Analytic moments for GJR-GARCH (1,1) processes

Article (Other)

Analytic Moments for GJR-GARCH(1,1) Processes

TECHNICAL APPENDICES

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It is assumed that the financial return r_t follows a GJR process with the s-step-ahead conditional variance given by:

$$h_{t+s} = \omega + \left(\alpha + \lambda I^{-}_{t+s-1}\right)\varepsilon^2_{t+s-1} + \beta h_{t+s-1} \tag{1}$$

The aim of this appendix is to calculate the conditional moments of the forward return r_{t+s} and of its conditional variance h_{t+s}, as well as of the aggregated return and of its aggregated conditional variance. Specifically, for $i = 1$, 2, 3 and 4, and for $x = r$ and h, we compute:

$$\tilde{\mu}^{(i)}_{x,s} = E_t\left(x^i_{t+s}\right), \quad \mu^{(i)}_{x,s} = E_t\left(\left(x_{t+s} - \tilde{\mu}^{(1)}_{x,s}\right)^i\right),$$

$$\tilde{M}^{(i)}_{x,n} = E_t\left[\left(\sum_{s=1}^{n} x_{t+s}\right)^i\right], \quad M^{(i)}_{x,n} = E_t\left(\left(\sum_{s=1}^{n}\left(x_{t+s} - \tilde{\mu}^{(1)}_{x,s}\right)\right)^i\right),$$

and the corresponding standardized moments:

$$\tau_{x,s} = \frac{\mu^{(3)}_{x,s}}{\left(\mu^{(2)}_{x,s}\right)^{3/2}}, \quad \mathrm{T}_{x,n} = \frac{M^{(3)}_{x,n}}{\left(M^{(2)}_{x,n}\right)^{3/2}}, \quad \kappa_{x,s} = \frac{\mu^{(4)}_{x,s}}{\left(\mu^{(2)}_{x,s}\right)^2}, \quad \mathrm{K}_{x,n} = \frac{M^{(4)}_{x,n}}{\left(M^{(2)}_{x,n}\right)^2}.$$

We focus on calculating the un-centred conditional moments; the centred moments will follow using simple formulae. We denote by F_0 the distribution function for $D(0,1)$, evaluated at zero, and set:

$$\varphi = \alpha + \lambda F_0 + \beta \tag{2}$$

and $\bar{h} = \omega(1-\varphi)^{-1}$. For both the normal and the standardized Student t, $F_0 = \frac{1}{2}$, since the two distributions are symmetric, thus for these two special cases φ becomes:

$$\varphi = \alpha + \frac{\lambda}{2} + \beta \tag{3}$$

For $s, u, v, w > 0$, we define $\tilde{\mu}_{h,suvw} = E_t\left(h_{t+s}h_{t+s+u}h_{t+s+u+v}h_{t+s+u+v+w}\right)$ and:

$$\tilde{\mu}^{(i,j)}_{h,su} = E_t\left(h^i_{t+s}h^j_{t+s+u}\right), \qquad \tilde{\mu}^{(i,j,k)}_{h,suv} = E_t\left(h^i_{t+s}h^j_{t+s+u}h^k_{t+s+u+v}\right), \qquad \theta^{(j)}_{su} = E_t\left(\varepsilon_{t+s}h^j_{t+s+u}\right).$$

After deriving the formulae for the generic model, we allow the innovations distribution $D(0,1)$ to take two specific functional forms that are commonly used in practice: the standard normal and the (standardized) Student t.[4] When $D(0,1)$ is the standard normal distribution and the even moments are given by $\mu^{(2r)}_z = \prod_{i=1}^{r}(2i-1)$. When $D(0,1)$ is the standardized Student t distribution, the odd order moments are again all zero (provided the number of degrees of freedom $\nu >$ the order of the moment) and the even moments are given by: $\mu^{(2r)}_z = (\nu-2)^r \frac{\Gamma\left(r+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\nu - r\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\nu\right)}$, if $\nu > 2r$.

[4]We shall only derive the expressions for the two special cases when a particular expression derived for the generic case differs for one (or both) of the special cases; when no formulae for the special cases are mentioned, the generic formulae apply.

A1: Conditional Moments of Forward and Aggregated Returns

(a) *First Conditional Moments of Forward and Aggregated Returns*

Set $R_{tn} = \sum_{s=1}^{n} r_{t+s}$ for the aggregated return for $s \geq 1$. By the tower law of expectations:

$$E_t\left(r_{t+s}\right) = E_t\left(\mu + \varepsilon_{t+s}\right) = \mu + E_t\left(\underbrace{E_{t+s-1}\left(\varepsilon_{t+s}\right)}_{0}\right) = \mu,$$
$$E_t\left(R_{tn}\right) = E_t\left(\sum_{s=1}^{n} r_{t+s}\right) = \sum_{s=1}^{n} E_t\left(r_{t+s}\right) = n\mu.$$

(b) *Second Conditional Moments of Forward and Aggregated Returns*

The second moment of the forward return is:

$$E_t\left(r_{t+s}^2\right) = E_t\left[\left(\mu + \varepsilon_{t+s}\right)^2\right] = \mu^2 + \bar{h} + \varphi^{s-1}\left(h_{t+1} - \bar{h}\right),$$

where we used the expression for the second moment of variance $\tilde{\mu}_{h,s}^{(1)} = \mu_{r,s}^{(2)} = \bar{h} + \varphi^{s-1}\left(h_{t+1} - \bar{h}\right)$ derived in Appendix A2 below. The second moment of the aggregated return is:

$$E_t\left(R_{tn}^2\right) = E_t\left(\sum_{s=1}^{n} r_{t+s}\right)^2 = E_t\left(\sum_{s=1}^{n} r_{t+s}^2 + 2\sum_{s=1}^{n}\sum_{u=1}^{n-s} r_{t+s} r_{t+s+u}\right). \qquad \text{But}$$
$$E_t\left(\sum_{s=1}^{n} r_{t+s}^2\right) = \sum_{s=1}^{n}\left(\mu^2 + \bar{h} + \varphi^{s-1}\left(h_{t+1} - \bar{h}\right)\right) = n\left(\mu^2 + \bar{h}\right) + \left(h_{t+1} - \bar{h}\right)\left(1 - \varphi\right)^{-1}\left(1 - \varphi^n\right),$$
$$E_t\left(\sum_{s=1}^{n}\sum_{u=1}^{n-s} r_{t+s} r_{t+s+u}\right) = \sum_{s=1}^{n}\sum_{u=1}^{n-s} E_t\left(r_{t+s} r_{t+s+u}\right) = \sum_{s=1}^{n}\sum_{u=1}^{n-s} E_t\left(\left(\mu + \varepsilon_{t+s}\right)\left(\mu + \varepsilon_{t+s+u}\right)\right) = {}^1\!/_2\, n\left(n - 1\right)\mu^2.$$

Hence, the expression for the second moment of aggregated returns becomes:

$$E_t\left(R_{tn}^2\right) = n^2\mu^2 + n\bar{h} + \left(h_{t+1} - \bar{h}\right)\left(1 - \varphi\right)^{-1}\left(1 - \varphi^n\right).$$

(c) *Third Conditional Moments of Forward and Aggregated Returns*

For the third moment of forward returns we write:

$$E_t\left(r_{t+s}^3\right) = E_t\left[\left(\mu + \varepsilon_{t+s}\right)^3\right] = E_t\left(\mu^3 + 3\mu^2\varepsilon_{t+s} + 3\mu\varepsilon_{t+s}^2 + \varepsilon_{t+s}^3\right)$$
$$= \mu^3 + 3\mu\tilde{\mu}_{h,s}^{(1)} + E_t\left(z_{t+s}^3 h_{t+s}^{3/2}\right) = \mu^3 + 3\mu\tilde{\mu}_{h,s}^{(1)} + \tau_z E_t\left(h_{t+s}^{3/2}\right).$$

For any GJR model with a symmetric innovations distribution the third moment of returns is:

$$E_t\left(r_{t+s}^3\right) = \mu^3 + 3\mu\tilde{\mu}_{h,s}^{(1)} = \mu^3 + 3\mu\left(\bar{h} + \varphi^{s-1}\left(h_{t+1} - \bar{h}\right)\right)$$

However, for the generic, skewed model, we need to approximate $E_t\left(h_{t+s}^{3/2}\right)$ using a second order Taylor series expansion. In general, for a smooth function $g\left(X\right)$:

$$g(X) \approx g(E_t(X)) + g'(E_t(X))(X - E_t(X)) + 1/2 g''(E_t(X))(X - E_t(X))^2.$$

Taking expectations we have: $E_t(g(X)) \approx g(E_t(X)) + 1/2 g''(E_t(X)) V_t(X)$. Setting $g(X) = X^{3/2}$, so $g'(X) = \frac{3}{2}X^{1/2}$ and $g''(X) = \frac{3}{4}X^{-1/2}$ and setting $X = h_{t+s}$ yields:

$$E_t\left(h_{t+s}^{3/2}\right) \simeq \tfrac{5}{8}\left(\tilde{\mu}_{h,s}^{(1)}\right)^{3/2} + \tfrac{3}{8}\tilde{\mu}_{h,s}^{(2)}\left(\tilde{\mu}_{h,s}^{(1)}\right)^{-1/2},$$

where the expressions for $\tilde{\mu}_{h,s}^{(1)}$ and $\tilde{\mu}_{h,s}^{(2)}$ are given in the Appendix A2 below.

We now compute the third moment of the aggregated returns:

$$\begin{aligned} E_t\left(R_{tn}^3\right) &= E_t\left(\sum_{s=1}^{n} r_{t+s}\right)^3 \\ &= \sum_{s=1}^{n} E_t\left(r_{t+s}^3\right) + 3\sum_{s=1}^{n}\sum_{u=1}^{n-s}\left[E_t\left(r_{t+s}^2 r_{t+s+u}\right) + E_t\left(r_{t+s} r_{t+s+u}^2\right)\right] \\ &+ 6\sum_{s=1}^{n}\sum_{u=1}^{n-s}\sum_{v=1}^{n-s-u} E_t\left(r_{t+s} r_{t+s+u} r_{t+s+u+v}\right) \end{aligned}$$

where

$$\begin{aligned} \sum_{s=1}^{n} E_t\left(r_{t+s}^3\right) &= n\mu\left(\mu^2 + 3\bar{h}\right) + 3\mu(1-\varphi)^{-1}\left(1-\varphi^n\right)\left(h_{t+1} - \bar{h}\right) + \tau_z \sum_{s=1}^{n} E_t\left(h_{t+s}^{3/2}\right), \\ \sum_{s=1}^{n}\sum_{u=1}^{n-s} E_t\left(r_{t+s}^2 r_{t+s+u}\right) &= \sum_{s=1}^{n}\sum_{u=1}^{n-s} E_t\left[\left(\mu^2 + 2\mu\varepsilon_{t+s} + \varepsilon_{t+s}^2\right)\left(\mu + \varepsilon_{t+s+u}\right)\right] \\ &= \sum_{s=1}^{n}\sum_{u=1}^{n-s}\left(\mu^3 + \mu\mu_{h,s}^{(1)}\right), \\ &= \mu\left[\frac{n(n-1)}{2}\left(\mu^2 + \bar{h}\right) + (1-\varphi)^{-1}\left[n - (1-\varphi)^{-1}\left(1-\varphi^n\right)\right]\left(h_{t+1} - \bar{h}\right)\right] \end{aligned}$$

and

$$\begin{aligned} \sum_{s=1}^{n}\sum_{u=1}^{n-s} E_t\left(r_{t+s} r_{t+s+u}^2\right) &= \sum_{s=1}^{n}\sum_{u=1}^{n-s} E_t\left(\left(\mu + \varepsilon_{t+s}\right)\left(\mu^2 + 2\mu\varepsilon_{t+s+u} + \varepsilon_{t+s+u}^2\right)\right) \\ &= \sum_{s=1}^{n}\sum_{u=1}^{n-s}\left(\mu^3 + \mu\tilde{\mu}_{h,s+u}^{(1)} + E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}^2\right)\right) \end{aligned}$$

and, letting f denote the density of the innovation distribution:

$$\begin{aligned} E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}^2\right) &= E_t\left(\varepsilon_{t+s}E_{t+s+u-1}\left(\varepsilon_{t+s+u}^2\right)\right) = \theta_{su}^{(1)} \\ &= E_t\left(\varepsilon_{t+s}\left(\omega + \left(\alpha + \lambda I_{t+s+u-1}\right)\varepsilon_{t+s+u-1}^2 + \beta h_{t+s+u-1}\right)\right) = \varphi E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u-1}^2\right) \\ &= \varphi^{u-1}E_t\left(\varepsilon_{t+s}h_{t+s+1}\right) = \varphi^{u-1}\left(\alpha\tau_z + \lambda\int\limits_{z=-\infty}^{0} z^3 f(z)\,dz\right)E_t\left(h_{t+s}^{3/2}\right). \end{aligned}$$

The final expression for $\sum\limits_{s=1}^{n}\sum\limits_{u=1}^{n-s}E_t\left(r_{t+s}r_{t+s+u}^2\right)$ becomes:

$$\begin{aligned} \sum_{s=1}^{n}\sum_{u=1}^{n-s}E_t\left(r_{t+s}r_{t+s+u}^2\right) &= \tfrac{n(n-1)}{2}\mu\left(\mu^2+\bar{h}\right) \\ &\quad +(1-\varphi)^{-1}\left(\begin{array}{l} \mu\left[\varphi(1-\varphi)^{-1}\left(1-\varphi^n\right) - n\varphi^n\right]\left(h_{t+1}-\bar{h}\right) \\ +\left(\alpha\tau_z + \lambda\int\limits_{z=-\infty}^{0} z^3 f(z)\,dz\right)\sum\limits_{s=1}^{n}(1-\varphi^{n-s})E_t\left(h_{t+s}^{3/2}\right) \end{array}\right) \end{aligned}$$

For the normal GJR we have $\tau_z = 0$ and it is easily shown that:

$$\int\limits_{z=-\infty}^{0} z^3 f(z)\,dz = \int\limits_{z=-\infty}^{0} \tfrac{1}{\sqrt{2\pi}} z^3 \exp\left(-\tfrac{z^2}{2}\right)dz = -\sqrt{\tfrac{2}{\pi}}.$$

Similarly, for the Student t GJR, we have $\tau_z = 0$ and easily get:

$$\int\limits_{z=-\infty}^{0} z^3 f(z) = \int\limits_{z=-\infty}^{0} z^3 \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi(\nu-2)}}\left(1+\frac{z^2}{\nu-2}\right)^{-\frac{\nu+1}{2}}dz = -\frac{2}{\sqrt{\pi}}\frac{(\nu-2)^{3/2}}{(\nu-1)(\nu-3)}\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)},$$

Finally, repeatedly applying the tower law, we have:

$$\sum_{s=1}^{n}\sum_{u=1}^{n-s}\sum_{v=1}^{n-s-u}E_t\left(r_{t+s}r_{t+s+u}r_{t+s+u+v}\right) = \sum_{s=1}^{n}\sum_{u=1}^{n-s}\sum_{v=1}^{n-s-u}\mu^3 = \frac{n(n-1)(n-2)}{6}\mu^3.$$

(d) *Fourth Conditional Moments of Forward and Aggregated Returns*

The fourth moment of the forward returns is

$$E_t\left(r_{t+s}^4\right) = E_t\left(\mu^4 + 4\mu^3\varepsilon_{t+s} + 6\mu^2\varepsilon_{t+s}^2 + 4\mu\varepsilon_{t+s}^3 + \varepsilon_{t+s}^4\right) = \mu^4 + 6\mu^2\tilde{\mu}_{h,s}^{(1)} + 4\mu\tau_z E_t\left(h_{t+s}^{3/2}\right) + \kappa_z\tilde{\mu}_{h,s}^{(2)},$$

where $\tilde{\mu}_{h,s}^{(1)}$ and $\tilde{\mu}_{h,s}^{(2)}$ are derived in the Appendix A2 below and $E_t\left(h_{t+s}^{3/2}\right)$ is given above as a function of these first two conditional moments of the forward variance. In the special case that the innovation distribution is the standard normal, $E_t\left(r_{t+s}^4\right)=\mu^4+6\mu^2\tilde{\mu}_{h,s}^{(1)}+3\tilde{\mu}_{h,s}^{(2)}$, while for the Student t GJR, $E_t\left(r_{t+s}^4\right)=\mu^4+6\mu^2\tilde{\mu}_{h,s}^{(1)}+3\frac{\nu-2}{\nu-4}\tilde{\mu}_{h,s}^{(2)}$. The fourth moment of aggregated returns is:

$$
\begin{aligned}
E_t\left(R_{tn}^4\right) &= \sum_{s=1}^{n} E_t\left(r_{t+s}^4\right)+\sum_{s=1}^{n}\sum_{u=1}^{n-s}\left[4\left(E_t\left(r_{t+s}^3 r_{t+s+u}\right)+E_t\left(r_{t+s} r_{t+s+u}^3\right)\right)+6E_t\left(r_{t+s}^2 r_{t+s+u}^2\right)\right] \\
&+ 12\sum_{s=1}^{n}\sum_{u=1}^{n-s}\sum_{v=1}^{n-s-u}\left(E_t\left(r_{t+s}^2 r_{t+s+u} r_{t+s+u+v}\right)+E_t\left(r_{t+s} r_{t+s+u}^2 r_{t+s+u+v}\right)+E_t\left(r_{t+s} r_{t+s+u} r_{t+s+u+v}^3\right)\right) \\
&+ 24\sum_{s=1}^{n}\sum_{u=1}^{n-s}\sum_{v=1}^{n-s-u}\sum_{w=1}^{n-s-u-v} E_t\left(r_{t+s} r_{t+s+u} r_{t+s+u+v} r_{t+s+u+v+w}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
\sum_{s=1}^{n} E_t\left(r_{t+s}^4\right) &= \sum_{s=1}^{n}\left(\mu^4+6\mu^2\tilde{\mu}_{h,s}^{(1)}+4\mu\tau_z E_t\left(h_{t+s}^{3/2}\right)+\kappa_z\tilde{\mu}_{h,s}^{(2)}\right) \\
&= n\mu^2\left(\mu^2+6\bar{h}\right)+6\mu^2(1-\varphi)^{-1}\left(1-\varphi^n\right)\left(h_{t+1}-\bar{h}\right)+\sum_{s=1}^{n}\left(4\mu\tau_z E_t\left(h_{t+s}^{3/2}\right)+\kappa_z\tilde{\mu}_{h,s}^{(2)}\right).
\end{aligned}
$$

For the normal GJR,

$$
\sum_{s=1}^{n} E_t\left(r_{t+s}^4\right)=n\mu^2\left(\mu^2+6\bar{h}\right)+6\mu^2(1-\varphi)^{-1}\left(1-\varphi^n\right)\left(h_{t+1}-\bar{h}\right)+3\sum_{s=1}^{n}\tilde{\mu}_{h,s}^{(2)},
$$

while for the Student t GJR the sum above becomes:

$$
\sum_{s=1}^{n} E_t\left(r_{t+s}^4\right)=n\mu^2\left(\mu^2+6\bar{h}\right)+6\mu^2(1-\varphi)^{-1}\left(1-\varphi^n\right)\left(h_{t+1}-\bar{h}\right)+3\frac{\nu-2}{\nu-4}\sum_{s=1}^{n}\tilde{\mu}_{h,s}^{(2)}.
$$

Note that

$$
\begin{aligned}
\sum_{s=1}^{n}\sum_{u=1}^{n-s} E_t\left(r_{t+s}^3 r_{t+s+u}\right) &= \sum_{s=1}^{n}\sum_{u=1}^{n-s} E_t\left(\left(\mu^3+3\left(\mu^2\varepsilon_{t+s}+\mu\varepsilon_{t+s}^2\right)+\varepsilon_{t+s}^3\right)\left(\mu+\varepsilon_{t+s+u}\right)\right) \\
&= \frac{n\left(n-1\right)}{2}\mu^2\left(\mu^2+3\bar{h}\right)+3\mu^2(1-\varphi)^{-1}\left[n-(1-\varphi)^{-1}\left(1-\varphi^n\right)\right]\left(h_{t+1}-\bar{h}\right) \\
&+ \mu\tau_z\sum_{s=1}^{n}\left(n-s\right)E_t\left(h_{t+s}^{3/2}\right),
\end{aligned}
$$

which for the normal and Student t GJR models becomes:

$$\sum_{s=1}^{n}\sum_{u=1}^{n-s} E_t\left(r_{t+s}^3 r_{t+s+u}\right) = \frac{n\,(n-1)}{2}\mu^2\left(\mu^2+3\bar{h}\right)+3\mu^2(1-\varphi)^{-1}\left[n-(1-\varphi)^{-1}\left(1-\varphi^n\right)\right]\left(h_{t+1}-\bar{h}\right),$$

$$\begin{aligned}\sum_{s=1}^{n}\sum_{u=1}^{n-s} E_t\left(r_{t+s} r_{t+s+u}^3\right) &= \sum_{s=1}^{n}\sum_{u=1}^{n-s} E_t\left(\left(\mu+\varepsilon_{t+s}\right)\left(\mu^3+3\left(\mu^2\varepsilon_{t+s+u}+\mu\varepsilon_{t+s+u}^2\right)+\varepsilon_{t+s+u}^3\right)\right)\\ &= \sum_{s=1}^{n}\sum_{u=1}^{n-s}\left(\mu^4+3\,\mu^2\tilde{\mu}_{h,s+u}^{(1)}+\mu\tau_z E_t\left(h_{t+s+u}^{3/2}\right)+3\mu E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}^2\right)+E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}^3\right)\right).\end{aligned}$$

Now, $E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}^3\right)=E_t\left(\varepsilon_{t+s}E_{t+s+u-1}\left(z_{t+s+u}^3 h_{t+s+u}^{3/2}\right)\right)=\tau_z E_t\left(\varepsilon_{t+s}h_{t+s+u}^{3/2}\right)=\tau_z\theta_{su}^{(3/2)}$, which for the normal and Student t GJR models reduces to $E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}^3\right)=0$, since $\tau_z=0$.[5] To solve for $\theta_{su}^{(3/2)}$ in the general case, we again use a second order Taylor expansion around $\tilde{\mu}_{h,s+u}^{(1)}$, obtaining:

$$h_{t+s+u}^{3/2}\simeq\left(\tilde{\mu}_{h,s+u}^{(1)}\right)^{3/2}+\frac{3}{2}\left(\tilde{\mu}_{h,s+u}^{(1)}\right)^{1/2}\left(h_{t+s+u}-\tilde{\mu}_{h,s+u}^{(1)}\right)+\frac{3}{8}\left(\tilde{\mu}_{h,s+u}^{(1)}\right)^{-1/2}\left(h_{t+s+u}-\tilde{\mu}_{h,s+u}^{(1)}\right)^2,$$

which yields:

$$E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}^3\right)=\tau_z\theta_{su}^{(3/2)}=\frac{3}{4}\tau_z\left(\tilde{\mu}_{h,s+u}^{(1)}\right)^{1/2}\left(E_t\left(\varepsilon_{t+s}h_{t+s+u}\right)+\frac{1}{2}\left(\tilde{\mu}_{h,s+u}^{(1)}\right)^{-1}E_t\left(\varepsilon_{t+s}h_{t+s+u}^2\right)\right).$$

Since, conditional on the information available at time t, the indicator function I_t^- is independent of all (contemporaneous) ε_t^{2k} for any natural number k, we have

$$\theta_{su}^{(2)}=E_t\left(\varepsilon_{t+s}\big(\omega+\left(\alpha+\lambda I_{t+s+u-1}^-\right)\varepsilon_{t+s+u-1}^2+\beta h_{t+s+u-1}\big)^2\right)=\gamma\theta_{s(u-1)}^{(2)}+2\omega\varphi\theta_{s(u-1)}^{(1)}.$$

We also set:[6]

$$\gamma=\left(\alpha^2+\left(2\alpha\lambda+\lambda^2\right)F_0\right)\kappa_z+\beta^2+2\beta\left(\alpha+\lambda F_0\right)=\varphi^2+\left(\kappa_z-1\right)\left(\alpha+\lambda F_0\right)^2+\kappa_z\lambda^2F_0\left(1-F_0\right).$$

If $D(0,1)$ is the standard normal distribution, $\gamma=\varphi^2+2\big(\alpha+\frac{\lambda}{2}\big)^2+\frac{3}{4}\lambda^2$. When $D(0,1)$ is the standardized Student t distribution, $\gamma=\varphi^2+\left(3\frac{\nu-2}{\nu-4}-1\right)\left(\alpha+\frac{\lambda}{2}\right)^2+\frac{3}{4}\left(\frac{\nu-2}{\nu-4}\right)\lambda^2$.

[5] Even though for the normal and Student t GJR $\tau_z=0$, $\theta_{su}^{(3/2)}=E_t\left(\varepsilon_{t+s}h_{t+s+u}^{3/2}\right)$ is generally non-zero for these models (and enters the expressions of higher moments computed below) and this is why we still consider the normal and Student t special cases in the derivation of $\theta_{su}^{(3/2)}$.

[6] It can be shown, using the Cauchy – Buniakowsky – Schwarz inequality, that the kurtosis is always greater than or equal to 1, hence $\kappa_z\geq 1$ (see also Stuart and Ord, 1994, p. 109). Now it can be easily seen that $\gamma>0$.

Solving the recursion for $\theta_{su}^{(2)}$, we have $\theta_{su}^{(2)} = \gamma^{u-1}\theta_{s1}^{(2)} + 2\omega\varphi \sum\limits_{j=1}^{u-1} \gamma^{j-1}\theta_{s(u-j)}^{(1)}$. But

$$\sum_{j=1}^{u-1} \gamma^{j-1}\theta_{s(u-j)}^{(1)} = c_4 \sum_{j=1}^{u-1} \gamma^{j-1}\varphi^{u-j-1} E_t\left(h_{t+s}^{3/2}\right) = c_4(\varphi-\gamma)^{-1}\left(\varphi^{u-1}-\gamma^{u-1}\right) E_t\left(h_{t+s}^{3/2}\right),$$

where $c_4 = \left(\alpha\tau_z + \lambda \int\limits_{x=-\infty}^{0} x^3 f(x)\, dx\right)$. Thus,

$$\begin{aligned}
\theta_{s1}^{(2)} &= E_t\left(\varepsilon_{t+s}h_{t+s+1}^2\right) = E_t\left(\varepsilon_{t+s}\left(\omega + \left(\alpha + \lambda I_{t+s}^-\right)\varepsilon_{t+s}^2 + \beta h_{t+s}\right)^2\right) \\
&= \left(\alpha\left(\alpha\mu_z^{(5)} + 2\beta\tau_z\right) + \lambda\left(2\alpha+\lambda\right) \int\limits_{x=-\infty}^{0} x^5 f(x)\, dx + 2\lambda\beta \int\limits_{x=-\infty}^{0} x^3 f(x)\, dx\right) E_t\left(h_{t+s}^{5/2}\right) \\
&+ 2\left(\omega\alpha\tau_z + \lambda\omega \int\limits_{x=-\infty}^{0} x^3 f(x)\, dx\right) E_t\left(h_{t+s}^{3/2}\right).
\end{aligned}$$

For the normal GJR, $\tau_z = \mu_z^{(5)} = 0$, $f(z) = \varphi(z)$, and $\int\limits_{z=-\infty}^{0} z^5\varphi(z)\, dz = -4\sqrt{\frac{2}{\pi}}$. Similarly, for the Student t GJR, we again have have $\tau_z = \mu_z^{(5)} = 0$ and $f(z) = f_\nu(z)$. After some algebraic manipulation, we have: $\int\limits_{z=-\infty}^{0} z^5 f_v(z) = -\frac{8}{\sqrt{\pi}} \frac{(\nu-2)^{5/2}}{(\nu-1)(\nu-3)(\nu-5)} \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}$. Thus, the final expression for $\theta_{su}^{(3/2)}$ becomes:

$$\theta_{su}^{(3/2)} = \frac{3}{4}\left(\tilde{\mu}_{h,s+u}^{(1)}\right)^{1/2} \left(\begin{array}{l} c_4\varphi^{u-1}E_t\left(h_{t+s}^{3/2}\right) + \frac{1}{2}\left(\tilde{\mu}_{h,s+u}^{(1)}\right)^{-1} \\ \left(c_5\gamma^{u-1}E_t\left(h_{t+s}^{5/2}\right) + 2\omega c_4\left(\varphi(\varphi-\gamma)^{-1}\left(\varphi^{u-1}-\gamma^{u-1}\right) + \gamma^{u-1}\right)E_t\left(h_{t+s}^{3/2}\right)\right) \end{array}\right),$$

where

$$c_5 = \alpha\left(\alpha\mu_z^{(5)} + 2\beta\tau_z\right) + \lambda\left(2\alpha+\lambda\right) \int\limits_{x=-\infty}^{0} x^5 f(x)\, dx + 2\beta\lambda \int\limits_{x=-\infty}^{0} x^3 f(x)\, dx$$

and $E_t\left(h_{t+s}^{5/2}\right)$ is given approximately, using a second order Taylor expansion for $h_{t+s}^{5/2}$ around

$E_t\left(h_{t+s}\right)$, as: $E_t\left(h_{t+s}^{5/2}\right) \simeq \frac{1}{8}\left(\tilde{\mu}_{h,s}^{(1)}\right)^{1/2}\left(15\tilde{\mu}_{h,s}^{(2)} - 7\left(\tilde{\mu}_{h,s}^{(1)}\right)^2\right)$. We also have:

$$\begin{aligned}
\sum_{s=1}^{n}\sum_{u=1}^{n-s} E_t\left(r_{t+s}^2 r_{t+s+u}^2\right) &= \sum_{s=1}^{n}\sum_{u=1}^{n-s} E_t\left[(\mu+\varepsilon_{t+s})^2(\mu+\varepsilon_{t+s+u})^2\right] \\
&= \sum_{s=1}^{n}\sum_{u=1}^{n-s}\left[\mu^4 + \mu^2\left(\tilde{\mu}_{h,s}^{(1)} + \tilde{\mu}_{h,s+u}^{(1)}\right) + 2\mu E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}^2\right) + E_t\left(\varepsilon_{t+s}^2\varepsilon_{t+s+u}^2\right)\right], \\
E_t\left(\varepsilon_{t+s}^2\varepsilon_{t+s+u}^2\right) &= E_t\left(\varepsilon_{t+s}^2 h_{t+s+u}\right) = \omega\tilde{\mu}_{h,s}^{(1)} + \varphi E_t\left(\varepsilon_{t+s}^2\varepsilon_{t+s+u-1}^2\right) \\
&= \omega\tilde{\mu}_{h,s}^{(1)}\left((1-\varphi)^{-1}(1-\varphi^u) - \varphi^{u-1}\right) + \varphi^{u-1}E_t\left(\varepsilon_{t+s}^2\varepsilon_{t+s+1}^2\right), \\
E_t\left(\varepsilon_{t+s}^2\varepsilon_{t+s+1}^2\right) &= E_t\left(\varepsilon_{t+s}^2 h_{t+s+1}\right) = E_t\left(\varepsilon_{t+s}^2\left(\omega + \left(\alpha + \lambda I_{t+s}^-\right)\varepsilon_{t+s}^2 + \beta h_{t+s}\right)\right) \\
&= \omega\tilde{\mu}_{h,s}^{(1)} + \kappa_z\left(\alpha + \lambda F_0 + \kappa_z^{-1}\beta\right)\tilde{\mu}_{h,s}^{(2)}
\end{aligned}$$

Hence, the final expression for $E_t\left(\varepsilon_{t+s}^2\varepsilon_{t+s+u}^2\right)$ becomes:

$$E_t\left(\varepsilon_{t+s}^2\varepsilon_{t+s+u}^2\right) = \bar{h}\left(1-\varphi^u\right)\tilde{\mu}_{h,s}^{(1)} + \varphi^{u-1}\kappa_z\left(\alpha + \lambda F_0 + \kappa_z^{-1}\beta\right)\tilde{\mu}_{h,s}^{(2)}.$$

The expressions for the normal and Student t GJR are obtained by replacing $\kappa_z = 3$ and $\kappa_z = 3\frac{(\nu-2)}{(\nu-4)}$, respectively, and $F_0 = \frac{1}{2}$ in the expression above.

Next,

$$\begin{aligned}
E_t\left(r_{t+s}^2 r_{t+s+u} r_{t+s+u+v}\right) &= E_t\left(r_{t+s}^2 r_{t+s+u} E_{t+s+u+v-1}\left(r_{t+s+u+v}\right)\right) = \mu E_t\left(r_{t+s}^2 r_{t+s+u}\right) = \mu^2 E_t\left(r_{t+s}^2\right) \\
&= \mu^2\left(\mu^2 + \tilde{\mu}_{h,s}^{(1)}\right), \\
E_t\left(r_{t+s} r_{t+s+u}^2 r_{t+s+u+v}\right) &= \mu E_t\left(\left(\mu + \varepsilon_{t+s}\right)\left(\mu^2 + h_{t+s+u}\right)\right) = \mu^4 + \mu^2\tilde{\mu}_{h,s}^{(1)} + \mu\theta_{su}^{(1)} \\
E_t\left(r_{t+s} r_{t+s+u} r_{t+s+u+v}^2\right) &= E_t\left(r_{t+s} r_{t+s+u} E_{t+s+u+v-1}\left(\mu^2 + 2\mu\varepsilon_{t+s+u+v} + \varepsilon_{t+s+u+v}^2\right)\right) \\
&= E_t\left(r_{t+s} r_{t+s+u}\left(\mu^2 + h_{t+s+u+v}\right)\right) \\
&= \mu^4 + \mu^2\tilde{\mu}_{h,(s+u+v)}^{(1)} + \mu\theta_{s(u+v)}^{(1)} + \mu E_t\left(\varepsilon_{t+s+u} h_{t+s+u+v}\right) + E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u} h_{t+s+u+v}\right), \\
E_t\left(\varepsilon_{t+s+u} h_{t+s+u+v}\right) &= E_t\left(E_{t+s}\left(\varepsilon_{t+s+u} h_{t+s+u+v}\right)\right) = E_t\left(E_{t_1}\left(\varepsilon_{t_1+u} h_{t_1+u+v}\right)\right),
\end{aligned}$$

where $t_1 = t + s$.

We showed that $E_t\left(\varepsilon_{t+s} h_{t+s+u}\right) = \theta_{su}^{(1)} = c_4\varphi^{u-1}E_t\left(h_{t+s}^{3/2}\right)$. Hence $E_{t_1}\left(\varepsilon_{t_1+u} h_{t_1+u+v}\right) = c_4\varphi^{v-1}E_{t_1}\left(h_{t_1+u}^{3/2}\right)$

and thus we have: $E_t(\varepsilon_{t+s+u}h_{t+s+u+v}) = c_4\varphi^{v-1}E_t\left(E_{t_1}\left(h_{t_1+u}^{3/2}\right)\right) = c_4\varphi^{v-1}E_t\left(h_{t+s+u}^{3/2}\right)$. Also,

$$\begin{aligned} E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}\varepsilon^2_{t+s+u+v}\right) &= E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}h_{t+s+u+v}\right) = E_t\left(\varepsilon_{t+s}E_{t_1}\left(\varepsilon_{t_1+u}h_{t_1+u+v}\right)\right) \\ &= c_4\varphi^{v-1}E_t\left(\varepsilon_{t+s}E_{t_1}\left(h_{t_1+u}^{3/2}\right)\right) = c_4\varphi^{v-1}E_t\left(\varepsilon_{t+s}h_{t+s+u}^{3/2}\right) = c_4\varphi^{v-1}\theta_{su}^{(3/2)}. \end{aligned}$$

Finally, repeatedly applying the tower law, we have that $E_t\left(r_{t+s}r_{t+s+u}r_{t+s+u+v}r_{t+s+u+v+w}\right) = \mu^4$.

(e) *Centred Conditional Moments of Forward and Aggregated Returns*

The second conditional centred moment of the forward returns (i.e. the conditional variance of the forward return) is:

$$\mu_{r,s}^{(2)} = E_t\left(\varepsilon^2_{t+s}\right) = E_t\left(h_{t+s}\right) = \tilde{\mu}_{h,s}^{(1)} = \bar{h} + \varphi^{s-1}\left(h_{t+1} - \bar{h}\right),$$

and for the aggregated returns it is:

$$\begin{aligned} M_{r,n}^{(2)} &= E_t\left(\left(\sum_{s=1}^{n}\varepsilon_{t+s}\right)^2\right) = \sum_{s=1}^{n}\tilde{\mu}_{h,s}^{(1)} + 2\sum_{s=1}^{n}\sum_{u=1}^{n-s}\underbrace{E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}\right)}_{0} = \sum_{s=1}^{n}\tilde{\mu}_{h,s}^{(1)} \\ &= n\bar{h} + (1-\varphi)^{-1}(1-\varphi^n)\left(h_{t+1} - \bar{h}\right). \end{aligned}$$

The third conditional centred moment of the forward returns is:

$$\mu_{r,s}^{(3)} = E_t\left(\varepsilon^3_{t+s}\right) = \tau_z E_t\left(h_{t+s}^{3/2}\right) \simeq \tfrac{1}{8}\tau_z\left(5\left(\mu_{h,s}^{(1)}\right)^{3/2} + 3\,\mu_{h,s}^{(2)}\left(\mu_{h,s}^{(1)}\right)^{-1/2}\right),$$

so $\mu_{r,s}^{(3)} = 0$ in the special cases when the innovation distribution is either the standard normal, or the standardized Student t. In general:

$$\begin{aligned} M_{r,n}^{(3)} &= E_t\left(\left(\sum_{s=1}^{n}\varepsilon_{t+s}\right)^3\right) = \sum_{s=1}^{n}E_t\left(\varepsilon^3_{t+s}\right) + 3\sum_{s=1}^{n}\sum_{u=1}^{n-s}\left[E_t\left(\varepsilon^2_{t+s}\varepsilon_{t+s+u}\right) + E_t\left(\varepsilon_{t+s}\varepsilon^2_{t+s+u}\right)\right] \\ &+ 6\sum_{s=1}^{n}\sum_{u=1}^{n-s}\sum_{v=1}^{n-s-u}E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}\varepsilon_{t+s+u+v}\right) \\ &= \tau_z\sum_{s=1}^{n}E_t\left(h_{t+s}^{3/2}\right) + 3\sum_{s=1}^{n}\sum_{u=1}^{n-s}E_t\left(\varepsilon_{t+s}\varepsilon^2_{t+s+u}\right) \\ &\simeq \frac{1}{8}\tau_z\sum_{s=1}^{n}\left(5\left(\tilde{\mu}_{h,s}^{(1)}\right)^{3/2} + 3\tilde{\mu}_{h,s}^{(2)}\left(\tilde{\mu}_{h,s}^{(1)}\right)^{-1/2}\right) + 3\sum_{s=1}^{n}\sum_{u=1}^{n-s}E_t\left(\varepsilon_{t+s}\varepsilon^2_{t+s+u}\right). \end{aligned}$$

For the normal and Student t GJR models, this expression simplifies to $M_{r,n}^{(3)} = 3\sum_{s=1}^{n}\sum_{u=1}^{n-s}E_t\left(\varepsilon_{t+s}\varepsilon^2_{t+s+u}\right)$.

The fourth conditional centred moment of the forward returns is:

$$\mu_{r,s}^{(4)} = E_t\left(\varepsilon_{t+s}^4\right) = \kappa_z \tilde{\mu}_{h,s}^{(2)},$$

so in the special case that the innovation distribution is the standard normal, $\mu_{r,s}^{(4)} = 3\tilde{\mu}_{h,s}^{(2)}$, while for the Student t GJR we have: $\mu_{r,s}^{(4)} = 3\frac{\nu-2}{\nu-4}\tilde{\mu}_{h,s}^{(2)}$.

The fourth conditional centred moment of the aggregated returns is:

$$\begin{aligned}
M_{r,n}^{(4)} &= E_t\left(\left(\sum_{s=1}^{n}\varepsilon_{t+s}\right)^4\right) \\
&= \sum_{s=1}^{n}\varepsilon_{t+s}^4 + \sum_{s=1}^{n}\sum_{u=1}^{n-s}\left(4\left(E_t\left(\varepsilon_{t+s}^3\varepsilon_{t+s+u}\right) + E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}^3\right)\right) + 6E_t\left(\varepsilon_{t+s}^2\varepsilon_{t+s+u}^2\right)\right) \\
&+ 12\sum_{s=1}^{n}\sum_{u=1}^{n-s}\sum_{v=1}^{n-s-u} E_t\left(\varepsilon_{t+s}^2\varepsilon_{t+s+u}\varepsilon_{t+s+u+v}\right) + E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}^2\varepsilon_{t+s+u+v}\right) + E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}\varepsilon_{t+s+u+v}^2\right) \\
&+ 24\sum_{s=1}^{n}\sum_{u=1}^{n-s}\sum_{v=1}^{n-s-u}\sum_{w=1}^{n-s-u-v} E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}\varepsilon_{t+s+u+v}\varepsilon_{t+s+u+v+w}\right) \\
&= \kappa_z\sum_{s=1}^{n}\tilde{\mu}_{h,s}^{(2)} + \sum_{s=1}^{n}\sum_{u=1}^{n-s}\left(4E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}^3\right) + 6E_t\left(\varepsilon_{t+s}^2\varepsilon_{t+s+u}^2\right)\right) + 12\sum_{s=1}^{n}\sum_{u=1}^{n-s}\sum_{v=1}^{n-s-u} E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}\varepsilon_{t+s+u+v}^2\right).
\end{aligned}$$

In the special case that the conditional distribution is the standard normal,

$$M_{r,n}^{(4)} = 3\sum_{s=1}^{n}\tilde{\mu}_{h,s}^{(2)} + 6\sum_{s=1}^{n}\sum_{u=1}^{n-s}E_t\left(\varepsilon_{t+s}^2\varepsilon_{t+s+u}^2\right) + 12\sum_{s=1}^{n}\sum_{u=1}^{n-s}\sum_{v=1}^{n-s-u}E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}\varepsilon_{t+s+u+v}^2\right),$$

while for a Student t GJR we obtain,

$$M_{r,n}^{(4)} = 3\frac{\nu-2}{\nu-4}\sum_{s=1}^{n}\tilde{\mu}_{h,s}^{(2)} + 6\sum_{s=1}^{n}\sum_{u=1}^{n-s}E_t\left(\varepsilon_{t+s}^2\varepsilon_{t+s+u}^2\right) + 12\sum_{s=1}^{n}\sum_{u=1}^{n-s}\sum_{v=1}^{n-s-u}E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}\varepsilon_{t+s+u+v}^2\right).$$

(f) *Standardized Conditional Moments of Forward and Aggregated Returns*

The skewness of the forward returns is:

$$\begin{aligned}
\tau_{r,s} &= \mu_{r,s}^{(3)}\left(\mu_{r,s}^{(2)}\right)^{-3/2} = \tau_z E_t\left(h_{t+s}^{3/2}\right)\left(\tilde{\mu}_{h,s}^{(1)}\right)^{-3/2} \simeq \frac{1}{8}\tau_z\left(5\left(\tilde{\mu}_{h,s}^{(1)}\right)^{3/2} + 3\tilde{\mu}_{h,s}^{(2)}\left(\tilde{\mu}_{h,s}^{(1)}\right)^{-1/2}\right)\left(\tilde{\mu}_{h,s}^{(1)}\right)^{-3/2} \\
&= \frac{1}{8}\tau_z\left(5 + 3\tilde{\mu}_{h,s}^{(2)}\left(\tilde{\mu}_{h,s}^{(1)}\right)^{-2}\right).
\end{aligned}$$

Note that, if we used only a first order Taylor series expansion, we would obtain $\tau_{r,s} \approx \tau_z$ and that $\tau_{r,s} = 0$ for both the normal and Student t GJR.

The kurtosis of the forward returns is:

$$\kappa_{r,s} = \mu_{r,s}^{(4)} \left(\mu_{r,s}^{(2)}\right)^{-2} = \kappa_z \tilde{\mu}_{h,s}^{(2)} \left(\tilde{\mu}_{h,s}^{(1)}\right)^{-2},$$

so in the special cases that the conditional distribution is standard normal $\kappa_{h,s} = 3\tilde{\mu}_{h,s}^{(2)} \left(\tilde{\mu}_{h,s}^{(1)}\right)^{-2}$, while for the Student t GJR we obtain: $\kappa_{h,s} = 3\frac{\nu-2}{\nu-4}\tilde{\mu}_{h,s}^{(2)} \left(\tilde{\mu}_{h,s}^{(1)}\right)^{-2}$. Finally, the skewness and kurtosis of the aggregated returns are:

$$\mathrm{T}_{r,n} = M_{r,n}^{(3)} \left(M_{r,n}^{(2)}\right)^{-3/2} \text{ and } \mathrm{K}_{r,n} = M_{r,n}^{(4)} \left(M_{r,n}^{(2)}\right)^{-2}.$$

A2: Conditional Moments of Forward and Aggregated Variances

(a) *First Conditional Moments of Forward and Aggregated Variances*

Applying the conditional expectation operator to (1), the first un-centred conditional moment of the forward variance may be written:

$$\tilde{\mu}_{h,s}^{(1)} = \mu_{r,s}^{(2)} = \bar{h} + \varphi^{s-1}\left(h_{t+1} - \bar{h}\right).$$

Similarly, the first un-centred conditional moment of the aggregated variance becomes:

$$\tilde{M}_{h,n}^{(1)} = \sum_{s=1}^{n} \tilde{\mu}_{h,s}^{(1)} = n\bar{h} + (1-\varphi)^{-1}(1-\varphi^n)\left(h_{t+1} - \bar{h}\right)$$

or equivalently, in recursive form: $\tilde{M}_{h,n}^{(1)} = \tilde{M}_{h,n-1}^{(1)} + \bar{h} + \varphi^{n-1}\left(h_{t+1} - \bar{h}\right)$.

(b) *Second Conditional Moments of Forward and Aggregated Variances*

The second moment of the forward variance is:

$$\begin{aligned}
\tilde{\mu}_{h,s}^{(2)} &= E_t\left(\left(\omega + \left(\alpha + \lambda I_{t+s-1}^{-}\right)\varepsilon_{t+s-1}^2 + \beta h_{t+s-1}\right)^2\right) \\
&= \omega^2 + 2\omega\varphi\tilde{\mu}_{h,s-1}^{(1)} + \left(\varphi^2 + (\kappa_z - 1)(\alpha + \lambda F_0)^2 + \kappa_z \lambda^2 F_0 (1 - F_0)\right)\tilde{\mu}_{h,s-1}^{(2)} \\
&= \sum_{i=1}^{s-1} \gamma^{i-1}\left(\omega^2 + 2\omega\varphi\left(\bar{h} + \varphi^{s-i-1}\left(h_{t+1} - \bar{h}\right)\right)\right) + \gamma^{s-1}h_{t+1}^2.
\end{aligned}$$

When $\gamma = 1$, the expression for the second moment of the forward variance becomes:

$$\tilde{\mu}_{h,s}^{(2)} = (s-1)\left(\omega^2 + 2\omega\varphi\bar{h}\right) + 2\varphi\bar{h}\left(1 - \varphi^{s-1}\right)\left(h_{t+1} - \bar{h}\right) + h_{t+1}^2.$$

For $\gamma \neq 1$ (and $\gamma \neq \varphi$), we introduce the following additional notation:

$$c_1 = \left(\omega^2 + 2\omega\varphi\bar{h}\right)(1-\gamma)^{-1}, \quad c_2 = 2\omega\varphi\left(h_{t+1} - \bar{h}\right)(\varphi-\gamma)^{-1} \text{and} \quad c_3 = c_1 + c_2.$$

Now the expression for the second moment of variance may be written:

$$\tilde{\mu}^{(2)}_{h,s} = c_1 + \left(h^2_{t+1} - c_3\right)\gamma^{s-1} + c_2\varphi^{s-1}.$$

The second moment of the aggregated variance is given by:

$$\tilde{M}^{(2)}_{h,n} = E_t\left[\left(\sum_{s=1}^{n} h_{t+s}\right)^2\right] = \sum_{s=1}^{n}\tilde{\mu}^{(2)}_{h,s} + 2\sum_{s=1}^{n}\sum_{u=1}^{n-s}\tilde{\mu}^{(1,1)}_{h,su},$$

where

$$\begin{aligned}
\tilde{\mu}^{(1,1)}_{h,su} &= E_t\left(h_{t+s}\left(\omega + \left(\alpha + \lambda I^-_{t+s+u-1}\right)\varepsilon^2_{t+s+u-1} + \beta h_{t+s+u-1}\right)\right) \\
&= \omega\tilde{\mu}^{(1)}_{h,s} + \varphi\tilde{\mu}^{(1,1)}_{h,s(u-1)} = \bar{h}\tilde{\mu}^{(1)}_{h,s} + \varphi^u\left(\tilde{\mu}^{(2)}_{h,s} - \bar{h}\tilde{\mu}^{(1)}_{h,s}\right).
\end{aligned}$$

Hence

$$\tilde{M}^{(2)}_{h,n} = \sum_{s=1}^{n}\left(\tilde{\mu}^{(2)}_{h,s} + 2\bar{h}(n-s)\tilde{\mu}^{(1)}_{h,s}\right) + 2\sum_{s=1}^{n}\sum_{u=1}^{n-s}\left(\varphi^u\left(\tilde{\mu}^{(2)}_{h,s} - \bar{h}\tilde{\mu}^{(1)}_{h,s}\right)\right). \tag{4}$$

Consider the first sum in (4). For $\gamma \neq 1$ and $\gamma \neq \varphi$, we have:

$$\begin{aligned}
\sum_{s=1}^{n}\tilde{\mu}^{(2)}_{h,s} &= \sum_{s=1}^{n}\left(c_1 + \left(h^2_{t+1} - c_3\right)\gamma^{s-1} + c_2\varphi^{s-1}\right) \\
&= nc_1 + \left(h^2_{t+1} - c_3\right)(1-\gamma)^{-1}(1-\gamma^n) + c_2(1-\varphi)^{-1}(1-\varphi^n),
\end{aligned} \tag{5}$$

and

$$\begin{aligned}
\sum_{s=1}^{n}(n-s)\,\tilde{\mu}^{(1)}_{h,s} &= \sum_{s=1}^{n}(n-s)\left(\bar{h} + \varphi^{s-1}\left(h_{t+1} - \bar{h}\right)\right) \\
&= \tfrac{n(n-1)}{2}\bar{h} + (1-\varphi)^{-1}\left[n - (1-\varphi)^{-1}(1-\varphi^n)\right]\left(h_{t+1} - \bar{h}\right).
\end{aligned} \tag{6}$$

Next we evaluate the double sum term in (4). We have:

$$\begin{aligned}
\sum_{s=1}^{n}\sum_{u=1}^{n-s}\varphi^u\tilde{\mu}^{(2)}_{h,s} &= \varphi(1-\varphi)^{-1}\sum_{s=1}^{n}\tilde{\mu}^{(2)}_{h,s}\left(1-\varphi^{n-s}\right) \\
&= \varphi(1-\varphi)^{-1}\left[\begin{array}{c} n\left(c_1 - c_2\varphi^{n-1}\right) + (c_2 - c_1)(1-\varphi)^{-1}(1-\varphi^n) + \left(h^2_{t+1} - c_3\right) \\ \left[(1-\gamma)^{-1}(1-\gamma^n) - (\varphi-\gamma)^{-1}(\varphi^n - \gamma^n)\right] \end{array}\right]
\end{aligned}$$

and

$$\begin{aligned}
\sum_{s=1}^{n}\sum_{u=1}^{n-s}\varphi^u\tilde{\mu}^{(1)}_{h,s} &= \varphi(1-\varphi)^{-1}\sum_{s=1}^{n}\left(1-\varphi^{n-s}\right)\left(\bar{h} + \varphi^{s-1}\left(h_{t+1} - \bar{h}\right)\right) \\
&= \varphi(1-\varphi)^{-1}\left(n\left(\bar{h} - \varphi^{n-1}\left(h_{t+1} - \bar{h}\right)\right) + (1-\varphi)^{-1}\left(h_{t+1} - 2\bar{h}\right)(1-\varphi^n)\right).
\end{aligned} \tag{7}$$

Thus the final expression for $\tilde{M}_{h,n}^{(2)}$ is:

$$\begin{aligned}
\tilde{M}_{h,n}^{(2)} &= nc_1 + \left(h_{t+1}^2 - c_3\right)(1-\gamma)^{-1}(1-\gamma^n) + c_2(1-\varphi)^{-1}(1-\varphi^n) \\
&+ 2\bar{h}\left(\tfrac{n(n-1)}{2}\bar{h} + (1-\varphi)^{-1}\left[n - (1-\varphi)^{-1}(1-\varphi^n)\right]\left(h_{t+1} - \bar{h}\right)\right) \\
&+ 2\varphi(1-\varphi)^{-1}\left[\begin{array}{c} n\left(c_1 - c_2\varphi^{n-1}\right) + (c_2 - c_1)(1-\varphi)^{-1}(1-\varphi^n) + \left(h_{t+1}^2 - c_3\right) \\ \left[(1-\gamma)^{-1}(1-\gamma^n) - (\varphi-\gamma)^{-1}(\varphi^n - \gamma^n)\right] \end{array}\right] \\
&- 2\bar{h}\varphi(1-\varphi)^{-1}\left(n\left(\bar{h} - \varphi^{n-1}\left(h_{t+1} - \bar{h}\right)\right) + (1-\varphi)^{-1}\left(h_{t+1} - 2\bar{h}\right)(1-\varphi^n)\right).
\end{aligned} \tag{8}$$

(c) *Third Conditional Moments of Forward and Aggregated Variances*

We now consider the third moment of the forward variance:

$$\begin{aligned}
\tilde{\mu}_{h,s}^{(3)} &= E_t\left[\left(\omega + \left(\alpha + \lambda I_{t+s-1}^{-}\right)\varepsilon_{t+s-1}^2 + \beta h_{t+s-1}\right)^3\right] \\
&= \omega^3 + 3\omega^2\varphi\tilde{\mu}_{h,s-1}^{(1)} + 3\omega\underbrace{\left[\kappa_z\left(\alpha^2 + \lambda(2\alpha+\lambda)F_0\right) + \beta^2 + 2\beta(\alpha + \lambda F_0)\right]}_{\gamma}\tilde{\mu}_{h,s-1}^{(2)} \\
&+ \left[\mu_z^{(6)}\left(\alpha^3 + 3\alpha\lambda(\alpha+\lambda)F_0 + \lambda^3 F_0\right) + 3\beta\kappa_z\left(\alpha^2 + \lambda(2\alpha+\lambda)F_0\right) + 3\beta^2(\alpha+\lambda F_0) + \beta^3\right]\tilde{\mu}_{h,s-1}^{(3)} \\
&= \sum_{i=0}^{s-2} c_6^i\left(\omega^3 + 3\omega^2\varphi\tilde{\mu}_{h,s-i-1}^{(1)} + 3\omega\gamma\tilde{\mu}_{h,s-i-1}^{(2)}\right) + c_6^{s-1}h_{t+1}^3,
\end{aligned}$$

where

$$c_6 = \mu_z^{(6)}\left(\alpha^3 + 3\alpha\lambda(\alpha+\lambda)F_0 + \lambda^3 F_0\right) + 3\beta\gamma - \beta^2\left(2\beta + 3(\alpha + \lambda F_0)\right).$$

For the special case where innovations are normally distributed, we have $F_0 = \frac{1}{2}$ and $\mu_z^{(6)} = 15$. Similarly, when innovations are Student t distributed, $F_0 = \frac{1}{2}$ again, and $\mu_z^{(6)} = 15\frac{(\nu-2)^2}{(\nu-4)(\nu-6)}$.

For $c_6 \neq 1$ (and $c_6 \neq \gamma$, $c_6 \neq \varphi$, and $\gamma \neq 1$), we have that:

$$\begin{aligned}
\tilde{\mu}_{h,s}^{(3)} &= \omega\left(\omega^2 + 3\omega\varphi\bar{h} + 3\gamma c_1\right)(1-c_6)^{-1} + \left(3\omega^2\varphi\left(h_{t+1} - \bar{h}\right) + 3\omega\gamma c_2\right)(\varphi - c_6)^{-1}\varphi^{s-1} \\
&+ c_{16}c_6^{s-1} + 3\omega\gamma\left(-c_3 + h_{t+1}^2\right)(\gamma - c_6)^{-1}\gamma^{s-1}.
\end{aligned}$$

where

$$c_{16} = h_{t+1}^3 - \omega\left(\omega^2 + 3\omega\varphi\bar{h} + 3\gamma c_1\right)(1-c_6)^{-1} - \left(3\omega^2\varphi\left(h_{t+1} - \bar{h}\right) + 3\omega\gamma\left(-c_1 + h_{t+1}^2\right)\right)(\varphi - c_6)^{-1}.$$

For $c_6 = 1$ and $\gamma \neq 1$, the expression for $\tilde{\mu}^{(3)}_{h,s}$ becomes:

$$
\begin{aligned}
\tilde{\mu}^{(3)}_{h,s} &= \sum_{i=0}^{s-2} \left(\omega^3 + 3\omega^2\varphi\tilde{\mu}^{(1)}_{h,s-i-1} + 3\omega\gamma\tilde{\mu}^{(2)}_{h,s-i-1}\right) + h^3_{t+1} \\
&= h^3_{t+1} + (s-1)\,\omega\left(\omega^2 + 3\bar{h}\varphi\omega + 3\gamma c_1\right) + 3\omega\left[\omega\varphi\left(h_{t+1} - \bar{h}\right) + \gamma c_2\right](1-\varphi)^{-1}\left(1-\varphi^{s-1}\right) \\
&+ 3\omega\gamma\left(h^2_{t+1} - c_3\right)(1-\gamma)^{-1}\left(1-\gamma^{s-1}\right)\,.
\end{aligned}
$$

For $c_6 \neq 1$ and $\gamma = 1$, we have that:

$$
\begin{aligned}
\tilde{\mu}^{(3)}_{h,s} &= \omega^3(1-c_6)^{-1}\left(1-c_6^{s-1}\right) + 3\omega^2\varphi\sum_{i=0}^{s-2}\left(c_6^i\left(\bar{h} + \varphi^{s-i-2}\left(h_{t+1} - \bar{h}\right)\right)\right) \\
&+ 3\omega\gamma\sum_{i=0}^{s-2}\left(c_6^i\left((s-i-2)\left(\omega^2 + 2\omega\varphi\bar{h}\right) + 2\varphi\bar{h}\left(1-\varphi^{s-i-2}\right)\left(h_{t+1} - \bar{h}\right) + h^2_{t+1}\right)\right) + c_6^{s-1}h^3_{t+1} \\
&= \left(\omega^3 + 3\omega^2\varphi\bar{h} + 6\omega\gamma\varphi\bar{h}\left(h_{t+1} - \bar{h}\right) + 3\omega\gamma h^2_{t+1}\right)(1-c_6)^{-1}\left(1-c_6^{s-1}\right) \\
&+ \left[3\omega^2\varphi\left(h_{t+1} - \bar{h}\right) - 6\omega\gamma\varphi\bar{h}\left(h_{t+1} - \bar{h}\right)\right](\varphi - c_6)^{-1}\left(\varphi^{s-1} - c_6^{s-1}\right) \\
&+ 3\omega\gamma\left(\omega^2 + 2\omega\varphi\bar{h}\right)\left((1-c_6)^{-1}(s-2) - c_6(1-c_6)^{-2}\left(1-c_6^{s-2}\right)\right) + c_6^{s-1}h^3_{t+1}.
\end{aligned}
$$

For the third moment of aggregated variance we write:

$$
\begin{aligned}
\tilde{M}^{(3)}_{h,n} &= E_t\left(\left(\sum_{s=1}^{n} h_{t+s}\right)^3\right) = \sum_{s=1}^{n} E_t\left(h^3_{t+s}\right) + 3\sum_{s=1}^{n}\sum_{u=1}^{n-s}\left[E_t\left(h^2_{t+s}h_{t+s+u}\right) + E_t\left(h_{t+s}h^2_{t+s+u}\right)\right] \\
&+ 6\sum_{s=1}^{n}\sum_{u=1}^{n-s}\sum_{v=1}^{n-s-u} E_t\left(h_{t+s}h_{t+s+u}h_{t+s+u+v}\right) \\
&+ \sum_{s=1}^{n}\tilde{\mu}^{(3)}_{h,s} + 3\sum_{s=1}^{n}\sum_{u=1}^{n-s}\left(\tilde{\mu}^{(1,2)}_{h,su} + \tilde{\mu}^{(2,1)}_{h,su}\right) + 6\sum_{s=1}^{n}\sum_{u=1}^{n-s}\sum_{v=1}^{n-s-u}\tilde{\mu}^{(1,1,1)}_{h,suv}
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{\mu}^{(2,1)}_{h,su} &= E_t\left(h^2_{t+s}\left(\omega + \left(\alpha + \lambda I^-_{t+s+u-1}\right)\varepsilon^2_{t+s+u-1} + \beta h_{t+s+u-1}\right)\right) = \omega\tilde{\mu}^{(2)}_{h,s} + \varphi\tilde{\mu}^{(2,1)}_{h,s(u-1)} \qquad (9) \\
&= \bar{h}\tilde{\mu}^{(2)}_{h,s} + \varphi^u\left(\tilde{\mu}^{(3)}_{h,s} - \bar{h}\tilde{\mu}^{(2)}_{h,s}\right),
\end{aligned}
$$

and

$$\tilde{\mu}_{h,su}^{(1,2)} = E_t\left(h_{t+s}\left(\begin{array}{l} \omega^2+\left(\alpha^2+2\alpha\lambda I_{t+s+u-1}^{-}+\lambda^2 I_{t+s+u-1}^{-}\right)\varepsilon_{t+s+u-1}^4+\beta^2 h_{t+s+u-1}^2 \\ +2\omega\left(\alpha+\lambda I_{t+s+u-1}^{-}\right)\varepsilon_{t+s+u-1}^2+2\omega\beta h_{t+s+u-1} \\ +2\beta\left(\alpha+\lambda I_{t+s+u-1}^{-}\right)\varepsilon_{t+s+u-1}^2 h_{t+s+u-1} \end{array}\right)\right) \tag{10}$$
$$= \omega^2\tilde{\mu}_{h,s}^{(1)}+2\omega\varphi\tilde{\mu}_{h,s(u-1)}^{(1,1)}+\gamma\tilde{\mu}_{h,s(u-1)}^{(1,2)}. = \sum_{j=0}^{u-1}\gamma^j\left(\omega^2\tilde{\mu}_{h,s}^{(1)}+2\omega\varphi\tilde{\mu}_{h,s(u-j-1)}^{(1,1)}\right)+\gamma^u\tilde{\mu}_{h,s}^{(3)}$$

Finally,

$$\tilde{\mu}_{h,suv}^{(1,1,1)} = E_t\left(h_{t+s}E_{t+s}\left(h_{t+s+u}h_{t+s+u+v}\right)\right) = E_t\left(h_{t+s}E_{t+s}\left(h_{t+s+u}h_{t+s+u+v}\right)\right).$$

We have already shown that

$$\tilde{\mu}_{h,su}^{(1,1)} = E_t\left(h_{t+s}h_{t+s+u}\right) = \bar{h}\tilde{\mu}_{h,s}^{(1)}+\varphi^u\left(\tilde{\mu}_{h,s}^{(2)}-\bar{h}\tilde{\mu}_{h,s}^{(1)}\right),$$

thus:

$$E_{t_1}\left(h_{t_1+u}h_{t_1+u+v}\right) = \bar{h}\tilde{\tilde{\mu}}_{h,u}^{(1)}+\varphi^v\left(\tilde{\tilde{\mu}}_{h,u}^{(2)}-\bar{h}\tilde{\tilde{\mu}}_{h,u}^{(1)}\right)$$

where $\tilde{\tilde{\mu}}_{h,u}^{(1)} = E_{t_1}\left(h_{t_1+u}\right)$ and $\tilde{\tilde{\mu}}_{h,u}^{(2)} = E_{t_1}\left(h_{t_1+u}^2\right)$. Since

$$\begin{aligned} \bar{h}\tilde{\mu}_{h,s}^{(1)}+\varphi^u\left(\tilde{\mu}_{h,s}^{(2)}-\bar{h}\tilde{\mu}_{h,s}^{(1)}\right) &= \left(\varphi^u\gamma^{s-1}\right)h_{t+1}^2+\left(\bar{h}\varphi^{s-1}\left(1-\varphi^u\right)+2\omega\varphi^{u+1}(\varphi-\gamma)^{-1}\left(\varphi^{s-1}-\gamma^{s-1}\right)\right)h_{t+1} \\ &+ \bar{h}^2\left(1-\varphi^{s-1}\right) \\ &+ \varphi^u\left(c_1+\left(2\omega\varphi\bar{h}(\varphi-\gamma)^{-1}-c_1\right)\gamma^{s-1}-2\omega\varphi^s\bar{h}(\varphi-\gamma)^{-1}-\bar{h}^2\left(1-\varphi^{s-1}\right)\right) \end{aligned}$$

we have:

$$\begin{aligned} E_{t_1}\left(h_{t_1+u}h_{t_1+u+v}\right) &= \left(\varphi^v\gamma^{u-1}\right)h_{t+s+1}^2+\left(\bar{h}\varphi^{u-1}\left(1-\varphi^v\right)+2\omega\varphi^{v+1}(\varphi-\gamma)^{-1}\left(\varphi^{u-1}-\gamma^{u-1}\right)\right)h_{t+s+1} \\ &+ \bar{h}^2\left(1-\varphi^{u-1}\right)+\varphi^v\left(\begin{array}{l} c_1+\left(2\omega\varphi\bar{h}(\varphi-\gamma)^{-1}-c_1\right)\gamma^{u-1} \\ -2\omega\varphi^u\bar{h}(\varphi-\gamma)^{-1}-\bar{h}^2\left(1-\varphi^{u-1}\right) \end{array}\right), \end{aligned}$$

and the expression for $\tilde{\mu}_{h,suv}^{(1,1,1)}$ now becomes:

$$\begin{aligned}\tilde{\mu}_{h,suv}^{(1,1,1)} &= \left(\varphi^{v}\gamma^{u-1}\right) E_t\left(h_{t+s}h_{t+s+1}^2\right) \\ &+ \left(\bar{h}\varphi^{u-1}\left(1-\varphi^{v}\right) + 2\omega\varphi^{v+1}(\varphi-\gamma)^{-1}\left(\varphi^{u-1}-\gamma^{u-1}\right)\right) E_t\left(h_{t+s}h_{t+s+1}\right) \\ &+ \left[\bar{h}^2\left(1-\varphi^{u-1}\right) + \varphi^{v}\begin{pmatrix} c_1 + \left(2\omega\varphi\bar{h}(\varphi-\gamma)^{-1} - c_1\right)\gamma^{u-1} \\ -2\omega\varphi^{u}\bar{h}(\varphi-\gamma)^{-1} - \bar{h}^2\left(1-\varphi^{u-1}\right)\end{pmatrix}\right]\tilde{\mu}_{h,s}^{(1)}.\end{aligned}$$

But $E_t\left(h_{t+s}h_{t+s+1}^2\right) = E_t\left(h_{t+s}\left(\omega + \left(\alpha+\lambda I_{t+s}^{-}\right)\varepsilon_{t+s}^2 + \beta h_{t+s}\right)^2\right) = \omega^2\tilde{\mu}_{h,s}^{(1)} + 2\omega\varphi\tilde{\mu}_{h,s}^{(2)} + \gamma\tilde{\mu}_{h,s}^{(3)}$ and $E_t\left(h_{t+s}h_{t+s+1}\right) = E_t\left(h_{t+s}\left(\omega + \left(\alpha+\lambda I_{t+s}^{-}\right)\varepsilon_{t+s}^2 + \beta h_{t+s}\right)\right) = \omega\tilde{\mu}_{h,s}^{(1)} + \varphi\tilde{\mu}_{h,s}^{(2)}$.

Hence the final expression for $\tilde{\mu}_{h,suv}^{(1,1,1)}$ is:

$$\begin{aligned}\tilde{\mu}_{h,suv}^{(1,1,1)} &= \left(\varphi^{v}\gamma^{u-1}\right)\left(\omega^2\tilde{\mu}_{h,s}^{(1)} + 2\omega\varphi\tilde{\mu}_{h,s}^{(2)} + \gamma\tilde{\mu}_{h,s}^{(3)}\right) \\ &+ \left(\bar{h}\varphi^{u-1}\left(1-\varphi^{v}\right) + 2\omega\varphi^{v+1}(\varphi-\gamma)^{-1}\left(\varphi^{u-1}-\gamma^{u-1}\right)\right)\left(\omega\tilde{\mu}_{h,s}^{(1)} + \varphi\tilde{\mu}_{h,s}^{(2)}\right) \\ &+ \left[\bar{h}^2\left(1-\varphi^{u-1}\right) + \varphi^{v}\begin{pmatrix} c_1 + \left(2\omega\varphi\bar{h}(\varphi-\gamma)^{-1} - c_1\right)\gamma^{u-1} \\ -2\omega\varphi^{u}\bar{h}(\varphi-\gamma)^{-1} - \bar{h}^2\left(1-\varphi^{u-1}\right)\end{pmatrix}\right]\tilde{\mu}_{h,s}^{(1)}.\end{aligned}$$

(d) *Fourth Conditional Moments of Forward and Aggregated Variances*

For the fourth moment of the forward variance we write:

$$\begin{aligned}\tilde{\mu}_{h,s}^{(4)} &= E_t\left[\left(\omega + \left(\alpha+\lambda I_{t+s-1}^{-}\right)\varepsilon_{t+s-1}^2 + \beta h_{t+s-1}\right)^4\right] \\ &= \omega^4 + 4\omega^3\varphi\tilde{\mu}_{h,s-1}^{(1)} + 6\omega^2\gamma\tilde{\mu}_{h,s-1}^{(2)} + 4\omega c_6\tilde{\mu}_{h,s-1}^{(3)} + c_7\tilde{\mu}_{h,s-1}^{(4)} \\ &= \sum_{j=0}^{s-2} c_7^{j}\left(\omega^4 + 4\omega^3\varphi\tilde{\mu}_{h,s-j-1}^{(1)} + 6\omega^2\gamma\tilde{\mu}_{h,s-j-1}^{(2)} + 4\omega c_6\tilde{\mu}_{h,s-j-1}^{(3)}\right) + c_7^{s-1}h_{t+1}^4\end{aligned}$$

where

$$\begin{aligned}c_7 &= \mu_z^{(8)}\left(\alpha^4 + F_0\left(\lambda^4 + 4\left(\alpha^3\lambda + \alpha\lambda^3\right) + 6\alpha^2\lambda^2\right)\right) + \beta^4 \\ &+ 4\left[\mu_z^{(6)}\beta\left(\alpha^3 + F_0\left(\lambda^3 + 3\left(\alpha^2\lambda + \alpha\lambda^2\right)\right)\right) + \beta^3\left(\alpha + \lambda F_0\right)\right] \\ &+ 6\kappa_z\beta^2\left(\alpha^2 + \lambda^2 F_0 + 2\alpha\lambda F_0\right).\end{aligned}$$

When the innovations are normally distributed, $\mu_z^{(8)} = 105$, and when they are Student t distributed $\mu_z^{(8)} = 105\frac{(\nu-2)^3}{(\nu-4)(\nu-6)(\nu-8)}$.

Finally, for the fourth moment of aggrregated variance we write:

$$
\begin{aligned}
\tilde{M}_{h,n}^{(4)} &= \sum_{s=1}^{n}\tilde{\mu}_{h,s}^{(4)}+\sum_{s=1}^{n}\sum_{u=1}^{n-s}\left(4\left(\tilde{\mu}_{h,su}^{(3,1)}+\tilde{\mu}_{h,su}^{(1,3)}\right)+6\tilde{\mu}_{h,su}^{(2,2)}\right)\\
&+ 12\sum_{s=1}^{n}\sum_{u=1}^{n-s}\sum_{v=1}^{n-s-u}\left(\tilde{\mu}_{h,suv}^{(2,1,1)}+\tilde{\mu}_{h,suv}^{(1,2,1)}+\tilde{\mu}_{h,suv}^{(1,1,2)}\right)+24\sum_{s=1}^{n}\sum_{u=1}^{n-s}\sum_{v=1}^{n-s-u}\sum_{w=1}^{n-s-u-v}\tilde{\mu}_{h,suvw}^{(1,1,1,1)},
\end{aligned}
$$

with

$$
\begin{aligned}
\tilde{\mu}_{h,su}^{(3,1)} &= E_t\left(h_{t+s}^3\left(\omega+\left(\alpha+\lambda I_{t+s+u-1}^-\right)\varepsilon_{t+s+u-1}^2+\beta h_{t+s+u-1}\right)\right)=\bar{h}\tilde{\mu}_{h,s}^{(3)}+\varphi^u\left(\tilde{\mu}_{h,s}^{(4)}-\bar{h}\tilde{\mu}_{h,s}^{(3)}\right),\\
\tilde{\mu}_{h,su}^{(1,3)} &= E_t\left(h_{t+s}\left(\omega+\left(\alpha+\lambda I_{t+s+u-1}^-\right)\varepsilon_{t+s+u-1}^2+\beta h_{t+s+u-1}\right)^3\right),\\
&= \omega^3\tilde{\mu}_{h,s}^{(1)}+3\omega\left(\omega\varphi\tilde{\mu}_{h,s(u-1)}^{(1,1)}+\gamma\tilde{\mu}_{h,s(u-1)}^{(1,2)}\right)+c_6\tilde{\mu}_{h,s(u-1)}^{(1,3)}\\
&= \sum_{j=0}^{u-1}c_6^j\left(\omega^3\tilde{\mu}_{h,s}^{(1)}+3\omega\left(\omega\varphi\tilde{\mu}_{h,s(u-j-1)}^{(1,1)}+\gamma\tilde{\mu}_{h,s(u-j-1)}^{(1,2)}\right)\right)+c_6^u\tilde{\mu}_{h,s}^{(4)},\\
\tilde{\mu}_{h,su}^{(2,2)} &= E_t\left(h_{t+s}^2\left(\omega+\left(\alpha+\lambda I_{t+s+u-1}^-\right)\varepsilon_{t+s+u-1}^2+\beta h_{t+s+u-1}\right)^2\right)\\
&= \omega^2\tilde{\mu}_{h,s}^{(2)}+2\omega\varphi\tilde{\mu}_{h,s(u-1)}^{(2,1)}+\gamma\tilde{\mu}_{h,s(u-1)}^{(2,2)}=\sum_{j=0}^{u-1}\gamma^j\left(\omega^2\tilde{\mu}_{h,s}^{(2)}+2\omega\varphi\tilde{\mu}_{h,s(u-j-1)}^{(2,1)}\right)+\gamma^u\tilde{\mu}_{h,s}^{(4)},\\
\tilde{\mu}_{h,suv}^{(2,1,1)} &= E_t\left(h_{t+s}^2E_{t+s}\left(h_{t+s+u}h_{t+s+u+v}\right)\right)=E_t\left(h_{t+s}^2E_{t+s}\left(h_{t+s+u}h_{t+s+u+v}\right)\right).
\end{aligned}
$$

By analogy with $\tilde{\mu}_{h,suv}^{(1,1,1)}$ we obtain the following expression for $\tilde{\mu}_{h,suv}^{(2,1,1)}$:

$$
\begin{aligned}
\tilde{\mu}_{h,suv}^{(2,1,1)} &+ \left(\varphi^v\gamma^{u-1}\right)\left(\omega^2\tilde{\mu}_{h,s}^{(2)}+2\omega\varphi\tilde{\mu}_{h,s}^{(3)}+\gamma\tilde{\mu}_{h,s}^{(4)}\right)\\
&+ \left(\bar{h}\varphi^{u-1}\left(1-\varphi^v\right)+2\omega\varphi^{v+1}(\varphi-\gamma)^{-1}\left(\varphi^{u-1}-\gamma^{u-1}\right)\right)\left(\omega\tilde{\mu}_{h,s}^{(2)}+\varphi\tilde{\mu}_{h,s}^{(3)}\right)\\
&+ \left[\bar{h}^2\left(1-\varphi^{u-1}\right)+\varphi^v\left(\begin{array}{c} c_1+\left(2\omega\varphi\bar{h}(\varphi-\gamma)^{-1}-c_1\right)\gamma^{u-1}\\ -2\omega\varphi^u\bar{h}(\varphi-\gamma)^{-1}-\bar{h}^2\left(1-\varphi^{u-1}\right)\end{array}\right)\right]\tilde{\mu}_{h,s}^{(2)}.
\end{aligned}
$$

Using the tower law we can also solve for $\tilde{\mu}_{h,suv}^{(1,2,1)}$, $\tilde{\mu}_{h,suv}^{(1,1,2)}$ and $\tilde{\mu}_{h,suv}^{(1,1,1,1)}$.

(e) *Centred Conditional Moments of Forward and Aggregated Variances*

The second centred moment of the forward variance, i.e. the conditional variance of the forward conditional variance is:

$$\mu^{(2)}_{h,s} = E_t\left(\left(h_{t+s} - \tilde{\mu}^{(1)}_{h,s}\right)^2\right) = \tilde{\mu}^{(2)}_{h,s} - \left(\tilde{\mu}^{(1)}_{h,s}\right)^2.$$

The second centred moment of the aggregated variance, i.e. the conditional variance of the aggregated conditional variance is:

$$\begin{aligned} M^{(2)}_{h,n} &= E_t\left(\left(\sum_{s=1}^{n}\left(h_{t+s} - \tilde{\mu}^{(1)}_{h,s}\right)\right)^2\right) = \sum_{s=1}^{n}\left(\tilde{\mu}^{(2)}_{h,s} - \left(\tilde{\mu}^{(1)}_{h,s}\right)^2\right) + 2\sum_{s=1}^{n}\sum_{u=1}^{n-s}\left(\tilde{\mu}^{(1,1)}_{h,su} - \tilde{\mu}^{(1)}_{h,s}\tilde{\mu}^{(1)}_{h,s+u}\right) \\ &= \sum_{s=1}^{n}\tilde{\mu}^{(2)}_{h,s} + 2\sum_{s=1}^{n}\sum_{u=1}^{n-s}\varphi^u\tilde{\mu}^{(2)}_{h,s} - \sum_{s=1}^{n}\left(\tilde{\mu}^{(1)}_{h,s}\right)^2 + 2\sum_{s=1}^{n}\sum_{u=1}^{n-s}\left(1-\varphi^u\right)\bar{h}\tilde{\mu}^{(1)}_{h,s} - 2\sum_{s=1}^{n}\sum_{u=1}^{n-s}\tilde{\mu}^{(1)}_{h,s}\tilde{\mu}^{(1)}_{h,s+u} \end{aligned}$$

or

$$M^{(2)}_{h,n} = \tilde{M}^{(2)}_{h,n} - \sum_{s=1}^{n}\left(\tilde{\mu}^{(1)}_{h,s}\right)^2 - 2\sum_{s=1}^{n}\sum_{u=1}^{n-s}\tilde{\mu}^{(1)}_{h,s}\tilde{\mu}^{(1)}_{h,s+u}, \tag{11}$$

where

$$\begin{aligned} \sum_{s=1}^{n}\left(\tilde{\mu}^{(1)}_{h,s}\right)^2 &= \sum_{s=1}^{n}\left(\bar{h} + \varphi^{s-1}\left(h_{t+1} - \bar{h}\right)\right)^2 \qquad (12) \\ &= n\bar{h}^2 + \left(h_{t+1} - \bar{h}\right)^2\left(1-\varphi^2\right)^{-1}\left(1-\varphi^{2n}\right) + 2\bar{h}\left(h_{t+1} - \bar{h}\right)\left(1-\varphi\right)^{-1}\left(1-\varphi^n\right) \end{aligned}$$

and

$$\begin{aligned} \sum_{s=1}^{n}\sum_{u=1}^{n-s}\tilde{\mu}^{(1)}_{h,s}\tilde{\mu}^{(1)}_{h,s+u} &= \sum_{s=1}^{n}\sum_{u=1}^{n-s}\left(\bar{h} + \varphi^{s-1}\left(h_{t+1} - \bar{h}\right)\right)\left(\bar{h} + \varphi^{s+u-1}\left(h_{t+1} - \bar{h}\right)\right) \qquad (13) \\ &= \frac{1}{2}n\left(n-1\right)\bar{h}^2 + \bar{h}\left(h_{t+1} - \bar{h}\right)\left(1-\varphi\right)^{-1}\left[n - \left(1-\varphi\right)^{-1}\left(1-\varphi^n\right)\right] \\ &+ \bar{h}\left(h_{t+1} - \bar{h}\right)\left(1-\varphi\right)^{-1}\left[\varphi(1-\varphi)^{-1}\left(1-\varphi^n\right) - n\varphi^n\right] \\ &+ \left(h_{t+1} - \bar{h}\right)^2(1-\varphi)^{-1}\left[\varphi\left(1-\varphi^2\right)^{-1}\left(1-\varphi^{2n}\right) - (1-\varphi)^{-1}\varphi^n\left(1-\varphi^n\right)\right] \end{aligned}$$

For $\gamma \neq 1$, the expression for the second moment of the aggregated variance is given by (8). For $\gamma = 1$, consider the first formula above for $M^{(2)}_{h,n}$. The expressions for the last three sums do not depend on γ and hence remain the same as in the $\gamma \neq 1$ case, while

$$\begin{aligned} \sum_{s=1}^{n}\tilde{\mu}^{(2)}_{h,s} &= \sum_{s=1}^{n}\left[(s-1)\left(\omega^2 + 2\omega\varphi\bar{h}\right) + 2\varphi\bar{h}\left(1-\varphi^{s-1}\right)\left(h_{t+1} - \bar{h}\right) + h^2_{t+1}\right] \qquad (14) \\ &= \frac{1}{2}n\left(n-1\right)\left(\omega^2 + 2\omega\varphi\bar{h}\right) + 2\varphi\bar{h}\left(h_{t+1} - \bar{h}\right)\left(n - \left(1-\varphi\right)^{-1}\left(1-\varphi^n\right)\right) + nh^2_{t+1} \end{aligned}$$

and

$$\sum_{s=1}^{n}\sum_{u=1}^{n-s}\varphi^{u}\tilde{\mu}_{h,s}^{(2)}=\sum_{s=1}^{n}\tilde{\mu}_{h,s}^{(2)}\sum_{u=1}^{n-s}\varphi^{u}=\varphi(1-\varphi)^{-1}\left[\sum_{s=1}^{n}\tilde{\mu}_{h,s}^{(2)}-\sum_{s=1}^{n}\varphi^{n-s}\tilde{\mu}_{h,s}^{(2)}\right] \tag{15}$$

and

$$\begin{aligned}\sum_{s=1}^{n}\varphi^{n-s}\tilde{\mu}_{h,s}^{(2)} &= \sum_{s=1}^{n}\varphi^{n-s}\left[(s-1)\left(\omega^{2}+2\omega\varphi\bar{h}\right)+2\varphi\bar{h}\left(1-\varphi^{s-1}\right)\left(h_{t+1}-\bar{h}\right)+h_{t+1}^{2}\right] \\ &= \left(\omega^{2}+2\omega\varphi\bar{h}\right)(1-\varphi)^{-1}\left[n-(1-\varphi)^{-1}\left(1-\varphi^{n}\right)\right]-2n\bar{h}\left(h_{t+1}-\bar{h}\right)\varphi^{n} \\ &+ \left[2\varphi\bar{h}\left(h_{t+1}-\bar{h}\right)+h_{t+1}^{2}\right](1-\varphi)^{-1}\left(1-\varphi^{n}\right).\end{aligned} \tag{16}$$

The third centred moment of the forward variance is:

$$\mu_{h,s}^{(3)}=E_{t}\left(\left(h_{t+s}-\tilde{\mu}_{h,s}^{(1)}\right)^{3}\right)=\tilde{\mu}_{h,s}^{(3)}-3\tilde{\mu}_{h,s}^{(2)}\tilde{\mu}_{h,s}^{(1)}+2\left(\tilde{\mu}_{h,s}^{(1)}\right)^{3} \tag{17}$$

and the third centred moment of the aggregated variance is:

$$\begin{aligned}M_{h,n}^{(3)} &= E_{t}\left(\left(\sum_{s=1}^{n}\left(h_{t+s}-\tilde{\mu}_{h,s}^{(1)}\right)\right)^{3}\right) \\ &= \sum_{s=1}^{n}\left(\tilde{\mu}_{h,s}^{(3)}-3\tilde{\mu}_{h,s}^{(2)}\tilde{\mu}_{h,s}^{(1)}+2\left(\tilde{\mu}_{h,s}^{(1)}\right)^{3}\right)+3\sum_{s=1}^{n}\sum_{u=1}^{n-s}\left(\begin{array}{l}\tilde{\mu}_{h,su}^{(2,1)}+\tilde{\mu}_{h,su}^{(1,2)}+2\left(\tilde{\mu}_{h,s}^{(1)}+\tilde{\mu}_{h,s+u}^{(1)}\right) \\ \left(\tilde{\mu}_{h,s}^{(1)}\tilde{\mu}_{h,s+u}^{(1)}-\tilde{\mu}_{h,su}^{(1,1)}\right)-\tilde{\mu}_{h,s}^{(1)}\tilde{\mu}_{h,s+u}^{(2)}-\tilde{\mu}_{h,s+u}^{(1)}\tilde{\mu}_{h,s}^{(2)}\end{array}\right) \\ &+ 6\sum_{s=1}^{n}\sum_{u=1}^{n-s}\sum_{v=1}^{n-s-u}\left(\begin{array}{l}\tilde{\mu}_{h,suv}^{(1,1,1)}-\tilde{\mu}_{h,s}^{(1)}\tilde{\mu}_{h,(s+u)v}^{(1,1)}-\tilde{\mu}_{h,(s+u)}^{(1)}\tilde{\mu}_{h,s(u+v)}^{(1,1)} \\ -\tilde{\mu}_{h,(s+u+v)}^{(1)}\tilde{\mu}_{h,su}^{(1,1)}+2\tilde{\mu}_{h,s}^{(1)}\tilde{\mu}_{h,(s+u)}^{(1)}\tilde{\mu}_{h,(s+u+v)}^{(1)}\end{array}\right).\end{aligned}$$

The fourth centred moment of the forward variance is:

$$\mu_{h,s}^{(4)}=\tilde{\mu}_{h,s}^{(4)}-4\tilde{\mu}_{h,s}^{(1)}\tilde{\mu}_{h,s}^{(3)}+6\left(\tilde{\mu}_{h,s}^{(1)}\right)^{2}\tilde{\mu}_{h,s}^{(2)}-3\left(\tilde{\mu}_{h,s}^{(1)}\right)^{4}.$$

Finally, the fourth centred moment of the aggregated variance is:

$$
\begin{aligned}
M_{h,n}^{(4)} &= E_t\left(\left(\sum_{s=1}^{n}\left(h_{t+s}-\tilde{\mu}_{h,s}^{(1)}\right)\right)^4\right) = \sum_{s=1}^{n} E_t\left(\left(h_{t+s}-\tilde{\mu}_{h,s}^{(1)}\right)^4\right) \\
&+ \sum_{s=1}^{n}\sum_{u=1}^{n-s}\left(\begin{array}{l} 4\left(E_t\left(\left(h_{t+s}-\tilde{\mu}_{h,s}^{(1)}\right)^3\left(h_{t+s+u}-\tilde{\mu}_{h,s+u}^{(1)}\right)\right)+E_t\left(\left(h_{t+s}-\tilde{\mu}_{h,s}^{(1)}\right)\left(h_{t+s+u}-\tilde{\mu}_{h,s+u}^{(1)}\right)^3\right)\right) \\ +6E_t\left(\left(h_{t+s}-\tilde{\mu}_{h,s}^{(1)}\right)^2\left(h_{t+s+u}-\tilde{\mu}_{h,s+u}^{(1)}\right)^2\right) \end{array}\right) \\
&+ 12\sum_{s=1}^{n}\sum_{u=1}^{n-s}\sum_{v=1}^{n-s-u}\left[\begin{array}{l} E_t\left(\left(h_{t+s}-\tilde{\mu}_{h,s}^{(1)}\right)^2\left(h_{t+s+u}-\tilde{\mu}_{h,s+u}^{(1)}\right)\left(h_{t+s+u+v}-\tilde{\mu}_{h,s+u+v}^{(1)}\right)\right) \\ +E_t\left(\left(h_{t+s}-\tilde{\mu}_{h,s}^{(1)}\right)\left(h_{t+s+u}-\tilde{\mu}_{h,s+u}^{(1)}\right)^2\left(h_{t+s+u+v}-\tilde{\mu}_{h,s+u+v}^{(1)}\right)\right) \\ +E_t\left(\left(h_{t+s}-\tilde{\mu}_{h,s}^{(1)}\right)\left(h_{t+s+u}-\tilde{\mu}_{h,s+u}^{(1)}\right)\left(h_{t+s+u+v}-\tilde{\mu}_{h,s+u+v}^{(1)}\right)^2\right) \end{array}\right] \\
&+ 24\sum_{s=1}^{n}\sum_{u=1}^{n-s}\sum_{v=1}^{n-s-u}\sum_{w=1}^{n-s-u-v} E_t\left[\begin{array}{l} \left(h_{t+s}-\tilde{\mu}_{h,s}^{(1)}\right)\left(h_{t+s+u}-\tilde{\mu}_{h,s+u}^{(1)}\right) \\ \left(h_{t+s+u+v}-\tilde{\mu}_{h,s+u+v}^{(1)}\right)\left(h_{t+s+u+v+w}-\tilde{\mu}_{h,s+u+v+w}^{(1)}\right) \end{array}\right].
\end{aligned}
$$

Performing the necessary algebaraic calculations yields the formula for $M_{h,n}^{(4)}$ from Theorem 2. The standardized moments (i.e. skewness and kurtosis) of the forward and aggregated variance distributions are now easily obtained from the central moments, as defined in Section 2.1.

A3: Limits of the Moments of Returns

This appendix derives the limits of the conditional moments of the forward and aggregated returns of the generic GJR model as the time horizon increases. When they differ from the generic result we state results for the two important special cases, namely the normal GJR (i.e. $D(0,1)$ is now the standard normal) and also the normal GARCH(1,1) (i.e. $D(0,1)$ is the standard normal and $\lambda = 0$). When $D(0,1)$ is the standard normal, $\tau_z = 0$, $F_0 = \frac{1}{2}$ and $\kappa_z = 3$ and φ and γ become (for the normal GJR):

$$\varphi = \alpha + \frac{\lambda}{2} + \beta \text{ and } \gamma = \varphi^2 + 2\left(\alpha + \frac{\lambda}{2}\right)^2 + \frac{3}{4}\lambda^2 \,. \tag{18}$$

Moreover, for the normal GARCH(1,1), $\lambda = 0$ and the two constants above simplify further:

$$\varphi = \alpha + \beta \text{ and } \gamma = \varphi^2 + 2\alpha^2 = (\alpha+\beta)^2 + 2\alpha^2 \,. \tag{19}$$

We assume $\varphi \in (0,1)$ and $\varphi \neq \gamma$.

(a) *Limits of the Forward and Aggregated Conditional Variance*

Both the forward and aggregated variance limits, expressed in daily units, are equal to the long-term variance, which we have denoted by $\bar{h}$. That is,

$$\lim_{s\to\infty}\mu_{r,s}^{(2)}=\lim_{s\to\infty}\tilde{\mu}_{h,s}^{(1)}=\lim_{s\to\infty}\left(\bar{h}+\varphi^{s-1}\left(h_{t+1}-\bar{h}\right)\right)=\bar{h}$$

and

$$\lim_{n\to\infty}\frac{M_{r,n}^{(2)}}{n}=\lim_{n\to\infty}\left[\frac{n\bar{h}+(1-\varphi)^{-1}(1-\varphi^{n})\left(h_{t+1}-\bar{h}\right)}{n}\right]=\bar{h}.$$

(b) *Limits of the Forward and Aggregated Conditional Skewness*

The forward skewness limit is:

$$\lim_{s\to\infty}\tau_{r,s}=\lim_{s\to\infty}\left[\frac{1}{8}\tau_z\left(5+3\tilde{\mu}_{h,s}^{(2)}\left(\tilde{\mu}_{h,s}^{(1)}\right)^{-2}\right)\right]=\frac{1}{8}\tau_z\left(5+3\lim_{s\to\infty}\tilde{\mu}_{h,s}^{(2)}\left(\lim_{s\to\infty}\tilde{\mu}_{h,s}^{(1)}\right)^{-2}\right)$$

where $\lim_{s\to\infty}\tilde{\mu}_{h,s}^{(2)}=\begin{cases}c_1 & \text{if } \gamma\in(0,1)\,,\\ \infty & \text{if } \gamma\in[1,\infty)\,.\end{cases}$

Hence:

$$\lim_{s\to\infty}\tau_{r,s}=\begin{cases}\frac{1}{8}\tau_z\left(5+3\left(\omega^2+2\omega\varphi\bar{h}\right)(1-\gamma)^{-1}\left(\bar{h}\right)^{-2}\right) & \text{if } \gamma\in(0,1)\,,\\ \operatorname{sgn}(\tau_z)\,\infty & \text{if } \gamma\in[1,\infty)\,.\end{cases}$$

For the normal GJR and the normal GARCH(1,1), $\tau_{r,s}=\lim_{s\to\infty}\tau_{r,s}=0$.

For the limit of the aggregated skewness set:

$$c_{12}=\tfrac{1}{8}\left(\tau_z+3\left(\alpha\tau_z+\lambda\int_{x=-\infty}^{0}x^3f(x)\,dx\right)(1-\varphi)^{-1}\right)$$

$$c_{13}=\tfrac{3}{8}\left(\left(\alpha\tau_z+\lambda\int_{x=-\infty}^{0}x^3f(x)\,dx\right)(1-\varphi)^{-1}\right)=c_{12}-\tfrac{\tau_z}{8}.$$

We have:

$$\begin{aligned}\lim_{n\to\infty}\mathrm{T}_{r,n}&=\lim_{n\to\infty}\tfrac{M_{r,n}^{(3)}}{n^{3/2}}\left(\tfrac{M_{r,n}^{(2)}}{n}\right)^{-3/2}=\lim_{n\to\infty}\tfrac{M_{r,n}^{(3)}}{n^{3/2}}\left[\lim_{n\to\infty}\left(\tfrac{M_{r,n}^{(2)}}{n}\right)\right]^{-3/2}\\&=c_{13}\bar{h}^{-3/2}\lim_{n\to\infty}\left[n^{-3/2}\sum_{s=1}^{n}\left(\tfrac{c_{12}}{c_{13}}-\varphi^{n-s}\right)\left(5\left(\tilde{\mu}_{h,s}^{(1)}\right)^{3/2}+3\tilde{\mu}_{h,s}^{(2)}\left(\tilde{\mu}_{h,s}^{(1)}\right)^{-1/2}\right)\right].\end{aligned}$$

Write $L = \lim_{n\to\infty} S_n$, $L_1 = \lim_{n\to\infty} S_{1,n}$, and $L_2 = \lim_{n\to\infty} S_{2,n}$ with

$$\begin{aligned}
S_n &= \frac{c_{13}\sum_{s=1}^{n}\left(\frac{c_{12}}{c_{13}} - \varphi^{n-s}\right)\left(5\left(\tilde{\mu}_{h,s}^{(1)}\right)^{3/2} + 3\tilde{\mu}_{h,s}^{(2)}\left(\tilde{\mu}_{h,s}^{(1)}\right)^{-1/2}\right)}{n^{3/2}}, \\
S_{1,n} &= \frac{c_{13}\sum_{s=1}^{n}\left(\frac{c_{12}}{c_{13}} - \varphi^{n-s}\right)\left(5\left(\max_{1\le s\le n}\left(\tilde{\mu}_{r,s}^{(1)}\right)\right)^{3/2} + 3\tilde{\mu}_{r,s}^{(2)}\left(\min_{1\le s\le n}\left(\tilde{\mu}_{r,s}^{(1)}\right)\right)^{-1/2}\right)}{n^{3/2}}, \\
S_{2,n} &= \frac{c_{13}\sum_{s=1}^{n}\left(\frac{c_{12}}{c_{13}} - \varphi^{n-s}\right)\left(5\left(\min_{1\le s\le n}\left(\tilde{\mu}_{r,s}^{(1)}\right)\right)^{3/2} + 3\tilde{\mu}_{r,s}^{(2)}\left(\max_{1\le s\le n}\left(\tilde{\mu}_{r,s}^{(1)}\right)\right)^{-1/2}\right)}{n^{3/2}}
\end{aligned}$$

<u>Case 1</u>

$c_{14} = \frac{c_{12}}{c_{13}} \le 0$, then $c_{14} - \varphi^{n-s} \le 0$ for any s. Also, if $\tau_z < 0$, then $S_{1,n} \le S_n \le S_{2,n}$ for any s.[7] Hence, if $L_1 = L_2$ then $L = L_1 = L_2$, by the squeeze theorem. We now prove that $L_1 = L_2$. Setting $L_{\max} = \lim_{n\to\infty}\left(\max_{1\le s\le n}\ \tilde{\mu}_{h,s}^{(1)}\right) = \max\left(\bar{h}, h_{t+1}\right)$ and $L_{\min} = \lim_{n\to\infty}\left(\min_{1\le s\le n}\ \tilde{\mu}_{h,s}^{(1)}\right) = \min\left(\bar{h}, h_{t+1}\right)$ we may write:

$$L_1 = c_{13}\left[5L_{\max}^{3/2}\lim_{n\to\infty}\left(n^{-1/2}c_{14} - n^{-3/2}(1-\varphi)^{-1}\left(1-\varphi^n\right)\right) + 3L_{\min}^{-1/2}\lim_{n\to\infty} n^{-3/2}\sum_{s=1}^{n}\left(c_{14} - \varphi^{n-s}\right)\tilde{\mu}_{h,s}^{(2)}\right]$$

and the first term above is zero. For $\gamma \neq 1$,

$$\begin{aligned}
\sum_{s=1}^{n}\left(c_{14} - \varphi^{n-s}\right)\tilde{\mu}_{h,s}^{(2)} &= \sum_{s=1}^{n}\left(c_{14} - \varphi^{n-s}\right)\left(c_1 + \left(h_{t+1}^2 - c_3\right)\gamma^{s-1} + c_2\varphi^{s-1}\right) \\
&= nc_1c_{14} + c_{14}\left(h_{t+1}^2 - c_3\right)(1-\gamma)^{-1}\left(1-\gamma^n\right) + \left(c_2c_{14} - c_1\right)(1-\varphi)^{-1}\left(1-\varphi^n\right) \\
&- \left(h_{t+1}^2 - c_3\right)(\varphi-\gamma)^{-1}\left(\varphi^n - \gamma^n\right) - nc_2\varphi^{n-1}\,.
\end{aligned}$$

For $\gamma = 1$,

$$\sum_{s=1}^{n}\left(c_{14} - \varphi^{n-s}\right)\tilde{\mu}_{h,s}^{(2)} = \sum_{s=1}^{n}\left(c_{14} - \varphi^{n-s}\right)\begin{bmatrix}(s-1)\left(\omega^2 + 2\omega\varphi\bar{h}\right) \\ +2\varphi\bar{h}\left(1-\varphi^{s-1}\right)\left(h_{t+1} - \bar{h}\right) + \tilde{\mu}_{h,1}^{(2)}\end{bmatrix}$$

$$\begin{aligned}
&= {}^1\!/\!_2 n(n-1)c_{14}\left(\omega^2 + 2\omega\varphi h_0\right)\bar{h} - \varphi\left(\omega^2 + 2\omega\varphi\bar{h}\right)(1-\varphi)^{-1} \\
&\left[(1-\varphi)^{-1}\left(1-\varphi^n\right) - \varphi^{-1}(n-1)\right] + 2c_{14}\varphi h_0\left(h_{t+1} - \bar{h}\right)\left(n - (1-\varphi)^{-1}\left(1-\varphi^n\right)\right) \\
&-2\varphi\bar{h}\left(h_{t+1} - \bar{h}\right)\left[(1-\varphi)^{-1}\left(1-\varphi^n\right) - n\varphi^{n-1}\right] + \tilde{\mu}_{h,1}^{(2)}\left(nc_{14} - (1-\varphi)^{-1}\left(1-\varphi^n\right)\right).
\end{aligned}$$

Hence

[7] If $\tau_z > 0$, then $S_{2,n} \le S_n \le S_{1,n}$. However, the limit does not change: the proof above still applies, only that $S_{2,n}$ and $S_{1,n}$ swap place above.

$$L_1 = \begin{cases} 0 & \text{if } \gamma \in (0,1)\,, \\ \operatorname{sgn}(c_{13}) \operatorname{sgn}\left(c_{14}\left(\omega^2 + 2\omega\varphi\bar{h}\right)\right)\infty & \text{if } \gamma = 1, \\ \operatorname{sgn}(c_{13}) \operatorname{sgn}\left[\left((\varphi-\gamma)^{-1} - c_{14}(1-\gamma)^{-1}\right)\left(h_{t+1}^2 - c_3\right)\right]\infty & \text{if } \gamma \in (1,\infty)\,. \end{cases}$$

Now, for $\gamma \in [1,\infty)$, $\operatorname{sgn}\left(h_{t+1}^2 - c_3\right) = 1$, $\operatorname{sgn}\left((\varphi-\gamma)^{-1} - c_{14}(1-\gamma)^{-1}\right) = -1$ and $\operatorname{sgn}\left(\omega^2 + 2\omega\varphi h_0\right) = 1$,

so

$$L_1 = \begin{cases} 0 & \text{if} \gamma \in (0,1)\,, \\ \operatorname{sgn}(c_{12})\infty & \text{if} \gamma \in [1,\infty)\,. \end{cases}$$

Analogously it can be shown that L_2 is the same limit, hence $L_1 = L_2$ and finally:

$$\lim_{n\to\infty} \mathrm{T}_{r,n} = \begin{cases} 0 & \text{if} \gamma \in (0,1)\,, \\ \operatorname{sgn}(c_{12})\infty & \text{if} \gamma \in [1,\infty)\,. \end{cases}$$

Case 2

$c_{14} > 0$: In this case $\exists$ an integer $\tilde{s} \geq 1$ such that $c_{14} - \varphi^{n-s} > 0$ for $\forall s > \tilde{s}$ and $c_{14} - \varphi^{n-s} \leq 0$ for $\forall\ \tilde{s} \geq s \geq 1$. It can be easily seen that if $c_{14} > \varphi$, then $\tilde{s} = 0$.

Write $S_n = S_n^{(1)} + S_n^{(2)}$, $\tilde{S}_{1,n} = S_{1,n}^{(1)} + S_{2,n}^{(2)}$, $\tilde{S}_{2,n} = S_{2,n}^{(1)} + S_{1,n}^{(2)}$ where

$$\begin{aligned} S_n^{(1)} &= \frac{c_{13}\sum_{s=1}^{\tilde{s}}\left(c_{14} - \varphi^{n-s}\right)\left(5\left(\tilde{\mu}_{h,s}^{(1)}\right)^{3/2} + 3\tilde{\mu}_{h,s}^{(2)}\left(\tilde{\mu}_{h,s}^{(1)}\right)^{-1/2}\right)}{n^{3/2}}, \\ S_n^{(2)} &= \frac{c_{13}\sum_{s=\tilde{s}+1}^{n}\left(c_{14} - \varphi^{n-s}\right)\left(5\left(\tilde{\mu}_{h,s}^{(1)}\right)^{3/2} + 3\tilde{\mu}_{h,s}^{(2)}\left(\tilde{\mu}_{h,s}^{(1)}\right)^{-1/2}\right)}{n^{3/2}}, \end{aligned}$$

and

$$\begin{aligned} S_{1,n}^{(1)} &= \frac{c_{13}\sum_{s=1}^{\tilde{s}}\left(c_{14} - \varphi^{n-s}\right)\left(5\left(\max\limits_{1\leq s\leq n}\left(\tilde{\mu}_{r,s}^{(1)}\right)\right)^{3/2} + 3\tilde{\mu}_{r,s}^{(2)}\left(\min\limits_{1\leq s\leq n}\left(\tilde{\mu}_{r,s}^{(1)}\right)\right)^{-1/2}\right)}{n^{3/2}}, \\ S_{2,n}^{(2)} &= \frac{c_{13}\sum_{s=\tilde{s}+1}^{n}\left(c_{14} - \varphi^{n-s}\right)\left(5\left(\min\limits_{1\leq s\leq n}\left(\tilde{\mu}_{r,s}^{(1)}\right)\right)^{3/2} + 3\tilde{\mu}_{r,s}^{(2)}\left(\max\limits_{1\leq s\leq n}\left(\tilde{\mu}_{r,s}^{(1)}\right)\right)^{-1/2}\right)}{n^{3/2}}. \end{aligned}$$

If we solve further for $S_{1,n}^{(1)}$, we have:

$$S_{1,n}^{(1)} = c_{13}\left[\begin{array}{l} 5\left(\max\limits_{1\leq s\leq n}\left(\tilde{\mu}_{h,s}^{(1)}\right)\right)^{3/2} n^{-3/2}\left(\tilde{s}c_{14} - \varphi^{n-\tilde{s}}\sum\limits_{s=1}^{\tilde{s}}\varphi^{\tilde{s}-s}\right) \\ +3\left(\min\limits_{1\leq s\leq n}\left(\tilde{\mu}_{h,s}^{(1)}\right)\right)^{-1/2} n^{-3/2}\left(c_{14}\sum\limits_{s=1}^{\tilde{s}}\tilde{\mu}_{h,s}^{(2)} - \varphi^{n-\tilde{s}}\sum\limits_{s=1}^{\tilde{s}}\varphi^{\tilde{s}-s}\tilde{\mu}_{h,s}^{(2)}\right)\end{array}\right].$$

Define: $f_1^{a,b} = \sum\limits_{s=a}^{b}\varphi^{b-s}$, $\quad f_2^{a,b} = \sum\limits_{s=a}^{b}\tilde{\mu}_{h,s}^{(2)}$ $\quad$ and $\quad f_3^{a,b} = \sum\limits_{s=a}^{b}\varphi^{b-s}\tilde{\mu}_{h,s}^{(2)}$. Then

$$S_{1,n}^{(1)} = c_{13}\left[5\left(\max_{1\leq s\leq n}\left(\tilde{\mu}_{h,s}^{(1)}\right)\right)^{3/2} n^{-3/2}\left[\tilde{s}c_{14} - \varphi^{n-\tilde{s}}f_1^{1,\tilde{s}}\right] + 3\left(\min_{1\leq s\leq n}\left(\tilde{\mu}_{h,s}^{(1)}\right)\right)^{-1/2} n^{-3/2}\left[f_2^{1,\tilde{s}} - \varphi^{n-\tilde{s}}f_3^{1,\tilde{s}}\right]\right],$$

and since $f_j^{1,\tilde{s}}$ $j = 1,2,3$ are all constant w.r.t. n we have $\lim\limits_{n\to\infty} S_{1,n}^{(1)} = 0$. Also

$$S_{2,n}^{(2)} = c_{13}\left[\begin{array}{l} 5\left(\min\limits_{1<s\leq n}\left(\tilde{\mu}_{h,s}^{(1)}\right)\right)^{3/2} n^{-3/2}\left(\left(n-\tilde{s}\right)c_{14} - f_1^{\tilde{s}+1,n}\right) \\ +3\left(\max\limits_{1<s\leq n}\left(\tilde{\mu}_{h,s}^{(1)}\right)\right)^{-1/2} n^{-3/2}\left(c_{14}f_2^{\tilde{s}+1,n} - f_3^{\tilde{s}+1,n}\right)\end{array}\right].$$

Now $f_1^{\tilde{s}+1,n} = (1-\varphi)^{-1}\left(1-\varphi^{n-\tilde{s}}\right)$. Also, for $\gamma \neq 1$, we have

$$\begin{aligned} f_2^{\tilde{s}+1,n} &= \sum_{s=\tilde{s}+1}^{n}\left(c_1 + \left(h_{t+1}^2 - c_3\right)\gamma^{s-1} + c_2\varphi^{s-1}\right) \\ &= \left(n-\tilde{s}\right)c_1 + \left(h_{t+1}^2 - c_3\right)\gamma^{\tilde{s}}(1-\gamma)^{-1}\left(1-\gamma^{n-\tilde{s}}\right) + c_2\varphi^{\tilde{s}}(1-\varphi)^{-1}\left(1-\varphi^{n-\tilde{s}}\right), \\ f_3^{\tilde{s}+1,n} &= \sum_{s=\tilde{s}+1}^{n}\varphi^{n-s}\left(c_1 + \left(h_{t+1}^2 - c_3\right)\gamma^{s-1} + c_2\varphi^{s-1}\right) \\ &= c_1(1-\varphi)^{-1}\left(1-\varphi^{n-\tilde{s}}\right) + \left(h_{t+1}^2 - c_3\right)\gamma^{\tilde{s}}(\varphi-\gamma)^{-1}\left(\varphi^{n-\tilde{s}} - \gamma^{n-\tilde{s}}\right) + c_2\left(n-\tilde{s}\right)\varphi^{n-1}. \end{aligned}$$

For $\gamma = 1$,

$$
\begin{aligned}
f_2^{\tilde{s}+1,n} &= \sum_{s=\tilde{s}+1}^{n} \left((s-1)\left(\omega^2 + 2\omega\varphi\bar{h}\right) + 2\varphi\bar{h}\left(1-\varphi^{s-1}\right)\left(h_{t+1}-\bar{h}\right) + h_{t+1}^2\right) \\
&= \frac{1}{2}\left(\omega^2 + 2\omega\varphi\bar{h}\right)(n-\tilde{s})(n-\tilde{s}-1) - 2\varphi\bar{h}\left(h_{t+1}-\bar{h}\right)\varphi^{\tilde{s}}(1-\varphi)^{-1}\left(1-\varphi^{n-\tilde{s}}\right) \\
&+ \left(2\varphi\bar{h}\left(h_{t+1}-\bar{h}\right) + h_{t+1}^2\right)(n-\tilde{s})\ , \\
f_3^{\tilde{s}+1,n} &= \sum_{s=\tilde{s}+1}^{n} \varphi^{n-s}\tilde{\mu}_{h,s}^{(2)} = \sum_{s=\tilde{s}+1}^{n} \varphi^{n-s}\left((s-1)\left(\omega^2 + 2\omega\varphi\bar{h}\right) + 2\varphi h_0\left(1-\varphi^{s-1}\right)\left(h_{t+1}-\bar{h}\right) + h_{t+1}^2\right) \\
&= \left(\omega^2 + 2\omega\varphi\bar{h}\right)(1-\varphi)^{-1}\left(\varphi(1-\varphi)^{-1}\left(1-\varphi^{n-\tilde{s}}\right) - \tilde{s}\varphi^{n-\tilde{s}}\right) \\
&+ \left[2\varphi\bar{h}\left(h_{t+1}-\bar{h}\right) + h_{t+1}^2\right](1-\varphi)^{-1}\left(1-\varphi^{n-\tilde{s}}\right) - \left(2\varphi\bar{h}\left(h_{t+1}-\bar{h}\right)\right)(n-\tilde{s})\varphi^{n-1}\,.
\end{aligned}
$$

Hence:

$$
\begin{aligned}
\lim_{n\to\infty}\tilde{S}_{1,n} = \lim_{n\to\infty}S_{2,n}^{(2)} &= \begin{cases} 0 & \text{if } \gamma \in (0,1)\,, \\ \operatorname{sgn}(c_{13})\,\infty & \text{if } \gamma = 1, \\ \operatorname{sgn}\left[c_{13}\left(h_{t+1}^2 - c_3\right)\left(-c_{14}(1-\gamma)^{-1} + (\varphi-\gamma)^{-1}\right)\right]\infty & \text{if } \gamma \in (1,\infty)\,, \end{cases} \\
&= \begin{cases} 0 & \text{if } \gamma \in (0,1)\,, \\ \operatorname{sgn}\left(\tau_z\left(\alpha + \frac{\gamma-\varphi}{3}\right) + \lambda \int\limits_{x=-\infty}^{0} x^3 f(x)\,dx\right)\infty & \text{if } \gamma \in [1,\infty)\,. \end{cases}
\end{aligned}
$$

Analogously, it can be shown that $\lim_{n\to\infty}\tilde{S}_{2,n}$ is identical to $\lim_{n\to\infty}\tilde{S}_{1,n}$. Finally, using $\min\left(\tilde{S}_{1,n},\tilde{S}_{2,n}\right) \leq S_n \leq \max\left(\tilde{S}_{1,n},\tilde{S}_{2,n}\right)$ and the squeeze theorem,[8]

$$
\lim_{n\to\infty}\mathrm{T}_{r,n} = \begin{cases} 0 & \text{if } \gamma \in (0,1)\,, \\ \operatorname{sgn}\left(\tau_z\left(\alpha + \frac{\gamma-\varphi}{3}\right) + \lambda \int\limits_{x=-\infty}^{0} x^3 f(x)\,dx\right)\infty & \text{if } \gamma \in [1,\infty)\,. \end{cases}
$$

Normal GJR: $\tau_z = 0$ and $\int\limits_{x=-\infty}^{0} x^3 f(x)\,dx = -\sqrt{\frac{2}{\pi}}$. Hence $\lim_{n\to\infty}\mathrm{T}_{r,n} = \begin{cases} 0 & \text{if } \gamma \in (0,1)\,, \\ -\operatorname{sgn}(\lambda)\,\infty & \text{if } \gamma \in [1,\infty)\,. \end{cases}$

Normal GARCH(1,1): $\tau_z = \lambda = 0$, and thus $\mathrm{T}_{r,n} = \lim_{n\to\infty}\mathrm{T}_{r,n} = 0$.

[8] If $c_{13} > 0$, then $\tilde{S}_{1,n} \leq S_n \leq \tilde{S}_{2,n}$, whereas if $c_{13} < 0$, the inequality is reversed. However, this does not change the proof above. Also $\operatorname{sgn}(-c_{12}(\varphi-\gamma) + c_{13}(1-\gamma)) = \operatorname{sgn}(c_{12})$, for $c_{14} < 0$. Hence the limits of aggregated skewness are the same, regardless of c_{14} being greater than or less than 0.

(c) *Limits of Forward and Aggregated Conditional Kurtosis*

The forward kurtosis limit is:

$$\lim_{s\to\infty}\kappa_{r,s}=\kappa_z\lim_{s\to\infty}\left(\tilde{\mu}_{h,s}^{(1)}\right)^{-2}\lim_{s\to\infty}\tilde{\mu}_{h,s}^{(2)}=\begin{cases}\kappa_z\omega\bar{h}^{-2}\left(\omega+2\varphi\bar{h}\right)(1-\gamma)^{-1} & \text{if } \gamma\in(0,1)\,,\\ \infty & \text{if } \gamma\in[1,\infty)\,.\end{cases}$$

For the normal GJR and normal GARCH(1,1) the limit of the kurtosis of forward returns becomes:

$$\lim_{s\to\infty}\kappa_{r,s}=\begin{cases}3\omega\bar{h}^{-2}\left(\omega+2\varphi\bar{h}\right)(1-\gamma)^{-1} & \text{if } \gamma\in(0,1)\,,\\ \infty & \text{if } \gamma\in[1,\infty)\,,\end{cases}$$

where φ and γ are now given by (18) and (19) for the normal GJR and normal GARCH(1,1), respectively.

For the limit of the aggregated kurtosis, we write:

$$\lim_{n\to\infty}\mathrm{K}_{r,n}=\lim_{n\to\infty}\left(M_{r,n}^{(2)}\right)^{-2}M_{r,n}^{(4)}=\lim_{n\to\infty}\left(n^{-1}M_{r,n}^{(2)}\right)^{-2}\left(n^{-2}M_{r,n}^{(4)}\right)=\bar{h}^{-2}\lim_{n\to\infty}\left(n^{-2}M_{r,n}^{(4)}\right).$$

$$\lim_{n\to\infty}\left(n^{-2}M_{r,n}^{(4)}\right)=\kappa_zA_1+6A_2+4A_3, \tag{20}$$

where

$$A_1=\lim_{n\to\infty}\left(n^{-2}\sum_{s=1}^{n}\tilde{\mu}_{h,s}^{(2)}\right),\qquad A_2=\lim_{n\to\infty}\left(n^{-2}\sum_{s=1}^{n}\sum_{u=1}^{n-s}E_t\left(\varepsilon_{t+s}^2\varepsilon_{t+s+u}^2\right)\right),$$

and

$$A_3=\lim_{n\to\infty}\left(n^{-2}\sum_{s=1}^{n}\sum_{u=1}^{n-s}E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}^3\right)\right)+3\lim_{n\to\infty}\left(n^{-2}\sum_{s=1}^{n}\sum_{u=1}^{n-s}\sum_{v=1}^{n-s-u}E_t\left(\varepsilon_{t+s}\varepsilon_{t+s+u}\varepsilon_{t+s+u+v}^2\right)\right).$$

Now, if $\gamma\neq 1$,

$$\begin{aligned}A_1 &= \lim_{n\to\infty}n^{-2}\left(nc_1+\left(\tilde{\mu}_{h,1}^{(2)}-c_3\right)(1-\gamma)^{-1}(1-\gamma^n)+c_2(1-\varphi)^{-1}(1-\varphi^n)\right)\\ &= c_1\lim_{n\to\infty}n^{-1}+\left(h_{t+1}^2-c_3\right)(1-\gamma)^{-1}\lim_{n\to\infty}n^{-2}(1-\gamma^n)+c_2(1-\varphi)^{-1}\lim_{n\to\infty}n^{-2}(1-\varphi^n)\,.\end{aligned}$$

Otherwise, when $\gamma=1$:

$$
\begin{aligned}
\sum_{s=1}^{n} \tilde{\mu}_{h,s}^{(2)} &= \sum_{s=1}^{n} \left((s-1)\left(\omega^2 + 2\omega\varphi\bar{h}\right) + 2\varphi h_0 \left(1-\varphi^{s-1}\right)\left(h_{t+1}-\bar{h}\right) + h_{t+1}^2 \right) \\
&= \frac{1}{2}\left(\omega^2 + 2\omega\varphi\bar{h}\right) n^2 + \left(2\varphi\bar{h}\left(h_{t+1}-\bar{h}\right) - \frac{1}{2}\left(\omega^2 + 2\omega\varphi\bar{h}\right) + h_{t+1}^2 \right) n \\
&\quad -2\varphi\bar{h}\left(h_{t+1}-\bar{h}\right)(1-\varphi)^{-1}\left(1-\varphi^n\right).
\end{aligned}
$$

So for $\gamma = 1$, we obtain that $A_1 = \lim_{n\to\infty} \frac{1}{2}\left(\omega^2 + 2\omega\varphi\bar{h}\right) + \lim_{n\to\infty} n^{-1}\left(2\varphi\bar{h}\left(h_{t+1}-\bar{h}\right) - \frac{1}{2}\left(\omega^2 + 2\omega\varphi\bar{h}\right) + h_{t+1}^2\right) - \lim_{n\to\infty} n^{-2} 2\varphi\bar{h}\left(h_{t+1}-\bar{h}\right)(1-\varphi)^{-1}\left(1-\varphi^n\right)$. Hence

$$
A_1 = \begin{cases} 0 & \text{if } \gamma \in (0,1), \\ 1/2\left(\omega^2 + 2\omega\varphi\bar{h}\right) & \text{if } \gamma = 1, \\ \infty & \text{if } \gamma \in (1,\infty). \end{cases} \tag{21}
$$

For A_2, using the derivations from the Appendix A1, we can write:

$$
\begin{aligned}
A_2 &= \lim_{n\to 0}\left(n^{-2} \sum_{s=1}^{n}\sum_{u=1}^{n-s} E_t\left(\varepsilon_{t+s}^2 \varepsilon_{t+s+u}^2\right)\right) \\
&= \lim_{n\to 0}\left(n^{-2}\left[\sum_{s=1}^{n}\sum_{u=1}^{n-s}\left[\bar{h}\left(1-\varphi^u\right)\tilde{\mu}_{h,s}^{(1)}\right] + \kappa_z\left(\alpha + \lambda F_0 + \kappa_z^{-1}\beta\right)\sum_{s=1}^{n}\sum_{u=1}^{n-2}\varphi^{u-1}\tilde{\mu}_{h,s}^{(2)}\right]\right).
\end{aligned}
$$

For $\gamma \neq 1$, the expressions for the two double sums in the expressions above were derived in the Appendix A1. Using those results, we have:

$$
A_2 = \lim_{n\to\infty} n^{-2} \left[\begin{array}{l}
\frac{1}{2} n(n-1)\bar{h}^2 + (1-\varphi)^{-1}\bar{h}\left(h_{t+1}-\bar{h}\right) \\
\left(\left(n - (1-\varphi)^{-1}\left(1-\varphi^n\right)\right) - \varphi \left(\begin{array}{l} n\bar{h}\left(h_{t+1}-\bar{h}\right)^{-1} + (1-\varphi)^{-1}\left(1-\varphi^n\right) - h_0 \\ \left(h_{t+1}-h_0\right)^{-1}(1-\varphi)^{-1}\left(1-\varphi^n\right) - n\varphi^{n-1} \end{array} \right) \right) \\
+\kappa_z\left(\alpha + \lambda F_0 + {\kappa_z}^{-1}\beta\right)(1-\varphi)^{-1} \\
\left(\begin{array}{l} nc_1 + \left(\tilde{\mu}_{h,1}^{(2)} - c_3\right)(1-\gamma)^{-1}\left(1-\gamma^n\right) + c_2(1-\varphi)^{-1}\left(1-\varphi^n\right) \\ -c_1(1-\varphi)^{-1}\left(1-\varphi^n\right) - \left(h_{t+1}^2 - c_3\right)(\varphi-\gamma)^{-1}\left(\varphi^n - \gamma^n\right) - nc_2\varphi^{n-1} \end{array} \right)
\end{array} \right].
$$

For $\gamma = 1$,

$$A_2 = \lim_{n\to\infty} n^{-2} \left[\begin{array}{l} 1/2 n\,(n-1)\,\bar{h}^2 + (1-\varphi)^{-1}\bar{h}\left(h_{t+1}-\bar{h}\right) \\ \left(\left(n-(1-\varphi)^{-1}(1-\varphi^n)\right) - \varphi \left(\begin{array}{l} n\bar{h}\left(h_{t+1}-\bar{h}\right)^{-1} + (1-\varphi)^{-1}(1-\varphi^n) - h_0 \\ (h_{t+1}-h_0)^{-1}(1-\varphi)^{-1}(1-\varphi^n) - n\varphi^{n-1} \end{array} \right) \right) \\ +\kappa_z\left(\alpha+\lambda F_0+\kappa_z{}^{-1}\beta\right)(1-\varphi)^{-1} \\ \left(\begin{array}{l} 1/2\left(\omega^2+2\omega\varphi\bar{h}\right)n^2 + \left(2\varphi\bar{h}\left(h_{t+1}-\bar{h}\right) - 1/2\left(\omega^2+2\omega\varphi\bar{h}\right) + h_{t+1}^2\right)n \\ -2\varphi\bar{h}\left(h_{t+1}-\bar{h}\right)(1-\varphi)^{-1}(1-\varphi^n) \\ +\varphi\left(\omega^2+2\omega\varphi\bar{h}\right)(1-\varphi)^{-1}\left[(1-\varphi)^{-1}(1-\varphi^n) - \varphi^{-1}(n-1)\right] \\ -2\varphi\bar{h}\left(h_{t+1}-\bar{h}\right)\left[(1-\varphi)^{-1}(1-\varphi^n) - n\varphi^{n-1}\right] - \tilde{\mu}_{h,1}^{(2)}(1-\varphi)^{-1}(1-\varphi^n) \end{array} \right) \end{array} \right].$$

Thus:

$$A_2 = \begin{cases} 1/2\,\bar{h}^2 & \text{if } \gamma \in (0,1)\,, \\ 1/2\left[\bar{h}^2 + \kappa_z\left(\alpha+\lambda F_0+\kappa_z^{-1}\beta\right)(1-\varphi)^{-1}\left(\omega^2+2\omega\varphi\bar{h}\right)\right] & \text{if } \gamma = 1, \\ \infty & \text{if } \gamma \in (1,\infty)\,. \end{cases} \tag{22}$$

Now

$$A_3 = \left[\tau_z + 3(1-\varphi)^{-1}c_4\right] \lim_{n\to\infty} n^{-2} \sum_{s=1}^{n}\sum_{u=1}^{n-s} \theta_{su}^{(3/2)} - 3(1-\varphi)^{-1}c_4 \lim_{n\to\infty} n^{-2} \sum_{s=1}^{n}\sum_{u=1}^{n-s} \varphi^{n-s-u}\theta_{su}^{(3/2)}$$

where

$$\begin{aligned} \theta_{su}^{(3/2)} &= \frac{3}{4}c_4\left[\left(\tilde{\mu}_{h,s+u}^{(1)}\right)^{1/2} + \omega\varphi(\varphi-\gamma)^{-1}\left(\tilde{\mu}_{h,s+u}^{(1)}\right)^{-1/2}\right]\varphi^{u-1}E_t\left(h_{t+s}^{3/2}\right) \\ &+ \frac{3}{8}\left(\tilde{\mu}_{h,s+u}^{(1)}\right)^{-1/2}\gamma^{u-1}\left(c_5E_t\left(h_{t+s}^{5/2}\right) + 2\omega\gamma(\gamma-\varphi)^{-1}c_4E_t\left(h_{t+s}^{3/2}\right)\right), \end{aligned}$$

and

$$\begin{aligned} \varphi^{n-s-u}\theta_{su}^{(3/2)} &= \frac{3}{4}c_4\left[\left(\tilde{\mu}_{h,s+u}^{(1)}\right)^{1/2} + \omega\varphi(\varphi-\gamma)^{-1}\left(\tilde{\mu}_{h,s+u}^{(1)}\right)^{-1/2}\right]\varphi^{n-s-1}E_t\left(h_{t+s}^{3/2}\right) \\ &+ \frac{3}{8}\left(\tilde{\mu}_{h,s+u}^{(1)}\right)^{-1/2}\varphi^{n-s-1}(\gamma/\varphi)^{u-1}\left(c_5E_t\left(h_{t+s}^{5/2}\right) + 2\omega\gamma(\gamma-\varphi)^{-1}c_4E_t\left(h_{t+s}^{3/2}\right)\right). \end{aligned}$$

We can now write: $b_{l,s,n} \le \sum_{u=1}^{n-s} \theta_{su}^{(3/2)} \le b_{u,s,n}$, where, for $\gamma \neq 1$

$$
\begin{aligned}
b_{l,s,n} &= \frac{3}{4}\left(c_4\left[q_{u,1,n-s}\left(\tilde{\mu}^{(1)}_{h,s+u},-c_4\right)\right]^{1/2}+\omega\varphi(\varphi-\gamma)^{-1}c_4\left[q_{u,1,n-s}\left(\tilde{\mu}^{(1)}_{h,s+u},(\varphi-\gamma)\,c_4\right)\right]^{-1/2}\right) \\
&\quad E_t\left(h^{3/2}_{t+s}\right)(1-\varphi)^{-1}\left(1-\varphi^{n-s}\right) \\
&+ \frac{3}{8}\left(\begin{array}{l} c_5\left[q_{u,1,n-s}\left(\tilde{\mu}^{(1)}_{h,s+u},c_5\right)\right]^{-1/2}E_t\left(h^{5/2}_{t+s}\right) \\ +2\omega\gamma(\gamma-\varphi)^{-1}c_4\left[q_{u,1,n-s}\left(\tilde{\mu}^{(1)}_{h,s+u},(\gamma-\varphi)\,c_4\right)\right]^{-1/2}E_t\left(h^{3/2}_{t+s}\right)\end{array}\right)(1-\gamma)^{-1}\left(1-\gamma^{n-s}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
b_{u,s,n} &= \frac{3}{4}\left(\begin{array}{l} c_4\left[q_{u,1,n-s}\left(\tilde{\mu}^{(1)}_{h,s+u},c_4\right)\right]^{1/2}+\omega\varphi(\varphi-\gamma)^{-1} \\ c_4\left[q_{u,1,n-s}\left(\tilde{\mu}^{(1)}_{h,s+u},-(\varphi-\gamma)\,c_4\right)\right]^{-1/2}\end{array}\right)E_t\left(h^{3/2}_{t+s}\right)(1-\varphi)^{-1}\left(1-\varphi^{n-s}\right) \\
&+ \frac{3}{8}\left(\begin{array}{l} c_5\left[q_{u,1,n-s}\left(\tilde{\mu}^{(1)}_{h,s+u},-c_5\right)\right]^{-1/2}E_t\left(h^{5/2}_{t+s}\right)+2\omega\gamma(\gamma-\varphi)^{-1} \\ c_4\left[q_{u,1,n-s}\left(\tilde{\mu}^{(1)}_{h,s+u},-(\gamma-\varphi)\,c_4\right)\right]^{-1/2}E_t\left(h^{3/2}_{t+s}\right)\end{array}\right)(1-\gamma)^{-1}\left(1-\gamma^{n-s}\right),
\end{aligned}
$$

and, for $\gamma = 1$

$$
\begin{aligned}
b_{l,s,n} &= \frac{3}{4}\left(c_4\left[q_{u,1,n-s}\left(\tilde{\mu}^{(1)}_{h,s+u},-c_4\right)\right]^{1/2}+\omega\varphi(\varphi-1)^{-1}c_4\left[q_{u,1,n-s}\left(\tilde{\mu}^{(1)}_{h,s+u},-c_4\right)\right]^{-1/2}\right) \\
&\quad E_t\left(h^{3/2}_{t+s}\right)(1-\varphi)^{-1}\left(1-\varphi^{n-s}\right) \\
&+ \frac{3}{8}(n-s)\left(\begin{array}{l} c_5\left[q_{u,1,n-s}\left(\tilde{\mu}^{(1)}_{h,s+u},c_5\right)\right]^{-1/2}E_t\left(h^{5/2}_{t+s}\right) \\ +2\omega(1-\varphi)^{-1}c_4\left[q_{u,1,n-s}\left(\tilde{\mu}^{(1)}_{h,s+u},c_4\right)\right]^{-1/2}E_t\left(h^{3/2}_{t+s}\right)\end{array}\right), \\
b_{u,s,n} &= \frac{3}{4}\left(c_4\left[q_{u,1,n-s}\left(\tilde{\mu}^{(1)}_{h,s+u},c_4\right)\right]^{1/2}+\omega\varphi(\varphi-1)^{-1}c_4\left[q_{u,1,n-s}\left(\tilde{\mu}^{(1)}_{h,s+u},c_4\right)\right]^{-1/2}\right) \\
&\quad E_t\left(h^{3/2}_{t+s}\right)(1-\varphi)^{-1}\left(1-\varphi^{n-s}\right) \\
&+ \frac{3}{8}(n-s)\left(\begin{array}{l} c_5\left[q_{u,1,n-s}\left(\tilde{\mu}^{(1)}_{h,s+u},-c_5\right)\right]^{-1/2}E_t\left(h^{5/2}_{t+s}\right) \\ +2\omega(1-\varphi)^{-1}c_4\left[q_{u,1,n-s}\left(\tilde{\mu}^{(1)}_{h,s+u},-c_4\right)\right]^{-1/2}E_t\left(h^{3/2}_{t+s}\right)\end{array}\right),
\end{aligned}
$$

with

$$
q_{z,a,b}\left(x_z,y\right)=\begin{cases}\max\limits_{a\le z\le b} x_z & \text{if } y\ge 0,\\ \min\limits_{a\le z\le b} x_z & \text{if } y<0,\end{cases}
$$

where $z = u$ or $z = s$.

Also, $b_{ul,n} \leq \sum_{s=1}^{n} b_{u,s,n} \leq b_{uu,n}$ where for $\gamma \neq 1$

$$\begin{aligned} b_{ul,n} &= \tfrac{3}{4}(1-\varphi)^{-1} \sum_{s=1}^{n} (1-\varphi^{n-s}) \\ &\quad \left[\begin{array}{l} c_4 q_{s,1,n}\left(E_t\left(h_{t+s}^{3/2}\right), -c_4\right)\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, c_4\right), -c_4\right)\right]^{1/2} \\ +\omega\varphi(\varphi-\gamma)^{-1} c_4 q_{s,1,n}\left(E_t\left(h_{t+s}^{3/2}\right), -(\varphi-\gamma)\, c_4\right) \\ \left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, -(\varphi-\gamma)\, c_4\right), (\varphi-\gamma)\, c_4\right)\right]^{-1/2} \end{array} \right] \\ &\quad +\tfrac{3}{8}(1-\gamma)^{-1} \sum_{s=1}^{n} (1-\gamma^{n-s}) \\ &\quad \left[\begin{array}{l} c_5 q_{s,1,n}\left(E_t\left(h_{t+s}^{5/2}\right), -c_5\right)\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, -c_5\right), c_5\right)\right]^{-1/2} \\ +2\omega\gamma(\gamma-\varphi)^{-1} c_4 q_{s,1,n}\left(E_t\left(h_{t+s}^{3/2}\right), -(\gamma-\varphi)\, c_4\right) \\ \left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, -(\gamma-\varphi)\, c_4\right), (\gamma-\varphi)\, c_4\right)\right]^{-1/2} \end{array} \right] \\ &= \tfrac{3}{4} l_{1,ul,n}(1-\varphi)^{-1}\left(n-(1-\varphi)^{-1}(1-\varphi^n)\right) + \tfrac{3}{8} l_{2,ul,n}(1-\gamma)^{-1}\left(n-(1-\gamma)^{-1}(1-\gamma^n)\right) \end{aligned}$$

with

$$\begin{aligned} l_{1,ul,n} &= c_4 q_{s,1,n}\left(E_t\left(h_{t+s}^{3/2}\right), -c_4\right)\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, c_4\right), -c_4\right)\right]^{1/2} \\ &\quad +\omega\varphi(\varphi-\gamma)^{-1} c_4 q_{s,1,n}\left(E_t\left(h_{t+s}^{3/2}\right), -(\varphi-\gamma)\, c_4\right) \\ &\quad \left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, -(\varphi-\gamma)\, c_4\right), (\varphi-\gamma)\, c_4\right)\right]^{-1/2} \end{aligned}$$

and

$$\begin{aligned} l_{2,ul,n} &= c_5 q_{s,1,n}\left(E_t\left(h_{t+s}^{5/2}\right), -c_5\right)\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, -c_5\right), c_5\right)\right]^{-1/2} \\ &\quad +2\omega\gamma(\gamma-\varphi)^{-1} c_4 q_{s,1,n}\left(E_t\left(h_{t+s}^{3/2}\right), -(\gamma-\varphi)\, c_4\right) \\ &\quad \left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, -(\gamma-\varphi)\, c_4\right), (\gamma-\varphi)\, c_4\right)\right]^{-1/2}. \end{aligned}$$

Also

$$\begin{aligned} b_{uu,n} &= \tfrac{3}{4}(1-\varphi)^{-1} \sum_{s=1}^{n} (1-\varphi^{n-s}) \\ &\quad \left[\begin{array}{l} c_4 q_{s,1,n}\left(E_t\left(h_{t+s}^{3/2}\right), c_4\right)\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, c_4\right), c_4\right)\right]^{1/2} + \omega\varphi(\varphi-\gamma)^{-1} c_4 \\ q_{s,1,n}\left(E_t\left(h_{t+s}^{3/2}\right), (\varphi-\gamma)\, c_4\right)\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, -(\varphi-\gamma)\, c_4\right), -(\varphi-\gamma)\, c_4\right)\right]^{-1/2} \end{array} \right] \\ &\quad +\tfrac{3}{8}(1-\gamma)^{-1} \sum_{s=1}^{n} (1-\gamma^{n-s}) \\ &\quad \left[\begin{array}{l} c_5 q_{s,1,n}\left(E_t\left(h_{t+s}^{5/2}\right), c_5\right)\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, -c_5\right), -c_5\right)\right]^{-1/2} + 2\omega\gamma(\gamma-\varphi)^{-1} c_4 \\ q_{s,1,n}\left(E_t\left(h_{t+s}^{3/2}\right), (\gamma-\varphi)\, c_4\right)\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, -(\gamma-\varphi)\, c_4\right), -(\gamma-\varphi)\, c_4\right)\right]^{-1/2} \end{array} \right] \\ &= \tfrac{3}{4} l_{1,uu,n}(1-\varphi)^{-1}\left(n-(1-\varphi)^{-1}(1-\varphi^n)\right) + \tfrac{3}{8} l_{2,uu,n}(1-\gamma)^{-1}\left(n-(1-\gamma)^{-1}(1-\gamma^n)\right) \end{aligned}.$$

with

$$l_{1,uu,n} = c_4 q_{s,1,n}\left(E_t\left(h_{t+s}^{3/2}\right), c_4\right)\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, c_4\right), c_4\right)\right]^{1/2}$$
$$+\omega\varphi(\varphi-\gamma)^{-1} c_4 q_{s,1,n}\left(E_t\left(h_{t+s}^{3/2}\right), (\varphi-\gamma)\, c_4\right)$$
$$\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, -(\varphi-\gamma)\, c_4\right), -(\varphi-\gamma)\, c_4\right)\right]^{-1/2}$$

and

$$l_{2,uu,n} = c_5 q_{s,1,n}\left(E_t\left(h_{t+s}^{5/2}\right), c_5\right)\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, -c_5\right), -c_5\right)\right]^{-1/2}$$
$$+2\omega\gamma(\gamma-\varphi)^{-1} c_4 q_{s,1,n}\left(E_t\left(h_{t+s}^{3/2}\right), (\gamma-\varphi)\, c_4\right) .$$
$$\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, -(\gamma-\varphi)\, c_4\right), -(\gamma-\varphi)\, c_4\right)\right]^{-1/2}.$$

For $\gamma = 1$:

$$b_{ul,n} = \frac{3}{4}\left[\begin{array}{l} c_4 q_{s,1,n}\left(E_t\left(h_{t+s}^{3/2}\right), -c_4\right)\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, c_4\right), -c_4\right)\right]^{1/2} + \omega\varphi(\varphi-1)^{-1} c_4 \\ q_{s,1,n}\left(E_t\left(h_{t+s}^{3/2}\right), c_4\right)\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, c_4\right), -c_4\right)\right]^{-1/2} \end{array}\right]$$
$$(1-\varphi)^{-1}\sum_{s=1}^{n}\left(1-\varphi^{n-s}\right) + \frac{3}{16} n\,(n-1)$$
$$\left[\begin{array}{l} c_5 q_{s,1,n}\left(E_t\left(h_{t+s}^{5/2}\right), -c_5\right)\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, -c_5\right), c_5\right)\right]^{-1/2} \\ +2\omega(1-\varphi)^{-1} c_4 q_{s,1,n}\left(E_t\left(h_{t+s}^{3/2}\right), -c_4\right)\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, -c_4\right), c_4\right)\right]^{-1/2} \end{array}\right]$$
$$= \frac{3}{4}\tilde{l}_{1,ul,n}(1-\varphi)^{-1}\left(n - (1-\varphi)^{-1}\left(1-\varphi^{n}\right)\right) + \frac{3}{16}\tilde{l}_{2,ul,n} n\,(n-1)$$

with

$$\tilde{l}_{1,ul,n} = c_4 q_{s,1,n}\left(E_t\left(h_{t+s}^{3/2}\right), -c_4\right)\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, c_4\right), -c_4\right)\right]^{1/2}$$
$$+\omega\varphi(\varphi-1)^{-1} c_4 q_{s,1,n}\left(E_t\left(h_{t+s}^{3/2}\right), c_4\right)\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, c_4\right), -c_4\right)\right]^{-1/2}$$

and

$$\tilde{l}_{2,ul,n} = c_5 q_{s,1,n}\left(E_t\left(h_{t+s}^{5/2}\right), -c_5\right)\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, -c_5\right), c_5\right)\right]^{-1/2}$$
$$+2\omega(1-\varphi)^{-1} c_4 q_{s,1,n}\left(E_t\left(h_{t+s}^{3/2}\right), -c_4\right)\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, -c_4\right), c_4\right)\right]^{-1/2}.$$

Also,

$$b_{uu,n} = \frac{3}{4}(1-\varphi)^{-1}\sum_{s=1}^{n}\left(1-\varphi^{n-s}\right)$$
$$\left[\begin{array}{l} c_4 q_{s,1,n}\left(E_t\left(h_{t+s}^{3/2}\right), c_4\right)\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, c_4\right), c_4\right)\right]^{1/2} + \omega\varphi(\varphi-1)^{-1} c_4 \\ q_{s,1,n}\left(E_t\left(h_{t+s}^{3/2}\right), -c_4\right)\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, c_4\right), c_4\right)\right]^{-1/2} \end{array}\right]$$
$$+\frac{3}{16} n\,(n-1)$$
$$\left[\begin{array}{l} c_5 q_{s,1,n}\left(E_t\left(h_{t+s}^{5/2}\right), c_5\right)\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, -c_5\right), -c_5\right)\right]^{-1/2} \\ +2\omega(1-\varphi)^{-1} c_4 q_{s,1,n}\left(E_t\left(h_{t+s}^{3/2}\right), c_4\right)\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, -c_4\right), -c_4\right)\right]^{-1/2} \end{array}\right]$$
$$= \frac{3}{4}\tilde{l}_{1,uu,n}(1-\varphi)^{-1}\left(n - (1-\varphi)^{-1}\left(1-\varphi^{n}\right)\right) + \frac{3}{16}\tilde{l}_{2,uu,n} n\,(n-1)$$

where

$$\tilde{l}_{1,uu,n} = c_4 q_{s,1,n}\left(E_t\left(h_{t+s}^{3/2}\right), c_4\right)\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, c_4\right), c_4\right)\right]^{1/2}$$
$$+\omega\varphi(\varphi-1)^{-1} c_4 q_{s,1,n}\left(E_t\left(h_{t+s}^{3/2}\right), -c_4\right)\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, c_4\right), c_4\right)\right]^{-1/2}$$

and

$$\tilde{l}_{2,uu,n} = c_5 q_{s,1,n}\left(E_t\left(h_{t+s}^{5/2}\right), c_5\right)\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, -c_5\right), -c_5\right)\right]^{-1/2}$$
$$+2\omega(1-\varphi)^{-1} c_4 q_{s,1,n}\left(E_t\left(h_{t+s}^{3/2}\right), c_4\right)\left[q_{s,1,n}\left(q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, -c_4\right), -c_4\right)\right]^{-1/2}.$$

We have previously shown that

$$L_{\max} = \lim_{n\to\infty}\left(\max_{1\le s\le n}\ \tilde{\mu}_{h,s}^{(1)}\right) = \max\left(\bar{h}, h_{t+1}\right); \quad L_{\min} = \lim_{n\to\infty}\left(\min_{1\le s\le n}\ \tilde{\mu}_{h,s}^{(1)}\right) = \min\left(\bar{h}, h_{t+1}\right).$$

Also, we have:

$$\begin{aligned}
\lim_{s\to\infty} E_t\left(h_{t+s}^{3/2}\right) &= \frac{1}{8}\lim_{s\to\infty}\left(5\left(\tilde{\mu}_{h,s}^{(1)}\right)^{3/2} + 3\tilde{\mu}_{h,s}^{(2)}\left(\tilde{\mu}_{h,s}^{(1)}\right)^{-1/2}\right) \\
&= \begin{cases} \frac{1}{8}\left(5\bar{h}^{3/2} + 3\bar{h}^{-1/2}\left(c_1 + \left(h_{t+1}^2 - c_3\right)\lim_{s\to\infty}\gamma^{s-1} + c_2\lim_{s\to\infty}\varphi^{s-1}\right)\right) & \text{if } \gamma \neq 1, \\ \frac{1}{8}\left(5\bar{h}^{3/2} + 3\bar{h}^{-1/2}\lim_{s\to\infty}\begin{pmatrix} (s-1)\left(\omega^2 + 2\omega\varphi\bar{h}\right) \\ +2\varphi\bar{h}\left(1-\varphi^{s-1}\right)\left(h_{t+1} - \bar{h}\right) + h_{t+1}^2 \end{pmatrix}\right) & \text{if } \gamma = 1, \end{cases} \\
&= \begin{cases} \frac{1}{8}\left(5\bar{h}^{3/2} + 3\bar{h}^{-1/2}c_1\right) & \text{if } \gamma \in (0,1), \\ \infty & \text{if } \gamma \in [1,\infty). \end{cases}
\end{aligned}$$

$$\lim_{s\to\infty} E_t\left(h_{t+s}^{5/2}\right) = \frac{1}{8}\lim_{s\to\infty}\left(\left(15\left(\tilde{\mu}_{h,s}^{(1)}\right)^{1/2}\tilde{\mu}_{h,s}^{(2)} - 7\left(\tilde{\mu}_{h,s}^{(1)}\right)^{5/2}\right)\right) = \begin{cases} \frac{1}{8}\left(-7\bar{h}^{5/2} + 15\bar{h}^{1/2}c_1\right) & \text{if } \gamma \in (0,1), \\ \infty & \text{if } \gamma \in [1,\infty). \end{cases}$$

Hence, when $\gamma \in (0,1)$, $\lim_{s\to\infty} E_t\left(h_{t+s}^{3/2}\right)$ and $\lim_{s\to\infty} E_t\left(h_{t+s}^{5/2}\right)$ exist and are finite.

Furthermore, we have that $\max_{1\le s\le n} E_t\left(h_{t+s}^{i/2}\right)$ and $\min_{1\le s\le n} E_t\left(h_{t+s}^{i/2}\right)$, $i = 3$ and 5 are bounded. Thus, $\lim_{n\to\infty}\left(l_{j,uu,n}\right)$ and $\lim_{n\to\infty}\left(l_{j,ul,n}\right)$, $j = 1$ and 2 exist and are finite. We get:

$$\lim_{n\to\infty} n^{-2} b_{ul,n} = \lim_{n\to\infty} n^{-2}\begin{pmatrix} \frac{3}{4}l_{1,ul,n}(1-\varphi)^{-1}\left(n - (1-\varphi)^{-1}\left(1-\varphi^n\right)\right) \\ +\frac{3}{8}l_{2,ul,n}(1-\gamma)^{-1}\left(n - (1-\gamma)^{-1}\left(1-\gamma^n\right)\right) \end{pmatrix} = 0$$

$$\lim_{n\to\infty} n^{-2} b_{uu,n} = \lim_{n\to\infty} n^{-2}\begin{bmatrix} \frac{3}{4}l_{1,uu,n}(1-\varphi)^{-1}\left(n - (1-\varphi)^{-1}\left(1-\varphi^n\right)\right) \\ +\frac{3}{8}l_{2,uu,n}(1-\gamma)^{-1}\left(n - (1-\gamma)^{-1}\left(1-\gamma^n\right)\right) \end{bmatrix} = 0.$$

Thus $\lim_{n\to\infty} n^{-2} \sum_{s=1}^{n} b_{u,s,n} = 0$. This translates into:

$$\lim_{n\to\infty} n^{-2} \sum_{s=1}^{n} \sum_{u=1}^{n-s} \theta_{su}^{(3/2)} \le \lim_{n\to\infty} n^{-2} \sum_{s=1}^{n} b_{u,s,n} = 0.$$

Similarly, it can be shown that:

$$0 = \lim_{n\to\infty} n^{-2} \sum_{s=1}^{n} b_{l,s,n} \le \lim_{n\to\infty} n^{-2} \sum_{s=1}^{n} \sum_{u=1}^{n-s} \theta_{su}^{(3/2)}.$$

Finally, we obtain that $\lim_{n\to\infty} n^{-2} \sum_{s=1}^{n} \sum_{u=1}^{n-s} \theta_{su}^{(3/2)} = 0$ for $\gamma \in (0,1)$.

Similarly, it can be shown that $\lim_{n\to\infty} n^{-2} \sum_{s=1}^{n} \sum_{u=1}^{n-s} \varphi^{n-s-u} \theta_{su}^{(3/2)} = 0$ for $\gamma \in (0,1)$.

Consider now: $B_{l,s,n} \le \sum_{u=1}^{n-s} \varphi^{n-s-u} \theta_{su}^{3/2} \le B_{u,s,n}$ where, for any γ:

$$\begin{aligned}
B_{l,s,n} &= \frac{3}{4}\left(c_4 \left[q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, -c_4\right)\right]^{1/2} + \omega\varphi(\varphi-\gamma)^{-1} c_4 \left[q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, (\varphi-\gamma)\, c_4\right)\right]^{-1/2} \right) \\
&\quad E_t\left(h_{t+s}^{3/2}\right)(n-s)\,\varphi^{n-s-1} \\
&+ \frac{3}{8}\left(\begin{array}{l} c_5 \left[q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, c_5\right)\right]^{-1/2} E_t\left(h_{t+s}^{5/2}\right) + 2\omega\gamma(\gamma-\varphi)^{-1} c_4 \\ \left[q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, (\gamma-\varphi)\, c_4\right)\right]^{-1/2} E_t\left(h_{t+s}^{3/2}\right) \end{array} \right) (\varphi-\gamma)^{-1}\left(\varphi^{n-s} - \gamma^{n-s}\right), \\
B_{u,s,n} &= \frac{3}{4}\left(c_4 \left[q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, c_4\right)\right]^{1/2} + \omega\varphi(\varphi-\gamma)^{-1} c_4 \left[q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, -(\varphi-\gamma)\, c_4\right)\right]^{-1/2} \right) \\
&\quad E_t\left(h_{t+s}^{3/2}\right)(n-s)\,\varphi^{n-s-1} \\
&+ \frac{3}{8}\left(\begin{array}{l} c_5 \left[q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, -c_5\right)\right]^{-1/2} E_t\left(h_{t+s}^{5/2}\right) + 2\omega\gamma(\gamma-\varphi)^{-1} c_4 \\ \left[q_{u,1,n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}, -(\gamma-\varphi)\, c_4\right)\right]^{-1/2} E_t\left(h_{t+s}^{3/2}\right) \end{array} \right) (\varphi-\gamma)^{-1}\left(\varphi^{n-s} - \gamma^{n-s}\right).
\end{aligned}$$

Also,

$$B_{ul,n} \le \sum_{s=1}^{n} B_{u,s,n} \le B_{uu,n}$$

where, for $\gamma \neq 1$:

$$\begin{aligned} B_{ul,n} &= \frac{3}{4} l_{1,ul,n} \left[(1-\varphi)^{-2} (1-\varphi^n) - (1-\varphi)^{-1} n \varphi^{n-1} \right] \\ &+ \frac{3}{8} l_{2,ul,n} \left[(\varphi-\gamma)^{-1} \Big((1-\varphi)^{-1} (1-\varphi^n) - (1-\gamma)^{-1} (1-\gamma^n) \Big) \right] \end{aligned}$$

$$\begin{aligned} B_{uu,n} &= \frac{3}{4} l_{1,uu,n} \left[(1-\varphi)^{-2} (1-\varphi^n) - (1-\varphi)^{-1} n \varphi^{n-1} \right] \\ &+ \frac{3}{8} l_{2,uu,n} \left[(\varphi-\gamma)^{-1} \Big((1-\varphi)^{-1} (1-\varphi^n) - (1-\gamma)^{-1} (1-\gamma^n) \Big) \right]. \end{aligned}$$

And for $\gamma = 1$:

$$\begin{aligned} B_{ul,n} &= \frac{3}{4} \tilde{l}_{1,ul,n} \left[(1-\varphi)^{-2} (1-\varphi^n) - (1-\varphi)^{-1} n \varphi^{n-1} \right] \\ &+ \frac{3}{8} \tilde{l}_{2,ul,n} (1-\varphi)^{-1} \Big(n - (1-\varphi)^{-1} (1-\varphi^n) \Big) \end{aligned}$$

$$\begin{aligned} B_{uu,n} &= \frac{3}{4} \tilde{l}_{1,uu,n} \left[(1-\varphi)^{-2} (1-\varphi^n) - (1-\varphi)^{-1} n \varphi^{n-1} \right] \\ &+ \frac{3}{8} \tilde{l}_{2,uu,n} (1-\varphi)^{-1} \Big(n - (1-\varphi)^{-1} (1-\varphi^n) \Big). \end{aligned}$$

For $\gamma \in (0,1)$ we obtained that $\lim_{n\to\infty} (l_{j,uu,n})$, $\lim_{n\to\infty} (l_{j,ul,n})$, $j = 1, 2$ exist and are finite. Thus:

$$\lim_{n\to\infty} n^{-2} B_{ul,n} = \lim_{n\to\infty} n^{-2} \left[\begin{array}{c} \frac{3}{4} l_{1,ul,n} \left[(1-\varphi)^{-2} (1-\varphi^n) - (1-\varphi)^{-1} n \varphi^{n-1} \right] + \frac{3}{8} l_{2,ul,n} \\ \left[(\varphi-\gamma)^{-1} \Big((1-\varphi)^{-1} (1-\varphi^n) - (1-\gamma)^{-1} (1-\gamma^n) \Big) \right] \end{array} \right] = 0$$

and

$$\lim_{n\to\infty} n^{-2} B_{uu,n} = \lim_{n\to\infty} n^{-2} \left[\begin{array}{c} \frac{3}{4} l_{1,uu,n} \left[(1-\varphi)^{-2} (1-\varphi^n) - (1-\varphi)^{-1} n \varphi^{n-1} \right] + \frac{3}{8} l_{2,uu,n} \\ \left[(\varphi-\gamma)^{-1} \Big((1-\varphi)^{-1} (1-\varphi^n) - (1-\gamma)^{-1} (1-\gamma^n) \Big) \right] \end{array} \right] = 0.$$

Thus $\lim_{n\to\infty} n^{-2} \sum_{s=1}^{n} B_{u,s,n} = 0$, yielding:

$$\lim_{n\to\infty} n^{-2} \sum_{s=1}^{n} \sum_{u=1}^{n-s} \varphi^{n-s-u} \theta_{su}^{(3/2)} \leq \lim_{n\to\infty} n^{-2} \sum_{s=1}^{n} B_{u,s,n} = 0.$$

Similarly, it can be shown that:

$$0=\lim_{n\to\infty} n^{-2}\sum_{s=1}^{n} B_{l,s,n}\leq \lim_{n\to\infty} n^{-2}\sum_{s=1}^{n}\sum_{u=1}^{n-s}\varphi^{n-s-u}\theta_{su}^{(3/2)}.$$

Finally, we have:

$$\lim_{n\to\infty} n^{-2}\sum_{s=1}^{n}\sum_{u=1}^{n-s}\varphi^{n-s-u}\theta_{su}^{(3/2)}=\lim_{n\to\infty} n^{-2}\sum_{s=1}^{n}\sum_{u=1}^{n-s}\theta_{su}^{(3/2)}=0,$$

therefore, $A_3=0$ for $\gamma\in(0,1)$.

For $\gamma\in(1,\infty)$, we showed that $A_1=A_2=\infty$. Thus the kurtosis of aggregated returns will diverge to plus infinity if we can show that A_3 is bounded below. Setting $c_{15}=\tau_z+3(1-\varphi)^{-1}c_4$, we have:

$$A_3=\lim_{n\to\infty} n^{-2}\sum_{s=1}^{n}\sum_{u=1}^{n-s}\left[\begin{array}{l}\frac{3}{4}\left(\tilde{\mu}_{h,s+u}^{(1)}\right)^{-1/2}c_4\left(c_{15}+(\tau_z-c_{15})\,\varphi^{n-s-u}\right)\\ \left[\tilde{\mu}_{h,s+u}^{(1)}\varphi^{u-1}+\omega(\varphi-\gamma)^{-1}\left(\varphi^u-\gamma^u\right)\right]E_t\left(h_{t+s}^{3/2}\right)\\ +\frac{3}{8}\left(\tilde{\mu}_{h,s+u}^{(1)}\right)^{-1/2}\gamma^{u-1}c_5\left(c_{15}+(\tau_z-c_{15})\,\varphi^{n-s-u}\right)E_t\left(h_{t+s}^{5/2}\right)\end{array}\right].$$

It is reasonable to assume that $\operatorname{sgn}(\tau_z)=-\operatorname{sgn}(\lambda)$ when $\tau_z\neq0$ and $\lambda\neq0$, since a positive λ means that volatility is more responsive to negative shocks rather than positive shocks of the same magnitude and translates into a negative skew of the aggregated returns. If $\operatorname{sgn}(\tau_z)=-\operatorname{sgn}(\lambda)\neq0$, then $\operatorname{sgn}(\tau_z)=\operatorname{sgn}(c_9)=\operatorname{sgn}(c_{17})$, and consequently

$$\operatorname{sgn}\left(c_4\left(c_{15}+(\tau_z-c_{15})\,\varphi^{n-s-u}\right)\right)=\operatorname{sgn}\left(c_4\left(\tau_z+3(1-\varphi)^{-1}c_4\left(1-\varphi^{n-s-u}\right)\right)\right)=1.$$

For $\operatorname{sgn}(\tau_z)=\operatorname{sgn}\left(\mu_z^{(5)}\right)$,[9] $\operatorname{sgn}(c_9)=\operatorname{sgn}(c_{10})$ and

$$\operatorname{sgn}\left(c_5\left[c_{15}+\varphi^{n-s-u}\left(\tau_z-c_{15}\right)\right]\right)=\operatorname{sgn}\left(c_5\left(\tau_z+3(1-\varphi)^{-1}c_4\left(1-\varphi^{n-s-u}\right)\right)\right)=1.$$

Furthermore, for either $\{\tau_z=0$ and $\lambda\neq0\}$ or $\{\tau_z\neq0$ and $\lambda=0\}$, we still have

$$\operatorname{sgn}\left(c_4\left(c_{15}+(\tau_z-c_{15})\,\varphi^{n-s-u}\right)\right)=\operatorname{sgn}\left(c_5\left(c_{15}+(\tau_z-c_{15})\,\varphi^{n-s-u}\right)\right)=1$$

[9] This is a sufficient but not necessary condition.

if $\operatorname{sgn}(\tau_z) = \operatorname{sgn}\left(\mu_z^{(5)}\right)$; hence all terms in A_3 are positive if $\operatorname{sgn}(|\lambda| + |\tau_z|) \neq 0$. Finally, for $\tau_z = \lambda = 0$, $A_3 = 0$. We have thus shown that A_3 is bounded below (by 0) and hence the limit $\lim_{n\to\infty} \mathrm{K}_{r,n} = \infty$ for $\gamma \in (1, \infty)$. For $\gamma = 1$, we can write:

$$\begin{aligned}
\theta_{su}^{(3/2)} &= \frac{3}{4}c_4\left[\left(\tilde{\mu}_{h,s+u}^{(1)}\right)^{1/2} + \omega\varphi(\varphi-1)^{-1}\left(\tilde{\mu}_{h,s+u}^{(1)}\right)^{-1/2}\right]\varphi^{u-1}E_t\left(h_{t+s}^{3/2}\right) \\
&+ \frac{3}{8}\left(\tilde{\mu}_{h,s+u}^{(1)}\right)^{-1/2}\left(c_5E_t\left(h_{t+s}^{5/2}\right) + 2\omega(1-\varphi)^{-1}c_4E_t\left(h_{t+s}^{3/2}\right)\right), \quad \text{and} \\
\varphi^{n-s-u}\theta_{su}^{(3/2)} &= \frac{3}{4}c_4\left[\left(\tilde{\mu}_{h,s+u}^{(1)}\right)^{1/2} + \omega\varphi(\varphi-1)^{-1}\left(\tilde{\mu}_{h,s+u}^{(1)}\right)^{-1/2}\right]\varphi^{n-s-1}E_t\left(h_{t+s}^{3/2}\right) \\
&+ \frac{3}{8}\left(\tilde{\mu}_{h,s+u}^{(1)}\right)^{-1/2}\varphi^{n-s-u}\left(c_5E_t\left(h_{t+s}^{5/2}\right) + 2\omega(1-\varphi)^{-1}c_4E_t\left(h_{t+s}^{3/2}\right)\right).
\end{aligned}$$

Thus, for $\gamma = 1$, the expression for A_3 is:

$$A_3 = \lim_{n\to\infty} n^{-2}\sum_{s=1}^{n}\sum_{u=1}^{n-s}\left(\begin{array}{l}
\frac{3}{4}c_4\left(c_{15} + (\tau_z - c_{15})\varphi^{n-s-u}\right)\left(\tilde{\mu}_{h,s+u}^{(1)}\right)^{-1/2} \\
\left[\tilde{\mu}_{h,s+u}^{(1)}\varphi^{u-1} + \omega(1-\varphi)^{-1}(1-\varphi^u)\right]E_t\left(h_{t+s}^{3/2}\right) \\
+\frac{3}{8}c_5\left(c_{15} + (\tau_z - c_{15})\varphi^{n-s-u}\right)\left(\tilde{\mu}_{h,s+u}^{(1)}\right)^{-1/2}E_t\left(h_{t+s}^{5/2}\right)
\end{array}\right).$$

Since c_4, c_5, and c_{15} do not depend on γ, we still have that $\operatorname{sgn}(c_4) = \operatorname{sgn}(c_5) = \operatorname{sgn}(c_{15})$ and that:

$$\operatorname{sgn}\left(c_4\left(c_{15} + (\tau_z - c_{15})\varphi^{n-s-u}\right)\right) = \operatorname{sgn}\left(c_5\left(c_{15} + (\tau_z - c_{15})\varphi^{n-s-u}\right)\right) = 1.$$

Now we write:

$$\begin{aligned}
A_3 &\geq \lim_{n\to\infty} n^{-2}\sum_{s=1}^{n}\sum_{u=1}^{n-s}\frac{3}{8}c_5\left(c_{15} + (\tau_z - c_{15})\varphi^{n-s-u}\right)\left(\tilde{\mu}_{h,s+u}^{(1)}\right)^{-1/2}E_t\left(h_{t+s}^{5/2}\right) \\
&\geq \frac{3}{8}\lim_{n\to\infty}\left[\max_{1\leq s\leq n}\left(\tilde{\mu}_{h,s+u}^{(1)}\right)\right]^{-1/2}\lim_{n\to\infty} n^{-2}\sum_{s=1}^{n}\sum_{u=1}^{n-s}c_5\left(c_{15} + (\tau_z - c_{15})\varphi^{n-s-u}\right)E_t\left(h_{t+s}^{5/2}\right) \\
&\geq \frac{3}{64}\lim_{n\to\infty}\left[\max_{1\leq s\leq n}\left(\tilde{\mu}_{h,s+u}^{(1)}\right)\right]^{-1/2}\lim_{n\to\infty}\left[\min_{1\leq s\leq n}\left[\min_{1\leq u\leq n-s}\left[c_5\left(c_{15} + (\tau_z - c_{15})\varphi^{n-s-u}\right)\right]\right]\right] \\
&\quad \lim_{n\to\infty} n^{-2}\sum_{s=1}^{n}\sum_{u=1}^{n-s}\left(\left(15\left(\tilde{\mu}_{h,s}^{(1)}\right)^{1/2}\tilde{\mu}_{h,s}^{(2)} - 7\left(\tilde{\mu}_{h,s}^{(1)}\right)^{5/2}\right)\right)
\end{aligned}$$

Since $\varphi \in (0,1)$, both $\lim_{n\to\infty}\left[\min_{1\leq s\leq n}\left[\min_{1\leq u\leq n-s}\left[c_5\left(c_{15} + (\tau_z - c_{15})\varphi^{n-s-u}\right)\right]\right]\right]$ and $\lim_{n\to\infty}\left[\max_{1\leq s\leq n}\left(\tilde{\mu}_{h,s+u}^{(1)}\right)\right]$

are finite and we have shown above that they are positive, if $\tau_z \neq 0$ or $\lambda \neq 0$. In this case,

$$
\begin{aligned}
& \lim_{n\to\infty} n^{-2} \sum_{s=1}^{n} \sum_{u=1}^{n-s} \left(\left(15 \left(\tilde{\mu}_{h,s}^{(1)} \right)^{1/2} \tilde{\mu}_{h,s}^{(2)} - 7 \left(\tilde{\mu}_{h,s}^{(1)} \right)^{5/2} \right) \right) \\
= & \lim_{n\to\infty} n^{-2} \sum_{s=1}^{n} \sum_{u=1}^{n-s} \left(\left(15 \left(\tilde{\mu}_{h,s}^{(1)} \right)^{1/2} \left(\begin{array}{l} (s-1)\left(\omega^2 + 2\omega\varphi\bar{h}\right) \\ +2\varphi\bar{h}\left(1-\varphi^{s-1}\right)\left(h_{t+1}-\bar{h}\right) + h_{t+1}^2 \end{array} \right) - 7 \left(\tilde{\mu}_{h,s}^{(1)} \right)^{5/2} \right) \right) \\
= & \; 15\left(\omega^2 + 2\omega\varphi\bar{h}\right) \lim_{n\to\infty} n^{-2} \sum_{s=1}^{n} \left(\tilde{\mu}_{h,s}^{(1)} \right)^{1/2} (n-s)(s-1) \\
\geq & \; 15\left(\omega^2 + 2\omega\varphi\bar{h}\right) \lim_{n\to\infty} \left[\min_{1\leq s\leq n} \left(\tilde{\mu}_{h,s+u}^{(1)} \right) \right] \lim_{n\to\infty} n^{-2} \left(\frac{n^2(n-5)}{6} \right),
\end{aligned}
$$

where $15\left(\omega^2 + 2\omega\varphi\bar{h}\right) \lim_{n\to\infty} \left[\min_{1\leq s\leq n} \left(\tilde{\mu}_{h,s+u}^{(1)} \right) \right]$ is positive and finite. Thus $A_3 = \infty$ for $\gamma = 1$ and $\tau_z \neq 0$ and\or $\lambda \neq 0$, while for $\gamma = 1$ and $\tau_z = \lambda = 0$, $c_4 = c_5 = 0$ and $A_3 = 0$. This completes our proof for the final expression for the limit of aggregated kurtosis given in Theorem 3.

For the normal GJR, $\tau_z = 0$ (but $\lambda \neq 0$), thus the expression for the limit of the aggregated kurtosis in his case simplifies to:

$$
\lim_{n\to\infty} \mathrm{K}_{r,n} = \begin{cases} 3 & \text{if } \gamma \in (0,1), \\ \infty & \text{if } \gamma \in [1,\infty). \end{cases}
$$

For the normal GARCH(1,1), we have $\tau_z = \lambda = 0$ and the limit of the aggregated kurtosis is now finite for $\gamma = 1$, although different from the normal value of 3. Thus, for the normal GARCH(1,1), we have:

$$
\lim_{n\to\infty} \mathrm{K}_{r,n} = \begin{cases} 3 & \text{if } \gamma \in (0,1), \\ 3\left[1 + \frac{1}{2}(1+\alpha+\beta)(1+5\alpha+\beta)\right] & \text{if } \gamma = 1, \\ \infty & \text{if } \gamma \in (1,\infty). \end{cases}
$$

A4: Limits of the Moments of Variances

This appendix derives the limits of the conditional moments of forward and aggregated variances of the generic GJR model as the time horizon increases. In what follows we use the notation and assumptions defined at the start of the Appendix; additionally, we assume $c_6 \neq \varphi$, $c_6 \neq \gamma$.

(a) *Limit of the Variance of Forward Variance*

$$\begin{aligned}
\lim_{s\to\infty} \mu^{(2)}_{h,s} &= \lim_{s\to\infty}\left(\tilde{\mu}^{(2)}_{h,s} - \left(\tilde{\mu}^{(1)}_{h,s}\right)^2\right) \\
&= \begin{cases} \lim\limits_{s\to\infty}\left[\begin{array}{l} \left(c_1-\bar{h}^2\right) + \left(-c_3 + h^2_{t+1}\right)\gamma^{s-1} \\ + \left[c_2 - 2\bar{h}\left(h_{t+1}-\bar{h}\right)\right]\varphi^{s-1} - \varphi^{2(s-1)}\left(h_{t+1}-\bar{h}\right)^2 \end{array}\right] & \text{if } \gamma \neq 1, \\ \lim\limits_{s\to\infty}\left[\begin{array}{l} \left(2\omega\varphi\bar{h}\left(h_{t+1}-\bar{h}\right) + h^2_{t+1} - \bar{h}^2\right) + (s-1)\left(\omega^2 + 2\omega\varphi\bar{h}\right) \\ -2\bar{h}\left(\omega\varphi - 1\right)\left(h_{t+1}-\bar{h}\right)\varphi^{s-1} - \varphi^{2(s-1)}\left(h_{t+1}-\bar{h}\right)^2 \end{array}\right] & \text{if } \gamma = 1, \end{cases} \\
&= \begin{cases} \left(c_1 - \bar{h}^2\right) & \text{if } \gamma \in (0,1)\,, \\ \infty & \text{if } \gamma \in [1,\infty)\,, \end{cases}
\end{aligned}$$

where $c_1 - \bar{h}^2 > 0$ and $h^2_{t+1} - c_3 > 0$.

(b) *Limit of the Variance of Aggregated Variance*

For $\gamma \neq 1$, using (8) and (11) – (14), the expression for the variance of the aggregated variance can be written as:[10]

$$M^{(2)}_{h,n} = An + B\gamma^n + C_n$$

where

$$\begin{aligned}
A &= \left(c_1 - \bar{h}^2\right)\left(1 + 2\varphi(1-\varphi)^{-1}\right), \\
B &= -\left(\tilde{\mu}^{(2)}_{h,1} - c_3\right)(1-\gamma)^{-1} - 2\varphi(1-\varphi)^{-1}\left[\left(\tilde{\mu}^{(2)}_{h,1} - c_3\right)(1-\gamma)^{-1} - (\varphi-\gamma)^{-1}\right], \quad \text{and} \\
C_n &= f\left(n\varphi^n, \varphi^n\right).
\end{aligned}$$

Since $\varphi \in (0,1)$, $\lim\limits_{n\to\infty}\varphi^n = \lim\limits_{n\to\infty} n\varphi^n = 0$ and $\exists\, C$ finite such that: $\lim\limits_{n\to\infty} C_n = \text{C}$.[11] Thus, the limit

[10] The n^2 terms cancel out.

[11] Showing that $\lim\limits_{n\to\infty} n\varphi^n = 0$ for $\varphi \in (0,1)$ is rather immediate. If we define $y = 1/\varphi$, then $y > 1$. We now have $\lim\limits_{n\to\infty} n\varphi^n = \lim\limits_{n\to\infty} \frac{n}{y^n} = 0$.

of the conditional variance of the aggregated conditional variance becomes:

$$\lim_{n\to\infty} M^{(2)}_{h,n} = \begin{cases} \operatorname{sgn}(A)\,\infty, & \text{if } \gamma \in (0,1)\,, \\ \operatorname{sgn}(B)\,\infty & \text{if } \gamma \in (1,\infty)\,, \end{cases} \tag{23}$$

where it can be easily seen that $\operatorname{sgn}(A) = 1$ and $\operatorname{sgn}(B) = 1$.

For $\gamma = 1$, the expression for the variance of the aggregated variance becomes:

$$M^{(2)}_{h,n} = A'n^2 + B'n + C'$$

where $A' = \frac{1}{2}\left[2\varphi(1-\varphi)^{-1} + 1\right]\left(\omega^2 + 2\omega\varphi\bar{h}\right) > 0.$[12] Hence: $\lim\limits_{n\to\infty} M^{(2)}_{h,n} = \infty$ for any γ.

As with the variance of aggregated returns, the conditional variance of the aggregated conditional variance diverges to infinity when we increase the time horizon infinitely. It is meaningful to compute the limit of the daily variance, i.e. the variance divided by time. However, unlike the daily (one-period) variance of aggregated returns which converges to the level of (daily) unconditional variance, the daily conditional variance of aggregated conditional variance diverges to infinity under certain parameter conditions:[13]

$$\lim_{n\to\infty} \frac{M^{(2)}_{h,n}}{n} = \begin{cases} \lim\limits_{n\to\infty}\left(A + B\frac{\gamma^n}{n} + \frac{C_n}{n}\right) & \text{if } \gamma \neq 1, \\ \lim\limits_{n\to\infty}\left(A'n + B' + \frac{C'}{n}\right) & \text{if } \gamma = 1, \end{cases} = \begin{cases} A & \text{if } \gamma \in (0,1)\,, \\ \infty & \text{if } \gamma \in [1,\infty)\,. \end{cases}$$

(c) *Limit of the Skewness of Forward Variance*

$$\lim_{s\to\infty} \tau_{h,s} = \lim_{s\to\infty}\left(\mu^{(3)}_{h,s}\left(\mu^{(2)}_{h,s}\right)^{-3/2}\right)$$

For $\gamma \in (0,1)$, we can write:

$$\lim_{s\to\infty} \tau_{h,s} = \frac{\lim\limits_{s\to\infty}\left(\mu^{(3)}_{h,s}\right)}{\lim\limits_{s\to\infty}\left(\mu^{(2)}_{h,s}\right)^{3/2}},$$

where $\lim\limits_{s\to\infty} \mu^{(2)}_{h,s} = c_1 - \bar{h}^2$. Using (17), we have:

[12] As in the $\gamma \neq 1$ case above, $C' = f\left(n\varphi^n, \varphi^n\right)$ and $\lim\limits_{n\to\infty} C' = \text{constant}$. B' is a constant, but its value and sign are irrelevant for the limit above.

[13] Unlike the conditional variance of the aggregated returns, which only depends on (powers of) the φ parameter which takes values only between 0 and 1, the conditional variance of the aggregated variance also depends on (powers of) the γ parameter, which can take any positive value.

$$\lim_{s\to\infty} \tau_{h,s} = \begin{cases} M_1 & \text{if } \gamma \in (0,1) \text{ and } c_6 \in (0,1), \\ \infty & \text{if } \gamma \in (0,1) \text{ and } c_6 \in [1,\infty), \end{cases} \tag{24}$$

where

$$M_1 = \frac{\omega\left(\omega^2 + 3\omega\varphi\bar{h} + 3\gamma c_1\right)(1-c_6)^{-1} - 3\bar{h}c_1 + 2\bar{h}^3}{\left(c_1 - \bar{h}^2\right)^{3/2}}.$$

For $\gamma \in (1,\infty)$, we can write:

$$\lim_{s\to\infty} \tau_{h,s} = \frac{\lim\limits_{s\to\infty}\left(\gamma^{-(3/2)s}\mu_{h,s}^{(3)}\right)}{\left[\lim\limits_{s\to\infty}\left(\gamma^{-s}\mu_{h,s}^{(2)}\right)\right]^{3/2}},$$

where $\lim\limits_{s\to\infty}\left(\gamma^{-s}\mu_{h,s}^{(2)}\right) = \gamma^{-1}\left(-c_3 + h_{t+1}^2\right)$. Using (17), and observing that $c_6 > \gamma^{3/2}$, we have that, for $\gamma \in (1,\infty)$:

$$\lim_{s\to\infty} \tau_{h,s} = \infty \tag{25}$$

For $\gamma = 1$, we can write:

$$\lim_{s\to\infty} \tau_{h,s} = \frac{\lim\limits_{s\to\infty}\left(s^{-3/2}\mu_{h,s}^{(3)}\right)}{\left[\lim\limits_{s\to\infty}\left(s^{-1}\mu_{h,s}^{(2)}\right)\right]^{3/2}},$$

where $\lim\limits_{s\to\infty}\left(s^{-1}\mu_{h,s}^{(2)}\right) = \left(\omega^2 + 2\omega\varphi\bar{h}\right)$. In this case, the limit becomes:

$$\lim_{s\to\infty} \tau_{h,s} = \lim_{s\to\infty}\left(s^{-3/2}\tilde{\mu}_{h,s}^{(3)}\right) = \infty \tag{26}$$

Now, (24) - (26) give the limit of the conditional skewness of the forward conditional variance in Theorem 4.

(d) *Limit of the Skewness of Aggregated Variance*

$$\lim_{n\to\infty} \mathrm{T}_{h,n} = \lim_{n\to\infty}\left[M_{h,n}^{(3)}\left(M_{h,n}^{(2)}\right)^{-3/2}\right].$$

For $\gamma \in (0,1)$, we can write:

$$\lim_{n\to\infty} \mathrm{T}_{h,n} = \frac{\lim\limits_{n\to\infty}\left(n^{-3/2}M_{h,n}^{(3)}\right)}{\left[\lim\limits_{n\to\infty}\left(n^{-1}M_{h,n}^{(2)}\right)\right]^{3/2}},$$

where $\lim_{n\to\infty}\left(n^{-1}M_{h,n}^{(2)}\right)=\left(c_1-\bar{h}^2\right)\left(1+2\varphi(1-\varphi)^{-1}\right)$. Also, we write:

$$\begin{aligned} n^{-3/2}M_{h,n}^{(3)} &= L_1-3L_2+2L_3+3\left(L_4+L_5+2\left(L_6+L_7-L_8-L_9\right)-L_{10}-L_{11}\right)\\ &+ 6\left(L_{12}-L_{13}-L_{14}-L_{15}+2L_{16}\right), \end{aligned}$$

where

$$L_1=\frac{\sum_{s=1}^{n}\tilde{\mu}_{h,s}^{(3)}}{n^{3/2}},\qquad L_2=\frac{\sum_{s=1}^{n}\tilde{\mu}_{h,s}^{(2)}\tilde{\mu}_{h,s}^{(1)}}{n^{3/2}},\qquad L_3=\frac{\sum_{s=1}^{n}\left(\tilde{\mu}_{h,s}^{(1)}\right)^3}{n^{3/2}}$$

$$L_4=\frac{\sum_{s=1}^{n}\sum_{u=1}^{n-s}\tilde{\mu}_{h,su}^{(2,1)}}{n^{3/2}},\qquad L_5=\frac{\sum_{s=1}^{n}\sum_{u=1}^{n-s}\tilde{\mu}_{h,su}^{(1,2)}}{n^{3/2}},\qquad L_6=\frac{\sum_{s=1}^{n}\sum_{u=1}^{n-s}\left(\tilde{\mu}_{h,s}^{(1)}\right)^2\tilde{\mu}_{h,s+u}^{(1)}}{n^{3/2}},$$

$$L_7=\frac{\sum_{s=1}^{n}\sum_{u=1}^{n-s}\left(\tilde{\mu}_{h,s+u}^{(1)}\right)^2\tilde{\mu}_{h,s}^{(1)}}{n^{3/2}},\qquad L_8=\frac{\sum_{s=1}^{n}\sum_{u=1}^{n-s}\tilde{\mu}_{h,s}^{(1)}\tilde{\mu}_{h,su}^{(1,1)}}{n^{3/2}},\qquad L_9=\frac{\sum_{s=1}^{n}\sum_{u=1}^{n-s}\tilde{\mu}_{h,s+u}^{(1)}\tilde{\mu}_{h,su}^{(1,1)}}{n^{3/2}},$$

$$L_{10}=\frac{\sum_{s=1}^{n}\sum_{u=1}^{n-s}\tilde{\mu}_{h,s}^{(1)}\tilde{\mu}_{h,s+u}^{(2)}}{n^{3/2}},\qquad L_{11}=\frac{\sum_{s=1}^{n}\sum_{u=1}^{n-s}\tilde{\mu}_{h,s+u}^{(1)}\tilde{\mu}_{h,s}^{(2)}}{n^{3/2}},\qquad L_{12}=\frac{\sum_{s=1}^{n}\sum_{u=1}^{n-s}\sum_{v=1}^{n-s-u}\tilde{\mu}_{h,suv}^{(1,1,1)}}{n^{3/2}},$$

$$L_{13}=\frac{\sum_{s=1}^{n}\sum_{u=1}^{n-s}\sum_{v=1}^{n-s-u}\tilde{\mu}_{h,s}^{(1)}\tilde{\mu}_{h,(s+u)v}^{(1,1)}}{n^{3/2}},\qquad L_{14}=\frac{\sum_{s=1}^{n}\sum_{u=1}^{n-s}\sum_{v=1}^{n-s-u}\tilde{\mu}_{h,(s+u)}^{(1)}\tilde{\mu}_{h,s(u+v)}^{(1,1)}}{n^{3/2}},$$

$$L_{15}=\frac{\sum_{s=1}^{n}\sum_{u=1}^{n-s}\sum_{v=1}^{n-s-u}\tilde{\mu}_{h,(s+u+v)}^{(1)}\tilde{\mu}_{h,su}^{(1,1)}}{n^{3/2}},\qquad L_{16}=\frac{\sum_{s=1}^{n}\sum_{u=1}^{n-s}\sum_{v=1}^{n-s-u}\tilde{\mu}_{h,s}^{(1)}\tilde{\mu}_{h,(s+u)}^{(1)}\tilde{\mu}_{h,(s+u+v)}^{(1)}}{n^{3/2}}.$$

Performing the necessary (tedious but straightforward) calculations and using the notation R_i, $\tilde{R}_i$, $i\in\{1,2,3....16\}$ with $\lim_{n\to\infty}R_i=\lim_{n\to\infty}\tilde{R}_i=0$, we have $L_2=R_2$, $L_3=R_3$,

$$L_6=\frac{\bar{h}^3}{2}\frac{n^2}{n^{3/2}}+R_6,\qquad L_7=\frac{\bar{h}^3}{2}\frac{n^2}{n^{3/2}}+R_7,\qquad L_8=\frac{\bar{h}^3}{2}\frac{n^2}{n^{3/2}}+R_8,$$

$$L_9=\frac{\bar{h}^3}{2}\frac{n^2}{n^{3/2}}+R_9,\qquad L_{10}=\frac{c_1\bar{h}}{2}\frac{n^2}{n^{3/2}}+R_{10},\qquad L_{11}=\frac{c_1\bar{h}}{2}\frac{n^2}{n^{3/2}}+R_{11},$$

and

$$L_1=\begin{cases} c_{16}(c_6-1)^{-1}\frac{c_6^n}{n^{3/2}}+R_1 & \text{if } c_6\neq 1,\\ \frac{\omega\left(\omega^2+3\omega\varphi\bar{h}+3\gamma c_1\right)}{2}\frac{n^2}{n^{3/2}}+\tilde{R}_1 & \text{if } c_6=1,\end{cases}$$

$$L_4 = \begin{cases} \frac{\varphi c_{16}}{(1-c_6)(\varphi-c_6)} \frac{c_6^n}{n^{3/2}} + \frac{\bar{h}}{2} c_1 \frac{n^2}{n^{3/2}} + R_4 & \text{if } c_6 \neq 1, \\ \frac{\bar{h}}{2} \left(c_1 + \varphi \left(\omega^2 + 3\bar{h}\varphi\omega + 3\gamma c_1 \right) \right) \frac{n^2}{n^{3/2}} + \tilde{R}_4 & \text{if } c_6 = 1, \end{cases}$$

$$L_5 = \begin{cases} \frac{\gamma c_{16}}{(1-c_6)(\gamma-c_6)} \frac{c_6^n}{n^{3/2}} + c_1 \frac{\bar{h}}{2} \frac{n^2}{n^{3/2}} + R_5 & \text{if } c_6 \neq 1, \\ \begin{bmatrix} \gamma(1-\gamma)^{-1} \frac{\omega\left(\omega^2+3\bar{h}\varphi\omega+3\gamma c_1\right)}{2} \\ \omega^2(1-\gamma)^{-1}\frac{\bar{h}}{2} + \omega\varphi\bar{h}^2 \end{bmatrix} \frac{n^2}{n^{3/2}} + \tilde{R}_5 & \text{if } c_6 = 1, \end{cases}$$

$$L_{12} = \begin{cases} \varphi\gamma c_{16}(c_6-1)^{-1}(c_6-\varphi)^{-1}(c_6-\gamma)^{-1}\frac{c_6^n}{n^{3/2}} + \frac{\bar{h}^3}{6}\frac{n^3}{n^{3/2}} \\ +\frac{\bar{h}}{2}(1-\varphi)^{-1}\left(2\varphi\left(c_1-\bar{h}^2\right) + \bar{h}\left(\left(h_{t+1}-\bar{h}\right)-\omega\right)\right)\frac{n^2}{n^{3/2}} + R_{12} & \text{if } c_6 \neq 1, \\ \frac{\bar{h}^3}{6}\frac{n^3}{n^{3/2}} + \frac{\bar{h}}{2}\begin{bmatrix} (1-\varphi)^{-1} \\ \left(c_1\left(1+\varphi\right) - 2\varphi\bar{h}^2 + \bar{h}\left(\left(h_{t+1}-\bar{h}\right)-\omega\right)\right) \\ +\varphi\gamma(1-\gamma)^{-1}\left(\omega^2+3\omega\varphi\bar{h}+3\gamma c_1\right) \end{bmatrix}\frac{n^2}{n^{3/2}} + \tilde{R_{12}} & \text{if } c_6 = 1, \end{cases}$$

$$L_{13} = \frac{\bar{h}^3}{6}\frac{n^3}{n^{3/2}} + \frac{\bar{h}}{2}\left(-\bar{h}^2 + (1-\varphi)^{-1}\left(\bar{h}\left(h_{t+1}-\bar{h}\right) + \varphi\left(c_1-\bar{h}^2\right)\right)\right)\frac{n^2}{n^{3/2}} + R_{13},$$

$$L_{14} = \frac{\bar{h}^3}{6}\frac{n^3}{n^{3/2}} + \frac{\bar{h}^2}{2}\left(-\bar{h} + \left(h_{t+1}-\bar{h}\right)(1-\varphi)^{-1}\right)\frac{n^2}{n^{3/2}} + R_{14},$$

$$L_{15} = \frac{\bar{h}^3}{6}\frac{n^3}{n^{3/2}} + \frac{\bar{h}}{2}(1-\varphi)^{-1}\left(-2\bar{h}^2 + \bar{h}h_{t+1} + c_1\varphi\right)\frac{n^2}{n^{3/2}} + R_{15},$$

$$L_{16} = \frac{\bar{h}^3}{6}\frac{n^3}{n^{3/2}} + \frac{\bar{h}^2}{2}\left(-\bar{h} + \left(h_{t+1}-\bar{h}\right)(1-\varphi)^{-1}\right)\frac{n^2}{n^{3/2}} + R_{16}.$$

Performing the necessary calculations, we obtain the expression in Theorem 4 for the limit of the conditional skewness of the aggregated conditional variance, where:

$$\begin{aligned} N &= \frac{\omega\left(\omega^2+3\omega\varphi\bar{h}+3\gamma c_1\right)}{2} + 3\frac{\bar{h}}{2}\left(c_1 + \varphi\left(\omega^2+3\omega\varphi\bar{h}+3\gamma c_1\right) + \omega^2(1-\gamma)^{-1} + 2\omega\varphi\bar{h}\right) \\ &+ 3\gamma(1-\gamma)^{-1}\frac{\left(\omega^2+3\omega\varphi\bar{h}+3\gamma c_1\right)}{2}\left(\omega+2\varphi\bar{h}\right) \\ &+ 3(1-\varphi)^{-1}\bar{h}\left[c_1\left(1+\varphi\right) - 2\varphi\bar{h}^2 + \bar{h}\left(\left(h_{t+1}-\bar{h}\right)-\omega\right) - \varphi\left(2c_1-\bar{h}^2\right)\right]. \end{aligned}$$

For $\gamma \in (1, \infty)$, we can write:

$$\lim_{n\to\infty} \mathrm{T}_{h,n} = \frac{\lim\limits_{n\to\infty}\left(\gamma^{-3/2n} M_{h,n}^{(3)}\right)}{\left[\lim\limits_{n\to\infty}\left(\gamma^{-n} M_{h,n}^{(2)}\right)\right]^{3/2}},$$

where

$$\lim_{n\to\infty}\left(\gamma^{-n} M_{h,n}^{(2)}\right) = -\left(\tilde{\mu}_{h,1}^{(2)} - c_3\right)(1-\gamma)^{-1} - 2\varphi(1-\varphi)^{-1}\left[\left(\tilde{\mu}_{h,1}^{(2)} - c_3\right)(1-\gamma)^{-1} - (\varphi-\gamma)^{-1}\right].$$

For $\gamma = 1$, we can write:

$$\lim_{n\to\infty} \mathrm{T}_{h,n} = \frac{\lim\limits_{n\to\infty}\left(n^{-3} M_{h,n}^{(3)}\right)}{\left[\lim\limits_{n\to\infty}\left(n^{-2} M_{h,n}^{(2)}\right)\right]^{3/2}},$$

where

$$\lim_{n\to\infty}\left(n^{-2} M_{h,n}^{(2)}\right) = \frac{1}{2}\left[2\varphi(1-\varphi)^{-1} + 1\right]\left(\omega^2 + 2\omega\varphi\bar{h}\right).$$

Reference

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