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Optimal Matterwave Gravimetry

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We calculate quantum and classical Fisher informations for gravity sensors based on matterwave interference, and find that current Mach-Zehnder interferometry is not optimally extracting the full metrological potential of these sensors. We show that by making measurements that resolve either the momentum or the position we can considerably improve the sensitivity. We also provide a simple modification that is capable of more than doubling the sensitivity.

Atom interferometry is a leading inertial-sensing technology, having demonstrated state-of-the-art gravimetry [1–7] and gradiometry [8–14] measurements. Nevertheless, orders of magnitude improvement in sensitivity is required for applications in navigation [15] and mineral exploration [16], as well as improved tests of the equivalence principle [17–19] and quantum gravity [20, 21]. For the commonly-used Mach-Zehnder [i.e. Kasevich-Chu (KC)] configuration [22, 23], semiclassical calculations [24–27] reveal that the matterwave accrues relative phase \( \phi = g \cdot k_L T^2 \), where \( g \) is the gravitational acceleration, \( \hbar k_L \) is the momentum separation of the two arms, and \( 2T_\pi \) is the total interrogation time. Assuming \( N \) uncorrelated particles, a population-difference measurement at the interferometer output yields sensitivity

\[
\Delta g = \frac{1}{\sqrt{N k_0 T^2}} , \tag{1}
\]

where \( k_0 \) is the component of \( k_L \) aligned with \( g \). Equation (1) implies only four routes to improved sensitivity: (1) increase interrogation time, (2) increase the momentum separation of the arms (e.g. via large momentum transfer beam splitters [28–32]), (3) increase the atom flux, and/or (4) surpass the shot-noise limit with quantum correlations [33–37]. Although all routes are worth pursuing, each has unique limitations. For instance, size, weight, and power constraints limit both \( T_\pi \) and the maximum momentum transferable via laser pulses. Additionally, evaporative-cooling losses and momentum width requirements constrain atom fluxes [38–42]. Increases with number-conserving feedback cooling are possible, but untested [43–45]. Finally, quantum-correlated states must be compatible with the requirements of high-precision metrology [5, 46–57] (e.g. high atom flux, low phase diffusion), and will only be advantageous if classical noise sources (e.g. [58, 59]) are sufficiently controlled to yield shot-noise-limited operation prior to quantum enhancement.

This assessment assumes that Eq. (1) is the optimal sensitivity. In this Letter, we prove this conventional wisdom false by showing that matterwave interferometers can attain better sensitivities than Eq. (1). Ultimately, the gravitational field affects the quantum state beyond the creation of a simple phase shift. We show this additional metrological potential via the quantum Fisher information (QFI), which determines the best possible sensitivity. We further determine the set of measurements required to attain this optimal sensitivity via the classical Fisher information (CFI). Our analysis reveals additional routes to improved sensitivity, such as variations in the measurement procedure and input source, and these should be considered when designing future matterwave gravimeters. We also present a modified interferometer that more than doubles the sensitivity for the same interrogation time and momentum separation.

We briefly review KC interferometry based on state-changing Raman transitions [Fig. 1(a)], although our results also hold for Bragg transitions [5] and Bloch oscillations [29] in the appropriate regime. At time \( t = 0 \) atoms with two internal states \( \ket{a} \) and \( \ket{b} \), initially in \( \ket{a} \), are excited to an equal superposition of \( \ket{a} \) and \( \ket{b} \) via a coherent \( \pi/2 \) pulse. Atoms transferred to \( \ket{b} \) also receive a momentum kick \( \hbar k_0 \). At \( t = T_\pi \), a \( \pi \) pulse acts as a mirror, before the two matterwaves are interfered at \( t = T = 2T_\pi \) by a second \( \pi/2 \) pulse.

FIG. 1. Spacetime diagrams for (a) KC interferometry and (b) Ramsey interferometry (no mirror pulse), which are both sensitive to gravitational fields and accelerations.

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I. QFI FOR A PARTICLE IN A GRAVITATIONAL FIELD

The quantum Cramer-Rao bound (QCRB) gives a lower bound on the sensitivity [60]. For $N$ uncorrelated particles this is $\Delta g^2 \geq 1/(NF_Q)$, where $F_Q$ is the single-particle QFI [61–63], which for a pure single-particle state $|\Psi\rangle$ is

$$F_Q = 4 \left( (\partial_\Psi \Psi | \partial_\Psi \Psi)^2 - |\Psi | \partial_\Psi \Psi |^2 \right).$$

(2)

For the KC interferometer, semiclassical arguments give $|\Psi \rangle = \frac{1}{\sqrt{2}}(|a\rangle + e^{ig_0} e^{i\theta} |b\rangle)$ before the final beam splitter and a QFI $F_Q^{sc} = k_0^2 T^4_\pi$ [64], consistent with Eq. (1). However, this derivation treats the particle’s motion semiclassically, neglecting the noncommutability of position and momentum. We account for this here. For the moment we consider only the centre of mass degrees of freedom. In the presence of a uniform gravitational field $g$ acting along the $z$-axis, a particle of mass $m$ in state $|\psi_0\rangle$ evolves to $|\psi(T)\rangle = \hat{U}_g |\psi_0\rangle$ after time $T$, where

$$\hat{U}_g = \exp[-\frac{m}{\hbar} \left( \frac{\hat{E}^2}{2m} + mg\hat{z} \right)].$$

As shown in Appendix A, we can rewrite

$$\hat{U}_g = e^{-\frac{i}{\hbar} \frac{\hat{E}^2}{2m} e^{-ig\hat{G}_0(T)} e^{im\frac{\pi^2}{2m}}},$$

(3)

where

$$\hat{G}_0(T) = \frac{T}{\hbar} \left( \frac{T}{2} \hat{p}_z + m\hat{z} \right).$$

(4)

The QFI is

$$F_Q(T) = 4 \text{Var}(G_0(T))$$

$$= \frac{T^4_\pi}{2} \text{Var}(p_z) + \frac{4m^2 \pi^2}{\hbar^2} \text{Var}(z) + \frac{4mT^3}{\hbar^2} \text{Cov}(p_z, z),$$

(5a)

where the variances and covariance are evaluated with respect to $|\psi_0\rangle$. To compare Eq. (5a) and $F_Q^{sc}$, consider a state $|\psi_0\rangle$ with two well-defined peaks in momentum space separated by $\hbar k_0$, giving $\text{Var}(p_z) \approx (\hbar k_0)^2$. For sufficiently large $k_0$ and $T$ such that $(\hbar k_0 T/2)^2 \gg m^2 \text{Var}(z)$, $mT \text{Cov}(p_z, z)$, the first term of Eq. (5b) dominates, and $F_Q(2T_\pi) \approx k_0^2 T^{4}_\pi = F_Q^{sc}$. However, the additional terms in Eq. (5b) potentially allow sensitivities better than Eq. (1).

II. QFI FOR KC INTERFEROMETRY

Equation (5a) is not the QFI for a KC interferometer, as we must account for the internal state degrees of freedom as well as the action of the mirror pulse. The evolution is given by

$$\hat{U}_{KC} = \hat{U}_g^{\phi_0} \hat{U}_g(T_2) \hat{U}_g^{\phi_2} \hat{U}_g(T_1) \hat{U}_g^{\phi_1},$$

(6)

where

$$\hat{U}_g^{\phi} = \hat{I} \cos \left( \frac{\phi}{2} \right) - i |b\rangle \langle a| e^{i(k_0 z - \phi)} + \text{h.c.} \sin \left( \frac{\phi}{2} \right).$$

(7)

governs the beam splitter and mirror dynamics. As shown in Appendix D, Eq. (7) is an excellent approximation to the beam splitting and mirror dynamics when pulse duration is much shorter than the timescale for atomic motional dynamics. Here $T_{1(2)}$ are evolution times before (after) the $\pi$ pulse and $\phi$ is the pulse phase, controlled via the relative phase of the two Raman lasers. The first $\pi/2$ pulse maps the initial state $|\Psi_0\rangle = |a\rangle |\psi_0\rangle$ to $|\Psi'_0\rangle = (|a\rangle - i e^{i(k_0 z - \phi_1)} |b\rangle) |\psi_0\rangle$, where $|\psi_0\rangle$ contains the initial state’s motional degrees of freedom. As detailed in Appendix B,

$$|\Psi(T)\rangle = \hat{U}_{KC} |\Psi_0\rangle = \hat{U}_0 e^{-iG_0(T) + G_\phi} |\Psi'_0\rangle,$$

(8)

where

$$\hat{G}_c = \hat{S}_z k_0^2 z^2,$$

(9a)

$$\hat{S}_z = \frac{1}{2} (|a\rangle \langle a| - |b\rangle \langle b|),$$

(9b)

$$\hat{U}_0 = \hat{U}_g^{\phi_0} e^{-i \frac{\pi}{2} \frac{g_0^2}{2m} \hat{E}^2 \hat{z}^2} \hat{U}_g^{\phi_2} e^{-i \frac{\pi}{2} \frac{g_0^2}{2m} \hat{E}^2 \hat{z}^2},$$

(9c)

and $T = T_1 + T_2$, giving QFI

$$F_Q^{KC}(T) = 4 \text{Var}(G_0(T)) + \frac{1}{2} k_0^2 (T^2 - 2T^2_\pi)^2,$$

(10)

where $\text{Var}(G_0(T))$ is taken with respect to $|\psi_0\rangle$. For $T_1 = T_2 = T_\pi$,

$$F_Q^{KC}(T) = 4 \text{Var}(G_0(T)) + k_0^2 T^4_\pi.$$

(11)

Since $\text{Var}(G_0(T)) \geq 0$, this implies $F_Q^{KC} \geq F_Q^{sc}$, thereby permitting sensitivities better than Eq. (1).

III. CLASSICAL FISHER INFORMATION

Although the QFI gives the best possible sensitivity, it is silent on how to achieve this sensitivity. The attainable sensitivity for a particular measurement choice is given by the CFI, which quantifies the information contained in the probability distribution constructed from measurements of a particular observable, and necessarily depends upon this choice of observable. We calculate the CFI via

$$F_C(\lambda) = \int d\lambda \frac{[\partial_\lambda P(\lambda)]^2}{P(\lambda)},$$

(12)

where $P(\lambda)$ is the probability of obtaining result $\lambda$ when the observable $\lambda$ is measured [61, 62]. The CFI is bounded by the QCRB $F_C \leq F_Q$, so a measurement that saturates this bound is the optimal measurement.

A. CFI for population-difference measurement

For the standard population-difference measurement at the KC interferometer output, $\lambda = \hat{S}_z$ and $F_C(\hat{S}_z) = \sum_{s = a, b} (\partial_\lambda P_s)^2 / P_s$, where $P_s = \int dz |\langle z | \Psi(T)\rangle|^2$. As
detailed in Appendix C, an analytic solution exists in this case. Specifically,

\[
P_a = \frac{1}{2}(1 + |C| \sin \alpha),
\]

\[
P_b = \frac{1}{2}(1 - |C| \sin \alpha),
\]

yielding

\[
F_C(\hat{S}_z) = \frac{|C|^2 \cos^2 \alpha}{1 - |C|^2 \sin^2 \alpha} k_0^2 \left( T_2^2 - T_1^2 \right)^2,
\]

where

\[
C = \langle \psi_0 | e^{i \frac{k_0}{m}(T_2 - T_1) \hat{p} \cdot 1} | \psi_0 \rangle = |C| e^{i \phi},
\]

\[
\alpha = \phi_f - \phi_g + \theta,
\]

with \( \phi_f = \frac{\hbar k}{mt} (T_2 - T_1) \) and \( \phi_g = k_0 g (T_2^2 - T_1^2) \). The contrast \( |C| \) is determined by the spatial overlap of the two output wavepackets, since \( \frac{\hbar k_0}{mt} (T_2 - T_1) \) is the spatial separation. This depends strongly on the time difference \( T_2 - T_1 \). For an initial Gaussian state \( |\psi(0)\rangle = \exp(-z^2/2\sigma^2)/(\pi \sigma)^{1/4} \), \( |C| = \exp[-\frac{\hbar k_0^2}{2m^2} (T_2 - T_1)^2] \).

Figure 2(a) shows the time dependence of the QFI and \( F_C(\hat{S}_z) \) for this initial Gaussian state. Here \( t = T_1 + T_2 \), we fix \( T_2 \) so the mirror pulse always occurs at \( t = T_\pi \), and the second beam splitter occurs instantaneously before measurement. Explicitly, if \( t \leq T_\pi \), then \( T_1 = t \), \( T_2 = 0 \), and the mirror pulse has no meaningful effect; if \( t > T_\pi \) then \( T_1 = T_\pi \) and \( T_2 = t - T_\pi \). When \( T_1 \) and \( T_2 \) are significantly different, the spatial overlap of the two modes at the interferometer output is poor, so both the contrast and CFI are close to zero. However, \( |C| = 1 \) when \( T_1 = T_2 \) and \( F_C(\hat{S}_z) = F_Q^{ KC} = k_0^2 T_\pi^4 \), giving the same sensitivity as Eq. (1). This is still less than \( F_Q^{ KC} \), indicating that a different measurement could yield improved sensitivities.

**B. CFI for momentum-distribution measurement**

Now consider a measurement that distinguishes internal states and fully resolves the \( z \)-component of the final momentum distribution, as reported in Ref. [65]. This measurement yields CFI

\[
F_C(\hat{S}_z, \hat{p}_z) = \sum_{s,a,b} \int dp_z \frac{[\partial_s P_s(p_z)]^2}{P_s(p_z)},
\]

where \( P_s(p_z) = |\langle s | (p_z | \Psi(T) \rangle|^2 \). Although no analytic formula exists for \( F_C(\hat{S}_z, \hat{p}_z) \), the probabilities can be determined by numerically solving the Schrödinger equation, and the CFI computed from finite differences of these probabilities [63]. This requires an explicit choice of \( g \); although we consider the sensitivity near \( g = 0 \) for all numerical calculations, a large offset in \( g \) is easily accounted for by adjusting the beam splitter phases, as in typical atomic gravimeters [41].

Figure 2(a) shows that \( F_C(\hat{S}_z, \hat{p}_z) \) is significantly larger than \( F_C(\hat{S}_z) \) and very close to \( F_Q^{ KC} \). Additionally, \( F_C(\hat{S}_z, \hat{p}_z) \approx F_Q^{ KC} \) even when \( T_1 \) and \( T_2 \) are vastly different. This is because \( P_s(p_z) \) displays interference fringes that are not present in \( P_s = \int dp_z P_s(p_z) \) when spatial overlap is poor.

The origin of the increased information in \( F_C(\hat{S}_z, \hat{p}_z) \) compared with \( F_C(\hat{S}_z) \) is easily understood. Additional to the CFI associated with population exchange (generated by \( \hat{G}_e \)), there is information due to a shift in the momentum distribution. Concretely, consider initial momentum distribution \( F_0(p_z) \). Under gravity, \( \hat{p}_z(t) = p_z(0) + mg t \), so \( P(p_z, t) = P_0(p_z - mg t) \), giving

\[
F_C(p_z) = \int dp_z \frac{[\partial_s P_s(p_z)]^2}{P_s(p_z)} = \int dp_z \frac{[\partial_s P_0(p_z)]^2}{P_0(p_z)} = (mt)^2 P_C^{ P_0},
\]

where \( F_C^{ P_0} \) is the CFI associated with resolvable small shifts in the momentum distribution. For the initial Gaussian considered in Fig. 2(a), adding this additional CFI to \( F_C(\hat{S}_z) \) gives \( F_C(\hat{S}_z, \hat{p}_z)|_{2T_\pi} = F_Q^{ KC} + 8(mT_\pi \sigma \hbar)^2 \), in perfect agreement with our numerics. Note that this additional information is not the result of a phase shift so, unlike a standard KC interferometer, it is not affected by additional phase noise.

Our simulations also find near-perfect correlations between internal and momentum states, so a measurement that only resolves momentum (and not \( z \)-states) also has CFI approximating \( F_C(\hat{S}_z, \hat{p}_z)|_{2T_\pi} \), since an atom’s internal state is inferred from its final momentum. Our analysis therefore holds for interferometers that do not change internal states, such as Bragg-scattering-based interferometers, provided \( \hbar k_0 \gg \delta p \), the wavepacket’s initial momentum width [5, 28]. In our simulations \( \hbar k_0 \approx 14 \delta p \).

**C. CFI for position-distribution measurement**

Although the momentum distribution cannot always be resolved, a measurement of the position distribution might be possible. Here the CFI is

\[
F_C(\hat{S}_z, \hat{z}) = \sum_{s,a,b} \int dz \frac{[\partial_s P_s(z)]^2}{P_s(z)},
\]

where \( P_s(z) = |\langle s | (z | \Psi(t) \rangle|^2 \). Figure 2(a) shows this is slightly better than the population-difference measurement, although significantly worse than the momentum measurement. Arguing as before, since the position distribution shifts due to \( \hat{z}(t) = \hat{z}(0) + \hat{p}_z(0) t/m + \frac{1}{2} g t^2 \), the additional CFI is \( (t^2/2) F_C^{ SC} \), where \( F_C^{ SC} = \int dz [\partial_s P(z)]^2 / P(z) \) is the CFI associated with resolvable
shifts in the position distribution. Since

\[ \text{Var}(z(t)) = \text{Var}(z(0)) + \frac{t^2}{m^2} \text{Var}(p_z(0)) + \frac{t}{2m} \text{Cov}(p_z(0), z(0)), \]

and \( F_\tilde{z} = 1/\text{Var}(z) \) for Gaussian states, we obtain

\[ F_C(\tilde{S}_z, \tilde{z})_{2T_\pi} = F_Q^{\text{sc}} + 8(\sigma m T_\pi^2)^2/[(\sigma^2 m)^2 + (2\hbar T_\pi^2)] \]

for the initial Gaussian considered in Fig. 2(a), in agreement with numerics.

We can increase \( F_C(\tilde{S}_z, \tilde{z}) \) with an initial state that decreases \( \text{Var}(z(2T_\pi)) \) at the interferometer output. This is not achieved by reducing \( \text{Var}(z(0)) \), but rather via an initial state with nontrivial correlations between position and momentum such that \( \text{Cov}(p_z, z) \) counteracts the wavepacket’s ballistic expansion. Figure 2(b) shows the QFI and CFI for initial state \( \langle z|\psi_0\rangle = e^{-(1+i)/2\sigma^2}/[\pi(2\sigma^2)]^{1/4} \). The imaginary term provides the position-momentum correlations and doubling the spatial width increases the ability of the wavepacket to be focused. This initial state could be engineered by applying a harmonic potential for a short duration (compared to motional dynamics), creating phase gradient \( \psi(z) \to \psi(z)e^{-iz^2/\sigma^2} \), for constant \( \sigma \), which depends on trap frequency and duration \( [66] \). Then \( F_C(\tilde{S}_z, \tilde{z}) \) saturates the QCRB at \( T_1 = T_2 \), at the cost of reduced \( F_C(\tilde{S}_z, \tilde{p}) \).

IV. OPTIMUM MEASUREMENTS

Since measurements in different bases yield different sensitivities, is there an accessible measurement basis that saturates the QCRB? Our above analysis suggests yes and, depending on the initial state, this optimum basis lies somewhere between position and momentum. We confirm this intuition by revisiting a particle in a gravitational field. We rewrite

\[ |\psi(t)\rangle = \hat{U}_g|\psi_0\rangle = \exp(-ig\hat{G}_0(t)|\psi_0\rangle), \]

where

\[ \hat{G}_0(t) = \hat{U}_p\hat{G}_0(t)\hat{U}_p^\dagger = \frac{t}{\hbar}(m\tilde{z} - \frac{1}{2}\tilde{p}z), \]

\( \hat{U}_p = \exp[-ipt^2/(2m\hbar)] \), and \( |\psi_0(t)\rangle = \hat{U}_p|\psi_0\rangle \) describes free-particle evolution. We can interpret \( \hat{G}_0(t) \) as the generator of displacements in \( \hat{Q} = c_1\hat{z} + c_2\hat{p}z \), where the coefficients \( c_i \) are real and chosen such that \( \hat{G}_0(t), \hat{Q} \rangle = i \). Hence, the probability distribution \( |\langle q|\psi(t)\rangle|^2 = |\langle q - g|\psi(t)\rangle|^2 \), where \( \hat{Q}|q\rangle = q|q\rangle \). If \( |\langle q|\psi(t)\rangle|^2 \) is Gaussian, then measurements of \( Q \) saturate the QCRB, since \( \hat{G}_0(t), \hat{Q} \rangle = i \) implies

\[ F_C(\hat{Q}) = \frac{1}{\text{Var}(\hat{Q})} = 4\text{Var}(\hat{G}_0(t)) = F_Q. \]

To measure \( \hat{Q} \), we mix \( \hat{z} \) and \( \hat{p}z \) by applying the potential \( V(z) = \frac{1}{2}m\omega^2z^2 \), since \( \tilde{z}(t) = \tilde{z}(0)\cos\omega t + \tilde{z}(0)\sin\omega t \). Subsequently measuring position yields a combination of position and momentum information. This scheme could be implemented using the following procedure:

1. At \( t = 2T_\pi \), apply the unitary \( \hat{U}_s = |a\rangle\langle a| + |b\rangle\langle b|e^{-i\kappa_0\hat{z}} \), which removes any momentum mismatch between the two modes. A state-selective Bragg transition achieves this.

2. Then apply the potential \( V(z) = \frac{1}{2}m\omega^2(z - z_0)^2 \), where \( z_0 = \hbar k_0 T_\pi/m \) is the matterwave’s centre-of-mass displacement at the interferometer output.

3. Finally, at some later time, we apply a beam splitter \( \hat{U}_\text{BS} = \frac{1}{\sqrt{2}}[1 + \langle a|\langle b|(-\text{h.c.})] \) immediately before measurement.

Figure 3 shows \( F_C(\tilde{S}_z, \tilde{z}) \) and \( F_C(\tilde{S}_z, \tilde{p}) \) for this scheme. Both CFIIs oscillate between \( F_Q^{\text{sc}} \) and the QFI, so a measurement in either the position or momentum basis saturates the QCRB if made at the appropriate time. This improved sensitivity does increase the interferometer time. However, the period of CFI oscillations is negligible compared to \( T_\pi \) for sufficiently large \( \omega \).

V. IMPROVED INTERFEROMETRY

In KC interferometry, the \( \pi \) pulse ensures that the wavepackets spatially overlap at \( t = 2T_\pi \). However, Fig. 2 and Fig. 3 reveal that spatial overlap is not required for a momentum measurement, making the mirror pulse unnecessary. More interestingly, removing the
FIG. 3. Fisher information (FI) $|\Psi(t)\rangle = U_{KC}(t) |\Psi_0\rangle$, where $T_1 = t$ and $T_2 = 0$ for $t \leq T_\pi$, otherwise $T_1 = T_\pi$ and $T_2 = t - T_\pi$, with a harmonic potential applied at $t = 2T_\pi$ and initial Gaussian motional state $|\Psi(t)\rangle = \exp(-z^2/2\sigma_t^2)/(\pi\sigma_t)^{1/4}$. We artificially turned off gravity at $t = 2T_\pi$ (which holds $F_Q^{KC}$ constant) to clearly show the effect of harmonic trapping. Specifically, the application of this harmonic potential can be used to saturate the QCRB with either a position-distribution or momentum-distribution measurement. Here $\sigma = 10L$, $T_\pi = 100t_0$, and $\omega = 3\pi/(2T_\pi)$. FI has units $k_0^2 T_\pi^4$, and length ($L = k_0^{-1}$) and time ($t_0 = m/\hbar k_0^2$) units depend on $k_0$.

We numerically solved the Schrödinger equation for the mirrorless Mach-Zehnder (i.e. Ramsey) configuration [Fig. 1(b)]. Figure 4(a) shows that a momentum measurement is always nearly optimal, and at $t = 2T_\pi$, $F_C(\hat{S}_z,\hat{p}_z)/F_Q^{KC} \approx 4.4$. Unfortunately, this improved sensitivity has a price: A lack of spatial overlap means that information is encoded in high-frequency interference fringes in the momentum distribution, requiring high-resolution momentum measurements. Following Refs. [67–71], we model imperfect resolution by convolving the momentum distribution at $t = 2T_\pi$ with a Gaussian of width $\sigma_p$ before constructing $F_C(\hat{S}_z,\hat{p}_z)$ [Fig. 4(b)]. This imperfect resolution may be due to limitations on the detection system, or other sources of classical noise. The mirrorless configuration is considerably more sensitive to imperfect momentum resolution than KC interferometry, where $F_C(\hat{S}_z,\hat{p}_z)$ begins to degrade only when $\sigma_p$ is comparable to the initial wavepacket’s momentum width. Furthermore, in the limit of a “bad” momentum measurement ($\sigma_p \rightarrow \infty$), the CFI goes to zero, whereas the CFI for KC interferometry approaches $F_Q^{Sc}$. Nevertheless, if high-resolution measurements are available (or actively developed), as reported in Ref. [72] for instance, our result suggests that pursuing a mirrorless configuration could yield substantial sensitivity gains.

VI. DISCUSSION AND OUTLOOK

An important experimental consideration is achieving high-resolution momentum measurements. Time-off-flight imaging is a standard technique, where ballistic expansion converts the momentum distribution into a position distribution [73, 74]. However, the expansion time needed for sufficient momentum resolution might be significantly longer than the interrogation time, in which case longer interrogation times are a better route to improved sensitivities. Bragg spectroscopy is perhaps a more promising approach [75, 76].

Reference [7] reports state-of-the-art gravimetry with a Bose-Einstein condensate (BEC), well-described by a pure motional state, and parameters: $\sigma = 40\mu m$, $T_\pi = 130ns$, $k_0 = 1.6 \times 10^7 m^{-1}$, $\delta p_z = 0.18h_0$. We estimate that $4\text{Var}(G_0(T)) \sim 7%$ of $F_Q^{Sc}$, so there is little gain in making optimal measurements [Eq. (11)]. However, $4\text{Var}(G_0(T)) \sim F_Q^{Sc}$ if $\sigma$ or $\delta p_z$ were increased by an order of magnitude. This suggests that creating initial (pure) states with large spatial extent, such as quasi-continuous atom lasers [42, 77], could yield substantial sensitivity gains. Additionally, compact and/or high-bandwidth devices could benefit from optimal measurements, since shorter interrogation times increase Var($G_0(T)$) relative to $F_Q^{Sc}$.

For KC interferometers with thermal (mixed) states, Eq. (11) is only an upper bound for the QFI [61]. A calcu-
loration of $F_Q$ and $F_C$ for thermal sources gives values substantially greater than $F_Q^{\text{sm}}$ [78], in qualitative agreement with our above analysis, showing that current thermal-atom gravimetry is suboptimal. However, the QFI and CFI are also smaller than Eq. (11) for thermal sources, suggesting that BECs possess metrological potential beyond what is possible with thermal sources.

Our approach to evaluating matterwave interferometry could significantly influence the design of future state-of-the-art gravimeters. Typical interferometer design assumes a particular form for the measurement signal (e.g., the population difference at the output varies sinusoidally with $g$) and looks no further if there is agreement with simple 'best case' formulae such as Eq. (1). In contrast, a Fisher analysis gives the full metrological potential of any given dynamical scheme without enforcing such a priori assumptions by simply considering the available data. Our matterwave gravimetry analysis opens up new routes to improved sensitivity – beyond those few implied by Eq. (1). This includes engineering states with high QFI [i.e. large Var($G_0(T)$)] and improving information extraction at the interferometer output. Our mirrorless scheme gives a substantial sensitivity boost if high-resolution momentum measurements are available. For [7], this momentum resolution is $10^{-2}h\kappa_0$, achievable by further developing the $2 \times 10^{-4}h\kappa_0$ resolution measurement of [72]. A Fisher analysis could prove beneficial for evaluating other atom-interferometer-based sensors which produce a complicated output signal, such as schemes utilizing Kapitza-Dirac scattering [79–84] or propagation in crossed waveguides [85].

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Appendix A: QFI of a particle in a gravitational field

Here we give a more detailed derivation of Eq. (5a). Approximating the gravitational field as a linear potential $mg\hat{z}$, the state of the particle after time $T$ is $\hat{U}_g(T)|\Psi_0\rangle$, where

$$\hat{U}_g(T) = \exp\left[-i\frac{T}{\hbar}\left(\frac{p_z^2}{2m} + mg\hat{z}\right)\right].$$  

(A1)

In order to isolate the contribution due to the gravitational field $g$, we make use of the Baker-Campbell-Hausdorff (BCH) lemma:

$$e^{X+Y} = e^X e^Y e^{-\frac{i}{2}[X,Y]} e^{\frac{i}{2}(Y,[X,Y])} + [X,[X,Y]] \right\},$$

(A2)

where $X$ and $Y$ are operators satisfying the commutation relations

$$[[[X, Y], X], X] = [[[X, Y], Y], Y] = [[[X, Y], Y], \hat{Y}] = 0.$$  

(A3)

This is true for $X = -\frac{iT}{\hbar}p_z$ and $Y = -\frac{iT}{\hbar}mg\hat{z}$, where

$$[\hat{X}, \hat{Y}] = \frac{iT^2g}{\hbar}p_z,$$  

(A4a)

$$[\hat{Y}, [\hat{X}, \hat{Y}]] = \frac{imT^2g^2\hbar}{h},$$  

(A4b)

$$[\hat{X}, [\hat{X}, \hat{Y}]] = 0.$$  

(A4c)

Thus, Eq. (A2) gives:

$$e^{-\frac{iT}{\hbar}(\frac{p_z^2}{2m} + mg\hat{z})} = e^{-\frac{iT}{\hbar}\frac{p_z^2}{2m}} e^{-i\frac{T}{\hbar}mg\hat{z}} e^{-i\frac{T^2g^2\hbar}{2m}},$$  

(A5)

We use Eq. (A2) again with the choice $\hat{X} = \frac{T}{\hbar}mg\hat{z}$ and $\hat{Y} = -\frac{iT^2g}{2h}p_z$, where $[\hat{X}, \hat{Y}] = -\frac{imT^2g^2\hbar}{2h}$, which allows us to combine $\exp[-i(T/\hbar)mg\hat{z}]$ and $\exp[-igT^2\hat{p}_z/(2\hbar)]$ into a single exponential:

$$e^{-\frac{iT}{\hbar}mg\hat{z}} e^{-i\frac{T^2g^2\hbar}{2m}} = e^{-i\hat{G}_0(T)} e^{-i\frac{mg^2T^2\hbar}{2}},$$

(A6)

Thus, the evolution operator $\hat{U}_g(T)$ can be written as:

$$\hat{U}_g(T) = e^{-\frac{iT}{\hbar}g^2\hat{z}\hat{p}_z} e^{-i\hat{G}_0(T)}.$$  

(A7)

We can ignore $\exp[img^2T^3/(12\hbar)]$, since this is just a global phase factor, and so the state of the particle after time $T$ is

$$|\Psi(T)\rangle = e^{-\frac{iT}{\hbar}g^2\hat{z}\hat{p}_z} e^{-i\hat{G}_0(T)} |\Psi_0\rangle.$$  

(A8)

It is now simple to compute the derivative of $|\Psi(T)\rangle$ with respect to $g$:

$$\langle \partial_g |\Psi(T)\rangle = -ie^{-\frac{iT}{\hbar}\frac{p_z^2}{2m}} \hat{G}_0(T) e^{-i\hat{G}_0(T)} |\Psi_0\rangle.$$  

(A9)

Consequently,

$$\langle \partial_g |\Psi(T)\rangle |\partial_g |\Psi(T)\rangle = \langle \Psi_0 |\hat{G}_0(T)^2 |\Psi_0\rangle,$$  

(A10a)

$$\langle \Psi(T) |\partial_g |\Psi(T)\rangle = -i\langle \Psi_0 |\hat{G}_0(T) |\Psi_0\rangle.$$  

(A10b)

Substituting these into Eq. (2) gives our final expression for the QFI, Eq. (5a).
Appendix B: QFI of a particle after KC interferometry

Here we provide a derivation of Eq. (10). The total evolution of a particle due to KC interferometry is given by the unitary operator

\[ \hat{U}_{KC} = \hat{U}^{\phi_3} \hat{U}_g(T_2) \hat{U}^{\phi_2} \hat{U}_g(T_1) \hat{U}^{\phi_1}, \]  

(B1)

where \( \hat{U}^{\phi_i} \) and \( \hat{U}^{\phi'_i} \) denote \( \pi/2 \) (50/50 beam splitting) and \( \pi \) (mirror) pulses, respectively, and the evolution due to the gravitational field, \( \hat{U}_g(T) \), was derived above [see Eq. (A7)]. This assumes that the \( \pi/2 \) and \( \pi \) pulses are instantaneous (strictly, occur on times much shorter than the interrogation times \( T_1 \) and \( T_2 \)).

To begin, the final \( \pi/2 \) pulse does not change the QFI, whilst the first \( \pi/2 \) pulse simply gives a new initial state for the particle [see Eq. (7)]:

\[ |\Psi_0\rangle = \hat{U}^{\phi_1} |\psi_0\rangle = \frac{1}{\sqrt{2}} \left( |\psi\rangle_a - i e^{i(k_0 \hat{z} - \phi_1)} |\psi\rangle_b \right), \]  

(B2)

where \( |\Psi_0\rangle = |a\rangle |\psi_0\rangle \) and \( \phi_1 \) is the phase of this first laser pulse. Consequently, the QFI can be computed from the product of operators \( \hat{U}_g(T_2) \hat{U}^{\phi_2}_\pi \hat{U}_g(T_1) \), provided expectations are taken with respect to the state \( |\Psi'_0\rangle \).

As in Appendix A, our goal is to isolate the \( g \)-dependence of the evolution. We first consider the product \( \hat{U}_g(T_2) \hat{U}^{\phi_2}_\pi \), where [see Eq. (7)]

\[ \hat{U}^{\phi_2}_\pi = -i \left( e^{-i(k_0 \hat{z} - \phi_2)} |a\rangle \langle b| + e^{i(k_0 \hat{z} - \phi_2)} |b\rangle \langle a| \right), \]  

(B3)

and \( \phi_2 \) is the phase of this mirror pulse. The BCH lemma Eq. (A2) implies that

\[ e^\hat{X} e^{\frac{i}{\hbar} \hat{Y}} e^{-\frac{i}{\hbar} \hat{X}} = e^{\frac{i}{\hbar} \hat{Y}} e^\hat{X} e^{\frac{i}{\hbar} \hat{Y}} e^{-\frac{i}{\hbar} \hat{X}} \]

(B4)

The application of Eq. (B4) with \( \hat{X} = -ig\hat{G}_0(T_2) \) and \( \hat{Y}_\pm = \pm ik_0 \hat{z} \) gives

\[ e^{-ig\hat{G}_0(T_2)} e^{\pm ik_0 \hat{z}} = e^{\pm ik_0 \hat{z}} e^{-ig\hat{G}_0(T_2)} e^{\mp \frac{i}{\hbar} k_0 T_2^2}, \]  

(B5)

where we have used \( [\hat{X}, \hat{Y}_\pm] = \mp igk_0 T_2^2/2 \). Therefore, after neglecting the global phase factor \( \exp[\mp ig^2 T_2^2/(12\hbar)] \) in \( \hat{U}_g(T) \):

\[ \hat{Y} = -\frac{\tau T_2}{\hbar} \frac{\vec{p}_m^2}{2m}, \]

for which

\[ [\hat{X}, \hat{Y}] = -\frac{ig T_1 T_2}{\hbar} \hat{p}_z, \]  

(B10a)

\[ [\hat{X}, [\hat{X}, \hat{Y}]] = -\frac{ig^2 T_1 T_2^2}{\hbar}, \]  

(B10b)

to obtain

\[ e^{-ig\hat{G}_0(T_2)} e^{-i \frac{\tau T_2}{\hbar} \frac{\vec{p}_m^2}{2m}} = e^{-i \frac{\tau T_2}{\hbar} \frac{\vec{p}_m^2}{2m}} e^{-ig\hat{G}_0(T_2)} e^{-ig T_1 T_2^2/2 \hbar} e^{i \frac{\vec{p}_m^2}{2m} \gamma T_1 T_2^2}, \]  

(B11)

and therefore (ignoring the global phase factor \( \exp[\mp ig^2 T_2^2/(12\hbar)] \))

\[ \hat{U}_g(T_2) \hat{U}^{\phi_2}_\pi \hat{U}_g(T_1) = e^{-i \frac{\tau T_2}{\hbar} \frac{\vec{p}_m^2}{2m}} \hat{U}^{\phi_2}_\pi e^{-ig \hat{G}_c} e^{-i \frac{\tau T_2}{\hbar} \frac{\vec{p}_m^2}{2m}} e^{-ig \hat{G}_0(T_2)} \times e^{-i \frac{\tau T_1 T_2}{\hbar} \hat{p}_z} e^{-ig \hat{G}_c} \]  

(B12)

We combine the final three exponentials into one using Eq. (A2):

\[ \hat{U}_g(T_2) \hat{U}^{\phi_2}_\pi \hat{U}_g(T_1) \]

\[ = e^{-i \frac{\tau T_2}{\hbar} \frac{\vec{p}_m^2}{2m}} \hat{U}^{\phi_2}_\pi e^{-i \frac{\tau T_2}{\hbar} \frac{\vec{p}_m^2}{2m}} e^{-ig(\hat{G}_0(T)+\hat{G}_c)}, \]  

(B13)
where $T = T_1 + T_2$ and we have neglected all the global phases produced during the calculation.

Including the first and second $\pi/2$ pulses (although the second pulse is not needed for calculating the QFI), we arrive at the following simplified expression for the full KC interferometer evolution:

$$
\hat{U}_{KC} = \hat{U}_0 e^{-i g (\hat{G}_o(T) + \hat{G}_e)} \hat{U}_\frac{\phi_1}{2},
$$

(B14)

where $\hat{U}_0 = \hat{U}_\frac{\phi_3}{2} e^{-i \frac{T_2}{m} \hat{p}_z^2} \hat{U}_\frac{\phi_2}{2} e^{-i \frac{T_1}{m} \hat{p}_z^2}$ is independent of $g$. The state of the particle after interrogation time $T$ is therefore

$$
|\Psi(T)\rangle = \hat{U}_{KC} |\Psi_0\rangle = \hat{U}_0 e^{-i g (\hat{G}_o(T) + \hat{G}_e)} |\Psi_0\rangle,
$$

(B15)

which is Eq. (8). Taking the derivative with respect to $g$ gives

$$
\langle \partial_g |\Psi(T)\rangle |\partial_g |\Psi(T)\rangle = (\langle \Psi_0\rangle (\hat{G}_o(T) + \hat{G}_e)^2 |\Psi_0\rangle,
$$

(B16a)

$$
\langle |\Psi(T)\rangle |\partial_g |Psi(T)\rangle = -i (\langle \Psi_0\rangle (\hat{G}_o(T) + \hat{G}_e) |\Psi_0\rangle.
$$

(B16b)

The QFI is therefore

$$
F_Q^{KC} = 4 \text{Var} \left( \hat{G}_o(T) + \hat{G}_e \right),
$$

(B17)

where the variance is taken with respect to $|\Psi_0\rangle$. We use Eq. (B2) to relate this to expectations taken with respect to the initial state $|\Psi_0\rangle$:

$$
F_Q^{KC} = 4 \text{Var} \hat{G}_o(T) + \frac{1}{4} \frac{\hbar^2}{T^2 - 2T_2^2} ,
$$

(B18)

which is Eq. (10).

**Appendix C: FC(\hat{S}_z) of KC interferometer**

To calculate the CFI $F_C(\hat{S}_z)$ [Eq. (14)], we need to determine expressions for the probabilities $P_a(T)$ and $P_b(T)$ that the particle is detected in state $|a\rangle$ and $|b\rangle$, respectively, at the interferometer output. This first requires expressing $\hat{U}_{KC}$ in a more convenient form. To begin, we use Eq. (B4) with $\hat{X} = -i \frac{T_2}{\hbar} \hat{p}_z^2$ and $\hat{Y}_\pm = \pm i k_0 \hat{z}$ to obtain

$$
e^{-i \frac{T_2}{\hbar} \hat{p}_z^2} e^{\pm i k_0 \hat{z}} = e^{\pm i k_0 \hat{z}} e^{-i \frac{T_2}{\hbar} \hat{p}_z^2} e^{\pm i \frac{\hbar^2 T_2}{2m} \hat{p}_z} e^{-i \frac{\hbar^2 T_2}{2m}},
$$

(C1)

where we used $[\hat{X}, \hat{Y}_\pm] = \mp \frac{i \hbar T_2}{m} \hat{p}_z$ and $[\hat{Y}_\pm, [\hat{X}, \hat{Y}_\pm]] = \frac{i \hbar k_0^2 T_2}{m}$. This allows us to commute $e^{-i \frac{T_2}{\hbar} \hat{p}_z^2}$ and $\hat{U}_{\phi_2}$:

$$
e^{-i \frac{T_2}{\hbar} \hat{p}_z^2} \hat{U}_{\phi_2} = \hat{U}_{\phi_2} e^{-i \frac{T_2}{\hbar} \hat{p}_z^2} e^{-2i \frac{\hbar k_0 T_2}{m} \hat{p}_z} e^{-\frac{\hbar^2 T_2}{2m}},
$$

(C2)

where we have again used Eq. (B7). Neglecting the global phase factor $\exp[-i \frac{\hbar^2 T_2}{2m}]$, we can therefore write Eq. (B14) in the convenient form

$$
\hat{U}_{KC} = \hat{U}_{int} \hat{U}_{ext} \hat{U}_\frac{\phi_1}{2},
$$

(C3)

$$
\hat{U}_{int} \equiv \hat{U}_{\phi_2} \hat{U}_{\phi_3} e^{-2i \frac{\hbar k_0 T_2}{m} \hat{p}_z} e^{-i g \hat{G}_e},
$$

(C4)

$$
\hat{U}_{ext} \equiv e^{-i \frac{T_2}{\hbar} \hat{p}_z^2} e^{-i g \hat{G}_o(T)}.
$$

(C5)

$\hat{U}_{ext}$ only acts on the external (i.e., motional) degrees of freedom, whereas $\hat{U}_{int}$ acts on both the internal and motional degrees of freedom. Note that $\hat{U}_{int}$ and $\hat{U}_{ext}$ do not commute.

The state of the particle at the output of the interferometer after interrogation time $T$ is therefore

$$
|\Psi(T)\rangle = \hat{U}_{int} \hat{U}_{ext} \hat{U}_\frac{\phi_1}{2} |\Psi_0\rangle
$$

$$
= \frac{1}{\sqrt{2}} \left( \hat{U}_{int} |a\rangle \hat{U}_{ext} |\psi_0\rangle - i \hat{U}_{int} |b\rangle \hat{U}_{ext} e^{i (k_0 \hat{z} - \phi_1)} |\psi_0\rangle \right).
$$

(C6)

From Eq. (7) we get:

$$
\hat{U}_{\phi_2} \hat{U}_{\phi_3} = -\frac{1}{\sqrt{2}} \left( e^{-i (\phi_2 - \phi_3)} |a\rangle \langle a| + e^{i (\phi_2 - \phi_3)} |b\rangle \langle b| \right)
$$

$$
- \frac{i}{\sqrt{2}} \left( e^{-i (k_0 \hat{z} - \phi_2)} |a\rangle \langle b| + e^{i (k_0 \hat{z} - \phi_2)} |b\rangle \langle a| \right),
$$

(C7)

where $\phi_2$ and $\phi_3$ are the phases of the second and the third laser pulses, respectively. Using this and Eq. (B7), we obtain

$$
\hat{U}_{int} |a\rangle = -\frac{1}{\sqrt{2}} \left[ e^{-i (\phi_2 - \phi_3)} |a\rangle + ie^{i (k_0 \hat{z} - \phi_2)} |b\rangle \right]
$$

$$
\times e^{-i \frac{\hbar k_0 T_2}{m} \hat{p}_z} e^{-i g \frac{\hbar k_0^2 T_2}{m}},
$$

(C8a)

$$
\hat{U}_{int} |b\rangle = -\frac{1}{\sqrt{2}} \left[ e^{i (\phi_2 - \phi_3)} |b\rangle + i e^{-i (k_0 \hat{z} - \phi_2)} |a\rangle \right]
$$

$$
\times e^{i \frac{\hbar k_0 T_2}{m} \hat{p}_z} e^{i g \frac{\hbar k_0^2 T_2}{m}}.
$$

(C8b)

Substituting Eqs. (C8) into Eq. (C6) gives

$$
|\Psi(T)\rangle = -\frac{1}{2} \left[ e^{-i (\phi_2 - \phi_3)} e^{-i \frac{\hbar k_0 T_2}{m} \hat{p}_z} e^{-i g \frac{\hbar k_0^2 T_2}{m} \hat{p}_z} \hat{U}_{ext} |\psi_0\rangle + e^{-i (k_0 \hat{z} - \phi_2)} e^{i \frac{\hbar k_0 T_2}{m} \hat{p}_z} e^{i g \frac{\hbar k_0^2 T_2}{m} \hat{p}_z} \hat{U}_{ext} e^{i (k_0 \hat{z} - \phi_1)} |\psi_0\rangle \right] |a\rangle
$$

$$
+ i \left( e^{i (k_0 \hat{z} - \phi_2)} e^{-i \frac{\hbar k_0 T_2}{m} \hat{p}_z} e^{-i g \frac{\hbar k_0^2 T_2}{m} \hat{p}_z} \hat{U}_{ext} |\psi_0\rangle - e^{i (\phi_2 - \phi_3)} e^{i \frac{\hbar k_0 T_2}{m} \hat{p}_z} e^{i g \frac{\hbar k_0^2 T_2}{m} \hat{p}_z} \hat{U}_{ext} e^{i (k_0 \hat{z} - \phi_1)} |\psi_0\rangle \right) |b\rangle.
$$

(C9)
Defining \(|\Psi_a(T)\rangle \equiv \langle a|\Psi(T)\rangle\), the probability of finding the particle in the internal state \(|a\rangle\) at the output port of the interferometer is

\[
P_a(T) = \langle \Psi_a(T)|\Psi_a(T)\rangle = \frac{1}{2}\left[1 + \frac{1}{2}(e^{i(gk_0T_2^2 - \Phi})(\psi_0|\psi_0\rangle + \text{h.c.})\right],
\]

where \(\Phi \equiv \phi_1 - 2\phi_2 + \phi_3\) and

\[
\hat{Q} \equiv e^{igG_0(T)}e^{\frac{igk_0}{2m}T_2}e^{ik_0z}e^{-ik_0z},
\]

\[
Q \equiv e^{igG_0(T)}e^{\frac{igk_0}{2m}T_2} \hat{p}_z e^{-ik_0z},
\]

\[
= e^{igG_0(T)}e^{-i(gk_0(T_2-T_1) + igk_0(T^2 - T_1^2)})e^{\frac{igk_0}{2m}T_1}\hat{p}_z.
\]

This final simplification follows from repeated application of Eq. (B4), and allows us to express the probability as

\[
P_a(T) = \frac{1}{2}\left[1 + \frac{1}{2}\left(e^{-i\Phi}e^{\frac{igk_0}{2m}(T_2-T_1)}e^{-igk_0(T^2 - T_1^2)}\times \langle \psi_0|e^{\frac{igk_0}{2m}(T_2-T_1)}\hat{p}_z|\psi_0\rangle + \text{h.c.}\right)\right].
\]

If we choose the phases of our laser pulses such that \(\phi_1 = \phi_2 = 0\), \(\phi_3 = \pi/2\), thereby operating at the point of maximum sensitivity, we can express the probabilities in the following way:

\[
P_a(T) = \frac{1}{2}\left[1 + \frac{1}{2}\left(C e^{i(\phi_f - \phi_g)} - C^* e^{-i(\phi_f - \phi_g)}\right)\right],
\]

\[
P_b(T) = \frac{1}{2}\left[1 + \frac{1}{2}\left(C e^{i(\phi_f - \phi_g)} - C^* e^{-i(\phi_f - \phi_g)}\right)\right],
\]

where

\[
\phi_f \equiv \frac{\hbar k_0^2}{2m}(T_2 - T_1),
\]

\[
\phi_g \equiv k_0\left(\frac{T_2}{2} - T_1^2\right),
\]

\[
C \equiv \langle \psi_0|e^{\frac{igk_0}{2m}(T_2-T_1)}\hat{p}_z|\psi_0\rangle.
\]

\(\phi_f\) represents the phase difference due to the non-symmetrical free evolution of the wavepackets in the two arms of the interferometer, while \(\phi_g\) is the phase difference due to gravity. Expressing \(C = |C|e^{i\alpha}\) allows us to write Eq. (C13) in the simplified form of Eqs. (13). Here \(|C|\) is interpreted as a fringe contrast and \(\alpha = \phi_f - \phi_g + \theta\) denotes the total phase shift.

If we measure the population difference of the two internal states, \(S_z\), at the output of the interferometer, the CFI is given by

\[
F_C(S_z) = \sum_{j=a,b} \frac{\left(\partial_j P_j\right)^2}{P_j} = \frac{\left(\partial_g P_a\right)^2}{P_a P_b},
\]

where the last equality follows from the relation \(P_a + P_b = 1\) \(\Rightarrow \partial_g P_a = -\partial_g P_b\). Noting that

\[
P_a(T)P_b(T) = \frac{1}{4}\left(1 - |C|^2 \sin^2 \alpha\right),
\]

\[
\partial_g P_a(T) = \frac{1}{2}|C|k_0 \left(\frac{T_2^2 - T_1^2}{2}\right) \cos \alpha,
\]

we arrive at Eq. (14).

**Appendix D: Beam splitter transformation: Derivation of Eq. (7).**

A Raman beam splitter is typically modelled by the Hamiltonian

\[
\hat{H}_{BS} = \frac{\hat{p}^2}{2m} - \hbar\delta |\langle b|\rangle + \frac{\hbar \Omega}{2} (|b\rangle\langle a|e^{i(k_0 z - \phi)} + \text{h.c.}),
\]

where \(\delta\) is the two-photon detuning and \(\Omega = \Omega_1\Omega_2/\Delta\) is the effective two-photon Rabi frequency, which depends on the single-photon Rabi frequencies \(\Omega_1\) and the single-photon detuning \(\Delta\) [86, 87]. The two-photon detuning is typically set to the two-photon resonance condition \(\delta = \hbar k_0^2/(2m)\). Evolution under this Hamiltonian for a duration \(\Delta t\) is given by the unitary time-evolution operator

\[
U^\theta_\phi = \exp\left[\frac{-i\Delta t}{\hbar}\hat{H}_{BS}\right] = e^{-i\left(\frac{\hat{p}^2}{2m} - \frac{\hbar k_0^2}{2m}(T_2 - T_1)\right)} e^{-i\frac{\pi}{4}\left(|b\rangle\langle a|e^{i(k_0 z - \phi)} + \text{h.c.}\right)},
\]

where we have defined \(\theta = \Omega \Delta t\). If \(\hbar \Omega\) is significantly greater than the spread in kinetic energy of the initial state, we can ignore the first term and obtain

\[
U^\theta_\phi = \exp\left[\frac{-i\theta}{2}\left(|b\rangle\langle a|e^{i(k_0 z - \phi)} + \text{h.c.}\right)\right] = \hat{I} \cos\left(\frac{\theta}{2}\right) - i(|b\rangle\langle a|e^{i(k_0 z - \phi)} + \text{h.c.}) \sin\left(\frac{\theta}{2}\right),
\]

which is Eq. (7).

Figure 5 shows the QFI and CFI when the evolution due to the beam splitter and mirror pulses is treated as Schrödinger evolution under Hamiltonian Eq. (D1). This evolution was solved numerically for different values of \(\Delta t\). We used the same initial state as Fig. 2(a). We set \(\Omega\) such that \(\Omega \Delta t = \pi/2\) for the two beam splitter pulses, and the duration of the interaction was doubled for the mirror pulse, resulting in \(\Omega(2\Delta t) = \pi\). We find excellent agreement with the ideal beam splitter case as long as \(\Delta t \ll T_{\pi}\). In the regime \(\Delta t \sim T_{\pi}\), there is significant motional dynamics during the beam splitter period, and our approximation is no longer valid. For example, for the maximum value of \(\Delta t\) simulated (\(\Delta t = 0.4T_{\pi}\), the total interferometer sequence time, which is the time from the commencement of the first beam splitter to the conclusion of the second beam splitter, is \(3.6T_{\pi}\) (compared to \(2T_{\pi}\) for instantaneous beam splitters). For typical experiments, such as Ref. [7], \(\Delta t/T_{\pi} \sim 10^{-4}\).
FIG. 5. (a) QFI and CFI computed using Eq. (D1) rather than Eq. (D3) as a function of $\Delta t$. Provided $\Delta t/T_\pi \ll 1$, Eq. (D3) (shown by dashed lines of the appropriate colour) is an excellent approximation to the true dynamics. Fisher information is presented in units of $k_0^2 T_\pi^4$.

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