On the uniqueness and monotonicity of energy minimisers in the homotopy classes of incompressible mappings and related problems

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On the Uniqueness and Monotonicity of Energy Minimisers in the Homotopy Classes of Incompressible Mappings and Related Problems

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Abstract

The goal of this paper is to prove the existence and uniqueness of the so-called energy minimisers in homotopy classes for the variational energy integral

$$\mathscr{F}[u;\mathbf{X}] = \int_{\mathbf{X}} F(|x|^2, |u|^2)|\nabla u|^2/2 \, \mathrm{d}x,$$

with $F \geq c > 0$ of class $\mathscr{C}^2$ and satisfying suitable conditions and u lying in the Sobolev space of weakly differentiable incompressible mappings of a finite open symmetric plane annulus $\mathbf{X}$ onto itself, specifically, lying in $\mathcal{A}(\mathbf{X}) = \{u \in \mathscr{W}^{1,2}(\mathbf{X}, \mathbb{R}^2) : \det \nabla u = 1 \ a.e. \text{ in } \mathbf{X}, \text{ and } u \equiv x \text{ on } \partial\mathbf{X}\}$. It is well known that the space $\mathcal{A}(\mathbf{X})$ admits a countably infinite homotopy class decomposition $\mathcal{A}(\mathbf{X}) = \bigcup \mathscr{A}_k$ (with $k \in \mathbb{Z}$). We prove that the energy integral $\mathscr{F}$ has a *unique* minimiser in each of these homotopy classes $\mathscr{A}_k$. Furthermore we show that each minimiser is a homeomorphic, monotone, radially symmetric twist mapping of class $\mathscr{C}^3(\overline{\mathbf{X}}, \overline{\mathbf{X}})$ or as smooth as F allows thereafter whilst also being a local minimiser of $\mathscr{F}$ over $\mathcal{A}(\mathbf{X})$ with respect to the L^1-metric. To our best knowledge this is the first uniqueness result for minimisers in homotopy classes in the context of incompressible mappings.

1 Statement of the result

Let $\mathbf{X} \subset \mathbb{R}^2$ be a bounded smooth domain and consider the variational energy integral given by

$$\mathscr{F}[u;\mathbf{X}] = \int_{\mathbf{X}} F(|x|^2, |u|^2)|\nabla u|^2/2 \, \mathrm{d}x, \tag{1.1}$$

with $F \geq c > 0$ sufficiently smooth and satisfying suitable conditions (see below) and u in the Sobolev space of weakly differentiable, incompressible (equivalently here area-preserving) mappings on $\mathbf{X}$ agreeing with identity on $\partial\mathbf{X}$, specifically, in $\mathcal{A}(\mathbf{X}) = \{u \in \mathscr{W}^{1,2}(\mathbf{X}, \mathbb{R}^2) : \det \nabla u = 1 \ a.e. \text{ in } \mathbf{X} \text{ and } u \equiv x \text{ on } \partial\mathbf{X}\}$. Here ∇u is the gradient of u – a 2×2 matrix-field in $\mathbf{X}$ – and the boundary condition

$u \equiv x$ on $\partial\mathbf{X}$ is interpreted in the usual sense of traces. As a consequence of the L^2-integrability of ∇u on $\mathbf{X}$, the incompressibility constraint $\det \nabla u = 1$ *a.e.* in $\mathbf{X}$, and the boundary condition $u \equiv x$ on $\partial\mathbf{X}$, each $u \in \mathcal{A}(\mathbf{X})$ has a representative in the space of continuous self-mappings of $\overline{\mathbf{X}}$, namely, [1]

$$\mathscr{C}_{id}(\mathbf{X}) = \{u \in \mathscr{C}(\overline{\mathbf{X}}, \overline{\mathbf{X}}) : u \equiv x \text{ on } \partial\mathbf{X}\} = \bigcup_{k \in \mathcal{K}} \mathscr{C}_k, \tag{1.2}$$

where referring to the union on the right, $\mathcal{K}$ denotes the index set of the family of pairwise disjoint path-connected components (or homotopy classes) of $\mathscr{C}_{id}(\mathbf{X})$, whilst $\mathscr{C}_k$ denotes the kth homotopy class in the family. As a result one can then consider an associated partitioning of the space of admissible mappings $\mathcal{A}(\mathbf{X})$ through $\mathcal{A}(\mathbf{X}) = \bigcup \mathscr{A}_k = \bigcup \{u \in \mathcal{A}(\mathbf{X}) : u \in \mathscr{C}_k\}$ (the unions over $k \in \mathcal{K}$) where $u \in \mathscr{C}_k$ stands for the continuous representative of $u \in \mathcal{A}(\mathbf{X})$. We shall henceforth refer to $\mathscr{A}_k$ as the kth homotopy class of $\mathcal{A}(\mathbf{X})$. It can be shown that on the sub-level sets of the energy each of these homotopy classes are sequentially weakly $\mathscr{W}^{1,2}$-closed and therefore by virtue of $\mathscr{F}[\cdot; \mathbf{X}]$ being sequentially weakly $\mathscr{W}^{1,2}$-lower semicontinuous it follows by an application of the direct methods of the calculus of variations that $\mathscr{F}$ admits a minimiser u_k in each $\mathscr{A}_k$. Furthermore each such u_k is an L^1-local minimiser of $\mathscr{F}$ in $\mathcal{A}(\mathbf{X})$ in the sense that for each $k \in \mathbb{Z}$ there exists $\delta = \delta[u_k] > 0$ such that $v \in \mathcal{A}(\mathbf{X})$ and $\|u_k - v\|_{L^1} < \delta$ gives the energy inequality $\mathscr{F}[u_k; \mathbf{X}] \le \mathscr{F}[v; \mathbf{X}]$.

The goal of this paper is to tackle the question of *uniqueness* for minimisers of $\mathscr{F}$ in the homotopy classes $\mathscr{A}_k$ and to describe certain other qualitative features of these minimisers. For the sake of this paper we confine to the geometric setup where $\mathbf{X}$ is a finite open plane annulus whose corresponding space $\mathscr{C}_{id}(\mathbf{X})$ has an infinite (countable) number of connected components with $\mathcal{K} = \mathbb{Z}$ ([30]-[33]). As for the energy integral $\mathscr{F}$ in (1.1) we introduce the class $\mathfrak{F}$ of integrands $\mathfrak{F} = \{F \in \mathscr{C}^2(\mathscr{R}) : F \ge c > 0$ satisfies either (**H1**) or (**H2**)$\}$, where $\mathscr{R} = [a^2, b^2] \times [a^2, b^2] \subset \mathbb{R}^2_+$, $\mathbf{X} = \mathbf{X}[a,b] = \{(x_1, x_2) : a < |x| < b\}$ with $b > a > 0$ and

(**H1**) $F(x,y) = F(1,y)/x$ and $\partial_y^2[F(x^2,y^2)^{1/2}] \ge -\partial_y[F(x^2,y^2)^{1/2}/y]$ on $\mathscr{R}$,

(**H2**) $F(x,y) = F(x,1)/y$ and $\partial_x^2[F(x^2,y^2)^{1/2}] \ge -\partial_x[F(x^2,y^2)^{1/2}/x]$ on $\mathscr{R}$.

Main Theorem. (Uniqueness and Symmetries of Minimisers in $\mathscr{A}_k$ – $n = 2$): *Let $\mathbf{X} = \mathbf{X}[a,b]$ and for $F \in \mathfrak{F}$ consider the energy integral* [2]

$$\mathscr{F}[u; \mathbf{X}] = \int_{\mathbf{X}} F(|x|^2, |u|^2)|\nabla u|^2/2 \, dx, \tag{1.3}$$

over the space of admissible mappings $\mathcal{A}(\mathbf{X})$. Then $\mathscr{F}$ has a unique minimiser u_k in each homotopy class $\mathscr{A}_k$ (with $k \in \mathbb{Z}$). Furthermore $u_k \in \mathscr{C}^3(\overline{\mathbf{X}}, \overline{\mathbf{X}})$ is a twist mapping of the form $u_k(x) = \mathbf{Q}[g_k](|x|)x$ with $x \in \overline{\mathbf{X}}$ where $\mathbf{Q}[g] \in \mathbf{SO}(2)$

[1] *See* Section 3 for further discussion and details.

[2] Throughout the paper $|\mathbf{F}|$ denotes the Hilbert-Schmidt norm of $\mathbf{F} \in \mathbb{R}^{2\times 2}$. In particular $|\nabla u|^2 = tr\{[\nabla u]^t[\nabla u]\} = tr\{[\nabla u][\nabla u]^t\}$.

is the rotation matrix by angle g [see (4.1)] while the angle of rotation function $g_k = g(r;k)$ is given explicitly by

$$g(r;k) = 2\pi k \left\{ \int_a^r \frac{ds}{s^3 F(s^2,s^2)} \right\} \left\{ \int_a^b \frac{ds}{s^3 F(s^2,s^2)} \right\}^{-1} \qquad a \le r \le b. \quad (1.4)$$

Here $g_k \in \mathscr{C}^3[a,b]$ is (strictly) monotone and depending on the sign of k is either increasing or decreasing.

Before moving forward let us pause briefly to make a few remarks and discuss some points regarding the above theorem. Firstly the existence of the infinite scale of minimisers in the homotopy classes $(\mathscr{A}_k : k \in \mathbb{Z})$ does *not* require (**H1**) or (**H2**) and all that is needed here is for F to be of class $\mathscr{C}$ and to satisfy the bound $F \ge c > 0$. Secondly each twist mapping $u_k = \mathbf{Q}[g_k](|x|)x$ as described in the theorem lies in $\mathscr{A}_k \cap \mathscr{C}^3$ and as is shown in Section 4 is a solution to the Euler-Lagrange system associated with $\mathscr{F}$ over $\mathcal{A}(\mathbf{X})$ given by

$$\mathbb{EL}[u, \mathscr{P}; \mathbf{X}] = \begin{cases} \operatorname{div} \mathfrak{S}(x,u,\nabla u) = \partial_\eta F |\nabla u|^2 u & \text{in } \mathbf{X}, \\ \det \nabla u = 1 & \text{in } \mathbf{X}, \\ u \equiv x & \text{on } \partial\mathbf{X}. \end{cases} \quad (1.5)$$

Here $F = F(\mu,\eta)$ with $\partial_\eta F = \partial_\eta F(\mu,\eta)$ for short and $\mathfrak{S}$ is the 2×2 matrix-field

$$\mathfrak{S}(x,y,\mathbf{F}) = F(|x|^2,|y|^2)\mathbf{F} - \mathscr{P}(x)\mathbf{F}^{-t}, \quad (1.6)$$

where $\mathscr{P}$ is a suitable and *a priori* unknown Lagrange multiplier associated with the incompressibility constraint. Thirdly as a result of u_k being a minimiser of $\mathscr{F}$ over $\mathscr{A}_k$ it follows that u_k is also an L^1-local minimiser of $\mathscr{F}$ over $\mathcal{A}(\mathbf{X})$ (see Section 3 for details). Therefore the main theorem proves the existence of a countable family of symmetric L^1-local minimisers of $\mathscr{F}$ over $\mathcal{A}(\mathbf{X})$ in the form of monotone twists. Fourthly it is only in the uniqueness and structure part of the proof that one of (**H1**) or (**H2**) is needed. Indeed speaking of technical details the proof of the main theorem in the case where the integrand $F \in \mathfrak{F}$ satisfies (**H1**) relies on the mappings $u \in \mathcal{A}(\mathbf{X})$ being a Sobolev homeomorphisms. This observation which is of independent interest is proved in Section 2 Theorem 2.1. The proof under (**H2**) is based on lifting, energy bounds and symmetrisation.

Let us now proceed on to highlighting some examples of energies that satisfy the assumptions (**H1**) or (**H2**) in the main theorem above.

- Let $\alpha^2 \ge 1$ and $F(x,y) = x^\alpha y^{-1}$ then the energy,

$$\mathscr{F}[u;\mathbf{X}] = \int_{\mathbf{X}} \frac{|x|^{2\alpha}|\nabla u|^2}{2|u|^2}\, \mathrm{d}x, \quad (1.7)$$

  satisfies (**H2**) but not (**H1**). [3]

[3] Here we point out that when $\alpha = 0$ by invoking the isoperimetric inequality and a variation

- Let $\alpha^2 \geq 1$ and $F(x,y) = y^\alpha x^{-1}$ then the energy,

$$\mathscr{F}[u;\mathbf{X}] = \int_{\mathbf{X}} \frac{|u|^{2\alpha}|\nabla u|^2}{2|x|^2}\,\mathrm{d}x, \tag{1.8}$$

  satisfies (**H1**) but not (**H2**).

- Let $\alpha \geq (\sqrt{2}-1)/a^2$ or $\alpha \leq -(\sqrt{2}+1)/a^2$ and $F(x,y) = e^{\alpha x} y^{-1}$ then

$$\mathscr{F}[u;\mathbf{X}] = \int_{\mathbf{X}} \frac{e^{\alpha|x|^2}|\nabla u|^2}{2|u|^2}\,\mathrm{d}x, \tag{1.9}$$

  satisfies (**H2**) but not (**H1**).

As a non-example we point out that the Dirichlet energy (i.e., $F \equiv 1$) is not covered by the main theorem as the technical apparatus developed here does not immediately apply ($F \notin \mathfrak{F}$). However it is our strong belief that the uniqueness result here would hold for the Dirichlet energy and possibly even a wider class of integrands than $\mathfrak{F}$. For motivation and background material on the problems considered here *see* [1]-[4], [9] and [31, 32, 33]. For related results and directions *see* [5, 10, 14, 16, 24] along with [21, 22, 23, 32]. In [21] by using the planar form of the isoperimetric inequality and the coarea formula for Sobolev functions it is shown that for the explicit choice of integrands $F(|x|^2, |u|^2)|\nabla u|^2 = |\nabla u|^2/|u|^2$, twists are amongst the minimisers in their respective homotopy class. For other related uniqueness results (not in the homotopy class context) *see* [17, 26] as well as [16, 27, 29]. For a description of the homotopy classes of $\mathcal{A}(\mathbf{X})$ for more general plane domains $\mathbf{X}$ using braid groups *see* [30].

Let us end this introduction by outlining the plan of the paper. In Section 2 we show that any admissible Sobolev mapping [in $\mathcal{A}(\mathbf{X})$] is a homeomorphism of class $\mathscr{C}_{id}(\mathbf{X})$. This result plays a key role in the proof of the main theorem in Section 5. In Section 3 we recall the necessary topological apparatus leading to $\mathcal{A}(\mathbf{X}) = \bigcup \mathscr{A}_k$ and justify the existence of a minimiser in each of the homotopy classes $\mathscr{A}_k$. We end the section with an interesting result on the L^p-distances between the classes $(\mathscr{A}_k : k \in \mathbb{Z})$ for $1 \leq p \leq \infty$. In Section 4 we introduce and study twists as solutions to the Euler-Lagrange system (1.5) (*see* Theorem 4.1). In Section 5 which is the heart of the paper we establish that the extremising twists from Section 4 are the unique minimisers of $\mathscr{F}$ over the classes $\mathscr{A}_k$ and so in particular are the same local minimisers found in Section 3.

2 Sobolev mappings in $\mathcal{A}(\mathbf{X})$ are global homeomorphisms in $\mathscr{C}_{id}(\mathbf{X})$

The aim of this section is to prove the statement that every mapping $u \in \mathcal{A}(\mathbf{X})$ is a homeomorphism with its inverse $u^{-1} \in \mathcal{A}(\mathbf{X})$. This conclusion is essentially

of the argument here it can be shown that the twist mappings u_k are minimisers of $\mathscr{F}$ in $\mathscr{A}_k$ for each $k \in \mathbb{Z}$ (*cf.* [21]). However in the presence of an x dependence in F more stringent assumptions have to be put in place in order to guarantee minimality and the *uniqueness* of minimisers.

a small extension of Theorem 2 in [3], however, as the result here does not follow directly from [3] we give a complete proof for the convenience of the reader.

Before proceeding on to this however we pause to collect some properties of mappings $u \in \mathcal{A}(\mathbf{X})$ needed later on. Firstly we recall from the introduction that any $u \in \mathcal{A}(\mathbf{X})$ has a representative in $\mathscr{C}(\overline{\mathbf{X}}, \mathbb{R}^2)$ again denoted by u (*see*, e.g., [20] or [18, 28]). Additionally since $u \equiv x$ on $\partial\mathbf{X}$ we have $\deg(u, \mathbf{X}, p) = \deg(x, \mathbf{X}, p)$ (here $\deg(u, \mathbf{X}, p)$ denotes the Brouwer degree of u at p with respect to $\mathbf{X}$) and therefore $\deg(u, \mathbf{X}, p) = 1$ for all $p \in \mathbf{X}$ and $\deg(u, \mathbf{X}, p) = 0$ for $p \in \mathbb{R}^2 \backslash \overline{\mathbf{X}}$. This now gives $\overline{\mathbf{X}} \subset u(\overline{\mathbf{X}})$. Conversely it can been seen that $u(\overline{\mathbf{X}}) \subset \overline{\mathbf{X}}$. Indeed by applying the integral formula for the degree (*see*, e.g. Theorem 5.35 in [12]) we have for any $p \in \mathbb{R}^2 \backslash \partial\mathbf{X}$,

$$\deg(u, \mathbf{X}, p) = \int_{\mathbf{X}} f(u(x)) \det \nabla u(x) \, \mathrm{d}x = \int_{\mathbf{X}} f(u(x)) \, \mathrm{d}x, \qquad (2.1)$$

where $f \in \mathscr{C}_c(\mathbb{R}^2)$ is any functions satisfying $\int_{\mathbb{R}^2} f(x) \mathrm{d}x = 1$ with support in the connected component of $\mathbb{R}^2 \backslash \partial\mathbf{X}$ containing p. Now suppose for the sake of a contradiction that $\exists x \in \mathbf{X}$ such that $u(x) = p \in \mathbb{R}^2 \backslash \overline{\mathbf{X}}$. Then we can pick $\delta > 0$ small enough such that $\overline{\mathbb{B}}_\delta(p) \subset \mathbb{R}^2 \backslash \overline{\mathbf{X}}$ and a suitable $f \geq 0$ with support in $\overline{\mathbb{B}}_\delta(p)$. The continuity of u means that $\exists \varepsilon > 0$ such that $u(\mathbb{B}_\varepsilon(x)) \subset \mathbb{B}_\delta(p)$ and therefore,

$$\deg(u, \mathbf{X}, p) = \int_{\mathbf{X}} f(u(x)) \, \mathrm{d}x > 0. \qquad (2.2)$$

Hence we have reached a contradiction since $\deg(u, \mathbf{X}, p) = 0$ for $p \in \mathbb{R}^2 \backslash \overline{\mathbf{X}}$. Therefore $\overline{\mathbf{X}} = u(\overline{\mathbf{X}})$. Now set $N(y, u, \mathbf{X}) = \sharp\{x \in \mathbf{X} : x \in u^{-1}(y)\}$ for $y \in \mathbb{R}^2$. Then by the above $N(u, \mathbf{X}, y) \geq 1$ for $y \in \mathbf{X}$ and $N(y, u, \mathbf{X}) = 0$ for $y \in \mathbb{R}^2 \backslash \overline{\mathbf{X}}$. Moreover by the area formula (*see*, e.g., Theorem 2.3 of p. 285 in [13])

$$|\mathbf{X}| = \int_{\mathbf{X}} |\det \nabla u| \, \mathrm{d}x = \int_{\mathbb{R}^2} N(y, u, \mathbf{X}) \, \mathrm{d}y,$$

which when combined with the earlier observation $N(y, u, \mathbf{X}) \geq 1$ on $\mathbf{X}$ leads to the conclusion $N(y, u, \mathbf{X}) = \sharp\{x \in \mathbf{X} : x \in u^{-1}(y)\} = 1$ for *a.e.* $y \in \mathbf{X}$. Therefore the mappings $u \in \mathcal{A}(\mathbf{X})$ are injective almost everywhere in $\mathbf{X}$. Furthermore the injectivity *a.e.* of $u \in \mathcal{A}(\mathbf{X})$ gives the following change of variables formula, *see* [13] p. 285-6 Theorem 2.4,

$$\int_G \varphi \circ u(x) \, \mathrm{d}x = \int_{u(G)} \varphi(y) \, \mathrm{d}y, \qquad (2.3)$$

for all $G \subset\subset \mathbf{X}$ and $\varphi \in L^1(\mathbb{R}^n)$. Note also that u satisfies the Luzin N and N^{-1} properties which means that the image and pre-image of sets of measure zero under u are also of measure zero. This is a consequence of $u \in \mathscr{W}^{1,2}(\mathbf{X}, \mathbb{R}^2)$ and $\det \nabla u = 1 > 0$ almost everywhere in $\mathbf{X}$. For a proof of this result the reader is referred to [12] p. 141 Theorem 5.32. We are now in a position to state and prove the aforementioned global invertibility result.

Theorem 2.1. *Let* $\mathcal{A}(\mathbf{X}) = \{u \in \mathscr{W}^{1,2}(\mathbf{X}, \mathbb{R}^2) : \det \nabla u = 1$ *a.e.* $u \equiv x$ *on* $\partial\mathbf{X}\}$. *Then any* $u \in \mathcal{A}(\mathbf{X})$ *is a Sobolev homeomorphism with its inverse mapping* $u^{-1} \in \mathcal{A}(\mathbf{X})$. *Furthermore*

$$\nabla u^{-1}(y) = (\nabla u)^{-1}(u^{-1}(y)), \tag{2.4}$$

for almost every $y \in \mathbf{X}$.

Proof. Let $\mathcal{B} = \mathbb{B}_r(0) \supset \overline{\mathbf{X}}$ denote the open ball with radius $r = b + \delta$ and $\delta > 0$ fixed. We extend u by identity off $\mathbf{X}$ and into $\mathcal{B}$, and so, $u \equiv x$ in $\mathcal{B}\backslash\mathbf{X}$. It is therefore clear that $u \in \mathcal{A}(\mathcal{B})$. Now let $\rho_\varepsilon \geq 0$ be a standard radially symmetric smooth mollifier with compact support inside $\mathbb{B}_\varepsilon(0)$ and with $\int_{\mathbb{R}^2} \rho_\varepsilon(v)\,\mathrm{d}v = 1$. As in [3] p. 320-3 we let for $\varepsilon > 0$

$$x_\varepsilon(v) = \int_{\mathcal{B}} \rho_\varepsilon(v - u(y))y\,\mathrm{d}y, \qquad v \in \mathbf{X}, \tag{2.5}$$

which will play the role of an approximation of the desired inverse mapping for u. Then by taking partial derivatives a basic calculation gives

$$\begin{aligned}\frac{\partial x_\varepsilon^\alpha}{\partial v^i}(v) &= \int_{\mathcal{B}} \rho_{\varepsilon,i}(v - u(y))y^\alpha\,\mathrm{d}y \\ &= \int_{\mathcal{B}} -\frac{\partial \rho_\varepsilon}{\partial y^\beta}(v - u(y))\,(\nabla u)^{-1}_{\beta,i}\,y^\alpha\,\mathrm{d}y.\end{aligned} \tag{2.6}$$

Now recalling the Sobolev integrability and uniform continuity of the extended mapping u on the compact set $\overline{\mathcal{B}} \subset \mathbb{R}^2$ we can pick a sequence of smooth mappings (u_s), say, via mollification, converging to u strongly in $\mathscr{W}^{1,2}(\mathcal{B}, \mathbb{R}^2)$ and uniformly in $\mathscr{C}(\overline{\mathcal{B}}, \overline{\mathcal{B}})$. Furthermore as $u \equiv x$ on $\mathcal{B}\backslash\mathbf{X}$ we can arrange this so that $u_s \equiv x$ on $\{y \in \mathcal{B} : \mathrm{dist}(y, \partial\mathcal{B}) < \delta/2\}$ whilst $\sup_{x\in\overline{\mathcal{B}}}|u_s - u| < \delta/4$ for $s > N(\delta)$. Thus for any $v \in \mathbf{X}$ we have $\mathbb{B}_\varepsilon(v) \cap \mathbb{B}_{\delta/4}(u_s(y)) = \varnothing$ for all $y \in \mathcal{B}$ satisfying $\mathrm{dist}(y, \partial\mathbf{X}) < \delta/4$ provided that $s > N(\delta)$ and $\varepsilon < \delta/4$. Hence, in particular, for all such y we have

$$\rho_\varepsilon(v - u_s(y)) = 0, \qquad \forall v \in \mathbf{X}. \tag{2.7}$$

Next by invoking the divergence free structure of the minors, here, adj, which by definition is the transpose of the cofactor matrix, it follows via an application of divergence theorem that,

$$-\int_{\mathcal{B}} \frac{\partial \rho_\varepsilon}{\partial y^\beta}(v - u_s(y))\,(\mathrm{adj}\nabla u_s)_{\beta,i}\,y^\alpha\,\mathrm{d}y = \int_{\mathcal{B}} \rho_\varepsilon(v - u_s(y))\,(\mathrm{adj}\nabla u_s)_{\alpha,i}\,\mathrm{d}y \tag{2.8}$$

for all $v \in \mathbf{X}$ provided $s > N(\delta)$. Now since $\det \nabla u = 1$ *a.e.* and $u_s \to u$ in $\mathscr{W}^{1,2}(\mathcal{B}, \mathbb{R}^2)$ it follows that $\mathrm{adj}\nabla u_s \to \mathrm{adj}\nabla u = (\nabla u)^{-1}$ in $L^2(\mathcal{B}, \mathbb{R}^{2\times 2})$. (Note that by basic considerations here $\mathrm{adj}\nabla u = (\nabla u)^{-1}$.) Therefore by passing to the limit in the above and referring to the description of the gradient of x_ε it is plain that

$$\frac{\partial x^\alpha_\varepsilon}{\partial v^i}(v) = \int_{\mathcal{B}} \rho_\varepsilon(v - u(y))\, (\nabla u)^{-1}_{\alpha,i}\, \mathrm{d}y, \qquad \forall v \in \mathbf{X}, \tag{2.9}$$

provided that $\varepsilon < \delta/4$. The aim is now to bound $(x_\varepsilon : \varepsilon > 0)$ in $\mathscr{W}^{1,2}(\mathbf{X}, \mathbb{R}^2)$. Towards this end it is firstly seen that that $\|x_\varepsilon\|_{L^2} \leq c < \infty$ and by the above,

$$\begin{aligned}
|\nabla x_\varepsilon|^2 &\leq \left(\int_{\mathcal{B}} \rho_\varepsilon(v - u(y)) |(\nabla u)^{-1}|\, \mathrm{d}y \right)^2 \\
&\leq \left(\int_{\mathcal{B}} \rho_\varepsilon(v - u(y))\, \mathrm{d}y \right) \left(\int_{\mathcal{B}} \rho_\varepsilon(v - u(y)) |(\nabla u)^{-1}|^2\, \mathrm{d}y \right) \\
&\leq \int_{\mathcal{B}} \rho_\varepsilon(v - u(y)) |(\nabla u)^{-1}|^2\, \mathrm{d}y.
\end{aligned}$$

Now as a result of ∇u being a 2×2 matrix-field satisfying $\det \nabla u = 1$ *a.e.* we have $|(\nabla u)^{-1}|(x) = |\nabla u|(x)$ for *a.e.* $x \in \mathcal{B}$. Hence referring to the above, upon substitution, for $v \in \mathbf{X}$ with $\varepsilon < \delta/4$ we have

$$|\nabla x_\varepsilon(v)|^2 \leq \int_{\mathcal{B}} \rho_\varepsilon(v - u(y)) |\nabla u|^2\, \mathrm{d}y. \tag{2.10}$$

Therefore integrating both sides, applying Fubini's theorem and noting that the mollifier ρ_ε has unit mass together give

$$\int_{\mathbf{X}} |\nabla x_\varepsilon(v)|^2\, \mathrm{d}v \leq \int_{\mathbf{X}} \int_{\mathcal{B}} \rho_\varepsilon(v - u(y)) |\nabla u|^2\, \mathrm{d}y \leq \int_{\mathcal{B}} |\nabla u|^2 \mathrm{d}y. \tag{2.11}$$

Therefore as envisaged $(x_\varepsilon : \varepsilon > 0)$ is bounded in $\mathscr{W}^{1,2}(\mathbf{X}, \mathbb{R}^2)$ if $\varepsilon < \delta/4$ and so the segment $(x_\varepsilon : 0 < \varepsilon < \delta/4)$ has a subsequence (not re-labelled) such that $x_\varepsilon \rightharpoonup x$ as $\varepsilon \searrow 0$ in $\mathscr{W}^{1,2}(\mathbf{X}, \mathbb{R}^2)$. Hence in particular for any compact set $A \subset \mathbf{X}$ we have

$$\int_A \frac{\partial x^\alpha_\varepsilon}{\partial v^i}(v)\, \mathrm{d}v \to \int_A \frac{\partial x^\alpha}{\partial v^i}(v)\, \mathrm{d}v, \qquad \varepsilon \searrow 0. \tag{2.12}$$

Now to utilise the above we integrate (2.9) over A and apply Fubini's theorem to obtain

$$\begin{aligned}
\int_A \frac{\partial x^\alpha_\varepsilon}{\partial v^i}(v)\, \mathrm{d}v &= \int_{\mathcal{B}} \int_A \rho_\varepsilon(v - u(y))\, \mathrm{d}v\, (\nabla u)^{-1}_{\alpha,i}\, \mathrm{d}y \\
&= \int_{\mathcal{B}} \rho_\varepsilon \star \chi_A(u(y))\, (\nabla u)^{-1}_{\alpha,i}\, \mathrm{d}y.
\end{aligned} \tag{2.13}$$

Thus combining with (2.12) results in the identity,

$$\int_{\mathcal{B}} (\nabla u)^{-1}{}_{i,\alpha}(y) \chi_A(u(y))\, \mathrm{d}y = \int_A \frac{\partial x^\alpha}{\partial v^i}(v)\, \mathrm{d}v. \tag{2.14}$$

Let us now pick $A = u(\overline{\mathbb{B}}_s(\zeta))$ where $\zeta \in \mathbf{X}$ and $s > 0$ is such that $\overline{\mathbb{B}}_s(\zeta) \subset \mathbf{X}$. Then as u is injective *a.e.* and satisfies the Luzin N^{-1} property we know that $u^{-1}(A) = \overline{\mathbb{B}}_s(\zeta)$ up to a set of measure zero. Thus,

$$\int_{\mathbb{B}_s(\zeta)} (\nabla u)^{-1}{}_{i,\alpha}(y)\,\mathrm{d}y = \int_{u(\overline{\mathbb{B}}_s(\zeta))} \frac{\partial x^\alpha}{\partial v^i}(v)\,\mathrm{d}v = \int_{\mathbb{B}_s(\zeta)} \frac{\partial x^\alpha}{\partial v^i}(u(y))\,\mathrm{d}y. \tag{2.15}$$

The last equality here comes from (2.3) and noting that $G = \overline{\mathbb{B}}_s(\zeta)$. Hence,

$$\int_{\mathbb{B}_s(\zeta)} (\nabla u)^{-1}{}_{i,\alpha}(y)\,\mathrm{d}y = \int_{\mathbb{B}_s(\zeta)} \frac{\partial x^\alpha}{\partial v^i}(u(y))\,\mathrm{d}y, \tag{2.16}$$

for all $s < \mathrm{dist}(\zeta, \partial\mathbf{X})$ and any $\zeta \in \mathbf{X}$. Then by an application of the Lebesgue differentiation theorem we have that,

$$(\nabla u)^{-1}{}_{i,\alpha}(y) = \frac{\partial x^\alpha}{\partial v^i}(u(y)), \tag{2.17}$$

for *a.e.* $y \in \mathbf{X}$ and hence $(\nabla u)^{-1}(y) = \nabla x(u(y))$ for *a.e.* $y \in \mathbf{X}$. Now let us set $\mathcal{S} = \{y \in \mathbf{X} : (2.17) \text{ fails}\}$. Then by virtue of the Luzin N property $u(\mathcal{S}) \subset \mathbf{X}$ has measure zero. Hence combining this with the almost everywhere injectivity of u implies that for *a.e.* $y \in \mathbf{X}$,

$$(\nabla u)^{-1}{}_{i,\alpha}(u^{-1}(y)) = \frac{\partial x^\alpha}{\partial v^i}(y). \tag{2.18}$$

Moreover $\det \nabla x(y) = \det (\nabla u)^{-1}(u^{-1}(y)) = \det \nabla u(u^{-1}(y)) = 1$ almost everywhere in $\mathbf{X}$ which gives that $x(\cdot)$ is continuous.

Recall that $x_\varepsilon \rightharpoonup x$ in $\mathscr{W}^{1,2}(\mathbf{X}, \mathbb{R}^2)$ then on a subsequence, again not relabelling, $x_\varepsilon \to x$ for almost every $v \in \mathbf{X}$. Additionally since $u(\cdot)$ is injective almost everywhere we let $N \subset \mathbf{X}$ denote the set of points where u fails to be injective or x_ε fails to converge pointwise. Then for $v \in \mathbf{X}\backslash N$ we have that $\exists! z \in \mathbf{X}$ s.t $u(z) = v$ and,

$$x_\varepsilon(v) - z = \int_{\mathcal{B}} \rho_\varepsilon(u(z) - u(y))(y - z)\,\mathrm{d}y. \tag{2.19}$$

Then by the uniqueness of z and the uniform continuity of u on $\overline{\mathbf{X}}$ we have that $\forall \mu > 0, \exists \delta(\mu)$ such that $|z - y| < \mu$ if $|u(z) - u(y)| \leq \delta$. Take $\varepsilon < \delta(\mu)$ then,

$$|x_\varepsilon(v) - z| \leq \mu \int_{\mathcal{B}} \rho_\varepsilon(u(z) - u(y))\,\mathrm{d}y = \mu, \tag{2.20}$$

which therefore gives that $x_\varepsilon(v) \to x(v) = z$ as $\varepsilon \searrow 0$ for all $v \in \mathbf{X}\backslash N$. Hence $u(x(v)) = u(z) = v$ for all $v \in \mathbf{X}\backslash N$ and therefore by the continuity of $x(\cdot)$ and $u(\cdot)$ we have that $u(x(v)) = v$ for all $v \in \mathbf{X}$. To see that it is a left inverse let $y \in \mathbf{X}$ and take $(y_n : n \geq 1) \subset \mathbf{X}$ such that $y_n \to y$ with $z_n = u(y_n) \in \mathbf{X}\backslash N$. Then we have that $u(v) = z_n$ if and only if $v = y_n$ and therefore using this with $u(x(v)) = v$ for all $v \in \mathbf{X}$ we conclude that,

$$x(u(y_n)) = y_n \implies x(u(y)) = y, \tag{2.21}$$

by the continuity of both $x(\cdot)$ and $u(\cdot)$. Thus $x(u(y)) = y$ for all $y \in \mathbf{X}$. One immediate consequence of this is that (2.18) gives,

$$\nabla x(v) = (\nabla u)^{-1}(x(v)), \tag{2.22}$$

for almost every $v \in \mathbf{X}$. Additionally if we let $y \in \partial\mathbf{X}$ and take $(y_n : n \geq 1) \subset \mathbf{X}$ such that $y_n \to y$ as $n \nearrow \infty$ then we have the following consequences

$$\begin{aligned} \lim_{n \nearrow \infty} u(y_n) &= u(y) = y, \\ \lim_{n \nearrow \infty} x(u(y_n)) &= x(u(y)), \\ \lim_{n \nearrow \infty} x(u(y_n)) &= \lim_{n \nearrow \infty} y_n = y. \end{aligned}$$

Therefore $x(y) = y$ on $\partial\mathbf{X}$. Hence x serves as the global inverse of $u \in \mathcal{A}(\mathbf{X})$ and as seen $x = u^{-1} \in \mathcal{A}(\mathbf{X})$. This completes the proof. □

Let us finish this section by outlining an alternative proof for Theorem 2.1 by invoking results from complex function theory. Given $u \in \mathcal{A}(\mathbf{X})$ we begin by extending u by identity off $\mathbf{X}$ and into a larger ball $\mathbb{B}_{2R}$ where $R > b$. Therefore $\overline{\mathbf{X}} \subset \mathbb{B}_R$ and as a result $u \in \mathcal{A}(\mathbb{B}_{2R})$. Now by virtue of the L^1-integrability of the dilatation $|\nabla u|^2/\det\nabla u$ over $\mathbb{B}_{2R}$ here it follows that u admits a Stoilow's type factorisation; namely, that there exists a homeomorphism $h \in \mathscr{W}^{1,2}(\Omega, \mathbb{B}_{2R})$ and a holomorphic mapping $\varphi \in \mathscr{W}^{1,2}(\Omega, \mathbb{C})$, where $\Omega \subset \mathbb{C}$ is some open set (*cf.* Theorem 1 in [15]) such that,

$$u = \varphi \circ h^{-1} \qquad \text{in } \mathbb{B}_{2R} \supset \mathbf{X}. \tag{2.23}$$

Now as $h^{-1}(\overline{\mathbb{B}}_R) \subset \Omega$ is compact and the zeros of φ' are isolated, it follows that φ restricted to $h^{-1}(\overline{\mathbb{B}}_R)$ is locally conformal at all but finitely many points. Thus, since h is a homeomorphism, it follows that u is a local homeomorphism at all but finitely many points in $\mathbb{B}_R$. Furthermore (*see* [19]) since $\partial\mathbb{B}_R$ is connected, u is one-to-one on $\partial\mathbb{B}_R$ and a local homeomorphism at all but finitely many points in $\mathbb{B}_R$ we obtain that u is a global homeomorphism on $\mathbb{B}_R$, which in turn implies that u is a global homeomorphism on $\overline{\mathbf{X}}$ as claimed. Note however that the proof presented above is more direct and avoids Stoilow's factorisation, the machinery complex function theory and the topological results in [19].

Remark 2.1. A basic adjustment of the proof of Theorem 2.1 leads to a similar conclusion for $n \geq 3$: If $u \in \mathscr{W}^{1,n}(\Omega, \mathbb{R}^n)$ with $\det\nabla u = 1$ *a.e.* in Ω and $u \equiv x$ on $\partial\Omega$ satisfies $(\nabla u)^{-1} \in L^n(\Omega, \mathbb{R}^{n\times n})$ [equivalently here $\mathrm{adj}\nabla u \in L^n(\Omega, \mathbb{R}^{n\times n})$] then u is a homeomorphism of $\overline{\Omega}$ onto itself. Although we do not use this results here but it is evident that this higher dimensional conclusion is outside the reach of complex function theory. (Note also that for $n = 2$ the L^2-integrability of $\mathrm{adj}\nabla u$ follows for free from the L^2-integrability of ∇u.)

3 Energy minimisers in $\mathscr{A}_k$, L^1-local minimisers in $\mathcal{A}(\mathbf{X})$ and the L^p-distances between $\mathscr{A}_k$

The purpose of this section is to put together some results that were eluded to in the introduction regarding the homotopy structure of $\mathcal{A}(\mathbf{X})$. These results lead to the existence of minimisers for $\mathscr{F}$ in each of the homotopy classes $\mathscr{A}_k$ with $k \in \mathbb{Z}$ that are then L^1-local minimisers of $\mathscr{F}$ over the whole space $\mathcal{A}(\mathbf{X})$. The proof of Theorem 3.1 below follows closely the arguments in [31, 32] and for this we will remain brief and focus mainly on the differences as much as needed.

Definition 3.1. *Let $\mathbf{X} = \mathbf{X}[a,b] = \{x \in \mathbb{R}^2 : a < |x| < b\}$ with $0 < a < b < \infty$. We denote by $\mathscr{C}_{id}(\mathbf{X})$ the space of continuous self-mappings of $\overline{\mathbf{X}}$ onto itself that agree with the identity mapping on $\partial\mathbf{X}$, that is,*

$$\mathscr{C}_{id} = \mathscr{C}_{id}(\mathbf{X}) := \{f \in \mathscr{C}(\overline{\mathbf{X}}, \overline{\mathbf{X}}) : f \equiv x \ \text{ on } \partial\mathbf{X}\}. \tag{3.1}$$

We equip $\mathscr{C}_{id}(\mathbf{X})$ with the uniform metric, i.e., metric of uniform convergence.

The intrinsic interest in this space of continuous self-mappings of $\overline{\mathbf{X}}$ comes from the fact that $\mathcal{A}(\mathbf{X})$ 'embeds' into $\mathscr{C}_{id}(\mathbf{X})$. Now note that in Section 2 we proved that each $u \in \mathcal{A}(\mathbf{X})$ has a continuous representative that is additionally a homeomorphism of $\overline{\mathbf{X}}$ onto itself. Thus every $u \in \mathcal{A}(\mathbf{X})$ has a representative (again denoted by u) in the space $\mathscr{H}_{id}(\mathbf{X}) \subset \mathscr{C}_{id}(\mathbf{X})$ defined by

$$\mathscr{H}_{id}(\mathbf{X}) = \{f \in \mathscr{C}_{id}(\mathbf{X}) : f \text{ is a homeomorphism on } \overline{\mathbf{X}}\} \tag{3.2}$$

Moreover associated to each $u \in \mathscr{C}_{id}(\mathbf{X})$ [therefore also $u \in \mathscr{H}_{id}(\mathbf{X})$] is the topological invariant $\mathbf{deg}(u)$ that denotes the index or winding number of the closed plane curve $r \mapsto \gamma_u(r) = u/|u|(r,\theta) : [a,b] \to \mathbb{S}^1$ with fixed $\theta \in [0, 2\pi]$. Note that by continuity of u this $\mathbb{Z}$-valued invariant is independent of the choice of θ. More importantly $\mathbf{deg}(u)$ provides an enumeration of the homotopy classes of $\mathscr{C}_{id}(\mathbf{X})$ that is summarised in the following proposition.

Proposition 3.1. *Let $\mathscr{H}_k = \{u \in \mathscr{H}_{id}(\mathbf{X}) : \mathbf{deg}(u) = k\}$ for $k \in \mathbb{Z}$. Then $\mathscr{H}_k$ are pairwise disjoint, $\mathscr{H}_k \subset \mathscr{C}_k$ and $\mathscr{H}_{id}(\mathbf{X}) = \bigcup_{k\in\mathbb{Z}} \mathscr{H}_k$. Here the degree map in either form $\mathbf{deg} : \{[u] : u \in \mathscr{H}_{id}(\mathbf{X})\} \to \mathbb{Z}$ or $\mathbf{deg} : \{[u] : u \in \mathscr{C}_{id}(\mathbf{X})\} \to \mathbb{Z}$ is a bijection.*

Note that if we impose further differentiability on the mapping $u \in \mathscr{C}_{id}(\mathbf{X})$ then we have the following integral representation of $\mathbf{deg}$: [4]

$$\mathbf{deg}(u) = \frac{1}{2\pi}\int_a^b \frac{u \times \partial_r u}{|u|^2}(r\omega)\,\mathrm{d}r, \qquad \omega = x/|x| \in \mathbb{S}^1, \quad r = |x| \in [a,b]. \tag{3.3}$$

Let us emphasise here that $\mathbf{deg}(u)$ should not be confused with the classical Brouwer degree of u. Indeed in this context as a result of $u \equiv x$ on $\partial\mathbf{X}$ every

[4] Note that for $u = (u_1, u_2)$, $v = (v_1, v_2)$ we write $u \times v = u_1 v_2 - v_1 u_2$.

mapping $u \in \mathscr{C}_{id}(\mathbf{X})$ and so $u \in \mathcal{A}(\mathbf{X})$ would have Brouwer degree $+1$. The degree formula (3.3) identifies the homotopy class membership of $u \in \mathscr{C}_{id}(\mathbf{X})$ in (1.2) and as seen can take any value in $\mathbb{Z}$. Utilising that each mapping $u \in \mathcal{A}(\mathbf{X})$ has a continuous representative [indeed a homeomorphism by Theorem 2.1] it is possible to partition the space $\mathcal{A}(\mathbf{X})$ by setting

$$\mathcal{A}(\mathbf{X}) = \bigcup_{k \in \mathbb{Z}} \mathscr{A}_k \quad \text{where} \quad \mathscr{A}_k = \{u \in \mathcal{A}(\mathbf{X}) : \mathbf{deg}\,(u) = k\}. \tag{3.4}$$

The outcome of this rich homotopy structure is that we gain the existence of countably many L^1-local minimisers of $\mathscr{F}$ as described below.

Theorem 3.1. *Consider $\mathscr{F}$ as in (1.1) with $F \in \mathscr{C}(\mathscr{R})$ satisfying $F \geq c > 0$. Then for each $k \in \mathbb{Z}$ there exists $u_k \in \mathscr{A}_k \subset \mathcal{A}(\mathbf{X})$ that minimises $\mathscr{F}$ over $\mathscr{A}_k$, that is,*

$$\mathscr{F}[u_k; \mathbf{X}] = \inf_{v \in \mathscr{A}_k} \mathscr{F}[v; \mathbf{X}]. \tag{3.5}$$

Moreover for each such minimiser u_k there exists an associated $\delta = \delta(u_k) > 0$ such that $\mathscr{F}[u_k; \mathbf{X}] \leq \mathscr{F}[v; \mathbf{X}]$ for all $v \in \mathcal{A}(\mathbf{X})$ satisfying $||v - u_k||_{L^1} < \delta$. Thus for each $k \in \mathbb{Z}$ the mapping u_k is a local minimiser of $\mathscr{F}$ in $\mathcal{A}(\mathbf{X})$ with respect to the L^1-metric.

Proof. Firstly fix $k \in \mathbb{Z}$ and pick $(v_j) \subset \mathscr{A}_k$ to be an *infimizing* sequence, i.e., $\mathscr{F}[v_j] \downarrow \alpha := \inf_{\mathscr{A}_k} \mathscr{F}[\cdot]$ where $\alpha < \infty$ as $a \leq |x|, |u| \leq b$ for $u \in \mathcal{A}(\mathbf{X})$. Now by passing to a subsequence (not re-labeled) this gives $v_j \rightharpoonup u$ in $\mathscr{W}^{1,2}(\mathbf{X}, \mathbb{R}^2)$. Moreover it can be shown (*see* the proof of Proposition 4.3 on page 404 in [31]) that we can extract a further subsequence [still denoted (v_j)] such that $v_j \to u$ uniformly in $\overline{\mathbf{X}}$ and so in particular $u \in \mathscr{A}_k$. Finally to justify the existence of a minimiser in $\mathscr{A}_k$ it remains to demonstrate that $\mathscr{F}$ is sequentially weakly $\mathscr{W}^{1,2}$ lower semicontinuous. To this end note that for (v_j) and u as above we have

$$\begin{aligned} \left| \int_{\mathbf{X}} F(|x|^2, |v_j|^2)|\nabla v_j|^2 - \int_{\mathbf{X}} F(|x|^2, |u|^2)|\nabla v_j|^2 \right| & \\ \leq \int_{\mathbf{X}} |\nabla v_j|^2 \left| F(|x|^2, |v_j|^2) - F(|x|^2, |u|^2) \right| & \\ \leq \sup_{\overline{\mathbf{X}}} \left| F(|x|^2, |v_j|^2) - F(|x|^2, |u|^2) \right| \int_{\mathbf{X}} |\nabla v_j|^2 \to 0 & \end{aligned} \tag{3.6}$$

as $j \nearrow \infty$ as a result of the $\mathscr{W}^{1,2}$ boundedness of (v_j), the uniform convergence $v_j \to u$ on $\overline{\mathbf{X}}$ and the uniform continuity of F over $\mathscr{R}$. Combining this with

$$\int_{\mathbf{X}} F(|x|^2, |u|^2)|\nabla u|^2 \leq \underline{\lim} \int_{\mathbf{X}} F(|x|^2, |u|^2)|\nabla v_j|^2 \tag{3.7}$$

gives the desired lower semicontinuity of the $\mathscr{F}$ energy on $\mathcal{A}(\mathbf{X})$ as claimed. Next to justify the L^1-local minimiser claim we argue by contradiction. Indeed fix k and suppose that $u = u_k$ is not a L^1-local minimiser; then $\exists (v_n : n \geq 1) \subset \mathcal{A}(\mathbf{X})$

so that $||v_n - u||_{L^1} \to 0$ as $n \nearrow \infty$ whilst $\mathscr{F}[v_n; \mathbf{X}] < \mathscr{F}[u; \mathbf{X}]$ for every $n \geq 1$. Now recall that $0 < c \leq F$ and therefore on the level of energy integrals we have

$$c\|\nabla v_n\|_2^2 \leq \int_{\mathbf{X}} F(|x|^2, |v_n|^2)|\nabla v_n|^2 \, \mathrm{d}x < \int_{\mathbf{X}} F(|x|^2, |u|^2)|\nabla u|^2 \, \mathrm{d}x = 2\mathscr{F}[u; \mathbf{X}],$$

which in turn results in (v_n) being bounded in $\mathscr{W}^{1,2}$. Thus as earlier it is possible to extract a subsequence such that (v_n) converges weakly in $\mathscr{W}^{1,2}$ and uniformly in $\overline{\mathbf{X}}$ to u. Hence as $\mathbf{deg}(v_n)$ is integer-valued $\exists N > 0$ such that $v_n \in \mathscr{A}_k$ for $n \geq N$. This however is a contradiction as $\mathscr{F}[v_n; \mathbf{X}] < \mathscr{F}[u; \mathbf{X}] = \inf_{\mathscr{A}_k} \mathscr{F}$. Hence $u = u_k$ must be an L^1-local minimiser as claimed. □

In the final section of the paper we improve this by showing that subject to further structural assumptions on F (specifically assuming $F \in \mathfrak{F}$) the minimiser u_k in the homotopy class $\mathscr{A}_k$ is unique and indeed a monotone twist mapping. Thus in particular the energy integral $\mathscr{F}$ in (1.1) has a countably infinite family of symmetric L^1-local minimisers. (For a formal definition of twist mappings *see* Definition 4.1.) We end this section with an interesting observation on the distances between the homotopy classes $\mathscr{A}_k$ described and used above in the familiar context of L^p-norms with $1 \leq p \leq \infty$.

Proposition 3.2. *The uniform distance between any pair of homotopy classes $\mathscr{A}_k$, $\mathscr{A}_m$ with $k \neq m$ is given by the inner diameter of $\mathbf{X}$, that is,*

$$d_U(k,m) := \inf_{\substack{u \in \mathscr{A}_k \\ v \in \mathscr{A}_m}} ||u - v||_{L^\infty(\mathbf{X};\mathbb{R}^2)} = 2a. \tag{3.8}$$

Proof. Firstly by restricting to the subclass of twist mappings, as in Definition 4.1, we have the upper bound

$$d_U(k,m) \leq \inf_{\substack{u \in \mathscr{A}_k \\ v \in \mathscr{A}_m \\ u,v \text{ twists}}} ||u - v||_{L^\infty(\mathbf{X};\mathbb{R}^2)}. \tag{3.9}$$

Now focusing on the term on the right a straightforward calculation using $u = \mathbf{Q}[g_u]x$, $v = \mathbf{Q}[g_v]x$ with $g_u = g_u(r)$, $g_v = g_v(r)$ gives

$$\begin{aligned} ||u - v||_{L^\infty(\mathbf{X};\mathbb{R}^2)} &= \sup_{x \in \mathbf{X}} |u(x) - v(x)| \\ &= \sup_{x \in \mathbf{X}} |(\mathbf{Q}[g_u] - \mathbf{Q}[g_v])x| \\ &= \sup_{a \leq r \leq b} r\sqrt{2 - 2\cos(g_u - g_v)}, \end{aligned} \tag{3.10}$$

where we have taken advantage of $|(\mathbf{Q}[g_u] - \mathbf{Q}[g_v])x|x|^{-1}| = \sqrt{2 - 2\cos(g_u - g_v)}$. Next suppose that we consider the particular twists $u = u^\varepsilon$, $v = v^\varepsilon$ with angle of rotation functions defined for $0 < \varepsilon < b - a$ by

$$g_u^\varepsilon(r) = \begin{cases} 0 & a \leq r \leq a + \varepsilon, \\ 2\pi k(r - a - \varepsilon)(b - a - \varepsilon)^{-1} & a + \varepsilon \leq r \leq b, \end{cases} \tag{3.11}$$

$$g_v^\varepsilon(r) = \begin{cases} 2\pi(m - k)(r - a - \varepsilon)\varepsilon^{-1} & a \leq r \leq a + \varepsilon, \\ 2\pi k(r - a - \varepsilon)(b - a - \varepsilon)^{-1} & a + \varepsilon \leq r \leq b. \end{cases} \tag{3.12}$$

Referring to earlier formulae one checks that $\mathbf{deg}(u^\varepsilon) = k$ and $\mathbf{deg}(v^\varepsilon) = m$ and so $u^\varepsilon \in \mathscr{A}_k$, $v^\varepsilon \in \mathscr{A}_m$. Moreover $\sqrt{1 - \cos[2\pi(k-m)\varepsilon^{-1}(r-a-\varepsilon)]} \le \sqrt{2}$ for $a \le r \le a + \varepsilon$ and so

$$\begin{aligned} ||u^\varepsilon - v^\varepsilon||_{L^\infty(\mathbf{X};\mathbb{R}^2)} &= \sup_{a \le r \le b} \sqrt{2} r \sqrt{1 - \cos(g_u^\varepsilon - g_v^\varepsilon)} \\ &= \sup_{a \le r \le a+\varepsilon} \sqrt{2} r \sqrt{1 - \cos[2\pi(k-m)\varepsilon^{-1}(r-a-\varepsilon)]} \\ &\le 2(a+\varepsilon). \end{aligned} \tag{3.13}$$

Thus $d_U(k,m) \le 2(a+\varepsilon)$ for all $\varepsilon \in (0, b-a]$ and hence $d_U(k,m) \le 2a$. Now to obtain a lower bound we first note the pointwise and elementary geometric inequality

$$|u - v| = \left| |u| \frac{u}{|u|} - |v| \frac{v}{|v|} \right| \ge \min\{|u|, |v|\} \left| \frac{u}{|u|} - \frac{v}{|v|} \right| \ge a \left| \frac{u}{|u|} - \frac{v}{|v|} \right|. \tag{3.14}$$

Therefore by virtue of $|u|^{-1}u$ and $|v|^{-1}v$ being $\mathbb{S}^1$-valued we obtain

$$\sup_{x \in \mathbf{X}} |u(x) - v(x)| \ge a \sup_{x \in \mathbf{X}} \left| \frac{u}{|u|}(x) - \frac{v}{|v|}(x) \right| = 2a, \tag{3.15}$$

where the last equality results from the fact that for each fixed $0 \le \theta \le 2\pi$ the continuous $\mathbb{S}^1$-valued curves $r \mapsto u/|u|(r,\theta)$ and $r \mapsto v/|v|(r,\theta)$ with $a \le r \le b$ have unequal indexes. Indeed by virtue of $k \ne m$, at least one of these curves covers all of $\mathbb{S}^1$ and so there exists $a < \bar{r} < b$ such that $u/|u|(\bar{r},\theta) = -v/|v|(\bar{r},\theta)$ and thus $\bar{x} = \bar{r}\theta \in \mathbf{X}$ such that $||u|^{-1}u(\bar{x}) - |v|^{-1}v(\bar{x})| = 2$. Therefore as claimed $d_U(k,m) = 2a$ for any pair of distinct integers $k, m \in \mathbb{Z}$. □

For the remaining L^p-norms we can again calculate the distance between homotopy classes but with a completely different outcome as is described below.

Proposition 3.3. *For $k, m \in \mathbb{Z}$ and $1 \le p < \infty$ the corresponding L^p-distance between the homotopy classes $\mathscr{A}_k$ and $\mathscr{A}_m$ is zero, that is,*

$$d_p(k,m) = \inf_{\substack{u \in \mathscr{A}_k \\ v \in \mathscr{A}_m}} \|u - v\|_{L^p(\mathbf{X};\mathbb{R}^2)} = 0. \tag{3.16}$$

Proof. As in the case of uniform distance treated in Proposition 3.2 above we can bound this distance from above by restricting to the class of twist mappings contained in the corresponding homotopy classes, that is,

$$d_p(k,m) \le \inf_{\substack{u \in \mathscr{A}_k \\ v \in \mathscr{A}_m \\ u,v \text{ twists}}} \|u - v\|_{L^p(\mathbf{X};\mathbb{R}^2)}. \tag{3.17}$$

Now regarding the L^p-distance between a pair of twists u, v, using the notation as in the previous proposition, we can easily calculate

$$\|u - v\|^p_{L^p(\mathbf{X};\mathbb{R}^2)} = \int_{\mathbf{X}} |(\mathbf{Q}[g_u] - \mathbf{Q}[g_v])x|^p \, dx = \int_{\mathbf{X}} 2^{\frac{p}{2}} |1 - \cos(g_u - g_v)|^{\frac{p}{2}} |x|^p \, dx. \tag{3.18}$$

Next specialising to the case where $g_u = g_u^\varepsilon = 2\pi k f_\varepsilon$ and $g_v = g_v^\varepsilon = 2\pi m f_\varepsilon$ with $0 < \varepsilon < b - a$ and f_ε defined by,

$$f_\varepsilon(r) = \begin{cases} 0 & a \le r \le b - \varepsilon, \\ (r - b + \varepsilon)/\varepsilon & b - \varepsilon \le r \le b, \end{cases} \tag{3.19}$$

it is easily seen that firstly $\mathbf{deg}(u^\varepsilon) = k$ and $\mathbf{deg}(v^\varepsilon) = m$ therefore $u^\varepsilon \in \mathscr{A}_k$, $v^\varepsilon \in \mathscr{A}_m$; and secondly that

$$\begin{aligned} \|u - v\|^p_{L^p(\mathbf{X};\mathbb{R}^2)} &= \int_0^{2\pi} \int_{b-\varepsilon}^b 2^{\frac{p}{2}} |1 - \cos\big(2\pi(k-m)(r - b + \varepsilon)/\varepsilon\big)|^{\frac{p}{2}} r^{p+1}\, dr d\theta \\ &\le \int_0^{2\pi} \int_{b-\varepsilon}^b 2^p r^{p+1}\, dr d\theta = \frac{\pi 2^{p+1}}{p+2} \big(b^{p+2} - (b-\varepsilon)^{p+2}\big). \end{aligned} \tag{3.20}$$

As a result it is seen that $\|u - v\|_{L^p(\mathbf{X};\mathbb{R}^2)}$ can be made arbitrarily small and so the sought L^p-distance between the components $\mathscr{A}_k$ and $\mathscr{A}_m$ is zero, that is, $d_p(k, m) = 0$ for $1 \le p < \infty$. □

4 Monotone twist mappings and a countably infinite family of solutions to the nonlinear system (1.5)

The aim of this section is to prove the existence of an infinite family of solutions in the form of twists to the Euler-Lagrange system associated $\mathscr{F}$ over $\mathcal{A}(\mathbf{X})$. In Section 5 we shall demonstrate that each of these twists are indeed the unique minimisers of $\mathscr{F}$ in their respective homotopy class $\mathscr{A}_k$. Let us proceed by first defining twist mappings (or twists for short) and examining some of their main properties.

Definition 4.1. *A mapping $u \in \mathscr{C}(\overline{\mathbf{X}}, \overline{\mathbf{X}})$ is a called a twist mapping iff it can be represented in the form*

$$u(x) = \begin{bmatrix} u_1(x) \\ u_2(x) \end{bmatrix} = \begin{bmatrix} \cos g(|x|) & -\sin g(|x|) \\ \sin g(|x|) & \cos g(|x|) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad a \le |x| \le b, \tag{4.1}$$

for some angle of rotation function $g = g(r) \in \mathscr{C}[a, b]$ with $r = |x| = \sqrt{x_1^2 + x_2^2}$. For the sake of brevity we often write $u = \mathbf{Q}[g](r)x$ where $\mathbf{Q}[g]$ is the $\mathbf{SO}(2)$-valued matrix on the right in (4.1). Furthermore upon identifying $x = (x_1, x_2)$ with $z = x_1 + ix_2$ and $u = (u_1, u_2)$ with $w = u_1 + iu_2$, (4.1) is seen to have the complex form [5]

$$w(z) = e^{ig(r)} z, \qquad r = \sqrt{z\bar{z}}. \tag{4.2}$$

[5] The advantage of (4.1) over (4.2) is that it immediately lends itself to generalisations to higher dimensions (*see* [25, 32, 33] for more).

Clearly the twist $u = \mathbf{Q}[g]x$ will lie in $\mathscr{C}_{id}(\mathbf{X})$ *iff* $g(a), g(b) \in 2\pi\mathbb{Z}$. Moreover subject to an additional differentiability assumption on the angle of rotation function g it is seen that

$$\nabla u = \mathbf{Q}[g] + r\dot{g}(r)\dot{\mathbf{Q}}[g]\theta \otimes \theta, \tag{4.3}$$

$$|\nabla u|^2 = tr\{[\nabla u][\nabla u]^t\} = 2 + r^2|\dot{g}|^2, \tag{4.4}$$

$$\det \nabla u = \det \Big(\mathbf{Q}[g] + r\dot{g}(r)\dot{\mathbf{Q}}[g]\theta \otimes \theta\Big) = 1. \tag{4.5}$$

Thus setting $\mathscr{G} = \mathscr{G}[(a,b); k_1, k_2] = \{g \in \mathscr{W}^{1,2}(a,b) : g(a) = 2\pi k_1, g(b) = 2\pi k_2\}$ with $k_1, k_2 \in \mathbb{Z}$ we have

$$g \in \mathscr{G} \implies u = \mathbf{Q}[g](r)x \in \mathcal{A}(\mathbf{X}). \tag{4.6}$$

Whilst the end-point conditions here on g result in $u \equiv x$ on $\partial\mathbf{X}$ a straightforward calculation gives

$$\mathbf{deg}(u) = \frac{1}{2\pi}\int_a^b \frac{u \times \partial_r u}{|u|^2}(r\omega)\,\mathrm{d}r = \frac{1}{2\pi}\int_a^b \dot{g}(r)\,\mathrm{d}r = k_2 - k_1. \tag{4.7}$$

Hence we can refine (4.6) to the level of components (homotopy classes) of $\mathcal{A}(\mathbf{X})$, namely,

$$g \in \mathscr{G}[(a,b); k_1, k_2] \implies u = \mathbf{Q}[g](r)x \in \mathscr{A}_{k_2-k_1} \subset \mathcal{A}(\mathbf{X}). \tag{4.8}$$

Note also that a twist mapping as defined by (4.1) is invariant under $\mathbf{SO}(2)$ in the sense that,

$$u(x) = \mathbf{R}^t u(\mathbf{R}x), \qquad \forall \mathbf{R} \in \mathbf{SO}(2), \quad \forall x \in \mathbf{X}, \tag{4.9}$$

which follows by a direct substitution. Therefore twist mappings by their nature posses an inherent $\mathbf{SO}(2)$-invariant rotational symmetry.

We now wish to prove that there are countably many twist mappings serving as solutions to the Euler-Lagrange system associated with the energy $\mathscr{F}$ over $\mathcal{A}(\mathbf{X})$. To this end recall that the Euler-Lagrange system here is given by [6]

$$\mathbb{EL}[u, \mathscr{P}; \mathbf{X}] = \begin{cases} \operatorname{div} \mathfrak{S}(x, u, \nabla u) = \partial_\eta F |\nabla u|^2 u & \text{in } \mathbf{X}, \\ \det \nabla u = 1 & \text{in } \mathbf{X}, \\ u \equiv x & \text{on } \partial\mathbf{X}, \end{cases} \tag{4.10}$$

where $F = F(\mu, \eta)$ with $\partial_\eta F = \partial_\eta F(\mu, \eta)$ and referring to the first line in the system above

$$\mathfrak{S}(x, u, \nabla u) = F(|x|^2, |u|^2)\nabla u - \mathscr{P}(x)(\nabla u)^{-t}. \tag{4.11}$$

[6] This follows by an application of the Lagrange multiplier method that in the interest of brevity will not be presented here. *See*, e.g., [23, 32].

In particular note that for mappings u of class $\mathbf{C}^2$ by taking advantage of the Piola identity we have

$$\operatorname{div}\mathfrak{S}(x,u,\nabla u)=\partial_\eta F|\nabla u|^2 u \iff (\nabla u)^t\left[\operatorname{div}(F\nabla u)-\partial_\eta F|\nabla u|^2 u\right]=\nabla\mathscr{P}. \tag{4.12}$$

Now for the sake of clarity by a classical solution we mean a pair $(u,\mathscr{P})$ with $u\in\mathscr{C}^2(\mathbf{X},\mathbb{R}^2)\cap\mathscr{C}(\overline{\mathbf{X}},\mathbb{R}^2)$ and $\mathscr{P}\in\mathscr{C}^1(\mathbf{X})\cap\mathscr{C}(\overline{\mathbf{X}})$ satisfying $\mathbb{EL}[u,\mathscr{P};\mathbf{X}]$. In order to achieve our aim we begin by restricting the energy to the class of twist mappings $u=\mathbf{Q}[g](r)x$ in $\mathcal{A}(\mathbf{X})$. The Euler-Lagrange equation associated with this restricted energy is a two-point boundary value problem (an ODE) for $\mathbf{Q}[g]$ whose solutions give candidate twist mappings u that can serve as solutions to the system $\mathbb{EL}[u,\mathscr{P};\mathbf{X}]$ in (4.10). A selection of these twists is then based on directly examining the latter against the full system $\mathbb{EL}[u,\mathscr{P};\mathbf{X}]$. Now to move forward note firstly that the energy $\mathscr{F}$ over the sub-class of twist mappings takes the form

$$\begin{aligned}\mathscr{F}[u;\mathbf{X}]&=\int_{\mathbf{X}}2^{-1}F(r^2,r^2)|\mathbf{Q}[g]+r\dot g(r)\dot{\mathbf{Q}}[g]\theta\otimes\theta|^2\,\mathrm{d}x\\&=\int_{\mathbf{X}}\mathrm{H}(r)\left(2+2r\dot g\langle\mathbf{Q}^t\dot{\mathbf{Q}}\theta,\theta\rangle+r^2\dot g^2|\mathbf{Q}^t\dot{\mathbf{Q}}\theta|\right)\,\mathrm{d}x\\&=2\pi\int_a^b\mathrm{H}(r)\left(2+r^2\dot g^2\right)\,r\mathrm{d}r=\mathsf{H}+2\pi\mathscr{E}[g;a,b].\end{aligned}\tag{4.13}$$

Note that in the last line above we have set $\mathrm{H}(r)=F(r^2,r^2)/2\ge c/2>0$, $\mathsf{H}=4\pi||r\mathrm{H}(r)||_{L^1(a,b)}=4\pi\int_a^b r\mathrm{H}(r)\,\mathrm{d}r$ and introduced the restricted energy by setting

$$\mathscr{E}[g;(a,b)]=\int_a^b r^3\mathrm{H}(r)\dot g^2(r)\,\mathrm{d}r,\qquad g\in\mathscr{G}[(a,b);k_1,k_2]. \tag{4.14}$$

It is then straightforward to verify that the Euler-Lagrange equation associated with the energy $\mathscr{F}$ restricted to the sub-class of twist mappings in $\mathcal{A}(\mathbf{X})$ is given by,

$$\mathbb{EL}[g,k_1,k_2;(a,b)]=\begin{cases}d/dr\left[r^3\mathrm{H}(r)\dot g(r)\right]=0, & a<r<b\\ g(a)=2\pi k_1,\quad g(b)=2\pi k_2.\end{cases}\tag{4.15}$$

Upon noting that by assumption $2\mathrm{H}(r)=F(r^2,r^2)\ge c>0$ on $[a,b]$ it follows that solutions to (4.15) can be expressed as

$$g(r)=\int_a^r\frac{c_1\mathrm{d}s}{s^3\mathrm{H}(s)}+c_2,\qquad a\le r\le b, \tag{4.16}$$

where $c_1=2\pi(k_2-k_1)/\mathscr{H}$ with $\mathscr{H}=\int_a^b r^{-3}\mathrm{H}(r)^{-1}\mathrm{d}r$ and $c_2=2\pi k_1$. Now as adding or subtracting an integer multiple of 2π to g does not affect the ODE and the twist mapping $u=\mathbf{Q}[g](r)x$ resulting from g, without loss of generality we set $k_1 = 0$ with $k_2 = k$. Consequently for each $k \in \mathbb{Z}$ there exists an angle of rotation function $g_k = g(r;k)$ serving as solution to the Euler-Lagrange equation associated with the restricted $\mathscr{E}$ which is explicitly given by

$$g_k(r) = \frac{1}{\mathscr{H}} \int_a^r \frac{2\pi k \mathrm{d}s}{s^3 \mathrm{H}(s)}, \qquad \mathscr{H} = \int_a^b r^{-3} \mathrm{H}(r)^{-1} \, \mathrm{d}r. \tag{4.17}$$

We shall from now on denote by $u_k \in \mathscr{A}_k$ the twist mapping with corresponding *angle of rotation* function $g(r;k)$ given by (4.17). The next task as described earlier is to show that these twists are solutions to the Euler-Lagrange system (4.10).

Theorem 4.1. *Let $F \in \mathscr{C}^2(\mathscr{R})$ satisfy $F \geq c > 0$ and consider the energy integral $\mathscr{F}$ defined by,*

$$\mathscr{F}[u;\mathbf{X}] = \int_{\mathbf{X}} F(|x|^2, |u|^2) |\nabla u|^2/2 \, \mathrm{d}x, \tag{4.18}$$

over the space of admissible incompressible mappings $\mathcal{A}(\mathbf{X})$. Then the infinite family of monotone twist mappings $u_k = \mathbf{Q}[g_k](r)x$ with g_k given by (4.17) *for $k \in \mathbb{Z}$ are solutions to the Euler-Lagrange system* (4.10).

Proof. Firstly by inspection it is seen that the angle of rotation functions $g(r;k)$ is of class $\mathscr{C}^3[a,b]$ and is monotone in r, that is, increasing when $k > 0$ and decreasing when $k < 0$. Hence each u_k is a monotone twist of class $\mathscr{C}^3(\overline{\mathbf{X}}, \overline{\mathbf{X}})$. Now by direct verification we can see that each twist mapping $u = \mathbf{Q}[g](r)x$ with an at least twice continuously differentiable angle of rotation function g satisfies the differential identities:

$$\nabla u = \mathbf{Q} + r\dot{g}\dot{\mathbf{Q}}\theta \otimes \theta, \quad |\nabla u|^2 = 2 + r^2 \dot{g}^2, \quad \Delta u = \left[3\dot{g}\dot{\mathbf{Q}} + r\ddot{g}\dot{\mathbf{Q}} - r\dot{g}^2 \mathbf{Q}\right]\theta.$$

A further set of direct and straightforward calculations based on the above also shows that for the quantities $\mathscr{I}_1, \mathscr{I}_2, \mathscr{I}_3$ defined below we have

$$\mathscr{I}_1 = (\nabla u)^t \nabla u (\nabla u)^t u = (1 + r^2 \dot{g}^2)x + r\dot{g}\mathbf{J}x, \tag{4.19}$$

$$\mathscr{I}_2 = |\nabla u|^2 (\nabla u)^t u = (2 + r^2 \dot{g}^2)x, \tag{4.20}$$

$$\mathscr{I}_3 = (\nabla u)^t \Delta u = \left[\left(3r^{-1}\dot{g} + \ddot{g}\right)\mathbf{J} + (2\dot{g}^2 + r\dot{g}\ddot{g})\mathbf{I}\right] x. \tag{4.21}$$

Now by taking advantage of the Piola identity and the divergence free structure of the cofactor matrix of a gradient field we can rewrite the Euler-Lagrange system (4.10) in the more suggestive form [*cf.* (4.12)]

$$\begin{aligned} \nabla \mathscr{P} &= (\nabla u)^t \left[\operatorname{div}(F\nabla u) - \partial_\eta F |\nabla u|^2 u\right] \\ &= 2\left(\partial_\mu F + \partial_\eta F\right) \mathscr{I}_1 - \partial_\eta F \mathscr{I}_2 + F \mathscr{I}_3. \end{aligned} \tag{4.22}$$

Note that here $F = F(r^2, r^2) = 2\mathrm{H}(r)$ and therefore $\dot{\mathrm{H}}(r)/r = \partial_\mu F + \partial_\eta F$. Thus,

$$\nabla \mathscr{P} = 2r^{-1}\dot{\mathrm{H}}\mathscr{I}_1 - \partial_\eta F \mathscr{I}_2 + 2\mathrm{H}\mathscr{I}_3. \tag{4.23}$$

Assuming now that the angle of rotation function g satisfies the ODE (4.15) it can be seen that $\mathrm{H}\mathscr{I}_3$ reduces to

$$\mathrm{H}\mathscr{I}_3 = r^{-3}\left[3r^2\mathrm{H}\dot{g} + r^3\ddot{g}\mathrm{H}\right]\mathbf{J}x + (2\dot{g}^2 + r\dot{g}\ddot{g})\mathrm{H}x = -\dot{g}\dot{\mathrm{H}}\mathbf{J}x + (2\dot{g}^2 + r\dot{g}\ddot{g})\mathrm{H}x. \tag{4.24}$$

Moreover,

$$r^{-1}\dot{\mathrm{H}}\mathscr{I}_1 + \mathrm{H}\mathscr{I}_3 = r^{-1}\dot{\mathrm{H}}\left(1 + r^2\dot{g}^2\right)x + (2\dot{g}^2 + r\dot{g}\ddot{g})\mathrm{H}x. \tag{4.25}$$

Therefore substituting (4.25) into (4.23) and writing hereafter $c = 2\pi k/\mathscr{H}$ for short and setting g from (4.17) gives

$$\begin{aligned}\nabla\mathscr{P} =& 2r^{-1}\dot{\mathrm{H}}\left(1 + r^2\dot{g}^2\right)x + 2(2\dot{g}^2 + r\dot{g}\ddot{g})\mathrm{H}x - \partial_\eta F(2 + r^2\dot{g}^2)x \\ =& -2c^2r^{-6}\mathrm{H}^{-2}(\mathrm{H} + r\dot{\mathrm{H}})x + 2r^{-1}\dot{\mathrm{H}}\left(1 + c^2r^{-4}\mathrm{H}^{-2}\right)x \\ & - \partial_\eta F\left(2 + c^2r^{-4}\mathrm{H}^{-2}\right)x.\end{aligned} \tag{4.26}$$

As $\nabla\mathrm{H}(|x|) = \dot{\mathrm{H}}x|x|^{-1}$ we are left to show that the vector fields $2c^2r^{-6}\mathrm{H}^{-2}(\mathrm{H} + r\dot{\mathrm{H}})x$, $2c^2r^{-5}\dot{\mathrm{H}}\mathrm{H}^{-2}x$ and $\partial_\eta F\left(2 + c^2r^{-4}\mathrm{H}^{-2}\right)x$ are gradients of suitable functions in $\mathbf{X}$ and thus conclude that $u = \mathbf{Q}[g](r)x$ is a solution to (4.10). This can be done by letting for $a \le r \le b$,

$$G_1(r) = 2\int_a^r c^2s^{-4}\mathrm{H}^{-2}\dot{\mathrm{H}}\,\mathrm{d}s, \tag{4.27}$$

and noting that $\nabla G_1(|x|) = 2c^2|x|^{-5}\dot{\mathrm{H}}\mathrm{H}^{-2}x$ and in a similar fashion letting

$$G_2(r) = \int_a^r s\,\partial_\eta F\left(2 + c^2s^{-4}\mathrm{H}^{-2}\right)\mathrm{d}s, \qquad G_3(r) = \int_a^r 2c^2s^{-5}\mathrm{H}^{-2}(\mathrm{H} + s\dot{\mathrm{H}})\,\mathrm{d}s, \tag{4.28}$$

giving $\nabla G_2(|x|) = \partial_2 F(|x|^2, |x|^2)(2+c^2|x|^{-4}\mathrm{H}^{-2})x$, $\nabla G_3(|x|) = 2c^2|x|^{-6}\mathrm{H}^{-2}(\mathrm{H}+|x|\dot{\mathrm{H}})x$. Thus by putting all the above together it follows that the twist $u = u_k$ with the angle of rotation function $g = g(r;k)$ as in (4.17) is a solution to the Euler-Lagrange system $\mathbb{EL}[u, \mathscr{P}; \mathbf{X}]$ in (4.10). □

5 Unique minimality of twist mappings in the homotopy classes $\mathscr{A}_k \subset \mathcal{A}(\mathbf{X})$

In this final section we prove the main result of the paper as announced earlier, namely, that each twist mapping u_k (with $k \in \mathbb{Z}$), as formulated in Section 4, is the *unique* minimiser of $\mathscr{F}$ in the homotopy class $\mathscr{A}_k$. Here the integrand F of $\mathscr{F}$ is assumed to satisfy $F \in \mathfrak{F}$ (a full description of $\mathfrak{F}$ is given below).

Our approach here is to show that a symmetrisation $R : u \in \mathscr{A}_k \mapsto \bar{u} \in \mathscr{A}_k$ strictly decreases the $\mathscr{F}$ energy when u is not a twist mapping. To achieve this we rely on a lifting result for mappings $u \in \mathcal{A}(\mathbf{X})$ given in Theorem 5.1. This result allows one to decompose the energy $\mathscr{F}[u;\mathbf{X}]$ into distinct terms involving $|u|$ and a function $g \in \mathscr{W}^{1,2}(\mathbf{X})\cap\mathscr{C}(\overline{\mathbf{X}})$ related to u (see below). Focusing on the terms containing $|u|$ we prove, using the coarea formula, in Proposition 5.1 and Proposition 5.2, that the associated part of the energy is smallest when $|u| = |x|$ and the task is essentially reduced to proving energy inequalities between the two mappings $|u|$ and $|x|$ and over almost every circle. It is then a consequence of the incompressibility constraint that equality can occur only if u is a twist. The final stage of the proof is to show, via a suitable averaging and symmetrisation argument, that there exists a $\bar{g} \in \mathscr{G}_k$ that lowers the part of the energy involving the g term. Therefore by combining the two fragments we conclude that the $\mathscr{F}$-energy of the twist mapping $\bar{u} = \mathbf{Q}[\bar{g}]x \in \mathscr{A}_k$ is strictly less than the $\mathscr{F}$-energy of $u \in \mathscr{A}_k$. A further consequence of this is that any minimiser of $\mathscr{F}$ over $\mathscr{A}_k$ must be a twist mapping. Hence we can deduce that u_k is the unique minimiser of $\mathscr{F}$ in $\mathscr{A}_k$ as u_k by formulation is the unique minimiser of the energy amongst twist mappings in $\mathscr{A}_k$.

Before we proceed to the main result let us firstly prove a preliminary result alluded to above which is needed in the proof of the symmetrisation argument. This show how the integral over the level set $\{x \in \mathbf{X} : |u| = t\}$ for a $u \in \mathcal{A}(\mathbf{X})$ and *a.e.* $t \in [a, b]$ relates to an integral of the inverse mapping $w = u^{-1}$.

Proposition 5.1. *Let $u \in \mathcal{A}(\mathbf{X})$ and let $w = u^{-1} \in \mathcal{A}(\mathbf{X})$ denote the inverse mapping of u. Then for $\varphi \in \mathscr{C}[a^2, b^2]$ and a.e. $a < t < b$ we have the identity,*

$$\int_0^{2\pi} \varphi(|w|^2)\left|\frac{\partial w}{\partial \theta}\right| \mathrm{d}\theta\bigg|_{r=t} = \int_{\{|u|=t\}} \varphi(|x|^2)\,\mathrm{d}\mathcal{H}^1, \tag{5.1}$$

where $\{|u| = t\} = \{x \in \mathbf{X} : |u| = t\}$ and $\partial w/\partial\theta = \nabla w\, x^\perp$ with $x^\perp = (-x_2, x_1)^t$.

Proof. To prove this result we rely on a version of the coarea formula for Sobolev functions as in [7] Proposition 2.1. Firstly let us fix a $t \in (a, b)$ and let us also take $0 < \delta < b - t$. Now define the open set $A = w(\mathbf{X}_t^{t+\delta}) \subset \mathbf{X}$ where $w = u^{-1}$ is the inverse of some fixed $u \in \mathcal{A}(\mathbf{X})$ (*see* Theorem 2.1) and $\mathbf{X}_t^{t+\delta} = \{x \in \mathbf{X} : t < |x| < t + \delta\}$. Furthermore we shall let χ_A denote the characteristic function of A and therefore $\varphi_A(x) = \varphi(|x|^2)\chi_A(x)$ is a Borel measurable function. Hence by an application of the coarea formula for Sobolev function we have

$$\int_{\mathbf{X}} \varphi_A|\nabla|u||\,\mathrm{d}x = \int_a^b \int_{\{|u|=s\}} \varphi_A\,\mathrm{d}\mathcal{H}^1\,\mathrm{d}s. \tag{5.2}$$

Moreover focusing further we note that the left hand side of (5.2) is given by,

$$\begin{aligned}\int_{\mathbf{X}} \varphi_A|\nabla|u||\,\mathrm{d}x &= \int_{w(\mathbf{X}_t^{t+\delta})} \varphi(|x|^2)|\nabla|u||\,\mathrm{d}x \\ &= \int_{\mathbf{X}_t^{t+\delta}} |y|^{-1}\varphi(|w|^2)|(\nabla w)^{-t}y|\,\mathrm{d}y. \end{aligned}\tag{5.3}$$

Note that in the above the last equality results from an application of the change of variable formula given by (2.3) and the fact that

$$|u|(w(y)) = |y| \implies \nabla w(y)^t \nabla |u|(w) = |y|^{-1} y, \tag{5.4}$$

for *a.e.* $y \in \mathbf{X}$. Furthermore specifically in two dimensions we have the identity $|(\nabla w)^{-t} y| = |\partial w/\partial\theta|$ and when combined with (5.2) this gives

$$\int_t^{t+\delta} \int_0^{2\pi} \varphi(|w|^2) \left| \frac{\partial w}{\partial \theta} \right| \mathrm{d}\theta \, \mathrm{d}r = \int_a^b \int_{\{|u|=s\}} \varphi_A \, \mathrm{d}\mathcal{H}^1 \, \mathrm{d}s. \tag{5.5}$$

Now as by earlier considerations u is a homeomorphism we also have that $\{|u| = s\} = w(\{|x| = s\})$ by virtue of $\{x \in \mathbf{X} : |u| > s\} = w(\mathbf{X}_s)$ where we have set $\mathbf{X}_s = \{x \in \mathbf{X} : |x| > s\}$. Next from the uniform continuity of w on $\overline{\mathbf{X}}_s$ it follows that

$$w(\overline{\mathbf{X}}_s) = \overline{w(\mathbf{X}_s)}, \quad w(\partial \mathbf{X}_s) = \partial w(\mathbf{X}_s). \tag{5.6}$$

This, therefore, upon invoking the identity boundary conditions, gives

$$\partial\{x \in \mathbf{X} : |u| > s\} = \{x \in \mathbf{X} : |x| = b\} \cup w(\{x \in \mathbf{X} : |x| = s\}), \tag{5.7}$$

and so in particular,

$$w(\{x \in \mathbf{X} : |x| = s\}) = \{x \in \mathbf{X} : |u| = s\}. \tag{5.8}$$

Using this identity it follows that for any $y \in \{|u| = t\}$ the characteristic function χ_A is non-zero at y, i.e., $\chi_A(y) \neq 0$, *iff* $t < s < t + \delta$. Hence (5.5) becomes,

$$\int_t^{t+\delta} \int_0^{2\pi} \varphi(|w|^2) \left| \frac{\partial w}{\partial \theta} \right| \mathrm{d}\theta \, \mathrm{d}r = \int_t^{t+\delta} \int_{\{|u|=s\}} \varphi \, \mathrm{d}\mathcal{H}^1 \, \mathrm{d}s. \tag{5.9}$$

As the integrands on both sides are L^1-summable on (a, b) the conclusion follows by an application of Lebesgue differentiation theorem. □

In the proof of the main result we make use of the following proposition that was proved earlier by the authors (*cf.* Proposition 7.4 in [21]).

Proposition 5.2. *Let $\Gamma \in \mathscr{C}^2[a, b]$ be such that $\dot{\Gamma}(s)/s$ is a monotone increasing function on $[a, b]$. Then for $u \in \mathcal{A}(\mathbf{X})$ and almost every $r \in [a, b]$ we have*

$$\int_0^{2\pi} \Gamma(|u|) \frac{u(r,\theta) \times u_\theta(r,\theta)}{|u|^2} \, \mathrm{d}\theta \geq 2\pi \Gamma(r). \tag{5.10}$$

The final ingredient before moving on to the main theorem is the following lifting result for mappings $u \in \mathcal{A}(\mathbf{X})$.

Theorem 5.1. *For each $u \in \mathscr{A}_k$ with $k \in \mathbb{Z}$ there exists a corresponding function $g \in \mathscr{W}^{1,2}(\mathbf{X}) \cap \mathscr{C}(\overline{\mathbf{X}})$ such that u has the following lifting,*

$$u(x) = |u|(x)\mathbf{Q}[g]x|x|^{-1}, \qquad x \in \mathbf{X}. \tag{5.11}$$

Here $g = 0$ and $g = 2\pi k$ on the boundary components $\partial\mathbf{X}_a$ and $\partial\mathbf{X}_b$ respectively. Furthermore we have

$$|\nabla u|^2 = |\nabla|u||^2 + |x|^{-2}|u|^2 \left(1 + 2\partial_\theta g\right) + |u|^2|\nabla g|^2, \tag{5.12}$$

for almost every $x \in \mathbf{X}$. Note that here $\partial_\theta g = \nabla g \cdot x^\perp$ with $x^\perp = (-x_2, x_1)$.

Proof. As $u \in \mathcal{A}(\mathbf{X})$ it admits a representative $u \in \mathscr{C}_{id}(\mathbf{X})$ and thus by basic considerations $u|u|^{-1} \in \mathscr{W}^{1,2}(\mathbf{X}, \mathbb{S}^1) \cap \mathscr{C}(\overline{\mathbf{X}}, \mathbb{S}^1)$. Now set

$$v(r,\theta) = \frac{u}{|u|}(r\cos\theta, r\sin\theta) : [a,b] \times [0, 2\pi] \to \mathbb{S}^1. \tag{5.13}$$

Then in view of $u/|u| \in \mathscr{W}^{1,2}(\mathbf{X}, \mathbb{S}^1) \cap \mathscr{C}(\overline{\mathbf{X}}, \mathbb{S}^1)$ it is plain that $v \in \mathscr{W}^{1,2}(\mathcal{R}, \mathbb{S}^1) \cap \mathscr{C}(\mathcal{R}, \mathbb{S}^1)$ where $\mathcal{R} = [a,b] \times [0,2\pi]$ and $v(r,0) = v(r,2\pi)$. Now since $\mathcal{R} \subset \mathbb{R}^2$ is simply-connected from Theorem 3 in [6] it follows that v admits a lifting (with the same regularity), i.e., $\exists h \in \mathscr{W}^{1,2}(\mathcal{R}) \cap \mathscr{C}(\mathcal{R})$ such that $v(r,\theta) = e^{i(h(r,\theta)+\theta)}$. Since the fibre of $z = 1$ under the covering map $e^{ix} : \mathbb{R} \to \mathbb{S}^1 \subset \mathbb{C}$ is the lattice $2\pi\mathbb{Z} \subset \mathbb{R}$ it follows by a basic continuity argument that $h(r,2\pi) - h(r,0) = 2\pi m$ for all $a \le r \le b$ and some $m \in \mathbb{Z}$. As a result for each fixed r, the plane curve $\gamma_r(\theta) = v(r,\theta) = u|u|^{-1}(r\cos\theta, r\sin\theta)$ with $0 \le \theta \le 2\pi$ has the winding number $m+1$ about the origin. However since the winding number of either γ_a or γ_b is $+1$ due to $\gamma_a(\theta) = \gamma_b(\theta) = (\cos\theta, \sin\theta)$ [recall $u \equiv x$ on $\partial\mathbf{X}$] it follows at once that $m = 0$ and so $h(r,2\pi) = h(r,0)$. Now let $g(x) = g(r\cos\theta, r\sin\theta) = h(r,\theta)$ which is well-defined by virtue of $h(r,2\pi) = h(r,0)$. Then,

$$v(r,\theta) = e^{i(h(r,\theta)+\theta)} = \mathbf{Q}[g(r\cos\theta, r\sin\theta)]\frac{x}{|x|}, \tag{5.14}$$

and so $u|u|^{-1} = \mathbf{Q}[g]x|x|^{-1}$ or $u = |u|\mathbf{Q}[g]|x|^{-1}x$ where $|u|, g \in \mathscr{W}^{1,2}(\mathbf{X}) \cap \mathscr{C}(\overline{\mathbf{X}})$. Note also that $u \equiv x$ on $\partial\mathbf{X}$ results in $g = 2\pi k_a$ and $g = 2\pi k_b$ on $\partial\mathbf{X}_a$ and $\partial\mathbf{X}_b$ respectively, where $k_a, k_b \in \mathbb{Z}$ and $k = k_b - k_a$. Now without loss of generality we take $k_a = 0$ and $k_b = k$ as taking away an integer multiple of 2π from g does not affect $\mathbf{Q}[g]$. This in turn results in $g \equiv 0$ on $\partial\mathbf{X}_a$ and $g \equiv 2\pi k$ on $\partial\mathbf{X}_b$ respectively. Finally (5.12) comes from a direct calculation which completes the proof. □

The significance of this formulation for the purpose of proving the main result lies in the decomposition of $|\nabla u|^2$. Indeed this allows one to write the energy $\mathscr{F}$ into a sum of of sub-energies which themselves are easier to tame. In particular, we are able to apply an averaging argument to the angle of rotation function g in order to obtain a corresponding radial function $\bar{g}$, which in turn defines a corresponding twist mapping $\bar{u}$. With these propositions now at our disposal we are in a position to prove the main result of this section. However let us first state the assumption we place on F.

The class $\mathfrak{F}$ and the integrand F. For the sake of clarity and convenience we recall the assumptions on the integrand F. Firstly we assume that $F \in \mathscr{C}^2(\mathscr{R})$ and $0 < c \leq F$ where $\mathscr{R} = [a^2, b^2] \times [a^2, b^2] \subset \mathbb{R}^2_+$. Furthermore we assume that one of the following holds:

(**H1**) $F(x, y) = F(1, y)/x$ and $\partial_y^2[\sqrt{F}] + \partial_y[\sqrt{F}/y] \geq 0$ on $\mathscr{R}$ with $F = F(x^2, y^2)$.

(**H2**) $F(x, y) = F(x, 1)/y$ and $\partial_x^2[\sqrt{F}] + \partial_x[\sqrt{F}/x] \geq 0$ on $\mathscr{R}$ with $F = F(x^2, y^2)$.

The type of integrands we are considering here prompts us to introduce the class $\mathfrak{F} = \{F \in \mathscr{C}^2(\mathscr{R}) : F \geq c > 0$ on $\mathscr{R}$ and satisfies either (**H1**) or (**H2**)$\}$. [7] In the remainder of this section we consider energy integrals in the form

$$\mathscr{F}[u; \mathbf{X}] = \frac{1}{2} \int_{\mathbf{X}} F(|x|^2, |u|^2) |\nabla u|^2 \, \mathrm{d}x, \tag{5.15}$$

where the integrand F lies in $\mathfrak{F}$.

Theorem 5.2. (*Unique minimality*) *For any $u \in \mathscr{A}_k$ (with $k \in \mathbb{Z}$) there exists a twist mapping $\bar{u} \in \mathscr{A}_k$ such that*

$$\mathscr{F}[\bar{u}; \mathbf{X}] \leq \mathscr{F}[u; \mathbf{X}]. \tag{5.16}$$

Furthermore the inequality is strict iff u is not a twist itself. As a result for each $k \in \mathbb{Z}$ the twist $u = u_k = \mathbf{Q}[g_k](r)x$ from Section 4 is the unique minimiser of $\mathscr{F}$ over $\mathscr{A}_k$ and an L^1-local minimiser of $\mathscr{F}$ over $\mathcal{A}(\mathbf{X})$.

Proof. Firstly note that in proving the theorem it is enough to assume only one of (**H1**) or (**H2**) as the result under the other assumption follows by invoking Theorem 2.1. Indeed in view of the latter if $u \in \mathscr{A}_k$ then it's inverse exists and $w = u^{-1} \in \mathscr{A}_{-k}$ where in particular

$$|\nabla u(w(y))|^2 = |\nabla w(y)|^2, \tag{5.17}$$

for almost every $y \in \mathbf{X}$. Therefore an application of Theorem 1.8 in [13] gives

$$\begin{aligned} 2\mathscr{F}[u; \mathbf{X}] &= \int_{\mathbf{X}} F(|x|^2, |u|^2) |\nabla u(x)|^2 \, \mathrm{d}x \\ &= \int_{\mathbf{X}} F(|w(y)|^2, |y|^2) |\nabla w(y)|^2 \, \mathrm{d}y. \end{aligned} \tag{5.18}$$

Now let us define $F^\star(\xi, \eta) = F(\eta, \xi)$; then clearly if F satisfies (**H1**) then $F^\star$ will satisfy (**H2**). Moreover upon denoting $\mathscr{F}^\star$ to be the energy integral

$$\mathscr{F}^\star[w; \mathbf{X}] = \int_{\mathbf{X}} F^\star(|x|^2, |w|^2) |\nabla w(x)|^2 / 2 \, \mathrm{d}x. \tag{5.19}$$

[7] For examples and further discussion see Section 1 following the statement of the main theorem.

it is seen using (5.18) that $\mathscr{F}[u;\mathbf{X}] = \mathscr{F}^\star[w;\mathbf{X}]$. Hence proving (5.16) amounts to showing $\mathscr{F}^\star[w;\mathbf{X}] \geq \mathscr{F}^*[\bar{u}^{-1};\mathbf{X}]$ that holds as $\bar{u}^{-1}$ is a twist. In light of this we shall assume hereafter for the sake of definiteness that $F \in \mathfrak{F}$ satisfies (**H2**) only. Now to proceed forward recall from Theorem 5.1 that for any $u \in \mathscr{A}_k$ there exists a corresponding $g \in \mathscr{W}^{1,2}(\mathbf{X}) \cap \mathscr{C}(\overline{\mathbf{X}})$ such that

$$|\nabla u|^2 = |\nabla|u||^2 + |x|^{-2}|u|^2\,(1 + 2\partial_\theta g) + |u|^2|\nabla g|^2. \tag{5.20}$$

By taking advantage of this formulation we can write the energy decomposition

$$\begin{aligned} 2\mathscr{F}[u;\mathbf{X}] &= \int_{\mathbf{X}} F(|x|^2, |u|^2)|\nabla u(x)|^2\,\mathrm{d}x \\ &= \int_{\mathbf{X}} F(|x|^2, 1)|u|^{-2}|\nabla u(x)|^2\,\mathrm{d}x \\ &= \int_{\mathbf{X}} F(|x|^2, 1)\left(|u|^{-2}|\nabla|u||^2 + |x|^{-2}\,(1 + 2\partial_\theta g) + |\nabla g|^2\right)\mathrm{d}x. \end{aligned} \tag{5.21}$$

Clearly for the middle term in the last line the following identity is seen to hold

$$\int_{\mathbf{X}} F(|x|^2, 1)|x|^{-2}\,(1 + 2\partial_\theta g)\;\mathrm{d}x = \int_{\mathbf{X}} F(|x|^2, |x|^2)\,\mathrm{d}x. \tag{5.22}$$

Therefore we are left with the following two terms to consider, that is,

$$\mathbf{I} = \int_{\mathbf{X}} F(|x|^2, |u|^2)|\nabla|u||^2\,\mathrm{d}x, \quad \mathbf{II} = \int_{\mathbf{X}} F(|x|^2, 1)|\nabla g|^2\,\mathrm{d}x, \tag{5.23}$$

which for the rest of the proof we shall deal with individually in three steps on the completion of which we shall conclude the uniqueness of twist minimiser $u_k \in \mathscr{A}_k$ in one last strike.

Step 1. **Bounding I from below:** Here we are considering **I** with the aim of proving the inequality,

$$\int_{\mathbf{X}} F(|x|^2, |x|^2)\,\mathrm{d}x \leq \int_{\mathbf{X}} F(|x|^2, |u|^2)|\nabla|u||^2\,\mathrm{d}x. \tag{5.24}$$

The idea is to use the coarea formula which enables one to take advantage of the inequalities established earlier on the boundaries of the level sets. To this end for $u \in \mathcal{A}(\mathbf{X})$ using the coarea formula for Sobolev functions (*see*, e.g., [7]) we can write

$$\int_{\mathbf{X}} F(|x|^2, |u|^2)^{1/2}|\nabla|u||\,\mathrm{d}x = \int_a^b \int_{\{|u|=t\}} F(|x|^2, t^2)^{1/2}\,\mathrm{d}\mathcal{H}^1\,\mathrm{d}t. \tag{5.25}$$

Now as u is a homeomorphism by Theorem 2.1 upon denoting its inverse by $w = u^{-1}$ and using Proposition 5.1 we can write

$$\int_0^{2\pi} F(|w|^2, t^2)^{1/2}\left|\frac{\partial w}{\partial \theta}\right|\mathrm{d}\theta\bigg|_{r=t} = \int_{\{|u|=t\}} F(|x|^2, t^2)^{1/2}\,\mathrm{d}\mathcal{H}^1, \tag{5.26}$$

for $a.e.$ $t \in [a,b]$. Next a straightforward calculation shows that

$$\left|\frac{\partial w}{\partial \theta}\right|^2 = \frac{(w \cdot \partial_\theta w)^2}{|w|^2} + \frac{(w \times \partial_\theta w)^2}{|w|^2}, \tag{5.27}$$

and so in particular $|\partial_\theta w| \geq (w \times \partial_\theta w)/|w|$. Therefore,

$$\int_0^{2\pi} F(|w|^2, t^2)^{1/2} \frac{w \times \partial_\theta w}{|w|} \, \mathrm{d}\theta \bigg|_{r=t} \leq \int_{\{|u|=t\}} F(|x|^2, t^2)^{1/2} \, \mathrm{d}\mathcal{H}^1, \tag{5.28}$$

for $a.e.$ $t \in [a,b]$. Now for $t \in [a,b]$ fixed let us introduce $\Gamma(s) = F(s^2, t^2)^{1/2} s$. Then as a direct differentiation gives $\dot\Gamma(s)/s = F(s^2,t^2)^{1/2}/s + \partial_s[F(s^2,t^2)^{1/2}]$ it follows without much difficulty and in light of the assumption (**H2**) that

$$\begin{aligned} \frac{d}{ds}\left(\frac{\dot\Gamma}{s}\right) &= \frac{d}{ds}\left(\frac{F(s^2,t^2)^{1/2}}{s} + \frac{d}{ds}\left[F(s^2,t^2)^{1/2}\right]\right) \\ &= \frac{d}{ds}\left(\frac{F(s^2,t^2)^{1/2}}{s}\right) + \frac{d^2}{ds^2}\left[F(s^2,t^2)^{1/2}\right] \geq 0. \end{aligned} \tag{5.29}$$

Thus in particular $\dot\Gamma(s)/s$ is monotone increasing and as $\Gamma \in \mathscr{C}^2[a,b]$ we have by Proposition 5.2 that,

$$\int_0^{2\pi} \Gamma(|w|) \frac{w \times \partial_\theta w}{|w|^2} \, \mathrm{d}\theta \bigg|_{r=t} \geq 2\pi\Gamma(t) = 2\pi t F(t^2, t^2)^{1/2}, \tag{5.30}$$

for $a.e.$ $t \in [a,b]$. This therefore allows us to conclude from (5.28) that for $a.e.$ $t \in [a,b]$ we have,

$$2\pi t F(t^2,t^2)^{1/2} \leq \int_{\{|u|=t\}} F(|x|^2,t^2)^{1/2} \, \mathrm{d}\mathcal{H}^1. \tag{5.31}$$

Let us denote by $\alpha_{|u|}$ the classical distribution function of $|u|$. It is not hard to see that for $a \leq t \leq b$ we have $\alpha_{|u|}(t) = |\{x \in \mathbf{X} : |u(x)| > t\}| = \pi(b^2 - t^2)$. Let us also recall the incompressibility constraint $1 = \det \nabla u = \partial_1 u \times \partial_2 u$ for $a.e.$ $x \in \mathbf{X}$ where each weak derivative of u can be written as,

$$\partial_1 u = \partial_1|u| \frac{u}{|u|} + |u|\partial_1\left(\frac{u}{|u|}\right), \quad \partial_2 u = \partial_2|u| \frac{u}{|u|} + |u|\partial_2\left(\frac{u}{|u|}\right). \tag{5.32}$$

Then an explicit calculation shows that,

$$\begin{aligned} \partial_1 u \times \partial_2 u &= \left[\partial_1|u|\frac{u}{|u|} + |u|\partial_1\left(\frac{u}{|u|}\right)\right] \times \left[\partial_2|u|\frac{u}{|u|} + |u|\partial_2\left(\frac{u}{|u|}\right)\right] \\ &= \partial_1|u|\left[u \times \partial_2\left(\frac{u}{|u|}\right)\right] - \partial_2|u|\left[u \times \partial_1\left(\frac{u}{|u|}\right)\right] \\ &= \langle\left[u \times \partial_2\left(\frac{u}{|u|}\right), -u \times \partial_1\left(\frac{u}{|u|}\right)\right], \nabla|u|\rangle, \end{aligned} \tag{5.33}$$

for *a.e.* $x \in \mathbf{X}$. Thus in particular the constraint $\det \nabla u = 1$ *a.e.* in $\mathbf{X}$ implies that $|\{x \in \mathbf{X} : |\nabla |u|| = 0\}| = 0$. Therefore a further application of the coarea formula for Sobolev functions as in Lemma 2.3 of [7] or Lemma 3.1 of [8] gives

$$\int_{\{x\in\mathbf{X}:|u|=t\}} \frac{\mathrm{d}\mathcal{H}^1}{|\nabla|u||} = -\frac{d}{dt}\alpha_{|u|}(t) = 2\pi t, \tag{5.34}$$

for *a.e.* $t \in [a, b]$. (Note that a Sobolev function $f \in W^{1,p}_{loc}(\mathbb{R}^n)$ for $1 \le p \le \infty$ is approximately differentiable *a.e.* and its approximate derivative equals its weak derivative *a.e.* with this approximate derivative being a Borel function. Thus in the lines (5.34) and (5.35) $\nabla|u|$ should, with a slight abuse of notation, be thought of as the approximate derivative.) Then by Holder's inequality and an application of (5.31) it follows that

$$2\pi F(t^2, t^2)t \le \int_{\{x\in\mathbf{X}:|u|=t\}} F(|x|^2, t^2)|\nabla|u||\,\mathrm{d}\mathcal{H}^1, \tag{5.35}$$

for almost every $t \in [a, b]$. A further application of the coarea formula together with (5.35) now results in

$$\int_{\mathbf{X}} F(|x|^2, |x|^2)\,\mathrm{d}x = \int_a^b 2\pi F(t^2, t^2)\,t\,\mathrm{d}t \le \int_{\mathbf{X}} F(|x|^2, |u|^2)|\nabla|u||^2\,\mathrm{d}x, \tag{5.36}$$

and therefore (5.24) follows at once.

Step 2. **Case of equality with I:** We now claim that equality in (5.24) can occur only if u is a twist mapping. The basic idea here is that the steps taken above necessitate that $|u| = |x|$ *a.e.* on $\mathbf{X}$ if we have equality, which in virtue of the determinant constraint and the identity boundary conditions result in u being a twist. With this in mind we note that by the above calculations equality can only occur if

$$\left|\frac{\partial w}{\partial \theta}\right| = \frac{w \times \partial_\theta w}{|w|}, \tag{5.37}$$

for *a.e.* $x \in \mathbf{X}$, which in turn implies that for almost every $x \in \mathbf{X}$ it must be that,

$$\frac{\partial}{\partial\theta}|w|^2 = 2w \cdot \frac{\partial w}{\partial\theta} = 0. \tag{5.38}$$

Now recall from (5.4) that

$$\frac{\partial w}{\partial\theta}(y) = |y|\mathbf{J}\nabla|u|(w), \quad \mathbf{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tag{5.39}$$

which by an application of (5.38) results in

$$2w \cdot \mathbf{J}\nabla|u|(w) = 0 \implies \frac{\partial}{\partial\theta}|u|(x) = 0, \tag{5.40}$$

for $a.e.$ $x \in \mathbf{X}$. Note that the last equality uses $w = u^{-1}$ and $y = u(x)$. We now deduce from this that $|u|(x) = h(|x|)$ for some $h \in L^2[a,b]$ and $\partial_r |u| = l(|x|)$ for some $l \in L^2[a,b]$. Indeed to see this we first transfer into polar co-ordinates by letting $f(r,\theta) = |u|(r\cos\theta, r\sin\theta)$. Then $f \in W^{1,2}(\mathcal{R})$ where $\mathcal{R} = [a,b] \times [0, 2\pi]$ whilst

$$\frac{\partial f}{\partial r} = \frac{\partial |u|}{\partial r}, \qquad \frac{\partial f}{\partial \theta} = \frac{\partial |u|}{\partial \theta} = 0, \tag{5.41}$$

for almost every $(r,\theta) \in \mathcal{R}$. From this it is straightforward to conclude, using the definition of weak derivative, that $f(r,\theta) = h(r)$ for almost every $(r,\theta) \in \mathcal{R}$ and $\partial_r f = l(r)$ for some $l \in L^2(a,b)$ and almost every $(r,\theta) \in \mathcal{R}$ that we abbreviate. Next let us note that

$$\partial_r u \times \partial_\theta u = |x| \det \nabla u = |x|, \tag{5.42}$$

for almost every $x \in \mathbf{X}$ where additionally the partial derivatives can be further expressed as

$$\partial_r u = \partial_r |u| \frac{u}{|u|} + |u| \partial_r \left(\frac{u}{|u|}\right), \tag{5.43}$$

$$\partial_\theta u = \partial_\theta |u| \frac{u}{|u|} + |u| \partial_\theta \left(\frac{u}{|u|}\right) = |u| \partial_\theta \left(\frac{u}{|u|}\right). \tag{5.44}$$

Therefore using the lifting result of Theorem 5.1 for $u \in \mathcal{A}(\mathbf{X})$ we can write the determinant constraint as,

$$2r = \frac{\partial |u|^2}{\partial r}\left(1 + \frac{\partial g}{\partial \theta}\right) \tag{5.45}$$

for $a.e.$ $x \in \mathbf{X}$. Integrating both sides over $\mathbf{X}_r^{r+\delta}$ and using the fact that $\partial_r |u|^2$ is constant with respect to θ leads to

$$\begin{aligned} \frac{4\pi}{3}[(r+\delta)^3 - r^3] &= \int_r^{r+\delta} r \partial_r |u|^2 \int_0^{2\pi} \left(1 + \frac{\partial g}{\partial \theta}\right) \mathrm{d}\theta \, \mathrm{d}r \\ &= 2\pi \int_r^{r+\delta} r \partial_r |u|^2 \, \mathrm{d}r. \end{aligned} \tag{5.46}$$

An application of Lebesgue's differentiation theorem gives $\partial_r |u|^2 = 2r$ for almost every $r \in [a,b]$ and so $\partial_r |u|^2 = 2|x|$ for $a.e.$ $x \in \mathbf{X}$. Now by invoking the boundary conditions on u this gives that $|u| = |x|$ for all $x \in \mathbf{X}$ as $|u|$ and $|x|$ agree almost everywhere and are both continuous. Thus by (5.45) and for $a.e.$ $x \in \mathbf{X}$ we have $\partial g / \partial \theta = 0$ from which it is possible to conclude, similar to what was done earlier, that $g(x) = g(|x|)$ where $g \in W^{1,2}[a,b] \cap \mathscr{C}[a,b]$. Hence we have shown for equality to occur it must be that $u(x) = \mathbf{Q}[g(|x|)]x$ or in other words that u must be a twist mapping. In conclusion we have shown that if $u \in \mathcal{A}(\mathbf{X})$ is not a twist mapping then

$$\int_{\mathbf{X}} F(|x|^2, |x|^2) \, \mathrm{d}x < \int_{\mathbf{X}} F(|x|^2, |u|^2) |\nabla |u||^2 \, \mathrm{d}x. \tag{5.47}$$

Step 3. **Bounding II from below:** We are now left to deal with the **II** term. Towards this end we aim to show that,

$$\int_{\mathbf{X}} F(|x|^2,1)|\nabla \bar{g}|^2 \,\mathrm{d}x \leq \int_{\mathbf{X}} F(|x|^2,1)|\nabla g|^2 \,\mathrm{d}x,$$

for some $\bar{g} \in \mathscr{W}^{1,2}(a,b)$ with $\bar{g}(a)=0$ and $\bar{g}(b)=2\pi k$. To achieve this we use an averaging argument. Indeed let us define for $r \in (a,b)$,

$$\bar{g}(r) = \frac{1}{2\pi}\int_a^r \int_0^{2\pi} \frac{\partial g}{\partial r}(s,\theta)\,\mathrm{d}\theta\, ds, \tag{5.48}$$

where here we have written g in polar co-ordinates. It is straightforward to see that $\bar{g}(a)=0$, $\bar{g}(b)=2\pi k$ and $\bar{g} \in \mathscr{W}^{1,2}(a,b)$ since g satisfies these boundary conditions and $g \in \mathscr{W}^{1,2}(\mathbf{X})$. Moreover by an application of Jensen's inequality we have

$$|\nabla \bar{g}|^2 = \left|\frac{d\bar{g}}{dr}\right|^2 \leq \frac{1}{2\pi}\int_0^{2\pi} |\nabla g|^2 \,\mathrm{d}\theta, \tag{5.49}$$

for almost every $r \in (a,b)$. Thus we have that,

$$\int_{\mathbf{X}} F(|x|^2,1)|\nabla \bar{g}|^2 \,\mathrm{d}x \leq \int_{\mathbf{X}} F(|x|^2,1)|\nabla g|^2 \,\mathrm{d}x. \tag{5.50}$$

Hence by combining together all the estimates we have proved so far we arrive at the energy inequality

$$\begin{aligned}\mathscr{F}[u;\mathbf{X}] &= \int_{\mathbf{X}} F(|x|^2,1)[|u|^{-2}|\nabla|u||^2 + |x|^{-2}\,(1+2g_\theta) + |\nabla g|^2]/2\,\mathrm{d}x \\ &\geq \int_{\mathbf{X}} F(|x|^2,|x|^2)|\nabla \bar{u}|^2/2\mathrm{d}x = \mathscr{F}[\bar{u};\mathbf{X}],\end{aligned} \tag{5.51}$$

where due to (5.47) the inequality is strict if u is a non-twist mapping. Moreover by recalling $\bar{g}$ satisfies $\bar{g}(a)=0$, $\bar{g}(b)=2\pi k$ and $\bar{g} \in \mathscr{W}^{1,2}(a,b)$ implies that the twist mapping $\bar{u} = \mathbf{Q}[\bar{g}]x \in \mathscr{A}_k$.

Step 4. **Uniqueness of twist minimisers:** Finally to conclude the proof we need to show that u_k is the unique minimiser of $\mathscr{F}$ in $\mathscr{A}_k$. To do this we argue indirectly and for the sake of a contradiction assume this not to be the case, i.e., that $\exists u \in \mathscr{A}_k$ with $u \neq u_k$ such that $\mathscr{F}[u_k] = \mathscr{F}[u]$. Then as by construction and a basic convexity argument u_k is the unique minimiser of $\mathscr{F}$ amongst twist mapping in $\mathscr{A}_k$ [*cf.* (4.13)-(4.14)] it must be that u is not a twist. However by the above, the symmetrisation of u denoted $\bar{u} \in \mathscr{A}_k$, satisfies

$$\mathscr{F}[u_k] \leq \mathscr{F}[\bar{u}] < \mathscr{F}[u] = \mathscr{F}[u_k], \tag{5.52}$$

where the strict inequality in the middle is a consequence of the first part of the proof and u not being twist. Therefore we have reached the desired contradiction and this finishes the proof. □

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