\theta(t)}(\mathbf{x}+\delta\mathbf{v}) - S_t\mathbf{x}\right) \tag{2.4}$$

and take the derivative. We obtain for small $\delta > 0$, using Taylor expansion up to order δ^2

$$\begin{aligned}
\frac{d}{dt}d(t)\Big|_{t=0} &= \left(\dot\theta(0)\mathbf{f}(\mathbf{x}+\delta\mathbf{v}) - \mathbf{f}(\mathbf{x})\right)^T M(\mathbf{x})\delta\mathbf{v} + \delta^2\mathbf{v}^T M'(\mathbf{x})\mathbf{v} \\
&\quad +\delta\mathbf{v}^T M(\mathbf{x})\left(\dot\theta(0)\mathbf{f}(\mathbf{x}+\delta\mathbf{v}) - \mathbf{f}(\mathbf{x})\right) \\
&\approx \delta(\dot\theta(0)-1)[\mathbf{f}(\mathbf{x})^T M(\mathbf{x})\mathbf{v} + \mathbf{v}^T M(\mathbf{x})\mathbf{f}(\mathbf{x})] \\
&\quad +\delta^2\dot\theta(0)[(D\mathbf{f}(\mathbf{x})\mathbf{v})^T M(\mathbf{x})\mathbf{v} + \mathbf{v}^T M(\mathbf{x})D\mathbf{f}(\mathbf{x})\mathbf{v}] + \delta^2\mathbf{v}^T M'(\mathbf{x})\mathbf{v} \\
&\approx \delta^2\Bigg[-\frac{\mathbf{v}^T(D\mathbf{f}(\mathbf{x})^T + D\mathbf{f}(\mathbf{x}))\mathbf{f}(\mathbf{x})}{\|\mathbf{f}(\mathbf{x})\|^2}[\mathbf{f}(\mathbf{x})^T M(\mathbf{x})\mathbf{v} + \mathbf{v}^T M(\mathbf{x})\mathbf{f}(\mathbf{x})] \\
&\quad +(D\mathbf{f}(\mathbf{x})\mathbf{v})^T M(\mathbf{x})\mathbf{v} + \mathbf{v}^T M(\mathbf{x})D\mathbf{f}(\mathbf{x})\mathbf{v} + \mathbf{v}^T M'(\mathbf{x})\mathbf{v}\Bigg] \text{ by (2.3)} \\
&= \delta^2\mathbf{v}^T\left[V(\mathbf{x})^T M(\mathbf{x}) + M(\mathbf{x})V(\mathbf{x}) + M'(\mathbf{x})\right]\mathbf{v} = 2\delta^2 L_M(\mathbf{x};\mathbf{v}),
\end{aligned}$$

where $V(\mathbf{x})$ and $L_M(\mathbf{x};\mathbf{v})$ are defined in Theorem 2.1 below. If $L_M(\mathbf{x};\mathbf{v})$ is bounded above by a negative constant $-\nu$, then $d(t)$ is exponentially decreasing.

After providing the heuristic connection between L_M and the exponential decrease between adjacent solutions, we state Theorem 2.1. For a sketch of the proof see Appendix B.

Theorem 2.1 *Let $K \subset \mathbb{R}^n$ be a compact, connected and positively invariant set which does not contain any equilibrium. Let $M \in C^1(K, \mathbb{S}^n)$ be a Riemannian metric and let $L_M(\mathbf{x}) \le -\nu < 0$ for all $\mathbf{x} \in K$ where*

$$\begin{aligned}
L_M(\mathbf{x}) &= \max_{\mathbf{v}\in\mathbb{R}^n, \mathbf{v}^T M(\mathbf{x})\mathbf{v}=1, \mathbf{v}^T\mathbf{f}(\mathbf{x})=0} L_M(\mathbf{x};\mathbf{v}), \\
L_M(\mathbf{x};\mathbf{v}) &= \frac{1}{2}\mathbf{v}^T\left(M'(\mathbf{x}) + V(\mathbf{x})^T M(\mathbf{x}) + M(\mathbf{x})V(\mathbf{x})\right)\mathbf{v} \text{ and} \\
V(\mathbf{x}) &= D\mathbf{f}(\mathbf{x}) - \frac{\mathbf{f}(\mathbf{x})\mathbf{f}(\mathbf{x})^T(D\mathbf{f}(\mathbf{x}) + D\mathbf{f}(\mathbf{x})^T)}{\|\mathbf{f}(\mathbf{x})\|^2}.
\end{aligned} \tag{2.5}$$

Here, $(M'(\mathbf{x}))_{i,j=1,\dots,n} = (\nabla M_{ij}(\mathbf{x}))^T\mathbf{f}(\mathbf{x})$ is the matrix of the orbital derivatives of M_{ij} along solutions of (2.1).

Then there is one and only one periodic orbit $\Omega \subset K$, it is exponentially stable and the real part of all Floquet exponents apart from the trivial one is at most $-\nu$. Moreover, $K \subset A(\Omega)$.

A Riemannian metric $M \in C^1(K, \mathbb{S}^n)$ satisfying $L_M(\mathbf{x}) \le -\nu < 0$ for all $\mathbf{x} \in K$ is called a contraction metric.

Let us explain how we have formulated the contraction condition in this paper, as there are different versions of the contraction condition in the literature. In particular, the main difference concerns the synchronization of the times of the two solutions $S_t(\mathbf{x})$ and $S_\theta(\mathbf{x}+\delta\mathbf{v})$: either this is done such that the difference vector $\mathbf{v}$ is

(a) perpendicular to the flow $\mathbf{f}(\mathbf{x})$ with respect to the metric M, i.e., $\mathbf{v}^T M(\mathbf{x})\mathbf{f}(\mathbf{x}) = 0$, see e.g. [5],

(b) perpendicular to the flow $\mathbf{f}(\mathbf{x})$ with respect to the Euclidean metric, i.e., $\mathbf{v}^T\mathbf{f}(\mathbf{x}) = 0$, see this paper, or, more generally

(c) perpendicular to $\mathbf{q}(\mathbf{x})$, i.e., $\mathbf{v}^T\mathbf{q}(\mathbf{x}) = 0$, where $\mathbf{q}(\mathbf{x})$ is not perpendicular to $\mathbf{f}(\mathbf{x})$, see e.g. [12].

In each of these cases, the operator L_M will take a different form.

While the form of L_M is simpler, when considering version (a), in this paper, see Theorem 2.1, we have chosen version (b) where $\mathbf{v}$ is perpendicular to $\mathbf{f}(\mathbf{x})$, see (2.2), while the distance is measured with respect to the Riemannian metric M, i.e. $\mathbf{v}^T M(\mathbf{x})\mathbf{v}$, see (2.4). The reason is that version (b) is most suitable for computations, since in the version (a) the unknown metric M also appears in the condition for $\mathbf{v}$; note that version (b) is a special case of version (c) with $\mathbf{q}(\mathbf{x}) = \mathbf{f}(\mathbf{x})$, but it is sufficiently general. The proof of Theorem 2.1 is very similar to the corresponding statement for version (a), see [5, Theorem 5], hence, we only outline the necessary adaptations in Appendix B.

We intend to determine a matrix-valued function M as in Theorem 2.1 through a matrix-valued PDE. For $M \in C^1(\mathbb{R}^n, \mathbb{S}^n)$ and $\mathbf{x} \in \mathbb{R}^n$ with $\mathbf{f}(\mathbf{x}) \neq \mathbf{0}$ define the first-order linear differential operator

$$\begin{aligned} LM(\mathbf{x}) &:= M'(\mathbf{x}) + V(\mathbf{x})^T M(\mathbf{x}) + M(\mathbf{x})V(\mathbf{x}) \\ &= M'(\mathbf{x}) + D\mathbf{f}(\mathbf{x})^T M(\mathbf{x}) + M(\mathbf{x})D\mathbf{f}(\mathbf{x}) \\ &\quad - \frac{M(\mathbf{x})\mathbf{f}(\mathbf{x})\mathbf{f}(\mathbf{x})^T(D\mathbf{f}(\mathbf{x}) + D\mathbf{f}(\mathbf{x})^T)}{\|\mathbf{f}(\mathbf{x})\|^2} \\ &\quad - \frac{(D\mathbf{f}(\mathbf{x}) + D\mathbf{f}(\mathbf{x})^T)\mathbf{f}(\mathbf{x})\mathbf{f}(\mathbf{x})^T M(\mathbf{x})}{\|\mathbf{f}(\mathbf{x})\|^2}. \end{aligned} \tag{2.6}$$

For all $\mathbf{x} \in \mathbb{R}^n$ with $\mathbf{f}(\mathbf{x}) \neq \mathbf{0}$ we also define

$$P_{\mathbf{x}} := I - \frac{\mathbf{f}(\mathbf{x})\mathbf{f}(\mathbf{x})^T}{\|\mathbf{f}(\mathbf{x})\|^2}. \tag{2.7}$$

It is easy to see that $P_{\mathbf{x}}$ is a projection onto the hyperplane perpendicular to $\mathbf{f}(\mathbf{x})$, i.e. $P_{\mathbf{x}}\mathbf{f}(\mathbf{x}) = \mathbf{0}$ and $P_{\mathbf{x}}P_{\mathbf{x}} = P_{\mathbf{x}}$. Moreover, we have $P_{\mathbf{x}}\mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^n$ with $\mathbf{f}(\mathbf{x})^T\mathbf{v} = 0$.

In the next proposition we will show that the solution of the matrix-valued PDE (2.8) is a contraction metric. In Theorem 3.1 we will show that if M also satisfies an extra condition at one point, then we can conclude the positive definiteness of $M(\mathbf{x})$ for all $\mathbf{x} \in A(\Omega)$.

Proposition 2.2 *Let $K \subset \mathbb{R}^n$ be a compact set which does not contain any equilibrium. Let $B \in C^0(K, \mathbb{S}^n)$ and $M \in C^1(K, \mathbb{S}^n)$ be such that both $B(\mathbf{x})$ and $M(\mathbf{x})$ are positive definite for each $\mathbf{x} \in K$. Let M satisfy*

$$LM(\mathbf{x}) = -P_{\mathbf{x}}^T B(\mathbf{x}) P_{\mathbf{x}} \tag{2.8}$$

for all $\mathbf{x} \in K$.

Then there are constants $\Lambda, \lambda > 0$ such that $\mathbf{v}^T B(\mathbf{x})\mathbf{v} \geq \lambda \|\mathbf{v}\|^2$ and $\mathbf{v}^T M(\mathbf{x})\mathbf{v} \leq \Lambda \|\mathbf{v}\|^2$ hold for all $\mathbf{x} \in K$ and all $\mathbf{v} \in \mathbb{R}^n$. Moreover,

$$L_M(\mathbf{x}) \leq -\frac{\lambda}{2\Lambda} =: -\nu < 0.$$

Proof: The existence of $\lambda, \Lambda > 0$ follows from the fact that B and M are positive definite and continuous on the compact set K. We have

$$\begin{aligned}
2L_M(\mathbf{x}) &= \max_{\mathbf{v} \in \mathbb{R}^n, \mathbf{v}^T M(\mathbf{x})\mathbf{v}=1, \mathbf{v}^T \mathbf{f}(\mathbf{x})=0} \mathbf{v}^T LM(\mathbf{x})\mathbf{v} \\
&= -\min_{\mathbf{v} \in \mathbb{R}^n, \mathbf{v}^T M(\mathbf{x})\mathbf{v}=1, \mathbf{v}^T \mathbf{f}(\mathbf{x})=0} \mathbf{v}^T P_{\mathbf{x}}^T B(\mathbf{x}) P_{\mathbf{x}} \mathbf{v} \\
&= -\min_{\mathbf{v} \in \mathbb{R}^n, \mathbf{v}^T M(\mathbf{x})\mathbf{v}=1, \mathbf{v}^T \mathbf{f}(\mathbf{x})=0} \mathbf{v}^T B(\mathbf{x})\mathbf{v} \\
&\leq -\lambda \min_{\mathbf{v} \in \mathbb{R}^n, \mathbf{v}^T M(\mathbf{x})\mathbf{v}=1, \mathbf{v}^T \mathbf{f}(\mathbf{x})=0} \|\mathbf{v}\|^2 \\
&\leq -\frac{\lambda}{\Lambda}.
\end{aligned}$$

□

3 Existence

Given an exponentially stable periodic orbit, we will now show the existence and uniqueness of the solution of (3.1) in its basin of attraction. We need to fix one value in (3.2) to guarantee that M is positive definite and to obtain uniqueness in Section 4.

Let us explain the ideas and compare them to the situation of an equilibrium: the positive definite matrix $B(\mathbf{x})$ denotes the contraction at each point and will later often be chosen as $B(\mathbf{x}) = I$. However, in contrast to the case of an equilibrium, we apply the projection $P_{\mathbf{x}}$ onto the hyperplane perpendicular to $\mathbf{f}(\mathbf{x})$, and consider $C(\mathbf{x}) = P_{\mathbf{x}}^T B(\mathbf{x}) P_{\mathbf{x}}$. The equation $LM(\mathbf{x}) = -C(\mathbf{x})$ thus guarantees contraction in all directions perpendicular to $\mathbf{f}(\mathbf{x})$. Finally, we consider vectors $\mathbf{v}$, which are collinear to $\mathbf{f}(\mathbf{x})$. Here, no condition regarding the contraction needs to be satisfied. However, to ensure that $M(\mathbf{x})$ is positive definite, we require that $\mathbf{f}(\mathbf{x})^T M(\mathbf{x})\mathbf{f}(\mathbf{x})$ is positive. This will be achieved by fixing this quantity as a positive value at an arbitrary point $\mathbf{x}_0$, see (3.2), and $\mathbf{f}(\mathbf{x})^T M(\mathbf{x})\mathbf{f}(\mathbf{x}) > 0$ will then hold for all $\mathbf{x} \in A(\Omega)$ due to (3.1). This can also be seen in the formula for $M(\mathbf{x})$: the first term, later called $M_1(\mathbf{x})$, is similar to the case of an equilibrium, but results in $\mathbf{f}(\mathbf{x})^T M_1(\mathbf{x})\mathbf{f}(\mathbf{x}) = 0$; the second term will guarantee that $\mathbf{f}(\mathbf{x})^T M(\mathbf{x})\mathbf{f}(\mathbf{x}) = c_0 \|f(\mathbf{x})\|^4 > 0$.

Theorem 3.1 *Let Ω be an exponentially stable periodic orbit of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{f} \in C^\sigma(\mathbb{R}^n, \mathbb{R}^n)$, where $\sigma \geq 2$, with basin of attraction $A(\Omega)$. Fix $\mathbf{x}_0 \in A(\Omega)$ and $c_0 \in \mathbb{R}^+$. Let $B \in C^{\sigma-1}(A(\Omega), \mathbb{S}^n)$ be such that $B(\mathbf{x})$ is positive definite for all $\mathbf{x} \in A(\Omega)$ and define $C \in C^{\sigma-1}(A(\Omega), \mathbb{S}^n)$ by (see (2.7))*

$$C(\mathbf{x}) = P_{\mathbf{x}}^T B(\mathbf{x}) P_{\mathbf{x}}.$$

Then there exists a solution $M \in C^{\sigma-1}(A(\Omega), \mathbb{S}^n)$ of the linear matrix-valued PDE (see (2.6))

$$\begin{aligned} LM(\mathbf{x}) &= -C(\mathbf{x}) \text{ for all } \mathbf{x} \in A(\Omega) & (3.1)\\ \text{satisfying } \mathbf{f}(\mathbf{x}_0)^T M(\mathbf{x}_0)\mathbf{f}(\mathbf{x}_0) &= c_0\|\mathbf{f}(\mathbf{x}_0)\|^4. & (3.2)\end{aligned}$$

The solution $M(\mathbf{x})$ is positive definite for all $\mathbf{x} \in A(\Omega)$ and it is of the form

$$M(\mathbf{x}) = \int_0^\infty \Phi(t, 0; \mathbf{x})^T C(S_t\mathbf{x})\Phi(t, 0; \mathbf{x})\, dt + c_0\mathbf{f}(\mathbf{x})\mathbf{f}(\mathbf{x})^T,$$

where $\Phi(t, 0; \mathbf{x})$ denotes the principal fundamental matrix solution of $\dot{\boldsymbol{\phi}}(t) = D\mathbf{f}(S_t\mathbf{x})\boldsymbol{\phi}(t)$ with $\Phi(0, 0; \mathbf{x}) = I$.

Proof: Denote

$$M_1(\mathbf{x}) = \int_0^\infty \Phi(t, 0; \mathbf{x})^T C(S_t\mathbf{x})\Phi(t, 0; \mathbf{x})\, dt \qquad (3.3)$$

and $M_2(\mathbf{x}) = \mathbf{f}(\mathbf{x})\mathbf{f}(\mathbf{x})^T$, so that $M(\mathbf{x}) = M_1(\mathbf{x}) + c_0 M_2(\mathbf{x})$. It is clear that $M_2 \in C^\sigma(A(\Omega), \mathbb{S}^n)$.

We will first show (3.2) in Step 1. In Step 2 we will show $LM_2(\mathbf{x}) = 0$. In Step 3 we will prove estimates on $P_{S_t\mathbf{x}}\Phi(t, 0; \mathbf{x})$ which will then enable us to show $LM_1(\mathbf{x}) = -C(\mathbf{x})$ in Step 4, proving (3.1). In Step 5 we will show that M_1 is well defined and $C^{\sigma-1}$. Finally, in Step 6, we show that M is positive definite.

Step 1: M satisfies (3.2)

To show (3.2), note that $\mathbf{f}(S_t\mathbf{x})$ solves $\dot{\boldsymbol{\phi}}(t) = D\mathbf{f}(S_t\mathbf{x})\boldsymbol{\phi}(t)$. Hence, $\Phi(t, 0; \mathbf{x})\mathbf{f}(\mathbf{x}) = \mathbf{f}(S_t\mathbf{x})$. This shows that for all $\mathbf{x} \in A(\Omega)$

$$\begin{aligned} \mathbf{f}(\mathbf{x})^T M_1(\mathbf{x})\mathbf{f}(\mathbf{x}) &= \int_0^\infty \mathbf{f}(S_t\mathbf{x})^T C(S_t\mathbf{x})\mathbf{f}(S_t\mathbf{x})\, dt\\ &= \int_0^\infty \mathbf{f}(S_t\mathbf{x})^T P_{S_t\mathbf{x}}^T B(S_t\mathbf{x}) P_{S_t\mathbf{x}}\mathbf{f}(S_t\mathbf{x})\, dt \;=\; 0\end{aligned}$$

since $P_{S_t\mathbf{x}}\mathbf{f}(S_t\mathbf{x}) = \mathbf{0}$. On the other hand we have

$$\mathbf{f}(\mathbf{x})^T M_2(\mathbf{x})\mathbf{f}(\mathbf{x}) = \|\mathbf{f}(\mathbf{x})\|^4.$$

This shows (3.2).

Step 2: $LM_2(\mathbf{x}) = 0$

We have, using $(\mathbf{f}(\mathbf{x}))' = D\mathbf{f}(\mathbf{x})\mathbf{f}(\mathbf{x})$,

$$\begin{aligned} LM_2(\mathbf{x}) &= D\mathbf{f}(\mathbf{x})\mathbf{f}(\mathbf{x})\mathbf{f}(\mathbf{x})^T + \mathbf{f}(\mathbf{x})\mathbf{f}(\mathbf{x})^T D\mathbf{f}(\mathbf{x})^T \\ &\quad + D\mathbf{f}(\mathbf{x})^T\mathbf{f}(\mathbf{x})\mathbf{f}(\mathbf{x})^T + \mathbf{f}(\mathbf{x})\mathbf{f}(\mathbf{x})^T D\mathbf{f}(\mathbf{x}) \\ &\quad - \frac{\mathbf{f}(\mathbf{x})\mathbf{f}(\mathbf{x})^T\mathbf{f}(\mathbf{x})\mathbf{f}(\mathbf{x})^T(D\mathbf{f}(\mathbf{x}) + D\mathbf{f}(\mathbf{x})^T)}{\|\mathbf{f}(\mathbf{x})\|^2} \\ &\quad - \frac{(D\mathbf{f}(\mathbf{x}) + D\mathbf{f}(\mathbf{x})^T)\mathbf{f}(\mathbf{x})\mathbf{f}(\mathbf{x})^T\mathbf{f}(\mathbf{x})\mathbf{f}(\mathbf{x})^T}{\|\mathbf{f}(\mathbf{x})\|^2} \\ &= 0. \end{aligned}$$

Step 3: $P_{S_t\mathbf{x}}\Phi(t, 0; \mathbf{x})$ **decreases exponentially**

To proceed with the proof, we will now show that $P_{S_t\mathbf{x}}\Phi(t, 0; \mathbf{x})$ decreases exponentially. This is done in several sub-steps. First we give an estimate for points $\mathbf{x} = \mathbf{p} \in \Omega$ on the periodic orbit in Lemma 3.2. Then we focus on points in a neighborhood U of the periodic orbit in Lemma 3.4; this will imply the estimate for all points $\mathbf{x} \in A(\Omega)$ in Step 4, see Lemma 3.5. The matrix norm in the following lemma and the rest of the paper is $\| \cdot \| = \| \cdot \|_2$, which is induced by the vector norm and sub-multiplicative.

Lemma 3.2 *Let $-\nu$ be the largest real part of the non-trivial Floquet exponents of the periodic orbit Ω and let $\varepsilon > 0$.*

Then there is a constant $c_1 > 0$ such that for all $\mathbf{p} \in \Omega$ and all $0 \leq s \leq t$ we have

$$\|P_{S_t\mathbf{p}}\Phi(t, 0; \mathbf{p})\Phi(s, 0; \mathbf{p})^{-1}\| \leq c_1 e^{(-\nu+\varepsilon)(t-s)} \quad (3.4)$$

$$\|\Phi(t, 0; \mathbf{p})\Phi(s, 0; \mathbf{p})^{-1}\| \leq c_1 \quad (3.5)$$

where $\Phi(t, 0; \mathbf{p})$ is the principal fundamental matrix solution of the first variational equation

$$\dot{\boldsymbol{\phi}}(t) = D\mathbf{f}(S_t\mathbf{p})\boldsymbol{\phi}(t) \quad (3.6)$$

with $\Phi(0, 0; \mathbf{p}) = I$.

Proof: Note that it is sufficient to prove the result for a fixed point $\mathbf{p} \in \Omega$. Indeed, let $\mathbf{q} = S_\theta\mathbf{p}$ with $\theta > 0$ be a different point on the periodic orbit. First note that we have

$$\Phi(t, 0; S_\theta\mathbf{p}) = \Phi(t + \theta, \theta; \mathbf{p})$$

for all $t \geq 0$ since both sides of the equation satisfy the same initial value problem. Hence,

$$\begin{aligned} \Phi(t, 0; S_\theta\mathbf{p}) &= \Phi(t + \theta, \theta; \mathbf{p}) \\ &= \Phi(t + \theta, 0; \mathbf{p})\Phi(\theta, 0; \mathbf{p})^{-1} \\ \Phi(t, 0; \mathbf{q})\Phi(s, 0; \mathbf{q})^{-1} &= \Phi(t + \theta, 0; \mathbf{p})\Phi(s + \theta, 0; \mathbf{p})^{-1} \end{aligned}$$

and the result for $\mathbf{q}$ follows from the result for $\mathbf{p}$.

Equation (3.6) is a T-periodic, linear equation for $\boldsymbol{\phi}$, where T is the period of the periodic orbit Ω, and $D\mathbf{f}$ is $C^{\sigma-1}$. By Floquet theory, the principal fundamental matrix solution $\Phi(t,0;\mathbf{p})$ of (3.6) with $\Phi(0,0;\mathbf{p}) = I$ can be written as

$$\Phi(t,0;\mathbf{p}) = Q(t)e^{Bt},$$

where $Q(\cdot) \in C^{\sigma-1}(\mathbb{R}, \mathbb{C}^{n\times n})$ is T-periodic with $Q(0) = Q(T) = I$, and $B \in \mathbb{C}^{n\times n}$. The eigenvalues of B are 0 with algebraic multiplicity one and the others have a real part of at most $-\nu < 0$. Let $S \in \mathbb{C}^{n\times n}$ be an invertible matrix such that $S^{-1}BS = A$ is in a special Jordan Normal Form, where the complex eigenvalues are on the diagonal and the 1 on the side diagonal is replaced by ε, and the first eigenvalue is 0.

Let $\mathbf{e}_1, \ldots, \mathbf{e}_n \in \mathbb{R}^n$ denote the standard basis of $\mathbb{R}^n$. We have

$$\begin{aligned} \|e^{At}\mathbf{x}\| &\leq e^{(-\nu+\varepsilon)t}\|\mathbf{x}\| \qquad \text{for all } \mathbf{x} \in \text{span}(\mathbf{e}_2, \ldots, \mathbf{e}_n) & (3.7) \\ \text{and } e^{At}\mathbf{e}_1 &= \mathbf{e}_1 \text{ for all } t \geq 0. & (3.8) \end{aligned}$$

Now we show that $\mathbf{f}(S_t\mathbf{p}) = \lambda Q(t)S\mathbf{e}_1$ holds for all $t \in \mathbb{R}$ with $\lambda \in \mathbb{C} \setminus \{0\}$. Indeed, since $\mathbf{f}(S_t\mathbf{p})$ solves (3.6), we have for all $s \in \mathbb{R}$

$$\begin{aligned} \mathbf{f}(S_t\mathbf{p}) &= \Phi(t,0;\mathbf{p})\Phi(s,0;\mathbf{p})^{-1}\mathbf{f}(S_s\mathbf{p}) \\ &= Q(t)e^{B(t-s)}Q(s)^{-1}\mathbf{f}(S_s\mathbf{p}) \\ &= Q(t)Se^{A(t-s)}S^{-1}Q(s)^{-1}\mathbf{f}(S_s\mathbf{p}). \qquad (3.9) \end{aligned}$$

For $t = s + T$ we have $\mathbf{f}(S_s\mathbf{p}) = \mathbf{f}(S_t\mathbf{p})$ and $Q(s) = Q(t)$ by the periodicity and thus

$$S^{-1}Q(t)^{-1}\mathbf{f}(S_t\mathbf{p}) = e^{AT}S^{-1}Q(t)^{-1}\mathbf{f}(S_t\mathbf{p}).$$

The form of A implies that $S^{-1}Q(t)^{-1}\mathbf{f}(S_t\mathbf{p}) = \lambda\mathbf{e}_1$ with $\lambda \neq 0$, and thus

$$\mathbf{f}(S_t\mathbf{p}) = \lambda Q(t)S\mathbf{e}_1 \qquad (3.10)$$

holds for all $t \in \mathbb{R}$.

We have for $t \geq s \geq 0$

$$\begin{aligned} \Phi(t,0;\mathbf{p})\Phi(s,0;\mathbf{p})^{-1} &= Q(t)e^{B(t-s)}Q(s)^{-1} \\ \|\Phi(t,0;\mathbf{p})\Phi(s,0;\mathbf{p})^{-1}\| &\leq \|Q(t)\|\|Q(s)^{-1}\|\|S\|\|S^{-1}\|\|e^{A(t-s)}\| \\ &\leq \max_{t'\in[0,T]}\|Q(t')\| \max_{s'\in[0,T]}\|Q(s')^{-1}\|\|S\|\|S^{-1}\| \end{aligned}$$

since Q is T-periodic. This shows (3.5).

Fix $s \geq 0$ and $\mathbf{c} \in \mathbb{R}^n$. Let us write

$$\begin{aligned} \Phi(t,0;\mathbf{p})\Phi(s,0;\mathbf{p})^{-1}\mathbf{c} &= Q(t)Se^{A(t-s)}S^{-1}Q(s)^{-1}\mathbf{c} \\ &= \sum_{i=1}^{n}\beta_i Q(t)Se^{A(t-s)}\mathbf{e}_i, \end{aligned}$$

where we have defined the $\beta_i \in \mathbb{C}$ by $\sum_{i=1}^n \beta_i \mathbf{e}_i = S^{-1}Q(s)^{-1}\mathbf{c}$. Note that $\sum_{i=1}^n |\beta_i|^2 = \|S^{-1}Q(s)^{-1}\mathbf{c}\|^2$. Using (2.7) and (3.8), we have

$$\begin{aligned}
P_{S_t\mathbf{p}}\Phi(t,0;\mathbf{p})\Phi(s,0;\mathbf{p})^{-1}\mathbf{c} &= \sum_{i=2}^n \beta_i Q(t)Se^{A(t-s)}\mathbf{e}_i + \beta_1 Q(t)S\mathbf{e}_1 \\
&\quad -\sum_{i=2}^n \beta_i \frac{\mathbf{f}(S_t\mathbf{p})^T Q(t)Se^{A(t-s)}\mathbf{e}_i}{\|\mathbf{f}(S_t\mathbf{p})\|^2}\mathbf{f}(S_t\mathbf{p}) \\
&\quad -\beta_1 \frac{\mathbf{f}(S_t\mathbf{p})^T Q(t)S\mathbf{e}_1}{\|\mathbf{f}(S_t\mathbf{p})\|^2}\mathbf{f}(S_t\mathbf{p}) \\
&= \sum_{i=2}^n \beta_i Q(t)Se^{A(t-s)}\mathbf{e}_i \\
&\quad -\sum_{i=2}^n \beta_i \frac{\mathbf{f}(S_t\mathbf{p})^T Q(t)Se^{A(t-s)}\mathbf{e}_i}{\|\mathbf{f}(S_t\mathbf{p})\|^2}\mathbf{f}(S_t\mathbf{p}). \qquad (3.11)
\end{aligned}$$

The two terms with β_1 cancel each other out since by (3.10)

$$\frac{\mathbf{f}(S_t\mathbf{p})^T Q(t)S\mathbf{e}_1}{\|\mathbf{f}(S_t\mathbf{p})\|^2}\mathbf{f}(S_t\mathbf{p}) = \frac{\mathbf{f}(S_t\mathbf{p})^T \lambda Q(t)S\mathbf{e}_1}{\|\mathbf{f}(S_t\mathbf{p})\|^2}Q(t)S\mathbf{e}_1 = Q(t)S\mathbf{e}_1.$$

In particular, at $t = s$, we have with (3.10)

$$\begin{aligned}
P_{S_t\mathbf{p}}\mathbf{c} &= \sum_{i=2}^n \beta_i Q(t)S\mathbf{e}_i - \sum_{j=2}^n \beta_j \frac{\mathbf{f}(S_t\mathbf{p})^T Q(t)S\mathbf{e}_j}{\|\mathbf{f}(S_t\mathbf{p})\|^2}\mathbf{f}(S_t\mathbf{p}) \\
&= Q(t)S\left[\sum_{i=2}^n \beta_i\mathbf{e}_i - \lambda\frac{\mathbf{f}(S_t\mathbf{p})^T\left(\sum_{j=2}^n \beta_j Q(t)S\mathbf{e}_j\right)}{\|\mathbf{f}(S_t\mathbf{p})\|^2}\mathbf{e}_1\right] \\
\|S^{-1}Q(t)^{-1}P_{S_t\mathbf{p}}\mathbf{c}\|^2 &\geq \sum_{i=2}^n |\beta_i|^2. \qquad (3.12)
\end{aligned}$$

We have from (3.11) for $0 \leq s \leq t$

$$\begin{aligned}
&\|P_{S_t\mathbf{p}}\Phi(t,0;\mathbf{p})\Phi(s,0;\mathbf{p})^{-1}\mathbf{c}\| \\
&\leq \|Q(t)\|\ \|S\| \left\|\sum_{i=2}^n \beta_i\mathbf{e}_i\right\| e^{(-\nu+\varepsilon)(t-s)} \\
&\quad + \frac{\|\mathbf{f}(S_t\mathbf{p})\|^2}{\|\mathbf{f}(S_t\mathbf{p})\|^2}\|Q(t)\|\ \|S\| \left\|\sum_{j=2}^n \beta_j\mathbf{e}_j\right\| e^{(-\nu+\varepsilon)(t-s)} \\
&\leq 2\,\|Q(t)\|\ \|S\| \sqrt{\sum_{i=2}^n |\beta_i|^2}\, e^{(-\nu+\varepsilon)(t-s)} \\
&\leq 2 \max_{t'\in[0,T]} \|Q(t')\|\,\|S\| \max_{s'\in[0,T]} \|Q(s')^{-1}\|\,\|S^{-1}\|\,\|\mathbf{c}\|\, e^{(-\nu+\varepsilon)(t-s)}
\end{aligned}$$

by (3.12). This shows (3.4) and the lemma. □

We use the following result from [8, Corollary 2]. In a neighborhood U of the periodic orbit we define a projection of a point $\mathbf{x} \in U$ onto a point $\pi(\mathbf{x}) \in \Omega$ on the periodic orbit. We can synchronize the times of trajectories through $\mathbf{x}$ (time t) and $\pi(\mathbf{x}) = \mathbf{p}$ (time θ) such that $\pi(S_t\mathbf{x}) = S_{\theta_{\mathbf{x}}(t)}\mathbf{p}$. Moreover, we define a distance of $S_t\mathbf{x}$ to the periodic orbit, in particular to $\pi(S_t\mathbf{x})$, which exponentially decreases along solutions. This notion of stability is also called Zhukovskii stability and its relation to Lyapunov stability has been studied, e.g. in [13].

Lemma 3.3 *Let Ω be an exponentially stable periodic orbit of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with $\mathbf{f} \in C^\sigma(\mathbb{R}^n, \mathbb{R}^n)$ and $\sigma \geq 2$ and denote by $-\nu < 0$ the maximal real part of all its non-trivial Floquet exponents.*

For $\varepsilon \in (0, \min(\nu, 1))$ there is a compact, positively invariant neighborhood U of Ω with $\Omega \subset U^\circ$ and $U \subset A(\Omega)$ and a map $\pi \in C^{\sigma-1}(U, \Omega)$ with $\pi(\mathbf{x}) = \mathbf{x}$ if and only if $\mathbf{x} \in \Omega$. Furthermore, for a fixed $\mathbf{x} \in U$ there is a bijective $C^{\sigma-1}$ map $\theta_{\mathbf{x}}\colon [0,\infty) \to [0,\infty)$ with inverse $t_{\mathbf{x}} = \theta_{\mathbf{x}}^{-1} \in C^{\sigma-1}([0,\infty), [0,\infty))$ such that $\theta_{\mathbf{x}}(0) = 0$ and

$$\pi(S_t\mathbf{x}) = S_{\theta_{\mathbf{x}}(t)}\pi(\mathbf{x})$$

for all $t \in [0,\infty)$. Moreover, $\dot{\theta}_{\mathbf{x}}(t) \in [1-\varepsilon, 1+\varepsilon]$ for all $t \geq 0$ and $\dot{t}_{\mathbf{x}}(\theta) \in [1-\varepsilon, 1+\varepsilon]$ for all $\theta \geq 0$.

Finally, there is a constant $C > 0$ such that

$$|\dot{t}_{\mathbf{x}}(\theta) - 1| \leq Ce^{-\mu_0\theta} \tag{3.13}$$

$$\|S_{t_{\mathbf{x}}(\theta)}\mathbf{x} - S_\theta\pi(\mathbf{x})\| \leq Ce^{-\mu_0\theta}\|\mathbf{x} - \pi(\mathbf{x})\| \tag{3.14}$$

for all $\theta \geq 0$ and all $\mathbf{x} \in U$, where $\mu_0 = \nu - \varepsilon$.

Using Lemma 3.3, we will now show Lemma 3.4.

Lemma 3.4 *Using the notation of Lemma 3.3 with $0 < \varepsilon < \min(1, \nu/2)$, there are constants $C > 0$ and $\kappa = \frac{\nu - 2\varepsilon}{1+\varepsilon} > 0$ such that for all $\mathbf{x} \in U$ we have*

$$\|P_{S_t\mathbf{x}}\boldsymbol{\phi}(t)\| \leq Ce^{-\kappa t}\|P_{\mathbf{x}}\boldsymbol{\phi}(0)\| \tag{3.15}$$

$$\|\boldsymbol{\phi}(t)\| \leq C\|\boldsymbol{\phi}(0)\| \tag{3.16}$$

for all $t \geq 0$. Here, $\boldsymbol{\phi}(t)$ is a solution of the first variational equation

$$\dot{\boldsymbol{\phi}}(t) = D\mathbf{f}(S_t\mathbf{x})\boldsymbol{\phi}(t). \tag{3.17}$$

Proof: Denote $\mu_0 = \nu - \varepsilon > 0$, let $\mathbf{x} \in U$ and denote the synchronized time by $\theta_{\mathbf{x}}(t) = \theta(t)$, see Lemma 3.3. We now drop the subscript.

Let $A(\theta) := D\mathbf{f}(S_\theta \mathbf{p})$ with $\mathbf{p} = \pi(\mathbf{x}) \in \Omega$, where π was defined in Lemma 3.3. Using the inverse of $\theta(t)$, namely $t = \theta^{-1}$, we define $D(\theta) := D\mathbf{f}(S_{t(\theta)}\mathbf{x})$ and $\boldsymbol{\psi}(\theta) := \boldsymbol{\phi}(t(\theta))$. Then we have by (3.17)

$$\frac{d}{d\theta}\boldsymbol{\psi}(\theta) = \frac{d}{dt}\boldsymbol{\phi}(t(\theta)) \cdot \dot{t}(\theta) = D(\theta)\boldsymbol{\psi}(\theta)\dot{t}(\theta). \tag{3.18}$$

Since $A(\theta) = D\mathbf{f}(S_\theta \mathbf{p})$ is T-periodic, we can use Floquet Theory to express the principal fundamental matrix solution $\Phi(\theta, 0; \mathbf{p})$ of $\dot{\mathbf{y}}(\theta) = A(\theta)\mathbf{y}(\theta)$ as in Lemma 3.2. In the following we will abbreviate it by $\Phi(\theta)$, where $\Phi(0) = I$.

As $\Phi(\theta)$ exists and is non-singular for all $\theta \in \mathbb{R}_0^+$, we have

$$\begin{aligned}
0 &= \frac{d}{d\theta}\left(\Phi(\theta)\Phi(\theta)^{-1}\right)\\
&= \left(\frac{d}{d\theta}\Phi(\theta)\right)\Phi(\theta)^{-1} + \Phi(\theta)\left(\frac{d}{d\theta}\Phi(\theta)^{-1}\right)\\
\frac{d}{d\theta}\Phi(\theta)^{-1} &= -\Phi(\theta)^{-1}\left(\frac{d}{d\theta}\Phi(\theta)\right)\Phi(\theta)^{-1}\\
&= -\Phi(\theta)^{-1}A(\theta).
\end{aligned} \tag{3.19}$$

Using (3.18) and (3.19) we have

$$\begin{aligned}
\frac{d}{d\theta}\left(\Phi(\theta)^{-1}\boldsymbol{\psi}(\theta)\right) &= -\Phi(\theta)^{-1}A(\theta)\boldsymbol{\psi}(\theta) + \Phi(\theta)^{-1}D(\theta)\boldsymbol{\psi}(\theta)\dot{t}(\theta)\\
&= \Phi(\theta)^{-1}\left(D(\theta) - A(\theta) + D(\theta)(\dot{t}(\theta) - 1)\right)\boldsymbol{\psi}(\theta).
\end{aligned}$$

Integrating both sides from 0 to $\theta \geq 0$ we obtain

$$\begin{aligned}
&\Phi(\theta)^{-1}\boldsymbol{\psi}(\theta) - \boldsymbol{\psi}(0)\\
&\quad= \int_0^\theta \Phi(s)^{-1}\left(D(s) - A(s) + D(s)(\dot{t}(s) - 1)\right)\boldsymbol{\psi}(s)\, ds\\
\boldsymbol{\psi}(\theta) &= \Phi(\theta)\boldsymbol{\psi}(0)\\
&\quad + \int_0^\theta \Phi(\theta)\Phi(s)^{-1}\left(D(s) - A(s) + D(s)(\dot{t}(s) - 1)\right)\boldsymbol{\psi}(s)\, ds.
\end{aligned} \tag{3.20}$$

Since $D\mathbf{f}$ is C^1 on the compact set U, there is a Lipschitz constant $L > 0$ such that

$$\begin{aligned}
\|D(s) - A(s)\| &= \|D\mathbf{f}(S_{t(s)}\mathbf{x}) - D\mathbf{f}(S_s\mathbf{p})\|\\
&\leq L\|S_{t(s)}\mathbf{x} - S_s\mathbf{p}\|\\
&\leq LCe^{-\mu_0 s}\|\mathbf{x} - \mathbf{p}\|
\end{aligned}$$

by (3.14). Hence, altogether we have with (3.13) and using that $D(s) = D\mathbf{f}(S_{t(s)}\mathbf{x})$ is bounded for all $s \in [0, \infty)$ and $\mathbf{x} \in U$ that there is a constant $d_1 > 0$ such that

$$\left\|D(s) - A(s) + D(s)(\dot{t}(s) - 1)\right\| \leq d_1 e^{-\mu_0 s}. \tag{3.21}$$

Estimate on $\|\boldsymbol{\psi}(\theta)\|$

From (3.20) we obtain

$$\begin{aligned} \|\boldsymbol{\psi}(\theta)\| &\leq \|\Phi(\theta)\|\|\boldsymbol{\psi}(0)\| + \int_0^\theta \|\Phi(\theta)\Phi(s)^{-1}\| \cdot \\ &\quad \left\|D(s) - A(s) + D(s)(\dot{t}(s) - 1)\right\| \cdot \|\boldsymbol{\psi}(s)\|\, ds \end{aligned} \tag{3.22}$$

for all $\theta \geq 0$. We have

$$\|\Phi(\theta)\| \leq c_1 \text{ for } \theta \geq 0 \tag{3.23}$$

$$\|\Phi(\theta)\Phi(s)^{-1}\| \leq c_1 \text{ for } \theta - s \geq 0, \tag{3.24}$$

see Lemma 3.2.

Using these estimates in (3.22), as well as (3.21) gives

$$\|\boldsymbol{\psi}(\theta)\| \leq c_1\|\boldsymbol{\psi}(0)\| + \int_0^\theta c_1 d_1 e^{-\mu_0 s}\|\boldsymbol{\psi}(s)\|\, ds.$$

Now we apply Lemma A.1 with $r(\theta) = \|\boldsymbol{\psi}(\theta)\|$, $a(\theta) = c_1\|\boldsymbol{\psi}(0)\|$, $K(\theta) = d_1c_1$ and $b(\theta) = e^{-\mu_0\theta}$, giving

$$\begin{aligned} \|\boldsymbol{\psi}(\theta)\| &\leq c_1\|\boldsymbol{\psi}(0)\| + d_1c_1^2\|\boldsymbol{\psi}(0)\| \int_0^\theta e^{-\mu_0 s}\, ds \cdot \exp\left(\int_0^\theta d_1c_1e^{-\mu_0 s}\, ds\right) \\ &\leq c_1\|\boldsymbol{\psi}(0)\| + \frac{d_1c_1^2}{\mu_0}\|\boldsymbol{\psi}(0)\| \cdot \exp\left(\frac{d_1c_1}{\mu_0}\right) \end{aligned}$$

using $\int_0^\theta e^{-\mu_0 s}\, ds = \frac{1}{\mu_0}(1 - e^{-\mu_0\theta}) \leq \frac{1}{\mu_0}$. Note that this holds for all $\theta \geq 0$ since $\boldsymbol{\psi}(\theta)$ is continuous. Using $\boldsymbol{\phi}(t(\theta)) = \boldsymbol{\psi}(\theta)$ and that $t(\theta)$ is bijective this shows (3.16).

Estimate on $\|P_{S_{t(\theta)}\mathbf{x}}\boldsymbol{\psi}(\theta)\|$

Note that by (2.7) we have

$$\begin{aligned} \frac{d}{dt}P_{S_t\mathbf{x}} &= -\frac{D\mathbf{f}(S_t\mathbf{x})\mathbf{f}(S_t\mathbf{x})\mathbf{f}(S_t\mathbf{x})^T + \mathbf{f}(S_t\mathbf{x})\mathbf{f}(S_t\mathbf{x})^T D\mathbf{f}(S_t\mathbf{x})^T}{\|\mathbf{f}(S_t\mathbf{x})\|^2} \\ &\quad + \frac{\mathbf{f}(S_t\mathbf{x})\mathbf{f}(S_t\mathbf{x})^T}{\|\mathbf{f}(S_t\mathbf{x})\|^4}\mathbf{f}(S_t\mathbf{x})^T(D\mathbf{f}(S_t\mathbf{x})^T + D\mathbf{f}(S_t\mathbf{x}))\mathbf{f}(S_t\mathbf{x}). \end{aligned}$$

Hence, using (3.18) we have

$$
\begin{aligned}
&\frac{d}{d\theta}(P_{S_{t(\theta)}\mathbf{x}}\boldsymbol{\psi}(\theta))\\
&= \dot{t}(\theta)\Bigg(-\frac{D\mathbf{f}(S_{t(\theta)}\mathbf{x})\mathbf{f}(S_{t(\theta)}\mathbf{x})\mathbf{f}(S_{t(\theta)}\mathbf{x})^T + \mathbf{f}(S_{t(\theta)}\mathbf{x})\mathbf{f}(S_{t(\theta)}\mathbf{x})^T D\mathbf{f}(S_{t(\theta)}\mathbf{x})^T}{\|\mathbf{f}(S_{t(\theta)}\mathbf{x})\|^2}\\
&\quad + \frac{\mathbf{f}(S_{t(\theta)}\mathbf{x})\mathbf{f}(S_{t(\theta)}\mathbf{x})^T}{\|\mathbf{f}(S_{t(\theta)}\mathbf{x})\|^4}\mathbf{f}(S_{t(\theta)}\mathbf{x})^T(D\mathbf{f}(S_{t(\theta)}\mathbf{x})^T + D\mathbf{f}(S_{t(\theta)}\mathbf{x}))\mathbf{f}(S_{t(\theta)}\mathbf{x})\\
&\quad + D\mathbf{f}(S_{t(\theta)}\mathbf{x}) - \frac{\mathbf{f}(S_{t(\theta)}\mathbf{x})\mathbf{f}(S_{t(\theta)}\mathbf{x})^T}{\|\mathbf{f}(S_{t(\theta)}\mathbf{x})\|^2}D\mathbf{f}(S_{t(\theta)}\mathbf{x})\Bigg)\boldsymbol{\psi}(\theta)\\
&= \dot{t}(\theta)\left(D\mathbf{f}(S_{t(\theta)}\mathbf{x}) - \mathbf{f}(S_{t(\theta)}\mathbf{x})\mathbf{r}(\theta)^T\right) P_{S_{t(\theta)}\mathbf{x}}\boldsymbol{\psi}(\theta) \qquad (3.25)
\end{aligned}
$$

where

$$
\mathbf{r}(\theta)^T = \frac{\mathbf{f}(S_{t(\theta)}\mathbf{x})^T(D\mathbf{f}(S_{t(\theta)}\mathbf{x})^T + D\mathbf{f}(S_{t(\theta)}\mathbf{x}))}{\|\mathbf{f}(S_{t(\theta)}\mathbf{x})\|^2}.
$$

Note that there is a constant $R > 0$ such that for all $\mathbf{x} \in U$ and all $\theta \geq 0$

$$
\|\mathbf{r}(\theta)\| \leq R \qquad (3.26)
$$

as $\mathbf{f} \in C^1$ in the compact, positively invariant set U. Using (3.19) we have

$$
\begin{aligned}
\frac{d}{d\theta}\left(\Phi(\theta)^{-1}P_{S_{t(\theta)}\mathbf{x}}\boldsymbol{\psi}(\theta)\right) &= \Phi(\theta)^{-1}\left(D(\theta) - A(\theta) + D(\theta)(\dot{t}(\theta) - 1)\right) P_{S_{t(\theta)}\mathbf{x}}\boldsymbol{\psi}(\theta)\\
&\quad - \Phi(\theta)^{-1}\mathbf{f}(S_{t(\theta)}\mathbf{x})\mathbf{r}(\theta)^T P_{S_{t(\theta)}\mathbf{x}}\boldsymbol{\psi}(\theta)\dot{t}(\theta).
\end{aligned}
$$

Integrating both sides from 0 to $\theta \geq 0$ we obtain

$$
\begin{aligned}
&\Phi(\theta)^{-1}P_{S_{t(\theta)}\mathbf{x}}\boldsymbol{\psi}(\theta) - P_{\mathbf{x}}\boldsymbol{\psi}(0)\\
&= \int_0^\theta \Phi(s)^{-1}\left[D(s) - A(s) + D(s)(\dot{t}(s) - 1)\right] P_{S_{t(s)}\mathbf{x}}\boldsymbol{\psi}(s)\, ds\\
&\quad - \int_0^\theta \Phi(s)^{-1}\mathbf{f}(S_{t(s)}\mathbf{x})\mathbf{r}(s)^T P_{S_{t(s)}\mathbf{x}}\boldsymbol{\psi}(s)\dot{t}(s)\, ds\\
P_{S_{t(\theta)}\mathbf{x}}\boldsymbol{\psi}(\theta) &= \Phi(\theta)P_{\mathbf{x}}\boldsymbol{\psi}(0)\\
&\quad + \int_0^\theta \Phi(\theta)\Phi(s)^{-1}\left[D(s) - A(s) + D(s)(\dot{t}(s) - 1)\right] P_{S_{t(s)}\mathbf{x}}\boldsymbol{\psi}(s)\, ds\\
&\quad - \int_0^\theta \Phi(\theta)\Phi(s)^{-1}\mathbf{f}(S_{t(s)}\mathbf{x})\mathbf{r}(s)^T P_{S_{t(s)}\mathbf{x}}\boldsymbol{\psi}(s)\dot{t}(s)\, ds. \qquad (3.27)
\end{aligned}
$$

We now multiply with $P_{S_{t(\theta)}\mathbf{x}}$ from the left, noting that P is a projection.

$$
\begin{aligned}
P_{S_{t(\theta)}\mathbf{x}}\boldsymbol{\psi}(\theta) &= P_{S_{t(\theta)}\mathbf{x}}\Phi(\theta)P_{\mathbf{x}}\boldsymbol{\psi}(0)\\
&\quad + \int_0^\theta P_{S_{t(\theta)}\mathbf{x}}\Phi(\theta)\Phi(s)^{-1}\left[D(s) - A(s) + D(s)(\dot{t}(s) - 1)\right] P_{S_{t(s)}\mathbf{x}}\boldsymbol{\psi}(s)\, ds\\
&\quad - \int_0^\theta P_{S_{t(\theta)}\mathbf{x}}\Phi(\theta)\Phi(s)^{-1}\mathbf{f}(S_{t(s)}\mathbf{x})\mathbf{r}(s)^T P_{S_{t(s)}\mathbf{x}}\boldsymbol{\psi}(s)\dot{t}(s)\, ds. \qquad (3.28)
\end{aligned}
$$

Let us focus on the term $P_{S_{t(\theta)}\mathbf{x}}\Phi(\theta)\Phi(s)^{-1}\mathbf{f}(S_{t(s)\mathbf{x}})$. Define $\mathbf{y}(\tau) := \mathbf{f}(S_{t(\tau)}\mathbf{x})$. We have $\frac{d\mathbf{y}}{d\tau}(\tau) = D\mathbf{f}(S_{t(\tau)}\mathbf{x})\mathbf{y}(\tau)\dot{t}(\tau) = D(\tau)\mathbf{y}(\tau)\dot{t}(\tau)$. Hence,

$$\begin{aligned}
\frac{d}{d\tau}\left(\Phi(\tau)^{-1}\mathbf{y}(\tau)\right) &= \Phi(\tau)^{-1}[D(\tau) - A(\tau) + D(\tau)(\dot{t}(\tau) - 1)]\mathbf{y}(\tau) \\
\Phi(\theta)^{-1}\mathbf{y}(\theta) - \Phi(s)^{-1}\mathbf{y}(s) &= \int_s^\theta \Phi(\tau)^{-1}[D(\tau) - A(\tau) + D(\tau)(\dot{t}(\tau) - 1)]\mathbf{y}(\tau)\, d\tau \\
\mathbf{y}(\theta) - \Phi(\theta)\Phi(s)^{-1}\mathbf{y}(s) &= \int_s^\theta \Phi(\theta)\Phi(\tau)^{-1}[D(\tau) - A(\tau) + D(\tau)(\dot{t}(\tau) - 1)]\mathbf{y}(\tau)\, d\tau.
\end{aligned}$$

Applying $P_{S_{t(\theta)}\mathbf{x}}$ from the left and noting that $P_{S_{t(\theta)}\mathbf{x}}\mathbf{y}(\theta) = P_{S_{t(\theta)}\mathbf{x}}\mathbf{f}(S_{t(\theta)}\mathbf{x}) = \mathbf{0}$ we have

$$\begin{aligned}
&-P_{S_{t(\theta)}\mathbf{x}}\Phi(\theta)\Phi(s)^{-1}\mathbf{f}(S_{t(s)}\mathbf{x}) \\
&= P_{S_{t(\theta)}\mathbf{x}}\int_s^\theta \Phi(\theta)\Phi(\tau)^{-1}[D(\tau) - A(\tau) + D(\tau)(\dot{t}(\tau) - 1)]\mathbf{f}(S_{t(\tau)}\mathbf{x})\, d\tau. \qquad (3.29)
\end{aligned}$$

We have from Lemma 3.3

$$\|P_{S_{t(\theta)}\mathbf{x}} - P_{S_\theta\mathbf{p}}\| \leq \tilde{L}\|S_{t(\theta)}\mathbf{x} - S_\theta\mathbf{p}\| \leq \tilde{L}Ce^{-\mu_0\theta}$$

since $P_\mathbf{x}$ is continuously differentiable with respect to $\mathbf{x}$ and hence Lipschitz continuous in the compact set U.

Define $\rho_0 = \nu - 2\varepsilon$ such that $0 < \rho_0 < \mu_0$. With $\|P_{S_\theta\mathbf{p}}\Phi(\theta)\| \leq c_1e^{-\mu_0\theta}$ for $\theta \geq 0$ from Lemma 3.2

$$\begin{aligned}
\|P_{S_{t(\theta)}\mathbf{x}}\Phi(\theta)\| &\leq \|P_{S_{t(\theta)}\mathbf{x}} - P_{S_\theta\mathbf{p}}\| \cdot \|\Phi(\theta)\| + \|P_{S_\theta\mathbf{p}}\Phi(\theta)\| \\
&\leq c_1\tilde{L}Ce^{-\mu_0\theta} + c_1e^{-\mu_0\theta} \text{ by (3.23)} \\
&\leq c_3e^{-\rho_0\theta} \text{ for } \theta \geq 0. \qquad (3.30)
\end{aligned}$$

We also have for all $\theta \geq \tau \geq 0$

$$\begin{aligned}
\|P_{S_{t(\theta)}\mathbf{x}}\Phi(\theta)\Phi(\tau)^{-1}\| &\leq \|P_{S_\theta\mathbf{p}}\Phi(\theta)\Phi(\tau)^{-1}\| + \|P_{S_{t(\theta)}\mathbf{x}} - P_{S_\theta\mathbf{p}}\| \cdot \|\Phi(\theta)\Phi(\tau)^{-1}\| \\
&\leq c_1e^{-\mu_0(\theta-\tau)} + c_1LCe^{-\mu_0\theta} \text{ by Lemma 3.2 and (3.24)} \\
&\leq c_4e^{-\rho_0(\theta-\tau)}. \qquad (3.31)
\end{aligned}$$

Since $\mathbf{f}$ is continuous on the compact set U, there exists a constant $F \geq 0$ such that $\|\mathbf{f}(\mathbf{x})\| \leq F$ for all $\mathbf{x} \in U$. Hence, we obtain with (3.29) and (3.21)

$$\|P_{S_{t(\theta)}\mathbf{x}}\Phi(\theta)\Phi(s)^{-1}\mathbf{f}(S_{t(s)}\mathbf{x})\| \leq \int_s^\theta c_4e^{-\rho_0(\theta-\tau)}d_1e^{-\mu_0\tau}F\, d\tau$$

and hence, using (3.26), $|\dot{t}(s)| \leq 1+\varepsilon$ and $\rho_0 < \mu_0$

$$\begin{aligned}
&\left\| \int_0^\theta P_{S_{t(\theta)}\mathbf{x}} \Phi(\theta)\Phi(s)^{-1}\mathbf{f}(S_{t(s)}\mathbf{x})\mathbf{r}(s)^T P_{S_{t(s)}\mathbf{x}} \boldsymbol{\psi}(s)\dot{t}(s)\, ds \right\| \\
&\leq (1+\varepsilon)Rc_4d_1F \int_0^\theta \int_s^\theta \|P_{S_{t(s)}\mathbf{x}}\boldsymbol{\psi}(s)\| e^{-\rho_0(\theta-\tau)}e^{-\mu_0\tau}\, d\tau\, ds \\
&= (1+\varepsilon)Rc_4d_1F \int_0^\theta \|P_{S_{t(s)}\mathbf{x}}\boldsymbol{\psi}(s)\| e^{-\rho_0\theta} \int_s^\theta e^{-(\mu_0-\rho_0)\tau}\, d\tau\, ds \\
&\leq (1+\varepsilon)Rc_4d_1F \int_0^\theta \|P_{S_{t(s)}\mathbf{x}}\boldsymbol{\psi}(s)\| e^{-\rho_0\theta} \frac{1}{\mu_0-\rho_0} e^{-(\mu_0-\rho_0)s}\, ds.
\end{aligned}$$

Hence, we have with (3.28), (3.30), (3.31) and (3.21)

$$\begin{aligned}
\|P_{S_{t(\theta)}\mathbf{x}}\boldsymbol{\psi}(\theta)\| \leq\ & c_3e^{-\rho_0\theta}\|P_{\mathbf{x}}\boldsymbol{\psi}(0)\| \\
&+ \underbrace{\left(1+(1+\varepsilon)\frac{RF}{\mu_0-\rho_0}\right)c_4d_1}_{=:c_5} \int_0^\theta e^{-\rho_0(\theta-s)}e^{-\mu_0 s}\|P_{S_{t(s)}\mathbf{x}}\boldsymbol{\psi}(s)\|\, ds.
\end{aligned}$$

Lemma A.1 with $r(\theta) = \|P_{S_{t(\theta)}\mathbf{x}}\boldsymbol{\psi}(\theta)\|$, $a(\theta) = c_3e^{-\rho_0\theta}\|P_{\mathbf{x}}\boldsymbol{\psi}(0)\|$, $K(\theta) = c_5e^{-\rho_0\theta}$ and $b(\theta) = e^{\theta(\rho_0-\mu_0)}$ gives

$$\begin{aligned}
&\|P_{S_{t(\theta)}\mathbf{x}}\boldsymbol{\psi}(\theta)\| \\
&\leq c_3e^{-\rho_0\theta}\|P_{\mathbf{x}}\boldsymbol{\psi}(0)\| \\
&\quad +c_5e^{-\rho_0\theta}\int_0^\theta c_3e^{-\rho_0 s}\|P_{\mathbf{x}}\boldsymbol{\psi}(0)\|e^{s(\rho_0-\mu_0)}\, ds \cdot \exp\left(\int_0^\theta c_5e^{-\rho_0 s}e^{s(\rho_0-\mu_0)}\, ds\right) \\
&= c_3e^{-\rho_0\theta}\|P_{\mathbf{x}}\boldsymbol{\psi}(0)\| + c_3c_5\|P_{\mathbf{x}}\boldsymbol{\psi}(0)\|e^{-\rho_0\theta}\int_0^\theta e^{-\mu_0 s}\, ds \cdot \exp\left(c_5\int_0^\theta e^{-\mu_0 s}\, ds\right) \\
&\leq c_3e^{-\rho_0\theta}\|P_{\mathbf{x}}\boldsymbol{\psi}(0)\| + \frac{c_3c_5}{\mu_0}\|P_{\mathbf{x}}\boldsymbol{\psi}(0)\|e^{-\rho_0\theta} \cdot \exp\left(\frac{c_5}{\mu_0}\right)
\end{aligned}$$

using $\int_0^\theta e^{-\mu_0 s}\, ds = \frac{1}{\mu_0}(1-e^{-\mu_0\theta}) \leq \frac{1}{\mu_0}$. Note that this holds for all $\theta \geq 0$ since $\boldsymbol{\psi}(\theta)$ is continuous.

This proves (3.15) with $\kappa := \rho_0/(1+\varepsilon)$ using $\boldsymbol{\phi}(t(\theta)) = \boldsymbol{\psi}(\theta)$ and that t is a bijection on $[0,\infty)$ as well as that we have for $\theta \geq 0$

$$t(\theta) = \int_0^\theta \dot{t}(s)\, ds \leq (1+\varepsilon)\theta,$$

since $\dot{t}(s) \leq 1+\varepsilon$ and $t(0) = 0$. □

Step 4: $LM_1(\mathbf{x}) = -C(\mathbf{x})$

Now fix $\mathbf{x} \in A(\Omega)$. Denote by $\Phi(\tau,\theta;\mathbf{x}) = \Phi(\tau,0;\mathbf{x})\Phi(\theta,0;\mathbf{x})^{-1}$ the state transition matrix. Note that for fixed $\mathbf{x}$ there exists a $\theta_0 > 0$ such that $S_\tau\mathbf{x}$, $D\mathbf{f}(S_\tau\mathbf{x})$ and thus also $\Phi(\tau,\theta;\mathbf{x})$ are defined for all $\tau,\theta \geq -\theta_0$, and $\Phi(\tau,\theta;\mathbf{x})$ is $C^{\sigma-1}$ with respect to $\mathbf{x}$, τ and θ.

By the Chapman-Kolmogorov identities, cf. e.g. [3], p. 151, we have

$$\begin{aligned}
\frac{d}{d\tau}\Phi(\tau,\theta;\mathbf{x}) &= D\mathbf{f}(S_\tau\mathbf{x})\Phi(\tau,\theta;\mathbf{x}), \\
\frac{d}{d\theta}\Phi(\tau,\theta;\mathbf{x}) &= -\Phi(\tau,\theta;\mathbf{x})D\mathbf{f}(S_\theta\mathbf{x}), && (3.32)\\
\Phi(\theta,\theta;\mathbf{x}) &= I, && (3.33)\\
\Phi(\tau,0;S_\theta\mathbf{x}) &= \Phi(\tau+\theta,\theta;\mathbf{x}). && (3.34)
\end{aligned}$$

for all $\tau, \tau+\theta \geq -\theta_0$. Also,

$$\Phi(\tau - T_0, -T_0; S_{\theta+T_0}\mathbf{x}) \;=\; \Phi(\tau,0;S_\theta\mathbf{x}) \tag{3.35}$$

for $\tau \geq T_0 \geq 0$ and $|\theta| \leq \theta_0$. The last two equations follow from the fact that both functions satisfy the same initial value problem. For example, both sides of (3.35) satisfy $\frac{d}{d\tau}\mathbf{y}(\tau) = D\mathbf{f}(S_{\tau+\theta}\mathbf{x})\mathbf{y}(\tau)$.

Define

$$g_T(\theta,\mathbf{x}) \;=\; \int_\theta^{T+\theta} \Phi(\tau,\theta;\mathbf{x})^T P^T_{S_\tau\mathbf{x}} B(S_\tau\mathbf{x}) P_{S_\tau\mathbf{x}} \Phi(\tau,\theta;\mathbf{x})\, d\tau. \tag{3.36}$$

We have for all $\theta \geq -\theta_0$ by a change of variables and (3.34)

$$\begin{aligned}
g_T(\theta,\mathbf{x}) &= \int_0^T \Phi(\tau+\theta,\theta;\mathbf{x})^T P^T_{S_{\tau+\theta}\mathbf{x}} B(S_{\tau+\theta}\mathbf{x}) P_{S_{\tau+\theta}\mathbf{x}} \Phi(\tau+\theta,\theta;\mathbf{x})\, d\tau && (3.37)\\
&= \int_0^T \Phi(\tau,0;S_\theta\mathbf{x})^T P^T_{S_{\tau+\theta}\mathbf{x}} B(S_{\tau+\theta}\mathbf{x}) P_{S_{\tau+\theta}\mathbf{x}} \Phi(\tau,0;S_\theta\mathbf{x})\, d\tau. && (3.38)
\end{aligned}$$

We will show that $g_T(\theta,\mathbf{x})$ converges pointwise and $\frac{d}{d\theta}g_T(\theta,\mathbf{x})$ converges uniformly in $|\theta| \leq \theta_0$ as $T \to \infty$ so that $\frac{d}{d\theta}\lim_{T\to\infty} g_T(\theta,\mathbf{x}) = \lim_{T\to\infty}\frac{d}{d\theta}g_T(\theta,\mathbf{x})$ for $|\theta| < \theta_0$. As the set $S = \overline{\bigcup_{t=-\theta_0}^{\infty}\{S_t\mathbf{x}\}}$ is compact and $B(\cdot)$ is continuous, there exists $B^* > 0$ such that

$$\|B(S_t\mathbf{x})\| \;\leq\; B^* \tag{3.39}$$

for all $t \geq -\theta_0$.

Lemma 3.5 *For a fixed* $\mathbf{x} \in A(\Omega)$ *there exists* $c > 0$ *such that*

$$\|\Phi(\tau,0;S_\theta\mathbf{x})^T P^T_{S_\tau(S_\theta\mathbf{x})} B(S_{\tau+\theta}\mathbf{x}) P_{S_\tau(S_\theta\mathbf{x})} \Phi(\tau,0;S_\theta\mathbf{x})\| \;\leq\; ce^{-2\kappa\tau} \tag{3.40}$$

for all $|\theta| \leq \theta_0$ *and all* $\tau \geq 0$.

Proof: Since $\mathbf{x} \in A(\Omega)$ and $\Omega \subset U^\circ$, where U is compact and positively invariant, there exists T_0 such that $S_{\tau+\theta}\mathbf{x} \in U$ for all $\tau \geq T_0$ and all $|\theta| \leq \theta_0$. Since all terms in (3.40) depend continuously on τ and θ, we can choose c such that the inequality holds for all $|\theta| \leq \theta_0$ and all $\tau \in [0, T_0]$.

We denote $\mathbf{y} = S_{\theta+T_0}\mathbf{x} \in U$. With $t = \tau - T_0$ we have

$$\|P_{S_t\mathbf{y}}\boldsymbol{\phi}(t)\| \leq Ce^{-\kappa t}\|P_{\mathbf{y}}\boldsymbol{\phi}(0)\|$$

by (3.15) of Lemma 3.4, where $\boldsymbol{\phi}(t)$ solves $\dot{\boldsymbol{\phi}}(t) = D\mathbf{f}(S_t\mathbf{y})\boldsymbol{\phi}(t)$. Note that by (3.35) $\Phi(\tau, 0; S_\theta\mathbf{x}) = \Phi(\tau - T_0, -T_0; \mathbf{y}) = \Phi(\tau - T_0, 0; \mathbf{y})\Phi(-T_0, 0; \mathbf{y})^{-1}$.

Taking each of the columns of $\Phi(-T_0, 0; \mathbf{y})^{-1}$ for $\boldsymbol{\phi}(0)$, we obtain, first with the matrix norm $\|\cdot\|_1$ and the vector norm $\|\cdot\|_1$, and then also for $\|\cdot\|_2$ with a different constant as all matrix and vector norms are equivalent,

$$\begin{aligned}\|P_{S_\tau(S_\theta\mathbf{x})}\Phi(\tau, 0; S_\theta\mathbf{x})\| &= \|P_{S_t\mathbf{y}}\Phi(t, 0; \mathbf{y})\Phi(-T_0, 0; \mathbf{y})^{-1}\| \\ &\leq C'e^{-\kappa\tau}.\end{aligned}$$

Using (3.39) completes the proof. □

The right-hand side of (3.40) is integrable over $\tau \in [0, \infty)$. Hence, by Lebesgue's dominated convergence theorem, the function $g_T(\theta, \mathbf{x})$, see (3.38), converges pointwise for $T \to \infty$ for $|\theta| \leq \theta_0$. This shows with (3.36) that $M_1(\mathbf{x}) = \lim_{T\to\infty} g_T(0, \mathbf{x})$ is well defined and symmetric.

Also, using (3.36), (3.33) and (3.32), we have

$$\begin{aligned}\frac{d}{d\theta}g_T(\theta, \mathbf{x}) &= \Phi(T+\theta, \theta; \mathbf{x})^T P^T_{S_{T+\theta}\mathbf{x}} B(S_{T+\theta}\mathbf{x}) P_{S_{T+\theta}\mathbf{x}} \Phi(T+\theta, \theta; \mathbf{x}) - P^T_{S_\theta\mathbf{x}} B(S_\theta\mathbf{x}) P_{S_\theta\mathbf{x}} \\ &\quad - D\mathbf{f}(S_\theta\mathbf{x})^T \int_\theta^{T+\theta} \Phi(\tau, \theta; \mathbf{x})^T P^T_{S_\tau\mathbf{x}} B(S_\tau\mathbf{x}) P_{S_\tau\mathbf{x}} \Phi(\tau, \theta; \mathbf{x})\, d\tau \\ &\quad - \int_\theta^{T+\theta} \Phi(\tau, \theta; \mathbf{x})^T P^T_{S_\tau\mathbf{x}} B(S_\tau\mathbf{x}) P_{S_\tau\mathbf{x}} \Phi(\tau, \theta; \mathbf{x})\, d\tau D\mathbf{f}(S_\theta\mathbf{x}) \\ &= \Phi(T, 0; S_\theta\mathbf{x})^T P^T_{S_T(S_\theta\mathbf{x})} B(S_{T+\theta}\mathbf{x}) P_{S_T(S_\theta\mathbf{x})} \Phi(T, 0; S_\theta\mathbf{x}) - P^T_{S_\theta\mathbf{x}} B(S_\theta\mathbf{x}) P_{S_\theta\mathbf{x}} \\ &\quad - D\mathbf{f}(S_\theta\mathbf{x})^T \int_0^T \Phi(\tau, 0; S_\theta\mathbf{x})^T P^T_{S_\tau(S_\theta\mathbf{x})} B(S_{\tau+\theta}\mathbf{x}) P_{S_\tau(S_\theta\mathbf{x})} \Phi(\tau, 0; S_\theta\mathbf{x})\, d\tau \\ &\quad - \int_0^T \Phi(\tau, 0; S_\theta\mathbf{x})^T P^T_{S_\tau(S_\theta\mathbf{x})} B(S_{\tau+\theta}\mathbf{x}) P_{S_\tau(S_\theta\mathbf{x})} \Phi(\tau, 0; S_\theta\mathbf{x})\, d\tau D\mathbf{f}(S_\theta\mathbf{x})\end{aligned}$$

by (3.34). The right-hand side converges uniformly for $|\theta| \leq \theta_0$ as $T \to \infty$ by (3.40). Hence, we can exchange limit and derivative, obtaining for $|\theta| < \theta_0$, again with (3.40),

$$\begin{aligned}&\frac{d}{d\theta}\lim_{T\to\infty} g_T(\theta, \mathbf{x}) \\ &= \lim_{T\to\infty}\frac{d}{d\theta}g_T(\theta, \mathbf{x}) \\ &= -P^T_{S_\theta\mathbf{x}} B(S_\theta\mathbf{x}) P_{S_\theta\mathbf{x}} \\ &\quad - D\mathbf{f}(S_\theta\mathbf{x})^T \int_0^\infty \Phi(\tau, 0; S_\theta\mathbf{x})^T P^T_{S_\tau(S_\theta\mathbf{x})} B(S_{\tau+\theta}\mathbf{x}) P_{S_\tau(S_\theta\mathbf{x})} \Phi(\tau, 0; S_\theta\mathbf{x})\, d\tau \\ &\quad - \int_0^\infty \Phi(\tau, 0; S_\theta\mathbf{x})^T P^T_{S_\tau(S_\theta\mathbf{x})} B(S_{\tau+\theta}\mathbf{x}) P_{S_\tau(S_\theta\mathbf{x})} \Phi(\tau, 0; S_\theta\mathbf{x})\, d\tau D\mathbf{f}(S_\theta\mathbf{x}). \quad (3.41)\end{aligned}$$

Altogether, we thus have

$$\begin{aligned}
M_1'(\mathbf{x}) &= \frac{d}{d\theta} M_1(S_\theta \mathbf{x})\Big|_{\theta=0} \\
&= \frac{d}{d\theta} \lim_{T\to\infty} \left[\int_0^T \Phi(\tau,0;S_\theta\mathbf{x})^T P^T_{S_\tau(S_\theta\mathbf{x})} B(S_\tau S_\theta \mathbf{x}) P_{S_\tau(S_\theta\mathbf{x})} \Phi(\tau,0;S_\theta\mathbf{x})\, d\tau \right]\Bigg|_{\theta=0} \\
&= \frac{d}{d\theta} \lim_{T\to\infty} g_T(\theta,\mathbf{x})\Big|_{\theta=0} \quad \text{by (3.38)} \\
&= -P_{\mathbf{x}}^T B(\mathbf{x}) P_{\mathbf{x}} - D\mathbf{f}(\mathbf{x})^T M_1(\mathbf{x}) - M_1(\mathbf{x}) D\mathbf{f}(\mathbf{x}) \text{ by (3.41)} \\
&= -C(\mathbf{x}) - D\mathbf{f}(\mathbf{x})^T M_1(\mathbf{x}) - M_1(\mathbf{x}) D\mathbf{f}(\mathbf{x}) \\
&\quad + \frac{M_1(\mathbf{x})\mathbf{f}(\mathbf{x})\mathbf{f}(\mathbf{x})^T (D\mathbf{f}(\mathbf{x}) + D\mathbf{f}(\mathbf{x})^T) + (D\mathbf{f}(\mathbf{x}) + D\mathbf{f}(\mathbf{x})^T)\mathbf{f}(\mathbf{x})\mathbf{f}(\mathbf{x})^T M_1(\mathbf{x})}{\|\mathbf{f}(\mathbf{x})\|^2}.
\end{aligned}$$

The last term is zero since $\Phi(t,0;\mathbf{x})\mathbf{f}(\mathbf{x}) = \mathbf{f}(S_t\mathbf{x})$ and thus

$$\begin{aligned}
M_1(\mathbf{x})\mathbf{f}(\mathbf{x}) &= \int_0^\infty \Phi(t,0;\mathbf{x})^T C(S_t\mathbf{x}) \Phi(t,0;\mathbf{x}) \mathbf{f}(\mathbf{x})\, dt \\
&= \int_0^\infty \Phi(t,0;\mathbf{x})^T P^T_{S_t\mathbf{x}} B(S_t\mathbf{x}) P_{S_t\mathbf{x}} \mathbf{f}(S_t\mathbf{x})\, dt \\
&= \mathbf{0}
\end{aligned}$$

using $P_{S_t\mathbf{x}}\mathbf{f}(S_t\mathbf{x}) = \mathbf{0}$. Similarly we have also $\mathbf{f}(\mathbf{x})^T M_1(\mathbf{x}) = \mathbf{0}^T$.

This shows the matrix equation $LM_1(\mathbf{x}) = -C(\mathbf{x})$ and thus (3.1).

Step 5: Smoothness of M_1

To prove that $M_1 \in C^{\sigma-1}(A(x_0), \mathbb{S}^n)$, we will define $\boldsymbol{\psi}(t,\mathbf{x}) := P_{S_t\mathbf{x}}\boldsymbol{\phi}(t)$, where $\boldsymbol{\phi}(t)$ is a solution of the first variational equation $\dot{\boldsymbol{\phi}}(t) = D\mathbf{f}(S_t\mathbf{x})\boldsymbol{\phi}(t)$. We will show by induction with respect to $|\boldsymbol{\alpha}|$ that

$$\|\partial_{\mathbf{x}}^{\boldsymbol{\alpha}}(\boldsymbol{\psi}(t,\mathbf{x}))\| \le c_{\boldsymbol{\alpha}} e^{-\kappa_0 t} \max_{\mathbf{0}\le\boldsymbol{\beta}\le\boldsymbol{\alpha}} \|\partial_{\mathbf{x}}^{\boldsymbol{\beta}}(\boldsymbol{\psi}(0,\mathbf{x}))\| \tag{3.42}$$

for all $|\boldsymbol{\alpha}| \le \sigma - 1$, $\mathbf{x} \in U$ and $t \ge 0$, where $\kappa_0 := \frac{\kappa}{2}$. For $\boldsymbol{\alpha} = \mathbf{0}$, (3.42) follows directly from Lemma 3.4.

For $\mathbf{x} \in U$ define $\boldsymbol{\phi}_0(t,\mathbf{x}) = \frac{\mathbf{f}(S_t\mathbf{x})}{\|\mathbf{f}(S_t\mathbf{x})\|^2}$ and $\mathbf{a} = \mathbf{f}(\mathbf{x})$. For $i = 1, \ldots, n$ let $\boldsymbol{\phi}_i(0,\mathbf{x}) = P_{\mathbf{x}}\mathbf{e}_i$ and let $\boldsymbol{\phi}_i(t,\mathbf{x})$ be a solution of $\dot{\boldsymbol{\phi}}(t) = D\mathbf{f}(S_t\mathbf{x})\boldsymbol{\phi}(t)$. Then

$$\Psi(t,\mathbf{x}) = (a_1\boldsymbol{\phi}_0(t,\mathbf{x}) + P_{S_t\mathbf{x}}\boldsymbol{\phi}_1(t,\mathbf{x}), \ldots, a_n\boldsymbol{\phi}_0(t,\mathbf{x}) + P_{S_t\mathbf{x}}\boldsymbol{\phi}_n(t,\mathbf{x}))$$

is the principal fundamental matrix solution of

$$\frac{d}{dt}\Psi(t,\mathbf{x}) = \widetilde{A}(t,\mathbf{x})\Psi(t,\mathbf{x}) \tag{3.43}$$

where $\widetilde{A}(t,\mathbf{x}) = D\mathbf{f}(S_t\mathbf{x}) - \frac{\mathbf{f}(S_t\mathbf{x})\mathbf{f}(S_t\mathbf{x})^T}{\|\mathbf{f}(S_t\mathbf{x})\|^2}(D\mathbf{f}(S_t\mathbf{x})^T + D\mathbf{f}(S_t\mathbf{x}))$. Indeed, it can be shown directly that $\boldsymbol{\phi}_0(t,\mathbf{x}) = \frac{\mathbf{f}(S_t\mathbf{x})}{\|\mathbf{f}(S_t\mathbf{x})\|^2}$ is a solution of (3.43) and in a similar way to (3.25) that $P_{S_t\mathbf{x}}\boldsymbol{\phi}_i(t,\mathbf{x})$ for $i = 1,\ldots,n$ are solutions of (3.43), see also [13]. Note that

$$\Psi(0,\mathbf{x}) = \boldsymbol{\phi}_0(0,\mathbf{x})\mathbf{a}^T + P_\mathbf{x} I = \frac{\mathbf{f}(\mathbf{x})\mathbf{f}(\mathbf{x})^T}{\|\mathbf{f}(\mathbf{x})\|^2} + P_\mathbf{x} I = I.$$

We have with Lemma 3.4

$$\begin{aligned}
\|P_{S_t\mathbf{x}}\Psi(t,\mathbf{x})\|_1 &= \max_{j=1,\ldots,n} \|a_j P_{S_t\mathbf{x}}\boldsymbol{\phi}_0(t,\mathbf{x}) + P_{S_t\mathbf{x}}\boldsymbol{\phi}_j(t,\mathbf{x})\|_1 \\
&= \max_{j=1,\ldots,n} \|P_{S_t\mathbf{x}}\boldsymbol{\phi}_j(t,\mathbf{x})\|_1 \\
&\leq Ce^{-2\kappa_0 t} \max_{j=1,\ldots,n} \|P_\mathbf{x}\boldsymbol{\phi}_j(0,\mathbf{x})\|_1 \\
&= Ce^{-2\kappa_0 t} \max_{j=1,\ldots,n} \|a_j P_\mathbf{x}\boldsymbol{\phi}_0(0,\mathbf{x}) + P_\mathbf{x}\boldsymbol{\phi}_j(0,\mathbf{x})\|_1 \\
&= Ce^{-2\kappa_0 t}\|P_\mathbf{x}\Psi(0,\mathbf{x})\|_1,
\end{aligned}$$

and, as all norms are equivalent and $\Psi(0,\mathbf{x}) = I$, with a different constant

$$\|P_{S_t\mathbf{x}}\Psi(t,\mathbf{x})\| \leq Ce^{-2\kappa_0 t} \tag{3.44}$$

for all $t \geq 0$ and all $\mathbf{x} \in U$.

We will now show the estimate

$$\|P_{S_t\mathbf{x}}\Psi(t,\mathbf{x})\Psi(s,\mathbf{x})^{-1}\| \leq Ce^{-2\kappa_0(t-s)} \tag{3.45}$$

for all $t \geq 0$, $0 \leq s \leq t$ and all $\mathbf{x} \in U$. From (3.44) we obtain by considering the point $S_s\mathbf{x}$ and the time $t-s$

$$\|P_{S_{t-s}S_s\mathbf{x}}\Psi(t-s,S_s\mathbf{x})\| \leq Ce^{-2\kappa_0(t-s)}. \tag{3.46}$$

Denoting the transition matrix from s to t for (3.43) by $\Psi(t,s;\mathbf{x})$, we have with (3.34)

$$\begin{aligned}
\Psi(t-s,S_s\mathbf{x}) &= \Psi(t-s,0;S_s\mathbf{x}) \\
&= \Psi(t,s;\mathbf{x}) \\
&= \Psi(t,\mathbf{x})\Psi(s,\mathbf{x})^{-1} \qquad (3.47) \\
\|P_{S_t\mathbf{x}}\Psi(t,\mathbf{x})\Psi(s,\mathbf{x})^{-1}\| &= \|P_{S_{t-s}S_s\mathbf{x}}\Psi(t-s,S_s\mathbf{x})\| \text{ by (3.47)} \\
&\leq Ce^{-2\kappa_0(t-s)}
\end{aligned}$$

by (3.46). This shows (3.45).

Now we assume (3.42) is true for all $\boldsymbol{\alpha}'$ with $|\boldsymbol{\alpha}'| \leq k-1$ and seek to show it for $|\boldsymbol{\alpha}| = k \leq \sigma - 1$. We will write

$$\begin{aligned}
\partial_\mathbf{x}^{\boldsymbol{\alpha}}\boldsymbol{\psi}(t,\mathbf{x}) &= \left[I - \frac{\mathbf{f}(S_t\mathbf{x})\mathbf{f}(S_t\mathbf{x})^T}{\|\mathbf{f}(S_t\mathbf{x})\|^2} + \frac{\mathbf{f}(S_t\mathbf{x})\mathbf{f}(S_t\mathbf{x})^T}{\|\mathbf{f}(S_t\mathbf{x})\|^2}\right]\partial_\mathbf{x}^{\boldsymbol{\alpha}}\boldsymbol{\psi}(t,\mathbf{x}) \\
&= P_{S_t\mathbf{x}}\partial_\mathbf{x}^{\boldsymbol{\alpha}}\boldsymbol{\psi}(t,\mathbf{x}) + \frac{\mathbf{f}(S_t\mathbf{x})\mathbf{f}(S_t\mathbf{x})^T}{\|\mathbf{f}(S_t\mathbf{x})\|^2}\partial_\mathbf{x}^{\boldsymbol{\alpha}}\boldsymbol{\psi}(t,\mathbf{x}) \qquad (3.48)
\end{aligned}$$

and show that each term satisfies the exponential bound in (3.42).

For the second term of (3.48), we have

$$\frac{\mathbf{f}(S_t\mathbf{x})\mathbf{f}(S_t\mathbf{x})^T}{\|\mathbf{f}(S_t\mathbf{x})\|^2}\boldsymbol{\psi}(t,\mathbf{x}) \;=\; \frac{\mathbf{f}(S_t\mathbf{x})\mathbf{f}(S_t\mathbf{x})^T}{\|\mathbf{f}(S_t\mathbf{x})\|^2}P_{S_t\mathbf{x}}\boldsymbol{\phi}(t) \;=\; \mathbf{0}$$

for all $t \geq 0$ and $\mathbf{x} \in U$ since $\mathbf{f}(S_t\mathbf{x})^T P_{S_t\mathbf{x}} = \mathbf{0}^T$. Hence,

$$\begin{aligned}
\mathbf{0} &= \partial_{\mathbf{x}}^{\boldsymbol{\alpha}}\left(\frac{\mathbf{f}(S_t\mathbf{x})\mathbf{f}(S_t\mathbf{x})^T}{\|\mathbf{f}(S_t\mathbf{x})\|^2}\boldsymbol{\psi}(t,\mathbf{x})\right)\\
&= \frac{\mathbf{f}(S_t\mathbf{x})\mathbf{f}(S_t\mathbf{x})^T}{\|\mathbf{f}(S_t\mathbf{x})\|^2}\partial_{\mathbf{x}}^{\boldsymbol{\alpha}}\boldsymbol{\psi}(t,\mathbf{x})\\
&\quad + \sum_{\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2=\boldsymbol{\alpha}, |\boldsymbol{\alpha}_1|\geq 1} c_{\boldsymbol{\alpha}_1}\partial_{\mathbf{x}}^{\boldsymbol{\alpha}_1}\left(\frac{\mathbf{f}(S_t\mathbf{x})\mathbf{f}(S_t\mathbf{x})^T}{\|\mathbf{f}(S_t\mathbf{x})\|^2}\right)\partial_{\mathbf{x}}^{\boldsymbol{\alpha}_2}\boldsymbol{\psi}(t,\mathbf{x})\\
\frac{\mathbf{f}(S_t\mathbf{x})\mathbf{f}(S_t\mathbf{x})^T}{\|\mathbf{f}(S_t\mathbf{x})\|^2}\partial_{\mathbf{x}}^{\boldsymbol{\alpha}}\boldsymbol{\psi}(t,\mathbf{x}) &= -\sum_{\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2=\boldsymbol{\alpha}, |\boldsymbol{\alpha}_1|\geq 1} c_{\boldsymbol{\alpha}_1}\partial_{\mathbf{x}}^{\boldsymbol{\alpha}_1}\left(\frac{\mathbf{f}(S_t\mathbf{x})\mathbf{f}(S_t\mathbf{x})^T}{\|\mathbf{f}(S_t\mathbf{x})\|^2}\right)\partial_{\mathbf{x}}^{\boldsymbol{\alpha}_2}\boldsymbol{\psi}(t,\mathbf{x}).
\end{aligned}$$

By induction assumption and smoothness of $\mathbf{f}$ in the compact, positively invariant set U, the norm of the right-hand side is smaller than $ce^{-\kappa_0 t}\max_{\mathbf{0}\leq\boldsymbol{\beta}<\boldsymbol{\alpha}}\|\partial_{\mathbf{x}}^{\boldsymbol{\beta}}\boldsymbol{\psi}(0,\mathbf{x})\|$, and thus so is the left-hand side. This shows the exponential bound on the norm of the second term of (3.48).

For the first term of (3.48), we have

$$\begin{aligned}
\frac{d}{dt}\partial_{\mathbf{x}}^{\boldsymbol{\alpha}}\boldsymbol{\psi}(t,\mathbf{x}) &= \partial_{\mathbf{x}}^{\boldsymbol{\alpha}}\frac{d}{dt}\boldsymbol{\psi}(t,\mathbf{x})\\
&= \partial_{\mathbf{x}}^{\boldsymbol{\alpha}}\left[\widetilde{A}(t,\mathbf{x})\boldsymbol{\psi}(t,\mathbf{x})\right]\\
&= \widetilde{A}(t,\mathbf{x})\partial_{\mathbf{x}}^{\boldsymbol{\alpha}}\boldsymbol{\psi}(t,\mathbf{x})\\
&\quad + \underbrace{\sum_{\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2=\boldsymbol{\alpha}, |\boldsymbol{\alpha}_1|\geq 1} c_{\boldsymbol{\alpha}_1}\partial_{\mathbf{x}}^{\boldsymbol{\alpha}_1}\widetilde{A}(t,\mathbf{x})\partial_{\mathbf{x}}^{\boldsymbol{\alpha}_2}\boldsymbol{\psi}(t,\mathbf{x})}_{=:g(t,\mathbf{x})}.
\end{aligned} \tag{3.49}$$

Note that we could exchange $\partial_{\mathbf{x}}^{\boldsymbol{\alpha}}$ and $\frac{d}{dt}$ above since $P_{S_t\mathbf{x}}$ and $\boldsymbol{\phi}$ are smooth enough, cf. e.g. [11], Chapter V, Theorem 4.1.

From the induction assumption we know that for all $|\boldsymbol{\alpha}_2| \leq k-1$

$$\|\partial_{\mathbf{x}}^{\boldsymbol{\alpha}_2}\boldsymbol{\psi}(t,\mathbf{x})\| \leq c_{\boldsymbol{\alpha}_2}e^{-\kappa_0 t}\max_{\mathbf{0}\leq\boldsymbol{\beta}\leq\boldsymbol{\alpha}_2}\|\partial_{\mathbf{x}}^{\boldsymbol{\beta}}\boldsymbol{\psi}(0,\mathbf{x})\|.$$

From the definition of $\widetilde{A}$, since $\mathbf{f} \in C^\sigma(\mathbb{R}^n,\mathbb{R}^n)$ and U is compact and positively invariant, there is a constant bounding $\|\partial_{\mathbf{x}}^{\boldsymbol{\alpha}_1}\widetilde{A}(t,\mathbf{x})\|$ for all $|\boldsymbol{\alpha}_1| \leq \sigma - 1$, $t \geq 0$ and $\mathbf{x} \in U$. This shows altogether that

$$\|g(t,\mathbf{x})\| \;\leq\; Ce^{-\kappa_0 t}\max_{\mathbf{0}\leq\boldsymbol{\beta}<\boldsymbol{\alpha}}\|\partial_{\mathbf{x}}^{\boldsymbol{\beta}}\boldsymbol{\psi}(0,\mathbf{x})\| \tag{3.50}$$

for all $t \geq 0$ and $\mathbf{x} \in U$.

Using the method of variation of constants, the solution $\mathbf{z}(t,\mathbf{x}) = \partial_{\mathbf{x}}^{\boldsymbol{\alpha}} \boldsymbol{\psi}(t,\mathbf{x})$ of

$$\frac{d}{dt}\mathbf{z}(t,\mathbf{x}) \quad = \quad \widetilde{A}(t,\mathbf{x})\mathbf{z}(t,\mathbf{x}) + g(t,\mathbf{x}), \tag{3.51}$$

see (3.49), satisfies

$$\mathbf{z}(t,\mathbf{x}) \quad = \quad \Psi(t,\mathbf{x})\mathbf{z}(0,\mathbf{x}) + \int_0^t \Psi(t,\mathbf{x})\Psi(s,\mathbf{x})^{-1} g(s,\mathbf{x})\, ds.$$

Application of the projection $P_{S_t\mathbf{x}}$ from the left on both sides gives

$$P_{S_t\mathbf{x}}\partial_{\mathbf{x}}^{\boldsymbol{\alpha}}\boldsymbol{\psi}(t,\mathbf{x}) \quad = \quad P_{S_t\mathbf{x}}\Psi(t,\mathbf{x})\partial_{\mathbf{x}}^{\boldsymbol{\alpha}}\boldsymbol{\psi}(0,\mathbf{x}) + \int_0^t P_{S_t\mathbf{x}}\Psi(t,\mathbf{x})\Psi(s,\mathbf{x})^{-1} g(s,\mathbf{x})\, ds.$$

Then we have with (3.44), (3.45) and (3.50)

$$\begin{aligned}
\|P_{S_t\mathbf{x}}\partial_{\mathbf{x}}^{\boldsymbol{\alpha}}\boldsymbol{\psi}(t,\mathbf{x})\| &\leq \left(Ce^{-2\kappa_0 t} + \int_0^t Ce^{-2\kappa_0(t-s)}e^{-\kappa_0 s}\, ds\right) \max_{\mathbf{0}\leq\boldsymbol{\beta}\leq\boldsymbol{\alpha}} \|\partial_{\mathbf{x}}^{\boldsymbol{\beta}}\boldsymbol{\psi}(0,\mathbf{x})\| \\
&= \left(Ce^{-2\kappa_0 t} + Ce^{-2\kappa_0 t}\int_0^t e^{\kappa_0 s}\, ds\right) \max_{\mathbf{0}\leq\boldsymbol{\beta}\leq\boldsymbol{\alpha}} \|\partial_{\mathbf{x}}^{\boldsymbol{\beta}}\boldsymbol{\psi}(0,\mathbf{x})\| \\
&\leq \left(Ce^{-2\kappa_0 t} + \frac{C}{\kappa_0}e^{-2\kappa_0 t}e^{\kappa_0 t}\right) \max_{\mathbf{0}\leq\boldsymbol{\beta}\leq\boldsymbol{\alpha}} \|\partial_{\mathbf{x}}^{\boldsymbol{\beta}}\boldsymbol{\psi}(0,\mathbf{x})\| \\
&\leq c_{\boldsymbol{\alpha}} e^{-\kappa_0 t} \max_{\mathbf{0}\leq\boldsymbol{\beta}\leq\boldsymbol{\alpha}} \|\partial_{\mathbf{x}}^{\boldsymbol{\beta}}\boldsymbol{\psi}(0,\mathbf{x})\|.
\end{aligned}$$

This shows the bound on the first term of (3.48) and thus (3.42).

Next, we show that $\int_0^T \partial_{\mathbf{x}}^{\boldsymbol{\alpha}}(\Phi(t,0;\mathbf{x})^T P_{S_t\mathbf{x}}^T B(S_t\mathbf{x}) P_{S_t\mathbf{x}}\Phi(t,0;\mathbf{x}))\, dt$ converges uniformly with respect to $\mathbf{x}$ as $T \to \infty$ for $1 \leq |\boldsymbol{\alpha}| \leq \sigma - 1$. Let $\mathbf{x} \in A(\Omega)$ and let O be a bounded, open neighborhood of $\mathbf{x}$, such that $\overline{O} \subset A(\Omega)$. Since $\overline{O}$ is compact, there is a $T_0 \in \mathbb{R}_0^+$ such that $S_{T_0+t}\overline{O} \subset U$ holds for all $t \geq 0$. Hence, it is sufficient to show the statement for all $\mathbf{x} \in U$.

We can write the i-th column of $P_{S_t\mathbf{x}}\Phi(t,0,\mathbf{x})$ as $\boldsymbol{\psi}(t,\mathbf{x}) = P_{S_t\mathbf{x}}\Phi(t,0;\mathbf{x})\mathbf{e}_i = P_{S_t\mathbf{x}}\boldsymbol{\phi}(t,\mathbf{x})$ with $\boldsymbol{\phi}(0,\mathbf{x}) = \mathbf{e}_i$. Thus, $\partial_{\mathbf{x}}^{\boldsymbol{\alpha}}\boldsymbol{\psi}(0,\mathbf{x}) = \partial_{\mathbf{x}}^{\boldsymbol{\alpha}} P_{\mathbf{x}}\mathbf{e}_i$, which can be bounded by a constant for all $\mathbf{x} \in U$ and $|\boldsymbol{\alpha}| \leq \sigma - 1$ by the smoothness of $\mathbf{f}$. Similarly, $\partial_{\mathbf{x}}^{\boldsymbol{\alpha}} B(S_t\mathbf{x})$ can be bounded by a constant for all $\mathbf{x} \in U$, $t \geq 0$ and $|\boldsymbol{\alpha}| \leq \sigma - 1$. Altogether, we have by (3.42)

$$\int_0^T \|\partial_{\mathbf{x}}^{\boldsymbol{\alpha}}(\Phi(t,0;\mathbf{x})^T P_{S_t\mathbf{x}}^T B(S_t\mathbf{x}) P_{S_t\mathbf{x}}\Phi(t,0;\mathbf{x}))\|\, dt \quad \leq \quad \int_0^T \tilde{c}e^{-2\kappa_0 t}\, dt$$

for all $\mathbf{x} \in U$ and all $T \geq 0$. Hence, $\int_0^T \partial_{\mathbf{x}}^{\boldsymbol{\alpha}}(\Phi(t,0;\mathbf{x})^T P_{S_t\mathbf{x}}^T B(S_t\mathbf{x}) P_{S_t\mathbf{x}}\Phi(t,0;\mathbf{x}))\, dt$ converges uniformly as $T \to \infty$. This proves that $M_1 \in C^{\sigma-1}(A(\Omega), \mathbb{S}^n)$.

Step 6: positive definiteness

To show the positive definiteness of M, fix $\mathbf{x} \in A(\Omega)$ and consider a general

$$\mathbb{R}^n \ni \mathbf{w} = \left(I - \frac{\mathbf{f}(\mathbf{x})\mathbf{f}(\mathbf{x})^T}{\|\mathbf{f}(\mathbf{x})\|^2} + \frac{\mathbf{f}(\mathbf{x})\mathbf{f}(\mathbf{x})^T}{\|\mathbf{f}(\mathbf{x})\|^2}\right)\mathbf{w} = \mathbf{v} + c\frac{\mathbf{f}(\mathbf{x})}{\|\mathbf{f}(\mathbf{x})\|^2}$$

with $\mathbf{v} = P_{\mathbf{x}}\mathbf{w}$, so $\mathbf{v} \perp \mathbf{f}(\mathbf{x})$, and $c = \mathbf{f}(\mathbf{x})^T\mathbf{w}$. Hence, using $C(\mathbf{x}) = P_{\mathbf{x}}^T B(\mathbf{x}) P_{\mathbf{x}}$, we have

$$\begin{aligned} \mathbf{w}^T M(\mathbf{x})\mathbf{w} &= \int_0^\infty [P_{S_t\mathbf{x}}\Phi(t,0;\mathbf{x})\mathbf{w}]^T B(S_t\mathbf{x})[P_{S_t\mathbf{x}}\Phi(t,0;\mathbf{x})\mathbf{w}]\,dt + c_0\mathbf{w}^T\mathbf{f}(\mathbf{x})\mathbf{f}(\mathbf{x})^T\mathbf{w} \\ &= \int_0^\infty [P_{S_t\mathbf{x}}\Phi(t,0;\mathbf{x})\mathbf{w}]^T B(S_t\mathbf{x})[P_{S_t\mathbf{x}}\Phi(t,0;\mathbf{x})\mathbf{w}]\,dt + c_0 c^2 \\ &\geq 0 \end{aligned}$$

due to the positive definiteness of B and $c_0 > 0$. We seek to show that the term is only 0 if $\mathbf{w} = \mathbf{0}$.

Let us assume that the term is zero, i.e. both summands are zero. The first term, since B is positive definite, is only zero if $P_{S_t\mathbf{x}}\Phi(t,0;\mathbf{x})\mathbf{w} = \mathbf{0}$ for all $t \geq 0$. In particular, for $t = 0$ we have $\mathbf{0} = P_{\mathbf{x}}\mathbf{w} = \mathbf{v}$. If the second term is zero, then, since $c_0 > 0$, $c = 0$, which together yields $\mathbf{w} = \mathbf{0}$.

This proves the theorem. □

4 Uniqueness

To show uniqueness of solutions to (3.1) and (3.2) in Theorem 4.2, let us first state the following lemma.

Lemma 4.1 *Denote by $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$ two solutions of $\dot{\boldsymbol{\phi}}(t) = D\mathbf{f}(S_t\mathbf{x})\boldsymbol{\phi}(t)$. Let $M \in C^1(\mathbb{R}^n, \mathbb{S}^n)$ such that $M(\mathbf{x})$ is positive definite for all $\mathbf{x} \in \mathbb{R}^n$.*

Then we have for any $\mathbf{x} \in \mathbb{R}^n$ with $\mathbf{f}(\mathbf{x}) \neq \mathbf{0}$ and all $t \geq 0$, denoting $\boldsymbol{\phi}_0(t) = \frac{\mathbf{f}(S_t\mathbf{x})}{\|\mathbf{f}(S_t\mathbf{x})\|^2}$,

$$\begin{aligned} \frac{d}{dt}\left[\boldsymbol{\phi}_1(t)^T P_{S_t\mathbf{x}}^T M(S_t\mathbf{x}) P_{S_t\mathbf{x}}\boldsymbol{\phi}_2(t)\right] &= \boldsymbol{\phi}_1(t)^T P_{S_t\mathbf{x}}^T L M(S_t\mathbf{x}) P_{S_t\mathbf{x}}\boldsymbol{\phi}_2(t), &(4.1)\\ \frac{d}{dt}\left[\boldsymbol{\phi}_0(t)^T M(S_t\mathbf{x})\boldsymbol{\phi}_0(t)\right] &= \boldsymbol{\phi}_0(t)^T L M(S_t\mathbf{x})\boldsymbol{\phi}_0(t), &(4.2)\\ \frac{d}{dt}\left[\boldsymbol{\phi}_1(t)^T P_{S_t\mathbf{x}}^T M(S_t\mathbf{x})\boldsymbol{\phi}_0(t)\right] &= \boldsymbol{\phi}_1(t)^T P_{S_t\mathbf{x}}^T L M(S_t\mathbf{x})\boldsymbol{\phi}_0(t), &(4.3)\\ \frac{d}{dt}\left[\boldsymbol{\phi}_0(t)^T M(S_t\mathbf{x}) P_{S_t\mathbf{x}}\boldsymbol{\phi}_1(t)\right] &= \boldsymbol{\phi}_0(t)^T L M(S_t\mathbf{x}) P_{S_t\mathbf{x}}\boldsymbol{\phi}_1(t). &(4.4) \end{aligned}$$

PROOF: Note that $\mathbf{f}(S_t\mathbf{x}) \neq \mathbf{0}$ for all $t \geq 0$ since $\mathbf{f}(\mathbf{x}) \neq \mathbf{0}$.

For the first statement we calculate

$$
\begin{aligned}
&\frac{d}{dt}\left[\boldsymbol{\phi}_1(t)^T P_{S_t\mathbf{x}}^T M(S_t\mathbf{x}) P_{S_t\mathbf{x}}\boldsymbol{\phi}_2(t)\right]\\
=\;& \boldsymbol{\phi}_1(t)^T D\mathbf{f}(S_t\mathbf{x})^T\left(I-\frac{\mathbf{f}(S_t\mathbf{x})\mathbf{f}(S_t\mathbf{x})^T}{\|\mathbf{f}(S_t\mathbf{x})\|^2}\right)M(S_t\mathbf{x})P_{S_t\mathbf{x}}\boldsymbol{\phi}_2(t)\\
&-\boldsymbol{\phi}_1(t)^T\frac{D\mathbf{f}(S_t\mathbf{x})\mathbf{f}(S_t\mathbf{x})\mathbf{f}(S_t\mathbf{x})^T+\mathbf{f}(S_t\mathbf{x})\mathbf{f}(S_t\mathbf{x})^T D\mathbf{f}(S_t\mathbf{x})^T}{\|\mathbf{f}(S_t\mathbf{x})\|^2}M(S_t\mathbf{x})P_{S_t\mathbf{x}}\boldsymbol{\phi}_2(t)\\
&+\boldsymbol{\phi}_1(t)^T\frac{\mathbf{f}(S_t\mathbf{x})\mathbf{f}(S_t\mathbf{x})^T(D\mathbf{f}(S_t\mathbf{x})+D\mathbf{f}(S_t\mathbf{x})^T)\mathbf{f}(S_t\mathbf{x})\mathbf{f}(S_t\mathbf{x})^T}{\|\mathbf{f}(S_t\mathbf{x})\|^4}M(S_t\mathbf{x})P_{S_t\mathbf{x}}\boldsymbol{\phi}_2(t)\\
&+\boldsymbol{\phi}_1(t)^T P_{S_t\mathbf{x}}^T M'(S_t\mathbf{x})P_{S_t\mathbf{x}}\boldsymbol{\phi}_2(t)\\
&-\boldsymbol{\phi}_1(t)^T P_{S_t\mathbf{x}}^T M(S_t\mathbf{x})\frac{D\mathbf{f}(S_t\mathbf{x})\mathbf{f}(S_t\mathbf{x})\mathbf{f}(S_t\mathbf{x})^T+\mathbf{f}(S_t\mathbf{x})\mathbf{f}(S_t\mathbf{x})^T D\mathbf{f}(S_t\mathbf{x})^T}{\|\mathbf{f}(S_t\mathbf{x})\|^2}\boldsymbol{\phi}_2(t)\\
&+\boldsymbol{\phi}_1(t)^T P_{S_t\mathbf{x}}^T M(S_t\mathbf{x})\frac{\mathbf{f}(S_t\mathbf{x})\mathbf{f}(S_t\mathbf{x})^T(D\mathbf{f}(S_t\mathbf{x})+D\mathbf{f}(S_t\mathbf{x})^T)\mathbf{f}(S_t\mathbf{x})\mathbf{f}(S_t\mathbf{x})^T}{\|\mathbf{f}(S_t\mathbf{x})\|^4}\boldsymbol{\phi}_2(t)\\
&+\boldsymbol{\phi}_1(t)^T P_{S_t\mathbf{x}}^T M(S_t\mathbf{x})\left(I-\frac{\mathbf{f}(S_t\mathbf{x})\mathbf{f}(S_t\mathbf{x})^T}{\|\mathbf{f}(S_t\mathbf{x})\|^2}\right)D\mathbf{f}(S_t\mathbf{x})\boldsymbol{\phi}_2(t)\\
=\;& \boldsymbol{\phi}_1(t)^T\left(I-\frac{\mathbf{f}(S_t\mathbf{x})\mathbf{f}(S_t\mathbf{x})^T}{\|\mathbf{f}(S_t\mathbf{x})\|^2}\right)\Bigg[M'(S_t\mathbf{x})+D\mathbf{f}(S_t\mathbf{x})^T M(S_t\mathbf{x})+M(S_t\mathbf{x})D\mathbf{f}(S_t\mathbf{x})\\
&-\frac{M(S_t\mathbf{x})\mathbf{f}(S_t\mathbf{x})\mathbf{f}(S_t\mathbf{x})^T(D\mathbf{f}(S_t\mathbf{x})+D\mathbf{f}(S_t\mathbf{x})^T)}{\|\mathbf{f}(S_t\mathbf{x})\|^2}\\
&-\frac{(D\mathbf{f}(S_t\mathbf{x})+D\mathbf{f}(S_t\mathbf{x})^T)\mathbf{f}(S_t\mathbf{x})\mathbf{f}(S_t\mathbf{x})^T M(S_t\mathbf{x})}{\|\mathbf{f}(S_t\mathbf{x})\|^2}\Bigg]\left(I-\frac{\mathbf{f}(S_t\mathbf{x})\mathbf{f}(S_t\mathbf{x})^T}{\|\mathbf{f}(S_t\mathbf{x})\|^2}\right)\boldsymbol{\phi}_2(t)
\end{aligned}
$$

which can be verified in a straight-forward calculation.

For the second statement we calculate

$$
\begin{aligned}
&\frac{d}{dt}\left[\frac{\mathbf{f}(S_t\mathbf{x})^T}{\|\mathbf{f}(S_t\mathbf{x})\|^2}M(S_t\mathbf{x})\frac{\mathbf{f}(S_t\mathbf{x})}{\|\mathbf{f}(S_t\mathbf{x})\|^2}\right]\\
=\;& \frac{\mathbf{f}(S_t\mathbf{x})^T D\mathbf{f}(S_t\mathbf{x})^T}{\|\mathbf{f}(S_t\mathbf{x})\|^2}M(S_t\mathbf{x})\frac{\mathbf{f}(S_t\mathbf{x})}{\|\mathbf{f}(S_t\mathbf{x})\|^2}\\
&-\frac{\mathbf{f}(S_t\mathbf{x})^T(D\mathbf{f}(S_t\mathbf{x})+D\mathbf{f}(S_t\mathbf{x})^T)\mathbf{f}(S_t\mathbf{x})\mathbf{f}(S_t\mathbf{x})^T}{\|\mathbf{f}(S_t\mathbf{x})\|^4}M(S_t\mathbf{x})\frac{\mathbf{f}(S_t\mathbf{x})}{\|\mathbf{f}(S_t\mathbf{x})\|^2}\\
&+\frac{\mathbf{f}(S_t\mathbf{x})^T}{\|\mathbf{f}(S_t\mathbf{x})\|^2}M'(S_t\mathbf{x})\frac{\mathbf{f}(S_t\mathbf{x})}{\|\mathbf{f}(S_t\mathbf{x})\|^2}\\
&+\frac{\mathbf{f}(S_t\mathbf{x})^T}{\|\mathbf{f}(S_t\mathbf{x})\|^2}M(S_t\mathbf{x})\frac{D\mathbf{f}(S_t\mathbf{x})\mathbf{f}(S_t\mathbf{x})}{\|\mathbf{f}(S_t\mathbf{x})\|^2}\\
&-\frac{\mathbf{f}(S_t\mathbf{x})^T}{\|\mathbf{f}(S_t\mathbf{x})\|^2}M(S_t\mathbf{x})\frac{\mathbf{f}(S_t\mathbf{x})\mathbf{f}(S_t\mathbf{x})^T(D\mathbf{f}(S_t\mathbf{x})+D\mathbf{f}(S_t\mathbf{x})^T)\mathbf{f}(S_t\mathbf{x})}{\|\mathbf{f}(S_t\mathbf{x})\|^4}
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{\mathbf{f}(S_t\mathbf{x})^T}{\|\mathbf{f}(S_t\mathbf{x})\|^2}\Bigg[M'(S_t\mathbf{x}) + D\mathbf{f}(S_t\mathbf{x})^T M(S_t\mathbf{x}) + M(S_t\mathbf{x})D\mathbf{f}(S_t\mathbf{x}) \\
& -\frac{M(S_t\mathbf{x})\mathbf{f}(S_t\mathbf{x})\mathbf{f}(S_t\mathbf{x})^T(D\mathbf{f}(S_t\mathbf{x}) + D\mathbf{f}(S_t\mathbf{x})^T)}{\|\mathbf{f}(S_t\mathbf{x})\|^2} \\
& -\frac{(D\mathbf{f}(S_t\mathbf{x}) + D\mathbf{f}(S_t\mathbf{x})^T)\mathbf{f}(S_t\mathbf{x})\mathbf{f}(S_t\mathbf{x})^T M(S_t\mathbf{x})}{\|\mathbf{f}(S_t\mathbf{x})\|^2}\Bigg]\frac{\mathbf{f}(S_t\mathbf{x})}{\|\mathbf{f}(S_t\mathbf{x})\|^2}
\end{aligned}
$$

which can be verified in a straight-forward calculation. The last statements can be proven in a similar way as the previous two. □

Theorem 4.2 *The solution M of* (3.1) *and* (3.2)*, see Theorem 3.1, is unique in $A(\Omega)$.*

Proof: Let M_1 and M_2 be two solutions of (3.1) and (3.2). Let $\mathbf{x} \in A(\Omega)$ and let $\Phi(t, 0; \mathbf{x})$ be the principal fundamental matrix solution of $\dot{\boldsymbol{\phi}}(t) = D\mathbf{f}(S_t\mathbf{x})\boldsymbol{\phi}(t)$ with $\Phi(0, 0; \mathbf{x}) = I$. We want to show that $\mathbf{u}_1^T[M_1(\mathbf{x}) - M_2(\mathbf{x})]\mathbf{u}_2 = 0$ for all $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{R}^n$. We write

$$\mathbf{u}_i = P_{\mathbf{x}}\mathbf{u}_i + c_i\frac{\mathbf{f}(\mathbf{x})}{\|\mathbf{f}(\mathbf{x})\|^2}$$

with $c_i = \mathbf{f}(\mathbf{x})^T\mathbf{u}_i$. Then we have, denoting $\boldsymbol{\phi}_0(t) = \frac{\mathbf{f}(S_t\mathbf{x})}{\|\mathbf{f}(S_t\mathbf{x})\|^2}$,

$$
\begin{aligned}
& \mathbf{u}_1^T[M_1(\mathbf{x}) - M_2(\mathbf{x})]\mathbf{u}_2 \\
& \quad = \mathbf{u}_1^T P_{\mathbf{x}}^T[M_1(\mathbf{x}) - M_2(\mathbf{x})]P_{\mathbf{x}}\mathbf{u}_2 + c_1\boldsymbol{\phi}_0(0)^T[M_1(\mathbf{x}) - M_2(\mathbf{x})]P_{\mathbf{x}}\mathbf{u}_2 \\
& \qquad + c_2\mathbf{u}_1^T P_{\mathbf{x}}^T[M_1(\mathbf{x}) - M_2(\mathbf{x})]\boldsymbol{\phi}_0(0) + c_1c_2\boldsymbol{\phi}_0(0)^T[M_1(\mathbf{x}) - M_2(\mathbf{x})]\boldsymbol{\phi}_0(0). \quad (4.5)
\end{aligned}
$$

We will show that each term in (4.5) is zero.

From (4.1) of Lemma 4.1 we have for $i = 1, 2$

$$
\begin{aligned}
\frac{d}{dt}\left[\Phi(t, 0; \mathbf{x})^T P_{S_t\mathbf{x}}^T M_i(S_t\mathbf{x})P_{S_t\mathbf{x}}\Phi(t, 0; \mathbf{x})\right] &= \Phi(t, 0; \mathbf{x})^T P_{S_t\mathbf{x}}^T LM_i(S_t\mathbf{x})P_{S_t\mathbf{x}}\Phi(t, 0; \mathbf{x}) \\
&= -\Phi(t, 0; \mathbf{x})^T P_{S_t\mathbf{x}}^T C(S_t\mathbf{x})P_{S_t\mathbf{x}}\Phi(t, 0; \mathbf{x}).
\end{aligned}
$$

Hence, by subtracting the equations for $M_1 - M_2$ we obtain

$$\frac{d}{dt}\left[\Phi(t, 0; \mathbf{x})^T P_{S_t\mathbf{x}}^T[M_1(S_t\mathbf{x}) - M_2(S_t\mathbf{x})]P_{S_t\mathbf{x}}\Phi(t, 0; \mathbf{x})\right] = 0$$

and by integrating

$$
\begin{aligned}
\|P_{\mathbf{x}}^T[M_1(\mathbf{x}) - M_2(\mathbf{x})]P_{\mathbf{x}}\| &= \left\|\Phi(t, 0; \mathbf{x})^T P_{S_t\mathbf{x}}^T[M_1(S_t\mathbf{x}) - M_2(S_t\mathbf{x})]P_{S_t\mathbf{x}}\Phi(t, 0; \mathbf{x})\right\| \\
&\leq \|P_{S_t\mathbf{x}}\Phi(t, 0; \mathbf{x})\|^2\|M_1(S_t\mathbf{x}) - M_2(S_t\mathbf{x})\| \\
&\to 0
\end{aligned}
$$

as $t \to \infty$ since M_i are continuous, $\overline{\bigcup_{t\geq 0} S_t\mathbf{x}}$ is compact and $\|P_{S_t\mathbf{x}}\Phi(t, 0; \mathbf{x})\|$ is exponentially decreasing to zero by Lemma 3.4 (note that there exists $T_0 \geq 0$ such that for all $t \geq T_0$ we have $S_t\mathbf{x} \in U$ as $\mathbf{x} \in A(\Omega)$).

Similarly, using (4.3) and (4.4), we have $P_{\mathbf{x}}^T[M_1(\mathbf{x}) - M_2(\mathbf{x})]\boldsymbol{\phi}_0(0) = \mathbf{0}$ as well as $\boldsymbol{\phi}_0(0)^T[M_1(\mathbf{x}) - M_2(\mathbf{x})]P_{\mathbf{x}} = \mathbf{0}$. This shows that the first three terms of (4.5) are zero.

For the last term of (4.5) we have for M_i satisfying $LM_i(\mathbf{x}) = -C(\mathbf{x})$ by (4.2) of Lemma 4.1

$$\begin{aligned}\frac{d}{dt}\left[\boldsymbol{\phi}_0(t)^T M_i(S_t\mathbf{x})\boldsymbol{\phi}_0(t)\right] &= \boldsymbol{\phi}_0(t)^T LM_i(S_t\mathbf{x})\boldsymbol{\phi}_0(t) \\ &= -\boldsymbol{\phi}_0(t)^T C(S_t\mathbf{x})\boldsymbol{\phi}_0(t) \\ &= -\boldsymbol{\phi}_0(t)^T P_{S_t\mathbf{x}}^T B(S_t\mathbf{x}) P_{S_t\mathbf{x}}\boldsymbol{\phi}_0(t) \\ &= 0\end{aligned}$$

since $P_{S_t\mathbf{x}}\boldsymbol{\phi}_0(t) = P_{S_t\mathbf{x}}\frac{\mathbf{f}(S_t\mathbf{x})}{\|\mathbf{f}(S_t\mathbf{x})\|^2} = \mathbf{0}$. Hence, $\boldsymbol{\phi}_0(t)^T M_i(S_t\mathbf{x})\boldsymbol{\phi}_0(t)$ is constant along trajectories. Restricting ourselves to the periodic orbit, this means in particular that there are constants μ_1, μ_2 such that $\frac{\mathbf{f}(\mathbf{p})^T}{\|\mathbf{f}(\mathbf{p})\|^2}M_i(\mathbf{p})\frac{\mathbf{f}(\mathbf{p})}{\|\mathbf{f}(\mathbf{p})\|^2} = \mu_i$ for all $\mathbf{p} \in \Omega$, $i = 1, 2$.

Since $\operatorname{dist}(S_t\mathbf{x}, \Omega) \to 0$ as $t \to \infty$, we have for a general $\mathbf{x} \in A(\Omega)$

$$\frac{\mathbf{f}(\mathbf{x})^T}{\|\mathbf{f}(\mathbf{x})\|^2}M_i(\mathbf{x})\frac{\mathbf{f}(\mathbf{x})}{\|\mathbf{f}(\mathbf{x})\|^2} = \frac{\mathbf{f}(S_t\mathbf{x})^T}{\|\mathbf{f}(S_t\mathbf{x})\|^2}M_i(S_t\mathbf{x})\frac{\mathbf{f}(S_t\mathbf{x})}{\|\mathbf{f}(S_t\mathbf{x})\|^2} \longrightarrow \mu_i$$

as $t \to \infty$. By (3.2), for $\mathbf{x} = \mathbf{x}_0$ we have for $i = 1, 2$

$$\mu_i = \frac{\mathbf{f}(\mathbf{x}_0)^T}{\|\mathbf{f}(\mathbf{x}_0)\|^2}M_i(\mathbf{x}_0)\frac{\mathbf{f}(\mathbf{x}_0)}{\|\mathbf{f}(\mathbf{x}_0)\|^2} = c_0$$

and thus $\mu_1 = \mu_2$. This means that

$$\boldsymbol{\phi}_0(\mathbf{x})^T[M_1(\mathbf{x}) - M_2(\mathbf{x})]\boldsymbol{\phi}_0(\mathbf{x}) = \mu_1 - \mu_2 = 0$$

for all $\mathbf{x} \in A(\Omega)$ and shows that the last term in (4.5) is zero. □

5 Examples

As a simple example, consider the ODE

$$\begin{cases}\dot{x} &= 1 \\ \dot{y} &= -y\end{cases} \tag{5.1}$$

which has the exponentially stable solution $(x(t), y(t)) = (t, 0)$. Note that this example does not have a periodic orbit, so Theorem 2.1 cannot be applied. However, it could be mapped to a system with an exponentially stable periodic orbit at the unit circle using $T\colon \mathbb{R} \times (-1, \infty) \to \mathbb{R}^2 \setminus \{(0,0)\}$, where $T(x, y) = (1 + y)\begin{pmatrix}\cos x \\ \sin x\end{pmatrix}$.

We will compute the solution $M(x, y)$ as in Theorem 3.1. The solution of (5.1) is $S_t(x, y) = (x + t, ye^{-t})$. We have $D\mathbf{f}(x, y) = \begin{pmatrix}0 & 0 \\ 0 & -1\end{pmatrix}$ and the principal fundamental matrix solution of $\dot{\phi}(t) = D\mathbf{f}(S_t\mathbf{x})\phi(t)$ with $\Phi(0,0;x,y) = I$ is given by

$$\Phi(t,0;x,y) = \begin{pmatrix} 1 & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

The projection is

$$P_{\mathbf{x}} = I - \frac{1}{1+y^2}\begin{pmatrix} 1 & -y \\ -y & y^2 \end{pmatrix} = \frac{1}{1+y^2}\begin{pmatrix} y^2 & y \\ y & 1 \end{pmatrix}.$$

To compute the metric with $B(\mathbf{x}) = I$ and $c_0 > 0$, we have with the formula given in Theorem 3.1, noting that $P_{\mathbf{x}}^T P_{\mathbf{x}} = P_{\mathbf{x}}$,

$$\begin{aligned}
M(x,y) &= \int_0^\infty \begin{pmatrix} 1 & 0 \\ 0 & e^{-t} \end{pmatrix} \frac{1}{1+y^2e^{-2t}} \begin{pmatrix} y^2e^{-2t} & ye^{-t} \\ ye^{-t} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{-t} \end{pmatrix} dt \\
&\quad + c_0 \begin{pmatrix} 1 & -y \\ -y & y^2 \end{pmatrix} \\
&= \int_0^\infty \frac{e^{-2t}}{1+y^2e^{-2t}} \begin{pmatrix} y^2 & y \\ y & 1 \end{pmatrix} dt + c_0 \begin{pmatrix} 1 & -y \\ -y & y^2 \end{pmatrix} \\
&= -\frac{1}{2}\ln\left(1+y^2e^{-2t}\right)\Big|_0^\infty \begin{pmatrix} 1 & \frac{1}{y} \\ \frac{1}{y} & \frac{1}{y^2} \end{pmatrix} + c_0 \begin{pmatrix} 1 & -y \\ -y & y^2 \end{pmatrix} \\
&= \frac{1}{2}\ln(1+y^2) \begin{pmatrix} 1 & \frac{1}{y} \\ \frac{1}{y} & \frac{1}{y^2} \end{pmatrix} + c_0 \begin{pmatrix} 1 & -y \\ -y & y^2 \end{pmatrix}.
\end{aligned}$$

Note that due to $\ln(1+y^2) = y^2 - \frac{y^4}{2} \pm \ldots$, the metric $M(x,y)$ is smooth and satisfies $M(x,0) = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + c_0 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, hence $M(x,0)$ is positive definite for all $x \in \mathbb{R}$. For $y \neq 0$ we have

$$\begin{aligned}
\det M(x,y) &= \frac{c_0}{2}\ln(1+y^2)\left(y + \frac{1}{y}\right)^2 > 0 \\
\operatorname{tr} M(x,y) &= \left[\frac{1}{2}\ln(1+y^2) + c_0 y^2\right]\left(1 + \frac{1}{y^2}\right) > 0,
\end{aligned}$$

which shows that $M(x,y)$ is positive definite for $y \neq 0$. It is a straight-forward calculation to check that $M(x,y)$ satisfies (3.1) and (3.2) for any $\mathbf{x}_0 \in \mathbb{R}^2$.

6 Conclusions

In this paper we have presented a matrix-valued PDE with a given value at one point; we have shown existence and uniqueness of a solution, we have established that the solution is of a specific form and that it is a positive definite matrix at each point.

In particular, this shows that the solution is a contraction metric, which implies the existence, uniqueness and exponential stability of a periodic orbit, and determines its basin of attraction. We have thus shown a converse theorem on the existence of a contraction metric for periodic orbits.

By characterizing the contraction metric as solution of a PDE, numerical methods can be employed for its explicit construction. For example, mesh-free collocation can be used to solve this linear matrix-valued PDE [9], and error estimates are available. A contraction metric is required to satisfy inequalities, and thus a sufficiently close numerical approximation to the solution of the PDE is a contraction metric, see [7].

A Gronwall

We cite the following lemma from [17, Lemma D.2].

Lemma A.1 *Let $r, K, a \in L^1_{loc}([0,\infty),\mathbb{R})$ be nonnegative functions and let $b \in L^\infty_{loc}([0,\infty),\mathbb{R})$ be a continuous nonnegative function such that*

$$r(\theta) \leq a(\theta) + K(\theta)\int_0^\theta b(s)r(s)\,ds$$

holds for almost all $\theta \geq 0$.

Then

$$r(\theta) \leq a(\theta) + K(\theta)\int_0^\theta a(s)b(s)\,ds \cdot \exp\left(\int_0^\theta K(s)b(s)\,ds\right)$$

holds for almost all $\theta \geq 0$.

B Proof of Theorem 2.1

In this section we give a sketch of the proof of Theorem 2.1. It is very similar to the proof of [5, Theorem 5], which considers adjacent solutions in direction $\mathbf{v}$ with $\mathbf{v}^T M(\mathbf{x})\mathbf{f}(\mathbf{x}) = 0$ while we consider $\mathbf{v}$ with $\mathbf{v}^T\mathbf{f}(\mathbf{x}) = 0$. Note that a similar result as in Theorem 2.1 with $M(\mathbf{x}) = I$, so the Euclidean metric, has been proven in [1].

We now use the notations as in [5, Theorem 5] and only highlight the necessary adaptations; all references are with respect to the proof of [5, Theorem 5]. In [5, Proposition 7], which defines the time synchronization $T_p^{p+\eta}$ of the solutions $S_\theta p$ and $S_{T_p^{p+\eta}(\theta)}(p+\eta)$, we replace the first equation by

$$\left(S_{T_p^{p+\eta}(\theta)}(p+\eta) - S_\theta p\right)^T f(S_\theta p) = 0.$$

In the proof we replace (12) by

$$\|Df(p) - Df(p+\xi)\| \leq C_1 := \frac{\lambda_m}{\lambda_M\left(1 + 2\frac{f_M^2}{f_m^2}\right)}\frac{k\nu}{2} \tag{B.1}$$

and (14) by

$$\delta' \quad := \quad \min\left(\delta_1, \frac{\sqrt{\lambda_m}}{f_M f_D}\frac{f_m^2}{2}, \frac{\lambda_m^{3/2} f_m^2}{4\lambda_M f_M f_D^2\left(1+\frac{f_M^2}{f_m^2}\right)}\frac{k\nu}{2}\right); \tag{B.2}$$

see [5] for the definition of the constants. (16) is replaced by

$$Q(T,\theta,\eta) = (S_T(p+\eta) - S_\theta p)^T f(S_\theta p) = 0.$$

We define

$$A(\theta) := \sqrt{(S_{T(\theta)}(p+\eta) - S_\theta p)^T M(S_\theta p)(S_{T(\theta)}(p+\eta) - S_\theta p)}$$

and define $v(\theta)$ by

$$A(\theta)v(\theta) = S_{T(\theta)}(p+\eta) - S_\theta p.$$

We replace (18) and (19) by

$$\begin{aligned}
\partial_\theta Q(T,\theta,\eta) &= -\|f(S_\theta p)\|^2 + A(\theta)v(\theta)^T Df(S_\theta p) f(S_\theta p) \\
\partial_T Q(T,\theta,\eta) &= f(S_T(p+\eta)) f(S_\theta p) \\
&= \|f(S_\theta p)\|^2 + A(\theta)\left(\int_0^1 Df(S_\theta p + \lambda A(\theta)v(\theta))\, d\lambda v(\theta)\right)^T f(S_\theta p)
\end{aligned}$$

$$\dot{T}(\theta) \quad = \quad 1 - A(\theta)\frac{\left(\left(\int_0^1 Df(S_\theta p + \lambda A(\theta)v(\theta))\, d\lambda v(\theta)\right)^T + v(\theta)^T Df(S_\theta p)\right) f(S_\theta p)}{\|f(S_\theta p)\|^2 + A(\theta)\left(\int_0^1 Df(S_\theta p + \lambda A(\theta)v(\theta))\, d\lambda v(\theta)\right)^T f(S_\theta p)}$$

The first equation in part III becomes thus

$$\begin{aligned}
\frac{d}{d\theta}A^2(\theta) &= (S_{T(\theta)}(p+\eta)-S_\theta p)^T M'(S_\theta p)(S_{T(\theta)}(p+\eta)-S_\theta p) \\
&\quad +2(S_{T(\theta)}(p+\eta)-S_\theta p)^T M(S_\theta p)\left(f(S_{T(\theta)}(p+\eta))\dot{T}(\theta)-f(S_\theta p)\right) \\
&= (S_{T(\theta)}(p+\eta)-S_\theta p)^T M'(S_\theta p)(S_{T(\theta)}(p+\eta)-S_\theta p) \\
&\quad +2(S_{T(\theta)}(p+\eta)-S_\theta p)^T M(S_\theta p)\cdot\Big([f(S_{T(\theta)}(p+\eta))-f(S_\theta p)] \\
&\quad +(\dot{T}(\theta)-1)f(S_\theta p)+(\dot{T}(\theta)-1)[f(S_{T(\theta)}(p+\eta))-f(S_\theta p)]\Big) \\
&= A^2(\theta)v(\theta)^T M'(S_\theta p)v(\theta) \\
&\quad +2A^2(\theta)v(\theta)^T M(S_\theta p)\int_0^1 Df(S_\theta p+\lambda A(\theta)v(\theta))\,d\lambda v(\theta) \\
&\quad -2A^2(\theta)v(\theta)^T M(S_\theta p)f(S_\theta p)\cdot \\
&\quad \frac{\left(\left(\int_0^1 Df(S_\theta p+\lambda A(\theta)v(\theta))\,d\lambda v(\theta)\right)^T+v(\theta)^T Df(S_\theta p)\right)f(S_\theta p)}{\|f(S_\theta p)\|^2+A(\theta)\left(\int_0^1 Df(S_\theta p+\lambda A(\theta)v(\theta))\,d\lambda v(\theta)\right)^T f(S_\theta p)} \\
&\quad -2A^3(\theta)v(\theta)^T M(S_\theta p)\int_0^1 Df(S_\theta p+\lambda A(\theta)v(\theta))\,d\lambda v(\theta) \\
&\quad \cdot\frac{\left(\left(\int_0^1 Df(S_\theta p+\lambda A(\theta)v(\theta))\,d\lambda v(\theta)\right)^T+v(\theta)^T Df(S_\theta p)\right)f(S_\theta p)}{\|f(S_\theta p)\|^2+A(\theta)\left(\int_0^1 Df(S_\theta p+\lambda A(\theta)v(\theta))\,d\lambda v(\theta)\right)^T f(S_\theta p)}.
\end{aligned}$$

To obtain a bound on the denominator of the last terms we use $A(\theta)\frac{f_M f_D}{\sqrt{\lambda_m}} \leq \frac{1}{2}f_m^2$ by (B.2). Also, using $\int_0^1 Df(S_\theta p+\lambda A(\theta)v(\theta))\,d\lambda = Df(S_\theta p)+\int_0^1[Df(S_\theta p+\lambda A(\theta)v(\theta))-Df(S_\theta p)]\,d\lambda$, we obtain with (B.1) and $\|v(\theta)\|\leq\frac{1}{\sqrt{\lambda_m}}$

$$\begin{aligned}
\frac{d}{d\theta}A^2(\theta) &\leq A^2(\theta)v(\theta)^T M'(S_\theta p)v(\theta) \\
&\quad +2A^2(\theta)v(\theta)^T M(S_\theta p)Df(S_\theta p)v(\theta)+2A^2(\theta)C_1\frac{\lambda_M}{\lambda_m} \\
&\quad -2A^2(\theta)v(\theta)^T M(S_\theta p)f(S_\theta p)\cdot \\
&\quad \frac{v(\theta)^T[Df(S_\theta p)^T+Df(S_\theta p)]f(S_\theta p)}{\|f(S_\theta p)\|^2+A(\theta)\left(\int_0^1 Df(S_\theta p+\lambda A(\theta)v(\theta))\,d\lambda v(\theta)\right)^T f(S_\theta p)} \\
&\quad +4A^2(\theta)\frac{\lambda_M f_M^2 C_1}{f_m^2\lambda_m}+8A^3(\theta)\frac{\lambda_M f_D^2 f_M}{f_m^2\lambda_m^{3/2}}.
\end{aligned}$$

Note that $\frac{1}{b+c}=\frac{1}{b}-\frac{c}{b(b+c)}$ holds for all $b, b+c>0$. Using this with $b=\|f(S_\theta p)\|^2$

and $c = A(\theta)\left(\int_0^1 Df(S_\theta p + \lambda A(\theta)v(\theta))\,d\lambda v(\theta)\right)^T f(S_\theta p)$, we have

$$\begin{aligned}
\frac{d}{d\theta}A^2(\theta) &\leq 2A^2(\theta)L_M(S_\theta p)\\
&\quad +2A^2(\theta)C_1\frac{\lambda_M}{\lambda_m} + 8A^3(\theta)\frac{\lambda_M f_D^2 f_M^3}{f_m^4\lambda_m^{3/2}}\\
&\quad +4A^2(\theta)\frac{\lambda_M f_M^2 C_1}{f_m^2\lambda_m} + 8A^3(\theta)\frac{\lambda_M f_D^2 f_M}{f_m^2\lambda_m^{3/2}}\\
&\leq -2A^2(\theta)\nu + 2A^2(\theta)C_1\frac{\lambda_M}{\lambda_m}\left(1+2\frac{f_M^2}{f_m^2}\right)\\
&\quad +8A^3(\theta)\frac{\lambda_M}{\lambda_m^{3/2}}\frac{f_M f_D^2}{f_m^2}\left(1+\frac{f_M^2}{f_m^2}\right)\\
&\leq 2A^2(\theta)\left[-\nu + \frac{k\nu}{2} + \frac{k\nu}{2}\right]\\
&= -2(1-k)\nu A^2(\theta)
\end{aligned}$$

using (B.1) and $4A(\theta)\frac{\lambda_M}{\lambda_m^{3/2}}\frac{f_M f_D^2}{f_m^2}\left(1+\frac{f_M^2}{f_m^2}\right) \leq \frac{k\nu}{2}$ because of (B.2). The rest of the proof is as in [5].

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