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Revealed time preference*

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Abstract

Suppose a researcher observes a *finite* number of preference comparisons between pairs (x, t) of rewards x delivered at time t . We provide necessary and sufficient conditions under which such a dataset can be rationalised with the most common specifications of the discounted utility model. The distinctive feature of our characterisation is that it refers to notions of stochastic dominance among empirical distributions over the observed rewards and time delays.

Keywords: revealed preference, testable restrictions, time preference, discounted utility, present bias, exponential discounting

JEL Classification: C14, C60, C61, D11, D12

1 Introduction

Suppose that in an experiment or an empirical study a subject is reporting her preference over delayed rewards, i.e., pairs (x, t) of prizes $x \in \mathbb{R}_+^\ell$ delivered at some date $t \in \mathbb{Z}_+$. Whenever she picks an option (x, t) over some alternative (y, s) , we say that the former is revealed preferred to the latter and denote it by $(x, t)R^*(y, s)$. Given a finite number of such observations, when can we rationalise this dataset with a discounted utility model? Specifically, we are interested in conditions on the revealed preference relation R^* under which there exists a felicity function u and a discounting function γ such that

$$(x, t)R^*(y, s) \text{ implies } u(x)\gamma(t) \geq u(y)\gamma(s).$$

Moreover, what can we say about the discounting function γ ? In particular, under what conditions on R^* it is weakly present-biased or exponential?

The objective of this paper is to establish the testable restrictions for different specifications of discounted utility, i.e., we provide tight conditions on the revealed preference R^* that allow us to falsify these models with directly observable data. Our approach is

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nonparametric and focuses on the most fundamental implications of discounted utility. However, we do not limit our attention to algorithms that enable us to perform such tests on empirical observations; rather, we explore the economic content of different models of time preference by analysing conditions on the data alone, without referring to unobservable concepts such as utility or marginal rate of substitution.

We characterise time preference in a framework where the domain of choices is restricted to prize-time pairs. There is an extensive decision theory literature that provides axiomatic foundations for choice over dated rewards, as discussed in the literature review below. The objective of those papers is to characterise complete preference relations that admit a particular utility representation. Notably, these studies assume that the researcher can observe comparisons between any two alternatives. In contrast, our analysis focuses on properties of a finite and incomplete revealed preference relation R^* . Since such a dataset provides limited information on consumer taste, answering the question of representation requires different techniques. In particular, apart from inferring the subject's preference nonparametrically, we ask when rationalising the data is even possible. Given that data is inherently finite in any empirical setting, we find our approach especially useful from the practical perspective.

The distinctive feature of our analysis is that it refers to notions of stochastic dominance among empirical distributions over observable rewards and time delays. Specifically, we focus on finite collections of elements $(x, t)R^*(y, s)$ in the revealed preference relation R^* , henceforth *samples*. In order to characterise a time preference of interest, we evaluate independently empirical distributions over rewards x, y and time delays t, s in the superior (i.e., on the left-hand side of R^*) and inferior pairs (on the right) that correspond to such a sample. In our axioms we determine the form of stochastic dominance among these distributions that is inadmissible by the particular model of discounted utility. In addition, we show that such a restriction is both necessary and sufficient for a revealed preference relation to be rationalisable in the specified sense. This technique allows us to present the axioms in a compact and transparent way, while generalising the existing characterisations of intertemporal choice that pertain to complete preference relations.

This paper is organised as follows. After the literature review below, in Section 2 we introduce our framework, necessary notation, and auxiliary definitions. In Section 3 we characterise the most general class of discounted utility models. Then, in Section 4 we focus on discounting that exhibits some level of present bias, including the most commonly known exponential discounting. We conclude in Section 5, where we discuss issues related to identification and indeterminacy of preferences. Proofs are presented in the [Appendix](#). The [Online supplement](#) includes auxiliary results omitted from the main text.

Related literature Stemming from the work by [Koopmans \(1960\)](#), a significant share of the economic literature was devoted to axiomatic characterisation of models of intertemporal choice. An important part of this research focuses on preferences defined over pairs (x, t) of reward vectors $x \in \mathbb{R}_+^\ell$ and time delays $t \in \mathbb{Z}_+$. This includes, e.g., [Fishburn and Rubinstein \(1982\)](#), [Prelec \(2004\)](#), [Ok and Masatlioglu \(2007\)](#), [Halevy \(2008\)](#), [Bleichrodt et al. \(2009\)](#), or [Noor \(2011\)](#).¹ Unlike in our analysis, the above papers adopt a complete preference relation as a premise. Hence, they assume that it is possible to monitor a binary comparison between any two prize-time pairs that belong to the

¹Alternatively, [Fishburn and Edwards \(1997\)](#), [Hayashi \(2003\)](#), [Olea and Strzalecki \(2014\)](#), [Galperti and Strulovici \(2017\)](#) characterise time preference over consumption streams rather than dated rewards.

domain. In contrast, we assume that only a finite number of comparisons is observable, which makes our approach particularly useful for practical applications.

In the past three decades there has been an abundance of experimental studies that investigated consumer preference over prize-time pairs. See Table 1 in [Frederick et al. \(2002\)](#) or Table 1 in [Takeuchi \(2011\)](#) for an extensive list of such experiments. It is also worth mentioning [Chabris et al. \(2008, 2009\)](#), [Dohmen et al. \(2012\)](#), [Andersen et al. \(2013, 2014\)](#), and [Halevy \(2015\)](#). Even though the above studies differed in their designs, the objective of each one of them was to determine preferences of individual subjects over dated rewards. In almost all of these experiments the data were used to estimate parametrically a particular model of time preference using econometric methods. A notable exception is [Halevy \(2015\)](#) where the study was tailored to test directly for particular axioms characterising different discounted utility models. Our nonparametric approach is largely applicable to data from all of the above studies. Moreover, it does not only allow to elicit time preference from the observable choices, but foremost determine whether rationalising the data is even possible.

We consider [Echenique et al. \(2017\)](#) to be the closest to our work. It provides a characterisation of various forms of time-separable preferences defined over streams of consumption, rather than prize-time pairs. This paper adopts an Afriat-type framework in which an observation consists of a path of one-dimensional consumption goods acquired at prevailing time-contingent prices. The main results introduce tight conditions under which such a finite dataset can be rationalised by several forms of discounted utility.

An obvious difference between this paper and ours follows from the distinction in domains over which a time preference is defined. However, also our approach to axiomatisation is fundamentally different. The testable restrictions developed in [Echenique et al.](#) are induced by first-order conditions corresponding to the consumer optimisation problem. This requires convexity of preferences and sets of alternatives from which subjects make their choices. By restricting our attention to dated rewards, we are able to characterise discounted utility without the additional conditions on time preference. Nevertheless, as the framework in the two papers is distinct and prescribed for different empirical applications, the results are not comparable.

2 Preliminaries

We begin with a formal specification of our framework, notation, and definitions.

2.1 Basic notation

In this paper we discuss preferences defined over delayed rewards, i.e., pairs (x, t) in which $x \in \mathbb{R}_+^\ell$ denotes a prize that is delivered at time $t \in \mathbb{Z}_+$.² Throughout the analysis we assume that the researcher observes a *finite* number of binary comparisons between such options. We summarise these observations with a *revealed preference relation* R^* defined as a finite subset of $(\mathbb{R}_+^\ell \times \mathbb{Z}_+) \times (\mathbb{R}_+^\ell \times \mathbb{Z}_+)$. That is, whenever we observe that the subject prefers option (x, t) to (y, s) then the pair $((x, t), (y, s))$ is an element of R^* . For convenience, we denote it by $(x, t)R^*(y, s)$.

²We endow space $\mathbb{R}_+^\ell$ with a natural product order $\geq$. That is, we denote $x \geq y$ whenever $x^i \geq y^i$, for all $i = 1, \dots, \ell$. In addition, let $x > y$ if $x \geq y$ and $x \neq y$. Finally, let $x \gg y$ if $x^i > y^i$, for all $i = 1, \dots, \ell$. Similarly, we endow $\mathbb{Z}_+$ with the natural order $\geq$.

In some instances, the researcher may possess additional information on preference revealed in the data. Specifically, if the subject *strictly* prefers (x, t) to (y, s) , then $((x, t), (y, s))$ is an element of a (possibly empty) *revealed strict preference relation* P^* , denoted by $(x, t)P^*(y, s)$. Clearly, we have $P^* \subseteq R^*$, by construction.³ The revealed preference pair (R^*, P^*) is the premise of our analysis.

Our framework can be applied to a wide variety of economic studies, including those cited in Section 1. In a majority of these experiments subjects are presented with multiple decision problems. In each one, they are asked to choose between prize-time pairs (x, t) and (y, s) . Whenever (x, t) is chosen over (y, s) , preference maximisation hypothesis implies that the former is revealed preferred to the latter, which we denote by $(x, t)R^*(y, s)$. Depending on the experimental design and/or information available to the researcher, the above comparison may be included in the strict relation P^* .

In an equally relevant class of studies, subjects are asked in multiple trials to propose a reward x that they would be willing to accept at a predetermined date t , in order to be indifferent to an alternative prize-time pair (y, s) . This reveals both $(x, t)R^*(y, s)$ and $(y, s)R^*(x, t)$, while neither of the comparisons is an element of P^* .

Let $\mathcal{X}$ be the set of *observable rewards* containing vectors $x \in \mathbb{R}_+^\ell$ such that either $(x, t)R^*(y, s)$ or $(y, s)R^*(x, t)$, for some $y \in \mathbb{R}_+^\ell$ and $t, s \in \mathbb{Z}_+$. Similarly, we denote the set of *observable time delays* by $\mathcal{T}$. Our characterisation evokes the notion of a *sample*, which is a finite collection $\{((x_i, t_i), (y_i, s_i))\}_{i \in I}$ of elements in R^* . In particular, it may be generated “with replacement”. That is, it may contain more than one copy of each element in R^* . We abuse the notation and denote its cardinality with I .

A revealed preference relation pair (R^*, P^*) is *rationalisable* by a utility function $v : \mathbb{R}_+^\ell \times \mathbb{Z}_+ \rightarrow \mathbb{R}_+$ if, for any $(x, t), (y, s)$ in $\mathbb{R}_+^\ell \times \mathbb{Z}_+$, we have

$$\text{if } (x, t)R^*(y, s) \text{ then } v(x, t) \geq v(y, s), \quad (1)$$

where the inequality is strict whenever $(x, t)P^*(y, s)$. The objective of this paper is to determine the necessary and sufficient conditions on the pair (R^*, P^*) under which it can be rationalised within a particular class of utility functions v .

Each model of time preference discussed in this paper is a special case of monotone preferences. Specifically, a utility function v is *monotone* whenever $x \geq y$ and $t \leq s$ implies $v(x, t) \geq v(y, s)$, where the inequality is strict if either $x > y$ or $t < s$. This is to say that the subject prefers greater rewards at earlier dates. For the necessary and sufficient conditions for this form of rationalisation see Nishimura et al. (2017).

2.2 Stochastic dominance

Here we introduce three notions of stochastic dominance that are crucial for the remainder of this paper. A more detailed characterisation of these stochastic orderings and their properties is presented in the [Online supplement](#).

Let Δ_S denote a set of probability distributions over a finite set $S \subset \mathbb{R}^\ell$. We define an element μ of Δ_S as a function that assigns a positive value $\mu(x)$ to each $x \in S$ and satisfies $\sum_{x \in S} \mu(x) = 1$. A function $f : \mathbb{R}^\ell \rightarrow \mathbb{R}$ is *increasing* if $x \geq y$ implies $f(x) \geq f(y)$, for any $x, y \in \mathbb{R}^\ell$, where $\geq$ is the natural product order on $\mathbb{R}^\ell$; it is *strictly* increasing whenever $x > y$ implies $f(x) > f(y)$.

³Note that P^* is induced by observations in the experiment. Specifically, since the relation is incomplete, it is not true that *not* $(y, s)R^*(x, t)$ implies $(x, t)P^*(y, s)$. So far we impose no further restrictions on (R^*, P^*) . In particular, they need not be transitive.

For arbitrary probability distributions $\mu, \nu \in \Delta_S$, we say that μ dominates ν with respect to the *first order stochastic dominance* whenever

$$\sum_{x \in S} f(x)\mu(x) \geq \sum_{x \in S} f(x)\nu(x), \quad (2)$$

for any increasing function $f : \mathbb{R}^\ell \rightarrow \mathbb{R}$. That is, the expected value of f has to be higher with respect to distribution μ than ν , for an arbitrary increasing function f . Roughly speaking, the former distribution assigns higher probabilities to greater rewards than the latter. Whenever S is a subset of $\mathbb{R}$, this condition implies that cumulative distribution corresponding to μ is dominated (pointwise) by the cumulative distribution of ν .

Take two distinct distributions μ and ν in Δ_S , i.e., $\mu \neq \nu$. Clearly, whenever μ first order stochastically dominates ν , then (2) must hold for any increasing f . However, it can be shown that, in addition, we have $\sum_{x \in S} f(x)\mu(x) > \sum_{x \in S} f(x)\nu(x)$, for any *strictly* increasing f . See Corollary 1 in the [Online supplement](#) for a proof of this claim.

We say that function $f : \mathbb{R}^\ell \rightarrow \mathbb{R}$ is *concave* if for any $x, y \in \mathbb{R}^\ell$ and $\alpha \in [0, 1]$, we have $f(\alpha x + (1 - \alpha)y) \geq \alpha f(x) + (1 - \alpha)f(y)$. Function f is convex if and only if $-f$ is concave. For any two probability distributions $\mu, \nu \in \Delta_S$, we say that ν is a *mean-preserving spread* of μ , whenever condition (2) holds for any concave function f . Hence, ν is formed by spreading out one or more portions of μ 's probability without changing its expected value. Whenever S is a subset of $\mathbb{R}$, this is to say that the means of the two distributions are equal, while the cumulative distribution corresponding to μ crosses the cumulative distribution of ν at most once and from below.

Finally, for any two probability distributions $\mu, \nu \in \Delta_S$, we say that μ dominates ν with respect to the *second order stochastic dominance*, whenever condition (2) is satisfied for any concave and increasing function f . This relation can be thought of as a combination of the first order stochastic and mean-preserving spread dominance. In fact, it is weaker than either of the two orderings.

There is a particular relation between the mean-preserving spread criterion and second order stochastic dominance that is crucial for our analysis. Suppose that distribution μ second order stochastically dominates ν , but ν is *not* a mean-preserving spread of μ . Clearly, condition (2) has to hold for any concave and increasing f . However, it can be shown that, in addition, we have $\sum_{x \in S} f(x)\mu(x) > \sum_{x \in S} f(x)\nu(x)$ for any concave and *strictly* increasing f . A formal proof of this claim is presented in the [Online supplement](#). In particular, if μ second order stochastically dominates ν and their means are equal then distribution ν is a mean-preserving spread of μ .

3 Discounted utility rationalisation

We commence our discussion on the discounted utility model by characterising its most general specification. A revealed preference pair (R^*, P^*) is rationalisable by a *discounted utility* if there is a strictly increasing felicity function $u : \mathbb{R}_+^\ell \rightarrow \mathbb{R}_{++}$ and a strictly decreasing discounting function $\gamma : \mathbb{Z}_+ \rightarrow (0, 1]$ with $\gamma(0) = 1$ such that utility $v(x, t) := u(x)\gamma(t)$ rationalises (R^*, P^*) according to (1) in Section 2.1. Clearly, any preference consistent with the above model must be monotone.

We find it instructive to begin the analysis with the following example. Suppose that we observe choices of an individual over prize-time pairs (x, t) , where rewards x are given by one-dimensional payments in $\mathbb{R}_+$. Let the revealed preference relation be given by

$$(20, 3)R^*(10, 0), (5, 0)R^*(15, 3), (15, 0)R^*(25, 1), \text{ and } (10, 2)R^*(5, 0),$$

with $P^* = \emptyset$. Observe that monotonicity is not violated directly in the above example. In fact, relation (R^*, P^*) is rationalisable with a monotone utility function.⁴ However, it can not be supported by discounted utility. Notice that, whenever the relation is rationalisable by a felicity u and discounting γ , then $(x, t)R^*(y, s)$ implies $\tilde{u}(x) + \tilde{\gamma}(t) \geq \tilde{u}(y) + \tilde{\gamma}(s)$, where $\tilde{u} := \log u$ and $\tilde{\gamma} := \log \gamma$. Once we sum up the above inequalities, we obtain

$$\tilde{u}(20) + \tilde{\gamma}(2) \geq \tilde{u}(25) + \tilde{\gamma}(1),$$

which yields a contradiction by strict monotonicity of functions $\tilde{u}$ and $\tilde{\gamma}$. In particular, even though the comparison between $(20, 2)$ and $(25, 1)$ was never observed by the researcher and monotonicity was never violated directly, the pair (R^*, P^*) is consistent with a discounted utility maximisation only if the subject prefers the lower payment at the more distant date. Thus, to guarantee that an incomplete revealed relation is consistent with discounted utility, a stronger monotonicity restriction is required.

The above violation is implied by a particular relation between distributions of monetary payments and time delays on both sides of R^* . Notice that the probability distribution over rewards in the superior pairs (i.e., on the left-hand side of R^*) that assigns probability $1/4$ to payments 5, 10, 15, and 20, is strictly first order stochastically dominated by the distribution of rewards in the inferior pairs (on the right-hand side), that attaches weight $1/4$ to 5, 10, 15, and 25.⁵ In addition, the distribution of time delays in the superior pairs strictly first order stochastically dominates the distribution of time delays in the inferior options. Thus, the relation R^* reveals that the subject prefers lower payments at greater time delays in this general, stochastic sense. However, as shown above, this can not be reconciled with discounted utility maximisation.

This observation generalises to arbitrary samples of the revealed preference relation. Formally, take any sample $\{(x_i, t_i), (y_i, s_i)\}_{i \in I}$ of R^* , defined as in Section 2.1. Denote the empirical distribution over rewards x_i by $\mu_x(z) := |\{i \in I : z = x_i\}|/I$. Similarly, let μ_y , τ_t , and τ_s denote distributions of prizes y_i and time delays t_i , s_i , respectively. Whenever the revealed preference relation R^* is rationalisable with a discounted utility function for some felicity u and discounting γ , then

$$\sum_{z \in \mathcal{X}} \tilde{u}(z) \mu_x(z) + \sum_{z \in \mathcal{T}} \tilde{\gamma}(z) \tau_t(z) \geq \sum_{z \in \mathcal{X}} \tilde{u}(z) \mu_y(z) + \sum_{z \in \mathcal{T}} \tilde{\gamma}(z) \tau_s(z), \quad (3)$$

where functions $\tilde{u}$ and $\tilde{\gamma}$ are defined as previously, while sets $\mathcal{X}$ and $\mathcal{T}$ consist of observable rewards and time delays, respectively. Suppose that, in the selected sample, the distribution μ_y of rewards y_i first order stochastically dominates distribution μ_x over prizes x_i , while the distribution τ_t of time delays t_i first order stochastically dominates distribution τ_s over dates s_i . By monotonicity of $\tilde{u}$ and $\tilde{\gamma}$, this implies that

$$\sum_{z \in \mathcal{X}} \tilde{u}(z) \mu_x(z) \leq \sum_{z \in \mathcal{X}} \tilde{u}(z) \mu_y(z) \quad \text{and} \quad \sum_{z \in \mathcal{T}} \tilde{\gamma}(z) \tau_t(z) \leq \sum_{z \in \mathcal{T}} \tilde{\gamma}(z) \tau_s(z). \quad (4)$$

Therefore, for (3) and (4) to be satisfied simultaneously, it must be that $\mu_x = \mu_y$ and $\tau_t = \tau_s$. Otherwise, following our discussion in Section 2.2, strict monotonicity of $\tilde{u}$ and $\tilde{\gamma}$ would imply that at least one inequality in (4) is strict, violating (3).

⁴Indeed, the relation admits a transitive closure (R, P) that is consistent with our notion of monotonicity, e.g., $(15, 0)R(25, 1)R(20, 3)R(10, 0)R(10, 2)R(5, 0)R(15, 3)$. Following Theorem 3 in Nishimura et al. (2017), this is both necessary and sufficient for this type of rationalisation.

⁵Clearly, for any increasing f the expected value of the function with respect to the latter distribution is at least as high as with respect to the former.

The above restriction is not the only implication of the discounted utility model. In particular, consider a revealed preference relation pair (R^*, P^*) given by

$$(20, 3)R^*(10, 0), (5, 0)R^*(15, 3), (10, 2)R^*(5, 0), \text{ and } (15, 0)P^*(20, 2).$$

Note that, probability distributions defined over monetary payments in pairs on the left and right-hand side of the relation are equivalent. The same applies to distributions of time delays. Nevertheless, (R^*, P^*) is not rationalisable by a discounted utility. If that were the case, there would be functions $\tilde{u}$ and $\tilde{\gamma}$ such that $(x, t)R^*(y, s)$ implied $\tilde{u}(x) + \tilde{\gamma}(t) \geq \tilde{u}(y) + \tilde{\gamma}(s)$ and $(15, 0)P^*(20, 2)$ implied $\tilde{u}(15) + \tilde{\gamma}(0) > \tilde{u}(20) + \tilde{\gamma}(2)$. However, summing up those inequalities would imply $0 > 0$.⁶

This example shows that, for any sample of R^* such that the distribution μ_y of rewards y_i dominates distribution μ_x over prizes x_i , while the distribution τ_t of time delays t_i dominates distribution τ_s over dates s_i , *none* of its elements may be selected from P^* . Otherwise, inequality (3) would be strict, which could not be reconciled with (4).

Axiom 1 (Dominance). *For any sample of R^* , if distribution μ_y dominates μ_x and τ_t dominates τ_s in the first order stochastic sense, then $\mu_x = \mu_y$ and $\tau_t = \tau_s$. Moreover, such a sample contains no elements of the revealed strict preference relation P^* .*

The role of the dominance axiom is two-fold. First of all, it generalises *monotonicity* of the revealed preference (R^*, P^*) . Since the relation is incomplete, it may not violate monotonicity directly. This more general condition guarantees that the pair admits a monotone extension. Second of all, as the condition is imposed on marginal distributions over rewards and time delays, it implies *separability* of (R^*, P^*) with respect to the two variables. In fact, Axiom 1 implies the well-known *cancellation law*, thoroughly discussed in Kraft et al. (1959), Scott (1964), or Fishburn (1967).

Theorem 1. *A revealed preference relation pair (R^*, P^*) is rationalisable by a discounted utility function if and only if it obeys dominance.*

The proof of the result is rather extensive, hence, we postpone it until the appendix. The argument is instructive and provides a method of constructing both the felicity function u and the discounting function γ that rationalise the revealed preference relation. Moreover, we show that verifying dominance can be reduced to a linear program. This guarantees that the test is implementable and can be applied to empirical analyses.

4 Testing for weak present bias

In this section we characterise the general class of preferences that exhibit some degree of present bias. Then, we focus on its border case: exponential discounting.

4.1 Weakly present-biased preferences

A strictly decreasing discounting function $\gamma : \mathbb{Z}_+ \rightarrow (0, 1]$ with $\gamma(0) = 1$ is *weakly present-biased* whenever the ratio $\gamma(t)/\gamma(t+1)$ is decreasing in t . Thus, the relative discounting between subsequent periods decreases as the two dates are farther away in the future. Equivalently, this is to say that the discounting function γ admits a *log-convex* extension

⁶Analogously to the previous example, this relation is rationalisable with a monotone function.

to the positive real numbers. This class contains the most commonly known forms of discounting, including hyperbolic, quasi-hyperbolic, and exponential.

A revealed preference relation (R^*, P^*) is rationalisable by a *weakly present-biased discounted utility* function if there is a strictly increasing felicity $u : \mathbb{R}_+^\ell \rightarrow \mathbb{R}_{++}$ and a weakly present-biased discounting $\gamma : \mathbb{Z}_+ \rightarrow (0, 1]$ such that function $v(x, t) := u(x)\gamma(t)$ rationalises (R^*, P^*) according to (1) in Section 2.1.

The above model of time preference captures present bias through the notion of *strong diminishing impatience* introduced in Prelec (2004) and Halevy (2008). A preference relation R satisfies this property if $(x, t)R(y, s)$ implies $(x, t+z)R(y, s+z)$, for any $s \leq t$ and a positive integer z . That is, if the agent is willing to wait $(t-s)$ periods to obtain prize x rather than y , starting from time s , then she is willing to wait the same number of periods starting from a later date $s+z$. Hence, the subject is more patient and willing to wait longer when choosing between more distant rewards.

Since the above model of time preference is a special case of discounted utility, it has to satisfy dominance. However, Axiom 1 is not sufficient for weakly present bias rationalisation. Consider a revealed preference relation R^* consisting of

$$(20, 3)R^*(10, 0), (5, 0)R^*(15, 4), (15, 0)R^*(20, 1), \text{ and } (10, 3)R^*(5, 0),$$

where $P^* = \emptyset$. Observe that the distributions of rewards on the left and right sides of the relation are equivalent. That is, we have $\mu_x = \mu_y$. What is more, distribution of time delays in the superior pairs is given by $\tau_t(0) = \tau_t(3) = 1/2$ and $\tau_t(1) = \tau_t(4) = 0$, while time delays in the inferior options are distributed according to $\tau_s(0) = 1/2$, $\tau_s(1) = \tau_s(4) = 1/4$, and $\tau_s(3) = 0$. In particular, neither is first order stochastically dominant. Nevertheless, the relation is not rationalisable with weak present bias.

If that were the case, there would be a felicity u and discounting γ such that $(x, t)R^*(y, s)$ would imply $\tilde{u}(x) + \tilde{\gamma}(t) \geq \tilde{u}(y) + \tilde{\gamma}(s)$, where $\tilde{u} := \log u$ and $\tilde{\gamma} := \log \gamma$. Summing up the above inequalities for all comparisons in R^* would imply

$$\tilde{\gamma}(3) - \tilde{\gamma}(4) \geq \tilde{\gamma}(1) - \tilde{\gamma}(3) > \tilde{\gamma}(2) - \tilde{\gamma}(3),$$

where the latter inequality follows from strict monotonicity of $\tilde{\gamma}$. However, this is equivalent to $\gamma(3)/\gamma(4) > \gamma(2)/\gamma(3)$, contradicting that γ is weakly present-biased.⁷

Even though τ_t and τ_s are not related in the first order stochastic sense, such a violation occurs whenever distribution τ_t second order stochastically dominates τ_s , but the latter is not a mean-preserving spread of the former.⁸ Indeed, suppose that a pair of revealed preference relations (R^*, P^*) can be rationalised with a weakly present-biased discounted utility model for some felicity u and discounting γ . Define functions $\tilde{u}$ and $\tilde{\gamma}$ as previously. Take any sample $\{(x_i, t_i), (y_i, s_i)\}_{i \in I}$ of R^* and let μ_x , μ_y and τ_t , τ_s denote probability distributions over rewards x_i , y_i and time delays t_i , s_i , respectively. Following the argument in Section 3, condition (3) must be satisfied.

Assume that in the above sample distribution μ_y of rewards y_i first order stochastically dominates distribution μ_x over prizes x_i , while the distribution τ_t of time delays t_i second order stochastically dominates distribution τ_s over dates s_i . Since function $\tilde{u}$ is strictly

⁷At the same time, the relation is rationalisable by any discounted utility function with a felicity function $u(x) = x/5$ and discounting satisfying $\gamma(0) = 1$, $\gamma(1) = 3/4$, $\gamma(3) = 1/2$, and $\gamma(4) = 1/4$.

⁸To show that τ_t second order stochastically dominates τ_s , note that for any increasing and concave f , we have $(1/2)f(0) + (1/4)f(1) + (1/4)f(4) \leq (1/2)f(0) + (1/2)f(3)$. Since the means of τ_t and τ_s are different, the latter is not a mean-preserving spread of the former.

increasing and $\tilde{\gamma}$ is strictly decreasing and convex, condition (4) must also hold.⁹ However, following our discussion in Section 2.2, inequalities in (3) and (4) may be satisfied simultaneously only if $\mu_x = \mu_y$ and τ_s is a mean-preserving spread of τ_t .

Although checking for the stochastic dominance is necessary for weakly present-biased rationalisation, it is not sufficient. Consider a revealed preference relation

$$(20, 3)R^*(10, 0), (5, 0)R^*(15, 4), (10, 3)R^*(5, 0), \text{ and } (15, 0)P^*(20, 2).$$

In particular, we have $\mu_x = \mu_y$, while the distribution τ_s of time delays on the right-hand side of the relation is a mean-preserving spread of the distribution τ_t over time delays on the left.¹⁰ However, the relation is not rationalisable in the above sense. If that were true, there would be functions $\tilde{u}$ and $\tilde{\gamma}$, specified as previously, such that if $(x, t)R^*(y, s)$ then $\tilde{u}(x) + \tilde{\gamma}(t) \geq \tilde{u}(y) + \tilde{\gamma}(s)$, while $\tilde{u}(15) + \tilde{\gamma}(0) > \tilde{u}(20) + \tilde{\gamma}(2)$. By summing up the inequalities, we would obtain $\tilde{\gamma}(3) - \tilde{\gamma}(4) > \tilde{\gamma}(2) - \tilde{\gamma}(3)$, or equivalently $\gamma(3)/\gamma(4) > \gamma(2)/\gamma(3)$, which would contradict that γ was weakly present-biased.¹¹

Axiom 2 (Cumulative dominance). *For any sample of R^* , if distribution μ_y first order stochastically dominates μ_x and τ_t second order stochastically dominates τ_s , then $\mu_x = \mu_y$ and τ_s is a mean-preserving spread of τ_t . Moreover, such a sample has no elements of P^* .*

It is easy to verify that this condition is stronger than Axiom 1. Apart from monotonicity and separability, Axiom 2 guarantees that the revealed preference obeys strong diminishing impatience introduced at the beginning of this subsection. Indeed, suppose that we observe $(x, t)R^*(y, s)$ and $(y, s + z)R^*(x, t + z)$, for $s \leq t$. By construction of this two-element sample, the induced distributions of rewards are equivalent, while time delays are distributed according to $\tau_t(t) = \tau_t(s + z) = \tau_s(s) = \tau_s(t + z) = 1/2$, hence, τ_s is a mean-preserving spread of τ_t . By Axiom 2, neither $(x, t)R^*(y, s)$ nor $(y, s + z)R^*(x, t + z)$ may be an element of P^* . Thus, we observe the two initial comparisons only if the agent is indifferent in either case, which is equivalent to strong diminishing impatience.

Theorem 2. *A revealed relation (R^*, P^*) is rationalisable by a weakly present-biased discounted utility function if and only if it obeys cumulative dominance.*

We postpone the proof until the appendix. As in the case of Theorem 1, the argument introduces a method of constructing a felicity u and a weakly present-biased discounting γ that rationalise the relation. Since such a construction can be reduced to a linear program, it is easily implementable to empirical analyses.

4.2 Exponential discounting

The logic applied to our characterisation of time preference in Sections 3 and 4.1 can be extended to other classes of models. We conclude this section by determining testable implications for *exponential discounting*. A pair (R^*, P^*) is rationalisable in this sense if there is a strictly increasing felicity $u : \mathbb{R}_+^\ell \rightarrow \mathbb{R}_{++}$ and a discount factor $\delta \in (0, 1)$ such that function $v(x, t) := \delta^t u(x)$ rationalises (R^*, P^*) as in (1) in Section 2.1.

As shown in Fishburn and Rubinstein (1982), the principal property of this class of time preference is *stationarity*. A relation R obeys this condition if $(x, t)R(y, s)$ implies

⁹Given that function γ is log-convex, it directly implies that $\tilde{\gamma}$ is convex.

¹⁰Clearly, distribution τ_t second order stochastically dominates τ_s and their means are equal.

¹¹This relation is rationalisable with the discounted utility from footnote 7.

$(x, t')R(y, s')$, for any rewards x, y and time delays t, t', s, s' such that $(t - s) = (t' - s')$. That is, as long as the delay between two prizes is unchanged, the dates of the two consumptions do not affect the choice of the agent.

Given that ratio $\delta^t/\delta^{t+1} = \delta^{-1}$ is constant with respect to t , exponential discounting is a special case of weak present bias. In particular, any relation (R^*, P^*) rationalisable in the above sense satisfies cumulative dominance. However, the condition is no longer sufficient. For example, consider a relation

$$(5, 0)R^*(10, 2), (10, 0)R^*(15, 1), (15, 4)R^*(5, 0),$$

with $P^* = \emptyset$. The distributions of rewards on both sides of the relation are equivalent. At the same time, distributions of time delays are given by $\tau_t(0) = 1 - \tau_t(4) = 2/3$ and $\tau_s(0) = \tau_s(1) = \tau_s(2) = 1/3$. In particular, τ_t does *not* dominate τ_s in the second order stochastic sense. If the relation was consistent with exponential discounting for some felicity u and discount factor δ , then $(x, t)R^*(y, s)$ would imply $\tilde{u}(x) + \tilde{\delta}t \geq \tilde{u}(y) + \tilde{\delta}s$, where $\tilde{u} := \log u$ and $\tilde{\delta} := \log \delta$. However, summing up the inequalities would imply

$$\tilde{\delta}(0 + 0 + 4) \geq \tilde{\delta}(0 + 1 + 2),$$

yielding a contradiction, since $\tilde{\delta} < 0$.¹² Note that this violation occurs only because the mean of τ_t is strictly greater than the mean of τ_s .

The general specification of this condition is summarised below. As previously, for any sample $\{((x_i, t_i), (y_i, s_i))\}_{i \in I}$ of R^* , denote distributions over rewards x_i, y_i and time delays t_i, s_i by μ_x, μ_y and τ_t, τ_s , respectively.

Axiom 3 (Mean dominance). *For any sample of R^* , distribution μ_y first order stochastically dominates μ_x and the mean of τ_t is greater than that of τ_s only if $\mu_x = \mu_y$ and the means of τ_t and τ_s are equal. Moreover, such a sample contains no elements of P^* .*

Axiom 3 generalises stationarity. Suppose we observe $(x, t)R^*(y, s)$ and $(y, s')R^*(x, t')$, where $(t - s) = (t' - s')$. Since the corresponding distributions of rewards in this two-element sample are equivalent, while $(t + s') = (t' + s)$, mean dominance requires that none of the above comparisons belongs to P^* . This is equivalent to stationarity.

Proposition 1. *A revealed preference relation (R^*, P^*) is rationalisable by an exponential discounted utility function if and only if it obeys mean dominance.*

We omit the proof as it is analogous to those supporting Theorems 1 and 2. As noted in Fishburn and Rubinstein (1982), in the prize-time framework it is impossible to identify the discount factor corresponding to the exponential discounted utility representation of a preference relation R .¹³ Nevertheless, it does not mean that this domain is unsuitable for a study of discounted utility. As shown in the example above, even though one can not identify particular values of the discounting parameter, the observable implications of each model are distinct even for a finite number of observations.

¹²However, the relation is rationalisable by a discounted utility with felicity function $u(x) = x/5$ and a weakly present-biased discounting satisfying $\gamma(0) = 1$ and $\gamma(t) := (3/4)(2/3)^t$, for $t > 0$.

¹³Indeed, if function $v(x, t) := \delta^t u(x)$ represents R , then so does $(v(x, t))^p := (\delta^p)^t (u(x))^p$, for any power $p > 0$. Varying p allow us to obtain any value of the discount factor δ^p .

5 On identification and indeterminacy

Thus far we focused on tests that determined if observations were consistent with a particular discounted utility model, while in the proofs we showed how to construct preferences that rationalise the data. A natural question is whether our methods allow us to estimate the original preferences that generated the observations. We address this issue in two ways. First, we show a general condition on data that is sufficient for identifying preferences as the number of observations becomes large. Then, we discuss an important class of revealed relations that conveys very little information on preferences.

5.1 Identification of time preference

Given that a revealed relation (R^*, P^*) is finite, there is a large number of preferences R that rationalise and extend it to the whole domain $\mathbb{R}_+^\ell \times \mathbb{Z}_+$ of prize-time pairs. This is true even within preferences consistent with discounted utility models. Therefore, it is important to specify conditions under which such an extension approximates the actual, true preference that generated the data.

To determine sufficient conditions under which estimating preferences is possible, we turn to the result in [Chambers et al. \(2017\)](#). Suppose that an agent is endowed with a complete, continuous, and monotone preference relation $\succeq$, where we define monotonicity as in [Section 2.1](#), i.e., if $x \geq y$ and $t \leq s$ then $(x, t) \succeq (y, s)$, with the relation being strict if at least one of the inequalities is strict. In such a case, a revealed preference relation R^* is a finite subset of elements in $\succeq$. By applying Theorem 8 in the aforementioned paper, it is possible to show that as the number of observations in R^* increases and becomes dense, any complete, continuous, and monotone relation R that rationalises R^* must converge to the original preference $\succeq$ that generated the data. Thus, it is possible to approximate the consumer preference given an increasingly complete dataset.¹⁴

Following our results from [Sections 3 and 4](#), whenever $\succeq$ is continuous and representable within a particular class of discounted utility models (e.g., the general, weakly present-biased, or exponential) any revealed preference relation R^* generated by $\succeq$ is rationalisable with a relation R that belongs to the same class of preferences. Following [Chambers et al. \(2017\)](#), as the researcher observes more comparisons from $\succeq$ so that R^* becomes dense, any such relation R converges to $\succeq$. Therefore, our methods allow to approximate consumer preference if the set of observations is large.

5.2 Anchored relations and indeterminacy

Here, we discuss an empirically relevant class of revealed preference relations that does not allow the researcher to distinguish among any of the discounted utility models. A revealed preference relation (R^*, P^*) is *anchored* if there is some prize-time pair (x^*, t^*) such that $(x, t) R^* (y, s)$ implies $(x, t) = (x^*, t^*)$ or $(y, s) = (x^*, t^*)$. In such cases the observer can monitor comparisons only with respect to a single fixed option (x^*, t^*) .

There are several notable experiments that induce anchored revealed preference relations, e.g., see [Pender \(1996\)](#), [Coller and Williams \(1999\)](#), [Harrison et al. \(2002\)](#), [Andersen et al. \(2008, 2014\)](#), or [Coller et al. \(2012\)](#).¹⁵ One advantage of these tests is that choices

¹⁴Precisely, relation R approaches $\succeq$ with respect to the closed convergence.

¹⁵It is crucial to point out that, apart from preference over prize-time pairs, the experiments in [Andersen et al. \(2008, 2014\)](#) were simultaneously studying choices over risky outcomes. Clearly, in this section

that subjects face in these studies are relatively simple, which minimises the chance of errors made by the agents. However, the simplicity of the design reduces empirical content of observations generated in those experiments.

Proposition 2 (Indeterminacy). *Suppose that rewards x are in $\mathbb{R}_+$. An anchored relation (R^*, P^*) is rationalisable by a monotone utility if and only if for any strictly decreasing discounting $\gamma : \mathbb{Z}_+ \rightarrow (0, 1]$ there is a strictly increasing felicity $u : \mathbb{R}_+ \rightarrow \mathbb{R}_{++}$ such that $v(x, t) := u(x)\gamma(t)$ rationalises (R^*, P^*) according to (1) in Section 2.1.*

Anchored revealed preference relations do not allow for a distinction between any specifications of the discounted utility model. In fact, Proposition 2 implies that any relation rationalisable by a monotone utility function can be supported by virtually *any* form of the discounted utility model. This is due to a significant level of incompleteness of the revealed preference. Since all observable comparisons are made with respect to a single fixed option (x^*, t^*) , such relations admit a great variety of transitive closures and allow to refute only the class of monotone time preference. Moreover, even as the number of observations tends to infinity, such datasets are never sufficiently spread out. Since Theorem 8 in Chambers et al. (2017) is not applicable in such a case, this class of experiments does not allow to approximate the preference generating the data.

A Appendix

We proceed with proofs of the results presented in the main body of the paper. Unless stated otherwise, we use the notation introduced earlier.

A.1 Proof of Theorem 1

Necessity of the axiom for discounted utility rationalisation is implied by the argument presented in Section 3. We prove that the condition is sufficient. Suppose that (R^*, P^*) obeys dominance. First, we claim that there are numbers $(\phi_x)_{x \in \mathcal{X}}$ and $(\varphi_t)_{t \in \mathcal{T}}$ such that (i) $x > y$ implies $\phi_x > \phi_y$, (ii) $t > s$ implies $\varphi_t < \varphi_s$, and (iii) $(x, t)R^*(y, s)$ implies $\phi_x + \varphi_t \geq \phi_y + \varphi_s$, where the inequality is strict if $(x, t)P^*(y, s)$.

For each $x \in \mathcal{X}$, let ϵ_x denote a vector with $|\mathcal{X}|$ entries equal to 0 apart from the one corresponding to x which is equal to 1. Define vector ϵ_t of length $|\mathcal{T}|$ analogously, for each $t \in \mathcal{T}$. In addition, for any two rewards x and y such that $x > y$, a *monotone shift* is a vector $\rho_{xy}^x := (\epsilon_x - \epsilon_y)$. Denote the set of all such vectors by $\mathcal{R}_{\mathcal{X}}$. Similarly, for any two time delays t, s such that $t > s$, the corresponding monotone shift is given by $\varrho_{st}^t := (\epsilon_t - \epsilon_s)$. Let $\mathcal{R}_{\mathcal{T}}$ be the set of all such vectors.

Label elements in R^* with indices $j \in J$ such that $R^* = \{((x_j, t_j), (y_j, s_j))\}_{j \in J}$. In order to prove our claim, we need to find vectors $\phi \in \mathbb{R}^{|\mathcal{X}|}$ and $\varphi \in \mathbb{R}^{|\mathcal{T}|}$ such that (i) $\rho \cdot \phi > 0$, for all $\rho \in \mathcal{R}_{\mathcal{X}}$, (ii) $-\varrho \cdot \varphi > 0$, for all $\varrho \in \mathcal{R}_{\mathcal{T}}$, while (iii) $(x_j, t_j)R^*(y_j, s_j)$ implies $(\epsilon_{x_j} - \epsilon_{y_j}) \cdot \phi + (\epsilon_{t_j} - \epsilon_{s_j}) \cdot \varphi \geq 0$, where the inequality is strict if $(x_j, t_j)P^*(y_j, s_j)$.

If such vectors do *not* exist, by Motzkin's Transposition (see Theorem 1 in the Online supplement) there are positive integers $(\lambda_j)_{j \in J}$, $(\theta_\rho)_{\rho \in \mathcal{R}_{\mathcal{X}}}$, and $(\vartheta_\varrho)_{\varrho \in \mathcal{R}_{\mathcal{T}}}$ such that

$$\sum_{j \in J} \lambda_j \epsilon_{x_j} + \sum_{\rho \in \mathcal{R}_{\mathcal{X}}} \theta_\rho \rho = \sum_{j \in J} \lambda_j \epsilon_{y_j} \quad \text{and} \quad \sum_{j \in J} \lambda_j \epsilon_{s_j} + \sum_{\varrho \in \mathcal{R}_{\mathcal{T}}} \vartheta_\varrho \varrho = \sum_{j \in J} \lambda_j \epsilon_{t_j}, \quad (\text{A1})$$

we refer only to the revealed preference relations defined over the domain of rewards and time delays, neglecting the additional information on preference over lotteries.

where either $\theta_\rho > 0$, for some $\rho \in \mathcal{R}_\mathcal{X}$, $\vartheta_\varrho > 0$, for some $\varrho \in \mathcal{R}_\mathcal{T}$, or $\lambda_j > 0$, for some $j \in J$ such that $(x_j, t_j)P^*(y_j, s_j)$. In either case, we have $\lambda_j > 0$, for some $j \in J$.

For each $x \in \mathcal{X}$, let $K_x := \{j \in J : x_j = x\}$ be the set of all indices $j \in J$ for which x_j is equal to x . Analogously, let $L_x := \{j \in J : y_j = x\}$. Define distributions

$$\mu_x(z) := \frac{\sum_{j \in K_x} \lambda_j}{\sum_{j \in J} \lambda_j} \quad \text{and} \quad \mu_y(z) := \frac{\sum_{j \in L_x} \lambda_j}{\sum_{j \in J} \lambda_j}. \quad (\text{A2})$$

Similarly, for each $t \in \mathcal{T}$, construct sets $M_t := \{j \in J : t_j = t\}$ and $N_t := \{j \in J : s_j = t\}$ containing all indices $j \in J$ for which t is equal to t_j and s_j , respectively. Moreover, let

$$\tau_t(z) := \frac{\sum_{j \in M_t} \lambda_j}{\sum_{j \in J} \lambda_j} \quad \text{and} \quad \tau_s(z) := \frac{\sum_{j \in N_t} \lambda_j}{\sum_{j \in J} \lambda_j}. \quad (\text{A3})$$

By construction of the distributions μ_x , μ_y and τ_t , τ_s , condition (A1) implies that

$$\mu_x + \sum_{\rho \in \mathcal{R}_\mathcal{X}} \hat{\theta}_\rho \rho = \mu_y \quad \text{and} \quad \tau_s + \sum_{\varrho \in \mathcal{R}_\mathcal{T}} \hat{\vartheta}_\varrho \varrho = \tau_t,$$

where $\hat{\theta}_\rho := \theta_\rho / \sum_{j \in J} \lambda_j$, for all $\rho \in \mathcal{R}_\mathcal{X}$, and $\hat{\vartheta}_\varrho := \vartheta_\varrho / \sum_{j \in J} \lambda_j$, for all $\varrho \in \mathcal{R}_\mathcal{T}$. Following Proposition 1 in the [Online supplement](#), the above condition implies that distribution μ_y dominates μ_x and τ_t dominates τ_s in the first order stochastic sense. Moreover, if $\hat{\theta}_\rho > 0$, for some $\rho \in \mathcal{R}_\mathcal{X}$, then $\mu_x \neq \mu_y$; if $\hat{\vartheta}_\varrho > 0$, for some $\varrho \in \mathcal{R}_\mathcal{T}$, then $\tau_t \neq \tau_s$. Otherwise, we have $\lambda_j > 0$ for some $j \in J$ such that $(x_j, t_j)P^*(y_j, s_j)$.

Construct a sample $\{((x_i, t_i), (y_i, s_i))\}_{i \in I}$ of R^* , where each element $((x_j, t_j), (y_j, s_j))$ of the relation is taken λ_j times, $j \in J$. By construction, the corresponding distributions of rewards x_i , y_i and time delays t_i , s_i in the collection are given by μ_x , μ_y and τ_t , τ_s , respectively. Our previous observation implies that either $\mu_x \neq \mu_y$, $\tau_t \neq \tau_s$, or the sample contains an element of P^* . This contradicts that (R^*, P^*) obeys dominance. Thus, there are numbers $(\phi_x)_{x \in \mathcal{X}}$ and $(\varphi_t)_{t \in \mathcal{T}}$ satisfying the inequalities specified at the beginning.

In order to complete the proof, take any such numbers. Define a strictly increasing function $\phi : \mathcal{X} \rightarrow \mathbb{R}$ by $\phi(x) := \phi_x$. By [Husseinov \(2010, Corollary 2\)](#), it admits a strictly increasing, continuous extension $\tilde{u}$ to $\mathbb{R}_+^\ell$. Similarly, function $\varphi : \mathcal{T} \rightarrow \mathbb{R}$, given by $\varphi(t) := \varphi_t$, has a strictly decreasing extension $\tilde{\gamma}$ to $\mathbb{Z}_+$. Define $u : \mathbb{R}_+^\ell \rightarrow \mathbb{R}_{++}$ by $u := \exp \tilde{u}$ which is strictly increasing and continuous. Analogously, let $\gamma : \mathbb{Z}_+ \rightarrow (0, 1]$ be given by $\gamma := \exp(\tilde{\gamma} - \tilde{\gamma}(0))$, which is strictly decreasing and satisfies $\gamma(0) = 1$. Since $u(x)\gamma(t) = \exp(\phi_x + \varphi_t - \tilde{\gamma}(0))$, for any pair $(x, t) \in \mathcal{X} \times \mathcal{T}$, function $v(x, t) := u(x)\gamma(t)$ rationalises the revealed preference relation (R^*, P^*) .

A.2 Proof of Theorem 2

The necessity of the axiom follows from the argument presented in the main body of the paper. We focus on the sufficiency part of the result. Suppose that (R^*, P^*) obeys cumulative dominance. We claim that there are numbers $(\phi_x)_{x \in \mathcal{X}}$, $(\varphi_t)_{t \in \mathcal{T}}$, and $(v_t)_{t \in \mathcal{T}}$ such that (i) $x > y$ implies $\phi_x > \phi_y$, (ii) v_t is strictly negative, for all $t \in \mathcal{T}$, (iii) for all $t \in \mathcal{T}$, we have $\varphi_t + v_t(s - t) \leq \varphi_s$, for all $s \in \mathcal{T}$, and (iv) $(x, t)R^*(y, s)$ implies $\phi_x + \varphi_t \geq \phi_y + \varphi_s$, where the inequality is strict if $(x, t)P^*(y, s)$.

As in the previous proof, we label elements in R^* with indices $j \in J$ such that $R^* = \{((x_j, t_j), (y_j, s_j))\}_{j \in J}$. Moreover, define vectors ϵ_x and ϵ_t as above, for each $x \in \mathcal{X}$

and $t \in \mathcal{T}$, and let $\mathcal{R}_{\mathcal{X}}$ be the set of monotone shifts ρ on $\mathcal{X}$. Finally, for all time delays $t \neq s$, a *shift* is a vector $\pi_s^t := (\varepsilon_t - \varepsilon_s)$. Denote the set of all such shifts on $\mathcal{T}$ by $\mathcal{P}_{\mathcal{T}}$.

In order to prove our claim, we need to find vectors $\phi \in \mathbb{R}^{|\mathcal{X}|}$ and $\varphi, v \in \mathbb{R}^{|\mathcal{T}|}$ such that (i) $\rho \cdot \phi > 0$, for all $\rho \in \mathcal{R}_{\mathcal{X}}$, (ii) $-v \gg 0$, (iii) $-\pi_s^t \cdot \varphi - (s - t)v_t \geq 0$, for all $t \neq s$, and (iv) $(x_j, t_j)R^*(y_j, s_j)$ implies $(\epsilon_{x_j} - \epsilon_{y_j}) \cdot \phi + (\varepsilon_{t_j} - \varepsilon_{s_j}) \cdot \varphi \geq 0$, where the inequality is strict whenever $(x_j, t_j)P^*(y_j, s_j)$. We show this by contradiction.

Suppose the vectors do not exist. By Motzkin's Transposition, there are some positive integers $(\lambda_j)_{j \in J}$, $(\theta_\rho)_{\rho \in \mathcal{R}_{\mathcal{X}}}$, $(\vartheta_\pi)_{\pi \in \mathcal{P}_{\mathcal{T}}}$, and $(\kappa_t)_{t \in \mathcal{T}}$ satisfying

$$\sum_{j \in J} \lambda_j \epsilon_{x_j} + \sum_{\rho \in \mathcal{R}_{\mathcal{X}}} \theta_\rho \rho = \sum_{j \in J} \lambda_j \epsilon_{y_j} \quad \text{and} \quad \sum_{j \in J} \lambda_j \varepsilon_{s_j} + \sum_{\pi \in \mathcal{P}_{\mathcal{T}}} \vartheta_\pi \pi = \sum_{j \in J} \lambda_j \varepsilon_{t_j}, \quad (\text{A4})$$

while $\sum_{s \in \mathcal{T}} \vartheta_\pi^s (s - t) = -\kappa_t$, for all $t \in \mathcal{T}$. Moreover, either $\theta_\rho > 0$, for some $\rho \in \mathcal{R}_{\mathcal{X}}$, $\kappa_t > 0$, for some $t \in \mathcal{T}$, or $\lambda_j > 0$, for some $j \in J$ such that $(x_j, t_j)P^*(y_j, s_j)$. Either way, we have $\lambda_j > 0$, for some $j \in J$. Construct distributions μ_x, μ_y and τ_t, τ_s as in (A2) and (A3), respectively. In addition, denote $\hat{\theta}_\rho := \theta_\rho / \sum_{j \in J} \lambda_j$, for all $\rho \in \mathcal{R}_{\mathcal{X}}$, $\hat{\vartheta}_\pi := \vartheta_\pi / \sum_{j \in J} \lambda_j$, for all $\pi \in \mathcal{P}_{\mathcal{T}}$, and $\hat{\kappa}_t := \kappa_t / \sum_{j \in J} \lambda_j$, for all $t \in \mathcal{T}$. By (A4):

$$\mu_x + \sum_{\rho \in \mathcal{R}_{\mathcal{X}}} \hat{\theta}_\rho \rho = \mu_y \quad \text{and} \quad \tau_s + \sum_{\pi \in \mathcal{P}_{\mathcal{T}}} \hat{\vartheta}_\pi \pi = \tau_t,$$

with $\sum_{s \in \mathcal{T}} \hat{\vartheta}_\pi^s (s - t) = -\hat{\kappa}_t$, for all $t \in \mathcal{T}$. By Propositions 1 and 3 in the [Online supplement](#), this implies that μ_y first order stochastically dominates μ_x and τ_t second order stochastically dominates τ_s . Moreover, if $\hat{\theta}_\rho > 0$, for some $\rho \in \mathcal{R}_{\mathcal{X}}$, then $\mu_x \neq \mu_y$; if $\hat{\kappa}_t > 0$, for some $t \in \mathcal{T}$, then τ_s is not a mean-preserving spread of τ_t . Otherwise, we have $\lambda_j > 0$ for some $j \in J$ such that $(x_j, t_j)P^*(y_j, s_j)$. By an argument analogous to the one in the proof of Theorem 1, there is a sample that violates cumulative dominance.

Take any numbers $(\phi_x)_{x \in \mathcal{X}}$, $(\varphi_t)_{t \in \mathcal{T}}$, and $(v_t)_{t \in \mathcal{T}}$ that solve the system of inequalities specified at the beginning of the proof. Define function u as in the previous subsection. Moreover, let $\tilde{\gamma} : \mathbb{Z}_+ \rightarrow \mathbb{R}$ be given by $\tilde{\gamma}(t) := \max_{s \in \mathcal{T}} \{\varphi_s + v_s(t - s)\}$, which is strictly decreasing, has a convex extension to $\mathbb{R}$, and $\tilde{\gamma}(t) = \varphi_t$, for all $t \in \mathcal{T}$. Define function $\gamma : \mathbb{Z}_+ \rightarrow (0, 1]$ by $\gamma := \exp(\tilde{\gamma} - \tilde{\gamma}(0))$. Clearly $v(x, t) := u(x)\gamma(t)$ rationalises (R^*, P^*) .

A.3 Proof of Proposition 2

Throughout this subsection we assume that (R^*, P^*) is an anchored revealed preference relation with a fixed option (x^*, t^*) . Implication $(\Leftarrow)$ is straightforward. To show $(\Rightarrow)$, suppose that (R^*, P^*) is rationalisable by a monotone function v . Since the revealed relation is finite, we may assume that the function is continuous, while $v(\cdot, t)$ is unbounded from above, for all $t \in \mathcal{T}$, without loss of generality. For each $t \in \mathcal{T}$, let number z_t satisfy $v(z_t, t) = v(x^*, t^*)$. By continuity, monotonicity, and unboundedness of v such a number exists and is unique. In addition, note that $t > s$ implies $z_t > z_s$, while $(x, t)R^*(x^*, t^*)$ implies $x \geq z_t$, where the inequality is strict if $(x, t)P^*(x^*, t^*)$. Finally, we have $z_{t^*} = x^*$.

Take any discounting function γ and a number $u^* > 0$. Define $u : \{z_t\}_{t \in \mathcal{T}} \rightarrow \mathbb{R}_{++}$ by $u(z_t) := u^* \gamma(t^*) / \gamma(t)$. As γ strictly decreases, u is strictly increasing. Clearly, it admits a continuous and strictly increasing extension $\bar{u}$ to $\mathbb{R}_+$.

Whenever $(x^*, t^*)R^*(x, t)$ then $\bar{u}(x) \leq \bar{u}(z_t) = u(z_t) = u^* \gamma(t^*) / \gamma(t)$, which implies that $u(x)\gamma(t) \leq u^* \gamma(t^*) = u(x^*)\gamma(t^*)$. Analogously, for any $(x, t)R^*(x^*, t^*)$ it must be that $u(x)\gamma(t) \geq u(x^*)\gamma(t^*)$. Finally, the inequalities are strict for any element of P^* .

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