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Arcs of Degree Four in a Finite Projective Plane

by

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Declaration

I hereby declare that this thesis has not been submitted in whole or in part to this or any other University for the award of a degree.

Signature:

Zainab Hamed

UNIVERSITY OF SUSSEX

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Abstract

The projective plane, $\mathrm{PG}(2,q)$, over a Galois field $\mathbf{F}_q$ is an incidence structure of points and lines. A $(k;n)$-arc $\mathscr{K}$ in $\mathrm{PG}(2,q)$ is a set of k points such that no $n+1$ of them are collinear but some n are collinear. A $(k;n)$-arc $\mathscr{K}$ in $\mathrm{PG}(2,q)$ is called *complete* if it is not contained in any $(k+1;n)$-arc. The existence of arcs for particular values of k and n pose interesting problems in finite geometry. It connects with coding theory and graph theory, with important applications in computer science. The main problem, known as the packing problem, is to determine the largest size $m_n(2,q)$ of $\mathscr{K}$ in $\mathrm{PG}(2,q)$. This problem has received much attention. Here, the work establishes complete arcs with a large number of points. In contrast, the problem to determine the smallest size $t_n(2,q)$ of a complete $(k;n)$-arc is mostly based on the lower bound arising from theoretical investigations.

This thesis has several goals.

The first goal is to classify certain $(k;4)$-arcs for $k = 6,\ldots,38$ in $\mathrm{PG}(2,13)$. This classification is established through an approach in Chapter 2. This approach uses a new geometrical method; it is a combination of projective inequivalence of $(k;4)$-arcs up to $k=6$ and certain *sd*-inequivalent $(k;4)$-arcs that have *sd*-inequivalent classes of secant distributions for $k=7,\ldots,38$. The part related to projectively inequivalent $(k;4)$-arcs up to $k=6$ starts by fixing the frame points $\{1,2,3,88\}$ and then classify the projectively inequivalent $(5;4)$-arcs. Among these $(5;4)$-arcs and $(6;4)$-arcs, the lexicographically least set are found. Now, the part regarding *sd*-inequivalent $(k;4)$-arcs in this method starts by choosing five *sd*-inequivalent $(7;4)$-arcs. This classification method may not produce all *sd*-inequivalent classes of $(k;4)$-arcs. However, it was necessary to employ this method due to the increasing number of $(k;4)$-arcs in $\mathrm{PG}(2,13)$ and the extreme computational difficulty of the problem. It reduces the constructed number of $(k;4)$-arcs in each process for large k. Consequently, it reduces the executed time for the computation which could last for years. Also, this method decreases the memory usage needed for the classification. The largest size of $(k;4)$-arc established through this method is $k = 38$.

The classification of certain $(k;4)$-arcs up to projective equivalence, for $k = 34, 35, 36, 37, 38$, is also established. This classification starts from the 77 incomplete $(34;4)$-arcs that are constructed from the *sd*-inequivalent $(33;4)$-arcs given in Section 2.29, Table 2.35. Here, the largest size of $(k;4)$-arc is still $k = 38$. In addition, the previous process is re-iterated with a different choice of five *sd*-inequivalent $(7;4)$-arcs. The purpose of this choice is to find a new size of complete $(k;4)$-arc for $k > 38$. This particular computation of $(k;4)$-arcs found no complete $(k;4)$-arc for $k > 38$. In contrast, a new size of complete $(k;4)$-arc in $\mathrm{PG}(2,13)$ is discovered. This size is $k = 36$ which is the largest complete $(k;4)$-arc in this computation. This result raises the second largest size of complete $(k;4)$-arc found in the first classification from $k = 35$ to $k = 36$.

The second goal is to discuss the incidence structure of the orbits of the groups of the projectively inequivalent $(6;4)$-arcs and also the incidence structures of the orbits of the groups other than the identity group of the *sd*-inequivalent $(k;4)$-arcs. In Chapter 3, these incidence structures are given for $k = 6, 7, 8, 9, 10, 11, 12, 13, 14, 38$. Also, the pictures of the geometric configurations of the lines and the points of the orbits are described.

The third goal is to find the sizes of certain *sd*-inequivalent complete $(k;4)$-arcs in $\mathrm{PG}(2,13)$. These sizes of complete $(k;4)$-arcs are given in Chapter 4 where the smallest size of complete $(k;4)$-arc is at most $k = 24$ and the largest size is at least $k = 38$.

The fourth goal is to give an example of an associated non-singular quartic curve $\mathscr{C}$ for each complete $(k;4)$-arc and to discuss the algebraic properties of each curve in terms of the number I of inflexion points, the number $|\mathscr{C} \cap \mathscr{K}|$ of rational points on the corresponding arc, and the number N_1 of rational points of $\mathscr{C}$. These curves are given in Chapter 5. Also, the algebraic properties of complete arcs of the most interesting sizes investigated in this thesis are studied. In addition, there are two examples of quartic curves $\mathscr{C}(g'_1)$ and $\mathscr{C}(g'_2)$ attaining the Hasse-Weil-Serre upper bound for the number N_1 of rational points on a curve over the finite field of order thirteen. This number is 32.

The fifth goal is to classify the $(k;4)$-arcs in $\mathrm{PG}(2,13)$ up to projective inequivalence for $k<10$. This classification is established in Chapter 6. It starts by fixing a triad, $\mathscr{U}_1$, on the projective line, $\mathrm{PG}(1,13)$. Here, the number of projectively inequivalent $(k;4)$-arcs are tested by using the tool given in Chapter 2. Then, among the number of the projectively inequivalent $(10;4)$-arcs found, the classification of *sd*-inequivalent $(k;4)$-arcs for $k=10$ is made. The number of these *sd*-inequivalent arcs is 36. Then, the 36 *sd*-inequivalent arcs are extended. The aim here is to investigate if there is a new size of *sd*-inequivalent $(k;4)$-arc for $k>38$ that can be obtained from these arcs. The largest size of *sd*-inequivalent $(k;4)$-arc in this process is the same as the largest size of the *sd*-inequivalent $(k;4)$-arc established in Chapter 2, that is, $k=38$.

In addition, the classification of $(k;n)$-arcs in $\mathrm{PG}(2,13)$ is extended from $n=4$ to $n=6$. This extension is given in Chapter 7 where some results of the classification of certain $(k;6)$-arcs for $k=9,\ldots,25$ are obtained using the same method as in Chapter 2 for $k=7,\ldots,38$. This process starts by fixing a certain $(8;6)$-arc containing six collinear points in $\mathrm{PG}(2,13)$.

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To the memory of my father

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Introduction

The theory of finite fields dates back to the seventeenth and eighteenth centuries. Many mathematicians have contributed to its development, such as Pierre de Fermat (1601-1665) and Leonard Euler (1707-1783) when they introduced the structure theory of special finite fields. However, the general theory of finite fields started with the French mathematician Evariste Galois (1811-1832) who determined the necessary and sufficient condition for a polynomial to be solvable by radicals. Also, the German mathematician Friedrich Gauss (1777-1855) who contributed significantly to many fields for instance number theory, algebra, differential geometry. Nowadays, the theory of finite fields has become very important and plays a significant role in applied mathematics, engineering science, computer science, and communication theory because of its many applications in these areas.[26]

Projective geometry is a branch of mathematics, one of the outstanding achievements of the nineteenth century, a century of remarkable mathematical achievements such as non-Euclidean geometry, abstract algebra, and the foundations of calculus. Projective geometry is as much a part of a general education in mathematics as differential equations and Galois theory. Moreover, it is a prerequisite for algebraic geometry, one of today's most vigorous and exciting branches of mathematics.

For more than fifty years projective geometry has been propelled in a new direction by its combinatorial connections. The challenge of describing a classical geometric structure by its parameters and properties that at first glance might seem superficial provided much of the impetus for finite geometry, another flourishing branch of mathematics.

In recent years new and important applications have been discovered. The structures of classical projective geometry are ideally suited for modern communications. In particular, projective geometry is applied to coding theory and to cryptography.[7]

Coding Theory dates back to 1948; it is used for multiple purposes in the military sphere including encryption of messages in wars and for the exchange of messages between the leaders of armies in a safe way by keeping the contents from leaking to the enemy. Also it means hide the true meaning of the message content in a manner to be agreed upon between the sender and the receiver and may be used for other purposes verify the identity of the sender.[24].

In general, coding theory is the study of methods for efficient and accurate diverse applications such as, the minimization of noise from compact disc recordings, the transmission of financial information across telephone lines, and data transfer from one computer to another.[23]

The main topic of this thesis is to classify certain $(k;4)$-arcs in $\mathrm{PG}(2,13)$ by a new method that is a combination of projectively inequivalent $(k;4)$-arcs and certain *sd*-inequivalent $(k;4)$-arcs that have *sd*-inequivalent classes N_c of secant distributions. The classification of certain $(k;4)$-arcs up to projective inequivalence for $k = 34, 35, 36, 37, 38$ is also given. The process to classify the *sd*-inequivalent $(k;4)$-arcs is re-iterated. The result of this particular computation is a new size of complete $(k;4)$-arc for $k = 36$. In addition, the incidence structure of the orbits for a $'$large$'$ group of *sd*-inequivalent $(k;4)$-arc is described. The sizes of certain *sd*-inequivalent complete $(k;4)$-arcs in $\mathrm{PG}(2,13)$ are established. Furthermore, for each complete $(k;4)$-arc, an associated quartic curve is defined and its properties described. Also, the classification of $(k;4)$-arcs up to projective inequivalence for $k < 10$ in $\mathrm{PG}(2,13)$ is given. Then these follows a process to discover whether there is a size of complete $(k;4)$-arc with $k > 38$. The classification of certain $(k;4)$-arcs in $\mathrm{PG}(2,13)$ is extended to classify certain $(k;6)$-arcs for $k = 9, \ldots, 25$. The computing tools that have been used to implement this work are GAP and Mathematica [13], [34].

Outline of the thesis

In this thesis, the following aspects are discussed in seven chapters.

Chapter one deals with the basic preliminaries to finite geometry and coding theory. Also, in this chapter some algebraic preliminaries of algebraic geometry related to the quartic curves corresponding to each complete $(k;4)$-arc in $\mathrm{PG}(2,13)$ are given.

Chapter two introduces a new approach to classify certain $(k;4)$-arcs in $\mathrm{PG}(2,13)$. The projectively inequivalent $(k;4)$-arcs up to $k = 6$ then the *sd*-inequivalent $(k;4)$-arcs that have *sd*-inequivalent classes N_c of the i-secant distributions are found for $k = 7, \ldots, 38$. Also, the corresponding linear codes of each $(k;4)$-arc are introduced via their parameters n,k,d and e.

Chapter three discusses the incidence structure of the orbits of the groups of the projectively inequivalent $(6;4)$-arcs given in Chapter 2. Also, the whole geometric picture of these structures of *sd*-inequivalent classes of *i*-secant distribution of the $(6;4)$-arcs is given. In addition, the incidence structures of the orbits of the groups other than the identity group of *sd*-inequivalent $(k;4)$-arcs for $k = 7,8,9,10,11,12,13,14,38$ are discussed.

Chapter four introduces the classification of certain *sd*-inequivalent complete $(k;4)$-arcs in $\mathrm{PG}(2,13)$ for $k = 24,25,26,27,28,29,30,31,32,33,34,35,38$. Furthermore, the complete $(36;4)$-arc established in Chapter 2 is discussed.

Chapter five presents examples of a non-singular quartic curves $\mathscr{C}(f_i)$ associated to complete $(k;4)$-arcs in $\mathrm{PG}(2,13)$ for $k = 24,25,26,27,28,29,30,31,32,33,34,35,36,38$. This chapter shows the algebraic properties for each quartic curve in terms of the number I of inflexion points, the number $|\mathscr{C} \cap \mathscr{K}|$ of rational points on the corresponding arc, and the number N_1 of rational points. In addition, two examples of quartic curves attaining the Hasse-Weil-Serre upper bound for the number N_1 of rational points on a curve over the finite field of order thirteen are given. This number is 32.

Chapter six discusses the classification of $(k;4)$-arcs up to projective inequivalence for $k < 10$. Then a process is done to discover if there is a new size of complete $(k;4)$-arc for $k > 38$. The size of complete $(k;4)$-arc investigated in this process is $k = 38$.

Chapter seven introduces some results of the classification of certain *sd*-inequivalent $(k;6)$-arcs in $\mathrm{PG}(2,13)$ for the values of $k = 9,\ldots,25$. This work is to be continued. The approach that has been used in this classification is based on the method used in Chapter 2, Section 2.3.

Chapter 1

Background

In this chapter, some essential and necessary principles regarding projective geometry, algebraic geometry, and coding theory are considered as a start of this work. The objective of this chapter is to introduce these basic principles which provide a nice structure and important information. They play a significant role in many problems.

The content of this chapter is based on the following standard references:
[26], [23], [40], [32], [7], [24], [18], [41], [16], [27], [30], [14], [17], [20], [21], [22], [33], [38], [35], [25], [19], [4], [39].

1.1 Finite fields

A field $\mathbf{F}$ is a set of elements with two operations, addition $(+)$ and multiplication $(\times)$, satisfying the following properties:

(a) $(\mathbf{F},+)$ is an abelian group with identity 0;

(b) $(\mathbf{F}\setminus\{0\},\times)$ is an abelian group with identity 1;

(c) $x(y+z)=xy+xz, \quad$ for all $x,y,z\in\mathbf{F}$.

A finite field of order q is also called a *Galois field*. It is denoted by $\mathbf{F}_q$. Note that a finite field is defined up to an isomorphism by the number q of its elements. Therefore, q must be an integer power p^h of a prime p. Here, p is the *characteristic* of the finite field. Then, every element

$x \in \mathbf{F}_q$ satisfies $x^q - x = 0$. When $q = p$, then $\mathbf{F}_p = \{0, 1, \dots, p-1\}$; when $q = p^h$, $h > 1$, then $\mathbf{F}_q = \{0, 1, \alpha, \alpha^2, \dots, \alpha^{q-2} \mid \alpha^{q-1} = 1\}$ for some $\alpha \in \mathbf{F}_q$. The non-zero elements of $\mathbf{F}_q$ form a group $\mathbf{F}_q^*$ of order $q-1$ such that $\mathbf{F}_q^* \cong Z_{q-1}$.

1.1.1 Primitive element

A *primitive* element of a finite field $\mathbf{F}_q$ is an element $x \in \mathbf{F}_q$ such that, for every non-zero element $y \in \mathbf{F}_q$, there is an integer n with $y = x^n$.

Theorem 1.1. *If* $\mathbf{F}$ *is a finite field, then the order of* $\mathbf{F}$ *is* $q = p^h$*, where* p *is the prime characteristic of* $\mathbf{F}$ *and* h *is a positive integer.*

Proof. See [12, Chapter X]. □

Lemma 1.2. *If* $\mathbf{F}$ *is a finite field with* q *elements, then every* $x \in \mathbf{F}$ *satisfies* $x^q = x$.

Proof. For $x = 0$, the identity $x^q = x$ is trivial. The nonzero elements of $\mathbf{F}$ form a group of order $q-1$ under multiplication. Thus $x^{q-1} = 1$ for all $x \in \mathbf{F}$ with $x \neq 0$. The multiplication by x yields the result. □

1.1.2 Polynomials

If $\mathbf{F}$ is a field, then the polynomial ring in the indeterminate X over $\mathbf{F}$ is the set

$$\mathbf{F}[X] = \{a_n X^n + a_{n-1} X^{n-1} + \cdots + a_0 \mid a_i \in \mathbf{F}, n \in \mathbf{N}\},$$

where $\mathbf{N}$ is the set of non-negative integers.

(1) If $a_n \neq 0$, then $\deg(f(X)) = n$.

(2) If $a_n = 1$, then $f(X)$ is *monic*.

(3) A polynomial $f(X)$ is *reducible* if

$$f(X) = g(X)h(X),$$

where $g(X), h(X) \in \mathbf{F}[X]$ and both $\deg(g(X))$ and $\deg(h(X))$ are less than $\deg(f(X))$. Otherwise it is *irreducible*.

Definition 1.3. An element $\beta \in \mathbf{F}$ is called a *root* or a *zero* of the polynomial $f \in \mathbf{F}[X]$ if $f(\beta) = 0$.

Theorem 1.4. *If $f(X) \in \mathbf{F}[X]$ and $c \in \mathbf{F}$, then $X - c$ is a factor of $f(X)$ if and only if $f(c) = 0$.*

Proof. See [12, Chapter VIII]. □

1.1.3 Primitive polynomials

An irreducible polynomial $f(X) \in \mathbf{F}_p[X]$ of degree m is *primitive* if the smallest n for which $f(X)$ divides $X^n - 1$ is $n = p^m - 1$.

1.2 Finite groups

Definition 1.5. A *group* is an ordered pair $(G, *)$, where G is a non-empty set and $*$ is a binary operation on G such that the following properties hold.

(1) For all a, b, c G, $a * (b * c) = (a * b) * c$.

(2) There exists $e \in G$ such that for all $a \in G$, $a * e = a = e * a$.

(3) For all $a \in G$, there exists $b \in G$ such that $a * b = e = b * a$.

Definition 1.6. A group G is *finite* if G contains only finitely many elements.

Definition 1.7. A group G is *cyclic* if every element of G is a power of a fixed element $g \in G$ such that, $G = \langle g \rangle = \{g, g^2, g^3, \ldots, g^n = 1\}$.

Definition 1.8. A group G is called trivial if it only contains the identity element.

A group G is called *abelian* or *commutative* if $ab = ba$ for any $a, b \in G$. The set Z of integers with the usual addition is an abelian group.

Theorem 1.9. *Any infinite cyclic group is isomorphic to the group Z, and any finite cyclic group of order n is isomorphic to the group Z_n.*

Proof. Let $\langle a\rangle$ be an infinite cyclic group. Define a mapping $\phi : Z \longrightarrow \langle a\rangle$ by the rule $\phi(i) = a^i$. Clearly, $\phi(i+j) = \phi(i)\phi(j)$ and ϕ is onto. Moreover, ϕ is injective if $a^i = a^j$ for some $i < j$, then $a^{j-i} = e$ and the group $\langle a\rangle$ would contain only the elements $e, a, \ldots, a^{j-i-1}$, which is impossible. Therefore ϕ is an isomorphism. If $\langle a\rangle$ is a cyclic group of order n, then the mapping $\phi : Z_n \longrightarrow \langle a\rangle$, given by the same rule $\phi(i) = a^i$, is an isomorphism.

□

Theorem 1.10. *Any subgroup of a cyclic group is cyclic.*

Proof. Let $\langle a\rangle$ be a cyclic group. Let H be a nontrivial subgroup of $\langle a\rangle$ and let m be the smallest positive integer, such that $a^m \in H$. Then $\langle a^m\rangle \leq H$. An arbitrary element of H has the form a^k. Dividing k by m, we get $k = mq + r, 0 \leq r < m$. Then $a^r = a^k(a^m)^{-q} \in H$. By the minimality of m it follows that $r = 0$. So, $a^k = (a^m)^q \in \langle a^m\rangle$.

□

1.2.1 Group action on a set

Let G be a group acts on a set X if for each $g \in G$ and $x \in X$ an element $gx \in X$ is defined, such that $g_2(g_1x) = (g_2g_1)x$ and $ex = x$ for all $x \in X,\ g_1, g_2 \in G$.
The set

$$\mathscr{O}rb(x) = \{gx \mid g \in G\},$$

is called the orbit of the element x. The stabilizer of an element x of X is the subgroup

$$S = \{g \in G \mid gx = x\}.$$

The fixed points set of an element g of G is the set defined as follows:

$$\mathscr{F}ix(g) = \{x \in X \mid gx = x\}.$$

1.3 Incidence structures

An incidence structure $\mathscr{I}$ is a triple $(\mathscr{P},\mathscr{B},I)$, where $\mathscr{P}$ is a set whose elements are called points, $\mathscr{B}$ is a set whose elements are called blocks or lines, and I is a symmetric incidence relation, such that $I \subseteq (\mathscr{P}\times\mathscr{B})\cup(\mathscr{B}\times\mathscr{P})$. If $(P,B)\in I$, we say that P is incident with B, or P lies on B, or B is incident with P, that B contains P. The points in an incidence structure that lie on the same line are collinear points. Also, The lines that pass through a point are concurrent lines. The dual of an incidence structure $(\mathscr{P},\mathscr{B},I)$ is also incidence structure denoted by $\mathscr{I}^D = (\mathscr{B},\mathscr{P},I)$. For instance, three points are collinear if they are contained in a line. Dually, we say that three lines are concurrent in a point. An isomorphism between incidence structures is a bijection between the point sets together with a bijection between the line sets such that preserved incidence. An automorphism of an incidence structure is an isomorphism from the incidence structure to itself. An incidence structure is said to be a self-dual if it is isomorphic to its dual.

1.4 Projective space over a field

Let $V(n+1,\mathbf{F})$ be the vector space of dimension $n+1$ over a field $\mathbf{F}$ with origin 0. Then the *n-dimensional projective space* is defined to be the set of the equivalence classes of $V(n+1,\mathbf{F})\setminus\{0\}$, where $(x_0,\ldots,x_n)\sim(\lambda x_0,\ldots,\lambda x_n), \forall\lambda\in\mathbf{F}\setminus\{0\}$, that is the set of all 1-dimensional subspaces of $V(n+1,\mathbf{F})$ and it is denoted by $\mathrm{PG}(n,\mathbf{F})$. When $\mathbf{F}=\mathbf{F}_q$, such that $V(n+1,q)=\mathbf{F}_q^{n+1}$, the projective space is denoted by $\mathrm{PG}(n,q)$. The 1-dimensional subspace $u=\{\lambda(x_0,x_1,\ldots,x_n) \mid \lambda\in\mathbf{F}_q\setminus\{0\}\}$ of $V(n+1,q)$ gives the point $\mathbf{P}(u)=\mathbf{P}(x_0,\ldots,x_n)$ of $\mathrm{PG}(n,q)$. The points $\mathbf{P}(x_0),\ldots,\mathbf{P}(x_n)$ are linearly independent if a set of vectors $x_0,\ldots,x_n$ representing them are linearly independent. The subspaces of $\mathrm{PG}(n,q)$ of dimension $0,1,2,3$, and $n-1$ are called *points*, *lines*, *planes*, *solids*, and *hyperplanes*.

Suppose that π_n and π_m are subspaces of $\mathrm{PG}(n,q)$; then the intersection of π_n and π_m denoted by $\pi_n\cap\pi_m$, that is the set of points common to π_n and π_m and is also a subspace. The span of π_n and π_m denoted by $\pi_n\pi_m$ is the smallest subspace containing the points of π_n and π_m. The number of points in $\mathrm{PG}(n,q)$ equals $\left(q^{n+1}-1\right)/(q-1)$. Also, the number of $(k-1)$-dimensional subspaces in $\mathrm{PG}(n-1,q)$ is given by the following:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n-1)(q^{n-1}-1)\cdots(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)\cdots(q-1)}$$

1.4.1 The projective plane $\mathrm{PG}(2,q)$

The projective plane $\mathrm{PG}(2,q)$ over $\mathbf{F}_q$ contains q^2+q+1 points and lines. There are $q+1$ points on each line and $q+1$ lines passing through each point. The value of q that has been used in this work is $q=13$. Therefore the projective plane $\mathrm{PG}(2,13)$ has 183 points and lines, with 14 points on each line and 14 lines passing through each point. The point $\mathbf{P}(x_0,x_1,x_2)$ in the projective plane, $\mathrm{PG}(2,q)$, can be represented as a vector of three coordinates over $\mathbf{F}_q$ as shown in Table 1.1.

Table 1.1: **The points in** $\mathrm{PG}(2,q)$

Point format	Number of points
$\mathbf{P}(x_0,x_1,1)$	q^2
$\mathbf{P}(x_0,1,0)$	q
$\mathbf{P}(1,0,0)$	1

A line in $\mathrm{PG}(2,q)$ is a set of points $\mathbf{P}(x_0,x_1,x_2)$ satisfying the homogeneous linear equation

$$ax_0+bx_1+cx_2=0,$$

with $a,b,c\in\mathbf{F}_q$ not all zero; it is denoted by $\mathbf{L}(a,b,c)$. Thus, a projective plane is an incidence structure of points and lines with the following properties:

(i) every two points are incident with a unique line;

(ii) every two lines are incident with a unique point;

(iii) there are four points, no three collinear.

1.4.2 General linear group of a vector space

Let $\mathbf{F}_q$ is a finite field and let $V(n,q)$ is a vector space of dimension n over $\mathbf{F}_q$, then the linear map $V(n,q)\longrightarrow V(n,q)$, such that $x\longrightarrow xA$, for $x\in V$ a row vector and A a non-singular $n\times n$

matrix over $\mathbf{F}_q$. The group consisting of all linear maps of $V(n,q)$, that is, the group consisting of all non-singular $n\times n$ matrices over $\mathbf{F}_q$, is called the general linear group and is denoted by $\mathrm{GL}(n,q)$. The order of $\mathrm{GL}(n,q)$ is as follows:

$$|\mathrm{GL}(n,q)| = (q^n-1)(q^n-q)(q^n-q^2)\cdots(q^n-q^{n-1}).$$

In addition, the subgroup $\mathrm{SL}(n,q)$ consisting of all matrices with determinant 1, and it is called the special linear group of degree n over $\mathbf{F}_q$. The group $\mathrm{SL}(n,q)$ contains a subgroup $\mathrm{UT}(n,q)$ consisting of those matrices with all entries below the main diagonal zero, and with the entries on the main diagonal equal to the identity. This subgroup is called the unitriangular group of degree n over $\mathbf{F}_q$.

1.4.3 Collineations and projectivities

(1) If $\mathscr{P}$ and $\mathscr{P}'$ are two projective spaces $\mathrm{PG}(k-1,q)$, then a *collineation* is a bijection mapping $\varphi:\mathscr{P}\longrightarrow\mathscr{P}'$ that preserves incidence. The collineations of $\mathrm{PG}(n,q)$ with the operation of composition, form a group, denoted by $\mathrm{P\Gamma L}(n+1,q)$ and called the collineation group of $\mathrm{PG}(n,q)$.

(2) For two spaces S and S', a *projectivity* is a bijection $\mathscr{T}:S\longrightarrow S'$ given by a non-singular $(n+1)\times(n+1)$ matrix H such that $\mathbf{P}(X)\mathscr{T}=\mathbf{P}(X')$ if $XH=tX'$ where $\mathbf{P}(X)$ and $\mathbf{P}(X')$ are points with coordinate vectors X and X', and $t\in\mathbf{F}_q\setminus\{0\}$.

    The group of the projectivities of $\mathrm{PG}(n,q)$ is the projective general linear group and denoted by $\mathrm{PGL}(n+1,q)$; it has order

$$|\mathrm{PGL}(n+1,q)| = q^{n(n+1)/2}\prod_{i=2}^{n+1}(q^i-1)\ .$$

Definition 1.11. A projectivity $\mathscr{T}$ is called *cyclic* if it permutes the points of $\mathrm{PG}(n,q)$ in a single cycle.

1.4.4 The fundamental theorem in $\mathrm{PG}(2,q)$

Theorem 1.12. *If $\phi : \mathscr{P} \longrightarrow \mathscr{P}'$ is a bijective mapping from one projective plane, $\mathrm{PG}(2,q)$, to another, then there is a unique projectivity shifting any quadrangle, that is, a set of four points no three collinear, to another quadrangle.*

The following section illustrates the construction of $\mathrm{PG}(2,q)$ in terms of points and lines.

1.5 The structure of $\mathrm{PG}(2,q)$

A cyclic projectivity $\mathscr{T}$ plays an important role in the construction of a projective plane. Let

$$f(x) = x^3 - a_2x^2 - a_1x - a_0$$

be a monic, primitive polynomial over $\mathbf{F}_q$; then its companion matrix M is

$$M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_0 & a_1 & a_2 \end{bmatrix}.$$

Let λ be a root of $f(x)$; then

$$\lambda^3 = a_2\lambda^2 + a_1\lambda + a_0 .$$

Therefore, for $i \in \left[0, q^2+q\right]$, there exist $y_0^i, y_1^i, y_2^i \in \mathbf{F}_q$ such that

$$\begin{aligned} \lambda^i &= y_2^{(i)}\lambda^2 + y_1^{(i)}\lambda + y_0^{(i)}, \\ \lambda^{(i+1)} &= y_2^{(i+1)}\lambda^2 + y_1^{(i+1)}\lambda + y_0^{(i+1)}. \end{aligned}$$

Thus, each point $\mathbf{P}(y_0^{(i)}, y_1^{(i)}, y_2^{(i)})$ in $\mathrm{PG}(n,q)$ can be generated as follows:

$$(y_0^{(i+1)}, y_1^{(i+1)}, y_2^{(i+1)}) = (y_0^{(i)}, y_1^{(i)}, y_2^{(i)})M;$$

here $y_0^{(i)}, y_1^{(i)}, y_2^{(i)}$ are the coordinates of the point P in $\mathrm{PG}(2,q)$.

Now, let $P(i) = \mathbf{P}(y_0^{(i)}, y_1^{(i)}, y_2^{(i)})$; then

$$P(i+1) = P(i)M.$$

In general, $P(j) = P(i)M^{j-i}$ for $0 \leq i < j \leq q^2+q$.

Let $P(0) = \mathbf{P}(1,0,0)$; then $\mathbf{P}(1) = (0,1,0)$, $\mathbf{P}(2) = (0,0,1)$. Since the order of the projectivity $\mathscr{T}$ is q^2+q+1, so

$$\mathrm{PG}(2,q) = \{P(0)M^i \mid i = 0,1,\ldots,q^2+q\}.$$

Also, the plane can always be represented via the array of dimension $(q^2+q+1) \times (q+1)$ as given below, with the entries reduced modulo q^2+q+1.

$$L = \begin{bmatrix} d_0 & d_1 & \ldots & d_q \\ d_0+1 & d_1+1 & \ldots & d_q+1 \\ \vdots & \vdots & & \vdots \\ d_0+q^2+q & d_1+q^2+q & \ldots & d_q+q^2+q \end{bmatrix};$$

here the first line comprises the points $P(0) = d_0$, $P(1) = d_1$, and the points $P(i)$ for $i = d_2,\ldots,d_q$ which are collinear with $P(0)$ and $P(1)$. The rows of L represent the lines of $\mathrm{PG}(2,q)$ while the columns represent the points. So, the cyclic projectivity provides a nice structure of the points and lines in $\mathrm{PG}(2,q)$.

Note that the companion matrix that has been used to construct the projective plane, $\mathrm{PG}(2,13)$ is as follows:

$$M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & 0 & 12 \end{bmatrix}.$$

Also, the first line of the 183 lines of $\mathrm{PG}(2,13)$ is $l_1 = \{1,2,9,21,36,83,89,115,119,128,133,144,161,182\}$. See Appendix A for tables of points and lines of $\mathrm{PG}(2,13)$.

1.6 Arcs and complete arcs over a projective space

Definition 1.13. (1) A $(k;n)$-*arc* $\mathscr{K}$ in $\mathrm{PG}(2,q)$ is a set of k points such that no $n+1$ of them are collinear but some n are collinear.

(2) A $(k;n)$-arc $\mathscr{K}$ in $\mathrm{PG}(2,q)$ is *complete* if it is not contained in any $(k+1;n)$-arc.

(3) If a line l of $\mathrm{PG}(2,q)$ meets $\mathscr{K}$ in i points, that is, $|l \cap \mathscr{K}| = i$, then it is an i-*secant* of $\mathscr{K}$. Thus, for $i = 0,1,2,3,4$, it is called 0-secant, 1-secant, 2-secant, 3-secant, and 4-secant.

(4) The smallest and the largest values of k for which a $(k;n)$-arc exists in $\mathrm{PG}(2,q)$ are denoted by $t_n(2,q)$ and $m_n(2,q)$.

Remark

Throughout Chapters 2 to Chapter 6 of this thesis, $n = 4$.

Corollary 1.14. *An upper bound for the size of an arc of degree n is*

$$m_n(2,q) \leq (n-1)q + n.$$

A $(k;n)$-arc which attains the above bound is called *maximal arc*.

Theorem 1.15. *If* $\mathscr{K}$ *is a maximal* $(k;n)$*-arc in* $\mathrm{PG}(2,q)$*, then one of the following holds:*

(1) $n = q+1$ and $\mathscr{K} = \mathrm{PG}(2,q)$;

(2) $n = q$ and $\mathscr{K} = \mathrm{AG}(2,q) = \mathrm{PG}(2,q) \setminus \ell$, for some line ℓ;

(3) $2 \leq n < q$, $n \mid q$, and the dual of the external lines of $\mathscr{K}$ forms a $((q+1-n)q/n; q/n)$-arc, also maximal.

Proof. See [18, Chapter 12]. □

Corollary 1.16. *A* $(k;n)$*-arc is a maximal if and only if every line in* $\mathrm{PG}(2,q)$ *is either an n-secant or an external line.*

Corollary 1.17. *If $2 < n < q$ and n does not divied q, then*

$$m_n(2,q) \leq (n-1)q + n - 2.$$

Theorem 1.18. *For q odd, a $(k;4)$-arc satisfies $k \leqslant 3q+1$.*

Proof. See [18, Chapter 12]. □

Corollary 1.19. *A $(k;n)$-arc with $n \geqslant 4$ in $\mathrm{PG}(2,q)$, $q \not\equiv 0 \pmod{n}$, satisfies*

$$k \leq (n-1)q + n - 3.$$

Theorem 1.20. *If $\mathscr{K}$ is a $(k;n)$-arc in $\mathrm{PG}(2,q)$ that has an external line and if $(n,q) = 1$, then $k \leq (n-1)q+1$.*

Proof. See [18, Chapter 12]. □

Theorem 1.21. *If $\mathscr{K}$ is a $(k;n)$-arc in $\mathrm{PG}(2,q)$ that has no external line, then*

$$k \leq (n-1)q + q - \sqrt{q^2 + q - nq}.$$

Proof. See [18, Chapter 12]. □

1.7 Plane algebraic curves

Let $\mathbf{F}$ be a field. Given a form f in the polynomial ring $\mathbf{F}[X,Y,Z]$, a *plane algebraic curve* $\mathscr{C} = \mathbf{v}(f) = (\mathbf{V}(f), f)$, where

$$\mathbf{V}(f) = \{\mathbf{P}(x,y,z) \in \mathrm{PG}(2,\mathbf{F}) \mid f(x,y,z) = 0\}.$$

(i) When $n = 1,2,3,4,5$ then $\mathscr{C}$ is a *line*, *conic*, *cubic*, *quartic*, *quintic*.

(ii) If $f(X,Y,Z)$ has a factorisation as $f(X,Y,Z) = g(X,Y,Z)h(X,Y,Z)$ where $deg(g) < deg(f)$ and $deg(h) < deg(f)$, then $\mathscr{C}$ is a *reducible* curve. Otherwise it is *irreducible*.

1.7.1 Quartic curves

From above, a plane quartic curve $\mathbf{v}(f)$ has degree four:

$$f = \sum_{i+j+k=4} a_{ijk} X^i Y^j Z^k ,$$

with not all $a_{ijk} = 0$.

1.7.2 Tangent line

Let $\mathscr{C} = \mathbf{v}(f)$ be an algebraic curve, and let $P = \mathbf{P}(x_0, y_0, z_0)$ be a point on $\mathscr{C}$. Then the tangent line ℓ to $\mathscr{C}$ at the point P has the following equation:

$$f_x(x - x_0) + f_y(y - y_0) + f_z(z - z_0) = 0.$$

1.7.3 The intersection multiplicity of a line and a curve

Let $\mathscr{C} = \mathbf{v}(f)$ be an algebraic curve, ℓ a line with equation $ax + by + cz = 0$, and $P = \mathbf{P}(x_0, y_0, z_0)$ a point on $\mathscr{C} \cap \ell$. Then the intersection multiplicity m of ℓ and $\mathscr{C}$ at P is calculated as follows. By substitution,

$$f((x_0, y_0, z_0) + t(x, y, z)) = t^m g(x, y, z),$$

where t does not divide g. Write $m = I(P; \ell, \mathscr{C})$.

1.7.4 The multiplicity of a point on a curve

The multiplicity of P on $\mathscr{C}$ is

$$m_P(\mathscr{C}) = \min\{I(P; \ell, \mathscr{C}) \mid \ell \subset \mathrm{PG}(2, q)\}.$$

1.7.5 Singular points

Let $\mathscr{C}$ be a plane curve given by the polynomial f and let $P = \mathbf{P}(x, y, z)$ be a point on $\mathscr{C}$. Then P is a *singular* point of $\mathscr{C}$ if it satisfies the following property:

$$f_x(x,y,z) = f_y(x,y,z) = f_z(x,y,z) = 0.$$

Therefore, the following are satisfied.

1. P is a *singular* point of $\mathscr{C}$ if every line through P has at least two-point intersection with $\mathscr{C}$ at P.

2. If the plane curve $\mathscr{C}$ has a *singular* point, then $\mathscr{C}$ is called a *singular* curve. Otherwise it is *non-singular* curve.

Inflexion points

Let $\mathscr{C}$ be a plane curve and let $P = \mathbf{P}(x,y,z)$ be a non-singular point of $\mathscr{C}$ with tangent ℓ to $\mathscr{C}$ at P. Then P is said to be an *inflexion* point of $\mathscr{C}$ if it satisfies the following property:

$$I(P;\ell,\mathscr{C}) \geq 3.$$

1.7.6 Hessian curve

Let $\mathscr{C}$ be a curve given by the polynomial f; then the Hessian $\mathscr{H}_{\mathscr{C}}$ of $\mathscr{C}$ is defined as follows: $\mathscr{H}_{\mathscr{C}} = \mathbf{v}(H)$, where

$$H = \det \begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix}.$$

Therefore, if the curve $\mathscr{C}$ has degree d then the second partial derivatives of f have degree $d-2$; this implies that the Hessian curve $\mathscr{H}_{\mathscr{C}}$ has degree $3(d-2)$. A non-singular point $P = \mathbf{P}(x,y,z)$ on $\mathscr{H}_{\mathscr{C}}$ is an inflexion point of $\mathscr{C}$.

1.7.7 The intersection of curves

Let $\mathscr{C}_1$ and $\mathscr{C}_2$ are two irreducible plane curves of degree n and m and if $P \in \mathscr{C}_1 \cap \mathscr{C}_2$, then the intersection number at P between $\mathscr{C}_1$ and $\mathscr{C}_2$ is an integer, denoted by $I(P;\mathscr{C}_1,\mathscr{C}_2)$.
Over an algebraically closed field, the following theorem is true.

Bézout's theorem

Suppose that $\mathscr{C}_1$ and $\mathscr{C}_2$ are two homogeneous curves of degree m_1 and m_2 which have no common component. Then the sum of the intersection numbers of $\mathscr{C}_1$ and $\mathscr{C}_2$ is the product of their degrees; that is, $\sum_{P\in\mathscr{C}_1\cap\mathscr{C}_2} I(P;\mathscr{C}_1,\mathscr{C}_2) = m_1m_2$.

1.7.8 The Hasse–Weil–Serre bound

Let $\mathscr{C}$ be a non-singular projective curve of genus g defined over the finite field $\mathbf{F}_q$, then the Hasse–Weil–Serre bound states that the number N_1 of rational points of the curve $\mathscr{C}$ satisfies the following inequality:

$$|N_1-(q+1)| \leq g\lfloor 2\sqrt{q}\rfloor \ ,$$

where g is the genus of the curve $\mathscr{C}$.

1.8 Coding theory

1.8.1 Error-correcting codes

The main task of error-correcting codes is to correct a message which transmitted through a noisy channel by adding redundant information to this message so that is possible to detect or correct errors.

1.8.2 Linear codes and their parameters

A linear $[n,k,d]$-code C over the finite field $\mathbf{F}_q$ is a subspace of dimension k of the n-dimensional vector space $V(n,q)=\mathbf{F}_q^n$ such that any two distinct vectors in C differ in at least d places. The elements of the code are called codewords. Also, the parameters n,k, and d are called the length, dimension, and minimum distance of C.

For any two code words $c_1,c_2\in C$, the minimum distance (Hamming distance) between c_1 and c_2 is denoted by $d(c_1,c_2)$ and it is defined to be the number of positions in which the corresponding coordinates differ. The minimum distance of C is

$$d(C) = min\{d(c_1, c_2) \mid c_1, c_2 \in C, c_1 \neq c_2\}.$$

Definition 1.22. Let $x, y \in V(n,q)$. Then

$$x \cdot y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

is the *scalar product* of x and y.

Definition 1.23. The weight $w(x)$ of $x \in V(n,q)$ is

$$w(x) = d(x, 0);$$

that is, $w(x)$ is the number of non-zero elements in x.

Remark

(1) If $x \cdot y = 0$, then x and y are *orthogonal* .

(2) The scalar product satisfies the following:

- (i) $(x+y) \cdot z = x \cdot z + y \cdot z$;
- (ii) $(\lambda x \cdot y) = \lambda (x \cdot y)$;
- (iii) $x \cdot y = y \cdot x$.

1.8.3 Generator and parity-check matrices

A *generator matrix* of a linear $[n,k,d]$-code C is a $k \times n$ matrix over the finite field $\mathbf{F}_q$ whose rows form a basis of C; it is denoted by G.

Let C be a linear $[n,k]$-code over $\mathbf{F}_q$. Then the dual code is defined as follows:

$$C^{\perp} = \{x \in V(n,q) \mid x \cdot y = 0, \text{for all } y \in C\}.$$

A *parity check matrix* of a linear $[n,k,d]$-code C is an $(n-k)\times n$ matrix over $\mathbf{F}_q$ whose rows form a basis of $C^{\perp}$; it is denoted by H.

Lemma 1.24. *If C is an $[n,k]$-code with generator matrix G, then*

(i) *$C^{\perp}$ is a linear code*;

(ii) *$C^{\perp}=\{x\in V(n,q)\mid xG^T=0\}$; that is, x is orthogonal to every row of G.*

Proof. (i) If $y,y_1\in C^{\perp}$, then

$$x\cdot y=x\cdot y_1=0 \text{ for all } x\in C$$

implies that

$$\begin{aligned} x\cdot(y+y_1) &= 0 \text{ for all } x\in C,\\ x\cdot(\lambda y) &= 0 \text{ for all } x\in C,\ \lambda\in\mathbf{F}_q. \end{aligned}$$

(ii)

$$\begin{aligned} xG^T=0 &\iff x(r_1^T,\dots,r_k^T)=0\\ &\iff x\cdot r_i=0 \text{ for all } i \end{aligned}$$

where $r_1,\dots,r_k$ are the rows of G.

□

Lemma 1.25. *If $d(C)=d$, then every $d-1$ columns of a parity check matrix H are linearly independent but some d columns are dependent.*

1.8.4 The Singleton bound

For an $[n,k,d]$-code C over the finite field $\mathbf{F}_q$, the Singleton bound states that

$$d(C)\le n-k+1.$$

1.8.5 MDS codes

An $[n,k,d]$-code C over the finite field $\mathbf{F}_q$ is said to be a maximum distance separable (MDS) code if d satisfies the following bound:

$$d(C) = n - k + 1.$$

Remark

In 1961, the packing problem, that is, the problem to determine the largest size $m_n(2,q)$ of a $(k;n)$-arc $\mathscr{K}$ in $\mathrm{PG}(2,q)$ showed an interesting connection with coding theory. This connection is between $(k;n)$-arcs in $\mathrm{PG}(2,q)$ and the $[n,k,d]_q$-codes in coding theory. This link gives the following theorem.

Theorem 1.26. [16] *There exists a projective* $[n,k,d]_q$*-code if and only if there exists a* $(k;k-d)$*-arc in* $\mathrm{PG}(k-1;q)$.

1.9 Lexicographically least set

Given the sets $A = \{a_1,\ldots,a_r\}$ and $B = \{b_1,\ldots,b_r\}$ of integers, with $a_1 < a_2 < \cdots < a_r$ and $b_1 < b_2 < \cdots < b_r$. Then $A \le B$ lexicographically if either $A = B$ or if, for some i with $1 \le i < r$, we have $a_1 = b_1,\ldots,a_i = b_i$, but $a_{i+1} < b_{i+1}$.

Chapter 2

The classification of certain $(k;4)$-arcs in $\mathrm{PG}(2,13)$

Introduction

This chapter contains a new approach to classify certain $(k;4)$-arcs in $\mathrm{PG}(2,13)$. This classification of $(k;4)$-arcs is for $k = 6,\ldots,38$. This method is a combination between the projectively inequivalent $(k;4)$-arcs and certain *sd*-inequivalent $(k;4)$-arcs that have *sd*-inequivalent classes of the *i*-secant distributions. Also, the $(k;4)$-arcs are classified up to projective inequivalence for certain values of k. This classification starts from the 77 incomplete $(34;4)$-arcs formed from the *sd*-inequivalent $(33;4)$-arcs constructed in the first classification. The previous combination process is re-iterated to give another classification of $(k;4)$-arcs; this gives a new size of $(k;4)$-arc for $k > 35$. In addition, the corresponding linear codes of each $(k;4)$-arc are introduced via their parameters n,k,d and e.

2.1 The approach

In this approach, the classification of $(k;4)$-arcs is established by classifying projectively inequivalent $(6;4)$-arcs. This process starts by fixing a frame, that is, the vertices of the quadrangle, in the plane, $\mathrm{PG}(2,13)$. This is the set $\mathscr{S}_0 = \{1,2,3,88\}$. Then, all the points from $\mathrm{PG}(2,13)$ which are not on any bisecant of $\mathscr{S}_0$ are added to form the $(5;3)$-arcs. These points are as follows:

$4,8,9,10,11,12,13,17,20,21,22,23,28,30,35,36,37,38,45,51,53,54,61,66,68,69,73,76,82,$
$83,84,85,89,90,91,103,114,115,116,117,118,119,120,121,127,128,129,130,132,133,134,$
$135,141,143,144,145,146,150,156,160,161,162,163,167,171,180,181,182,183.$

Hence, the constructed number of $(5;3)$-arcs is 69; this took 1834 milliseconds. The isomorphisms among the 69 $(5;3)$-arcs are tested according to the lexicographically least sets in the G-orbits [28] of $(5;3)$-arcs, where $G = PGL(2,13)$; it took 2330 msc. Then the number of the projectively inequivalent $(5;3)$-arcs is four. Also, the stabilisers of the four projectively inequivalent $(5;3)$-arcs are $Z_2 \times Z_2$, Z_6, Z_2, D_4; this took 2312 msec. These stabilisers partition the four projectively inequivalent $(5;3)$-arcs into a number of orbits. The i-secant distributions of the 69 $(5;3)$-arcs took 1780 msec. It shows that there are only two sd-inequivalent classes of i-secant distributions among the 69 i-secant distributions of $(5;3)$-arcs. The statistics of the projectively inequivalent $(5;3)$-arcs are shown in Table 2.1.

Table 2.1: **Projectively inequivalent** $(5;3)$**-arcs in** $\mathrm{PG}(2,13)$

Symbol	(5;3)-arc	Stabiliser	Orbits	$\{t_4,t_3,t_2,t_1,t_0\}$
$\mathscr{A}_1$	$\{1,2,3,88,9\}$	$Z_2 \times Z_2$	$\{1\},\{2,9\},\{3,88\}$	$\{0,1,7,53,122\}$
$\mathscr{A}_2$	$\{1,2,3,88,17\}$	Z_6	$\{1,2\},\{3,17,88\}$	$\{0,1,7,53,122\}$
$\mathscr{A}_3$	$\{1,2,3,88,22\}$	Z_2	$\{1,88\},\{2\},\{3\},\{22\}$	$\{0,1,7,53,122\}$
$\mathscr{A}_4$	$\{1,2,3,88,135\}$	D_4	$\{1,2,88,3\},\{135\}$	$\{0,2,4,56,121\}$

Theorem 2.1. *In* $\mathrm{PG}(2,13)$*, there are exactly four projectively inequivalent* $(5;3)$*-arcs.*

2.2 Projectively inequivalent $(6;4)$-arcs

The projectively inequivalent $(6;4)$-arcs in $\mathrm{PG}(2,13)$ are established as follows. All the points from the plane which are not on any 4-secant of each $(5;3)$-arc in Table 2.1 are added separately to $\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3$, and $\mathscr{A}_4$. The number of these points for each $(5;3)$-arc in Table 2.1 is $11,11,11,22$ respectively and they are given as follows.

(1) The added points for $\mathscr{A}_1$ are $21,36,83,89,115,119,128,133,144,161,182$.

(2) The added points for $\mathscr{A}_2$ are $12,28,45,66,68,69,76,103,150,156,182$.

(3) The added points for $\mathscr{A}_3$ are $10,37,84,90,116,120,129,134,145,162,183$.

(4) The added points for $\mathscr{A}_4$ are $4,11,13,23,30,38,51,53,54,61,73,85,91,117,121,130,141,146,163,167,171,180$.

The constructed number of $(6;4)$-arcs is 55. It took 1931 msec. Among these $(6;4)$-arcs, the canonical sets in the G-orbits of $(6;4)$-arcs that took 2257 msc show that the number of the projectively inequivalent $(6;4)$-arcs is ten having four stabiliser types; they are Z_2, $Z_2 \times Z_2$, Z_6, $Z_4 \times Z_2$, which took 2370 msec. In addition, the i-secant distrbutions of $(6;4)$-arcs are computed. The timing is 2005 msec. It shows that there are only two sd-inequivalent classes of i-secant distrbutions $\{t_4,t_3,t_2,t_1,t_0\}$ of $(6;4)$-arcs. The data of the projectively inequivalent $(6;4)$-arcs are given in Table 2.2.

Table 2.2: **Projectively inequivalent** $(6;4)$**-arcs in** PG$(2,13)$

Symbol	(6;4)-arc	Stabiliser	Orbits	$\{t_4,t_3,t_2,t_1,t_0\}$
$\mathscr{B}_1$	$\{1,2,3,88,9,21\}$	Z_2	$\{1\},\{2\},\{3,88\},\{9\},\{21\}$	$\{1,0,9,62,111\}$
$\mathscr{B}_2$	$\{1,2,3,88,9,83\}$	Z_2	$\{1\},\{2\},\{3,88\},\{9\},\{83\}$	$\{1,0,9,62,111\}$
$\mathscr{B}_3$	$\{1,2,3,88,9,89\}$	Z_2	$\{1\},\{2\},\{3,88\},\{9\},\{89\}$	$\{1,0,9,62,111\}$
$\mathscr{B}_4$	$\{1,2,3,88,9,115\}$	$Z_2 \times Z_2$	$\{1,9\},\{2,115\},\{3,88\}$	$\{1,0,9,62,111\}$
$\mathscr{B}_5$	$\{1,2,3,88,9,182\}$	$Z_2 \times Z_2$	$\{1\},\{2,9\},\{3,88\},\{182\}$	$\{1,1,6,65,110\}$
$\mathscr{B}_6$	$\{1,2,3,88,17,68\}$	Z_6	$\{1,2\},\{3,17,88\},\{68\}$	$\{1,0,9,62,111\}$
$\mathscr{B}_7$	$\{1,2,3,88,17,182\}$	Z_6	$\{1,2\},\{3,17,88\},\{182\}$	$\{1,1,6,65,110\}$
$\mathscr{B}_8$	$\{1,2,3,88,22,116\}$	$Z_4 \times Z_2$	$\{1,88\},\{2,22,3,116\}$	$\{1,0,9,62,111\}$
$\mathscr{B}_9$	$\{1,2,3,88,22,145\}$	$Z_2 \times Z_2$	$\{1,88\},\{2,145\},\{3,22\}$	$\{1,0,9,62,111\}$
$\mathscr{B}_{10}$	$\{1,2,3,88,22,183\}$	Z_2	$\{1,88\},\{2\},\{3\},\{22\},\{183\}$	$\{1,1,6,65,110\}$

Theorem 2.2. *In* PG$(2,13)$*, there are exactly ten projectively inequivalent* $(6;4)$*-arcs.*

Remark

(1) Let Φ_4 be the number of projectively inequivalent $(k;4)$-arcs in PG$(2,13)$. The memory issues and the increasing size of Φ_4 make the computation of Φ_4 too difficult. Therefore, an approach has been used to reduce the number of Φ_4 and also to classify the sets of $(k;4)$-arcs. This approach is for $k=7,\dots,38$.

(2) If two arcs $\mathscr{K}$ and $\mathscr{K}'$ have sd-inequivalent classes of i-secant distributions or sd-inequivalent stabiliser groups, they are *sd-inequivalent*.

2.3 sd-inequivalent $(7;4)$-arcs

Now, to classify the sd-inequivalent $(7;4)$-arcs $\mathscr{C}_i$ where $\mathscr{C}_i = \mathscr{B}_j \cup P_i$, for $j = 1,\ldots,10$, and $P_i \in \mathrm{PG}(2,13)$. This approach used another method to classify these sets. The method is based on using the points P_i of $\mathrm{PG}(2,13)$ which are not on any 4-secant to each of the 10 projectively inequivalent $(6;4)$-arcs constructed, and then adding these points to each projectively inequivalent $(6;4)$-arc; these are given in Table 2.2. Given a $(6;4)$-arc, say $\mathscr{B}_1$, the points P_i of $\mathrm{PG}(2,13)$ which are not on any 4-secant to $\mathscr{B}_1$ can be add separately to $\mathscr{B}_1$ to establish the $(7;4)$-arcs. The same is done for $\mathscr{B}_2,\ldots,\mathscr{B}_{10}$. Then, the constructed number of $(7;4)$-arcs is 1670. The i-secant distributions $\{t_4,t_3,t_2,t_1,t_0\}$ of the 1670 $(7;4)$-arcs are computed. Then, they are partitioned into a number of sd-inequivalent classes N_c of i-secant distributions. The number of these classes is five and they are listed in Table 2.4. So, from each sd-inequivalent class choose an associated $(7;4)$-arc to form the set of sd-inequivalent $(7;4)$-arcs. Thus, the number of sd-inequivalent $(7;4)$-arcs is five. The statistics are given in Tables 2.3, 2.4, 2.5.

Table 2.3: **Points added to the 10 projectively inequivalent** $(6;4)$**-arcs**

Arc	Points added
$\mathscr{B}_1$	{4,5,6, 7, 8, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 37, 38, 39,40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69,70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 84, 85, 86, 87, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101,102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 116, 117, 118, 120, 121, 122, 123, 124, 125,126, 127, 129, 130, 131, 132, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 145, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 162, 163, 164, 165, 166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 183}
$\mathscr{B}_2$	{4,5,6, 7, 8, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 37, 38, 39,40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69,70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 84, 85, 86, 87, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101,102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 116, 117, 118, 120, 121, 122, 123, 124, 125,126, 127, 129, 130, 131, 132, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 145, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 162, 163, 164, 165, 166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 183}

$\mathscr{B}_3$	{4,5,6, 7, 8, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 37, 38, 39,40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69,70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 84, 85, 86, 87, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101,102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 116, 117, 118, 120, 121, 122, 123, 124, 125,126, 127, 129, 130, 131, 132, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 145, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 162, 163, 164, 165, 166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 183}
$\mathscr{B}_4$	{4,5,6, 7, 8, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 37, 38, 39,40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69,70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 84, 85, 86, 87, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101,102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 116, 117, 118, 120, 121, 122, 123, 124, 125,126, 127, 129, 130, 131, 132, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 145, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 162, 163, 164, 165, 166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 183}
$\mathscr{B}_5$	{4,5,6, 7, 8, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 37, 38, 39,40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69,70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 84, 85, 86, 87, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101,102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 116, 117, 118, 120, 121, 122, 123, 124, 125,126, 127, 129, 130, 131, 132, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 145, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 162, 163, 164, 165, 166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 183}
$\mathscr{B}_6$	{4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65,67, 70, 71, 72, 73, 74, 75, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 89, 90, 91, 92,93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127,128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 144, 145, 146, 147, 148, 149, 151, 152, 153, 154, 155, 157, 158, 159, 160, 161, 162, 163, 164, 165, 166, 167,168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 183}
$\mathscr{B}_7$	{4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59,60, 61, 62, 63, 64, 65, 67, 70, 71, 72, 73, 74, 75, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127,128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 144, 145, 146, 147, 148, 149, 151,152,153, 154, 155, 157, 158, 159, 160, 161, 162, 163, 164, 165, 166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 183}

$\mathscr{B}_8$	{4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 85, 86, 87, 89, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 117, 118, 119, 121, 122, 123, 124, 125, 126, 127, 128, 130, 131, 132, 133, 135, 136, 137, 138, 139, 140, 141, 142, 143, 144, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 161, 163, 164, 165, 166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 182}
$\mathscr{B}_9$	{ 4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 85, 86, 87, 89, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 117, 118, 119, 121, 122, 123, 124, 125, 126, 127, 128, 130, 131, 132, 133, 135, 136, 137, 138, 139, 140, 141, 142, 143, 144, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 161, 163, 164, 165, 166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 182}
$\mathscr{B}_{10}$	{4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49,50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 85, 86, 87, 89, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 117, 118, 119, 121, 122, 123, 124, 125, 126, 127, 128, 130, 131, 132, 133, 135, 136, 137, 138, 139, 140, 141, 142, 143, 144, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 161, 163, 164, 165, 166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 182}

Table 2.4: N_c **of** $\{t_4,t_3,t_2,t_1,t_0\}$ **of** $(7;4)$**-arcs**

Number	N_c	Number of N_c
1	$\{1,1,12,67,102\}$	851
2	$\{1,0,15,64,103\}$	504
3	$\{1,2,9,70,101\}$	264
4	$\{2,0,9,72,100\}$	33
5	$\{1,3,6,73,100\}$	18

Table 2.5: sd**-inequivalent** $(7;4)$**-arcs in** $\mathrm{PG}(2,13)$

Symbol	$(7;4)$-arc	Stabiliser	Orbits	$\{t_4,t_3,t_2,t_1,t_0\}$
$\mathscr{C}_1$	$\{1,2,3,88,9,21,5\}$	I	$\{1\}$, $\{2\}$, $\{3\}$, $\{5\}$, $\{9\}$, $\{21\}$, $\{88\}$	$\{1,0,15,64,103\}$
$\mathscr{C}_2$	$\{1,2,3,88,9,21,35\}$	I	$\{1\}$, $\{2\}$, $\{3\}$, $\{9\}$, $\{21\}$, $\{35\}$, $\{88\}$	$\{1,2,9,70,101\}$
$\mathscr{C}_3$	$\{1,2,3,88,22,145,12\}$	I	$\{1\}$, $\{2\}$, $\{3\}$, $\{12\}$, $\{22\}$, $\{88\}$, $\{145\}$	$\{1,1,12,67,102\}$
$\mathscr{C}_4$	$\{1,2,3,88,9,182,12\}$	D_4	$\{1,12\}$, $\{2,9,88,3\}$, $\{182\}$	$\{2,0,9,72,100\}$
$\mathscr{C}_5$	$\{1,2,3,88,9,182,35\}$	Z_2	$\{1,182\}$, $\{2\}$, $\{3,35\}$, $\{9\}$, $\{88\}$	$\{1,3,6,73,100\}$

Theorem 2.3. *In* $\mathrm{PG}(2,13)$*, there are at least five sd-inequivalent* $(7;4)$*-arcs.*

Remark

(1) In this thesis, the term $'sd$-inequivalent$'$ sets of $(k;4)$-arcs means these sets having sd-inequivalent classes N_c of secant distributions $\{t_4,t_3,t_2,t_1,t_0\}$ of the sets of $(k;4)$-arcs.

(2) The sets of $(k;4)$-arcs for $k=8,9,\ldots,38$ will be established by adding all the points from the plane which are not on any 4-secant to each sd-inequivalent $(k;4)$-arc for $k=7,\ldots,37$.

(3) The timings of $(k;4)$-arcs for $k=7,\ldots,38$ are given in Table 2.44.

2.4 sd-inequivalent $(8;4)$-arcs

The $(8;4)$-arcs in PG$(2,13)$ are constructed by adding all the points from the plane which are not on any 4-secant to each sd-inequivalent $(7;4)$-arc listed in Table 2.5. Thus, the constructed number of $(8;4)$-arcs is 820. The i-secant distributions of 820 $(8;4)$-arcs are found, and then according to the number N_c of sd-inequivalent classes of i-secant distributions of $(8;4)$-arcs, the number of sd-inequivalent $(8;4)$-arcs is 11 having the groups I, $Z_2\times Z_2$, D_4. The analytic data are given in Tables 2.6, 2.7,2.8.

Table 2.6: **Points added to the sd-inequivalent $(7;4)$-arcs**

Arc	Points added
$\mathscr{C}_1$	{4, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 84, 85, 86, 87, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 116, 117, 118, 120, 121, 122, 123, 124, 125,126, 127, 129, 130, 131, 132, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 145, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 162, 163, 164, 165, 166, 167,168, 169, 170, 171, 172, 173, 174, 175,176, 177, 178, 179, 180, 181, 183 }
$\mathscr{C}_2$	{4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 37, 38, 39, 40, 41, 42, 43, 44, 45,46, 47, 48, 49, 50, 51,52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 84, 85,86, 87, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 116, 117, 118, 120, 121, 122, 123, 124, 125,126, 127, 129, 130, 131, 132, 134, 135,136, 137, 138, 139, 140, 141, 142, 143, 145, 146, 147, 148, 149, 150, 151, 152, 153,154, 155, 156, 157, 158, 159, 160, 162, 163, 164, 165, 166,167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179,180,181,183 }

$\mathscr{C}_3$	{4, 5, 6, 7, 8, 9, 11, 13, 14, 15, 16, 17, 18, 19, 20, 21, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 85, 86, 87, 89, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 117, 118, 119, 121, 122, 123, 124, 125, 126, 127, 128, 130, 131, 132, 133, 135, 136, 137,138, 139, 140, 141, 142, 143,144, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 161, 163, 164, 165, 166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 182}
$\mathscr{C}_4$	{4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 37, 38, 39, 40, 41, 42, 43, 44, 45,46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 84, 85, 86, 87, 90, 91, 92,93,94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 116, 117, 118, 120, 121, 122, 123, 124, 125, 126, 127, 129, 130, 131, 132, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 145, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 162, 163, 164, 165, 166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 183 }
$\mathscr{C}_5$	{4, 5, 6, 7, 8, 10, 11, 13, 14, 15, 16, 18, 19, 20, 22, 23, 24, 25, 26, 27, 29, 30, 31, 32, 33, 34, 35, 37, 38, 39, 40, 41, 42, 43, 44, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 67, 70, 71, 72, 73, 74, 75, 77, 78, 79, 80, 81, 82, 84, 85, 86, 87, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 104, 105, 106, 107, 108, 109, 110,111, 112, 113, 114, 116, 117, 118, 120, 121, 122, 123, 124, 125, 126, 127, 129, 130, 131, 132, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 145, 146, 147, 148, 149, 151, 152, 153, 154, 155, 157, 158, 159, 160, 162, 163, 164, 165, 166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179,180, 181, 183 }

Table 2.7: N_c **of** $\{t_4,t_3,t_2,t_1,t_0\}$ **of** $(8;4)$**-arcs**

Number	N_c	Number of N_c
1	$\{1,2,16,70,94\}$	176
2	$\{1,1,19,67,95\}$	136
3	$\{1,0,22,64,96\}$	34
4	$\{1,3,13,73,93\}$	174
5	$\{1,4,10,76,92\}$	71
6	$\{2,2,10,78,91\}$	46
7	$\{2,3,7,81,90\}$	5
8	$\{1,5,7,79,91\}$	6
9	$\{1,6,4,82,90\}$	1
10	$\{2,1,13,75,92\}$	99
11	$\{2,0,16,72,93\}$	72

Table 2.8: sd**-inequivalent** $(8;4)$**-arcs in** PG$(2,13)$

Symbol	(8;4)-arc	Stabiliser	$\{t_4,t_3,t_2,t_1,t_0\}$
$\mathscr{D}_1$	$\{1,2,3,88,9,21,5,4\}$	I	$\{1,2,16,70,94\}$
$\mathscr{D}_2$	$\{1,2,3,88,9,21,5,15\}$	I	$\{1,0,22,64,96\}$
$\mathscr{D}_3$	$\{1,2,3,88,9,21,5,39\}$	I	$\{1,3,13,73,93\}$
$\mathscr{D}_4$	$\{1,2,3,88,9,21,35,7\}$	I	$\{1,4,10,76,92\}$
$\mathscr{D}_5$	$\{1,2,3,88,9,21,35,39\}$	I	$\{2,2,10,78,91\}$
$\mathscr{D}_6$	$\{1,2,3,88,22,145,12,16\}$	I	$\{1,1,19,67,95\}$
$\mathscr{D}_7$	$\{1,2,3,88,22,145,12,17\}$	I	$\{2,1,13,75,92\}$
$\mathscr{D}_8$	$\{1,2,3,88,22,145,12,28\}$	I	$\{2,0,16,72,93\}$
$\mathscr{D}_9$	$\{1,2,3,88,9,182,35,53\}$	$Z_2\times Z_2$	$\{2,3,7,81,90\}$
$\mathscr{D}_{10}$	$\{1,2,3,88,9,182,35,56\}$	I	$\{1,5,7,79,91\}$
$\mathscr{D}_{11}$	$\{1,2,3,88,9,182,35,135\}$	D_4	$\{1,6,4,82,90\}$

Theorem 2.4. *In* PG$(2,13)$*, there are at least eleven* sd*-inequivalent* $(8;4)$*-arcs.*

Remark

In Table 2.8, the group $Z_2\times Z_2$ partitions the sd-inequivalent $(8;4)$-arc $\mathscr{D}_9=\{1,2,3,88,9,182,35,53\}$ into 4 orbits as the following:

$$\mathscr{O}rb_1(\mathscr{D}_9)=\{1,35,3,182\},\mathscr{O}rb_2(\mathscr{D}_9)=\{2,53\},\mathscr{O}rb_3(\mathscr{D}_9)=\{9\},\mathscr{O}rb_4(\mathscr{D}_9)=\{88\}.$$

Also, the dihedral group splits the set $\mathscr{D}_{11}=\{1,2,3,88,9,182,35,135\}$ into three orbits; they are as follows:

$$\mathscr{O}rb_1(\mathscr{D}_{11})=\{1,182\},\mathscr{O}rb_2(\mathscr{D}_{11})=\{2,9\},\mathscr{O}rb_3(\mathscr{D}_{11})=\{3,35,135,88\}.$$

2.5 sd-inequivalent $(9;4)$-arcs

From Table 2.8, there are 11 sd-inequivalent $(8;4)$-arcs. So, by adding all the points from the plane which are not on any 4-secant to each sd-inequivalent $(8;4)$-arc, the constructed number of $(9;4)$-arcs is 1775. Then, according to the number of sd-inequivalent classes of i-secant distribution of $(9;4)$-arcs, the number of sd-inequivalent $(9;4)$-arcs is 21. The statistics are given in Tables 2.9, 2.10.

Table 2.9: N_c **of** $\{t_4,t_3,t_2,t_1,t_0\}$ **of** $(9;4)$**-arcs**

Number	N_c	Number of N_c
1	$\{2,2,18,76,85\}$	192
2	$\{2,3,15,79,84\}$	201
3	$\{1,5,15,77,85\}$	151
4	$\{1,2,24,68,88\}$	162
5	$\{1,3,21,71,87\}$	178
6	$\{1,1,27,65,89\}$	80
7	$\{1,0,30,62,90\}$	13
8	$\{1,6,12,80,84\}$	137
9	$\{2,4,12,82,83\}$	139
10	$\{1,7,9,83,83\}$	42
11	$\{3,2,12,84,82\}$	35
12	$\{2,5,9,85,82\}$	74
13	$\{2,6,6,88,81\}$	11
14	$\{3,3,9,87,81\}$	5
15	$\{1,8,6,86,82\}$	5
16	$\{3,4,6,90,80\}$	2
17	$\{2,1,21,73,86\}$	132
18	$\{2,0,24,70,87\}$	30
19	$\{1,4,18,74,86\}$	159
20	$\{3,0,18,78,84\}$	7
21	$\{3,1,15,81,83\}$	20

Table 2.10: sd**-inequivalent** $(9;4)$**-arcs in** $\mathrm{PG}(2,13)$

Symbol	(9;4)-arc	Stabiliser	$\{t_4,t_3,t_2,t_1,t_0\}$
$\mathscr{E}_1$	$\{1,2,3,88,9,21,5,4,12\}$	I	$\{2,2,18,76,85\}$
$\mathscr{E}_2$	$\{1,2,3,88,9,21,5,4,39\}$	I	$\{2,3,15,79,84\}$
$\mathscr{E}_3$	$\{1,2,3,88,9,21,5,4,61\}$	I	$\{1,5,15,77,85\}$
$\mathscr{E}_4$	$\{1,2,3,88,9,21,5,15,33\}$	I	$\{1,0,30,62,90\}$
$\mathscr{E}_5$	$\{1,2,3,88,9,21,5,39,90\}$	I	$\{2,4,12,82,83\}$
$\mathscr{E}_6$	$\{1,2,3,88,9,21,5,39,61\}$	I	$\{1,6,12,80,84\}$
$\mathscr{E}_7$	$\{1,2,3,88,9,21,35,7,17\}$	I	$\{1,7,9,83,83\}$
$\mathscr{E}_8$	$\{1,2,3,88,9,21,35,7,160\}$	I	$\{3,2,12,84,82\}$
$\mathscr{E}_9$	$\{1,2,3,88,9,182,35,56,45\}$	I	$\{2,5,9,85,82\}$
$\mathscr{E}_{10}$	$\{1,2,3,88,9,182,35,56,53\}$	Z_2	$\{2,6,6,88,81\}$
$\mathscr{E}_{11}$	$\{1,2,3,88,9,182,35,56,66\}$	Z_3	$\{3,3,9,87,81\}$

$\mathscr{E}_{12}$	$\{1,2,3,88,9,182,35,56,135\}$	Z_2	$\{1,8,6,86,82\}$
$\mathscr{E}_{13}$	$\{1,2,3,88,9,182,35,135,53\}$	S_4	$\{3,4,6,90,80\}$
$\mathscr{E}_{14}$	$\{1,2,3,88,22,145,12,16,4\}$	I	$\{1,3,21,71,87\}$
$\mathscr{E}_{15}$	$\{1,2,3,88,22,145,12,16,17\}$	I	$\{2,1,21,73,86\}$
$\mathscr{E}_{16}$	$\{1,2,3,88,22,145,12,16,20\}$	I	$\{1,2,24,68,88\}$
$\mathscr{E}_{17}$	$\{1,2,3,88,22,145,12,16,28\}$	I	$\{2,0,24,70,87\}$
$\mathscr{E}_{18}$	$\{1,2,3,88,22,145,12,16,33\}$	I	$\{1,1,27,65,89\}$
$\mathscr{E}_{19}$	$\{1,2,3,88,22,145,12,16,49\}$	I	$\{1,4,18,74,86\}$
$\mathscr{E}_{20}$	$\{1,2,3,88,22,145,12,17,6\}$	I	$\{3,0,18,78,84\}$
$\mathscr{E}_{21}$	$\{1,2,3,88,22,145,12,17,34\}$	I	$\{3,1,15,81,83\}$

Theorem 2.5. *In* PG$(2,13)$*, there are at least* 21 *sd-inequivalent* $(9;4)$*-arcs.*

Remark

In this construction, among the 21 *sd*-inequivalent $(9;4)$-arcs, there are 4 types of the groups: I, Z_2, Z_3, S_4. These groups split their associated *sd*-inequivalent $(9;4)$-arcs into a number of orbits as follows:

(1) The set $\mathscr{E}_1 = \{1,2,3,88,9,21,5,4,12\}$ has the identity group that partitions $\mathscr{E}_1$ into 9 single orbits. They are as follows:

$$\mathscr{O}rb_1(\mathscr{E}_1) = \{1\}, \mathscr{O}rb_2(\mathscr{E}_1) = \{2\}, \mathscr{O}rb_3(\mathscr{E}_1) = \{3\},$$
$$\mathscr{O}rb_4(\mathscr{E}_1) = \{4\}, \mathscr{O}rb_5(\mathscr{E}_1) = \{5\}, \mathscr{O}rb_6(\mathscr{E}_1) = \{9\},$$
$$\mathscr{O}rb_7(\mathscr{E}_1) = \{12\}, \mathscr{O}rb_8(\mathscr{E}_1) = \{21\}, \mathscr{O}rb_9(\mathscr{E}_1) = \{88\}.$$

(2) The set $\mathscr{E}_{10} = \{1,2,3,88,9,182,35,56,53\}$ has the group Z_2 that partitions $\mathscr{E}_{10}$ into 6 orbits as follows:

$$\mathscr{O}rb_1(\mathscr{E}_{10}) = \{1,3\}, \mathscr{O}rb_2(\mathscr{E}_{10}) = \{2,53\}, \mathscr{O}rb_3(\mathscr{E}_{10}) = \{9\},$$
$$\mathscr{O}rb_4(\mathscr{E}_{10}) = \{35,182\}, \mathscr{O}rb_5(\mathscr{E}_{10}) = \{56\}, \mathscr{O}rb_6(\mathscr{E}_{10}) = \{88\}.$$

(3) The set $\mathscr{E}_{11} = \{1,2,3,88,9,182,35,56,66\}$ has the group Z_3 that partitions $\mathscr{E}_{11}$ into 3 orbits. They are

the following:

$$\mathscr{O}rb_1(\mathscr{E}_{11}) = \{1,3,56\}, \mathscr{O}rb_2(\mathscr{E}_{11}) = \{2,182,66\}, \mathscr{O}rb_3(\mathscr{E}_{11}) = \{9,88,35\}.$$

(4) The set $\mathscr{E}_{13} = \{1,2,3,88,9,182,35,135,53\}$ has the group S_4 that splits $\mathscr{E}_{13}$ into 2 orbits. They are as follows:

$$\mathscr{O}rb_1(\mathscr{E}_{13}) = \{1,3,35,135,182,88\}, \mathscr{O}rb_2(\mathscr{E}_{13}) = \{2,9,53\}.$$

2.6 *sd*-inequivalent $(10;4)$-arcs

In this process, the number of *sd*-inequivalent classes N_c of secant distributions of $(10;4)$-arcs is 34. So, there are 34 *sd*-inequivalent $(10;4)$-arcs having four types of stabiliser groups. The statistics are given in Table 2.11 and Table 2.12.

Table 2.11: N_c **of** $\{t_4,t_3,t_2,t_1,t_0\}$ **of** $(10;4)$**-arcs**

Number	N_c	Number of N_c
1	$\{4,1,18,85,75\}$	20
2	$\{3,3,18,83,76\}$	177
3	$\{2,7,12,87,75\}$	152
4	$\{2,6,15,84,76\}$	196
5	$\{1,9,12,85,76\}$	62
6	$\{1,2,33,64,83\}$	121
7	$\{1,3,30,67,82\}$	165
8	$\{1,1,36,61,84\}$	36
9	$\{1,4,27,70,81\}$	166
10	$\{1,0,39,58,85\}$	3
11	$\{3,4,15,86,75\}$	207
12	$\{4,2,15,88,74\}$	33
13	$\{3,5,12,89,74\}$	129
14	$\{2,8,9,90,74\}$	27
15	$\{4,3,12,91,73\}$	40
16	$\{4,4,9,94,72\}$	17
17	$\{1,10,9,88,75\}$	4
18	$\{3,7,6,95,72\}$	7

19	$\{3,6,9,92,73\}$	21
20	$\{1,11,6,91,74\}$	1
21	$\{1,6,21,76,79\}$	136
22	$\{2,4,21,78,78\}$	205
23	$\{1,7,18,79,78\}$	130
24	$\{3,0,27,74,79\}$	20
25	$\{3,1,24,77,78\}$	99
26	$\{3,2,21,80,77\}$	181
27	$\{2,3,24,75,79\}$	203
28	$\{2,2,27,72,80\}$	151
29	$\{2,0,33,66,82\}$	7
30	$\{2,1,30,69,81\}$	79
31	$\{1,5,24,73,80\}$	165
32	$\{1,8,15,82,77\}$	105
33	$\{2,5,18,81,77\}$	204
34	$\{4,0,21,82,76\}$	5

Table 2.12: sd**-inequivalent** $(10;4)$**-arcs in** $\mathrm{PG}(2,13)$

Symbol	(10;4)-arc	Stabiliser	$\{t_4,t_3,t_2,t_1,t_0\}$
$\mathscr{F}_1$	$\{1,2,3,88,9,21,5,4,61,38\}$	I	$\{3,3,18,83,76\}$
$\mathscr{F}_2$	$\{1,2,3,88,9,21,5,4,61,39\}$	I	$\{2,7,12,87,75\}$
$\mathscr{F}_3$	$\{1,2,3,88,9,21,5,4,61,68\}$	I	$\{2,6,15,84,76\}$
$\mathscr{F}_4$	$\{1,2,3,88,9,21,5,4,61,158\}$	I	$\{1,9,12,85,76\}$
$\mathscr{F}_5$	$\{1,2,3,88,9,21,5,4,39,91\}$	I	$\{4,1,18,85,75\}$
$\mathscr{F}_6$	$\{1,2,3,88,9,21,5,15,33,17\}$	I	$\{1,1,36,61,84\}$
$\mathscr{F}_7$	$\{1,2,3,88,9,21,5,15,33,55\}$	I	$\{1,0,39,58,85\}$
$\mathscr{F}_8$	$\{1,2,3,88,9,21,5,39,61,53\}$	I	$\{3,4,15,86,75\}$
$\mathscr{F}_9$	$\{1,2,3,88,9,21,5,39,90,105\}$	I	$\{4,2,15,88,74\}$
$\mathscr{F}_{10}$	$\{1,2,3,88,9,21,35,7,17,66\}$	Z_2	$\{3,5,12,89,74\}$
$\mathscr{F}_{11}$	$\{1,2,3,88,9,21,35,7,17,91\}$	I	$\{2,8,9,90,74\}$
$\mathscr{F}_{12}$	$\{1,2,3,88,9,182,35,56,53,66\}$	Z_2	$\{4,4,9,94,72\}$

$\mathscr{F}_{13}$	$\{1,2,3,88,9,182,35,56,135,22\}$	I	$\{1,10,9,88,75\}$
$\mathscr{F}_{14}$	$\{1,2,3,88,9,182,35,56,135,53\}$	S_3	$\{3,7,6,95,72\}$
$\mathscr{F}_{15}$	$\{1,2,3,88,9,182,35,56,135,66\}$	I	$\{3,6,9,92,73\}$
$\mathscr{F}_{16}$	$\{1,2,3,88,9,182,35,56,135,90\}$	$Z_2\times Z_2$	$\{1,11,6,91,74\}$
$\mathscr{F}_{17}$	$\{1,2,3,88,9,182,35,56,45,104\}$	I	$\{4,3,12,91,73\}$
$\mathscr{F}_{18}$	$\{1,2,3,88,22,145,12,16,4,7\}$	I	$\{1,6,21,76,79\}$
$\mathscr{F}_{19}$	$\{1,2,3,88,22,145,12,16,4,17\}$	I	$\{2,4,21,78,78\}$
$\mathscr{F}_{20}$	$\{1,2,3,88,22,145,12,16,4,141\}$	I	$\{1,7,18,79,78\}$
$\mathscr{F}_{21}$	$\{1,2,3,88,22,145,12,16,17,6\}$	I	$\{3,0,27,74,79\}$
$\mathscr{F}_{22}$	$\{1,2,3,88,22,145,12,16,17,34\}$	I	$\{3,1,24,77,78\}$
$\mathscr{F}_{23}$	$\{1,2,3,88,22,145,12,16,17,92\}$	I	$\{3,2,21,80,77\}$
$\mathscr{F}_{24}$	$\{1,2,3,88,22,145,12,16,20,8\}$	I	$\{2,1,30,69,81\}$
$\mathscr{F}_{25}$	$\{1,2,3,88,22,145,12,16,20,9\}$	I	$\{1,3,30,67,82\}$
$\mathscr{F}_{26}$	$\{1,2,3,88,22,145,12,16,20,17\}$	I	$\{2,2,27,72,80\}$
$\mathscr{F}_{27}$	$\{1,2,3,88,22,145,12,16,20,23\}$	I	$\{1,5,24,73,80\}$
$\mathscr{F}_{28}$	$\{1,2,3,88,22,145,12,16,20,31\}$	I	$\{1,4,27,70,81\}$
$\mathscr{F}_{29}$	$\{1,2,3,88,22,145,12,16,20,33\}$	I	$\{1,2,33,64,83\}$
$\mathscr{F}_{30}$	$\{1,2,3,88,22,145,12,16,20,69\}$	I	$\{2,3,24,75,79\}$
$\mathscr{F}_{31}$	$\{1,2,3,88,22,145,12,16,33,28\}$	I	$\{2,0,33,66,82\}$
$\mathscr{F}_{32}$	$\{1,2,3,88,22,145,12,16,49,61\}$	I	$\{1,8,15,82,77\}$
$\mathscr{F}_{33}$	$\{1,2,3,88,22,145,12,16,49,92\}$	I	$\{2,5,18,81,77\}$
$\mathscr{F}_{34}$	$\{1,2,3,88,22,145,12,17,34,14\}$	I	$\{4,0,21,82,76\}$

Theorem 2.6. *In* $\mathrm{PG}(2,13)$*, there are at least* 34 *sd-inequivalent* $(10;4)$*-arcs.*

Remark

In Table 2.12, there are three groups other than the identity group; they are $Z_2,\ S_3,\ Z_2\times Z_2$. These groups have the following action on the associated arcs.

(1) The group Z_2 divides the set $\mathscr{F}_{10} = \{1,2,3,88,9,21,35,7,17,66\}$ into 6 orbits. They are as follows:

$$\mathscr{O}rb_1(\mathscr{F}_{10}) = \{1,88\}, \mathscr{O}rb_2(\mathscr{F}_{10}) = \{2,66\}, \mathscr{O}rb_3(\mathscr{F}_{10}) = \{3,9\},$$
$$\mathscr{O}rb_4(\mathscr{F}_{10}) = \{7\}, \mathscr{O}rb_5(\mathscr{F}_{10}) = \{17,21\}, \mathscr{O}rb_6(\mathscr{F}_{10}) = \{35\}.$$

However, it divides the set $\mathscr{F}_{12} = \{1,2,3,88,9,182,35,56,53,66\}$ into 7 orbits. They are as follows:

$$\mathscr{O}rb_1(\mathscr{F}_{12}) = \{1\}, \mathscr{O}rb_2(\mathscr{F}_{12}) = \{2,9\}, \mathscr{O}rb_3(\mathscr{F}_{12}) = \{3,66\}, \mathscr{O}rb_4(\mathscr{F}_{12}) = \{35\},$$
$$\mathscr{O}rb_5(\mathscr{F}_{12}) = \{53,56\}, \mathscr{O}rb_6(\mathscr{F}_{12}) = \{88\}, \mathscr{O}rb_7(\mathscr{F}_{12}) = \{182\}.$$

(2) The group S_3 splits the set $\mathscr{F}_{14} = \{1,2,3,88,9,182,35,56,135,53\}$ into 4 orbits. They are as follows:

$$\mathscr{O}rb_1(\mathscr{F}_{14}) = \{1,3,135\}, \mathscr{O}rb_2(\mathscr{F}_{14}) = \{2,9,53\},$$
$$\mathscr{O}rb_3(\mathscr{F}_{14}) = \{35,88,182\}, \mathscr{O}rb_4(\mathscr{F}_{14}) = \{56\}.$$

(3) The group $Z_2 \times Z_2$ partitions the set $\mathscr{F}_{16} = \{1,2,3,88,9,182,35,56,135,90\}$ into 5 orbits. They are as follows:

$$\mathscr{O}rb_1(\mathscr{F}_{16}) = \{1\}, \mathscr{O}rb_2(\mathscr{F}_{16}) = \{2,9\}, \mathscr{O}rb_3(\mathscr{F}_{16}) = \{3,135,35,88\},$$
$$\mathscr{O}rb_4(\mathscr{F}_{16}) = \{56,90\}, \mathscr{O}rb_5(\mathscr{F}_{16}) = \{182\}.$$

2.7 *sd*-inequivalent $(11;4)$-arcs

Among the 5144 $(11;4)$-arcs found, the number of *sd*-inequivalent $(11;4)$-arcs is 52 as shown in Table 2.13. These $(11;4)$-arcs have four types of stabiliser groups, they are I, D_4, Z_2, $Z_2 \times Z_2$. The values of t_i of the secant distributions $\{t_4,t_3,t_2,t_1,t_0\}$ of the *sd*-inequivalent $(11;4)$-arcs are as follows:

$$t_4 \in \{1,\ldots,6\},\ t_3 \in \{0,\ldots,13\},\ t_2 \in \{7,\ldots,46\},\ t_1 \in \{55,\ldots,100\},\ t_0 \in \{64,\ldots,80\}.$$

Table 2.13: sd**-inequivalent** $(11;4)$**-arcs in** $\mathrm{PG}(2,13)$

Symbol	(11;4)-arc	Stabiliser
$\mathscr{G}_1$	$\{1,2,3,88,9,21,5,4,39,91,35\}$	I
$\mathscr{G}_2$	$\{1,2,3,88,9,21,5,4,39,91,64\}$	I
$\mathscr{G}_3$	$\{1,2,3,88,9,21,5,4,39,91,104\}$	I
$\mathscr{G}_4$	$\{1,2,3,88,9,21,5,4,61,39,38\}$	I
$\mathscr{G}_5$	$\{1,2,3,88,9,21,5,4,61,158,34\}$	I
$\mathscr{G}_6$	$\{1,2,3,88,9,21,5,4,61,158,39\}$	I
$\mathscr{G}_7$	$\{1,2,3,88,9,21,5,4,61,158,49\}$	I
$\mathscr{G}_8$	$\{1,2,3,88,9,21,5,4,61,158,68\}$	I
$\mathscr{G}_9$	$\{1,2,3,88,9,21,5,15,33,17,12\}$	I
$\mathscr{G}_{10}$	$\{1,2,3,88,9,21,5,15,33,17,45\}$	I
$\mathscr{G}_{11}$	$\{1,2,3,88,9,21,5,15,33,55,4\}$	I
$\mathscr{G}_{12}$	$\{1,2,3,88,9,21,5,15,33,55,8\}$	I
$\mathscr{G}_{13}$	$\{1,2,3,88,9,21,5,15,33,55,44\}$	I
$\mathscr{G}_{14}$	$\{1,2,3,88,9,21,35,7,17,66,127\}$	I
$\mathscr{G}_{15}$	$\{1,2,3,88,9,21,35,7,17,91,39\}$	I
$\mathscr{G}_{16}$	$\{1,2,3,88,9,21,35,7,17,91,92\}$	I
$\mathscr{G}_{17}$	$\{1,2,3,88,9,182,35,56,53,66,143\}$	D_4
$\mathscr{G}_{18}$	$\{1,2,3,88,9,182,35,56,135,22,53\}$	I
$\mathscr{G}_{19}$	$\{1,2,3,88,9,182,35,56,135,22,151\}$	Z_2
$\mathscr{G}_{20}$	$\{1,2,3,88,9,182,35,56,135,66,53\}$	Z_2
$\mathscr{G}_{21}$	$\{1,2,3,88,9,182,35,56,135,66,73\}$	I
$\mathscr{G}_{22}$	$\{1,2,3,88,9,182,35,56,135,66,117\}$	I
$\mathscr{G}_{23}$	$\{1,2,3,88,9,182,35,56,135,90,53\}$	$Z_2 \times Z_2$
$\mathscr{G}_{24}$	$\{1,2,3,88,22,145,12,16,20,23,32\}$	I
$\mathscr{G}_{25}$	$\{1,2,3,88,22,145,12,16,20,23,181\}$	I
$\mathscr{G}_{26}$	$\{1,2,3,88,22,145,12,16,20,31,33\}$	I
$\mathscr{G}_{27}$	$\{1,2,3,88,22,145,12,16,20,31,26\}$	I
$\mathscr{G}_{28}$	$\{1,2,3,88,22,145,12,16,20,31,5\}$	I
$\mathscr{G}_{29}$	$\{1,2,3,88,22,145,12,16,20,31,4\}$	I
$\mathscr{G}_{30}$	$\{1,2,3,88,22,145,12,16,20,31,32\}$	I
$\mathscr{G}_{31}$	$\{1,2,3,88,22,145,12,16,20,31,28\}$	I

$\mathscr{G}_{32}$	$\{1,2,3,88,22,145,12,16,20,31,6\}$	I
$\mathscr{G}_{33}$	$\{1,2,3,88,22,145,12,16,20,31,45\}$	I
$\mathscr{G}_{34}$	$\{1,2,3,88,22,145,12,16,20,31,181\}$	I
$\mathscr{G}_{35}$	$\{1,2,3,88,22,145,12,16,20,31,69\}$	I
$\mathscr{G}_{36}$	$\{1,2,3,88,22,145,12,16,20,8,76\}$	I
$\mathscr{G}_{37}$	$\{1,2,3,88,22,145,12,16,20,8,28\}$	I
$\mathscr{G}_{38}$	$\{1,2,3,88,22,145,12,16,20,9,182\}$	I
$\mathscr{G}_{39}$	$\{1,2,3,88,22,145,12,16,4,7,17\}$	I
$\mathscr{G}_{40}$	$\{1,2,3,88,22,145,12,16,4,7,92\}$	I
$\mathscr{G}_{41}$	$\{1,2,3,88,22,145,12,16,4,7,141\}$	I
$\mathscr{G}_{42}$	$\{1,2,3,88,22,145,12,16,4,7,146\}$	I
$\mathscr{G}_{43}$	$\{1,2,3,88,22,145,12,16,4,17,92\}$	I
$\mathscr{G}_{44}$	$\{1,2,3,88,22,145,12,16,4,141,146\}$	I
$\mathscr{G}_{45}$	$\{1,2,3,88,22,145,12,16,33,28,78\}$	I
$\mathscr{G}_{46}$	$\{1,2,3,88,22,145,12,16,17,34,14\}$	I
$\mathscr{G}_{47}$	$\{1,2,3,88,22,145,12,16,17,34,15\}$	I
$\mathscr{G}_{48}$	$\{1,2,3,88,22,145,12,16,17,34,49\}$	I
$\mathscr{G}_{49}$	$\{1,2,3,88,22,145,12,16,49,61,18\}$	I
$\mathscr{G}_{50}$	$\{1,2,3,88,22,145,12,16,49,61,24\}$	I
$\mathscr{G}_{51}$	$\{1,2,3,88,22,145,12,16,49,61,141\}$	I
$\mathscr{G}_{52}$	$\{1,2,3,88,22,145,12,16,49,92,164\}$	I

Theorem 2.7. *In* $\mathrm{PG}(2,13)$*, there are at least* 52 *sd-inequivalent* $(11;4)$*-arcs.*

Remark

In Table 2.13, the stabilisers of 52 *sd*-inequivalent $(11;4)$-arcs are the following:

$$I, D_4,\ Z_2,\ Z_2 \times Z_2.$$

(1) The group D_4 divides the set $\mathscr{G}_{17} = \{1,2,3,88,9,182,35,56,53,66,143\}$ into 4 orbits. They are the

following:

$$\mathscr{O}rb_1(\mathscr{G}_{17}) = \{1,3,66,143\}, \mathscr{O}rb_2(\mathscr{G}_{17}) = \{2,9,53,56\},$$
$$\mathscr{O}rb_3(\mathscr{G}_{17}) = \{35,182\}, \mathscr{O}rb_4(\mathscr{G}_{17}) = \{88\}.$$

(2) The group Z_2 divides the set $\mathscr{G}_{19} = \{1,2,3,88,9,182,35,56,135,22,151\}$ into 7 orbits. They are listed as follows:

$$\mathscr{O}rb_1(\mathscr{G}_{19}) = \{1\}, \mathscr{O}rb_2(\mathscr{G}_{19}) = \{2,9\}, \mathscr{O}rb_3(\mathscr{G}_{19}) = \{3,135\}, \mathscr{O}rb_4(\mathscr{G}_{19}) = \{22,151\},$$
$$\mathscr{O}rb_5(\mathscr{G}_{19}) = \{35,88\}, \mathscr{O}rb_6(\mathscr{G}_{19}) = \{56\}, \mathscr{O}rb_7(\mathscr{G}_{19}) = \{182\}.$$

However, it splits the set $\mathscr{G}_{20} = \{1,2,3,88,9,182,35,56,135,66,53\}$ into 8 orbits. They are given as follows:

$$\mathscr{O}rb_1(\mathscr{G}_{20}) = \{1,135\}, \mathscr{O}rb_2(\mathscr{G}_{20}) = \{2\}, \mathscr{O}rb_3(\mathscr{G}_{20}) = \{3\}, \mathscr{O}rb_4(\mathscr{G}_{20}) = \{9,53\},$$
$$\mathscr{O}rb_5(\mathscr{G}_{20}) = \{35\}, \mathscr{O}rb_6(\mathscr{G}_{20}) = \{56\}, \mathscr{O}rb_7(\mathscr{G}_{20}) = \{66\}, \mathscr{O}rb_8(\mathscr{G}_{20}) = \{88,182\}.$$

(3) The group $Z_2 \times Z_2$ partitions the set $\mathscr{G}_{23} = \{1,2,3,88,9,182,35,56,135,90,53\}$ into 6 orbits. They are the following:

$$\mathscr{O}rb_1(\mathscr{G}_{23}) = \{1\}, \mathscr{O}rb_2(\mathscr{G}_{23}) = \{2,9\}, \mathscr{O}rb_3(\mathscr{G}_{23}) = \{3,135,35,88\},$$
$$\mathscr{O}rb_4(\mathscr{G}_{23}) = \{53\}, \mathscr{O}rb_5(\mathscr{G}_{23}) = \{56,90\}, \mathscr{O}rb_6(\mathscr{G}_{23}) = \{182\}.$$

2.8 *sd*-inequivalent $(12;4)$-arcs

In this process, there are 7534 arcs found including 75 *sd*-inequivalent classes N_c of secant distribution. So that the number of *sd*-inequivalent (12,4)-arcs is 75 as given in Table 2.14. The values of t_i of the secant distributions $\{t_4,t_3,t_2,t_1,t_0\}$ of the *sd*-inequivalent $(12;4)$-arcs are as follows:

$$t_4 \in \{1,\ldots,7\},\ t_3 \in \{0,\ldots,16\},\ t_2 \in \{9,\ldots,51\},\ t_1 \in \{53,\ldots,105\},\ t_0 \in \{56,\ldots,75\}.$$

Table 2.14: sd**-inequivalent** $(12;4)$**-arcs in** $\mathrm{PG}(2,13)$

Symbol	(12;4)-arc	Stabiliser
$\mathscr{H}_1$	$\{1,2,3,88,9,21,5,4,61,158,49,10\}$	I
$\mathscr{H}_2$	$\{1,2,3,88,9,21,5,4,61,158,49,39\}$	I
$\mathscr{H}_3$	$\{1,2,3,88,9,21,5,4,61,158,49,68\}$	I
$\mathscr{H}_4$	$\{1,2,3,88,9,21,5,4,61,158,49,92\}$	I
$\mathscr{H}_5$	$\{1,2,3,88,9,21,5,4,61,158,34,39\}$	I
$\mathscr{H}_6$	$\{1,2,3,88,9,21,5,4,61,158,68,92\}$	I
$\mathscr{H}_7$	$\{1,2,3,88,9,21,5,4,39,91,35,8\}$	I
$\mathscr{H}_8$	$\{1,2,3,88,9,21,5,4,39,91,35,82\}$	I
$\mathscr{H}_9$	$\{1,2,3,88,9,21,5,15,33,55,44,6\}$	I
$\mathscr{H}_{10}$	$\{1,2,3,88,9,21,5,15,33,55,44,8\}$	I
$\mathscr{H}_{11}$	$\{1,2,3,88,9,21,5,15,33,17,12,4\}$	I
$\mathscr{H}_{12}$	$\{1,2,3,88,9,21,5,15,33,17,12,24\}$	I
$\mathscr{H}_{13}$	$\{1,2,3,88,9,21,5,15,33,17,12,147\}$	Z_2
$\mathscr{H}_{14}$	$\{1,2,3,88,9,21,5,15,33,55,44,19\}$	I
$\mathscr{H}_{15}$	$\{1,2,3,88,9,21,5,15,33,55,44,28\}$	I
$\mathscr{H}_{16}$	$\{1,2,3,88,9,21,35,7,17,91,92,39\}$	I
$\mathscr{H}_{17}$	$\{1,2,3,88,9,21,35,7,17,91,39,66\}$	I
$\mathscr{H}_{18}$	$\{1,2,3,88,9,21,35,7,17,91,39,146\}$	I
$\mathscr{H}_{19}$	$\{1,2,3,88,9,182,35,56,53,66,143,13\}$	Z_2
$\mathscr{H}_{20}$	$\{1,2,3,88,9,182,35,56,53,66,143,117\}$	Z_2
$\mathscr{H}_{21}$	$\{1,2,3,88,9,182,35,56,135,22,151,31\}$	I
$\mathscr{H}_{22}$	$\{1,2,3,88,9,182,35,56,135,22,151,55\}$	Z_2
$\mathscr{H}_{23}$	$\{1,2,3,88,9,182,35,56,135,22,53,90\}$	I
$\mathscr{H}_{24}$	$\{1,2,3,88,9,182,35,56,135,22,53,140\}$	Z_2
$\mathscr{H}_{25}$	$\{1,2,3,88,9,182,35,56,135,66,73,100\}$	Z_3
$\mathscr{H}_{26}$	$\{1,2,3,88,9,182,35,56,135,90,53,134\}$	S_3
$\mathscr{H}_{27}$	$\{1,2,3,88,9,182,35,56,135,66,117,53\}$	I
$\mathscr{H}_{28}$	$\{1,2,3,88,9,182,35,56,135,66,53,101\}$	$Z_2 \times Z_2$
$\mathscr{H}_{29}$	$\{1,2,3,88,22,145,12,16,20,8,28,9\}$	I
$\mathscr{H}_{30}$	$\{1,2,3,88,22,145,12,16,20,31,33,38\}$	I
$\mathscr{H}_{31}$	$\{1,2,3,88,22,145,12,16,20,31,33,80\}$	I
$\mathscr{H}_{32}$	$\{1,2,3,88,22,145,12,16,20,31,33,9\}$	I
$\mathscr{H}_{33}$	$\{1,2,3,88,22,145,12,16,20,31,33,4\}$	I
$\mathscr{H}_{34}$	$\{1,2,3,88,22,145,12,16,20,31,33,73\}$	I

$\mathscr{H}_{35}$	$\{1,2,3,88,22,145,12,16,20,31,5,30\}$	I
$\mathscr{H}_{36}$	$\{1,2,3,88,22,145,12,16,20,31,4,32\}$	I
$\mathscr{H}_{37}$	$\{1,2,3,88,22,145,12,16,20,31,4,73\}$	I
$\mathscr{H}_{38}$	$\{1,2,3,88,22,145,12,16,20,23,32,175\}$	I
$\mathscr{H}_{39}$	$\{1,2,3,88,22,145,12,16,20,31,33,28\}$	I
$\mathscr{H}_{40}$	$\{1,2,3,88,22,145,12,16,20,31,33,6\}$	I
$\mathscr{H}_{41}$	$\{1,2,3,88,22,145,12,16,20,31,33,8\}$	I
$\mathscr{H}_{42}$	$\{1,2,3,88,22,145,12,16,20,31,33,45\}$	I
$\mathscr{H}_{43}$	$\{1,2,3,88,22,145,12,16,20,31,26,45\}$	I
$\mathscr{H}_{44}$	$\{1,2,3,88,22,145,12,16,20,31,5,45\}$	I
$\mathscr{H}_{45}$	$\{1,2,3,88,22,145,12,16,20,31,4,71\}$	I
$\mathscr{H}_{46}$	$\{1,2,3,88,22,145,12,16,20,31,32,73\}$	I
$\mathscr{H}_{47}$	$\{1,2,3,88,22,145,12,16,20,23,32,117\}$	I
$\mathscr{H}_{48}$	$\{1,2,3,88,22,145,12,16,20,31,33,69\}$	I
$\mathscr{H}_{49}$	$\{1,2,3,88,22,145,12,16,20,31,26,69\}$	I
$\mathscr{H}_{50}$	$\{1,2,3,88,22,145,12,16,20,31,5,69\}$	I
$\mathscr{H}_{51}$	$\{1,2,3,88,22,145,12,16,20,31,4,69\}$	I
$\mathscr{H}_{52}$	$\{1,2,3,88,22,145,12,16,20,31,32,66\}$	I
$\mathscr{H}_{53}$	$\{1,2,3,88,22,145,12,16,20,23,32,181\}$	I
$\mathscr{H}_{54}$	$\{1,2,3,88,22,145,12,16,20,31,6,17\}$	I
$\mathscr{H}_{55}$	$\{1,2,3,88,22,145,12,16,20,31,6,175\}$	I
$\mathscr{H}_{56}$	$\{1,2,3,88,22,145,12,16,20,31,45,85\}$	I
$\mathscr{H}_{57}$	$\{1,2,3,88,22,145,12,16,20,31,181,69\}$	I
$\mathscr{H}_{58}$	$\{1,2,3,88,22,145,12,16,20,31,69,95\}$	I
$\mathscr{H}_{59}$	$\{1,2,3,88,22,145,12,16,20,23,181,11\}$	I
$\mathscr{H}_{60}$	$\{1,2,3,88,22,145,12,16,20,23,181,182\}$	I
$\mathscr{H}_{61}$	$\{1,2,3,88,22,145,12,16,4,7,141,79\}$	I
$\mathscr{H}_{62}$	$\{1,2,3,88,22,145,12,16,4,7,141,17\}$	I
$\mathscr{H}_{63}$	$\{1,2,3,88,22,145,12,16,4,7,141,49\}$	I
$\mathscr{H}_{64}$	$\{1,2,3,88,22,145,12,16,4,7,141,146\}$	I
$\mathscr{H}_{65}$	$\{1,2,3,88,22,145,12,16,4,7,17,71\}$	I
$\mathscr{H}_{66}$	$\{1,2,3,88,22,145,12,16,4,7,146,49\}$	I
$\mathscr{H}_{67}$	$\{1,2,3,88,22,145,12,16,4,7,17,92\}$	I
$\mathscr{H}_{68}$	$\{1,2,3,88,22,145,12,16,4,17,92,73\}$	I

$\mathscr{H}_{69}$	$\{1,2,3,88,22,145,12,16,4,141,146,164\}$	I
$\mathscr{H}_{70}$	$\{1,2,3,88,22,145,12,16,17,34,15,56\}$	I
$\mathscr{H}_{71}$	$\{1,2,3,88,22,145,12,16,17,34,14,148\}$	I
$\mathscr{H}_{72}$	$\{1,2,3,88,22,145,12,16,49,61,141,164\}$	I
$\mathscr{H}_{73}$	$\{1,2,3,88,22,145,12,16,49,61,18,7\}$	I
$\mathscr{H}_{74}$	$\{1,2,3,88,22,145,12,16,49,61,24,85\}$	I
$\mathscr{H}_{75}$	$\{1,2,3,88,22,145,12,16,49,92,164,141\}$	I

Theorem 2.8. *In* PG$(2;13)$*, there are at least* 75 *sd-inequivalent* $(12;4)$*-arcs.*

Remark

The stabiliser groups of the 75 *sd*-inequivalent $(12;4)$-arcs other than the identity group are the following:

$$Z_2\ ,Z_3\ ,S_3\ ,\ Z_2\times Z_2.$$

(1) The group Z_2 splits the sets $\mathscr{H}_{13}$, $\mathscr{H}_{19}$, $\mathscr{H}_{20}$, $\mathscr{H}_{22}$, $\mathscr{H}_{24}$ into $6,8,8,8,8$ orbits. They are given as follows:

$$\mathscr{O}rb(\mathscr{H}_{13}) = \{1,17\},\{2,12\},\{3,21\},\{5,147\},\{9,88\},\{15,33\}.$$

$$\mathscr{O}rb(\mathscr{H}_{19}) = \{1,66\},\{2\},\{3,143\},\{9,56\},\{13\},\{35,182\},\{53\},\{88\}.$$

$$\mathscr{O}rb(\mathscr{H}_{20}) = \{1,3\},\{2,53\},\{9\},\{35,182\},\{56\},\{66,143\},\{88\},\{117\}.$$

$$\mathscr{O}rb(\mathscr{H}_{22}) = \{1\},\{2,9\},\{3,135\},\{22,151\},\{35,88\},\{55\},\{56\},\{182\}.$$

$$\mathscr{O}rb(\mathscr{H}_{24}) = \{1,88\},\{2\},\{3\},\{9,53\},\{22\},\{35\},\{56,140\},\{135,182\}.$$

(2) The group Z_3 splits the set $\mathscr{H}_{25}$ into 4 orbits of size 3 as follows:

$$\mathscr{O}rb(\mathscr{H}_{25}) = \{1,3,56\},\{2,182,66\},\{9,88,35\},\{73,135,100\}.$$

(3) The group S_3 partitions the set $\mathscr{H}_{26}$ into 4 orbits of size 3 as follows:

$$\mathscr{O}rb(\mathscr{H}_{26}) = \{1,35,135\},\{2,9,53\},\{3,88,182\},\{56,90,134\}.$$

(4) The group $Z_2 \times Z_2$ partitions the set $\mathscr{H}_{28}$ into 6 orbits of sizes 1,2,4,2,2,1 as given below:

$$\mathscr{O}rb(\mathscr{H}_{28}) = \{1\},\{2,9\},\{3,135,101,66\},\{35,88\},\{53,56\},\{182\}.$$

2.9 *sd*-inequivalent $(13;4)$-arcs

In this process, the number of *sd*-inequivalent (13;4)-arcs is 101. The values of t_i of the secant distributions $\{t_4,t_3,t_2,t_1,t_0\}$ of the *sd*-inequivalent $(13;4)$-arcs are as follows:

$$t_4 \in \{1,\dots,9\}, t_3 \in \{0,\dots,19\}, t_2 \in \{9,\dots,57\}, t_1 \in \{49,\dots,110\}, t_0 \in \{48,\dots,71\}.$$

Table 2.15: *sd*-**inequivalent** $(13;4)$-**arcs in** $\mathrm{PG}(2,13)$

Symbol	(13;4)-arc	Stabiliser
$\mathscr{O}_1$	$\{1,2,3,88,9,21,5,4,61,158,49,10,92\}$	I
$\mathscr{O}_2$	$\{1,2,3,88,9,21,5,4,61,158,49,68,39\}$	I
$\mathscr{O}_3$	$\{1,2,3,88,9,21,5,4,61,158,68,92,171\}$	I
$\mathscr{O}_4$	$\{1,2,3,88,9,21,5,4,61,158,49,68,92\}$	I
$\mathscr{O}_5$	$\{1,2,3,88,9,21,5,4,61,158,49,39,81\}$	I
$\mathscr{O}_6$	$\{1,2,3,88,9,21,5,4,39,91,35,82,74\}$	I
$\mathscr{O}_7$	$\{1,2,3,88,9,21,5,15,33,55,44,8,82\}$	I
$\mathscr{O}_8$	$\{1,2,3,88,9,21,5,15,33,55,44,19,78\}$	I
$\mathscr{O}_9$	$\{1,2,3,88,9,21,5,15,33,17,12,24,40\}$	I
$\mathscr{O}_{10}$	$\{1,2,3,88,9,21,5,15,33,17,12,24,80\}$	I
$\mathscr{O}_{11}$	$\{1,2,3,88,9,21,5,15,33,55,44,8,6\}$	I
$\mathscr{O}_{12}$	$\{1,2,3,88,9,21,5,15,33,55,44,8,7\}$	I
$\mathscr{O}_{13}$	$\{1,2,3,88,9,21,5,15,33,55,44,8,18\}$	I
$\mathscr{O}_{14}$	$\{1,2,3,88,9,21,5,15,33,55,44,8,19\}$	I
$\mathscr{O}_{15}$	$\{1,2,3,88,9,21,5,15,33,55,44,8,23\}$	I
$\mathscr{O}_{16}$	$\{1,2,3,88,9,21,5,15,33,55,44,8,92\}$	I

$\mathscr{O}_{17}$	$\{1,2,3,88,9,21,5,15,33,55,44,8,121\}$	I
$\mathscr{O}_{18}$	$\{1,2,3,88,9,21,35,7,17,91,92,39,6\}$	I
$\mathscr{O}_{19}$	$\{1,2,3,88,9,21,35,7,17,91,92,39,12\}$	I
$\mathscr{O}_{20}$	$\{1,2,3,88,9,21,35,7,17,91,92,39,106\}$	Z_6
$\mathscr{O}_{21}$	$\{1,2,3,88,9,21,35,7,17,91,39,66,121\}$	I
$\mathscr{O}_{22}$	$\{1,2,3,88,9,21,35,7,17,91,39,146,20\}$	I
$\mathscr{O}_{23}$	$\{1,2,3,88,9,21,35,7,17,91,39,146,107\}$	I
$\mathscr{O}_{24}$	$\{1,2,3,88,9,182,35,56,53,66,143,117,135\}$	Z_2
$\mathscr{O}_{25}$	$\{1,2,3,88,9,182,35,56,135,22,151,55,57\}$	I
$\mathscr{O}_{26}$	$\{1,2,3,88,9,182,35,56,135,22,151,55,31\}$	I
$\mathscr{O}_{27}$	$\{1,2,3,88,9,182,35,56,135,90,53,134,8\}$	Z_2
$\mathscr{O}_{28}$	$\{1,2,3,88,9,182,35,56,135,90,53,134,140\}$	S_4
$\mathscr{O}_{29}$	$\{1,2,3,88,9,182,35,56,135,22,53,140,90\}$	Z_2
$\mathscr{O}_{30}$	$\{1,2,3,88,9,182,35,56,135,22,151,55,53\}$	Z_2
$\mathscr{O}_{31}$	$\{1,2,3,88,9,182,35,56,135,22,151,31,53\}$	I
$\mathscr{O}_{32}$	$\{1,2,3,88,9,182,35,56,135,22,53,140,31\}$	I
$\mathscr{O}_{33}$	$\{1,2,3,88,22,145,12,16,20,31,33,38,5\}$	I
$\mathscr{O}_{34}$	$\{1,2,3,88,22,145,12,16,20,31,33,38,7\}$	I
$\mathscr{O}_{35}$	$\{1,2,3,88,22,145,12,16,20,31,33,38,34\}$	I
$\mathscr{O}_{36}$	$\{1,2,3,88,22,145,12,16,20,31,33,38,158\}$	I
$\mathscr{O}_{37}$	$\{1,2,3,88,22,145,12,16,20,31,33,38,39\}$	I
$\mathscr{O}_{38}$	$\{1,2,3,88,22,145,12,16,20,31,33,38,15\}$	I
$\mathscr{O}_{39}$	$\{1,2,3,88,22,145,12,16,20,31,33,38,6\}$	I
$\mathscr{O}_{40}$	$\{1,2,3,88,22,145,12,16,20,31,33,38,4\}$	I
$\mathscr{O}_{41}$	$\{1,2,3,88,22,145,12,16,20,31,33,38,18\}$	I
$\mathscr{O}_{42}$	$\{1,2,3,88,22,145,12,16,20,31,33,38,69\}$	I
$\mathscr{O}_{43}$	$\{1,2,3,88,22,145,12,16,20,31,33,38,66\}$	I
$\mathscr{O}_{44}$	$\{1,2,3,88,22,145,12,16,20,31,33,38,79\}$	I
$\mathscr{O}_{45}$	$\{1,2,3,88,22,145,12,16,20,31,33,4,149\}$	I
$\mathscr{O}_{46}$	$\{1,2,3,88,22,145,12,16,20,31,33,73,136\}$	I

$\mathscr{O}_{47}$	$\{1,2,3,88,22,145,12,16,20,31,5,30,144\}$	I
$\mathscr{O}_{48}$	$\{1,2,3,88,22,145,12,16,20,31,4,73,44\}$	I
$\mathscr{O}_{49}$	$\{1,2,3,88,22,145,12,16,20,23,32,175,29\}$	I
$\mathscr{O}_{50}$	$\{1,2,3,88,22,145,12,16,20,31,33,4,73\}$	I
$\mathscr{O}_{51}$	$\{1,2,3,88,22,145,12,16,20,31,33,73,128\}$	I
$\mathscr{O}_{52}$	$\{1,2,3,88,22,145,12,16,20,31,4,32,23\}$	I
$\mathscr{O}_{53}$	$\{1,2,3,88,22,145,12,16,20,31,4,32,73\}$	I
$\mathscr{O}_{54}$	$\{1,2,3,88,22,145,12,16,20,23,32,175,30\}$	I
$\mathscr{O}_{55}$	$\{1,2,3,88,22,145,12,16,4,7,141,79,17\}$	I
$\mathscr{O}_{56}$	$\{1,2,3,88,22,145,12,16,49,61,24,85,87\}$	I
$\mathscr{O}_{57}$	$\{1,2,3,88,22,145,12,16,20,31,33,80,69\}$	I
$\mathscr{O}_{58}$	$\{1,2,3,88,22,145,12,16,20,31,33,4,71\}$	I
$\mathscr{O}_{59}$	$\{1,2,3,88,22,145,12,16,20,31,33,73,45\}$	I
$\mathscr{O}_{60}$	$\{1,2,3,88,22,145,12,16,20,31,5,30,69\}$	I
$\mathscr{O}_{61}$	$\{1,2,3,88,22,145,12,16,20,31,4,73,8\}$	I
$\mathscr{O}_{62}$	$\{1,2,3,88,22,145,12,16,20,31,4,73,71\}$	I
$\mathscr{O}_{63}$	$\{1,2,3,88,22,145,12,16,4,7,141,79,25\}$	I
$\mathscr{O}_{64}$	$\{1,2,3,88,22,145,12,16,4,7,141,79,69\}$	I
$\mathscr{O}_{65}$	$\{1,2,3,88,22,145,12,16,49,61,24,85,34\}$	I
$\mathscr{O}_{66}$	$\{1,2,3,88,22,145,12,16,20,31,33,6,17\}$	I
$\mathscr{O}_{67}$	$\{1,2,3,88,22,145,12,16,20,31,33,8,17\}$	I
$\mathscr{O}_{68}$	$\{1,2,3,88,22,145,12,16,20,31,33,8,66\}$	I
$\mathscr{O}_{69}$	$\{1,2,3,88,22,145,12,16,20,31,26,45,106\}$	I
$\mathscr{O}_{70}$	$\{1,2,3,88,22,145,12,16,20,31,5,30,169\}$	I
$\mathscr{O}_{71}$	$\{1,2,3,88,22,145,12,16,20,31,4,32,156\}$	I
$\mathscr{O}_{72}$	$\{1,2,3,88,22,145,12,16,20,31,4,71,131\}$	I
$\mathscr{O}_{73}$	$\{1,2,3,88,22,145,12,16,20,23,32,117,48\}$	I
$\mathscr{O}_{74}$	$\{1,2,3,88,22,145,12,16,4,7,141,79,49\}$	I
$\mathscr{O}_{75}$	$\{1,2,3,88,22,145,12,16,49,61,24,85,151\}$	I
$\mathscr{O}_{76}$	$\{1,2,3,88,22,145,12,16,20,8,28,9,133\}$	I

$\mathcal{O}_{77}$	$\{1,2,3,88,22,145,12,16,20,8,28,9,89\}$	I
$\mathcal{O}_{78}$	$\{1,2,3,88,22,145,12,16,20,31,33,69,95\}$	I
$\mathcal{O}_{79}$	$\{1,2,3,88,22,145,12,16,20,31,26,69,132\}$	I
$\mathcal{O}_{80}$	$\{1,2,3,88,22,145,12,16,20,31,5,69,99\}$	I
$\mathcal{O}_{81}$	$\{1,2,3,88,22,145,12,16,20,31,4,69,95\}$	I
$\mathcal{O}_{82}$	$\{1,2,3,88,22,145,12,16,20,31,4,69,85\}$	I
$\mathcal{O}_{83}$	$\{1,2,3,88,22,145,12,16,20,23,32,181,44\}$	I
$\mathcal{O}_{84}$	$\{1,2,3,88,22,145,12,16,4,7,141,49,69\}$	I
$\mathcal{O}_{85}$	$\{1,2,3,88,22,145,12,16,4,7,141,79,44\}$	I
$\mathcal{O}_{86}$	$\{1,2,3,88,22,145,12,16,4,7,141,17,73\}$	I
$\mathcal{O}_{87}$	$\{1,2,3,88,22,145,12,16,20,31,69,95,35\}$	I
$\mathcal{O}_{88}$	$\{1,2,3,88,22,145,12,16,20,31,69,95,8\}$	I
$\mathcal{O}_{89}$	$\{1,2,3,88,22,145,12,16,20,31,45,85,91\}$	I
$\mathcal{O}_{90}$	$\{1,2,3,88,22,145,12,16,20,31,181,69,64\}$	I
$\mathcal{O}_{91}$	$\{1,2,3,88,22,145,12,16,20,31,181,69,63\}$	I
$\mathcal{O}_{92}$	$\{1,2,3,88,22,145,12,16,4,7,17,71,6\}$	I
$\mathcal{O}_{93}$	$\{1,2,3,88,22,145,12,16,4,7,141,49,164\}$	I
$\mathcal{O}_{94}$	$\{1,2,3,88,22,145,12,16,4,7,17,92,23\}$	I
$\mathcal{O}_{95}$	$\{1,2,3,88,22,145,12,16,4,7,17,71,92\}$	I
$\mathcal{O}_{96}$	$\{1,2,3,88,22,145,12,16,20,23,181,11,182\}$	I
$\mathcal{O}_{97}$	$\{1,2,3,88,22,145,12,16,4,141,146,164,109\}$	I
$\mathcal{O}_{98}$	$\{1,2,3,88,22,145,12,16,49,61,18,7,66\}$	I
$\mathcal{O}_{99}$	$\{1,2,3,88,22,145,12,16,4,17,92,73,135\}$	I
$\mathcal{O}_{100}$	$\{1,2,3,88,22,145,12,16,49,92,164,141,17\}$	I
$\mathcal{O}_{101}$	$\{1,2,3,88,22,145,12,16,49,61,18,7,53\}$	I

Theorem 2.9. *In* $\mathrm{PG}(2;13)$*, there are at least* 101 *sd-inequivalent* $(13;4)$*-arcs.*

Remark

The groups of the 101 *sd*-inequivalent $(13;4)$-arcs other than the identity group are the following:

$$Z_6,\ Z_2,\ S_4.$$

(1) The group Z_2 splits the sets $\mathscr{O}_{24},\ \mathscr{O}_{27},\ \mathscr{O}_{29},\ \mathscr{O}_{30}$ into $9,9,9,9$ orbits. They are the following:

$$\mathscr{O}rb(\mathscr{O}_{24}) = \{1,3\},\{2,53\},\{9\},\{35,182\},\{56\},\{66,143\},\{88\},\{117\},\{135\};$$

$$\mathscr{O}rb(\mathscr{O}_{27}) = \{1,35\},\{2,53\},\{3,182\},\{8\},\{9\},\{56,134\},\{88\},\{90\},\{135\};$$

$$\mathscr{O}rb(\mathscr{O}_{29}) = \{1,88\},\{2\},\{3\},\{9,53\},\{22\},\{35\},\{56,140\},\{90\},\{135,182\};$$

$$\mathscr{O}rb(\mathscr{O}_{30}) = \{1\},\{2,9\},\{3,135\},\{22,151\},\{35,88\},\{53\},\{55\},\{56\},\{182\}.$$

(2) The group Z_6 splits the set $\mathscr{O}_{20}$ into 4 orbits of sizes 2,6,2,3 as follows:

$$\mathscr{O}rb(\mathscr{O}_{20}) = \{1,2\},\{3,106,88,7,17,39\},\{9,21\},\{35,92,91\}.$$

(3) The group S_4 divides the set $\mathscr{O}_{28}$ into 3 orbits of sizes 6,3,4 as follows:

$$\mathscr{O}rb(\mathscr{O}_{28}) = \{1,3,35,135,88,182\},\{2,9,53\},\{56,90,140,134\}.$$

2.10 *sd*-inequivalent $(14;4)$-arcs

In this process, the number of *sd*-inequivalent $(14;4)$-arcs is 133. The values of t_i of the secant distributions $\{t_4,t_3,t_2,t_1,t_0\}$ of the *sd*-inequivalent $(14;4)$-arcs are as follows:

$$t_4 \in \{1,\ldots,10\}, t_3 \in \{0,\ldots,22\}, t_2 \in \{13,\ldots,64\}, t_1 \in \{43,\ldots,112\}, t_0 \in \{42,\ldots,68\}.$$

The statistics of *sd*-inequivalent $(14;4)$-arcs are given in Table 2.16.

Table 2.16: sd**-inequivalent** $(14;4)$**-arcs in** $\mathrm{PG}(2,13)$

Symbol	(14;4)-arc	Stabiliser
$\mathscr{P}_1$	$\{1,2,3,88,9,21,5,4,61,158,49,10,92,135\}$	I
$\mathscr{P}_2$	$\{1,2,3,88,9,21,5,4,61,158,49,68,39,81\}$	I
$\mathscr{P}_3$	$\{1,2,3,88,9,21,5,4,61,158,68,92,171,34\}$	I
$\mathscr{P}_4$	$\{1,2,3,88,9,21,5,15,33,17,12,24,80,16\}$	I
$\mathscr{P}_5$	$\{1,2,3,88,9,21,5,15,33,55,44,8,18,46\}$	I
$\mathscr{P}_6$	$\{1,2,3,88,9,21,5,15,33,55,44,8,18,24\}$	I
$\mathscr{P}_7$	$\{1,2,3,88,9,21,5,15,33,55,44,8,18,13\}$	I
$\mathscr{P}_8$	$\{1,2,3,88,9,21,5,15,33,55,44,8,18,4\}$	I
$\mathscr{P}_9$	$\{1,2,3,88,9,21,5,15,33,55,44,8,121,124\}$	I
$\mathscr{P}_{10}$	$\{1,2,3,88,9,21,5,15,33,55,44,8,18,19\}$	I
$\mathscr{P}_{11}$	$\{1,2,3,88,9,21,5,15,33,55,44,8,18,27\}$	I
$\mathscr{P}_{12}$	$\{1,2,3,88,9,21,5,15,33,55,44,8,121,166\}$	I
$\mathscr{P}_{13}$	$\{1,2,3,88,9,21,5,15,33,55,44,8,23,121\}$	I
$\mathscr{P}_{14}$	$\{1,2,3,88,9,21,5,15,33,55,44,8,18,20\}$	I
$\mathscr{P}_{15}$	$\{1,2,3,88,9,21,5,15,33,55,44,8,23,85\}$	I
$\mathscr{P}_{16}$	$\{1,2,3,88,9,21,5,15,33,55,44,8,23,139\}$	I
$\mathscr{P}_{17}$	$\{1,2,3,88,9,21,5,15,33,55,44,8,18,82\}$	I
$\mathscr{P}_{18}$	$\{1,2,3,88,9,21,5,15,33,55,44,8,23,82\}$	I
$\mathscr{P}_{19}$	$\{1,2,3,88,9,21,5,15,33,55,44,8,7,160\}$	I
$\mathscr{P}_{20}$	$\{1,2,3,88,9,21,5,15,33,55,44,8,7,82\}$	I
$\mathscr{P}_{21}$	$\{1,2,3,88,9,21,35,7,17,91,92,39,6,68\}$	I
$\mathscr{P}_{22}$	$\{1,2,3,88,9,21,35,7,17,91,92,39,12,56\}$	I
$\mathscr{P}_{23}$	$\{1,2,3,88,9,21,35,7,17,91,92,39,6,4\}$	I
$\mathscr{P}_{24}$	$\{1,2,3,88,9,21,35,7,17,91,92,39,6,106\}$	I
$\mathscr{P}_{25}$	$\{1,2,3,88,9,21,35,7,17,91,39,66,121,160\}$	I
$\mathscr{P}_{26}$	$\{1,2,3,88,9,21,35,7,17,91,39,146,20,69\}$	I
$\mathscr{P}_{27}$	$\{1,2,3,88,9,21,35,7,17,91,39,146,20,106\}$	Z_4

$\mathscr{P}_{28}$	$\{1,2,3,88,9,21,35,7,17,91,92,39,106,68\}$	Z_6
$\mathscr{P}_{29}$	$\{1,2,3,88,9,182,35,56,135,22,151,55,57,31\}$	I
$\mathscr{P}_{30}$	$\{1,2,3,88,9,182,35,56,135,22,151,55,31,153\}$	I
$\mathscr{P}_{31}$	$\{1,2,3,88,9,182,35,56,135,22,151,55,57,38\}$	I
$\mathscr{P}_{32}$	$\{1,2,3,88,9,182,35,56,135,22,151,55,31,45\}$	Z_2
$\mathscr{P}_{33}$	$\{1,2,3,88,9,182,35,56,135,22,151,55,31,85\}$	I
$\mathscr{P}_{34}$	$\{1,2,3,88,9,182,35,56,135,22,151,55,31,53\}$	I
$\mathscr{P}_{35}$	$\{1,2,3,88,9,182,35,56,135,22,151,31,53,45\}$	Z_2
$\mathscr{P}_{36}$	$\{1,2,3,88,9,182,35,56,135,22,151,31,53,90\}$	I
$\mathscr{P}_{37}$	$\{1,2,3,88,9,182,35,56,53,66,143,117,135,5\}$	I
$\mathscr{P}_{38}$	$\{1,2,3,88,9,182,35,56,135,22,53,140,31,90\}$	I
$\mathscr{P}_{39}$	$\{1,2,3,88,9,182,35,56,135,90,53,134,8,12\}$	I
$\mathscr{P}_{40}$	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,80\}$	I
$\mathscr{P}_{41}$	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,7\}$	I
$\mathscr{P}_{42}$	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,27\}$	I
$\mathscr{P}_{43}$	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,34\}$	I
$\mathscr{P}_{44}$	$\{1,2,3,88,22,145,12,16,20,31,33,38,7,34\}$	I
$\mathscr{P}_{45}$	$\{1,2,3,88,22,145,12,16,20,31,33,38,34,27\}$	I
$\mathscr{P}_{46}$	$\{1,2,3,88,22,145,12,16,20,31,33,38,34,144\}$	I
$\mathscr{P}_{47}$	$\{1,2,3,88,22,145,12,16,20,31,33,73,136,139\}$	I
$\mathscr{P}_{48}$	$\{1,2,3,88,22,145,12,16,20,31,5,30,144,60\}$	I
$\mathscr{P}_{49}$	$\{1,2,3,88,22,145,12,16,20,23,32,175,29,152\}$	I
$\mathscr{P}_{50}$	$\{1,2,3,88,22,145,12,16,20,23,32,175,29,164\}$	I
$\mathscr{P}_{51}$	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40\}$	I
$\mathscr{P}_{52}$	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,158\}$	I
$\mathscr{P}_{53}$	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,26\}$	I
$\mathscr{P}_{54}$	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,5\}$	I
$\mathscr{P}_{55}$	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,6\}$	I
$\mathscr{P}_{56}$	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,11\}$	I

$\mathscr{P}_{57}$	$\{1,2,3,88,22,145,12,16,20,31,33,38,5,45\}$	I
$\mathscr{P}_{58}$	$\{1,2,3,88,22,145,12,16,20,31,33,38,7,4\}$	I
$\mathscr{P}_{59}$	$\{1,2,3,88,22,145,12,16,20,31,33,38,34,55\}$	I
$\mathscr{P}_{60}$	$\{1,2,3,88,22,145,12,16,20,31,33,4,149,73\}$	I
$\mathscr{P}_{61}$	$\{1,2,3,88,22,145,12,16,20,31,33,4,149,128\}$	I
$\mathscr{P}_{62}$	$\{1,2,3,88,22,145,12,16,20,31,4,73,44,49\}$	I
$\mathscr{P}_{63}$	$\{1,2,3,88,22,145,12,16,20,31,4,73,44,94\}$	I
$\mathscr{P}_{64}$	$\{1,2,3,88,22,145,12,16,20,23,32,175,29,106\}$	I
$\mathscr{P}_{65}$	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,18\}$	I
$\mathscr{P}_{66}$	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,4\}$	I
$\mathscr{P}_{67}$	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,118\}$	I
$\mathscr{P}_{68}$	$\{1,2,3,88,22,145,12,16,20,31,33,38,5,87\}$	I
$\mathscr{P}_{69}$	$\{1,2,3,88,22,145,12,16,20,31,33,38,7,66\}$	I
$\mathscr{P}_{70}$	$\{1,2,3,88,22,145,12,16,20,31,33,4,149,11\}$	I
$\mathscr{P}_{71}$	$\{1,2,3,88,22,145,12,16,20,31,33,4,149,136\}$	I
$\mathscr{P}_{72}$	$\{1,2,3,88,22,145,12,16,20,31,33,73,136,128\}$	I
$\mathscr{P}_{73}$	$\{1,2,3,88,22,145,12,16,20,31,4,73,44,32\}$	I
$\mathscr{P}_{74}$	$\{1,2,3,88,22,145,12,16,20,23,32,175,29,57\}$	I
$\mathscr{P}_{75}$	$\{1,2,3,88,22,145,12,16,4,7,141,79,44,17\}$	I
$\mathscr{P}_{76}$	$\{1,2,3,88,22,145,12,16,4,7,141,79,44,146\}$	I
$\mathscr{P}_{77}$	$\{1,2,3,88,22,145,12,16,20,31,33,38,158,66\}$	I
$\mathscr{P}_{78}$	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,66\}$	I
$\mathscr{P}_{79}$	$\{1,2,3,88,22,145,12,16,20,31,33,38,15,66\}$	I
$\mathscr{P}_{80}$	$\{1,2,3,88,22,145,12,16,20,31,33,38,34,85\}$	I
$\mathscr{P}_{81}$	$\{1,2,3,88,22,145,12,16,20,31,33,4,149,8\}$	I
$\mathscr{P}_{82}$	$\{1,2,3,88,22,145,12,16,20,31,33,4,73,45\}$	I
$\mathscr{P}_{83}$	$\{1,2,3,88,22,145,12,16,20,31,33,73,136,72\}$	I
$\mathscr{P}_{84}$	$\{1,2,3,88,22,145,12,16,20,31,5,30,144,118\}$	I
$\mathscr{P}_{85}$	$\{1,2,3,88,22,145,12,16,20,31,4,73,44,131\}$	I
$\mathscr{P}_{86}$	$\{1,2,3,88,22,145,12,16,4,7,141,79,44,25\}$	I

$\mathscr{P}_{87}$	$\{1,2,3,88,22,145,12,16,49,61,24,85,87,34\}$	I
$\mathscr{P}_{88}$	$\{1,2,3,88,22,145,12,16,20,31,33,80,69,107\}$	I
$\mathscr{P}_{89}$	$\{1,2,3,88,22,145,12,16,20,31,33,38,15,132\}$	I
$\mathscr{P}_{90}$	$\{1,2,3,88,22,145,12,16,20,31,33,38,69,95\}$	I
$\mathscr{P}_{91}$	$\{1,2,3,88,22,145,12,16,20,31,33,38,18,66\}$	I
$\mathscr{P}_{92}$	$\{1,2,3,88,22,145,12,16,20,31,33,38,66,8\}$	I
$\mathscr{P}_{93}$	$\{1,2,3,88,22,145,12,16,20,31,33,38,66,118\}$	I
$\mathscr{P}_{94}$	$\{1,2,3,88,22,145,12,16,20,31,33,4,71,131\}$	I
$\mathscr{P}_{95}$	$\{1,2,3,88,22,145,12,16,20,31,33,73,128,72\}$	I
$\mathscr{P}_{96}$	$\{1,2,3,88,22,145,12,16,20,31,4,32,73,8\}$	I
$\mathscr{P}_{97}$	$\{1,2,3,88,22,145,12,16,20,23,32,175,30,13\}$	I
$\mathscr{P}_{98}$	$\{1,2,3,88,22,145,12,16,4,7,141,79,17,49\}$	I
$\mathscr{P}_{99}$	$\{1,2,3,88,22,145,12,16,49,61,24,85,87,66\}$	I
$\mathscr{P}_{100}$	$\{1,2,3,88,22,145,12,16,20,8,28,9,89,25\}$	I
$\mathscr{P}_{101}$	$\{1,2,3,88,22,145,12,16,20,31,33,6,17,8\}$	I
$\mathscr{P}_{102}$	$\{1,2,3,88,22,145,12,16,20,31,33,8,66,48\}$	I
$\mathscr{P}_{103}$	$\{1,2,3,88,22,145,12,16,20,31,33,8,66,71\}$	I
$\mathscr{P}_{104}$	$\{1,2,3,88,22,145,12,16,20,31,33,73,45,107\}$	I
$\mathscr{P}_{105}$	$\{1,2,3,88,22,145,12,16,20,31,5,30,69,169\}$	I
$\mathscr{P}_{106}$	$\{1,2,3,88,22,145,12,16,20,31,4,71,131,8\}$	I
$\mathscr{P}_{107}$	$\{1,2,3,88,22,145,12,16,20,31,4,71,131,156\}$	I
$\mathscr{P}_{108}$	$\{1,2,3,88,22,145,12,16,20,31,4,73,71,131\}$	I
$\mathscr{P}_{109}$	$\{1,2,3,88,22,145,12,16,4,7,141,79,25,164\}$	I
$\mathscr{P}_{110}$	$\{1,2,3,88,22,145,12,16,49,61,24,85,34,151\}$	I
$\mathscr{P}_{111}$	$\{1,2,3,88,22,145,12,16,49,61,24,85,34,92\}$	I
$\mathscr{P}_{112}$	$\{1,2,3,88,22,145,12,16,20,31,33,69,95,127\}$	I
$\mathscr{P}_{113}$	$\{1,2,3,88,22,145,12,16,20,31,69,95,8,39\}$	I
$\mathscr{P}_{114}$	$\{1,2,3,88,22,145,12,16,20,31,26,69,132,95\}$	I
$\mathscr{P}_{115}$	$\{1,2,3,88,22,145,12,16,20,31,26,69,132,141\}$	I
$\mathscr{P}_{116}$	$\{1,2,3,88,22,145,12,16,20,31,5,69,99,118\}$	I

$\mathscr{P}_{117}$	$\{1,2,3,88,22,145,12,16,20,31,4,69,85,24\}$	I
$\mathscr{P}_{118}$	$\{1,2,3,88,22,145,12,16,20,23,32,181,44,170\}$	I
$\mathscr{P}_{119}$	$\{1,2,3,88,22,145,12,16,4,7,141,49,69,75\}$	I
$\mathscr{P}_{120}$	$\{1,2,3,88,22,145,12,16,4,7,141,49,69,61\}$	I
$\mathscr{P}_{121}$	$\{1,2,3,88,22,145,12,16,4,7,141,17,73,146\}$	I
$\mathscr{P}_{122}$	$\{1,2,3,88,22,145,12,16,20,23,181,11,182,9\}$	I
$\mathscr{P}_{123}$	$\{1,2,3,88,22,145,12,16,20,31,181,69,64,172\}$	I
$\mathscr{P}_{124}$	$\{1,2,3,88,22,145,12,16,20,31,181,69,63,44\}$	I
$\mathscr{P}_{125}$	$\{1,2,3,88,22,145,12,16,4,7,17,71,6,72\}$	I
$\mathscr{P}_{126}$	$\{1,2,3,88,22,145,12,16,4,7,17,71,6,146\}$	I
$\mathscr{P}_{127}$	$\{1,2,3,88,22,145,12,16,4,7,141,49,164,61\}$	I
$\mathscr{P}_{128}$	$\{1,2,3,88,22,145,12,16,49,61,18,7,53,146\}$	I
$\mathscr{P}_{129}$	$\{1,2,3,88,22,145,12,16,49,92,164,141,17,54\}$	I
$\mathscr{P}_{130}$	$\{1,2,3,88,22,145,12,16,49,92,164,141,17,13\}$	I
$\mathscr{P}_{131}$	$\{1,2,3,88,22,145,12,16,4,141,146,164,109,103\}$	I
$\mathscr{P}_{132}$	$\{1,2,3,88,22,145,12,16,4,7,17,71,92,146\}$	I
$\mathscr{P}_{133}$	$\{1,2,3,88,22,145,12,16,49,61,18,7,53,66\}$	I

Theorem 2.10. *In* PG$(2;13)$*, there are at least* 133 *sd-inequivalent* $(14;4)$*-arcs.*

Remark

The groups of the 133 *sd*-inequivalent $(14;4)$-arcs other than the identity group are Z_2, Z_4 ,Z_6. These groups partition their associated $(14;4)$-arcs into a number of orbits as follows:

(1) $\mathscr{O}rb(\mathscr{P}_{32}) = \{1\},\{2,9\},\{3,135\},\{22,151\},\{31,45\},\{35,88\},\{55\},\{56\},\{182\};$

(2) $\mathscr{O}rb(\mathscr{P}_{35}) = \{1\},\{2,9\},\{3,135\},\{22,151\},\{31,45\},\{35,88\},\{53\},\{56\},\{182\};$

(3) $\mathscr{O}rb(\mathscr{P}_{27}) = \{1,20,2,146\},\{3,88,106,7\},\{9,39,21,17\},\{35,91\};$

(4) $\mathscr{O}rb(\mathscr{P}_{28}) = \{1,2\},\{3,106,88,7,17,39\},\{9,21\},\{35,92,91\},\{68\}.$

2.11 sd-inequivalent $(15;4)$-arcs

There are 170 sd-inequivalent $(15;4)$-arcs. The statistics of the groups of sd-inequivalent $(15;4)$-arcs shows that there are three types of the stabiliser groups; they are I, Z_2, Z_3. Also, the values of t_i of the secant distributions $\{t_4,t_3,t_2,t_1,t_0\}$ of the sd-inequivalent $(15;4)$-arcs are as follows:

$$t_4 \in \{1,\ldots,12\}, t_3 \in \{0,\ldots,26\}, t_2 \in \{12,\ldots,69\}, t_1 \in \{38,\ldots,114\}, t_0 \in \{36,\ldots,65\}.$$

Theorem 2.11. *In* $\mathrm{PG}(2,13)$*, there are at least* 170 sd*-inequivalent* $(15;4)$*-arcs.*

Remark

Among the stabiliser groups of the 170 sd-inequivalent $(15;4)$-arcs, there are six sd-inequivalent $(15;4)$-arcs having the groups Z_2 and Z_3. These groups split their corresponding sd-inequivalent $(15;4)$-arcs into a number of orbits listed in Table 2.17.

Table 2.17: **Group orbits of the six** sd**-inequivalent** $(15;4)$**-arcs**

sd-inequivalent $(15;4)$-arcs	Stabilizer	Orbits
$\{1,2,3,88,9,182,35,56,135,22,151,55,31,45,53\}$	Z_2	$\{1\},\{2,9\},\{3,135\},\{22,151\},\{31,45\},\{35,88\},\{53\},\{55\},\{56\},\{182\}$
$\{1,2,3,88,9,182,35,56,135,22,151,31,53,45,90\}$	Z_2	$\{1\},\{2,9\},\{3,135\},\{22,151\},\{31,45\},\{35,88\},\{53\},\{56\},\{90\},\{182\}$
$\{1,2,3,88,9,182,35,56,135,90,53,134,8,12,146\}$	Z_2	$\{1\},\{2,9\},\{3,88\},\{8,146\},\{12\},\{35,135\},\{53\},\{56,90\},\{134\},\{182\}$
$\{1,2,3,88,9,21,35,7,17,91,92,39,6,4,106\}$	Z_2	$\{1,2\},\{3,106\},\{4,6\},\{7,88\},\{9,21\},\{17,39\},\{35\},\{91\},\{92\}$
$\{1,2,3,88,9,21,35,7,17,91,92,39,6,68,20\}$	Z_3	$\{1\},\{2,20,6\},\{3\},\{7,39,68\},\{9,88,92\},\{17,21,35\},\{91\}$
$\{1,2,3,88,9,21,35,7,17,91,92,39,6,68,106\}$	Z_3	$\{1,2,6\},\{3,39,68\},\{7,88,35\},\{9,106,17\},\{21,91,92\}$

2.12 sd-inequivalent $(16;4)$-arcs

In this process, among the 19780 (16;4)-arcs, the number of sd-inequivalent (16;4)-arcs is 213. The stabiliser groups of the 213 sd-inequivalent (16;4)-arcs are I, Z_2, Z_3, Z_4. The values of t_i of the secant distributions $\{t_4,t_3,t_2,t_1,t_0\}$ of the sd-inequivalent (16;4)-arcs are as follows:

$$t_4 \in \{1,\dots,15\}, t_3 \in \{0,\dots,30\}, t_2 \in \{12,\dots,75\}, t_1 \in \{34,\dots,116\}, t_0 \in \{30,\dots,62\}.$$

Theorem 2.12. *In* $\mathrm{PG}(2,13)$*, there are at least* 213 sd*-inequivalent* $(16;4)$*-arcs.*

Remark

Among the 213 sd-inequivalent $(16;4)$-arcs, there are only five sd-inequivalent $(16;4)$-arcs with the stabiliser groups Z_2, Z_3, Z_4. These stabilisers partition the associated sd-inequivalent $(16;4)$-arcs into a number of orbits listed in Table 2.18.

Table 2.18: **Group orbits of the five sd-inequivalent $(16;4)$-arcs**

sd-inequivalent $(16;4)$-arcs	Stabilizer	Orbits
$\{1,2,3,88,9,182,35,56,135,22,151,55,31,153,111,45\}$	Z_2	$\{1\},\{2,9\},\{3,135\},\{22,151\},\{31,45\},\{35,88\},\{55\},\{56\},\{111,153\},\{182\}$
$\{1,2,3,88,9,182,35,56,135,22,151,31,53,45,134,140\}$	Z_2	$\{1\},\{2,9\},\{3,135\},\{22,151\},\{31,45\},\{35,88\},\{53\},\{56\},\{134,140\},\{182\}$
$\{1,2,3,88,9,21,35,7,17,91,92,39,6,4,106,22\}$	Z_2	$\{1,2\},\{3,106\},\{4,6\},\{7,88\},\{9,21\},\{17,39\},\{22\},\{35\},\{91\},\{92\}$
$\{1,2,3,88,9,21,35,7,17,91,92,39,6,68,20,4\}$	Z_3	$\{1\},\{2,20,6\},\{3\},\{4\},\{7,39,68\},\{9,88,92\},\{17,21,35\},\{91\}$
$\{1,2,3,88,9,21,35,7,17,91,92,39,6,68,118,106\}$	Z_4	$\{1,2,118,6\},\{3,7\},\{9,39,68,17\},\{21,92\},\{35,106,88,91\}$

2.13 sd-inequivalent $(17;4)$-arcs

In this process, there are 256 sd-inequivalent $(17;4)$-arcs. The groups of these sd-inequivalent $(17;4)$-arcs are the identity group and Z_2. The values of t_i of the secant distributions $\{t_4,t_3,t_2,t_1,t_0\}$ of the sd-inequivalent $(17;4)$-arcs are as follows:

$$t_4 \in \{1,\dots,16\}, t_3 \in \{1,\dots,34\}, t_2 \in \{16,\dots,79\}, t_1 \in \{30,\dots,112\}, t_0 \in \{28,\dots,59\}.$$

Theorem 2.13. *In* $\mathrm{PG}(2,13)$*, there are at least* 256 sd*-inequivalent* $(17;4)$*-arcs.*

Remark

In Table 2.19, the group Z_2 splits the four corresponding sd-inequivalent $(17;4)$-arcs into a number of orbits.

Table 2.19: **Group orbits of the four sd-inequivalent $(17;4)$-arcs**

sd-inequivalent $(17;4)$-arcs	Stabilizer	Orbits
$\{1,2,3,88,9,182,35,56,135,22,151,55,31,153,111,45,53\}$	Z_2	$\{1\},\{2,9\},\{3,135\}$, $\{22,151\},\{31,45\},\{35,88\}$, $\{53\},\{55\},\{56\}$, $\{111,153\},\{182\}$
$\{1,2,3,88,9,182,35,56,135,22,151,55,31,45,53,165,172\}$	Z_2	$\{1\},\{2,9\},\{3,135\}$, $\{22,151\},\{31,45\},\{35,88\}$, $\{53\},\{55\},\{56\}$, $\{165,172\},\{182\}$
$\{1,2,3,88,9,182,35,56,135,22,151,31,53,45,134,140,169\}$	Z_2	$\{1\},\{2,9\},\{3,135\}$, $\{22,151\},\{31,45\},\{35,88\}$, $\{53\},\{56\},\{134,140\}$, $\{169\},\{182\}$
$\{1,2,3,88,9,21,35,7,17,91,92,39,6,4,106,66,143\}$	Z_2	$\{1,2\},\{3,106\},\{4,6\}$, $\{7,88\},\{9,21\},\{17,39\}$, $\{35\},\{66,143\},\{91\}$, $\{92\}$

2.14 *sd*-inequivalent $(18;4)$-arcs

In this process, there are 25677 (18;4)-arcs found including 293 *sd*-inequivalent (18;4)-arcs each having the identity group. The values of t_i of the secant distributions $\{t_4,t_3,t_2,t_1,t_0\}$ of the *sd*-inequivalent $(18;4)$-arcs are as follows:

$$t_4 \in \{1,\dots,18\}, t_3 \in \{1,\dots,39\}, t_2 \in \{18,\dots,81\}, t_1 \in \{27,\dots,110\}, t_0 \in \{24,\dots,56\}.$$

Remark

The notation S in Table 2.20 indicates the stabilisers of *sd*-inequivalent $(18;4)$-arcs while n indicates the number of these stabilisers and similarly in each table for $k = 19,\dots,37$.

Table 2.20: **Statistics of *sd*-inequivalent $(18;4)$-arcs**

sd-inequivalent (18;4)-arcs	$S:n$
293	$I:293$

Theorem 2.14. *In* $\mathrm{PG}(2,13)$*, there are at least* 293 *sd-inequivalent* $(18;4)$*-arcs.*

2.15 *sd*-inequivalent $(19;4)$-arcs

In this process, among the 27370 arcs found, there are 338 *sd*-inequivalent (19;4)-arcs each having the identity group. The values of t_i of the secant distributions $\{t_4,t_3,t_2,t_1,t_0\}$ of the *sd*-inequivalent $(19;4)$-arcs are as follows:

$$t_4 \in \{1,\dots,20\}, t_3 \in \{2,\dots,45\}, t_2 \in \{18,\dots,81\}, t_1 \in \{23,\dots,107\}, t_0 \in \{21,\dots,54\}.$$

Table 2.21: **Statistics of *sd*-inequivalent $(19;4)$-arcs**

sd-inequivalent (19;4)-arcs	$S:n$
338	$I:338$

Theorem 2.15. *In* $\mathrm{PG}(2,13)$*, there are at least* 338 *sd-inequivalent* $(19;4)$*-arcs.*

2.16 *sd*-inequivalent $(20;4)$-arcs

There are 28908 $(20;4)$-arcs found. Among them the number of *sd*-inequivalent (20;4)-arcs is 378 each having the identity group. The values of t_i of the secant distributions $\{t_4,t_3,t_2,t_1,t_0\}$ of the *sd*-inequivalent $(20;4)$-arcs are as follows:

$$t_4 \in \{1,\ldots,22\}, t_3 \in \{3,\ldots,50\}, t_2 \in \{19,\ldots,85\}, t_1 \in \{20,\ldots,104\}, t_0 \in \{18,\ldots,52\}.$$

Table 2.22: **Statistics of *sd*-inequivalent $(20;4)$-arcs**

sd-inequivalent (20;4)-arcs	$S:n$
378	$I:378$

Theorem 2.16. *In* $\mathrm{PG}(2,13)$*, there are at least* 378 *sd-inequivalent* $(20;4)$*-arcs.*

2.17 *sd*-inequivalent $(21;4)$-arcs

In this process, there are 28973 $(21;4)$-arcs found. Among them there are 411 *sd*-inequivalent (21;4)-arcs according to the number of *sd*-inequivalent classes of *i*-secant distribution. The values of t_i of these classes of $\{t_4,t_3,t_2,t_1,t_0\}$ are as follows:

$$t_4 \in \{1,\ldots,24\}, t_3 \in \{6,\ldots,55\}, t_2 \in \{18,\ldots,87\}, t_1 \in \{16,\ldots,101\}, t_0 \in \{16,\ldots,50\}.$$

Table 2.23: **Statistics of *sd*-inequivalent $(21;4)$-arcs**

sd-inequivalent (21;4)-arcs	$S:n$
411	$I:411$

Theorem 2.17. *In* $\mathrm{PG}(2,13)$*, there are at least* 411 *sd-inequivalent* $(21;4)$*-arcs.*

2.18 *sd*-inequivalent $(22;4)$-arcs

In this process, there are 27587 $(22;4)$-arcs found including 436 *sd*-inequivalent (22;4)-arcs all having the identity group. The values of t_i of the secant distributions $\{t_4,t_3,t_2,t_1,t_0\}$ of the *sd*-inequivalent $(22;4)$-arcs are as follows:

$$t_4 \in \{2,\ldots,27\}, t_3 \in \{7,\ldots,58\}, t_2 \in \{18,\ldots,87\}, t_1 \in \{15,\ldots,97\}, t_0 \in \{14,\ldots,47\}.$$

Table 2.24: **Statistics of** sd**-inequivalent** $(22;4)$**-arcs**

sd-inequivalent (22;4)-arcs	$S:n$
436	$I:436$

Theorem 2.18. *In* $\mathrm{PG}(2,13)$*, there are at least* 436 sd*-inequivalent* $(22;4)$*-arcs.*

2.19 sd-inequivalent $(23;4)$-arcs

In this process, the number of sd-inequivalent classes of i-secant distributions of the 24708 $(23;4)$-arcs found is 455. So the number of sd-inequivalent (23;4)-arcs is 455 where 454 of them have the identity group and one arc has the group Z_2. The values of t_i of the secant distributions $\{t_4,t_3,t_2,t_1,t_0\}$ of the sd-inequivalent $(23;4)$-arcs are as follows:

$$t_4 \in \{3,\ldots,30\}, t_3 \in \{8,\ldots,64\}, t_2 \in \{19,\ldots,85\}, t_1 \in \{13,\ldots,94\}, t_0 \in \{12,\ldots,46\}.$$

Table 2.25: **Statistics of** sd**-inequivalent** $(23;4)$**-arcs**

sd-inequivalent (23;4)-arcs	$S:n$
455	$I:454$, $Z_2:1$

Theorem 2.19. *In* $\mathrm{PG}(2,13)$*, there are at least* 455 sd*-inequivalent* $(23;4)$*-arcs.*

Remark

In Table 2.25, the group Z_2 partitions the associated sd-inequivalent $(23;4)$-arc $\mathscr{K}$ into 13 orbits as follows:

$\mathscr{K} = \{1,2,3,88,9,21,5,15,33,55,44,8,18,46,50,124,31,80,96,164,169,28,101\}$.

$\mathscr{O}rb(\mathscr{K}) = \{1,96\},\{2,28\},\{3,164\},\{5,88\},\{8,21\},\{9\},\{15,50\},\{18,101\},\{31\},$

$\{33,124\},\{44,169\},\{46\},\{55,80\}$.

2.20 *sd*-inequivalent $(24;4)$-arcs

In this process, there are four complete $(24;4)$-arcs and 458 *sd*-inequivalent $(24;4)$-arcs all have the identity group. The values of t_i of the secant distributions $\{t_4,t_3,t_2,t_1,t_0\}$ of the *sd*-inequivalent $(24;4)$-arcs are as follows:

$$t_4 \in \{5,\ldots,33\}, t_3 \in \{8,\ldots,69\}, t_2 \in \{18,\ldots,84\}, t_1 \in \{8,\ldots,90\}, t_0 \in \{11,\ldots,45\}.$$

The statistics are given in Table 2.26.

Table 2.26: **Statistics of *sd*-inequivalent $(24;4)$-arcs**

sd-inequivalent (24;4)-arcs	$S:n$
458	$I:458$

Theorem 2.20. *In* $\mathrm{PG}(2,13)$*, there are at least* 458 *sd-inequivalent* $(24;4)$*-arcs.*

2.21 *sd*-inequivalent $(25;4)$-arcs

Among the number of incomplete $(25;4)$-arcs found, there are 467 *sd*-inequivalent $(25;4)$-arcs all have the identity group. Also, the number of complete $(25;4)$-arcs is four. The values of t_i of the secant distributions $\{t_4,t_3,t_2,t_1,t_0\}$ of the *sd*-inequivalent $(25;4)$-arcs are as follows:

$$t_4 \in \{7,\ldots,36\}, t_3 \in \{11,\ldots,75\}, t_2 \in \{18,\ldots,84\}, t_1 \in \{7,\ldots,86\}, t_0 \in \{10,\ldots,43\}.$$

The statistics are shown in Table 2.27.

Table 2.27: **Statistics of *sd*-inequivalent $(25;4)$-arcs**

sd-inequivalent (25;4)-arcs	$S:n$
467	$I:467$

Theorem 2.21. *In* $\mathrm{PG}(2,13)$*, there are at least* 467 *sd-inequivalent* $(25;4)$*-arcs.*

2.22 *sd*-inequivalent $(26;4)$-arcs

According to the number of *sd*-inequivalent classes of the incomplete $(26;4)$-arcs found, there are 447 *sd*-inequivalent $(26;4)$-arcs all having the identity group and Z_2; also the number of complete $(26;4)$-arcs is 15. The values of t_i of the secant distributions $\{t_4,t_3,t_2,t_1,t_0\}$ of the *sd*-inequivalent $(26;4)$-arcs are as follows:

$$t_4 \in \{10,\dots,40\}, t_3 \in \{12,\dots,77\}, t_2 \in \{16,\dots,82\}, t_1 \in \{7,\dots,81\}, t_0 \in \{10,\dots,41\}.$$

The statistics are shown in Table 2.28.

Table 2.28: **Statistics of *sd*-inequivalent $(26;4)$-arcs**

sd-inequivalent (26;4)-arcs	$S:n$
447	$I:446,\ Z_2:1$

Theorem 2.22. *In* $\mathrm{PG}(2,13)$*, there are at least* 447 *sd-inequivalent* $(26;4)$*-arcs.*

Remark

In Table 2.28, the group Z_2 splits the associated *sd*-inequivalent $(26;4)$-arc $\mathscr{K}$ into 15 orbits as the following:

$\mathscr{K} = \{1,2,3,88,9,21,5,15,33,55,44,8,18,46,50,124,31,80,96,164,169,28,101,24,93,66\}$.

$$\begin{aligned}\mathscr{O}rb(\mathscr{K}) \;=\; & \{1,96\},\{2,28\},\{3,164\},\{5,88\},\{8,21\},\{9\},\{15,50\},\{18,101\},\\ & \{24,66\},\{31\},\{33,124\},\{44,169\},\{46\},\{55,80\},\{93\}.\end{aligned}$$

2.23 *sd*-inequivalent $(27;4)$-arcs

In this process, there are 43 complete $(27;4)$-arcs and according to the number N_c of secant distribution of incomplete $(27;4)$-arcs found, there are 403 *sd*-inequivalent arcs. These 403 arcs all have the identity group. The values of t_i of the secant distributions $\{t_4,t_3,t_2,t_1,t_0\}$ of the *sd*-inequivalent $(27;4)$-arcs are as follows:

$$t_4 \in \{13,\dots,43\}, t_3 \in \{16,\dots,79\}, t_2 \in \{18,\dots,78\}, t_1 \in \{7,\dots,76\}, t_0 \in \{10,\dots,40\}.$$

Table 2.29, presents the related statistics.

Table 2.29: **Statistics of** sd**-inequivalent** $(27;4)$**-arcs**

sd-inequivalent (27;4)-arcs	$S:n$
403	$I:403$

Theorem 2.23. *In* $\mathrm{PG}(2,13)$*, there are at least* 403 sd*-inequivalent* $(27;4)$*-arcs.*

2.24 sd-inequivalent $(28;4)$-arcs

There are 367 sd-inequivalent $(28;4)$-arcs all have the identity group. Also, there are 44 complete $(28;4)$-arcs. The values of t_i of the secant distributions $\{t_4,t_3,t_2,t_1,t_0\}$ of the sd-inequivalent $(28;4)$-arcs are as follows:

$$t_4 \in \{17,\dots,45\}, t_3 \in \{18,\dots,81\}, t_2 \in \{15,\dots,72\}, t_1 \in \{6,\dots,71\}, t_0 \in \{10,\dots,39\}.$$

The related statistics are given in Table 2.30.

Table 2.30: **Statistics of** sd**-inequivalent** $(28;4)$**-arcs**

sd-inequivalent (28;4)-arcs	$S:n$
367	$I:367$

Theorem 2.24. *In* $\mathrm{PG}(2,13)$*, there are at least* 367 sd*-inequivalent* $(28;4)$*-arcs.*

2.25 sd-inequivalent $(29;4)$-arcs

In this process, the number of sd-inequivalent $(29;4)$-arcs is 301. Also, there are 100 complete arcs. The stabiliser groups of the 301 sd-inequivalent $(29;4)$-arcs are the identity. The following are the values of t_i of the secant distributions $\{t_4,t_3,t_2,t_1,t_0\}$ of the sd-inequivalent $(29;4)$-arcs.

$$t_4 \in \{21,\dots,48\}, t_3 \in \{23,\dots,83\}, t_2 \in \{13,\dots,67\}, t_1 \in \{6,\dots,64\}, t_0 \in \{11,\dots,38\}.$$

The related statistics are given in Table 2.31.

Table 2.31: **Statistics of** sd**-inequivalent** $(29;4)$**-arcs**

sd-inequivalent (29;4)-arcs	$S:n$
301	$I:301$

Theorem 2.25. *In* $\mathrm{PG}(2,13)$*, there are at least* 301 sd*-inequivalent* $(29;4)$*-arcs.*

2.26 sd-inequivalent $(30;4)$-arcs

The number of complete $(30;4)$-arcs is 133 and the number of sd-inequivalent $(30;4)$-arcs is 226 all have the identity group. The following are the values of t_i of the secant distributions $\{t_4,t_3,t_2,t_1,t_0\}$ of the sd-inequivalent $(30;4)$-arcs.

$$t_4 \in \{25,\ldots,50\}, t_3 \in \{29,\ldots,86\}, t_2 \in \{15,\ldots,63\}, t_1 \in \{4,\ldots,58\}, t_0 \in \{13,\ldots,38\}.$$

The data are given in Table 2.32.

Table 2.32: **Statistics of** sd**-inequivalent** $(30;4)$**-arcs**

sd-inequivalent (30;4)-arcs	$S:n$
226	$I:226$

Theorem 2.26. *In* $\mathrm{PG}(2,13)$*, there are at least* 226 sd*-inequivalent* $(30;4)$*-arcs.*

2.27 sd-inequivalent $(31;4)$-arcs

There are 775 incomplete $(31;4)$-arcs containing 165 sd-inequivalent $(31;4)$-arcs each having the identity group. Also, the number of complete $(31;4)$-arcs is 123. The following are the values of t_i of the secant distributions $\{t_4,t_3,t_2,t_1,t_0\}$ of the sd-inequivalent $(31;4)$-arcs.

$$t_4 \in \{29,\ldots,53\}, t_3 \in \{29,\ldots,90\}, t_2 \in \{15,\ldots,60\}, t_1 \in \{5,\ldots,53\}, t_0 \in \{14,\ldots,37\}.$$

The related statistics are introduced in Table 2.33.

Table 2.33: **Statistics of** sd**-inequivalent** $(31;4)$**-arcs**

sd-inequivalent (31;4)-arcs	$S:n$
165	$I:165$

Theorem 2.27. *In* $\mathrm{PG}(2,13)$*, there are at least* 165 *sd-inequivalent* $(31;4)$*-arcs.*

2.28 sd-inequivalent $(32;4)$-arcs

In this process, the number of incomplete and complete $(32;4)$-arcs is 319 and 119 respectively. Among the 319 incomplete arcs there are 89 sd-inequivalent $(32;4)$-arcs each having the identity group. The values of t_i of the secant distributions $\{t_4,t_3,t_2,t_1,t_0\}$ of the sd-inequivalent $(32;4)$-arcs are as follows:

$$t_4 \in \{37,\ldots,55\}, t_3 \in \{41,\ldots,85\}, t_2 \in \{16,\ldots,49\}, t_1 \in \{4,\ldots,42\}, t_0 \in \{18,\ldots,35\}.$$

The statistics are given in Table 2.34.

Table 2.34: **Statistics of** sd**-inequivalent** $(32;4)$**-arcs**

sd-inequivalent (32;4)-arcs	$S:n$
89	$I:89$

Theorem 2.28. *In* $\mathrm{PG}(2,13)$*, there are at least* 89 *sd-inequivalent* $(32;4)$*-arcs.*

2.29 sd-inequivalent $(33;4)$-arcs

In this process, the constructed number of incomplete $(33;4)$-arcs is 129 including 36 sd-inequivalent $(33;4)$-arcs. Also, there are 77 complete $(33;4)$-arcs. The stabiliser group of each of the 36 arcs is the identity group. The values of t_i of the secant distributions $\{t_4,t_3,t_2,t_1,t_0\}$ of the sd-inequivalent $(33;4)$-arcs are as follows:

$$t_4 \in \{45,\ldots,58\}, t_3 \in \{46,\ldots 80\}, t_2 \in \{18,\ldots 42\}, t_1 \in \{4,\ldots 21\}, t_0 \in \{25,\ldots 34\}.$$

The statistics of this process are given in Table 2.35.

Table 2.35: **Statistics of** sd**-inequivalent** $(33;4)$**-arcs**

sd-inequivalent (33;4)-arcs	$S:n$
36	$I:36$

Theorem 2.29. *In* $\mathrm{PG}(2,13)$*, there are at least* 36 *sd-inequivalent* $(33;4)$*-arcs.*

2.30 sd-inequivalent $(34;4)$-arcs

In this process, there are 77 incomplete $(34;4)$-arcs and 24 complete $(34;4)$-arcs. In addition, there are 22 sd-inequivalent $(34;4)$-arcs all having the identity group. The values of t_i of the secant distributions $\{t_4,t_3,t_2,t_1,t_0\}$ of the sd-inequivalent $(34;4)$-arcs are as follows:

$$t_4 \in \{54,\dots,65\}, t_3 \in \{47,\dots,74\}, t_2 \in \{15,\dots,30\}, t_1 \in \{7,\dots,21\}, t_0 \in \{24,\dots,32\}.$$

The related statistics are given in Table 2.36.

Table 2.36: **Statistics of** sd**-inequivalent** $(34;4)$**-arcs**

sd-inequivalent (34;4)-arcs	$S:n$
22	$I:22$

Theorem 2.30. *In* $\mathrm{PG}(2,13)$*, there are at least* 22 *sd-inequivalent* $(34;4)$*-arcs.*

2.31 sd-inequivalent $(35;4)$-arcs

In this process, there are 40 incomplete $(35;4)$-arcs including eight sd-inequivalent $(35;4)$-arcs and 20 complete arcs. The stabilisers of the eight sd-inequivalent $(35;4)$-arcs are the identity group and Z_2. In addition, The values of t_i of the secant distributions $\{t_4,t_3,t_2,t_1,t_0\}$ of the sd-inequivalent $(35;4)$-arcs are as follows:

$$t_4 \in \{71,\dots,73\}, t_3 \in \{43,\dots,50\}, t_2 \in \{19,\dots,28\}, t_1 \in \{13,\dots,20\}, t_0 \in \{24,\dots,26\}.$$

The statistics are given in Table 2.37.

Table 2.37: **Statistics of** sd**-inequivalent** $(35;4)$**-arcs**

sd-inequivalent (35;4)-arcs	$S:n$
8	$I:7,\ Z_2:1$

Theorem 2.31. *In* $\mathrm{PG}(2,13)$*, there are at least eight sd-inequivalent* $(35;4)$*-arcs.*

Remark

In Table 2.37, the group Z_2 divides the associated sd-inequivalent $(35;4)$-arc $\mathscr{K}$ into 18 orbits as follows:

$\mathscr{K} = \{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,$

130, 141, 158, 164, 166, 168, 152 } .

$\mathscr{O}rb(\mathscr{K}) = \{1,2\},\{3,31\},\{12,16\},\{20,38\},\{22,40\},\{33,109\},\{39,141\},\{44,50\},\{46,70\},\{54,168\},$

$\{57,158\},\{82,101\},\{88,123\},\{94,145\},\{98,152\},\{102,166\},\{106,130\},\{164\}.$

2.32 sd-inequivalent $(36;4)$-arcs

This process reveals 24 incomplete arcs; among them there are six sd-inequivalent $(36;4)$-arcs. The stabilisers of these arcs are the identity group and Z_2 as shown in Table 2.39. The statistics of the sd-inequivalent classes of i-secant distributions of the sd-inequivalent $(36;4)$-arcs are given in Table 2.38.

Table 2.38: N_c **of** $\{t_4,t_3,t_2,t_1,t_0\}$

Number	N_c
1	{ 80, 44, 18, 16, 25 }
2	{ 81, 40, 24, 12, 26}
3	{ 81, 41, 21, 15, 25 }
4	{ 81, 42, 18, 18, 24 }
5	{ 82, 38, 24, 14, 25 }
6	{ 82, 39, 21, 17, 24 }

Table 2.39: **Statistics of** sd**-inequivalent** $(36;4)$**-arcs**

sd-inequivalent (36;4)-arcs	$S:n$
6	$I:3,\quad Z_2:3$

Theorem 2.32. *In* $\mathrm{PG}(2,13)$*, there are at least six sd-inequivalent* $(36;4)$*-arcs.*

Remark

In Table 2.39, the six *sd*-inequivalent $(36;4)$-arcs have two types of stabiliser groups, the identity group that splits the associated *sd*-inequivalent $(36;4)$-arcs into 1-point orbits and the group Z_2 that partitions the associated arcs into a number of orbits as below.

(1) $\mathscr{K}_1 = \{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,$ $123,130,141,158,164,175,154,152,166\}.$

$$\begin{aligned}\mathscr{O}rb(\mathscr{K}_1) \quad = \quad & \{1,2\},\{3,109\},\{12,40\},\{16,20\},\{22,54\},\{31,164\},\{33,154\},\{38,70\},\\ & \{39,94\},\{44,88\},\{46,175\},\{50\},\{57\},\{82,102\},\{98,145\},\{101,152\},\\ & \{106,141\},\{123,158\},\{130,166\}.\end{aligned}$$

(2) $\mathscr{K}_2 = \{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,$ $123,130,141,148,158,175,164,154,168\}.$

$$\begin{aligned}\mathscr{O}rb(\mathscr{K}_2) \quad = \quad & \{1,2\},\{3,44\},\{12,168\},\{16,38\},\{20,175\},\{22\},\{31,57\},\{33,50\},\\ & \{39\},\{40,46\},\{54,148\},\{70\},\{82\},\{88,154\},\{94,98\},\{101,141\},\\ & \{102,130\},\{106,145\},\{109,158\},\{123,164\}.\end{aligned}$$

(3) $\mathscr{K}_3 = \{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,$ $123,130,141,148,158,175,164,154,152\}.$

$$\begin{aligned}\mathscr{O}rb(\mathscr{K}_3) \quad = \quad & \{1,2\},\{3,57\},\{12,70\},\{16,148\},\{20\},\{22,175\},\{31,88\},\{33,123\},\\ & \{38,46\},\{39,152\},\{40,54\},\{44,154\},\{50,109\},\{82,106\},\{94,141\},\\ & \{98,130\},\{101,102\},\{145\},\{158,164\}.\end{aligned}$$

2.33 *sd*-inequivalent $(37;4)$-arcs

In this process, the number of *sd*-inequivalent $(37;4)$-arcs is three as given in Table 2.40. The stabilisers of these arcs are the group Z_2. The statistics of N_c are given as follows:

$\{91,33,21,13,25\},\{92,31,21,15,24\},\{91,34,18,16,24\}.$

Table 2.40: **Statistics of** ***sd*****-inequivalent** $(37;4)$**-arcs**

sd-inequivalent (37;4)-arcs	$S:n$
3	$Z_2:3$

Theorem 2.33. *In* $\mathrm{PG}(2,13)$*, there are at least three sd-inequivalent* $(37;4)$*-arcs.*

Remark

In Table 2.40, the three *sd*-inequivalent $(37;4)$-arcs have the group Z_2 that splits the associated arc into a number of orbits as follows:

(1) $\mathscr{K}_1=\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,$ $123,130,141,148,158,175,164,154,168,152\}.$

$$\begin{aligned}\mathscr{O}rb(\mathscr{K}_1) \quad = \quad & \{1,2\},\{3,57\},\{12,70\},\{16,148\},\{20\},\{22,175\},\{31,88\},\{33,123\},\{38,46\},\\ & \{39,152\},\{40,54\},\{44,154\},\{50,109\},\{82,106\},\{94,141\},\{98,130\},\\ & \{101,102\},\{145\},\{158,164\},\{168\}.\end{aligned}$$

(2) $\mathscr{K}_2=\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,$ $123,130,141,158,164,175,154,152,166,148\}.$

$$\begin{aligned}\mathscr{O}rb(\mathscr{K}_2) \quad = \quad & \{1,2\},\{3,57\},\{12,70\},\{16,148\},\{20\},\{22,175\},\{31,88\},\{33,123\},\{38,46\},\\ & \{39,152\},\{40,54\},\{44,154\},\{50,109\},\{82,106\},\{94,141\},\{98,130\},\\ & \{101,102\},\{145\},\{158,164\},\{166\}.\end{aligned}$$

(3) $\mathscr{K}_3=\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,$ $123,130,141,148,158,175,164,166,152,168\}.$

$$\mathscr{O}rb(\mathscr{K}_3) = \{1,2\},\{3,31\},\{12,16\},\{20,38\},\{22,40\},\{33,109\},\{39,141\},\{44,50\},\{46,70\}, \{54,168\},\{57,158\},\{82,101\},\{88,123\},\{94,145\},\{98,152\},\{102,166\}, \{106,130\},\{148,175\},\{164\}.$$

2.34 *sd*-inequivalent $(38;4)$-arcs

In this process, the number of $(38;4)$-arcs is three. These arcs are complete $(38;4)$-arcs having *sd*-equivalent classes of *i*-secant distributions $\{t_4,t_3,t_2,t_1,t_0\}$. Also, the stabiliser group of these complete $(38;4)$-arcs is the dihedral group, D_{12}. Therefore, the three complete $(38;4)$-arcs have equivalent properties for the stabiliser and the type of the *i*-secant distribution. The number of *sd*-inequivalent $(38;4)$-arcs in $\mathrm{PG}(2,13)$ is one, as shown in Table 2.42. The statistics of the three complete $(38;4)$-arcs are given in Table 2.41.

Table 2.41: **Complete** $(38;4)$**-arcs in** $\mathrm{PG}(2,13)$

Symbol	Complete (38;4)-arc	Stabiliser	$\{t_4,t_3,t_2,t_1,t_0\}$
$\mathscr{L}_1$	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,$ 46, 50, 54, 57, 70, 82, 94, 98, 101, 102, 106, 109, 123, $130,141,148,158,175,164,154,168,152,166\}$	D_{12}	$\{102,24,19,14,24\}$
$\mathscr{L}_2$	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,$ 46, 50, 54, 57, 70, 82, 94, 98, 101, 102, 106, 109, 123, $130,141,158,164,175,154,152,166,148,168\}$	D_{12}	$\{102,24,19,14,24\}$
$\mathscr{L}_3$	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,$ 46, 50, 54, 57, 70, 82, 94, 98, 101, 102, 106, 109, 123, $130,141,148,158,175,164,166,152,168,154\}$	D_{12}	$\{102,24,19,14,24\}$

Table 2.42: **The** *sd***-inequivalent complete** $(38;4)$**-arc in** $\mathrm{PG}(2,13)$

$(38;4)$-arcs	*sd*-inequivalent (38;4)-arc	Stabiliser	*sd*-equivalent class $\{t_4,t_3,t_2,t_1,t_0\}$
3	1	D_{12}	{102,24,19,14,24}

Theorem 2.34. *In* $\mathrm{PG}(2,13)$*, there is at least one sd-inequivalent complete* $(38;4)$*-arc.*

Remark

From Table 2.42, the dihedral group partitions the associated *sd*-inequivalent complete $(38;4)$-arc into 4 orbits of sizes 2, 12,12,12 as follows:

$\mathscr{L}_1 = \{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,$ 109, 123, 130, 141, 148, 158, 175, 164, 154, 168, 152, 166$\}$.

$\mathscr{O}rb_1(\mathscr{L}_1) = \{1,2\}$,

$\mathscr{O}rb_2(\mathscr{L}_1) = \{3,109,88,123,33,31,50,44,57,158,154,164\}$,

$\mathscr{O}rb_3(\mathscr{L}_1) = \{12,175,148,16,22,40,70,46,54,168,38,20\}$,

$\mathscr{O}rb_4(\mathscr{L}_1) = \{39,82,94,145,102,166,141,101,152,98,106,130\}$.

2.35 Linear codes from $(k;4)$-arcs

There is a connection between a $(k;n)$-arc in the projective plane, $\mathrm{PG}(2,q)$ and a linear code. From Chapter 1, Section 1.6, a $(k;n)$-arc $\mathscr{K}$ in $\mathrm{PG}(2,q)$ is a set of k points with no $n+1$ of them collinear but some n collinear. Also, from Section 1.8.2, a linear $[n,k,d]$-code C over the finite field $\mathbf{F}_q$ is a subspace of dimension k of the n-dimensional vector space $V(n,q) = \mathbf{F}_q^n$. The parameters n,k, and d of C are called the length, dimension, and minimum distance of C. The minimum distance is defined as the following:

$$d(C) = min\{d(c_1,c_2) \mid c_1,c_2 \in C, c_1 \neq c_2\}.$$

In general, an $[n,k,d]_q$ -code can be looked at in the projective space $\mathrm{PG}(k-1,q)$ of dimension $k-1$ as a set $\mathscr{S}$ of n points in $\mathrm{PG}(k-1,q)$ where the coordinates of the points are the columns of a $k \times n$ generator matrix G of C.

In $\mathrm{PG}(2,13)$, a $(k;4)$-arc is a set of k points no 5 points collinear and then a line contains at most $k-d$ points. Hence a $(k;4)$-arc gives a linear $[n,3,n-4]_{13}$-code of distance $d = n-4$.

Table 2.43 introduces the parameters n,k,d and e (errors corrected) via their values as below:

$n = 6,7,8,9,\ldots,38$, $k = 3$, $d = n-4$, and $e = \lfloor (d-1) \rfloor /2$.

Table 2.43: $[n,k,d]_{13}$**-codes in** $\mathrm{PG}(2,13)$

Arc size	$[n,k,d]_{13}$-code	e	Arc size	$[n,k,d]_{13}$-code	e
6	$[6,3,2]_{13}$	0	7	$[7,3,3]_{13}$	1
8	$[8,3,4]_{13}$	1	9	$[9,3,5]_{13}$	2
10	$[10,3,6]_{13}$	2	11	$[11,3,7]_{13}$	3
12	$[12,3,8]_{13}$	3	13	$[13,3,9]_{13}$	4
14	$[14,3,10]_{13}$	4	15	$[15,3,11]_{13}$	5
16	$[16,3,12]_{13}$	5	17	$[17,3,13]_{13}$	6
18	$[18,3,14]_{13}$	6	19	$[19,3,15]_{13}$	7
20	$[20,3,16]_{13}$	7	21	$[21,3,17]_{13}$	8
22	$[22,3,18]_{13}$	8	23	$[23,3,19]_{13}$	9
24	$[24,3,20]_{13}$	9	25	$[25,3,21]_{13}$	10
26	$[26,3,22]_{13}$	10	27	$[27,3,23]_{13}$	11
28	$[28,3,24]_{13}$	11	29	$[29,3,25]_{13}$	12
30	$[30,3,26]_{13}$	12	31	$[31,3,27]_{13}$	13
32	$[32,3,28]_{13}$	13	33	$[33,3,29]_{13}$	14
34	$[34,3,30]_{13}$	14	35	$[35,3,31]_{13}$	15
36	$[36,3,32]_{13}$	15	37	$[37,3,33]_{13}$	16
38	$[38,3,34]_{13}$	16			

Remark

In Table 2.44, the timings in milliseconds of $(k;4)$-arcs for $k=7,\ldots,38$ in $\mathrm{PG}(2,13)$ are given according to the construction of $(k;4)$-arcs, the i-secant distributions of $(k;4)$-arcs, and the stabilisers of the *sd*-inequivalent $(k;4)$-arcs. The same is done in Table 2.58.

Table 2.44: **Timing (msec) of** $(k;4)$**-arcs for** $k = 7,\ldots,38$

$(k;4)$-arcs	Construction	$\{t_4,t_3,t_2,t_1,t_0\}$	Stabilisers
$(7;4)$-arcs	2791	2787	2357
$(8;4)$-arcs	2344	2720	2303
$(9;4)$-arcs	2512	3003	2001
$(10;4)$-arcs	3205	3947	2819
$(11;4)$-arcs	4011	5258	3177
$(12;4)$-arcs	6389	7618	3950
$(13;4)$-arcs	8076	9629	4379
$(14;4)$-arcs	9636	13058	5660
$(15;4)$-arcs	12236	16058	29491
$(16;4)$-arcs	15945	20160	36968
$(17;4)$-arcs	16349	22956	53584
$(18;4)$-arcs	22071	25672	65548
$(19;4)$-arcs	25695	32794	92126
$(20;4)$-arcs	26627	34477	902330
$(21;4)$-arcs	27282	37594	90583
$(22;4)$-arcs	27892	31880	94274
$(23;4)$-arcs	24339	31393	82778
$(24;4)$-arcs	19469	31304	71014
$(25;4)$-arcs	15774	26615	74222
$(26;4)$-arcs	12626	22321	77038
$(27;4)$-arcs	7987	16538	75816
$(28;4)$-arcs	5819	12337	76610
$(29;4)$-arcs	4243	8084	60667
$(30;4)$-arcs	3253	4762	44944
$(31;4)$-arcs	2756	3231	31684
$(32;4)$-arcs	2507	2424	18548
$(33;4)$-arcs	2077	3508	9609
$(34;4)$-arcs	2101	2159	7056
$(35;4)$-arcs	2135	1999	3501
$(36;4)$-arcs	1970	2023	3916
$(37;4)$-arcs	2181	1948	2413
$(38;4)$-arcs	2088	1921	2062

2.36 The classification of certain $(k;4)$-arcs up to projective equivalence, for $k = 34, 35, 36, 37, 38$ in $\mathrm{PG}(2,13)$

Introduction

Throughout this section, the classification of certain $(k;4)$-arcs in $\mathrm{PG}(2,13)$ is established up to projective inequivalence for $k = 34, 35, 36, 37, 38$. Here, all the points from the plane which are not incident with any 4-secant are added to each projectively inequivalent $(k;4)$-arc to construct the sets of $(k+1;4)$-arcs. The isomorphisms between the sets of incomplete $(k;4)$-arcs are tested according to the lexicographically least sets in the G-orbits of incomplete $(k;4)$-arcs in each process.

2.36.1 Projectively inequivalent $(34;4)$-arcs

From Section 2.30, there are 77 incomplete $(34;4)$-arcs. These arcs are constructed from the number of sd-inequivalent $(33;4)$-arcs that given in Section 2.29. The lexicographically least sets in the G-orbits of the 77 incomplete $(34;4)$-arcs show that there are 51 incomplete $(34;4)$-arcs are projectively inequivalent. Also, the stabilisers of these arcs are the identity groups. The statistics of this construction are given in Table 2.45 and Table 2.46.

Table 2.45: **Projectively inequivalent (34;4)-arcs**

Number	(34;4)-arc
1	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,158,175,152\}$
2	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,158,175,154\}$
3	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,158,175,164\}$
4	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,158,175,166\}$
5	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,158,175,168\}$
6	$\{1,2,3,88,9,21,5,15,33,55,44,8,18,46,50,124,31,80,96,164,14,121,129,67,56,132,153,176,60,38,17,24,70,28\}$
7	$\{1,2,3,88,9,21,5,15,33,55,44,8,18,46,50,124,31,80,96,164,14,121,129,67,56,132,153,176,60,38,17,24,70,37\}$
8	$\{1,2,3,88,9,21,5,15,33,55,44,8,18,46,50,124,31,80,96,164,14,121,129,67,56,132,153,176,60,38,17,24,70,75\}$
9	$\{1,2,3,88,9,21,5,15,33,55,44,8,18,46,50,124,31,80,96,164,14,121,129,67,56,132,153,176,60,38,17,24,70,113\}$
10	$\{1,2,3,88,9,21,5,15,33,55,44,8,18,46,50,124,31,80,96,164,14,121,129,67,56,132,153,176,60,38,17,24,37,28\}$
11	$\{1,2,3,88,9,21,5,15,33,55,44,8,18,46,50,124,31,80,96,164,14,121,129,67,56,132,153,176,60,38,17,24,37,75\}$
12	$\{1,2,3,88,9,21,5,15,33,55,44,8,18,46,50,124,31,80,96,164,14,121,129,67,56,132,153,176,60,38,17,24,37,113\}$

13	$\{1,2,3,88,9,21,5,15,33,55,44,8,18,46,50,124,31,80,96,164,14,121,129,67,56,132,153,176,60,38,17,24,28,113\}$
14	$\{1,2,3,88,9,21,5,15,33,55,44,8,18,46,50,124,31,80,96,164,14,121,129,67,56,132,153,176,60,38,24,70,28,37\}$
15	$\{1,2,3,88,9,21,5,15,33,55,44,8,18,46,50,124,31,80,96,164,14,121,129,67,56,132,153,176,60,38,24,70,28,113\}$
16	$\{1,2,3,88,9,21,5,15,33,55,44,8,18,46,50,124,31,80,96,164,14,121,129,67,56,132,153,176,60,38,17,37,28,70\}$
17	$\{1,2,3,88,9,21,5,15,33,55,44,8,18,46,50,124,31,80,96,164,14,121,129,67,56,132,153,176,60,38,17,37,28,113\}$
18	$\{1,2,3,88,9,21,5,15,33,55,44,8,18,46,50,124,31,80,96,164,14,121,129,67,56,132,153,176,60,38,17,37,75,70\}$
19	$\{1,2,3,88,9,21,5,15,33,55,44,8,18,46,50,124,31,80,96,164,14,121,129,67,56,132,38,153,17,60,24,37,28,70\}$
20	$\{1,2,3,88,9,21,5,15,33,55,44,8,18,46,50,124,31,80,96,164,14,121,129,67,56,132,38,153,17,60,24,37,28,75\}$
21	$\{1,2,3,88,9,21,5,15,33,55,44,8,18,46,50,124,31,80,96,164,14,121,129,67,56,132,38,153,17,60,24,37,28,113\}$
22	$\{1,2,3,88,9,21,5,15,33,55,44,8,18,46,50,124,31,80,96,164,14,121,129,67,56,132,38,17,60,24,113,145,70,28\}$
23	$\{1,2,3,88,9,21,5,15,33,55,44,8,18,46,50,124,31,80,96,164,14,121,129,67,56,132,38,17,60,24,113,145,70,169\}$
24	$\{1,2,3,88,9,21,5,15,33,55,44,8,18,46,50,124,31,80,96,164,14,121,129,67,56,132,38,17,60,24,113,145,70,176\}$
25	$\{1,2,3,88,9,21,5,15,33,55,44,8,18,46,50,124,31,80,96,164,14,121,129,67,56,132,38,153,17,60,28,75,70,24\}$
26	$\{1,2,3,88,9,21,5,15,33,55,44,8,18,46,50,124,31,80,96,164,14,121,129,67,56,132,38,153,17,60,28,75,70,37\}$
27	$\{1,2,3,88,9,21,5,15,33,55,44,8,18,46,50,124,31,80,96,164,14,121,129,67,56,132,38,17,60,24,113,145,169,28\}$
28	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,158,164,175,152\}$
29	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,158,164,175,154\}$
30	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,158,164,175,166\}$
31	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,158,164,175,168\}$
32	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,158,164,166,148\}$
33	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,158,164,166,152\}$
34	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,158,164,166,154\}$
35	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,158,164,166,168\}$
36	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,158,164,152\}$
37	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,158,164,154\}$
38	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,158,164,168\}$
39	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,158,168,152\}$
40	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,158,168,154\}$
41	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,158,168,166\}$
42	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,158,152,154\}$
43	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,158,152,166\}$
44	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,166,152,154\}$
45	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,166,152,164\}$
46	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,166,152,168\}$
47	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,166,152,175\}$
48	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,158,154,166\}$
49	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,166,154,164\}$
50	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,166,154,168\}$
51	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,166,154,175\}$

Table 2.46: **Statistics of projectively inequivalent** $(34;4)$**-arcs in** $\mathrm{PG}(2,13)$

Number of incomplete $(34;4)$-arcs	Projectively inequivalent $(34;4)$-arcs	$S:n$
77	51	$I:51$

Theorem 2.35. *In* $\mathrm{PG}(2,13)$*, there are at least* 51 *projectively inequivalent* $(34;4)$*-arcs.*

2.36.2 Projectively inequivalent $(35;4)$-arcs

In this process, for each projectively inequivalent $(34;4)$-arc given in Table 2.45, all the points from the plane which are not on any 4-secant are added separately to establish the $(35;4)$-arcs. Here, there are 116 incomplete $(35;4)$-arcs and 33 complete $(35;4)$-arcs. Among the 116 incomplete $(35;4)$-arcs, the number of the projectively inequivalent $(35;4)$-arcs is 38. The stabilisers of the 38 projectively inequivalent $(35;4)$-arcs are the identity group and Z_2. The statistics of this construction are given in Table 2.47 and Table 2.48.

Table 2.47: **Projectively inequivalent (35;4)-arcs**

Number	(35;4)-arc
1	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,158,175,152,154\}$
2	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,166,152,164,175\}$
3	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,158,175,152,164\}$
4	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,158,175,164,166\}$
5	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,166,152,154,168\}$
6	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,166,152,154,164\}$
7	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,158,164,152,154\}$
8	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,158,164,175,154,166\}$
9	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,158,164,166,148,152\}$
10	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,158,164,175,166,168\}$
11	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,166,152,164,168\}$
12	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,158,164,152,168\}$
13	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,158,164,175,152,166\}$
14	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,158,152,154,166\}$
15	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,166,152,154,175\}$
16	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,166,154,168,175\}$
17	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,166,154,164,175\}$
18	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,158,164,175,152,168\}$
19	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,158,168,152,166\}$
20	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,158,175,152,168\}$

21	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,158,175,166,168\}$
22	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,166,152,168,175\}$
23	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,158,175,152,166\}$
24	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,158,168,152,154\}$
25	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,158,164,166,154,168\}$
26	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,158,164,166,148,168\}$
27	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,158,164,166,152,168\}$
28	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,158,168,154,166\}$
29	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,166,154,164,168\}$
30	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,158,164,154,168\}$
31	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,158,175,154,168\}$
32	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,158,175,164,168\}$
33	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,158,164,175,154,168\}$
34	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,158,164,166,152,154\}$
35	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,158,164,166,148,154\}$
36	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,158,175,154,166\}$
37	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,158,175,154,164\}$
38	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,158,164,175,152,154\}$

Table 2.48: **Statistics of projectively inequivalent** $(35;4)$**-arcs in** $\mathrm{PG}(2,13)$

Number of incomplete $(35;4)$-arcs	Projectively inequivalent $(35;4)$-arcs	$S:n$
116	38	$I:36,\ Z_2:2$

Theorem 2.36. *In* $\mathrm{PG}(2,13)$*, there are at least* 38 *projectively inequivalent* $(35;4)$*-arcs.*

Remark

The identity groups of the projectively inequivalent $(35;4)$-arcs partition the associated projectively inequivalent $(35;4)$-arcs into 35 orbits of length 1. Also, the groups Z_2 split the corresponding projectively inequivalent $(35;4)$-arcs into a number of orbits of sizes 1 and 2.

2.36.3 Projectively inequivalent $(36;4)$-arcs

In this process, the number of incomplete $(36;4)$-arcs that have been established is 114. Among these arcs there are 23 projectively inequivalent $(36;4)$-arcs. The stabilisers of the 23 projectively inequivalent $(36;4)$-arcs are the identity group, Z_2, $Z_2 \times Z_2$. The statistics of this construction are given in Table 2.49 and Table 2.50.

Table 2.49: **Projectively inequivalent (36;4)-arcs**

Number	(36;4)-arc
1	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,158,164,175,152,168,154\}$
2	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,158,175,152,154,166\}$
3	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,158,175,164,166,168\}$
4	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,158,175,152,154,164\}$
5	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,158,175,152,154,168\}$
6	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,158,175,164,166,154\}$
7	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,166,152,154,168,164\}$
8	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,158,164,175,154,166,168\}$
9	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,166,152,164,175,158\}$
10	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,166,152,164,175,168\}$
11	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,158,175,152,164,168\}$
12	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,166,152,154,168,175\}$
13	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,166,152,154,168,158\}$
14	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,166,152,154,164,158\}$
15	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,158,164,152,154,168\}$
16	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,158,164,175,154,166,152\}$
17	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,158,164,166,148,152,168\}$
18	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,166,154,168,175,158\}$
19	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,166,154,168,175,164\}$
20	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,158,168,152,166,175\}$
21	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,158,164,166,154,168,148\}$
22	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,158,164,166,154,168,152\}$
23	$\{1,2,3,88,22,145,12,16,20,31,33,38,39,40,44,46,50,54,57,70,82,94,98,101,102,106,109,123,130,141,148,158,164,154,168,175\}$

Table 2.50: **Statistics of projectively inequivalent** $(36;4)$**-arcs in** $\mathrm{PG}(2,13)$

Number of incomplete $(36;4)$-arcs	Projectively inequivalent $(36;4)$-arcs	$S:n$
114	23	$I:14,\ Z_2:8,\ Z_2\times Z_2:1$

Theorem 2.37. *In* $\mathrm{PG}(2,13)$*, there are at least* 23 *projectively inequivalent* $(36;4)$*-arcs.*

2.36.4 Projectively inequivalent $(37;4)$-arcs

In this process, the number of incomplete $(37;4)$-arcs is 46. Among the 46 incomplete $(37;4)$-arcs, there are three projectively inequivalent $(37;4)$-arcs. The stabiliser groups of the three projectively inequivalent

$(37;4)$-arcs are Z_2. The analytic statistics are given in the following tables.

Table 2.51: **Projectively inequivalent (37;4)-arcs**

Number	(37;4)-arc
1	{ 1, 2, 3, 88, 22, 145, 12, 16, 20, 31, 33, 38, 39, 40, 44, 46, 50, 54, 57, 70, 82, 94, 98, 101, 102, 106, 109, 123, 130, 141, 158, 164, 175, 152, 168, 154, 166 }
2	{ 1, 2, 3, 88, 22, 145, 12, 16, 20, 31, 33, 38, 39, 40, 44, 46, 50, 54, 57, 70, 82, 94, 98, 101, 102, 106, 109, 123, 130, 141, 158, 164, 175, 152, 168, 154, 148 }
3	{ 1, 2, 3, 88, 22, 145, 12, 16, 20, 31, 33, 38, 39, 40, 44, 46, 50, 54, 57, 70, 82, 94, 98, 101, 102, 106, 109, 123, 130, 141, 148, 158, 175, 152, 154, 166, 168 }

Table 2.52: **Statistics of projectively inequivalent** $(37;4)$**-arcs in** PG$(2,13)$

Number of incomplete $(37;4)$-arcs	Projectively inequivalent $(37;4)$-arcs	$S:n$
46	3	$Z_2:3$

Theorem 2.38. *In* PG$(2,13)$*, there are at least three projectively inequivalent* $(37;4)$*-arcs.*

2.36.5 Projectively inequivalent $(38;4)$-arcs

In Table 2.51, there are three projectively inequivalent $(37;4)$-arcs. So, all the points from the plane which do not lie on any 4-secant are added to each projectively inequivalent $(37;4)$-arc. Therefore, the number of $(38;4)$-arcs is three. The three $(38;4)$-arcs are complete and they are projectively equivalent. So there exists one projectively inequivalent $(38;4)$-arc having dihedral group, D_{12}. The analytic statistics of this process are given in Table 2.53.

Table 2.53: **The complete** $(38;4)$**-arcs in** PG$(2,13)$

Number	complete (38;4)-arc	Stabiliser
1	{1, 2, 3, 88, 22, 145, 12, 16, 20, 31, 33, 38, 39, 40, 44, 46, 50, 54, 57, 70, 82, 94, 98, 101, 102, 106, 109, 123, 130, 141, 158, 164, 175, 152, 168, 154, 166, 148 }	D_{12}
2	{1, 2, 3, 88, 22, 145, 12, 16, 20, 31, 33, 38, 39, 40, 44, 46, 50, 54, 57, 70, 82, 94, 98, 101, 102, 106, 109, 123, 130, 141, 158, 164, 175, 152, 168, 154, 148, 166}	D_{12}
3	{1, 2, 3, 88, 22, 145, 12, 16, 20, 31, 33, 38, 39, 40, 44, 46, 50, 54, 57, 70, 82, 94, 98, 101, 102, 106, 109, 123, 130, 141, 148, 158, 175, 152, 154, 166, 168, 164}	D_{12}

Theorem 2.39. *In* PG$(2,13)$*, there is at least one projectively inequivalent* $(38;4)$*-arc.*

Remark

In Table 2.54, the timings of the classification of projectively inequivalent $(k;4)$-arcs for $k=34,35,36,37,38$ are given.

Table 2.54: **Timing (msec) of the projectively inequivalent** $(k;4)$**-arcs for** $k = 34,\ldots,38$

$(k;4)$-arcs	Construction	Canonical sets of $(k;4)$-arcs	Stabilisers
$(34;4)$-arcs	2101	253575	20776
$(35;4)$-arcs	3458	632505	12257
$(36;4)$-arcs	3183	624960	6978
$(37;4)$-arcs	1877	170180	2310
$(38;4)$-arcs	1742	4086	1930

2.37 New size of complete $(k;4)$-arc from different choice of five *sd*-inequivalent $(7;4)$-arcs in $\mathrm{PG}(2,13)$

Introduction

In this section, a different choice of five *sd*-inequivalent $(7;4)$-arcs which have *sd*-inequivalent classes of i-secant distributions is made. These classes are introduced in Section 2.3, Table 2.4, where there are five *sd*-inequivalent classes of i-secant distributions. So, the approach introduced in Section 2.3 is re-iterated with these new *sd*-inequivalent $(7;4)$-arcs. The purpose of this choice is to investigate whether there exists a new size of complete $(k;4)$-arc for $k > 38$. The result of this process of *sd*-inequivalent $(k;4)$-arcs through this choice showed that there is no complete $(k;4)$-arc for $k > 38$. However, a new size of complete $(k;4)$-arc in $\mathrm{PG}(2,13)$ has been discovered. This size is $k = 36$ which is the largest complete $(k;4)$-arc established in this process. Since the approach of this process has followed the same method in Section 2.3. The results are summarized in three tables, where the different choice of the five *sd*-inequivalent $(7;4)$-arcs which have *sd*-inequivalent classes of i-secant distributions is given in Table 2.55. The statistics of *sd*-inequivalent $(k;4)$-arcs for $k = 8,\ldots,36$ are given in Table 2.56. Also, the statistics of the complete $(36;4)$-arc are given in Table 2.57.

Table 2.55: **The different choice of the five** *sd***-inequivalent** $(7;4)$**-arcs**

Symbol	(7;4)-arc	Stabiliser	$\{t_4,t_3,t_2,t_1,t_0\}$
$\mathscr{C}_1$	$\{1,2,3,88,17,182,132\}$	I	$\{1,2,9,70,101\}$
$\mathscr{C}_2$	$\{1,2,3,88,9,115,159\}$	I	$\{1,0,15,64,103\}$
$\mathscr{C}_3$	$\{1,2,3,88,9,89,8\}$	I	$\{1,1,12,67,102\}$
$\mathscr{C}_4$	$\{1,2,3,88,22,183,181\}$	Z_2	$\{2,0,9,72,100\}$
$\mathscr{C}_5$	$\{1,2,3,88,9,182,183\}$	Z_2	$\{1,3,6,73,100\}$

Table 2.56: **Statistics of** sd**-inequivalent** $(k;4)$**-arcs**

$(k;4)$-arcs	sd-inequivalent $(k;4)$-arcs	Stabilisers
$(8;4)$-arcs	11	$I:8, Z_2:1, Z_3:1, D_4:1$
$(9;4)$-arcs	21	$I:15, Z_2:4, Z_3:1, S_4:1$
$(10;4)$-arcs	34	$I:30, Z_2:2, Z_2\times Z_2:1, S_3:1$
$(11;4)$-arcs	52	$I:46, Z_2:3, Z_3:1, Z_2\times Z_2:1, D_4:1$
$(12;4)$-arcs	77	$I:70, Z_2:4, Z_2\times Z_2:2, S_3:1$
$(13;4)$-arcs	103	$I:97, Z_2:5, S_4:1$
$(14;4)$-arcs	136	$I:131, Z_2:5$
$(15;4)$-arcs	175	$I:171, Z_2:2, S_3:2$
$(16;4)$-arcs	215	$I:212, Z_2:3$
$(17;4)$-arcs	255	$I:251, Z_2:4$
$(18;4)$-arcs	301	$I:292, Z_2:8, S_3:1$
$(19;4)$-arcs	345	$I:337, Z_2:7, Z_3:1$
$(20;4)$-arcs	387	$I:376, Z_2:10, D_4:1$
$(21;4)$-arcs	424	$I:414, Z_2:10$
$(22;4)$-arcs	458	$I:442, Z_2:13, Z_2\times Z_2:3$
$(23;4)$-arcs	487	$I:472, Z_2:15$
$(24;4)$-arcs	501	$I:479, Z_2:19, Z_2\times Z_2:1, D_4:1, S_4:1$
$(25;4)$-arcs	499	$I:484, Z_2:14, S_3:1$
$(26;4)$-arcs	477	$I:464, Z_2:11, Z_2\times Z_2:2$
$(27;4)$-arcs	441	$I:427, Z_2:12, Z_3:1, S_3:1$
$(28;4)$-arcs	401	$I:389, Z_2:10, Z_3:1, S_4:1$
$(29;4)$-arcs	332	$I:327, Z_2:5$
$(30;4)$-arcs	254	$I:245, Z_2:7, Z_3:1, S_3:1$
$(31;4)$-arcs	155	$I:152, Z_2:2, Z_3:1$
$(32;4)$-arcs	75	$I:74, Z_2:1$
$(33;4)$-arcs	24	$I:23, Z_2:1$
$(34;4)$-arcs	1	$Z_2:1$
$(35;4)$-arcs	1	$I:1$
$(36;4)$-arcs	1	$S_3:1$

Table 2.57: **Complete** $(36;4)$**-arc** $\mathscr{K}$

$\mathscr{K}$	Stabiliser	$\{t_4,t_3,t_2,t_1,t_0\}$
$\{1,2,3,88,9,115,159,5,7,55,63,96,67,22,108,148,57,75,35,$ $122,50,121,61,138,157,70,97,24,72,43,84,91,99,155,166,171\}$	S_3	$\{78,49,15,15,26\}$

Theorem 2.40. *In* PG$(2,13)$*, there is at least one complete* $(36;4)$*-arc.*

Remark

The group S_3 in Table 2.57 partitions the complete $(36;4)$-arc into 8 orbits of sizes $6,6,3,3,6,3,3,6$ which are the following:

$\mathscr{O}rb(\mathscr{K}) = \{1,9,50,148,138,96\}, \{2,115,157,7,121,159\}, \{3,88,55\}, \{5,108,35\}, \{22,57,63,67,75,122\},$

$\{24,84,43\}, \{61,97,70\}, \{72,99,166,155,171,91\}.$

Table 2.58: **Timing (msec) of** $(k;4)$**-arcs for** $k = 7,\ldots,36$

$(k;4)$-arcs	Construction	$\{t_4,t_3,t_2,t_1,t_0\}$	Stabilisers
$(7;4)$-arcs	2791	2787	1886
$(8;4)$-arcs	2518	2510	1802
$(9;4)$-arcs	2846	3936	2866
$(10;4)$-arcs	3567	3902	2118
$(11;4)$-arcs	4689	4966	2462
$(12;4)$-arcs	6791	7361	3143
$(13;4)$-arcs	7678	9012	4159
$(14;4)$-arcs	12158	11620	5355
$(15;4)$-arcs	15213	14956	36312
$(16;4)$-arcs	16535	19729	50555
$(17;4)$-arcs	20591	24849	65609
$(18;4)$-arcs	24194	26614	77669
$(19;4)$-arcs	29959	30694	92199
$(20;4)$-arcs	37272	31812	101438
$(21;4)$-arcs	30237	34802	127633
$(22;4)$-arcs	31914	35913	159671
$(23;4)$-arcs	26611	32239	146252
$(24;4)$-arcs	28062	33093	136999
$(25;4)$-arcs	21255	30525	132499
$(26;4)$-arcs	16067	24790	127089
$(27;4)$-arcs	11089	17529	107378
$(28;4)$-arcs	7684	13643	89743
$(29;4)$-arcs	5359	7735	74802
$(30;4)$-arcs	3676	6837	55589
$(31;4)$-arcs	2733	3306	40820
$(32;4)$-arcs	2241	2461	15843
$(33;4)$-arcs	2223	1863	5420
$(34;4)$-arcs	2335	1660	1727
$(35;4)$-arcs	2111	1646	1873
$(36;4)$-arc	2070	1806	1813

Chapter 3

Incidence structures

Introduction

In this chapter, the incidence structure of the projectively inequivalent $(6;4)$-arcs, shown in Chapter 2, Section 2.2, is discussed. The whole picture of these structures of sd-inequivalent classes of i-secant distribution of the $(6;4)$-arcs $\mathscr{B}_6$, $\mathscr{B}_7$, and $\mathscr{B}_8$ is given. In addition, the incidence structure of the orbits of the groups other than the identity group of sd-inequivalent $(k;4)$-arcs, for $k = 7,8,9,10,11,12,13,14,38$, is given. Here, the geometric configurations of the lines and the points of the orbits of these groups are described.

3.1 Geometric configuration of the projectively inequivalent $(6;4)$-arcs

From Chapter 2, Section 2.2, the projectively inequivalent $(6,4)$-arcs $\mathscr{B}_1$, $\mathscr{B}_2$, $\mathscr{B}_3$, and $\mathscr{B}_{10}$ all have the stabiliser group Z_2. This group splits $\mathscr{B}_1$, $\mathscr{B}_2$, and $\mathscr{B}_3$ into 5 orbits of sizes 1,1,2,1,1 as follows:

$$\mathscr{O}rb(\mathscr{B}_1) = \{1\},\{2\},\{3,88\},\{9\},\{21\};$$

$$\mathscr{O}rb(\mathscr{B}_2) = \{1\},\{2\},\{3,88\},\{9\},\{83\};$$

$$\mathscr{O}rb(\mathscr{B}_3) = \{1\},\{2\},\{3,88\},\{9\},\{89\}.$$

Also, it partitions $\mathscr{B}_{10}$ into 5 orbits of sizes $2,1,1,1,1$. They are as follows:

$$\mathscr{O}rb(\mathscr{B}_{10}) = \{1,88\},\{2\},\{3\},\{22\},\{183\}.$$

The secant distribution of the orbits of the inequivalent $(6;4)$-arc $\mathscr{B}_1$ are $\{0,0,0,14,169\}$, $\{0,0,0,14,169\}$, $\{0,0,1,26,156\}$, $\{0,0,0,14,169\}$,$\{0,0,0,14,169\}$ respectively. So, the geometric configuration pictures of these orbits are single point, single point, two points, single point, and single point. This is the same as $\mathscr{B}_2$ and $\mathscr{B}_3$. In contrast, the secant distribution of the orbits of $\mathscr{B}_{10}$ are $\{0,0,1,26,156\}$, $\{0,0,0,14,169\}$, $\{0,0,0,14,169\}$, $\{0,0,0,14,169\}$,$\{0,0,0,14,169\}$. The geometric pictures of the orbits of $\mathscr{B}_{10}$ are two points and four single points. Also, the group $Z_2 \times Z_2$ divides the the projectively inequivalent $(6,4)$-arcs $\mathscr{B}_4$, $\mathscr{B}_5$, and $\mathscr{B}_9$ into 3,4,3 orbits as follows:

$$\begin{aligned}\mathscr{O}rb(\mathscr{B}_4) &= \{1,9\},\{2,115\},\{3,88\};\\ \mathscr{O}rb(\mathscr{B}_5) &= \{1\},\{2,9\},\{3,88\},\{182\};\\ \mathscr{O}rb(\mathscr{B}_9) &= \{1,88\},\{2,145\},\{3,22\}.\end{aligned}$$

The secant distribution of the orbits of $\mathscr{B}_4$ are $\{0,0,1,26,156\}$, $\{0,0,1,26,156\}$, $\{0,0,1,26,156\}$; for $\mathscr{B}_5$ they are $\{0,0,0,14,169\}$, $\{0,0,1,26,156\}$, $\{0,0,1,26,156\}$, $\{0,0,0,14,169\}$. However, the secant distribution of $\mathscr{B}_9$ is the same as $\mathscr{B}_4$. So, the geometric picture of the orbits of $\mathscr{B}_4$ is two points, two points, and two points. This is the same as $\mathscr{B}_9$. The picture of the orbits of $\mathscr{B}_5$ is single point, two points, two points, and single point. In addition, the projectively inequivalent $(6,4)$-arcs $\mathscr{B}_6$, $\mathscr{B}_7$, and $\mathscr{B}_8$ have the stabiliser groups Z_6, Z_6, and $Z_4 \times Z_2$. These groups partition the associated arcs into a number of orbits, where Z_6 divides $\mathscr{B}_6$ into 3 orbits of sizes 2,3,1 as follows:

$$\mathscr{O}rb_1(\mathscr{B}_6) = \{1,2\},\ \mathscr{O}rb_2(\mathscr{B}_6) = \{3,17,88\},\ \mathscr{O}rb_3(\mathscr{B}_6) = \{68\}.$$

The secant distribution $\{t_4,t_3,t_2,t_1,t_0\}$ of the orbits $\mathscr{O}rb_1$ and $\mathscr{O}rb_2$ are $\{0,0,1,26,156\}$ and $\{0,1,0,39,143\}$. Therefore, the geometric configuration picture of the associated inequivalent $(6;4)$-arc of each of these orbits is two points, three points incident with a unique line, and a single point. Also, the same group splits the $(6;4)$-arc $\mathscr{B}_7$ into following orbits:

$$\mathscr{O}rb_1(\mathscr{B}_7) = \{1,2\},\ \mathscr{O}rb_2(\mathscr{B}_7) = \{3,17,88\},\ \mathscr{O}rb_3(\mathscr{B}_7) = \{182\}.$$

This is the same as $\mathscr{B}_6$. However, the group $Z_4 \times Z_2$ divides the associated $(6;4)$-arc $\mathscr{B}_8$ into two orbits as follows:

$$\mathscr{O}rb_1(\mathscr{B}_8) = \{1,88\},\ \mathscr{O}rb_2(\mathscr{B}_8) = \{2,22,3,116\}.$$

The values of $\{t_4,t_3,t_2,t_1,t_0\}$ of these orbits are $\{0,0,1,26,156\}$ and $\{1,0,0,52,130\}$. These structures describe the geometric configuration of the associated inequivalent $(6;4)$-arc $\mathscr{B}_8$ as two lines, the first line with two incident points and the second line with four collinear points. The description given in Table 3.1.

Table 3.1: **Geometric configuration of the projectively inequivalent $(6;4)$-arcs**

Symbol	Group	$\{t_4,t_3,t_2,t_1,t_0\}$	Configuration
$\mathscr{B}_1$	Z_2	$\{0,0,0,14,169\}$	point
		$\{0,0,0,14,169\}$	point
		$\{0,0,1,26,156\}$	2-arc
		$\{0,0,0,14,169\}$	point
		$\{0,0,0,14,169\}$	point
$\mathscr{B}_2$	Z_2	$\{0,0,0,14,169\}$	point
		$\{0,0,0,14,169\}$	point
		$\{0,0,1,26,156\}$	2-arc
		$\{0,0,0,14,169\}$	point
		$\{0,0,0,14,169\}$	point
$\mathscr{B}_3$	Z_2	$\{0,0,0,14,169\}$	point
		$\{0,0,0,14,169\}$	point
		$\{0,0,1,26,156\}$	2-arc
		$\{0,0,0,14,169\}$	point
		$\{0,0,0,14,169\}$	point
$\mathscr{B}_4$	$Z_2 \times Z_2$	$\{0,0,1,26,156\}$	2-arc
		$\{0,0,1,26,156\}$	2-arc
		$\{0,0,1,26,156\}$	2-arc
$\mathscr{B}_5$	$Z_2 \times Z_2$	$\{0,0,0,14,169\}$	point
		$\{0,0,1,26,156\}$	2-arc
		$\{0,0,1,26,156\}$	2-arc
		$\{0,0,0,14,169\}$	point
$\mathscr{B}_6$	Z_6	$\{0,0,1,26,156\}$	2-arc
		$\{0,1,0,39,143\}$	3 collinear points
		$\{0,0,0,14,169\}$	point
$\mathscr{B}_7$	Z_6	$\{0,0,1,26,156\}$	2-arc
		$\{0,1,0,39,143\}$	3 collinear points
		$\{0,0,0,14,169\}$	point
$\mathscr{B}_8$	$Z_4 \times Z_2$	$\{0,0,1,26,156\}$	2-arc
		$\{1,0,0,52,130\}$	4 collinear points
$\mathscr{B}_9$	$Z_2 \times Z_2$	$\{0,0,1,26,156\}$	2-arc
		$\{0,0,1,26,156\}$	2-arc
		$\{0,0,1,26,156\}$	2-arc
$\mathscr{B}_{10}$	Z_2	$\{0,0,1,26,156\}$	2-arc
		$\{0,0,0,14,169\}$	point
		$\{0,0,0,14,169\}$	point
		$\{0,0,0,14,169\}$	point
		$\{0,0,0,14,169\}$	point

The whole pictures of the incidence structures of the $(6;4)$-arcs $\mathscr{B}_6$, $\mathscr{B}_7$, and $\mathscr{B}_8$ are described in Figure 3.1.

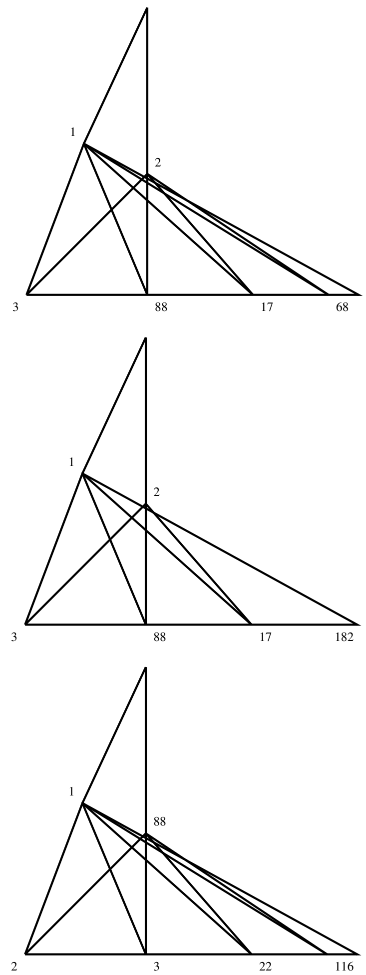

Figure 3.1: **Incidence structures of $\mathscr{B}_6$, $\mathscr{B}_7$, and $\mathscr{B}_8$**

3.2 Geometric configuration of the *sd*-inequivalent $(7;4)$-arc

In Chapter 2, Table 2.5, the *sd*-inequivalent $(7;4)$-arc $\mathscr{C}_4 = \{1,2,3,88,9,182,12\}$ has dihedral group D_4, which splits $\mathscr{C}_4$ into three orbits:

$$\mathscr{O}rb_1(\mathscr{C}_4) = \{1,12\},\ \mathscr{O}rb_2(\mathscr{C}_4) = \{2,9,88,3\},\ \mathscr{O}rb_3(\mathscr{C}_4) = \{182\}.$$

The values of $\{t_4,t_3,t_2,t_1,t_0\}$ of the $\mathscr{O}rb_1$, $\mathscr{O}rb_2$, and $\mathscr{O}rb_3$ are the following:

$$\{0,0,1,26,156\},\{0,0,6,44,133\},\{0,0,0,14,169\}.$$

Note that $\mathscr{O}rb_1$ is two points. However, $\mathscr{O}rb_2$ is a quadrangle whose vertices are the points $2,9,88,3$. Also, the third orbit is just a single point. The description given in Table 3.2.

Table 3.2: **Geometric configuration of the sd-inequivalent $(7;4)$-arc**

Symbol	Group	$\{t_4,t_3,t_2,t_1,t_0\}$	Configuration
$\mathscr{C}_4$	D_4	$\{0,0,1,26,156\}$	2-arc
		$\{0,0,6,44,133\}$	vertices of a quadrangle
		$\{0,0,0,14,169\}$	point

Remark

In Figure 3.2, the geometric pictures of the incidence structures of the orbits $\mathscr{O}rb_1$ and $\mathscr{O}rb_2$ of the $(7;4)$-arc $\mathscr{C}_4$ are given.

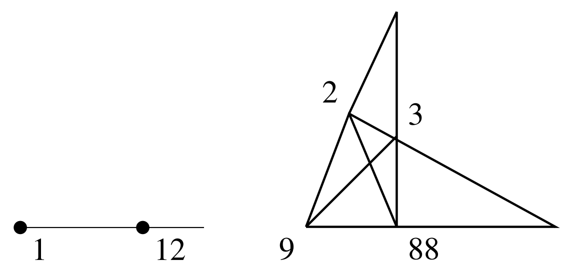

Figure 3.2: **Incidence structure of the orbits $\mathscr{O}rb_1$ and $\mathscr{O}rb_2$ of $\mathscr{C}_4$**

3.3 Geometric configuration of sd-inequivalent $(8;4)$-arcs

In Section 2.4, Table 2.8, the groups, $Z_2 \times Z_2$ and D_4 divide the associated sd-inequivalent $(8;4)$-arcs $\mathscr{D}_9$ and $\mathscr{D}_{11}$ into 4 and 3 orbits respectively. The values of $\{t_4,t_3,t_2,t_1,t_0\}$ and the geometric configuration of these orbits are given in Table 3.3.

Table 3.3: **Geometric configuration of sd-inequivalent $(8;4)$-arcs**

Symbol	Group	$\{t_4,t_3,t_2,t_1,t_0\}$	Configuration
$\mathscr{D}_9$	$Z_2 \times Z_2$	$\{0,0,6,44,133\}$	vertices of a quadrangle
		$\{0,0,1,26,156\}$	2-arc
		$\{0,0,0,14,169\}$	point
		$\{0,0,0,14,169\}$	point
$\mathscr{D}_{11}$	D_4	$\{0,0,1,26,156\}$	2-arc
		$\{0,0,1,26,156\}$	2-arc
		$\{0,0,6,44,133\}$	vertices of a quadrangle

Remark

The geometric pictures of the orbits $\mathscr{O}rb_1$ and $\mathscr{O}rb_2$ of the $(8;4)$-arc $\mathscr{D}_9$ are given in Figure 3.3. Also, the pictures of $\mathscr{O}rb_1$, $\mathscr{O}rb_2$, and $\mathscr{O}rb_3$ of $\mathscr{D}_{11}$ are drawn in Figure 3.4.

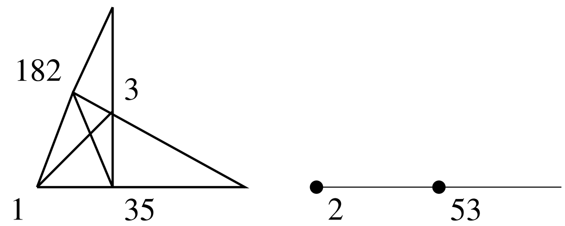

Figure 3.3: **Incidence structure of the orbits $\mathscr{O}rb_1$ and $\mathscr{O}rb_2$ of $\mathscr{D}_9$**

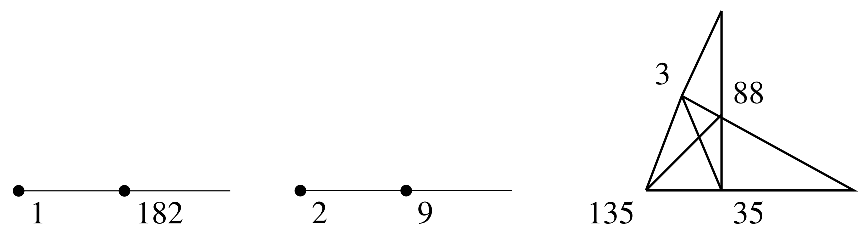

Figure 3.4: **Incidence structure of the orbits $\mathscr{O}rb_1, \mathscr{O}rb_2$, and $\mathscr{O}rb_3$ of $\mathscr{D}_{11}$**

3.4 Geometric configuration of sd-inequivalent $(9;4)$-arcs

In Section 2.5, the group Z_3 partitions the sd-inequivalent $(9;4)$-arc $\mathscr{E}_{11}$ into 3 orbits of size 3. Also, the group S_4 splits the arc $\mathscr{E}_{13}$ into two orbits of sizes $6,3$. The incidence structures of the orbits of $\mathscr{E}_{11}$ are triangles. In addition, the incidence structure of the two orbits of the group S_4 are a quadrilateral whose vertices belong to the first orbit, and a triangle whose vertices are the second orbit. The geometric configurations are given in Table 3.4.

Table 3.4: **Geometric configuration of sd-inequivalent $(9;4)$-arcs**

Symbol	Group	$\{t_4,t_3,t_2,t_1,t_0\}$	Configuration
$\mathscr{E}_{11}$	Z_3	$\{0,0,3,36,144\}$	vertices of a triangle
		$\{0,0,3,36,144\}$	vertices of a triangle
		$\{0,0,3,36,144\}$	vertices of a triangle
$\mathscr{E}_{13}$	S_4	$\{0,4,3,66,110\}$	a quadrilateral
		$\{0,0,3,36,144\}$	vertices of a triangle

Remark

In Figure 3.5, the geometric pictures of the incidence structures of the orbits $\mathscr{O}rb_1, \mathscr{O}rb_2,$ and $\mathscr{O}rb_3$ of $\mathscr{E}_{11}$ are described. In addition, the geometric pictures of $\mathscr{O}rb_1$and $\mathscr{O}rb_2$ of the $(9;4)$-arc $\mathscr{E}_{13}$ are given in Figure 3.6.

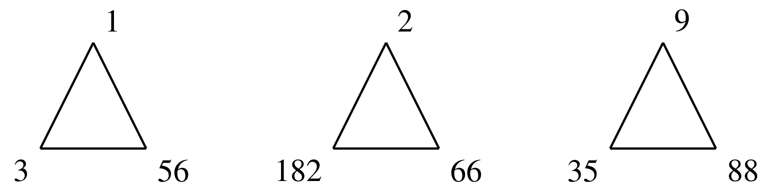

Figure 3.5: **Incidence structures of the orbits $\mathscr{O}rb_1, \mathscr{O}rb_2,$ and $\mathscr{O}rb_3$ of $\mathscr{E}_{11}$**

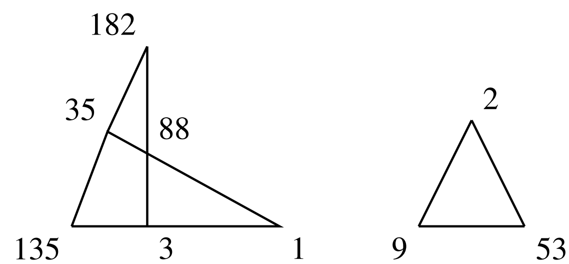

Figure 3.6: **Incidence structure of the orbits $\mathscr{O}rb_1$ and $\mathscr{O}rb_2$ of $\mathscr{E}_{13}$**

3.5 Geometric configuration of the *sd*-inequivalent $(10;4)$-arc

Table 2.12 in Chapter 2 shows that the large group among the groups of the *sd*-inequivalent $(10;4)$-arcs is the group S_3. This group divides the corresponding $(10;4)$-arc $\mathscr{F}_{14}$ into 4 orbits of sizes 3,3,3,1. The incidence structure of these orbits are three collinear points, two triangles, and a single point. The description is given in Table 3.5. Also, the geometric pictures of the $\mathscr{O}rb_1, \mathscr{O}rb_2,$ and $\mathscr{O}rb_3$ are given in Figure 3.7.

Table 3.5: **Geometric configuration of the *sd*-inequivalent $(10;4)$-arc**

Symbol	Group	$\{t_4,t_3,t_2,t_1,t_0\}$	Configuration
$\mathscr{F}_{14}$	S_3	$\{0,1,0,39,143\}$	3 collinear points
		$\{0,0,3,36,144\}$	vertices of a triangle
		$\{0,0,3,36,144\}$	vertices of a triangle
		$\{0,0,0,14,169\}$	point

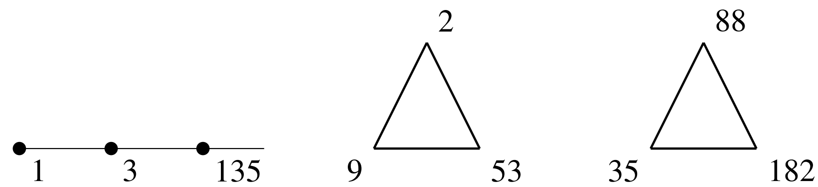

Figure 3.7: **Incidence structure of the orbits $\mathscr{O}rb_1, \mathscr{O}rb_2,$ and $\mathscr{O}rb_3$ of $\mathscr{F}_{14}$**

3.6 Geometric configuration of *sd*-inequivalent $(11;4)$-arcs

The dihedral group D_4 in Table 2.13 partitions the $(11;4)$-arc $\mathscr{G}_{17}$ into 4 orbits of sizes 4,4,2,1. Furthermore, the group $Z_2 \times Z_2$ partitions $\mathscr{G}_{23}$ into 6 orbits of sizes $1,2,4,1,2,1$. The description of the geometric configurations are given in Table 3.6. Also, the geometric pictures of these orbits are shown in Figure 3.8 and Figure 3.9.

Table 3.6: **Geometric configuration of *sd*-inequivalent $(11;4)$-arcs**

Symbol	Group	$\{t_4,t_3,t_2,t_1,t_0\}$	Configuration
$\mathscr{G}_{17}$	D_4	$\{0,0,6,44,133\}$	vertices of a quadrangle
		$\{0,0,6,44,133\}$	vertices of a quadrangle
		$\{0,0,1,26,156\}$	2-arc
		$\{0,0,0,14,169\}$	point
$\mathscr{G}_{23}$	$Z_2 \times Z_2$	$\{0,0,0,14,169\}$	point
		$\{0,0,1,26,156\}$	2-arc
		$\{0,0,6,44,133\}$	vertices of a quadrangle
		$\{0,0,0,14,169\}$	point
		$\{0,0,1,26,156\}$	2-arc
		$\{0,0,0,14,169\}$	point

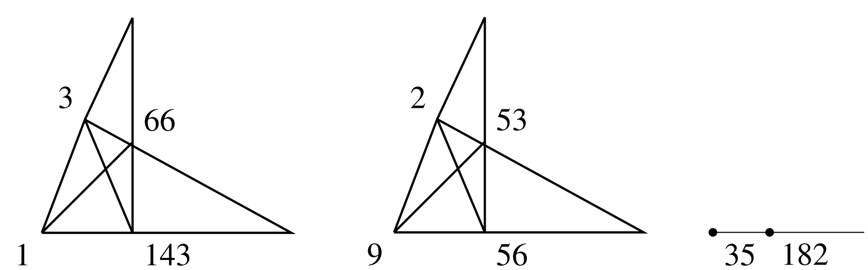

Figure 3.8: **Incidence structure of the orbits $\mathscr{O}rb_1, \mathscr{O}rb_2,$ and $\mathscr{O}rb_3$ of $\mathscr{G}_{17}$**

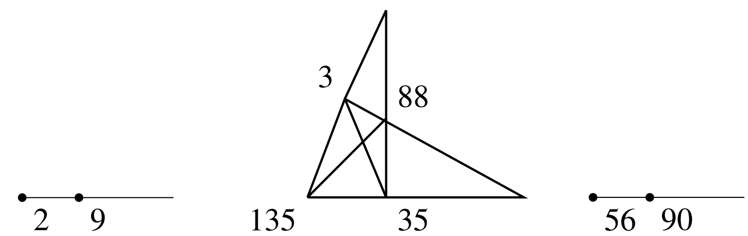

Figure 3.9: **Incidence structure of the orbits $\mathscr{O}rb_2, \mathscr{O}rb_3,$ and $\mathscr{O}rb_5$ of $\mathscr{G}_{23}$**

3.7 Geometric configuration of sd-inequivalent $(12;4)$-arcs

The sd-inequivalent $(12;4)$-arcs $\mathscr{H}_{25}$, $\mathscr{H}_{26}$, $\mathscr{H}_{28}$ have the groups Z_3, S_3, $Z_2 \times Z_2$ respectively. These group split the corresponding $(12;4)$-arcs into a number of orbits as shown in Section 2.8. The configurations of these orbits are described in Table 3.7 and the geometric pictures of the orbits are shown in Figures 3.10, 3.11, and 3.12.

Table 3.7: **Geometric configuration of sd-inequivalent $(12;4)$-arcs**

Symbol	Group	$\{t_4,t_3,t_2,t_1,t_0\}$	Configuration
$\mathscr{H}_{25}$	Z_3	$\{0,0,3,36,144\}$	vertices of a triangle
		$\{0,0,3,36,144\}$	vertices of a triangle
		$\{0,0,3,36,144\}$	vertices of a triangle
		$\{0,0,3,36,144\}$	vertices of a triangle
$\mathscr{H}_{26}$	S_3	$\{0,0,3,36,144\}$	vertices of a triangle
		$\{0,0,3,36,144\}$	vertices of a triangle
		$\{0,1,0,39,143\}$	three collinear points
		$\{0,0,3,36,144\}$	vertices of a triangle
$\mathscr{H}_{28}$	$Z_2 \times Z_2$	$\{0,0,0,14,169\}$	point
		$\{0,0,1,26,156\}$	2-arc
		$\{0,0,6,44,133\}$	vertices of a quadrangle
		$\{0,0,1,26,156\}$	2-arc
		$\{0,0,1,26,156\}$	2-arc
		$\{0,0,0,14,169\}$	point

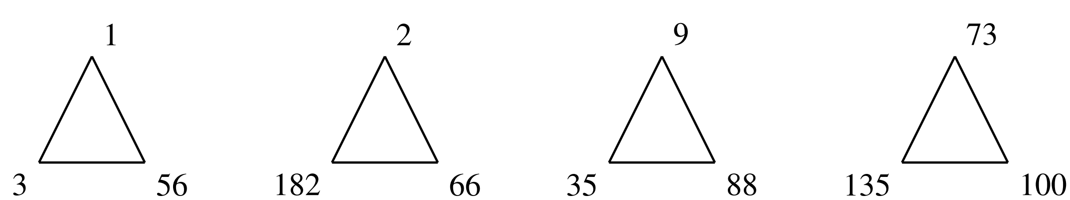

Figure 3.10: **Incidence structure of the orbits $\mathscr{O}rb_1, \mathscr{O}rb_2, \mathscr{O}rb_3,$ and $\mathscr{O}rb_4$ of $\mathscr{H}_{25}$**

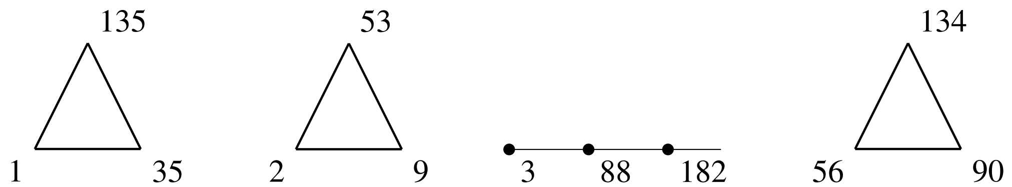

Figure 3.11: **Incidence structure of the orbits $\mathscr{O}rb_1, \mathscr{O}rb_2$, $\mathscr{O}rb_3$, and $\mathscr{O}rb_4$ of $\mathscr{H}_{26}$**

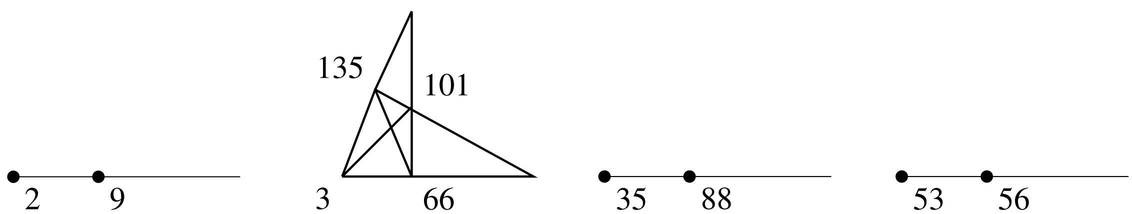

Figure 3.12: **Incidence structure of the orbits $\mathscr{O}rb_2, \mathscr{O}rb_3$, $\mathscr{O}rb_4$, and $\mathscr{O}rb_5$ of $\mathscr{H}_{28}$**

3.8 Geometric configuration of *sd*-inequivalent $(13;4)$-arcs

In Section 2.9, the groups Z_6 and S_4 partition the *sd*-inequivalent $(13;4)$-arcs $\mathscr{O}_{20}$ and $\mathscr{O}_{28}$ into a number of orbits. Table 3.8 introduces their geometric configurations. Also, the geometric pictures of these orbits are shown in Figure 3.13 and Figure 3.14.

Table 3.8: **Geometric configuration of *sd*-inequivalent $(13;4)$-arcs**

Symbol	Group	$\{t_4, t_3, t_2, t_1, t_0\}$	Configuration
$\mathscr{O}_{20}$	Z_6	$\{0,0,1,26,156\}$	2-arc
		$\{0,2,9,60,112\}$	two concurrent lines
		$\{0,0,1,26,156\}$	2-arc
		$\{0,1,0,39,143\}$	3 collinear points
$\mathscr{O}_{28}$	S_4	$\{0,4,3,66,110\}$	a quadrilateral
		$\{0,0,3,36,144\}$	vertices of a triangle
		$\{0,0,6,44,133\}$	vertices of a quadrangle

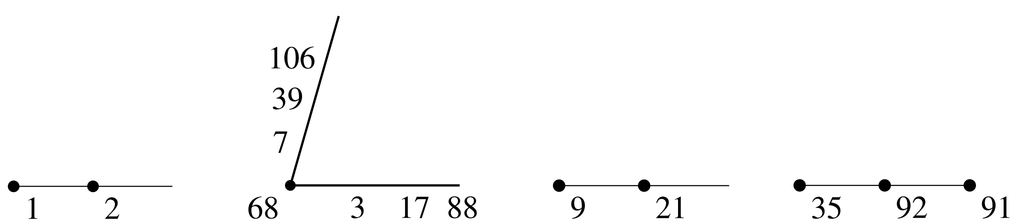

Figure 3.13: **Incidence structure of the orbits $\mathscr{O}rb_1, \mathscr{O}rb_2$, , $\mathscr{O}rb_3$, and $\mathscr{O}rb_4$ of $\mathscr{O}_{20}$**

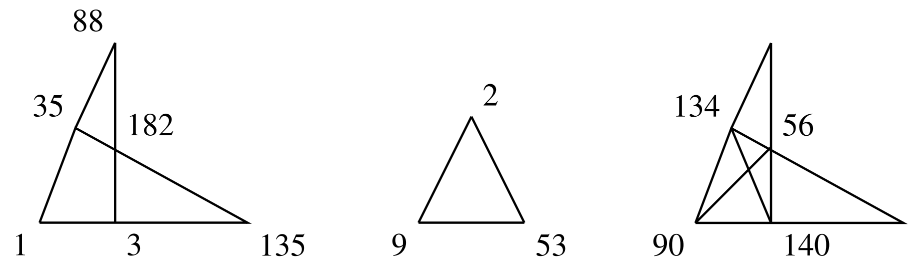

Figure 3.14: **Incidence structure of the orbits $\mathscr{O}rb_1, \mathscr{O}rb_2,$ and $\mathscr{O}rb_3$ of $\mathscr{O}_{28}$**

3.9 Geometric configuration of *sd*-inequivalent $(14;4)$-arcs

The geometric configurations of the orbits of the groups Z_4 and Z_6 of the *sd*-inequivalent $(14;4)$-arcs $\mathscr{P}_{27}$ and $\mathscr{P}_{28}$ given in Section 2.10, Table 2.16, are described in Table 3.9. In addition, the geometric pictures of these orbits are shown in Figure 3.15 and Figure 3.16.

Table 3.9: **Geometric configuration of *sd*-inequivalent $(14;4)$-arcs**

Symbol	Group	$\{t_4,t_3,t_2,t_1,t_0\}$	Configuration
$\mathscr{P}_{27}$	Z_4	$\{0,0,6,44,133\}$	vertices of a quadrangle
		$\{0,0,6,44,133\}$	vertices of a quadrangle
		$\{0,0,6,44,133\}$	vertices of a quadrangle
		$\{0,0,1,26,156\}$	2-arc
$\mathscr{P}_{28}$	Z_6	$\{0,0,1,26,156\}$	2-arc
		$\{0,2,9,60,112\}$	two concurrent lines
		$\{0,0,1,26,156\}$	2-arc
		$\{0,1,0,39,143\}$	3 collinear points
		$\{0,0,0,14,169\}$	point

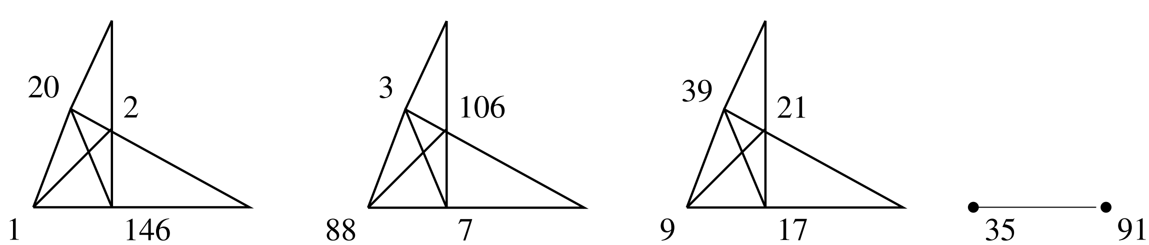

Figure 3.15: **Incidence structure of the orbits $\mathscr{O}rb_1, \mathscr{O}rb_2, \mathscr{O}rb_3,$ and $\mathscr{O}rb_4$ of $\mathscr{P}_{27}$**

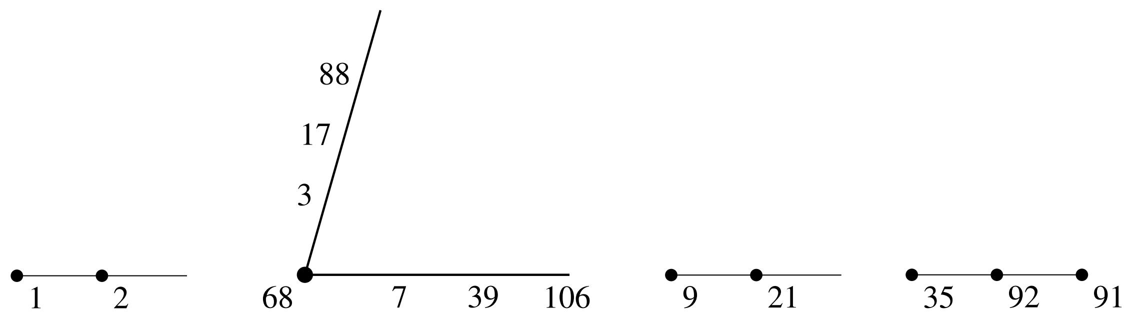

Figure 3.16: **Incidence structure of the orbits $\mathcal{O}rb_1, \mathcal{O}rb_2$, $\mathcal{O}rb_3$, and $\mathcal{O}rb_4$ of $\mathscr{P}_{28}$**

3.10 Geometric configuration of the *sd*-inequivalent $(38;4)$-arc

In Section 2.34, the size of the largest complete $(k;4)$-arc found is 38. From Table 2.41, the statistics of the three complete $(38;4)$-arcs are shown. There is one *sd*-inequivalent complete $(38;4)$-arc $\mathscr{X}_1$ having dihedral group D_{12}; this group partitions the $(38;4)$-arc into 4 orbits of sizes $2,12,12,12$. The incidence structures of these 4 orbits are the following:

(1) $\mathcal{O}rb_1(\mathscr{X}_1) = \{1,2\}$. The value of secant distribution $\{t_4,t_3,t_2,t_1,t_0\}$ of this orbit is $\{0,0,1,26,156\}$.

(2) $\mathcal{O}rb_2(\mathscr{X}_1) = \{3,109,88,123,33,31,50,44,57,158,154,164\}$. The value of secant distribution of this orbit is $\{0,0,66,36,81\}$.

(3) $\mathcal{O}rb_3(\mathscr{X}_1) = \{12,175,148,16,22,40,70,46,54,168,38,20\}$. The value of secant distribution of this orbit is $\{0,0,66,36,81\}$.

(4) $\mathcal{O}rb_4(\mathscr{X}_1) = \{39,82,94,145,102,166,141,101,152,98,106,130\}$. The value of secant distribution $\{t_4,t_3,t_2,t_1,t_0\}$ of this orbit is $\{0,0,66,36,81\}$.

Remark

From the above description of points and lines, the geometric configurations of the four orbits of complete $(38;4)$-arc are given in Table 3.10.

Table 3.10: **Geometric configuration of the sd-inequivalent $(38;4)$-arc**

Orbit number	$\{t_4, t_3, t_2, t_1, t_0\}$	Configuration	Conic
1	$\{0, 0, 1, 26, 156\}$	2-arc	
2	$\{0, 0, 66, 36, 81\}$	Incomplete 12-arc	$xy + xz + 11yz$
3	$\{0, 0, 66, 36, 81\}$	Incomplete 12-arc	$4xy + 4xz + 5yz + z^2$
4	$\{0, 0, 66, 36, 81\}$	Incomplete 12-arc	$5xy + 5xz + 3yz + z^2$

Conclusion

The conics in Table 3.10 intersect in two points $\{1, 2\}$. Hence, each of the complete $(38;4)$-arcs is a union of three conics.

Chapter 4

The classification of certain *sd*-inequivalent complete $(k;4)$-arcs in $\mathrm{PG}(2,13)$

Introduction

In this chapter, certain *sd*-inequivalent complete $(k;4)$-arcs for the values of $k = 24,25,26,27,28,29,30,31,$ 32,33, 34,35,38 are determined amongst certain *sd*-inequivalent $(k;4)$-arcs found, where for each k, the *sd*-inequivalent complete $(k;4)$-arcs are classified using the previous classification of *sd*-inequivalent $(k;4)$-arcs, for $k = 24,25,26,27,28,29,30,31,32,33,34,35,38$. Also, this chapter presents the statistics of the numbers, groups, and *sd*-inequivalent classes $\{t_4,t_3,t_2,t_1,t_0\}$ of i-secant distributions of the *sd*-inequivalent complete $(k;4)$-arcs. The new size of complete $(k;4)$-arc for $k = 36$ is discussed as well.

Historical review

One of the main problems in the study of projective planes, which is also of interest in coding theory, is finding the sizes of complete arcs in $\mathrm{PG}(2,q)$. Many investigations have been done to determine the values of the smallest and the largest sizes of complete $(k;n)$-arcs; these are denoted by $t_n(2,q)$ and $m_n(2,q)$.

In 1947, Bose [10] proved that

$$m_2(2,q) = \begin{cases} q+1, & \text{for } q \text{ odd;} \\ q+2, & \text{for } q \text{ even.} \end{cases}$$

In 1954, Segre [36], [37] proved that every $(q+1)$-arc is a conic for q odd. In 1956 Barlotti [3] showed that

$$m_n(2,q) \leq \begin{cases} (n-1)q+n \text{ for } (n,q)=1, \\ (n-1)q+n-2 \text{ for } n>2. \end{cases}$$

In 1964, Lunelli and Sce [29] improved these bounds as follows:

$$m_n(2,q) \leq \begin{cases} (n-1)q+n-3 \text{ for } 4 \leq n \leq q, \\ (n-1)q+n-4 \text{ for } 9 \leq n \leq q. \end{cases}$$

In 1993, Ali [1] showed that the smallest and the largest size of complete arcs in $\mathrm{PG}(2,13)$ for $n=2$ are 8 and 14. For $n=3$, Marcugini [31] showed that the minimum size is 15 and the maximum size is 23. Also, for $n=4$, Ball [2] gave an example of a large $(k;4)$-arc in $\mathrm{PG}(2,13)$, that is, a $(34;4)$-arc. Then this bound has been raised by computer search from a $(34;4)$-arc to a $(35;4)$-arc by Daskalov and Contreras [11]. In Table 4.1, the values of $m_n(2,q)$ for small values of n and q are introduced [22].

Table 4.1: **The values of** $m_n(2,q)$ **for small values of** n,q

q / n	2	3	4	5	7	8	9	11	13
2	4	4	6	6	8	10	10	12	14
3	7	9	9	11	15	15	17	21	23
4		13	16	16	22	28	28	32	38 – 40
5			21	25	29	33	37	43 – 45	49 – 53
6				31	36	42	48	56	64 – 66
7					49	49	55	67	79
8					57	64	65	77 – 78	92
9						73	81	89 – 90	105
10							91	100 – 102	118 – 119
11								121	132 – 133
12								133	145 – 147
13									169
14									183

4.1 *sd*-inequivalent complete $(24;4)$-arcs

The number of *sd*-inequivalent complete $(24;4)$-arcs is three, each having the identity group that partitions the associated arcs into 24 orbits of length 1. The timing of the groups is 1939 msec. The i-secant distributions of the three *sd*-inequivalent complete $(24;4)$-arcs are given as follows; it took 1982 msec.

$$\{32,12,48,76,15\},\{32,13,45,79,14\},\{33,11,45,81,13\}.$$

4.1.1 Remark

The statistics of *sd*-inequivalent complete $(24;4)$-arcs are written in Table 4.2. The symbol S in Table 4.2 indicates the stabiliser group type of each arc. Similarly, this notation is used for each table of *sd*-inequivalent complete $(k;4)$-arcs. Also, the next tables for $k > 25$, the notation S_t stands for the stabilisers time.

Table 4.2: ***sd*-inequivalent complete $(24;4)$-arcs**

Arc size	*sd*-inequivalent complete $(24;4)$-arcs	S
24	3	I

Throughout this study the smallest complete $(k;4)$-arc has size at most 24. So, the following theorem holds.

Theorem 4.1. *In* PG$(2,13)$*, there are at least three sd-inequivalent complete* $(24;4)$*-arcs.*

4.2 *sd*-inequivalent complete $(25;4)$-arcs

The number of *sd*-inequivalent complete $(25;4)$-arcs is four, each having the identity group; this took 1942 msec. Also, the four classes of *sd*-inequivalent secant distributions which took 1990 msec are as follows:

$$\{33,18,48,68,16\},\{34,14,54,64,17\},\{34,15,51,67,16\},\{35,14,48,72,14\}.$$

Table 4.3: ***sd*-inequivalent complete $(25;4)$-arcs**

Arc size	*sd*-inequivalent complete $(25;4)$-arcs	S
25	4	I

Theorem 4.2. *In* PG$(2,13)$*, there are at least four sd-inequivalent complete* $(25;4)$*-arcs.*

4.3 *sd*-inequivalent complete $(26;4)$-arcs

The number of *sd*-inequivalent complete $(26;4)$-arcs is 13. The stabilisers of *sd*-inequivalent complete $(26;4)$-arcs are the identity groups as introduced in Table 4.5. Also, the classes of *sd*-inequivalent secant distributions took 2245 msec. They are given in Table 4.4.

Table 4.4: sd**-inequivalent classes for** $(26;4)$**-arcs**

Number	$\{t_4,t_3,t_2,t_1,t_0\}$
1	$\{35,25,40,69,14\}$
2	$\{36,20,49,62,16\}$
3	$\{36,21,46,65,15\}$
4	$\{36,23,40,71,13\}$
5	$\{37,18,49,64,15\}$
6	$\{37,23,34,79,10\}$
7	$\{38,13,58,57,17\}$
8	$\{38,16,49,66,14\}$
9	$\{38,17,46,69,13\}$
10	$\{39,15,46,71,12\}$
11	$\{39,17,40,77,10\}$
12	$\{39,18,37,80,9\}$
13	$\{40,14,43,76,10\}$

Table 4.5: sd**-inequivalent complete** $(26;4)$**-arcs**

Arc size	sd-inequivalent complete $(26;4)$-arcs	S	S_t (msec)
26	13	I	2638

Theorem 4.3. *In* $\mathrm{PG}(2,13)$*, there are at least* 13 *sd-inequivalent complete* $(26;4)$*-arcs.*

4.4 sd-inequivalent complete $(27;4)$-arcs

The number of sd-inequivalent complete $(27;4)$-arcs is 33. The stabiliser of each of these arcs is the identity group. The values of secant distributions $\{t_4,t_3,t_2,t_1,t_0\}$ are given in Table 4.6. This took 2714 msec. The statistics of this process are shown in Table 4.7.

Table 4.6: sd**-inequivalent classes for** $(27;4)$**-arcs**

Number	$\{t_4,t_3,t_2,t_1,t_0\}$
1	$\{35,27,60,37,24\}$
2	$\{36,27,54,45,21\}$
3	$\{36,28,51,48,20\}$
4	$\{36,30,45,54,18\}$
5	$\{36,31,42,57,17\}$
6	$\{36,34,33,66,14\}$
7	$\{37,30,39,62,15\}$
8	$\{37,31,36,65,14\}$
9	$\{37,34,27,74,11\}$
10	$\{38,31,30,73,11\}$
11	$\{39,20,57,48,19\}$

12	$\{39,23,48,57,16\}$
13	$\{39,24,45,60,15\}$
14	$\{39,25,42,63,14\}$
15	$\{39,29,30,75,10\}$
16	$\{40,19,54,53,17\}$
17	$\{40,20,51,56,16\}$
18	$\{40,22,45,62,14\}$
19	$\{40,24,39,68,12\}$
20	$\{40,25,36,71,11\}$
21	$\{40,26,33,74,10\}$
22	$\{41,18,51,58,15\}$
23	$\{41,19,48,61,14\}$
24	$\{41,21,42,67,12\}$
25	$\{41,22,39,70,11\}$
26	$\{41,23,36,73,10\}$
27	$\{41,24,33,76,9\}$
28	$\{42,16,51,60,14\}$
29	$\{42,17,48,63,13\}$
30	$\{42,18,45,66,12\}$
31	$\{42,19,42,69,11\}$
32	$\{43,14,51,62,13\}$
33	$\{43,15,48,65,12\}$

Table 4.7: *sd*-**inequivalent complete** $(27;4)$**-arcs**

Arc size	*sd*-inequivalent complete $(27;4)$-arcs	S	S_t (msec)
27	33	I	5486

Theorem 4.4. *In* PG$(2,13)$*, there are at least* 33 *sd-inequivalent complete* $(27;4)$*-arcs.*

4.5 *sd*-inequivalent complete $(28;4)$-arcs

The number of *sd*-inequivalent complete $(28;4)$-arcs is 33, each having the identity group. The *sd*-inequivalent classes of secant distributions $\{t_4,t_3,t_2,t_1,t_0\}$ took 2743 msec. Also, the statistics of *sd*-inequivalent complete $(28;4)$-arcs are given in Table 4.8 and Table 4.9.

Table 4.8: *sd*-**inequivalent classes for** $(28;4)$**-arcs**

Number	$\{t_4,t_3,t_2,t_1,t_0\}$
1	$\{36,39,45,41,22\}$
2	$\{38,37,39,51,18\}$
3	$\{38,38,36,54,17\}$
4	$\{39,37,33,59,15\}$
5	$\{40,30,48,46,19\}$
6	$\{40,31,45,49,18\}$

7	$\{40,32,42,52,17\}$
8	$\{40,33,39,55,16\}$
9	$\{40,34,36,58,15\}$
10	$\{41,30,42,54,16\}$
11	$\{42,22,60,38,21\}$
12	$\{42,24,54,44,19\}$
13	$\{42,25,51,47,18\}$
14	$\{42,26,48,50,17\}$
15	$\{42,27,45,53,16\}$
16	$\{42,29,39,59,14\}$
17	$\{42,30,36,62,13\}$
18	$\{42,31,33,65,12\}$
19	$\{43,20,60,40,20\}$
20	$\{43,21,57,43,19\}$
21	$\{43,22,54,46,18\}$
22	$\{43,23,51,49,17\}$
23	$\{43,24,48,52,16\}$
24	$\{43,27,39,61,13\}$
25	$\{43,28,36,64,12\}$
26	$\{43,29,33,67,11\}$
27	$\{43,30,30,70,10\}$
28	$\{44,22,48,54,15\}$
29	$\{44,26,36,66,11\}$
30	$\{44,27,33,69,10\}$
31	$\{45,23,39,65,11\}$
32	$\{46,18,48,58,13\}$
33	$\{47,18,42,66,10\}$

Table 4.9: ***sd*-inequivalent complete** $(28;4)$**-arcs**

Arc size	*sd*-inequivalent complete $(28;4)$-arcs	S	S_t (msec)
28	33	I	6346

Theorem 4.5. *In* PG$(2,13)$*, there are at least* 33 *sd-inequivalent complete* $(28;4)$*-arcs.*

4.6 *sd*-inequivalent complete $(29;4)$-arcs

There are 60 *sd*-inequivalent complete $(29;4)$-arcs each having the identity group as shown in Table 4.11. The *i*-secants statistics are given in Table 4.10. It took 2944 msec.

Table 4.10: ***sd*-inequivalent classes for** $(29;4)$**-arcs**

Number	$\{t_4,t_3,t_2,t_1,t_0\}$
1	$\{38,47,37,39,22\}$
2	$\{38,49,31,45,20\}$
3	$\{39,38,58,20,28\}$

4	$\{39,43,43,35,23\}$
5	$\{39,44,40,38,22\}$
6	$\{39,46,34,44,20\}$
7	$\{40,41,43,37,22\}$
8	$\{40,44,34,46,19\}$
9	$\{41,34,58,24,26\}$
10	$\{41,39,43,39,21\}$
11	$\{41,40,40,42,20\}$
12	$\{41,42,34,48,18\}$
13	$\{41,43,31,51,17\}$
14	$\{42,32,58,26,25\}$
15	$\{42,36,46,38,21\}$
16	$\{42,37,43,41,20\}$
17	$\{42,39,37,47,18\}$
18	$\{42,41,31,53,16\}$
19	$\{42,42,28,56,15\}$
20	$\{43,29,61,25,25\}$
21	$\{43,30,58,28,24\}$
22	$\{43,32,52,34,22\}$
23	$\{43,35,43,43,19\}$
24	$\{43,37,37,49,17\}$
25	$\{43,38,34,52,16\}$
26	$\{43,39,31,55,15\}$
27	$\{43,41,25,61,13\}$
28	$\{44,31,49,39,20\}$
29	$\{44,32,46,42,19\}$
30	$\{44,35,37,51,16\}$
31	$\{44,37,31,57,14\}$
32	$\{44,38,28,60,13\}$
33	$\{45,25,61,29,23\}$
34	$\{45,26,58,32,22\}$
35	$\{45,27,55,35,21\}$
36	$\{45,28,52,38,20\}$
37	$\{45,29,49,41,19\}$
38	$\{45,30,46,44,18\}$
39	$\{45,32,40,50,16\}$
40	$\{45,33,37,53,15\}$
41	$\{45,37,25,65,11\}$
42	$\{45,38,22,68,10\}$
43	$\{46,25,55,37,20\}$
44	$\{46,26,52,40,19\}$
45	$\{46,28,46,46,17\}$
46	$\{46,31,37,55,14\}$
47	$\{46,32,34,58,13\}$
48	$\{46,33,31,61,12\}$
49	$\{47,26,46,48,16\}$
50	$\{47,27,43,51,15\}$
51	$\{47,28,40,54,14\}$
52	$\{47,29,37,57,13\}$
53	$\{47,30,34,60,12\}$
54	$\{47,31,31,63,11\}$
55	$\{47,32,28,66,10\}$
56	$\{48,23,49,47,16\}$
57	$\{48,24,46,50,15\}$
58	$\{48,26,40,56,13\}$
59	$\{48,27,37,59,12\}$
60	$\{48,28,34,62,11\}$

Table 4.11: sd**-inequivalent complete** $(29;4)$**-arcs**

Arc size	sd-inequivalent complete $(29;4)$-arcs	S	S_t (msec)
29	60	I	12120

Theorem 4.6. *In* $\mathrm{PG}(2,13)$*, there are at least* 60 *sd-inequivalent complete* $(29;4)$*-arcs.*

4.7 sd-inequivalent complete $(30;4)$-arcs

The number of sd-inequivalent complete (30;4)-arcs is 83, each having the identity group. The sd-inequivalent secants values $\{t_4,t_3,t_2,t_1,t_0\}$ are shown in Table 4.12. This took 3173 msec. The statistics of sd-inequivalent complete $(30;4)$-arcs are given in Table 4.13.

Table 4.12: sd**-inequivalent classes for** $(30;4)$**-arcs**

Number	$\{t_4,t_3,t_2,t_1,t_0\}$
1	$\{38,64,15,46,20\}$
2	$\{40,55,30,35,23\}$
3	$\{40,58,21,44,20\}$
4	$\{40,59,18,47,19\}$
5	$\{41,47,48,19,28\}$
6	$\{41,53,30,37,22\}$
7	$\{41,55,24,43,20\}$
8	$\{41,57,18,49,18\}$
9	$\{42,45,48,21,27\}$
10	$\{42,53,24,45,19\}$
11	$\{42,54,21,48,18\}$
12	$\{43,42,51,20,27\}$
13	$\{43,43,48,23,26\}$
14	$\{43,47,36,35,22\}$
15	$\{43,49,30,41,20\}$
16	$\{43,50,27,44,19\}$
17	$\{43,51,24,47,18\}$
18	$\{43,52,21,50,17\}$
19	$\{44,41,48,25,25\}$
20	$\{44,42,45,28,24\}$
21	$\{44,44,39,34,22\}$
22	$\{44,45,36,37,21\}$
23	$\{44,46,33,40,20\}$
24	$\{44,48,27,46,18\}$
25	$\{44,49,24,49,17\}$
26	$\{44,50,21,52,16\}$
27	$\{45,35,60,15,28\}$
28	$\{45,37,54,21,26\}$

29	$\{45,39,48,27,24\}$
30	$\{45,41,42,33,22\}$
31	$\{45,42,39,36,21\}$
32	$\{45,44,33,42,19\}$
33	$\{45,45,30,45,18\}$
34	$\{45,46,27,48,17\}$
35	$\{45,47,24,51,16\}$
36	$\{46,33,60,17,27\}$
37	$\{46,34,57,20,26\}$
38	$\{46,35,54,23,25\}$
39	$\{46,37,48,29,23\}$
40	$\{46,38,45,32,22\}$
41	$\{46,40,39,38,20\}$
42	$\{46,41,36,41,19\}$
43	$\{46,43,30,47,17\}$
44	$\{46,44,27,50,16\}$
45	$\{46,45,24,53,15\}$
46	$\{47,32,57,22,25\}$
47	$\{47,33,54,25,24\}$
48	$\{47,34,51,28,23\}$
49	$\{47,37,42,37,20\}$
50	$\{47,41,30,49,16\}$
51	$\{47,43,24,55,14\}$
52	$\{47,44,21,58,13\}$
53	$\{48,29,60,21,25\}$
54	$\{48,30,57,24,24\}$
55	$\{48,31,54,27,23\}$
56	$\{48,32,51,30,22\}$
57	$\{48,33,48,33,21\}$
58	$\{48,34,45,36,20\}$
59	$\{48,35,42,39,19\}$
60	$\{48,37,36,45,17\}$
61	$\{48,38,33,48,16\}$
62	$\{48,39,30,51,15\}$
63	$\{48,40,27,54,14\}$
64	$\{48,41,24,57,13\}$
65	$\{48,42,21,60,12\}$
66	$\{49,28,57,26,23\}$
67	$\{49,33,42,41,18\}$
68	$\{49,35,36,47,16\}$
69	$\{49,36,33,50,15\}$
70	$\{49,37,30,53,14\}$
71	$\{50,30,45,40,18\}$
72	$\{50,33,36,49,15\}$
73	$\{50,34,33,52,14\}$
74	$\{50,35,30,55,13\}$
75	$\{51,27,48,39,18\}$
76	$\{51,30,39,48,15\}$
77	$\{51,31,36,51,14\}$
78	$\{51,32,33,54,13\}$
79	$\{51,33,30,57,12\}$
80	$\{51,34,27,60,11\}$
81	$\{52,31,30,59,11\}$
82	$\{53,24,45,46,15\}$
83	$\{53,25,42,49,14\}$

Table 4.13: sd**-inequivalent complete** $(30;4)$**-arcs**

Arc size	sd-inequivalent complete $(30;4)$-arcs	S	S_t (msec)
30	83	I	17161

Theorem 4.7. *In* $\mathrm{PG}(2,13)$*, there are at least* 83 *sd-inequivalent complete* $(30;4)$*-arcs.*

4.8 sd-inequivalent complete $(31;4)$-arcs

The number of sd-inequivalent complete $(31;4)$-arcs is 77 as written in Table 4.15. The stabilisers of the 77 sd-inequivalent complete $(31;4)$-arcs are always the identity group. The sd-inequivalent secants distributions that took 3053 msec are given in Table 4.14.

Table 4.14: sd**-inequivalent classes for** $(31;4)$**-arcs**

Number	$\{t_4,t_3,t_2,t_1,t_0\}$
1	$\{40,65,30,19,29\}$
2	$\{41,63,30,21,28\}$
3	$\{41,64,27,24,27\}$
4	$\{41,65,24,27,26\}$
5	$\{42,60,33,20,28\}$
6	$\{42,61,30,23,27\}$
7	$\{42,62,27,26,26\}$
8	$\{43,57,36,19,28\}$
9	$\{43,58,33,22,27\}$
10	$\{43,59,30,25,26\}$
11	$\{43,60,27,28,25\}$
12	$\{44,50,51,6,32\}$
13	$\{44,54,39,18,28\}$
14	$\{44,55,36,21,27\}$
15	$\{44,56,33,24,26\}$
16	$\{44,57,30,27,25\}$
17	$\{44,58,27,30,24\}$
18	$\{44,60,21,36,22\}$
19	$\{45,51,42,17,28\}$
20	$\{45,54,33,26,25\}$
21	$\{45,56,27,32,23\}$
22	$\{45,57,24,35,22\}$
23	$\{45,58,21,38,21\}$
24	$\{46,45,54,7,31\}$
25	$\{46,48,45,16,28\}$
26	$\{46,51,36,25,25\}$
27	$\{46,54,27,34,22\}$
28	$\{46,55,24,37,21\}$

29	$\{47,45,48,15,28\}$
30	$\{47,51,30,33,22\}$
31	$\{47,52,27,36,21\}$
32	$\{47,53,24,39,20\}$
33	$\{47,54,21,42,19\}$
34	$\{48,41,54,11,29\}$
35	$\{48,43,48,17,27\}$
36	$\{48,44,45,20,26\}$
37	$\{48,46,39,26,24\}$
38	$\{48,47,36,29,23\}$
39	$\{48,48,33,32,22\}$
40	$\{48,49,30,35,21\}$
41	$\{48,50,27,38,20\}$
42	$\{48,52,21,44,18\}$
43	$\{49,39,54,13,28\}$
44	$\{49,41,48,19,26\}$
45	$\{49,42,45,22,25\}$
46	$\{49,43,42,25,24\}$
47	$\{49,44,39,28,23\}$
48	$\{49,45,36,31,22\}$
49	$\{49,46,33,34,21\}$
50	$\{49,48,27,40,19\}$
51	$\{49,49,24,43,18\}$
52	$\{49,52,15,52,15\}$
53	$\{50,36,57,12,28\}$
54	$\{50,37,54,15,27\}$
55	$\{50,38,51,18,26\}$
56	$\{50,39,48,21,25\}$
57	$\{50,40,45,24,24\}$
58	$\{50,44,33,36,20\}$
59	$\{50,46,27,42,18\}$
60	$\{51,34,57,14,27\}$
61	$\{51,35,54,17,26\}$
62	$\{51,37,48,23,24\}$
63	$\{51,39,42,29,22\}$
64	$\{51,44,27,44,17\}$
65	$\{51,45,24,47,16\}$
66	$\{51,46,21,50,15\}$
67	$\{52,37,42,31,21\}$
68	$\{52,38,39,34,20\}$
69	$\{52,43,24,49,15\}$
70	$\{53,40,27,48,15\}$
71	$\{53,42,21,54,13\}$
72	$\{54,36,33,44,16\}$
73	$\{54,37,30,47,15\}$
74	$\{55,28,51,28,21\}$
75	$\{55,34,33,46,15\}$
76	$\{55,36,27,52,13\}$
77	$\{56,33,30,51,13\}$

Table 4.15: sd**-inequivalent complete** $(31;4)$**-arcs**

Arc size	sd-inequivalent complete $(31;4)$-arcs	S	S_t (msec)
31	77	I	14381

Theorem 4.8. *In* PG$(2,13)$*, there are at least* 77 *sd-inequivalent complete* $(31;4)$*-arcs.*

4.9 sd-inequivalent complete $(32;4)$-arcs

The statistics of sd-inequivalent complete $(32;4)$-arcs showed that there are 66 sd-inequivalent complete $(32;4)$-arcs each having the identity group. The values of sd-inequivalent classes of secant distributions $\{t_4,t_3,t_2,t_1,t_0\}$ that took 2901 msec are given in Table 4.16.

Table 4.16: sd**-inequivalent classes for** $(32;4)$**-arcs**

Number	$\{t_4,t_3,t_2,t_1,t_0\}$
1	$\{44,69,25,15,30\}$
2	$\{44,71,19,21,28\}$
3	$\{44,72,16,24,27\}$
4	$\{45,66,28,14,30\}$
5	$\{45,70,16,26,26\}$
6	$\{46,61,37,7,32\}$
7	$\{46,65,25,19,28\}$
8	$\{47,61,31,15,29\}$
9	$\{47,62,28,18,28\}$
10	$\{47,64,22,24,26\}$
11	$\{47,65,19,27,25\}$
12	$\{47,66,16,30,24\}$
13	$\{48,58,34,14,29\}$
14	$\{48,60,28,20,27\}$
15	$\{48,63,19,29,24\}$
16	$\{49,53,43,7,31\}$
17	$\{49,55,37,13,29\}$
18	$\{49,57,31,19,27\}$
19	$\{49,58,28,22,26\}$
20	$\{49,59,25,25,25\}$
21	$\{49,60,22,28,24\}$
22	$\{49,62,16,34,22\}$
23	$\{50,53,37,15,28\}$
24	$\{50,55,31,21,26\}$
25	$\{50,56,28,24,25\}$
26	$\{50,57,25,27,24\}$
27	$\{50,58,22,30,23\}$
28	$\{50,59,19,33,22\}$

29	$\{51,47,49,5,31\}$
30	$\{51,48,46,8,30\}$
31	$\{51,50,40,14,28\}$
32	$\{51,51,37,17,27\}$
33	$\{51,53,31,23,25\}$
34	$\{51,54,28,26,24\}$
35	$\{51,55,25,29,23\}$
36	$\{51,57,19,35,21\}$
37	$\{52,45,49,7,30\}$
38	$\{52,46,46,10,29\}$
39	$\{52,48,40,16,27\}$
40	$\{52,50,34,22,25\}$
41	$\{52,51,31,25,24\}$
42	$\{52,52,28,28,23\}$
43	$\{52,53,25,31,22\}$
44	$\{53,43,49,9,29\}$
45	$\{53,50,28,30,22\}$
46	$\{53,51,25,33,21\}$
47	$\{53,52,22,36,20\}$
48	$\{53,53,19,39,19\}$
49	$\{54,43,43,17,26\}$
50	$\{54,45,37,23,24\}$
51	$\{54,47,31,29,22\}$
52	$\{54,49,25,35,20\}$
53	$\{55,41,43,19,25\}$
54	$\{55,43,37,25,23\}$
55	$\{55,45,31,31,21\}$
56	$\{56,38,46,18,25\}$
57	$\{56,41,37,27,22\}$
58	$\{56,42,34,30,21\}$
59	$\{56,44,28,36,19\}$
60	$\{56,45,25,39,18\}$
61	$\{56,46,22,42,17\}$
62	$\{57,37,43,23,23\}$
63	$\{57,39,37,29,21\}$
64	$\{58,32,52,16,25\}$
65	$\{58,40,28,40,17\}$
66	$\{59,40,22,48,14\}$

Table 4.17: ***sd*-inequivalent complete $(32;4)$-arcs**

Arc size	*sd*-inequivalent complete $(32;4)$-arcs	S	S_t (msec)
32	66	I	12930

Theorem 4.9. *In* $\mathrm{PG}(2,13)$*, there are at least* 66 *sd-inequivalent complete* $(32;4)$*-arcs.*

4.10 sd-inequivalent complete $(33;4)$-arcs

There are 33 sd-inequivalent complete (33;4)-arcs. The stabilisers of the 33 sd-inequivalent complete (33;4)-arcs are the identity group. The values of sd-inequivalent secant distributions $\{t_4,t_3,t_2,t_1,t_0\}$ that took 2730 msec are shown in Table 4.18. Also, the statistics of this process are given in Table 4.19.

Table 4.18: sd**-inequivalent classes for** $(33;4)$**-arcs**

Number	$\{t_4,t_3,t_2,t_1,t_0\}$
1	$\{51,68,18,18,28\}$
2	$\{52,63,27,11,30\}$
3	$\{52,64,24,14,29\}$
4	$\{52,66,18,20,27\}$
5	$\{53,61,27,13,29\}$
6	$\{53,62,24,16,28\}$
7	$\{53,63,21,19,27\}$
8	$\{54,58,30,12,29\}$
9	$\{54,59,27,15,28\}$
10	$\{54,60,24,18,27\}$
11	$\{54,61,21,21,26\}$
12	$\{55,54,36,8,30\}$
13	$\{55,56,30,14,28\}$
14	$\{55,58,24,20,26\}$
15	$\{55,60,18,26,24\}$
16	$\{56,52,36,10,29\}$
17	$\{56,57,21,25,24\}$
18	$\{57,49,39,9,29\}$
19	$\{57,53,27,21,25\}$
20	$\{57,55,21,27,23\}$
21	$\{58,47,39,11,28\}$
22	$\{58,49,33,17,26\}$
23	$\{58,54,18,32,21\}$
24	$\{59,47,33,19,25\}$
25	$\{59,49,27,25,23\}$
26	$\{59,50,24,28,22\}$
27	$\{60,42,42,12,27\}$
28	$\{60,44,36,18,25\}$
29	$\{60,45,33,21,24\}$
30	$\{61,43,33,23,23\}$
31	$\{61,47,21,35,19\}$
32	$\{61,48,18,38,18\}$
33	$\{63,36,42,18,24\}$

Table 4.19: sd**-inequivalent complete** $(33;4)$**-arcs**

Arc size	sd-inequivalent complete $(33;4)$-arcs	S	S_t (msec)
33	33	I	8308

Theorem 4.10. *In* $\mathrm{PG}(2,13)$*, there are at least* 33 *sd-inequivalent complete* $(33;4)$*-arcs.*

4.11 *sd*-inequivalent complete $(34;4)$-arcs

In Table 4.20, the number of *sd*-inequivalent complete $(34;4)$-arcs is 12 each having the identity group. In addition, the *sd*-inequivalent secants values which took 2105 msec are given as follows:

$$\{56,67,24,3,33\},\{56,69,18,9,31\},\{57,65,24,5,32\},\{57,67,18,11,30\}\{58,63,24,7,31\},$$
$$\{58,64,21,10,30\},\{60,58,27,8,30\},\{60,59,24,11,29\}\{61,55,30,7,30\},\{61,56,27,10,29\},$$
$$\{63,53,24,17,26\},\{65,45,36,9,28\}.$$

Table 4.20: *sd***-inequivalent complete** $(34;4)$**-arcs**

Arc size	*sd*-inequivalent complete $(34;4)$-arcs	S	S_t (msec)
34	12	I	4522

Theorem 4.11. *In* $\mathrm{PG}(2,13)$*, there are at least* 12 *sd-inequivalent complete* $(34;4)$*-arcs.*

4.12 *sd*-inequivalent complete $(35;4)$-arcs

There are six *sd*-inequivalent complete $(35;4)$-arcs each having the identity group as shown in Table 4.21. The classes of *sd*-inequivalent secant distribution that took 2013 msec are given as follows:

$$\{64,66,13,10,30\},\{65,63,16,9,30\},\{66,61,16,11,29\},$$
$$\{68,56,19,12,28\},\{69,53,22,11,28\},\{70,51,22,13,27\}.$$

Table 4.21: *sd***-inequivalent complete** $(35;4)$**-arcs**

Arc size	*sd*-inequivalent complete $(35;4)$-arcs	S	S_t (msec)
35	6	I	3375

Theorem 4.12. *In* $\mathrm{PG}(2,13)$*, there are at least six sd-inequivalent complete* $(35;4)$*-arcs.*

4.13 *sd*-inequivalent complete $(38;4)$-arc

In PG$(2,13)$, the largest size of complete $(k;4)$-arc discovered in this work is at least 38. Among the three complete arcs found there is one *sd*-inequivalent complete $(38;4)$-arc; it has dihedral group, D_{12}. Also, the values of t_i of secant distribution of the complete $(38;4)$-arc are the following:

$t_4 = 102, \quad t_3 = 24, \quad t_2 = 19, \quad t_1 = 14, \quad t_0 = 24.$

Table 4.22: ***sd*-inequivalent complete $(38;4)$-arc**

Arc size	*sd*-inequivalent complete $(38;4)$-arcs	S	S_t (msec)
38	1	D_{12}	2062

Theorem 4.13. *In* PG$(2,13)$*, there is at least one sd-inequivalent complete* $(38;4)$*-arc.*

4.14 Complete $(36;4)$-arc

In Chapter 2, Section 2.37, the classification of certain $(k;4)$-arcs that relies on the different choice of the five *sd*-inequivalent $(7;4)$-arcs is done. The process in this case found a complete arc of size 36 and none larger. Therefore, this new result raises the size of the second largest complete $(k;4)$-arc in PG(2,13) given in Section 4.12 from a $(35;4)$-arc to a $(36;4)$-arc. The statistics of this complete $(36;4)$-arc are written in Chapter 2, Section 2.37.

Chapter 5

Algebraic properties of quartic curves for complete $(k;4)$-arcs

5.1 Introduction

A plane curve C^n of order n is represented by an equation $f(x,y,z)=0$, in which f is a polynomial of degree n, homogeneous in x,y,z. Since the number of terms in f is $\frac{1}{2}(n+1)(n+2)$, so the curve depends on the ratios of that number of coefficients and therefore on the values of the $\frac{1}{2}n(n+3)$ parameters. Since to make C^n pass through a given point imposes a linear condition on the coefficients, it follows that there exists a unique curve C^n through $\frac{1}{2}n(n+3)$ given points. Therefore, curves of order n have freedom $\frac{1}{2}n(n+3)$. For $n=4$, a quartic curve is therefore determined by 14 points.

In this chapter, examples of non-singular quartic curves associated to complete $(k;4)$-arcs in $\mathrm{PG}(2,13)$ for $k=24,25,26,27,28,29,30,31,32,33,34,35,36,38$ are defined. The algebraic properties for each curve in terms of the number of inflexion points and rational points are given. In addition, two examples of quartic curves attaining the Hasse-Weil-Serre upper bound for the number N_1 of rational points on a curve over the finite field of order thirteen are given, where this number is 32. The algebraic properties of the two quartic curves such us the number of inflexion points, the number of rational points, the secant distributions, and the stabiliser groups are given.

5.2 Quartic curves for complete $(24;4)$-arcs

Table 5.1 shows the quartic curves $\mathscr{C}(f_1), \mathscr{C}(f_2), \mathscr{C}(f_3), \mathscr{C}(f_4)$ that took 1960 msc, none of which is singular. Also, these curves do not have inflexion points as the determinant of the Hessian matrix $\mathscr{H}_P(f_i)$ is non-zero at all rational points P of $\mathscr{C}(f_i)$ for $i = 1,2,3,4$. The properties of $\mathscr{C}(f_i)$, $i = 1,2,3,4$, are listed in Table 5.1 where, for each f_i, the number I of inflexion points, the number $|\mathscr{C} \cap \mathscr{K}|$ of rational points lying on the corresponding arc $\mathscr{K}$, the number N_1 of rational points that took 2386 msec, and the stabiliser group S that took 2249 msec are given.

Table 5.1: **Quartic curves for complete (24;4)-arcs**

Symbol	$\mathscr{C}(f_i)$	I	$\lvert\mathscr{C} \cap \mathscr{K}\rvert$	N_1	S
f_1	$x^4+10x^3z+3x^2y^2+7x^2yz+x^2z^2+4xy^3+8xy^2z+8xyz^2+2y^4+9y^3z+3y^2z^2+4yz^3+8z^4$	0	14	28	Z_2
f_2	$x^4+10x^3y-x^3z+11x^2y^2-x^2yz+5x^2z^2-xy^3+9xy^2z+4xyz^2+8xz^3+5y^4-y^3z+2y^2z^2+yz^3$	0	15	24	I
f_3	$x^4+x^3y+11x^3z+11x^2y^2+2x^2yz+4x^2z^2+5xy^3-xy^2z+8xyz^2+7xz^3+2y^4+2y^3z+8y^2z^2+8yz^3+5z^4$	0	14	24	I
f_4	$x^4+6x^3y+3x^3z+8x^2y^2+x^2yz+5x^2z^2+10xy^3+6xy^2z+10xyz^2+2y^4+6y^3z+10y^2z^2+yz^3+z^4$	0	14	24	I

Remark

Note that the choice of the fourteen points to define a non-singular quartic curve is not unique. In Table 5.1, the associated complete $(24;4)$-arc for f_1 is as follows:

$\mathscr{K}_1 = \{1,2,3,88,9,182,35,56,135,22,151,55,31,153,45,171,85,168,48,113,183,59,92,125\}$.

The points that defined the polynomial curve f_1 are as follows:

$$\{31,45,48,55,59,85,92,113,125,151,153,168,171,183\}.$$

So, if we choose another 14 points to define a quartic curve $\mathscr{C}(f_1)$, a new polynomial f_1^* is found; it may or may not have equivalent algebraic properties to those of $\mathscr{C}(f_1)$. Thus, let $\{1,9,135,22,151,55,45,171,85,168,48,113,183,59\}$ be the new set of 14 points of $\mathscr{K}_1$; then the associated quartic polynomial f_1^* is as follows:

$$f_1^* = 10x^3z + 10x^2y^2 - x^2yz + 2x^2z^2 - xy^3 + 4xy^2z + 11xz^3 + y^4 + 6y^2z^2 + 3yz^3 + 3z^4.$$

The curve $\mathscr{C}(f_1^*)$ has one inflexion point. Also, the number $|\mathscr{C} \cap \mathscr{K}|$ of rational points of $\mathscr{C}(f_1^*)$ lying on $\mathscr{K}_1$ is 14 and the number N_1 of rational points of $\mathscr{C}(f_1^*)$ is 25; this took 2295 msec. Thus the two quartic curves $\mathscr{C}(f_1)$ and $\mathscr{C}(f_1^*)$ have the equivalent property related to the number $|\mathscr{C} \cap \mathscr{K}|$ of rational points on the corresponding arc, where each has 14 points. Also, the curve $\mathscr{C}(f_1)$ does not have inflexions while $\mathscr{C}(f_1^*)$ has one inflexion point. In addition, the number $N_1(\mathscr{C}(f_1))$ is 28 whereas the number $N_1(\mathscr{C}(f_1^*))$ is 25 rational points. Furthermore, the groups of $\mathscr{C}(f_1)$ and $\mathscr{C}(f_1^*)$ are not equivalent; here $\mathscr{C}(f_1)$ has the group Z_2 and $\mathscr{C}(f_1^*)$ has the identity group; this took 1988 msec.

5.3 Quartic curves for complete $(25;4)$-arcs

In Table 5.2, the statistics of the associated quartic curves of the complete (25;4)-arcs that took 1972 msc are described. It gives the number I of inflexions, the number $|\mathscr{C} \cap \mathscr{K}|$ of intersection points the curve with the arc, and the number N_1 of rational points of the curves. This number took 2265 msec.

Table 5.2: **Quartic curves for complete (25;4)-arcs**

Symbol	$\mathscr{C}(f_i)$	I	$\lvert\mathscr{C} \cap \mathscr{K}\rvert$	N_1
f_1	$x^4 + 5x^3z + 11x^2y^2 - x^2yz - x^2z^2 - xy^3 - xy^2z + 6xyz^2 + 9xz^3 + 2y^4 + 10y^3z + 10y^2z^2 + 5yz^3 + 7z^4$	1	15	27
f_2	$x^4 - x^3y + 11x^3z + 7x^2y^2 + 3x^2yz + 5x^2z^2 + xy^3 + 8xy^2z + 6xyz^2 + xz^3 + 7y^4 + 4y^3z + 6y^2z^2 - yz^3 + 2z^4$	0	14	24
f_3	$x^4 + 7x^3y + 7x^3z + x^2y^2 + 11x^2yz - x^2z^2 + 11xy^3 + 11xy^2z - xyz^2 + 2xz^3 + 6y^4 + 5y^3z + y^2z^2 + 8yz^3 + 9z^4$	1	14	24
f_4	$x^4 + x^3y + x^3z + 5x^2y^2 + 6x^2yz + 4x^2z^2 + 9xy^3 + xy^2z + 8xyz^2 + 3xz^3 + 6y^4 + 11y^3z + 11yz^3 + 3z^4$	2	15	28

5.4 Quartic curves for complete $(26;4)$-arcs

In these statistics, for the fifteen quartic curves of the complete (26;4)-arcs that took 2015 msc, there are 5 quartic curves $\mathscr{C}(f_4), \mathscr{C}(f_5), \mathscr{C}(f_7), \mathscr{C}(f_8)$, and $\mathscr{C}(f_{12})$ which do not have inflexion points and ten quartic curves $\mathscr{C}(f_1), \mathscr{C}(f_2), \mathscr{C}(f_3), \mathscr{C}(f_6), \mathscr{C}(f_9), \mathscr{C}(f_{10}), \mathscr{C}(f_{11}), \mathscr{C}(f_{13}), \mathscr{C}(f_{14})$, and $\mathscr{C}(f_{15})$ whose number of inflexion points is $1,2$ or 3. The number $|\mathscr{C} \cap \mathscr{K}|$ is between 14 and 18 points. The number N_1 of rational points of these curves ranges between 19 and 28; it took 2659 msec. The statistics are given in Table 5.3.

Table 5.3: **Quartic curves for complete (26;4)-arcs**

Symbol	$\mathscr{C}(f_i)$	I	$\lvert\mathscr{C} \cap \mathscr{K}\rvert$	N_1
f_1	$x^3y+7x^3z+2x^2y^2+8x^2yz+7x^2z^2+7xy^3+10xy^2z+xyz^2+7xz^3+11y^3z+5y^2z^2-yz^3$	1	15	25
f_2	$x^4+2x^3y+5x^3z+6x^2y^2+3x^2yz+7x^2z^2+8xy^3+9xy^2z+4xyz^2+xz^3+8y^4+8y^3z+4y^2z^2+5yz^3+6z^4$	2	14	24
f_3	$x^4+6x^3y+6x^3z+8x^2y^2+9x^2yz+2x^2z^2+7xy^3+3xy^2z+xyz^2+8xz^3+4y^4+9y^3z+9yz^3+2z^4$	1	15	27
f_4	$x^4+9x^3y+8x^3z+9x^2y^2+x^2yz+8x^2z^2-xy^2z+8xyz^2+11xz^3+4y^4+9y^3z+5yz^3+z^4$	0	14	22
f_5	$x^4+3x^3z+3x^2y^2+11x^2yz+9x^2z^2+4xy^3+xy^2z+2xyz^2+11xz^3+5y^3z+3y^2z^2+10yz^3+z^4$	0	15	26
f_6	$x^4+2x^3y+5x^3z+6x^2yz+8x^2z^2+xy^3+3xy^2z+2xyz^2+7xz^3-y^4+y^3z+11yz^3+9z^4$	3	15	24
f_7	$x^4+x^3y+x^3z+3x^2yz+4xy^3+9xy^2z+4xyz^2+10xz^3+7y^4-y^3z+10y^2z^2+2yz^3+z^4$	0	14	27
f_8	$x^4+5x^3y+7x^3z+7x^2y^2+5x^2yz+9x^2z^2+9xy^2z+8xyz^2+9xz^3+11y^4+5y^3z-y^2z^2+8yz^3-z^4$	0	15	28
f_9	$x^4+9x^3y+4x^3z+8x^2y^2-x^2yz+5x^2z^2+5xy^3+3xy^2z-xyz^2+xz^3+8y^4+5y^3z+7yz^3+10z^4$	2	14	23
f_{10}	$x^4+3x^3y+2x^3z+6x^2y^2+2x^2yz+8xy^3+2xyz^2+5y^3z+7y^2z^2+11yz^3+z^4$	1	15	24
f_{11}	$x^4+2x^3y+10x^3z+9x^2y^2+10x^2yz+3x^2z^2+5xy^3+8xy^2z+5xyz^2+4xz^3+11y^4+y^3z+4y^2z^2+5yz^3$	1	18	25
f_{12}	$x^4+9x^3z+2x^2y^2+7x^2yz+x^2z^2-xy^3+7xy^2z+2xyz^2+10xz^3+2y^4+3y^3z+10y^2z^2+8yz^3+9z^4$	0	15	21
f_{13}	$x^4+5x^3y+5x^3z+6x^2y^2+8x^2yz-x^2z^2+8xy^3-xy^2z+11xyz^2+8xz^3+8y^4+10y^3z+9y^2z^2+9yz^3+3z^4$	2	14	24
f_{14}	$x^4+7x^3y+6x^3z+10x^2y^2+10x^2yz+8x^2z^2+3xy^2z+9xyz^2+2xz^3+10y^4+8y^3z+5y^2z^2+yz^3+4z^4$	1	15	20
f_{15}	$x^4+2x^3y+9x^3z+11x^2y^2+4x^2yz+x^2z^2+4xy^3-xy^2z+4xyz^2+3xz^3+7y^4+6y^3z+4y^2z^2-yz^3+8z^4$	2	15	19

5.5 Quartic curves for complete $(27;4)$-arcs

Among the 43 quartic curves of the complete $(27;4)$-arcs that took 2166 msc, there are 13 quartic curves that do not have any inflexion points. Also the number of quartic curves that have inflexion points is 30. The number $|\mathscr{C} \cap \mathscr{K}|$ of each of the 43 quartic curves on the corresponding arc ranges between 14 and 17 points. In addition, the number N_1 of these curves ranges from 20 to 30 rational points; it took 3914 msec. The statistics of these curves are shown in Table 5.4.

Table 5.4: **Quartic curves for complete (27;4)-arcs**

Symbol	$\mathscr{C}(f_i)$	I	$\lvert\mathscr{C} \cap \mathscr{K}\rvert$	N_1
f_1	$x^3y+10x^3z+2x^2y^2+x^2yz+2x^2z^2+2xy^3-xy^2z+6xyz^2+8xz^3+6y^3z+11y^2z^2+7yz^3$	0	16	26
f_2	$x^4+6x^3y+5x^3z+8x^2y^2+11x^2yz+3x^2z^2+4xy^3+6xy^2z-xyz^2+9xz^3+10y^4+7y^3z+2yz^3$	3	16	24
f_3	$x^3y+x^3z+6x^2y^2+4x^2yz+10x^2z^2+10xy^3+7xy^2z+9xyz^2+11xz^3+8y^3z+4y^2z^2+7yz^3$	2	15	28
f_4	$x^4+10x^3y+2x^3z+3x^2y^2+8x^2yz+9x^2z^2+7xy^3-xy^2z+8xyz^2+6xz^3+8y^4+10y^3z+10y^2z^2+5z^4$	3	14	25
f_5	$x^4+9x^3y+4x^3z+4x^2y^2+3x^2yz+x^2z^2+5xy^3+10xy^2z+5xz^3+9y^4+y^3z-y^2z^2+7yz^3+5z^4$	0	14	22
f_6	$x^4+10x^3y+4x^3z+5x^2y^2+9x^2z^2+7xy^3+8xy^2z+10xz^3+6y^3z+2y^2z^2+2yz^3-z^4$	1	17	26
f_7	$x^4+8x^3z+9x^2y^2+6x^2yz+5x^2z^2+10xy^3+11xy^2z+3xyz^2+6y^4+y^3z+5y^2z^2+5yz^3+7z^4$	3	14	27
f_8	$x^4+9x^3z+4x^2y^2+11x^2yz+8x^2z^2+11xy^3+xy^2z+8xyz^2+9xz^3+5y^3z+10y^2z^2+3yz^3+5z^4$	2	15	26
f_9	$x^4+x^3y+11x^3z+4x^2y^2+3x^2yz+4x^2z^2+10xy^3+3xy^2z-xyz^2+4xz^3+5y^4+y^3z+9y^2z^2+4yz^3+10z^4$	0	14	30
f_{10}	$x^4+7x^3y+8x^2y^2+2xy^3+7xy^2z+7xyz^2+5xz^3+6y^4+3y^3z+4y^2z^2+2z^4$	1	16	24
f_{11}	$10x^3y+5x^3z+6x^2y^2+4x^2yz+9x^2z^2+9xy^3+11xy^2z-xyz^2+3xz^3+y^4+6y^3z+7y^2z^2+9yz^3+8z^4$	0	15	21
f_{12}	$x^4+2x^3y+11x^3z+11x^2y^2+7x^2yz-x^2z^2+6xy^3+4xy^2z+7xyz^2+7xz^3-y^4+9y^2z^2+2yz^3$	1	15	22
f_{13}	$x^4+11x^3y+9x^3z+x^2y^2-x^2yz+6x^2z^2+8xy^3-xy^2z+6xyz^2+6xz^3+6y^4+3y^3z+6y^2z^2+8yz^3+5z^4$	2	14	21

f_{14}	$x^4+3x^3y+x^3z+5x^2y^2+9x^2yz-x^2z^2+9xy^3+5xy^2z+5xyz^2-xz^3+8y^4+10y^3z+5y^2z^2+yz^3+2z^4$	1	14	22
f_{15}	$x^4+x^3y+11x^3z+4x^2y^2+3x^2yz+4x^2z^2+10xy^3+3xy^2z-xyz^2+4xz^3+5y^4+y^3z+9y^2z^2+4yz^3+10z^4$	0	14	30
f_{16}	$x^4+7x^3y+3x^3z-x^2y^2+6x^2yz+x^2z^2+9xy^3+4xy^2z+3xyz^2+6xz^3+11y^4+y^3z+2y^2z^2+2yz^3+2z^4$	2	14	24
f_{17}	$x^4+11x^3z+9x^2y^2+2x^2yz+9x^2z^2+10xy^3+8xy^2z+3xyz^2+8xz^3+6y^4+9y^3z+4y^2z^2+yz^3-z^4$	2	15	26
f_{18}	$x^4+4x^3z+8x^2y^2+6x^2yz+8x^2z^2+6xy^3+11xy^2z+8xyz^2+11xz^3+11y^4+8y^3z+3yz^3+8z^4$	2	14	23
f_{19}	$x^4-x^3y+7x^3z+10x^2y^2-x^2yz-x^2z^2+10xy^3-xy^2z+10xyz^2+xz^3+8y^4+10y^3z+8y^2z^2+10z^4$	1	15	26
f_{20}	$x^4+3x^3z+10x^2y^2+6x^2yz+3x^2z^2+3xy^3+2xyz^2+4xz^3+y^4+6y^3z+4y^2z^2+4yz^3+3z^4$	0	14	23
f_{21}	$x^4+5x^3y+2x^2y^2+2x^2yz+6x^2z^2-xy^2z+xyz^2+4xz^3+3y^4+y^3z+5y^2z^2+11yz^3+11z^4$	2	14	27
f_{22}	$x^4+11x^3z+2x^2z^2+5xy^3+9xy^2z+2xyz^2+5xz^3+11y^4+11y^3z+9y^2z^2+9yz^3+2z^4$	1	14	28
f_{23}	$x^4+6x^3y+10x^3z+x^2y^2+3x^2yz+2x^2z^2+10xy^3+8xy^2z+9xyz^2+10xz^3+10y^4+8y^3z+5y^2z^2+11yz^3$	3	16	27
f_{24}	$x^4+10x^3y+11x^2y^2+x^2yz+x^2z^2+3xy^3+8xy^2z-xz^3+y^4+9y^3z+9yz^3$	1	16	27
f_{25}	$x^4+5x^3y+x^3z+8x^2y^2+x^2yz+9x^2z^2+10xy^3+5xy^2z+7xyz^2+xz^3+6y^4+11y^3z+9y^2z^2+9yz^3+z^4$	1	15	25
f_{26}	$x^4+9x^3y+8x^3z+6x^2y^2+8x^2yz+2x^2z^2+7xy^3-xy^2z+11xyz^2+8xz^3+8y^4+6y^3z+9y^2z^2+2yz^3+z^4$	0	16	28
f_{27}	$x^4+11x^3z+2x^2z^2+5xy^3+9xy^2z+2xyz^2+5xz^3+11y^4+11y^3z+9y^2z^2+9yz^3+2z^4$	1	14	28
f_{28}	$x^4+3x^3y+5x^3z+6x^2y^2+9x^2yz+10xy^2z+9xyz^2+5xz^3+3y^4+10y^3z+5y^2z^2+9yz^3+2z^4$	1	14	25
f_{29}	$x^4+2x^3y+11x^3z-x^2y^2+5x^2yz+8x^2z^2+6xy^3+10xy^2z+3xyz^2+8xz^3+5y^4-y^3z-y^2z^2+4yz^3+7z^4$	0	15	24
f_{30}	$x^4+4x^3y+2x^3z+7x^2y^2+8x^2yz+6x^2z^2+10xy^3+5xy^2z+8xyz^2+6xz^3+6y^4-y^3z+9y^2z^2-yz^3+z^4$	2	16	27
f_{31}	$6x^3y+5x^2y^2+10x^2yz+xy^2z+3xyz^2+2xz^3+y^4+8y^3z+6yz^3+10z^4$	3	16	29
f_{32}	$2x^3y+2x^3z+8x^2yz+10x^2z^2+10xy^3+8xy^2z+9xyz^2+y^4+2y^3z+11y^2z^2+9yz^3+z^4$	1	17	25
f_{33}	$x^4+6x^3y+3x^3z+8x^2y^2+4x^2yz+5x^2z^2+11xy^2z+9xyz^2+6xz^3+2y^4+y^3z+11y^2z^2+11yz^3+11z^4$	2	15	22

f_{34}	$x^3y+10x^3z+2x^2y^2+x^2yz+2x^2z^2+2xy^3-xy^2z+6xyz^2+8xz^3+6y^3z+11y^2z^2+7yz^3$	0	16	26
f_{35}	$x^3y+10x^3z+2x^2y^2+x^2yz+2x^2z^2+2xy^3-xy^2z+6xyz^2+8xz^3+6y^3z+11y^2z^2+7yz^3$	0	16	26
f_{36}	$x^4+9x^3y+6x^3z-x^2yz+10x^2z^2+xy^3+5xy^2z+3xyz^2+11y^4+9y^3z+8y^2z^2+3yz^3+9z^4$	1	15	24
f_{37}	$x^4+2x^3y+8x^3z+5x^2y^2+9x^2yz+7x^2z^2+4xy^3+11xy^2z+6xyz^2+10xz^3+y^4+3y^2z^2+10yz^3+5z^4$	0	14	26
f_{38}	$x^4+5x^3y+4x^2y^2+2x^2yz+x^2z^2+10xy^3+5xy^2z+5xyz^2+6xz^3+9y^4+2y^3z+3y^2z^2+6yz^3-z^4$	1	14	20
f_{39}	$x^4+4x^3z+4x^2y^2+8x^2z^2+10xy^3+9xy^2z+4xyz^2+10xz^3+4y^4+4y^3z+4y^2z^2+10yz^3+3z^4$	0	16	27
f_{40}	$x^4+6x^3y+3x^3z+2x^2y^2+5x^2yz+4x^2z^2-xy^3+10xy^2z+7xyz^2-xz^3+8y^4+8y^3z+10y^2z^2+8yz^3+11z^4$	1	14	26
f_{41}	$x^4+7x^3y+6x^3z+2x^2y^2+6x^2yz+3x^2z^2+2xy^3+10xy^2z+xyz^2+8xz^3+10y^4+3y^3z+6y^2z^2+7yz^3+8z^4$	1	14	24
f_{42}	$x^4+6x^3y+6x^3z+3x^2y^2+9x^2yz+x^2z^2+3xy^3+4xy^2z+4xyz^2+9xz^3+6y^3z+5y^2z^2+11yz^3+9z^4$	1	15	25
f_{43}	$x^4+2x^3y+11x^3z-x^2y^2+5x^2yz+8x^2z^2+6xy^3+10xy^2z+3xyz^2+8xz^3+5y^4-y^3z-y^2z^2+4yz^3+7z^4$	0	14	24

5.6 Quartic curves for complete $(28;4)$-arcs

There are 44 curves took 2226 msc. Among them the number of associated quartic curves of the complete $(28;4)$-arcs that do not have inflextions is 16. Here, there are 28 quartic curves whose number I of inflexion points is 1 or 2. Also, the number $|\mathscr{C}\cap\mathscr{K}|$ of the 44 curves on the corresponding arc ranges between 14 and 17. In addition, the number N_1 of rational points is between 20 and 28; it took 5103 msec. The statistics are shown in Table 5.5.

Table 5.5: **Quartic curves for complete (28;4)-arcs**

Symbol	$\mathscr{C}(f_i)$	I	$\lvert\mathscr{C}\cap\mathscr{K}\rvert$	N_1
f_1	$x^4+5x^3y+7x^3z+7x^2y^2+9x^2yz+8x^2z^2+8xy^3+11xy^2z+3xyz^2+11xz^3+4y^4+9y^3z+5y^2z^2+10yz^3+6z^4$	0	17	24
f_2	$x^4+3x^3y+x^2y^2-x^2yz+7x^2z^2+2xy^3+4xy^2z+3xyz^2+9xz^3+10y^4+11y^3z+7y^2z^2+5yz^3+10z^4$	1	15	23

f_3	$x^4+3x^3y+7x^3z+8x^2y^2+10x^2yz+10x^2z^2+11xy^3-xy^2z+3xyz^2+9xz^3+4y^4+10y^3z-y^2z^2+8yz^3+3z^4$	2	14	24
f_4	$x^4-x^3y+3x^3z+11x^2y^2+x^2yz+9x^2z^2+3xy^3-xy^2z+5xyz^2+10y^4+y^3z+9y^2z^2+6z^4$	2	15	28
f_5	$x^4+7x^3y+x^3z+7x^2y^2+11x^2yz+9x^2z^2+7xy^3+11xy^2z+3xyz^2+2xz^3+y^4+y^3z+6y^2z^2+9yz^3$	0	16	24
f_6	$x^4+x^3y+6x^3z+10x^2y^2+2x^2yz+3x^2z^2+4xy^3+4xy^2z+3xyz^2+6xz^3+5y^4+2y^3z+10y^2z^2+5yz^3+11z^4$	0	14	25
f_7	$x^4+6x^3y+10x^3z+3x^2y^2+2x^2yz+x^2z^2+10xy^3+8xy^2z+7xyz^2-xz^3+4y^4+10y^3z+y^2z^2+2yz^3+6z^4$	1	14	20
f_8	$x^4+5x^3y+5x^3z+5x^2y^2+10x^2yz+10x^2z^2+9xy^3-xy^2z+4xyz^2+8xz^3+10y^4+10y^3z+y^2z^2+9yz^3+8z^4$	0	14	28
f_9	$11x^3y+7x^3z+9x^2yz+8x^2z^2+6xy^3+xy^2z+4xyz^2+8xz^3+4y^3z+2y^2z^2+z^4$	0	17	25
f_{10}	$x^4+10x^3y+x^2y^2+4x^2yz+8xy^3+5xy^2z+8xyz^2+3xz^3+11y^4+3y^3z+10y^2z^2+2yz^3+9z^4$	1	15	23
f_{11}	$x^4+10x^3y+2x^3z+10x^2yz+3x^2z^2+3xy^3+7xy^2z+2xyz^2+2xz^3+5y^4+2y^3z+7y^2z^2+11yz^3+11z^4$	0	14	24
f_{12}	$x^4+6x^3y+3x^3z+7x^2y^2+2x^2yz+9x^2z^2+5xy^3+11xy^2z+8xyz^2+xz^3+2y^3z+y^2z^2+11yz^3+z^4$	1	15	25
f_{13}	$x^4+11x^3y+6x^3z+6x^2y^2+7x^2z^2+5xy^3+9xy^2z+9xyz^2+2xz^3+4y^3z+3y^2z^2+2yz^3$	1	15	26
f_{14}	$x^4+9x^3y+x^2y^2+4x^2yz+x^2z^2+11xy^2z+9xyz^2+10xz^3+10y^4+11yz^3-z^4$	1	15	25
f_{15}	$x^4+x^3y+6x^3z+6x^2y^2+3x^2yz+x^2z^2+9xy^3+8xy^2z+10xyz^2+9xz^3+4y^4+9y^3z+2y^2z^2+5yz^3$	1	16	26
f_{16}	$x^4+5x^3y+10x^3z+11x^2y^2+7x^2yz-x^2z^2+4xy^3+4xy^2z+11xyz^2+3xz^3-y^4+10y^3z+2y^2z^2+2yz^3+8z^4$	1	14	25
f_{17}	$x^4+3x^3y+9x^3z+11x^2y^2+3x^2yz+7xy^3+9xy^2z-xyz^2+5xz^3+4y^4+y^3z+2y^2z^2+5yz^3+5z^4$	0	14	25
f_{18}	$10x^3y+3x^3z+2x^2y^2+6x^2yz+4xy^3+10xy^2z+4xyz^2+4xz^3+y^4+4y^3z+6y^2z^2+7z^4$	2	15	25
f_{19}	$x^4+3x^3y+x^3z+2x^2y^2+x^2yz+9x^2z^2+4xy^3+4xy^2z+xyz^2+11xz^3-y^4+7y^3z+6y^2z^2+6yz^3+4z^4$	2	16	26
f_{20}	$x^4+3x^3y+10x^3z+7x^2y^2+x^2yz+5x^2z^2-xy^3+5xy^2z+11xyz^2+8xz^3-y^4+7y^3z+6y^2z^2+5yz^3+11z^4$	1	15	25
f_{21}	$x^4+2x^3y+8x^3z+10x^2y^2+10x^2yz-xy^3+6xy^2z+10xyz^2+8xz^3+6y^4+9y^3z+6yz^3+4z^4$	0	14	25

f_{22}	$x^4+6x^3y+5x^3z+7x^2y^2+9x^2yz+2x^2z^2-xy^3+3xy^2z+xz^3+5y^4+y^3z+5y^2z^2+9yz^3+3z^4$	0	14	23
f_{23}	$x^4-x^3y+4x^3z+6x^2y^2+11x^2yz+5x^2z^2-xy^3+8xy^2z+4xyz^2+10xz^3+10y^4+7y^3z+8y^2z^2+7yz^3+8z^4$	2	15	24
f_{24}	$x^4+8x^3y+x^3z+11x^2y^2+10x^2z^2+6xy^3+5xy^2z+2xyz^2+9xz^3+9y^2z^2+2yz^3+11z^4$	0	16	26
f_{25}	$x^4+3x^3y+9x^3z+11x^2y^2+3x^2yz+7xy^3+9xy^2z-xyz^2+5xz^3+4y^4+y^3z+2y^2z^2+5yz^3+5z^4$	0	14	25
f_{26}	$x^4+4x^3y+x^3z+4x^2y^2+3x^2yz+8x^2z^2+10xy^3+3xy^2z+2xyz^2+10xz^3+6y^4+3y^3z+11y^2z^2+10yz^3+11z^4$	2	14	24
f_{27}	$x^4+8x^3y-x^3z+5x^2y^2+2x^2yz+7x^2z^2+6xy^3+11xy^2z+7xyz^2+4xz^3-y^4+6y^3z+9y^2z^2+z^4$	0	14	22
f_{28}	$x^4-x^3y+4x^3z+6x^2y^2+11x^2yz+5x^2z^2-xy^3+8xy^2z+4xyz^2+10xz^3+10y^4+7y^3z+8y^2z^2+7yz^3+8z^4$	2	15	24
f_{29}	$x^4+3x^3y+9x^3z+11x^2y^2+3x^2yz+7xy^3+9xy^2z-xyz^2+5xz^3+4y^4+y^3z+2y^2z^2+5yz^3+5z^4$	0	15	25
f_{30}	$x^4+x^3y+6x^2yz+5x^2z^2+5xy^3+8xyz^2+xz^3+6y^4+8y^3z+2y^2z^2+2yz^3+11z^4$	0	14	23
f_{31}	$x^4+3x^3y+x^2y^2-x^2yz+7x^2z^2+2xy^3+4xy^2z+3xyz^2+9xz^3+10y^4+11y^3z+7y^2z^2+5yz^3+10z^4$	1	14	23
f_{32}	$x^4+3x^3y+x^2y^2-x^2yz+7x^2z^2+2xy^3+4xy^2z+3xyz^2+9xz^3+10y^4+11y^3z+7y^2z^2+5yz^3+10z^4$	1	14	23
f_{33}	$x^4+3x^3y+7x^3z+8x^2y^2+10x^2yz+10x^2z^2+11xy^3-xy^2z+3xyz^2+9xz^3+4y^4+10y^3z-y^2z^2+8yz^3+3z^4$	2	14	24
f_{34}	$x^4+3x^3y+7x^3z+8x^2y^2+10x^2yz+10x^2z^2+11xy^3-xy^2z+3xyz^2+9xz^3+4y^4+10y^3z-y^2z^2+8yz^3+3z^4$	2	15	24
f_{35}	$x^4-x^3y+3x^3z+11x^2y^2+x^2yz+9x^2z^2+3xy^3-xy^2z+5xyz^2+10y^4+y^3z+9y^2z^2+6z^4$	2	14	28
f_{36}	$x^4+5x^3y+5x^3z+5x^2y^2+10x^2yz+10x^2z^2+9xy^3-xy^2z+4xyz^2+8xz^3+10y^4+10y^3z+y^2z^2+9yz^3+8z^4$	0	14	28
f_{37}	$x^4+9x^3y+3x^3z+4x^2y^2+3x^2yz+x^2z^2+7xy^3+3xy^2z+2xyz^2+xz^3+5y^4+2y^3z+3y^2z^2+yz^3+3z^4$	1	16	24
f_{38}	$x^4+11x^3y+6x^3z+6x^2y^2+7x^2z^2+5xy^3+9xy^2z+9xyz^2+2xz^3+4y^3z+3y^2z^2+2yz^3$	1	14	26
f_{39}	$x^4+x^3y+6x^3z+6x^2y^2+3x^2yz+x^2z^2+9xy^3+8xy^2z+10xyz^2+9xz^3+4y^4+9y^3z+2y^2z^2+5yz^3$	1	15	26
f_{40}	$x^4+x^3y+6x^3z+6x^2y^2+3x^2yz+x^2z^2+9xy^3+8xy^2z+10xyz^2+9xz^3+4y^4+9y^3z+2y^2z^2+5yz^3$	1	15	26

f_{41}	$10x^3y+3x^3z+2x^2y^2+6x^2yz+4xy^3+10xy^2z+4xyz^2+4xz^3+y^4+4y^3z+6y^2z^2+7z^4$	2	15	25
f_{42}	$x^4+3x^3y+10x^3z+7x^2y^2+x^2yz+5x^2z^2-xy^3+5xy^2z+11xyz^2+8xz^3-y^4+7y^3z+6y^2z^2+5yz^3+11z^4$	1	14	25
f_{43}	$x^4+2x^3y+8x^3z+10x^2y^2+10x^2yz-xy^3+6xy^2z+10xyz^2+8xz^3+6y^4+9y^3z+6yz^3+4z^4$	0	14	25
f_{44}	$x^4-x^3y+4x^3z+6x^2y^2+11x^2yz+5x^2z^2-xy^3+8xy^2z+4xyz^2+10xz^3+10y^4+7y^3z+8y^2z^2+7yz^3+8z^4$	1	14	24

5.7 Quartic curves for complete $(29;4)$-arcs

In these statistics, there are 100 quartic curves. It took 2467 msc. The number of associated quartic curves of the complete $(29;4)$-arcs that do not have inflexion points is 37. There are 63 curves whose number I of inflexions is $1,2$ or 3. In addition, the number $|\mathscr{C}\cap\mathscr{K}|$ for the 100 quartic curves is between 14 and 18. Also, the number N_1 of rational points ranges from 18 to 30. It took 6102 msec. The statistics are shown in Table 5.6.

Table 5.6: **Quartic curves for complete (29;4)-arcs**

Symbol	$\mathscr{C}(f_i)$	I	$\lvert\mathscr{C}\cap\mathscr{K}\rvert$	N_1
f_1	$x^4-x^3y+11x^3z+5x^2y^2+6x^2yz+9x^2z^2+10xy^2z+4xyz^2+8xz^3+3y^4+10y^3z+4y^2z^2-yz^3+6z^4$	0	15	27
f_2	$x^4+5x^3y+4x^3z+5x^2y^2+11x^2yz+9x^2z^2+10xy^3+4xy^2z-xyz^2+4xz^3+3y^4-y^3z+y^2z^2+9yz^3+3z^4$	0	14	23
f_3	$x^4+11x^3y+6x^3z+3x^2y^2-x^2yz+2x^2z^2+5xy^3+9xy^2z+xyz^2-xz^3+11y^4-y^3z+y^2z^2+6yz^3+2z^4$	1	16	27
f_4	$x^4+3x^3y+4x^3z+5x^2y^2+4x^2yz+9x^2z^2+2xy^3+3xy^2z+9xyz^2+xz^3+y^4+y^3z+4y^2z^2+9yz^3+10z^4$	2	17	25
f_5	$x^4+10x^3z+6x^2y^2+2x^2z^2+4xy^3+3xy^2z+7xyz^2+3xz^3+6y^4+3y^3z+y^2z^2+4z^4$	1	15	27
f_6	$x^4-x^3y+2x^2y^2+2x^2yz+9x^2z^2+5xy^3+4xy^2z+2xyz^2+4y^3z+6y^2z^2-yz^3+7z^4$	0	17	24
f_7	$x^4+11x^3y+8x^3z+9x^2y^2+x^2z^2+2xy^3+8xy^2z+5xyz^2+3xz^3+4y^4+7y^3z+7y^2z^2+11z^4$	1	16	29

f_8	$x^4+9x^3y+6x^3z+9x^2y^2+3x^2yz+8x^2z^2+xy^3+6xyz^2+9xz^3+6y^4+4y^3z+3y^2z^2+9yz^3+3z^4$	0	15	25
f_9	$x^4+9x^3y-x^3z+10x^2y^2+9x^2yz+8x^2z^2+9xy^3+2xy^2z+8xyz^2+5xz^3+7y^4+7y^3z+11y^2z^2+10yz^3+8z^4$	0	14	26
f_{10}	$x^4+x^3z+5x^2y^2+5x^2yz-x^2z^2+2xy^3+11xy^2z+6xyz^2+9xz^3+9y^4-y^3z-y^2z^2+3yz^3+4z^4$	0	16	29
f_{11}	$x^4+x^3y+4x^3z+7x^2y^2+8x^2yz-x^2z^2+xy^3+xy^2z+7xz^3+3y^4+7y^3z+5y^2z^2+4yz^3+7z^4$	3	15	26
f_{12}	$x^4+9x^3y-x^3z+11x^2yz+4x^2z^2+7xy^3+4xy^2z+6xyz^2+7y^4-y^3z+8y^2z^2+7yz^3+z^4$	1	16	25
f_{13}	$x^4+5x^3y+7x^3z+9x^2y^2-x^2yz+x^2z^2+11xy^3+7xy^2z+3xyz^2+8xz^3+9y^4+5y^3z+9y^2z^2+11yz^3+4z^4$	0	15	25
f_{14}	$x^4+3x^3y+4x^3z+8x^2y^2+10x^2yz+8x^2z^2+6xy^3-xy^2z+8xyz^2+5xz^3+11y^4+7y^3z+5y^2z^2+8yz^3+z^4$	2	15	23
f_{15}	$x^4+4x^3y+5x^3z+9x^2y^2+11x^2yz+5xy^3+2xy^2z+5xyz^2+5xz^3+3y^3z+10yz^3+3z^4$	1	17	26
f_{16}	$x^4+7x^3y+2x^3z+6x^2y^2+10x^2yz+10x^2z^2+8xy^3+6xy^2z+7xyz^2+9xz^3+3y^3z-y^2z^2+11yz^3+8z^4$	0	16	28
f_{17}	$x^4+6x^3y+8x^3z+10x^2y^2+9x^2yz+7x^2z^2+3xy^3+4xy^2z+9xyz^2+9xz^3+11y^4+3y^3z+yz^3+8z^4$	2	15	25
f_{18}	$x^4+10x^3y+10x^2y^2+4x^2yz+10x^2z^2+2xy^2z+9xyz^2+9xz^3+7y^4+11y^3z+y^2z^2+2yz^3+2z^4$	0	14	30
f_{19}	$x^4+9x^3y+7x^2y^2+x^2yz+4x^2z^2+11xy^3-xy^2z+8xyz^2-xz^3+2y^4-y^3z+10y^2z^2+3z^4$	1	16	28
f_{20}	$x^4+10x^3y+3x^3z+6x^2y^2+7x^2z^2+xy^3+xy^2z-xyz^2+6xz^3+8y^4+10y^3z+7y^2z^2-yz^3+7z^4$	0	15	28
f_{21}	$x^4+7x^3y-x^3z+3x^2y^2+x^2yz+3x^2z^2+10xy^3+5xy^2z+2xyz^2+xz^3+11y^4+5y^3z+8y^2z^2+6yz^3+6z^4$	2	14	20
f_{22}	$x^4+10x^3y+9x^3z+8x^2y^2+10x^2yz+11x^2z^2+9xy^2z+10xyz^2+6xz^3+6y^3z+3y^2z^2+5yz^3+7z^4$	1	15	26
f_{23}	$x^4+2x^3y+10x^3z+3x^2y^2+11x^2yz-x^2z^2+4xy^3+8xy^2z+6xz^3+10y^4+4y^3z+5y^2z^2+2yz^3+4z^4$	0	15	24
f_{24}	$x^4+5x^3y+7x^3z+8x^2y^2+4x^2yz+3x^2z^2+6xy^3+2xy^2z+8xyz^2+5xz^3+4y^4+4y^3z+10y^2z^2+yz^3+z^4$	2	14	24
f_{25}	$x^4+5x^3z+10x^2y^2+10x^2yz+8x^2z^2+xy^3+9xy^2z+xyz^2+3xz^3+4y^4+2y^3z+2y^2z^2+7yz^3+10z^4$	1	15	23

f_{26}	$x^4+10x^3y+6x^3z+9x^2y^2+4x^2yz+x^2z^2+3xy^3+6xy^2z+3xyz^2+7xz^3+5y^4+10y^3z+4y^2z^2-yz^3+2z^4$	1	15	28
f_{27}	$x^4+3x^3y+8x^3z+11x^2y^2+8x^2yz+9x^2z^2+3xy^3+9xy^2z+7xyz^2+11xz^3-y^4+3y^3z+9yz^3+5z^4$	2	14	29
f_{28}	$x^4+7x^3y+6x^3z+10x^2y^2+5x^2yz+7x^2z^2+xy^3+10xy^2z+6xyz^2+xz^3+9y^3z-y^2z^2+5yz^3+9z^4$	1	15	24
f_{29}	$6x^3y+10x^3z+10x^2y^2+5x^2yz+3x^2z^2+11xy^3+2xy^2z+4xyz^2-xz^3+y^4+6y^3z+3y^2z^2+3yz^3+6z^4$	2	16	28
f_{30}	$x^4+7x^3z+4x^2y^2+10x^2z^2+10xy^3+xy^2z+11xyz^2+7xz^3+4y^4+8y^3z+yz^3+9z^4$	0	15	23
f_{31}	$x^4+7x^3z+4x^2y^2+10x^2z^2+10xy^3+xy^2z+11xyz^2+7xz^3+4y^4+8y^3z+yz^3+9z^4$	0	15	23
f_{32}	$x^4+5x^3y+6x^3z+9x^2y^2+3x^2yz+11x^2z^2+5xy^3+11xy^2z+9xyz^2+11xz^3+y^4+8y^3z+8y^2z^2+8yz^3+6z^4$	1	14	24
f_{33}	$5x^3y+2x^3z+5x^2y^2+5x^2yz-x^2z^2-xy^3+11xy^2z+7xyz^2+2xz^3+y^4+8y^3z+10y^2z^2+3yz^3+8z^4$	2	15	26
f_{34}	$x^4+9x^3y+4x^3z-x^2y^2+7x^2yz+4x^2z^2-xy^3+3xy^2z+5xyz^2+8xz^3+9y^4+y^3z+4y^2z^2-yz^3$	2	14	20
f_{35}	$x^4+5x^3y+7x^3z+9x^2y^2-x^2yz+x^2z^2+11xy^3+7xy^2z+3xyz^2+8xz^3+9y^4+5y^3z+9y^2z^2+11yz^3+4z^4$	0	15	25
f_{36}	$x^4+2x^3y+11x^3z+11x^2y^2+7x^2yz-x^2z^2+6xy^3+4xy^2z+7xyz^2+7xz^3-y^4+9y^2z^2+2yz^3$	1	14	22
f_{37}	$x^4+3x^3y+4x^3z+8x^2y^2+10x^2yz+10x^2z^2-xy^3+11xy^2z+11xyz^2+5xz^3+7y^4+8y^3z+8y^2z^2+6yz^3+8z^4$	0	16	27
f_{38}	$x^4+11x^3y+6x^3z+6x^2y^2+7x^2z^2+5xy^3+9xy^2z+9xyz^2+2xz^3+4y^3z+3y^2z^2+2yz^3$	1	15	26
f_{39}	$x^4+x^3y+8x^3z+5x^2y^2+5x^2yz+4x^2z^2-xy^3+7xy^2z+8xyz^2+xz^3+9y^4+10y^3z+2y^2z^2+3yz^3+2z^4$	2	15	24
f_{40}	$x^3y+6x^3z+8x^2y^2+4x^2yz-x^2z^2+2xy^3+2xy^2z+2xz^3+7y^3z+10y^2z^2+11yz^3$	1	15	27
f_{41}	$x^4+5x^3y+11x^3z+8x^2y^2+7x^2yz+5x^2z^2+xy^3+11xy^2z+11xyz^2+y^4+7y^3z+2y^2z^2-yz^3+9z^4$	3	15	27
f_{42}	$x^4+2x^3y+8x^3z+6x^2y^2+3x^2yz+3x^2z^2+xy^3+2xy^2z+8xyz^2+4xz^3+5y^4+2y^3z+9y^2z^2$	2	15	23
f_{43}	$x^4+3x^3y+11x^3z+6x^2y^2+8x^2yz+8x^2z^2+2xy^3+11xy^2z+7xyz^2+10xz^3+10y^4+9y^3z+y^2z^2+4yz^3$	2	14	25

f_{44}	$x^4+x^3y+x^3z+8x^2y^2+5x^2yz+4xy^2z+11xyz^2+9xz^3+8y^4+y^3z+7y^2z^2+yz^3+7z^4$	1	14	18
f_{45}	$x^4+9x^3y+x^3z+5x^2y^2+4x^2yz+5x^2z^2+9xy^3+8xy^2z+11xz^3+y^4+4y^3z+7y^2z^2$	0	16	26
f_{46}	$x^4+4x^3y+x^3z+9x^2y^2+9x^2z^2+7xy^3+7xy^2z+9xyz^2+3xz^3+5y^4+9y^3z+9y^2z^2+8yz^3+5z^4$	1	14	26
f_{47}	$x^4+4x^3y+6x^3z+8x^2y^2-x^2yz-x^2z^2+8xy^3+9xy^2z+6xyz^2+7xz^3+3y^3z+7y^2z^2+5yz^3$	0	16	28
f_{48}	$x^4+5x^3y+11x^3z+9x^2y^2+4x^2yz-x^2z^2+7xy^3+6xy^2z+10xyz^2+xz^3+5y^4+7y^3z+6y^2z^2+9yz^3$	3	18	30
f_{49}	$x^4+5x^3y+6x^2y^2+3x^2yz+2x^2z^2+3xy^3+11xy^2z+9xyz^2+8y^4+6y^3z+8y^2z^2+4yz^3+3z^4$	1	14	21
f_{50}	$x^3y+6x^3z+10x^2y^2+7x^2yz+10xy^3+9xyz^2+6xz^3+10y^3z+y^2z^2+5yz^3$	2	14	26
f_{51}	$x^3y+5x^3z+6x^2y^2+5x^2yz+6x^2z^2-xy^3+11xy^2z+2xz^3+9y^3z+10y^2z^2+11yz^3$	0	14	28
f_{52}	$x^4+10x^3y+10x^2y^2+4x^2yz+10x^2z^2+2xy^2z+9xyz^2+9xz^3+7y^4+11y^3z+y^2z^2+2yz^3+2z^4$	1	17	30
f_{53}	$x^4+11x^3y+6x^3z+6x^2y^2+7x^2z^2+5xy^3+9xy^2z+9xyz^2+2xz^3+4y^3z+3y^2z^2+2yz^3$	1	14	26
f_{54}	$x^4+11x^3y-x^3z+6x^2yz+3x^2z^2+xy^3+11xy^2z+9xz^3+11y^4-y^3z+9z^4$	1	14	25
f_{55}	$x^4+11x^3y+2x^3z+5x^2y^2+3x^2yz+6x^2z^2+9xy^3-xy^2z+2xyz^2+10xz^3+3y^4+2y^3z+yz^3+2z^4$	1	15	25
f_{56}	$x^4+x^3y+x^3z+4x^2y^2-x^2yz+6x^2z^2+9xy^2z+3xyz^2+5xz^3+8y^4+9y^3z+9y^2z^2-yz^3+5z^4$	0	16	28
f_{57}	$x^4+3x^3y+8x^3z+10x^2y^2+5x^2yz+10x^2z^2+9xy^3-xyz^2+xz^3+7y^3z+2y^2z^2+10yz^3$	1	14	25
f_{58}	$x^4+9x^3y+4x^3z+10x^2y^2+2x^2yz+7xy^3+4xy^2z+11xyz^2+4xz^3+6y^4-y^3z+3y^2z^2+11yz^3+7z^4$	2	15	28
f_{59}	$x^4+8x^3y+9x^3z+9x^2yz-x^2z^2+5xy^3-xy^2z+4xyz^2+10xz^3+10y^4+7y^3z+5y^2z^2+10yz^3+3z^4$	2	15	25
f_{60}	$x^4-x^3y+3x^3z+4x^2yz+4x^2z^2+10xy^3+9xy^2z+8xyz^2+2xz^3+9y^4+2y^3z+3y^2z^2+11yz^3+4z^4$	0	15	27
f_{61}	$x^4+11x^3y+8x^3z+7x^2y^2+3x^2yz+x^2z^2+7xy^3+8xy^2z+8xyz^2-xz^3+5y^4+11y^3z+4y^2z^2+7z^4$	3	16	28
f_{62}	$x^4-x^3y+4x^3z+8x^2y^2+2x^2yz+5x^2z^2+10xy^2z+8xyz^2+2xz^3+2y^4+5y^2z^2+9yz^3+7z^4$	1	15	25
f_{63}	$x^4+5x^3y+7x^2y^2+4x^2yz+11x^2z^2+6xy^3+10xy^2z+10xz^3+8y^4+4y^3z+5y^2z^2+9yz^3+11z^4$	0	14	24

f_{64}	$x^4+x^3y+3x^3z+10x^2y^2+9x^2yz+x^2z^2+10xy^3+7xy^2z-xyz^2-xz^3+11y^3z+8y^2z^2+11yz^3+7z^4$	0	16	23
f_{65}	$x^4+3x^3y+x^3z-x^2y^2+8x^2yz+2x^2z^2+2xy^3+2xy^2z+3xyz^2+10xz^3+2y^4+11y^3z+10y^2z^2+7yz^3+7z^4$	1	15	24
f_{66}	$x^4+11x^3y+11x^3z-x^2y^2+11x^2yz+6x^2z^2-xy^3+3xy^2z+4xyz^2+7xz^3+9y^4+6y^3z+3z^4$	0	15	29
f_{67}	$x^4+10x^3y+10x^3z-x^2y^2+10x^2yz+4x^2z^2+2xy^3+5xy^2z+5xyz^2+11xz^3+9y^4+8y^3z+6y^2z^2+6yz^3+9z^4$	3	16	26
f_{68}	$x^4-x^3y+11x^3z+5x^2y^2+6x^2yz+9x^2z^2+10xy^2z+4xyz^2+8xz^3+3y^4+10y^3z+4y^2z^2-yz^3+6z^4$	0	15	27
f_{69}	$x^4-x^3y+11x^3z+5x^2y^2+6x^2yz+9x^2z^2+10xy^2z+4xyz^2+8xz^3+3y^4+10y^3z+4y^2z^2-yz^3+6z^4$	0	15	27
f_{70}	$x^4+11x^3y+11x^3z+10x^2y^2+8x^2yz+5x^2z^2+4xy^3+8xy^2z+3xyz^2+8xz^3+y^4+5y^3z+3y^2z^2+4yz^3+9z^4$	0	16	20
f_{71}	$x^4+3x^3y+4x^3z+5x^2y^2+4x^2yz+9x^2z^2+2xy^3+3xy^2z+9xyz^2+xz^3+y^4+y^3z+4y^2z^2+9yz^3+10z^4$	2	16	25
f_{72}	$x^4+10x^3z+6x^2y^2+2x^2z^2+4xy^3+3xy^2z+7xyz^2+3xz^3+6y^4+3y^3z+y^2z^2+4z^4$	1	14	27
f_{73}	$x^4+11x^3y+8x^3z+9x^2y^2+x^2z^2+2xy^3+8xy^2z+5xyz^2+3xz^3+4y^4+7y^3z+7y^2z^2+11z^4$	1	16	29
f_{74}	$x^4+9x^3y+6x^3z+9x^2y^2+3x^2yz+8x^2z^2+xy^3+6xyz^2+9xz^3+6y^4+4y^3z+3y^2z^2+9yz^3+3z^4$	0	14	25
f_{75}	$x^4+9x^3y+6x^3z+9x^2y^2+3x^2yz+8x^2z^2+xy^3+6xyz^2+9xz^3+6y^4+4y^3z+3y^2z^2+9yz^3+3z^4$	0	14	25
f_{76}	$x^4+x^3z+5x^2y^2+5x^2yz-x^2z^2+2xy^3+11xy^2z+6xyz^2+9xz^3+9y^4-y^3z-y^2z^2+3yz^3+4z^4$	0	15	29
f_{77}	$x^4+x^3z+5x^2y^2+5x^2yz-x^2z^2+2xy^3+11xy^2z+6xyz^2+9xz^3+9y^4-y^3z-y^2z^2+3yz^3+4z^4$	0	15	29
f_{78}	$x^4+x^3z+5x^2y^2+5x^2yz-x^2z^2+2xy^3+11xy^2z+6xyz^2+9xz^3+9y^4-y^3z-y^2z^2+3yz^3+4z^4$	0	15	29
f_{79}	$x^4+4x^3y+5x^3z+9x^2y^2+11x^2yz+5xy^3+2xy^2z+5xyz^2+5xz^3+3y^3z+10yz^3+3z^4$	1	16	26
f_{80}	$x^4+6x^3y+8x^3z+10x^2y^2+9x^2yz+7x^2z^2+3xy^3+4xy^2z+9xyz^2+9xz^3+11y^4+3y^3z+yz^3+8z^4$	2	14	25
f_{81}	$x^4+10x^3y+9x^3z+8x^2y^2+10x^2yz+11x^2z^2+9xy^2z+10xyz^2+6xz^3+6y^3z+3y^2z^2+5yz^3+7z^4$	1	14	26

f_{82}	$x^4+7x^3y+6x^3z+10x^2y^2+5x^2yz+7x^2z^2+xy^3+10xy^2z+6xyz^2+xz^3+9y^3z-y^2z^2+5yz^3+9z^4$	0	14	24
f_{83}	$x^4+7x^3z+4x^2y^2+10x^2z^2+10xy^3+xy^2z+11xyz^2+7xz^3+4y^4+8y^3z+yz^3+9z^4$	0	15	23
f_{84}	$x^4+7x^3z+4x^2y^2+10x^2z^2+10xy^3+xy^2z+11xyz^2+7xz^3+4y^4+8y^3z+yz^3+9z^4$	0	15	23
f_{85}	$x^4+7x^3z+4x^2y^2+10x^2z^2+10xy^3+xy^2z+11xyz^2+7xz^3+4y^4+8y^3z+yz^3+9z^4$	0	15	23
f_{86}	$x^4+5x^3y+6x^3z+9x^2y^2+3x^2yz+11x^2z^2+5xy^3+11xy^2z+9xyz^2+11xz^3+y^4+8y^3z+8y^2z^2+8yz^3+6z^4$	1	14	24
f_{87}	$x^4+5x^3y+6x^3z+9x^2y^2+3x^2yz+11x^2z^2+5xy^3+11xy^2z+9xyz^2+11xz^3+y^4+8y^3z+8y^2z^2+8yz^3+6z^4$	1	14	24
f_{88}	$5x^3y+2x^3z+5x^2y^2+5x^2yz-x^2z^2-xy^3+11xy^2z+7xyz^2+2xz^3+y^4+8y^3z+10y^2z^2+3yz^3+8z^4$	2	16	26
f_{89}	$x^4-x^3y+6x^3z+5x^2y^2-x^2yz+7x^2z^2+10xy^3+5xy^2z-xyz^2+8xz^3+11y^4+4y^3z+5y^2z^2+10yz^3+7z^4$	1	14	27
f_{90}	$x^4+3x^3y+4x^3z+8x^2y^2+10x^2yz+10x^2z^2-xy^3+11xy^2z+11xyz^2+5xz^3+7y^4+8y^3z+8y^2z^2+6yz^3+8z^4$	0	16	27
f_{91}	$x^4+x^3y+8x^3z+5x^2y^2+5x^2yz+4x^2z^2-xy^3+7xy^2z+8xyz^2+xz^3+9y^4+10y^3z+2y^2z^2+3yz^3+2z^4$	2	15	24
f_{92}	$x^3y+6x^3z+8x^2y^2+4x^2yz-x^2z^2+2xy^3+2xy^2z+2xz^3+7y^3z+10y^2z^2+11yz^3$	1	15	27
f_{93}	$x^4+2x^3y+8x^3z+6x^2y^2+3x^2yz+3x^2z^2+xy^3+2xy^2z+8xyz^2+4xz^3+5y^4+2y^3z+9y^2z^2$	2	15	23
f_{94}	$x^4+3x^3y+11x^3z+6x^2y^2+8x^2yz+8x^2z^2+2xy^3+11xy^2z+7xyz^2+10xz^3+10y^4+9y^3z+y^2z^2+4yz^3$	2	14	25
f_{95}	$x^4+11x^3y+8x^3z+7x^2y^2+3x^2yz+x^2z^2+7xy^3+8xy^2z+8xyz^2-xz^3+5y^4+11y^3z+4y^2z^2+7z^4$	1	16	28
f_{96}	$x^4+5x^3y+11x^3z+9x^2y^2+4x^2yz-x^2z^2+7xy^3+6xy^2z+10xyz^2+xz^3+5y^4+7y^3z+6y^2z^2+9yz^3$	3	17	30
f_{97}	$x^4+5x^3y+11x^3z+9x^2y^2+4x^2yz-x^2z^2+7xy^3+6xy^2z+10xyz^2+xz^3+5y^4+7y^3z+6y^2z^2+9yz^3$	3	18	30
f_{98}	$x^4+5x^3y+11x^3z+9x^2y^2+4x^2yz-x^2z^2+7xy^3+6xy^2z+10xyz^2+xz^3+5y^4+7y^3z+6y^2z^2+9yz^3$	3	18	30
f_{99}	$x^4+5x^3y+6x^2y^2+3x^2yz+2x^2z^2+3xy^3+11xy^2z+9xyz^2+8y^4+6y^3z+8y^2z^2+4yz^3+3z^4$	1	14	21
f_{100}	$x^4+x^3y+x^3z+4x^2y^2-x^2yz+6x^2z^2+9xy^2z+3xyz^2+5xz^3+8y^4+9y^3z+9y^2z^2-yz^3+5z^4$	0	15	28

5.8 Quartic curves for complete $(30;4)$-arcs

The associated quartic curves of complete (30;4)-arcs that took 2653 msc are shown in Table 5.7. There are 45 curves among the 133 quartic curves that do not have inflexion points and the number of quartic curves that have inflexion points is 88. The number $|\mathscr{C} \cap \mathscr{K}|$ for the 133 quartic curves is between 14 and 19 and the number N_1 ranges between 19 and 30. It took 12310 msec.

Table 5.7: **Quartic curves for complete (30;4)-arcs**

Symbol	$\mathscr{C}(f_i)$	I	$\lvert\mathscr{C} \cap \mathscr{K}\rvert$	N_1
f_1	$3x^3y+2x^3z+10x^2y^2+3x^2z^2+7xy^3+7xy^2z+11xz^3+y^4+3y^3z+6y^2z^2+11yz^3+10z^4$	2	16	27
f_2	$x^4+3x^3y+2x^3z+5x^2y^2+3x^2yz+3x^2z^2-xy^3+2xy^2z+4xyz^2+xz^3+3y^3z+7y^2z^2+10yz^3+6z^4$	0	15	28
f_3	$x^4+9x^3y-x^3z+6x^2y^2+2x^2yz+7x^2z^2+5xy^3+4xy^2z+10xyz^2-xz^3-y^4+y^3z+11y^2z^2+2yz^3+10z^4$	1	16	24
f_4	$x^4+9x^3y-x^3z+2x^2y^2+7x^2yz+x^2z^2+3xy^2z+3xyz^2+xz^3+5y^4+y^3z+4y^2z^2+7yz^3+11z^4$	1	14	20
f_5	$x^4+6x^3y+2x^2y^2-x^2yz-x^2z^2+4xy^3+8xyz^2+6xz^3-y^4+7y^3z+5y^2z^2+11yz^3+11z^4$	1	15	27
f_6	$x^4+8x^3y+x^3z+5x^2y^2+8x^2yz+6x^2z^2+7xy^3-xyz^2+11y^4+4y^3z+10y^2z^2+7z^4$	2	14	21
f_7	$9x^3y-x^3z+5x^2y^2+7x^2yz+8xy^3+11xy^2z+6xyz^2+10xz^3+y^4+y^3z+5y^2z^2+3yz^3+3z^4$	0	15	25
f_8	$x^4+2x^3y+11x^3z-x^2y^2+x^2yz+5x^2z^2+11xy^3+2xy^2z-xyz^2+11xz^3+7y^4+y^3z+5y^2z^2+5yz^3+8z^4$	1	15	26
f_9	$9x^3y+8x^3z+2x^2y^2+9x^2yz+x^2z^2+11xy^3+11xy^2z+4xyz^2+3xz^3+y^4+10y^3z+2y^2z^2+7yz^3+9z^4$	1	17	30
f_{10}	$x^4+6x^3y+11x^2y^2+5x^2yz+9x^2z^2+10xy^3+5xy^2z+9xz^3+7y^3z+11y^2z^2+5yz^3-z^4$	3	18	23
f_{11}	$x^4+3x^3y+2x^2y^2+10x^2yz+10xy^3+10xy^2z+3xyz^2+7xz^3-y^4+y^3z+4y^2z^2+8yz^3+5z^4$	2	15	25
f_{12}	$x^4+9x^3z+10x^2y^2+8x^2yz+11x^2z^2+4xy^3+4xy^2z+4xyz^2+4xz^3-y^4+3y^3z+6y^2z^2+3yz^3+2z^4$	1	15	23
f_{13}	$x^4+8x^3y+4x^3z+10x^2y^2+8x^2yz+8x^2z^2-xy^3+8xz^3-y^4+y^3z+2y^2z^2+4yz^3+z^4$	0	15	24
f_{14}	$x^4+6x^3y+3x^3z+5x^2y^2+5x^2yz+10x^2z^2+3xy^3+4xy^2z+3xyz^2+7xz^3-y^4+5y^3z+5y^2z^2+11yz^3+9z^4$	0	15	26

f_{15}	$x^4+5x^3z+4x^2y^2+8x^2z^2+4xy^2z+6xyz^2+4xz^3+6y^4+3y^3z+7y^2z^2+2yz^3+9z^4$	0	14	25
f_{16}	$x^4+5x^3y+7x^3z+7x^2y^2+9x^2yz+8x^2z^2+8xy^3+11xy^2z+3xyz^2+11xz^3+4y^4+9y^3z+5y^2z^2+10yz^3+6z^4$	0	15	24
f_{17}	$x^4+6x^3y+2x^3z+5x^2y^2+11x^2yz+8x^2z^2+5xy^3+xy^2z+5xyz^2+10xz^3+4y^4+5y^3z-y^2z^2+yz^3$	1	16	26
f_{18}	$x^4+10x^3y+10x^3z+10x^2y^2+6x^2yz+11x^2z^2+8xy^3+4xy^2z+9xyz^2+4xz^3+8y^4+4y^3z+4y^2z^2-yz^3+2z^4$	2	14	25
f_{19}	$x^4+2x^3y+5x^2y^2+4x^2yz+11x^2z^2+2xy^3+4xy^2z+5xyz^2+10xz^3+7y^3z+9y^2z^2+5yz^3+6z^4$	2	15	25
f_{20}	$x^4+3x^3y+10x^3z+3x^2y^2+x^2yz+6x^2z^2+2xy^3+3xy^2z+9xyz^2+5xz^3+2y^4+4y^3z+5y^2z^2+yz^3+4z^4$	1	14	25
f_{21}	$x^4+5x^3y+6x^3z+10x^2yz+7x^2z^2+11xy^3+10xy^2z+4xyz^2+8xz^3+4y^4+5y^3z+2y^2z^2+5yz^3+3z^4$	1	14	24
f_{22}	$8x^3y+7x^3z+4x^2y^2+8x^2yz+4x^2z^2+8xy^3-xy^2z+xyz^2+2xz^3+y^4-y^3z+9y^2z^2+11yz^3+9z^4$	1	16	26
f_{23}	$x^3y+3x^3z+7x^2y^2-x^2yz+2x^2z^2+5xy^3+xy^2z+6xyz^2+10xz^3+y^4-y^3z-y^2z^2+yz^3+z^4$	1	16	22
f_{24}	$x^4+8x^3y+x^3z+5x^2y^2+8x^2yz+6x^2z^2+7xy^3-xyz^2+11y^4+4y^3z+10y^2z^2+7z^4$	2	14	21
f_{25}	$x^4+8x^3y+6x^3z+x^2y^2+3x^2yz+11xy^3+8xy^2z+xyz^2+8xz^3+9y^3z+y^2z^2+9yz^3+z^4$	1	18	24
f_{26}	$x^4+9x^3y+7x^3z+10x^2yz+6x^2z^2+11xy^3+10xy^2z+11xyz^2+7xz^3+6y^4+10y^3z+5y^2z^2+11yz^3+10z^4$	2	15	23
f_{27}	$x^4+x^3z+10x^2y^2+5x^2yz+x^2z^2+11xy^3+11xy^2z+11xyz^2+10y^4+8y^3z+8yz^3+5z^4$	0	14	19
f_{28}	$x^4+3x^3y+5x^2y^2+10x^2yz+6x^2z^2+5xy^3+7xy^2z+xyz^2+7xz^3+4y^4+3y^3z+5yz^3+2z^4$	0	15	24
f_{29}	$x^4+5x^3y+5x^3z+4x^2y^2+11x^2yz+x^2z^2+9xy^3+3xy^2z+8xyz^2+6xz^3+4y^4+8y^3z+2y^2z^2+3yz^3+5z^4$	1	15	26
f_{30}	$x^4+5x^3y+6x^3z+4x^2y^2+4x^2yz+x^2z^2+4xy^3+2xy^2z+10xyz^2+11xz^3+4y^4+8y^3z+y^2z^2+11yz^3-z^4$	0	14	23
f_{31}	$x^4+5x^3y+x^3z+2x^2y^2+5x^2z^2+5xy^3+10xy^2z+11xz^3+11y^4+4y^2z^2+4yz^3+6z^4$	1	16	27

f_{32}	$x^4+x^3y+7x^3z-x^2y^2+6x^2yz+2x^2z^2+xy^3+5xyz^2+3xz^3+11y^4+2y^3z+6y^2z^2+4yz^3+7z^4$	3	15	28
f_{33}	$x^4+9x^3y+2x^3z-x^2y^2+3x^2yz+xy^3+xy^2z+4xyz^2+3xz^3-y^4+4y^3z+7y^2z^2+4yz^3-z^4$	2	15	30
f_{34}	$x^4+11x^3y+x^3z-x^2y^2+2x^2yz-x^2z^2+11xy^2z+xyz^2+5xz^3+7y^4+6y^3z+7y^2z^2+5yz^3$	2	15	21
f_{35}	$x^4+7x^3y+10x^3z+x^2y^2+10x^2yz+3x^2z^2+4xy^3+5xy^2z+9xyz^2+7xz^3+8y^4+10y^3z+10y^2z^2+9yz^3$	1	15	25
f_{36}	$x^4+11x^3y+10x^3z-x^2y^2+7x^2yz+10x^2z^2+2xy^3+2xy^2z+8y^4+7y^3z+5y^2z^2+11z^4$	2	15	20
f_{37}	$x^4+11x^3y+10x^3z+5x^2y^2+8x^2yz+2x^2z^2+2xy^3+4xy^2z-xz^3+4y^3z+4y^2z^2+8yz^3-z^4$	0	16	25
f_{38}	$x^4+5x^3y+9x^2yz+2x^2z^2+10xy^2z+7xz^3+3y^4+11y^3z+y^2z^2+3yz^3$	2	16	29
f_{39}	$x^4+5x^3y+5x^3z-x^2y^2+3x^2yz-x^2z^2+10xy^3-xy^2z+8xyz^2+xz^3+y^4+11y^3z+8y^2z^2+7yz^3-z^4$	1	15	27
f_{40}	$x^3y+7x^3z+4x^2y^2+7x^2z^2+7xy^3+8xy^2z+10xyz^2+8xz^3+y^4+4y^3z+11y^2z^2+9yz^3+3z^4$	1	16	28
f_{41}	$x^4+x^3y+6x^3z+11x^2y^2+10x^2z^2+5xy^3+11xy^2z+6xyz^2+3xz^3+9y^4+4y^3z+6y^2z^2+11yz^3+11z^4$	1	16	26
f_{42}	$x^4+5x^3y+x^3z+5x^2y^2-x^2yz+8xy^3+8xy^2z+xyz^2+2xz^3+2y^4+11y^3z+6y^2z^2+4yz^3+2z^4$	1	16	28
f_{43}	$x^4+3x^3y+5x^3z+11x^2y^2+9x^2yz+4x^2z^2+2xy^2z+6xyz^2+10xz^3+4y^4+4y^3z+6y^2z^2+2yz^3+9z^4$	1	15	21
f_{44}	$x^4+4x^3y+2x^3z+3x^2y^2+7x^2yz+11x^2z^2+10xy^2z-xyz^2+3xz^3+3y^4+7y^3z+2y^2z^2+10yz^3+10z^4$	0	17	26
f_{45}	$x^4+2x^3y+8x^3z+x^2y^2+11x^2yz+10x^2z^2+2xy^3+11xy^2z+5xyz^2+4xz^3+4y^4+4y^3z+8y^2z^2+11yz^3+4z^4$	3	17	25
f_{46}	$x^4+x^3y+5x^3z+7x^2y^2-x^2yz+6x^2z^2+11xy^3+8xy^2z+10xyz^2+8xz^3+6y^4+4y^3z+4y^2z^2+6yz^3-z^4$	1	19	29
f_{47}	$x^4+6x^3y+6x^3z+7x^2y^2+x^2yz+2x^2z^2-xy^3+7xy^2z+5xyz^2+4xz^3+11y^4+2y^3z-y^2z^2+10yz^3+6z^4$	0	14	21
f_{48}	$x^4+5x^3y+10x^3z+x^2y^2+4x^2yz+2x^2z^2+3xy^3+7xy^2z-xyz^2+6xz^3+9y^4+10y^3z+6y^2z^2+8z^4$	2	16	26

f_{49}	$x^4+5x^3y+x^3z+4x^2y^2+x^2yz-x^2z^2+10xy^2z+6xyz^2+10xz^3+5y^4+5y^3z+2y^2z^2+5yz^3+3z^4$	0	14	28
f_{50}	$x^4+4x^3y+8x^3z+5x^2y^2+5x^2yz+5x^2z^2+11xy^3+xy^2z+11xyz^2+9xz^3+2y^4+6y^3z+5y^2z^2+z^4$	1	14	22
f_{51}	$x^4+2x^3y+2x^3z+7x^2yz+10x^2z^2-xy^3+3xy^2z+3xyz^2+5xz^3+11y^4+6y^3z+9y^2z^2+9yz^3+11z^4$	1	15	29
f_{52}	$x^4+4x^3z+4x^2y^2+6x^2yz+10x^2z^2+10xy^3+xy^2z+4xyz^2+4xz^3+6y^4+11y^3z+11y^2z^2+7yz^3+7z^4$	0	14	25
f_{53}	$10x^3y+4x^3z+x^2y^2+11x^2yz-x^2z^2+3xy^3+xy^2z-xyz^2+6xz^3+y^4+2y^3z+10y^2z^2+11yz^3+10z^4$	2	18	25
f_{54}	$x^4+3x^3y+11x^3z+10x^2y^2+9x^2yz+7x^2z^2+9xy^3+10xy^2z+10xyz^2+xz^3+10y^4+8y^3z+10y^2z^2+10yz^3+2z^4$	2	15	20
f_{55}	$x^4+4x^3y+4x^3z+5x^2y^2+5x^2yz+7x^2z^2+6xy^3+6xy^2z+10xyz^2+9xz^3+3y^4+9y^3z+2y^2z^2+11yz^3+4z^4$	0	15	27
f_{56}	$7x^3y+3x^3z+8x^2y^2+5x^2yz+5x^2z^2+9xy^3-xyz^2+3xz^3+y^4+2y^3z+3yz^3+5z^4$	2	16	24
f_{57}	$x^4+3x^3y+3x^3z-x^2y^2+6x^2yz+8x^2z^2-xy^3+7xy^2z+6xyz^2+xz^3+5y^4+4y^3z+11yz^3+10z^4$	2	16	29
f_{58}	$x^4+4x^3y+8x^3z+5x^2y^2+5x^2yz+5x^2z^2+11xy^3+xy^2z+11xyz^2+9xz^3+2y^4+6y^3z+5y^2z^2+z^4$	1	14	22
f_{59}	$x^4+6x^3y+x^3z+4x^2y^2+5x^2yz+6x^2z^2+5xy^3+11xy^2z+xyz^2+3xz^3+8y^4+2y^3z-y^2z^2+4yz^3+3z^4$	0	14	27
f_{60}	$x^4+3x^3y-x^3z+11x^2y^2+2x^2yz+7x^2z^2+8xy^3+5xy^2z-xyz^2+6xz^3+3y^3z+5y^2z^2+10yz^3+10z^4$	1	15	26
f_{61}	$x^4+3x^3z+2x^2z^2+7xy^3+6xy^2z+4xyz^2+8xz^3+3y^4+7y^3z+y^2z^2+8yz^3+z^4$	0	16	22
f_{62}	$x^4+5x^3y-x^3z-x^2y^2+x^2yz-x^2z^2+3xy^3+3xy^2z-xyz^2+9xz^3+6y^4+10y^3z+9y^2z^2+2yz^3+8z^4$	1	15	22
f_{63}	$x^4+2x^3y+9x^3z+3x^2y^2+10x^2yz+4x^2z^2+10xy^3+4xy^2z+11xz^3+y^4+3y^3z+11y^2z^2+3yz^3$	1	15	25
f_{64}	$x^4+2x^3y+7x^3z+x^2y^2+4x^2yz+2x^2z^2+9xy^3+8xy^2z+7xyz^2-xz^3+11y^4+3y^3z+2yz^3+9z^4$	0	17	27
f_{65}	$x^4+2x^3y+7x^3z+x^2y^2+4x^2yz+2x^2z^2+9xy^3+8xy^2z+7xyz^2-xz^3+11y^4+3y^3z+2yz^3+9z^4$	0	17	27
f_{66}	$x^4+7x^3y+7x^3z+8x^2y^2+6x^2yz+7x^2z^2+3xy^3+9xy^2z+10xyz^2+xz^3+6y^4-y^3z-y^2z^2+6yz^3+6z^4$	0	14	23

f_{67}	$x^4+7x^3y+x^3z+5x^2y^2-x^2yz+8x^2z^2-xy^3+9xy^2z+3xyz^2+6xz^3+8y^4+8y^3z+y^2z^2-yz^3+2z^4$	0	16	19
f_{68}	$x^4+3x^3y+11x^3z-x^2y^2+4x^2yz+6x^2z^2+3xy^3+5xyz^2+10xz^3+3y^4-y^3z-y^2z^2+7yz^3$	2	15	28
f_{69}	$x^4+7x^3y+2x^3z+6x^2y^2+10x^2yz+10x^2z^2+8xy^3+6xy^2z+7xyz^2+9xz^3+3y^3z-y^2z^2+11yz^3+8z^4$	0	16	28
f_{70}	$x^4+7x^3y+2x^3z+7x^2y^2+x^2yz+11x^2z^2+3xy^3+7xy^2z+5xyz^2+9xz^3-y^4+3y^3z+7y^2z^2+6yz^3+6z^4$	0	14	26
f_{71}	$x^4+7x^3y+3x^3z+5x^2y^2+3x^2yz+x^2z^2+3xy^3+9xy^2z+7xyz^2+xz^3+11y^4+5y^3z+8y^2z^2+3yz^3+7z^4$	0	16	25
f_{72}	$x^4+4x^3y+3x^2yz+3xy^3+8xy^2z-xyz^2+2xz^3+7y^4+3y^3z+6y^2z^2+4yz^3-z^4$	2	15	23
f_{73}	$x^4+11x^3y+4x^3z+10x^2y^2-xy^2z+xyz^2+3xz^3+10y^4+3y^3z+4y^2z^2+7yz^3+11z^4$	1	15	26
f_{74}	$x^4+11x^3y+4x^3z+10x^2y^2-xy^2z+xyz^2+3xz^3+10y^4+3y^3z+4y^2z^2+7yz^3+11z^4$	1	15	26
f_{75}	$x^4+7x^3y+x^3z+5x^2y^2-x^2yz+8x^2z^2-xy^3+9xy^2z+3xyz^2+6xz^3+8y^4+8y^3z+y^2z^2-yz^3+2z^4$	0	16	19
f_{76}	$7x^3y+11x^3z+x^2yz+9xy^2z+6xyz^2+3xz^3+y^4+4y^3z+8yz^3+11z^4$	2	16	29
f_{77}	$x^4+10x^3y+3x^3z+9x^2y^2+4x^2yz+5x^2z^2+10xy^3+9xyz^2-xz^3+8y^4+y^3z+10y^2z^2+5yz^3+10z^4$	0	14	27
f_{78}	$x^4+9x^3y+10x^3z+9x^2y^2+x^2yz+8x^2z^2+3xy^3+10xy^2z-xyz^2+6xz^3-y^4+3y^3z-y^2z^2+5yz^3+8z^4$	0	15	27
f_{79}	$x^4+x^3y+11x^3z+4x^2y^2+2x^2yz+2x^2z^2+10xy^3+2xy^2z+8xyz^2+8xz^3+10y^4+7y^3z+5y^2z^2+3yz^3+3z^4$	1	15	26
f_{80}	$x^4+10x^3y+x^3z+8x^2y^2+2xy^3+3xy^2z+xyz^2+11xz^3+6y^4+11y^3z+5y^2z^2+6yz^3+8z^4$	1	14	19
f_{81}	$x^4+6x^3y+x^3z+5x^2y^2+5x^2yz+10x^2z^2+9xy^2z+5xyz^2+10xz^3+10y^3z+4y^2z^2+6yz^3+4z^4$	1	15	25
f_{82}	$x^4-x^3y+11x^3z+4x^2y^2+2x^2yz+11xy^3+7xy^2z+3xyz^2+10xz^3+3y^4+2y^3z+5y^2z^2+2yz^3+9z^4$	0	14	26
f_{83}	$x^4+10x^3y+6x^3z+5x^2y^2+10x^2yz+8x^2z^2+3xy^3+6xyz^2+3xz^3+2y^3z+6yz^3+5z^4$	2	17	28
f_{84}	$x^4+6x^3y+5x^3z+3x^2y^2+11x^2yz+5x^2z^2+5xy^3+xy^2z+xyz^2+11xz^3+7y^4+10y^3z+4y^2z^2+7yz^3+3z^4$	0	16	27
f_{85}	$3x^3y+2x^3z+10x^2y^2+3x^2z^2+7xy^3+7xy^2z+11xz^3+y^4+3y^3z+6y^2z^2+11yz^3+10z^4$	2	15	27
f_{86}	$3x^3y+2x^3z+10x^2y^2+3x^2z^2+7xy^3+7xy^2z+11xz^3+y^4+3y^3z+6y^2z^2+11yz^3+10z^4$	2	15	27

f_{87}	$3x^3y+2x^3z+10x^2y^2+3x^2z^2+7xy^3+7xy^2z+11xz^3+y^4+3y^3z+6y^2z^2+11yz^3+10z^4$	2	15	27
f_{88}	$x^4+3x^3y+2x^3z+5x^2y^2+3x^2yz+3x^2z^2-xy^3+2xy^2z+4xyz^2+xz^3+3y^3z+7y^2z^2+10yz^3+6z^4$	0	14	28
f_{89}	$x^4+3x^3y+2x^3z+5x^2y^2+3x^2yz+3x^2z^2-xy^3+2xy^2z+4xyz^2+xz^3+3y^3z+7y^2z^2+10yz^3+6z^4$	0	15	28
f_{90}	$x^4+9x^3y-x^3z+6x^2y^2+2x^2yz+7x^2z^2+5xy^3+4xy^2z+10xyz^2-xz^3-y^4+y^3z+11y^2z^2+2yz^3+10z^4$	1	15	24
f_{91}	$x^4+x^3y+11x^3z+11x^2yz-x^2z^2+3xy^3+6xy^2z-xyz^2+7xz^3+6y^4+7y^3z-y^2z^2+5z^4$	0	15	24
f_{92}	$x^4+x^3y+11x^3z+11x^2yz-x^2z^2+3xy^3+6xy^2z-xyz^2+7xz^3+6y^4+7y^3z-y^2z^2+5z^4$	0	16	24
f_{93}	$x^4+6x^3y+2x^2y^2-x^2yz-x^2z^2+4xy^3+8xyz^2+6xz^3-y^4+7y^3z+5y^2z^2+11yz^3+11z^4$	1	14	27
f_{94}	$x^4+4x^3y+4x^3z+4x^2y^2+2x^2yz+5x^2z^2+4xy^3+6xy^2z+8xyz^2+7xz^3+11y^4+3y^3z+y^2z^2+yz^3+5z^4$	3	16	30
f_{95}	$x^4+4x^3y+4x^3z+4x^2y^2+2x^2yz+5x^2z^2+4xy^3+6xy^2z+8xyz^2+7xz^3+11y^4+3y^3z+y^2z^2+yz^3+5z^4$	3	16	30
f_{96}	$x^4+4x^3y+4x^3z+4x^2y^2+2x^2yz+5x^2z^2+4xy^3+6xy^2z+8xyz^2+7xz^3+11y^4+3y^3z+y^2z^2+yz^3+5z^4$	2	15	30
f_{97}	$x^4+2x^3y+11x^3z-x^2y^2+x^2yz+5x^2z^2+11xy^3+2xy^2z-xyz^2+11xz^3+7y^4+y^3z+5y^2z^2+5yz^3+8z^4$	1	15	26
f_{98}	$9x^3y+8x^3z+2x^2y^2+9x^2yz+x^2z^2+11xy^3+11xy^2z+4xyz^2+3xz^3+y^4+10y^3z+2y^2z^2+7yz^3+9z^4$	1	16	30
f_{99}	$9x^3y+8x^3z+2x^2y^2+9x^2yz+x^2z^2+11xy^3+11xy^2z+4xyz^2+3xz^3+y^4+10y^3z+2y^2z^2+7yz^3+9z^4$	1	16	30
f_{100}	$x^4+6x^3y+11x^2y^2+5x^2yz+9x^2z^2+10xy^3+5xy^2z+9xz^3+7y^3z+11y^2z^2+5yz^3-z^4$	3	17	23
f_{101}	$x^4+6x^3y+11x^2y^2+5x^2yz+9x^2z^2+10xy^3+5xy^2z+9xz^3+7y^3z+11y^2z^2+5yz^3-z^4$	3	17	23
f_{102}	$x^4+3x^3y+2x^2y^2+10x^2yz+10xy^3+10xy^2z+3xyz^2+7xz^3-y^4+y^3z+4y^2z^2+8yz^3+5z^4$	2	15	25
f_{103}	$x^4+8x^3y+4x^3z+10x^2y^2+8x^2yz+8x^2z^2-xy^3+8xz^3-y^4+y^3z+2y^2z^2+4yz^3+z^4$	0	15	24
f_{104}	$x^4+9x^3y+9x^3z+11x^2y^2+11x^2yz+9x^2z^2+9xy^3+xy^2z+xyz^2+xz^3+3y^4+3y^3z+4y^2z^2+yz^3+8z^4$	2	15	27
f_{105}	$x^4+9x^3y+9x^3z+11x^2y^2+11x^2yz+9x^2z^2+9xy^3+xy^2z+xyz^2+xz^3+3y^4+3y^3z+4y^2z^2+yz^3+8z^4$	2	15	27
f_{106}	$x^4+5x^3y+7x^3z+7x^2y^2+9x^2yz+8x^2z^2+8xy^3+11xy^2z+3xyz^2+11xz^3+4y^4+9y^3z+5y^2z^2+10yz^3+6z^4$	0	15	24

f_{107}	$x^4+6x^3y+2x^3z+5x^2y^2+11x^2yz+8x^2z^2+5xy^3+xy^2z+5xyz^2+10xz^3+4y^4+5y^3z-y^2z^2+yz^3$	1	15	26
f_{108}	$x^4+5x^3y+10x^3z+x^2yz+8x^2z^2+7xy^3-xy^2z+6xyz^2+7xz^3+7y^4+10y^3z+9y^2z^2+3yz^3+8z^4$	2	14	23
f_{109}	$x^4+5x^3y+10x^3z+x^2yz+8x^2z^2+7xy^3-xy^2z+6xyz^2+7xz^3+7y^4+10y^3z+9y^2z^2+3yz^3+8z^4$	2	14	23
f_{110}	$x^4-x^3y+11x^3z+6x^2y^2+4x^2yz-x^2z^2+xy^3+xy^2z-xyz^2+2xz^3+4y^4+5y^3z+10y^2z^2+3yz^3+4z^4$	0	14	23
f_{111}	$x^4+11x^3y+11x^2y^2+8x^2yz+3x^2z^2+9xy^3+9xy^2z+11xyz^2+2xz^3+10y^4+9y^3z+y^2z^2+11yz^3+2z^4$	2	16	26
f_{112}	$x^4+11x^3y+11x^2y^2+8x^2yz+3x^2z^2+9xy^3+9xy^2z+11xyz^2+2xz^3+10y^4+9y^3z+y^2z^2+11yz^3+2z^4$	2	16	26
f_{113}	$x^4+7x^3y+10x^3z+11x^2y^2+4x^2yz+7x^2z^2+5xy^3+3xy^2z+7y^4+8y^3z+9y^2z^2+4yz^3+2z^4$	2	16	26
f_{114}	$x^4+9x^3y+7x^3z+10x^2yz+6x^2z^2+11xy^3+10xy^2z+11xyz^2+7xz^3+6y^4+10y^3z+5y^2z^2+11yz^3+10z^4$	2	14	23
f_{115}	$x^4+3x^3y+8x^3z+4x^2y^2+2x^2yz+11x^2z^2+9xy^3+2xy^2z+10xz^3+9y^4+6y^3z+4y^2z^2+3yz^3$	2	16	28
f_{116}	$x^4+3x^3y+5x^2y^2+10x^2yz+6x^2z^2+5xy^3+7xy^2z+xyz^2+7xz^3+4y^4+3y^3z+5yz^3+2z^4$	0	14	24
f_{117}	$7x^3y+8x^3z+10x^2y^2+5x^2yz+4x^2z^2+2xy^3+11xyz^2-xz^3+y^4+9y^3z+10y^2z^2+9yz^3+11z^4$	0	16	26
f_{118}	$x^4+5x^3y+x^3z+2x^2y^2+5x^2z^2+5xy^3+10xy^2z+11xz^3+11y^4+4y^2z^2+4yz^3+6z^4$	1	15	27
f_{119}	$x^4+7x^3y+10x^3z+x^2y^2+10x^2yz+3x^2z^2+4xy^3+5xy^2z+9xyz^2+7xz^3+8y^4+10y^3z+10y^2z^2+9yz^3$	1	14	25
f_{120}	$x^4+11x^3y+10x^3z+5x^2y^2+8x^2yz+2x^2z^2+2xy^3+4xy^2z-xz^3+4y^3z+4y^2z^2+8yz^3-z^4$	0	15	25
f_{121}	$x^4+5x^3y+x^3z+5x^2y^2-x^2yz+8xy^3+8xy^2z+xyz^2+2xz^3+2y^4+11y^3z+6y^2z^2+4yz^3+2z^4$	1	15	28
f_{122}	$x^4+4x^3y+2x^3z+3x^2y^2+7x^2yz+11x^2z^2+10xy^2z-xyz^2+3xz^3+3y^4+7y^3z+2y^2z^2+10yz^3+10z^4$	0	16	26
f_{123}	$x^4+2x^3y+8x^3z+x^2y^2+11x^2yz+10x^2z^2+2xy^3+11xy^2z+5xyz^2+4xz^3+4y^4+4y^3z+8y^2z^2+11yz^3+4z^4$	3	16	25
f_{124}	$x^4+x^3y+5x^3z+7x^2y^2-x^2yz+6x^2z^2+11xy^3+8xy^2z+10xyz^2+8xz^3+6y^4+4y^3z+4y^2z^2+6yz^3-z^4$	1	18	29
f_{125}	$3x^3y-x^2y^2-x^2yz-x^2z^2+5xy^3+xy^2z+11xz^3+y^4-y^3z+7y^2z^2+2yz^3+3z^4$	2	16	23
f_{126}	$x^4+3x^3y-x^3z+11x^2y^2+2x^2yz+7x^2z^2+8xy^3+5xy^2z-xyz^2+6xz^3+3y^3z+5y^2z^2+10yz^3+10z^4$	1	15	26

f_{127}	$x^4+5x^3y-x^3z-x^2y^2+x^2yz-x^2z^2+3xy^3+3xy^2z-xyz^2+9xz^3+6y^4+10y^3z+9y^2z^2+2yz^3+8z^4$	1	14	22
f_{128}	$x^4+7x^3y+x^3z+5x^2y^2-x^2yz+8x^2z^2-xy^3+9xy^2z+3xyz^2+6xz^3+8y^4+8y^3z+y^2z^2-yz^3+2z^4$	0	15	19
f_{129}	$x^4+7x^3y+2x^3z+7x^2y^2+x^2yz+11x^2z^2+3xy^3+7xy^2z+5xyz^2+9xz^3-y^4+3y^3z+7y^2z^2+6yz^3+6z^4$	0	14	26
f_{130}	$x^4+9x^3y+10x^3z+9x^2y^2+x^2yz+8x^2z^2+3xy^3+10xy^2z-xyz^2+6xz^3-y^4+3y^3z-y^2z^2+5yz^3+8z^4$	0	14	27
f_{131}	$x^4+9x^3y+10x^3z+9x^2y^2+x^2yz+8x^2z^2+3xy^3+10xy^2z-xyz^2+6xz^3-y^4+3y^3z-y^2z^2+5yz^3+8z^4$	0	14	27
f_{132}	$x^4+x^3y+11x^3z+4x^2y^2+2x^2yz+2x^2z^2+10xy^3+2xy^2z+8xyz^2+8xz^3+10y^4+7y^3z+5y^2z^2+3yz^3+3z^4$	1	15	26
f_{133}	$x^4+x^3y+3x^3z+6x^2y^2+4x^2yz+10x^2z^2+4xy^3+8xy^2z+7xyz^2-xz^3+3y^4+8y^3z+2y^2z^2+yz^3$	0	15	26

5.9 Quartic curves for complete $(31;4)$-arcs

There are 123 quartic curves associated to complete $(31;4)$-arcs. It took 2584 msc. The number of quartic curves that do not have inflexion points is 57. Here, 66 quartic curves have between 1 and 3 inflexion points. The number $|\mathscr{C}\cap\mathscr{K}|$ of rational points for the 123 curves is between 14 and 18. Also, the number N_1 of these curves ranges between 17 and 31. It took 10198 msec. The statistics are given in Table 5.8.

Table 5.8: **Quartic curves for complete (31;4)-arcs**

Symbol	$\mathscr{C}(f_i)$	I	$\lvert\mathscr{C}\cap\mathscr{K}\rvert$	N_1
f_1	$x^4+9x^3y+2x^3z+9x^2y^2+9x^2yz+3x^2z^2+11xy^3+7xy^2z+5xyz^2+11xz^3+8y^4+2y^3z-y^2z^2+4yz^3+5z^4$	2	15	27
f_2	$x^4+2x^3y+x^3z+5x^2y^2+4x^2yz+6x^2z^2+10xy^3+9xy^2z+4xyz^2+4xz^3+10y^4+11y^2z^2+11yz^3+8z^4$	3	14	21
f_3	$x^4+2x^3y+10x^3z+2x^2y^2-x^2yz+7x^2z^2+7xy^3+9xy^2z+7xz^3+5y^4+y^3z+10y^2z^2+9yz^3+4z^4$	1	15	25
f_4	$x^4+4x^3y-x^3z+7x^2y^2+x^2yz+8x^2z^2+10xy^3+10xy^2z+4xyz^2+4xz^3+5y^4+6y^3z+4y^2z^2+11yz^3+z^4$	0	16	24

f_5	$x^4+6x^3y+6x^3z+6x^2y^2+9x^2yz+7x^2z^2+3xy^3+3xy^2z+11xz^3+8y^4-y^3z+11y^2z^2+2yz^3-z^4$	1	15	24
f_6	$3x^3y+8x^3z+4x^2y^2+11x^2yz+10x^2z^2+3xy^3+xy^2z+8xz^3+y^4+11y^3z+10y^2z^2+8yz^3+7z^4$	0	15	24
f_7	$x^4+6x^3y+4x^3z+7x^2y^2+x^2yz+4x^2z^2+10xy^3+2xyz^2+8xz^3+3y^4+6y^3z+11y^2z^2+9yz^3+2z^4$	0	16	26
f_8	$x^4+8x^3y+6x^3z-x^2y^2+3x^2yz+3x^2z^2+7xy^3+5xy^2z+7xyz^2+3xz^3+4y^4-y^3z+9y^2z^2+10yz^3+6z^4$	1	14	27
f_9	$x^4-x^3y-x^3z+3x^2y^2+4x^2yz+8xy^3+2xy^2z+8xyz^2-xz^3-y^4+4y^3z+3y^2z^2+7z^4$	0	16	28
f_{10}	$x^4+x^3y+7x^3z+9x^2yz+4x^2z^2+9xy^3+10xy^2z+6xyz^2+2xz^3+4y^4+2y^3z+6yz^3$	0	16	29
f_{11}	$x^4+3x^3y+10x^3z+9x^2y^2+x^2yz+11x^2z^2+xy^3+3xy^2z-xyz^2+11xz^3+11y^4+5y^3z+2y^2z^2-yz^3+2z^4$	1	14	25
f_{12}	$x^4+2x^3y+2x^3z+3x^2y^2+3x^2yz+2x^2z^2+10xy^3+10xy^2z+6xyz^2+8xz^3+8y^3z+3yz^3+9z^4$	0	15	23
f_{13}	$-x^3y+7x^2y^2-x^2yz+6xy^3+4xy^2z+5xyz^2+3xz^3+y^4+10y^3z+8y^2z^2+yz^3+9z^4$	2	17	27
f_{14}	$3x^3y+3x^3z+11x^2y^2+5x^2yz+6x^2z^2+4xy^3+2xy^2z+10xyz^2-xz^3+y^4+3y^3z+4yz^3+2z^4$	1	15	18
f_{15}	$x^4+3x^3y-x^3z-x^2y^2+8x^2yz+3x^2z^2+8xy^3+7xy^2z+3xyz^2+xz^3+2y^4+2y^3z+6y^2z^2+6z^4$	1	15	25
f_{16}	$x^4+9x^3y+2x^3z+10x^2y^2+8x^2yz+5x^2z^2-xy^3+4xy^2z+5xyz^2+8xz^3+5y^4+10y^3z+3yz^3+4z^4$	0	14	26
f_{17}	$x^4+10x^3z-x^2y^2+9x^2yz+4x^2z^2+xy^3+xy^2z+6xyz^2+9xz^3+9y^4+4y^3z+6y^2z^2+8yz^3+11z^4$	0	17	26
f_{18}	$11x^3y+x^3z+9x^2yz+3x^2z^2+11xy^3+5xy^2z+4xyz^2+3xz^3+y^4+4y^3z+9yz^3+6z^4$	1	15	23
f_{19}	$8x^3y+10x^3z+6x^2y^2+10x^2yz+6x^2z^2+11xy^3+11xy^2z+9xyz^2+xz^3+y^4+11y^3z+2y^2z^2+5yz^3+4z^4$	0	16	20
f_{20}	$x^4+11x^3y+2x^3z+11x^2y^2+4x^2yz+7x^2z^2-xy^3+2xy^2z-xyz^2+xz^3+10y^4+6y^3z+9y^2z^2+8yz^3+z^4$	0	15	27
f_{21}	$x^4-x^3y+4x^3z+x^2y^2+3x^2yz+2x^2z^2+xy^3+4xy^2z+8xyz^2+xz^3+10y^4+3y^3z+6y^2z^2+7yz^3+9z^4$	0	14	21
f_{22}	$x^4+7x^3y+4x^3z+9x^2y^2+x^2yz+5x^2z^2+11xy^3+11xy^2z+7xyz^2+3xz^3+3y^4+10y^3z+3y^2z^2+10yz^3+4z^4$	0	14	17

f_{23}	$x^4+9x^3y-x^3z+x^2yz+3x^2z^2+8xy^3+2xy^2z+3xyz^2-xz^3+y^4+y^3z+2y^2z^2+4yz^3+6z^4$	1	17	27
f_{24}	$x^4+2x^3z+11x^2y^2-x^2yz+3x^2z^2-xy^3+10xy^2z+7xyz^2+3xz^3+2y^4+2y^3z+9y^2z^2+3yz^3+8z^4$	0	17	26
f_{25}	$x^4+x^3y+9x^3z+3x^2y^2+4x^2yz+4x^2z^2+10xy^3+9xy^2z+3xyz^2+6xz^3+9y^4-y^3z+3y^2z^2+9yz^3+8z^4$	1	16	23
f_{26}	$x^4+10x^3y+11x^3z+7x^2y^2+x^2yz+9x^2z^2+5xy^3+10xy^2z-xyz^2+2xz^3+y^4-yz^3+3z^4$	0	15	25
f_{27}	$x^4+11x^3y+2x^3z+9x^2y^2+4x^2yz+7x^2z^2+5xy^3+7xy^2z+8xyz^2+10xz^3+5y^4+2y^3z+y^2z^2+9yz^3+5z^4$	0	14	24
f_{28}	$x^4-x^3y+11x^3z-x^2y^2+9x^2yz-x^2z^2+5xy^3+xy^2z+3xyz^2+xz^3+7y^4+4y^3z+7y^2z^2+11yz^3+7z^4$	0	16	24
f_{29}	$x^4+7x^3y+4x^3z+x^2y^2+6x^2yz+6x^2z^2+11xy^3+11xy^2z+11xyz^2+8xz^3-y^4+4y^3z+4y^2z^2+3yz^3+3z^4$	1	14	23
f_{30}	$x^4+3x^3y+6x^3z+3x^2y^2+4x^2yz+xy^3+9xyz^2+11xz^3+2y^4+11y^3z+10y^2z^2+11yz^3+5z^4$	0	16	23
f_{31}	$x^4+9x^3y-x^3z+x^2yz+3x^2z^2+8xy^3+2xy^2z+3xyz^2-xz^3+y^4+y^3z+2y^2z^2+4yz^3+6z^4$	1	17	27
f_{32}	$x^4+4x^3y+2x^3z+3x^2y^2+9x^2yz+5x^2z^2+2xy^3+9xy^2z+4xyz^2+8xz^3+y^4+11y^3z+10y^2z^2+4z^4$	2	16	22
f_{33}	$x^4+7x^3y+3x^3z+11x^2y^2+6x^2yz-x^2z^2+5xy^3+10xy^2z-xyz^2+2y^4+10y^3z+3yz^3-z^4$	1	14	22
f_{34}	$x^4-x^3y+9x^3z+6x^2y^2+8x^2yz+4x^2z^2+8xy^3+4xy^2z+8xyz^2+4xz^3+3y^4+8y^3z+3y^2z^2-yz^3+8z^4$	2	14	24
f_{35}	$x^4+x^3z+6x^2y^2-x^2yz+5x^2z^2+9xy^3+9xy^2z+8xyz^2+xz^3+3y^3z+10y^2z^2-yz^3+8z^4$	0	16	27
f_{36}	$x^4+11x^3y+3x^3z+6x^2y^2+6x^2yz+6x^2z^2+11xy^3+10xy^2z-xyz^2+4xz^3+8y^3z+10y^2z^2+yz^3+6z^4$	2	16	25
f_{37}	$x^4+3x^3y+4x^3z+5x^2y^2+10x^2yz+7x^2z^2+9xy^3-xy^2z+2xyz^2+5xz^3+2y^3z+9y^2z^2+9yz^3$	3	15	28
f_{38}	$x^4+11x^3y+11x^3z+3x^2y^2+3x^2yz+x^2z^2+3xy^3+6xy^2z+5xz^3+4y^4+6y^3z+y^2z^2+3yz^3+7z^4$	1	15	28
f_{39}	$x^4+5x^3y+x^3z+8x^2y^2+4x^2yz+8xy^3+11xy^2z+11xyz^2+4xz^3-y^4+y^3z+6y^2z^2+4yz^3+3z^4$	2	14	24
f_{40}	$x^4+7x^3y+9x^3z+4x^2yz+7x^2z^2+4xy^3+4xy^2z+11xyz^2+7xz^3+10y^4+9y^3z+4y^2z^2+5yz^3+8z^4$	2	14	22

f_{41}	$x^4+7x^3y+3x^3z+9x^2y^2+3x^2yz+9x^2z^2+9xy^3+7xy^2z+2xyz^2+6xz^3+9y^4-y^3z+4y^2z^2-yz^3-z^4$	1	14	25
f_{42}	$x^4+7x^3y+8x^3z+5x^2y^2+x^2yz+4x^2z^2+4xy^3+10xy^2z+8xyz^2+3xz^3+2y^4+11y^3z+y^2z^2+8yz^3+10z^4$	0	15	29
f_{43}	$x^4+10x^3z+6x^2y^2+4x^2z^2+9xy^3+2xy^2z+4xyz^2+3xz^3+6y^4+3y^3z-y^2z^2+2yz^3+10z^4$	2	15	26
f_{44}	$x^4+5x^3y+10x^3z+11x^2y^2+11x^2yz+7x^2z^2+11xy^3+7xy^2z+6xyz^2+8xz^3+5y^4+y^2z^2+2yz^3+7z^4$	1	14	26
f_{45}	$x^3y+4x^3z-x^2yz+8x^2z^2+2xy^3+5xy^2z+2xyz^2+7xz^3+y^4+2y^3z+5y^2z^2+4yz^3+5z^4$	0	16	26
f_{46}	$x^4+8x^3y+6x^3z-x^2y^2+3x^2yz+3x^2z^2+7xy^3+5xy^2z+7xyz^2+3xz^3+4y^4-y^3z+9y^2z^2+10yz^3+6z^4$	1	15	27
f_{47}	$x^4+10x^3y+11x^3z+8x^2y^2+4x^2yz-x^2z^2+11xy^3+2xy^2z+5xyz^2+10xz^3+7y^4-y^3z+y^2z^2+yz^3+8z^4$	0	15	24
f_{48}	$x^4+5x^3y+2x^3z+7x^2yz-x^2z^2+11xy^3+7xy^2z+9xyz^2+8xz^3+y^4+2y^3z+10y^2z^2+5yz^3+8z^4$	3	14	26
f_{49}	$x^4+4x^3y+2x^3z+11x^2y^2+9x^2z^2-xy^3+5xy^2z+2xyz^2+4xz^3-y^4-yz^3+2z^4$	1	14	22
f_{50}	$x^4+5x^3y+3x^3z+10x^2y^2+10x^2yz+7x^2z^2+4xy^3+2xy^2z+7xyz^2+5xz^3+8y^4+2y^3z+9y^2z^2+yz^3+10z^4$	2	17	28
f_{51}	$x^4+x^3y+7x^3z+8x^2y^2+10x^2z^2+9xy^3+8xy^2z+8xyz^2+9xz^3+10y^3z+6y^2z^2+6yz^3+5z^4$	0	15	26
f_{52}	$x^4+8x^3y+10x^3z+6x^2y^2-x^2yz+x^2z^2-xy^3+10xy^2z+7xyz^2-xz^3+8y^4+4y^3z+y^2z^2+8yz^3+7z^4$	0	15	27
f_{53}	$x^4+4x^3y-x^2y^2-x^2yz-xy^3+10xy^2z+6xyz^2+6xz^3+5y^4+9y^3z+11y^2z^2+9yz^3+5z^4$	1	15	23
f_{54}	$3x^3y+x^3z+3x^2y^2+7x^2yz+5x^2z^2+10xy^3+6xy^2z+11xyz^2+4xz^3+y^4+6y^3z+2y^2z^2+5yz^3+3z^4$	0	15	29
f_{55}	$7x^3y+7x^3z+2x^2y^2+8x^2yz+2xy^3+9xy^2z+4xyz^2+xz^3+y^4+11y^3z+9y^2z^2+6yz^3+10z^4$	0	18	27
f_{56}	$x^4+5x^3y+3x^3z+10x^2y^2+10x^2yz+7x^2z^2+4xy^3+2xy^2z+7xyz^2+5xz^3+8y^4+2y^3z+9y^2z^2+yz^3+10z^4$	2	17	28
f_{57}	$x^4+3x^3y+6x^3z+7x^2y^2+3x^2yz-x^2z^2+3xy^3+xy^2z+8xyz^2+3xz^3+y^4+y^3z+4y^2z^2+4yz^3+8z^4$	1	15	26

f_{58}	$x^4+x^3y+7x^3z+8x^2y^2+10x^2z^2+9xy^3+8xy^2z+8xyz^2+9xz^3+10y^3z+6y^2z^2+6yz^3+5z^4$	0	15	26
f_{59}	$x^4+8x^3y+10x^3z+6x^2y^2-x^2yz+x^2z^2-xy^3+10xy^2z+7xyz^2-xz^3+8y^4+4y^3z+y^2z^2+8yz^3+7z^4$	0	15	27
f_{60}	$x^4+10x^3y-x^3z+7x^2y^2+2x^2yz+4xy^3+2xy^2z+11xyz^2+6xz^3+3y^4+2y^3z+y^2z^2+10yz^3+6z^4$	0	14	25
f_{61}	$8x^3y+9x^3z+4x^2y^2+8x^2yz+10x^2z^2+4xy^3+6xy^2z+6xyz^2+3xz^3+y^4+5y^3z+4y^2z^2+yz^3+10z^4$	1	15	23
f_{62}	$x^4+3x^3y+6x^3z+5x^2y^2+11x^2yz+9x^2z^2+9xy^3+11xy^2z+3xyz^2+5xz^3+y^4+8y^3z+y^2z^2+9yz^3+5z^4$	1	14	25
f_{63}	$x^4+4x^3y+4x^3z+10x^2y^2+8x^2yz+6x^2z^2-xy^3+3xy^2z+2xyz^2+3y^4-y^3z+10y^2z^2+11yz^3-z^4$	2	14	22
f_{64}	$x^4+4x^3y+10x^3z+3x^2yz+7x^2z^2+7xy^3+2xy^2z+8xyz^2+10xz^3-y^4+7y^3z+10y^2z^2+7yz^3+5z^4$	2	14	27
f_{65}	$x^4+8x^3y+8x^3z-x^2y^2+9x^2yz+8x^2z^2+7xy^3+10xy^2z-xyz^2+11xz^3+10y^4+4y^3z+3y^2z^2+3yz^3+6z^4$	2	14	23
f_{66}	$x^4+11x^3y+5x^3z+11x^2y^2+9x^2yz+7x^2z^2+4xy^3+9xy^2z+2xyz^2+9xz^3+5y^4+2y^3z+2y^2z^2+10yz^3+6z^4$	1	14	27
f_{67}	$8x^3y+11x^3z+x^2y^2+11x^2yz+5x^2z^2+5xy^2z+6xyz^2+y^4+8y^3z+9y^2z^2+10yz^3+6z^4$	0	15	24
f_{68}	$x^4+x^3y+8x^3z+3x^2y^2+9x^2yz+11x^2z^2+4xy^3+8xy^2z+4xz^3+3y^4+11y^3z+9yz^3$	1	15	23
f_{69}	$x^4+5x^3y+x^3z+10x^2y^2+9x^2yz+11x^2z^2+4xy^3+5xy^2z+9xyz^2+6y^4+9y^3z+y^2z^2+5yz^3$	0	16	29
f_{70}	$x^4+8x^3y+8x^3z-x^2y^2+9x^2yz+8x^2z^2+7xy^3+10xy^2z-xyz^2+11xz^3+10y^4+4y^3z+3y^2z^2+3yz^3+6z^4$	2	14	23
f_{71}	$x^4+9x^3y+9x^3z+7x^2yz+3x^2z^2+xy^3+8xy^2z+10xyz^2+3y^4+3y^3z+9y^2z^2+8yz^3+9z^4$	0	14	26
f_{72}	$x^4+6x^3y+8x^3z+6x^2y^2+2x^2yz+4x^2z^2+4xy^3+8xy^2z+9xz^3+6y^4+4y^3z+6y^2z^2+10yz^3+7z^4$	1	16	24
f_{73}	$x^4+6x^3y-x^3z+8x^2y^2+5x^2yz+8x^2z^2+2xy^3-xy^2z+4xyz^2-xz^3+y^4+10y^3z+11y^2z^2+11yz^3+z^4$	1	16	28
f_{74}	$x^4+8x^3y+10x^3z+4x^2y^2+10x^2yz-x^2z^2+8xy^3+7xy^2z+xyz^2+5xz^3+5y^4+9y^3z+yz^3+11z^4$	0	14	25

f_{75}	$x^4+9x^3y+9x^3z+7x^2yz+3x^2z^2+xy^3+8xy^2z+10xyz^2+3y^4+3y^3z+9y^2z^2+8yz^3+9z^4$	0	14	26
f_{76}	$x^4+6x^3y+7x^3z+x^2y^2+x^2yz+3x^2z^2-xy^3+9xy^2z+4xyz^2+6xz^3+4y^4+7y^3z+y^2z^2-yz^3+10z^4$	1	15	24
f_{77}	$x^4+5x^3y+x^2y^2+7x^2yz+4xy^3+5xy^2z+4xz^3+5y^4+2y^3z+5y^2z^2+yz^3+11z^4$	0	14	24
f_{78}	$x^4+8x^3y-x^3z+5x^2y^2+2x^2yz+7x^2z^2+6xy^3+11xy^2z+7xyz^2+4xz^3-y^4+6y^3z+9y^2z^2+z^4$	0	14	22
f_{79}	$x^4+2x^3y+10x^3z+9x^2y^2+10x^2yz+2x^2z^2+7xy^3+8xy^2z+6xyz^2+8xz^3+9y^4+5y^3z+6y^2z^2+2yz^3-z^4$	1	15	21
f_{80}	$x^4+3x^3y+11x^3z+8x^2yz+3x^2z^2+3xy^3+8xy^2z+2xyz^2+6y^4-y^3z+9y^2z^2+7yz^3+10z^4$	1	16	25
f_{81}	$x^4+10x^3y+4x^3z-x^2y^2+8x^2yz+10x^2z^2+8xy^3+11xy^2z+7xyz^2+10xz^3+y^4+y^3z+7y^2z^2+9yz^3$	1	18	24
f_{82}	$x^4+11x^3y+3x^3z+x^2y^2+x^2yz+6x^2z^2+4xy^3+2xy^2z+7xyz^2+4xz^3+9y^4-y^3z+8y^2z^2+10yz^3+z^4$	0	15	27
f_{83}	$x^4+11x^3y+10x^3z+10x^2y^2+2x^2yz+9x^2z^2+3xy^3+6xy^2z+7xyz^2+5xz^3-y^4+5y^3z+y^2z^2+4yz^3+8z^4$	2	15	21
f_{84}	$x^4+3x^3y+11x^3z+8x^2yz+3x^2z^2+3xy^3+8xy^2z+2xyz^2+6y^4-y^3z+9y^2z^2+7yz^3+10z^4$	1	16	25
f_{85}	$x^4+3x^3y+11x^3z+8x^2yz+3x^2z^2+3xy^3+8xy^2z+2xyz^2+6y^4-y^3z+9y^2z^2+7yz^3+10z^4$	1	16	25
f_{86}	$x^4+11x^3y+4x^3z+7x^2y^2+2x^2yz+2x^2z^2+5xy^3+10xy^2z-xyz^2+6xz^3+9y^4+2y^3z+4y^2z^2+3yz^3+5z^4$	1	15	31
f_{87}	$x^4+9x^3y+2x^3z-x^2yz+11x^2z^2+xy^3+8xy^2z+4xyz^2+8xz^3-y^4+3y^3z+y^2z^2+5yz^3+z^4$	3	15	30
f_{88}	$x^4+3x^3y+10x^3z+4x^2y^2+7x^2yz+4xy^3+6xy^2z+6xyz^2+5xz^3+5y^4+3y^3z+y^2z^2+6yz^3$	0	15	22
f_{89}	$x^4+9x^3y+2x^3z+9x^2y^2+9x^2yz+3x^2z^2+11xy^3+7xy^2z+5xyz^2+11xz^3+8y^4+2y^3z-y^2z^2+4yz^3+5z^4$	2	14	27
f_{90}	$x^4+x^3y+x^3z+x^2y^2+5x^2yz+x^2z^2+2xy^3+8xy^2z+4xyz^2+4xz^3+11y^4+7y^3z+9y^2z^2+10yz^3+z^4$	0	16	23
f_{91}	$x^4+x^3y+x^3z+x^2y^2+5x^2yz+x^2z^2+2xy^3+8xy^2z+4xyz^2+4xz^3+11y^4+7y^3z+9y^2z^2+10yz^3+z^4$	0	15	23

f_{92}	$x^4+2x^3y+10x^3z+2x^2y^2-x^2yz+7x^2z^2+7xy^3+9xy^2z+7xz^3+5y^4+y^3z+10y^2z^2+9yz^3+4z^4$	1	14	25
f_{93}	$x^4+2x^3y+10x^3z+2x^2y^2-x^2yz+7x^2z^2+7xy^3+9xy^2z+7xz^3+5y^4+y^3z+10y^2z^2+9yz^3+4z^4$	1	14	25
f_{94}	$x^4+4x^3y-x^3z+7x^2y^2+x^2yz+8x^2z^2+10xy^3+10xy^2z+4xyz^2+4xz^3+5y^4+6y^3z+4y^2z^2+11yz^3+z^4$	0	16	24
f_{95}	$x^4+4x^3y-x^3z+7x^2y^2+x^2yz+8x^2z^2+10xy^3+10xy^2z+4xyz^2+4xz^3+5y^4+6y^3z+4y^2z^2+11yz^3+z^4$	0	16	24
f_{96}	$x^4+4x^3y-x^3z+7x^2y^2+x^2yz+8x^2z^2+10xy^3+10xy^2z+4xyz^2+4xz^3+5y^4+6y^3z+4y^2z^2+11yz^3+z^4$	0	16	24
f_{97}	$x^4+6x^3y+4x^3z+7x^2y^2+x^2yz+4x^2z^2+10xy^3+2xyz^2+8xz^3+3y^4+6y^3z+11y^2z^2+9yz^3+2z^4$	0	15	26
f_{98}	$x^4+6x^3y+4x^3z+7x^2y^2+x^2yz+4x^2z^2+10xy^3+2xyz^2+8xz^3+3y^4+6y^3z+11y^2z^2+9yz^3+2z^4$	0	15	26
f_{99}	$x^4+8x^3y+6x^3z-x^2y^2+3x^2yz+3x^2z^2+7xy^3+5xy^2z+7xyz^2+3xz^3+4y^4-y^3z+9y^2z^2+10yz^3+6z^4$	1	14	27
f_{100}	$x^4-x^3y-x^3z+3x^2y^2+4x^2yz+8xy^3+2xy^2z+8xyz^2-xz^3-y^4+4y^3z+3y^2z^2+7z^4$	0	15	28
f_{101}	$x^4-x^3y-x^3z+3x^2y^2+4x^2yz+8xy^3+2xy^2z+8xyz^2-xz^3-y^4+4y^3z+3y^2z^2+7z^4$	0	15	28
f_{102}	$x^4+x^3y+7x^3z+9x^2yz+4x^2z^2+9xy^3+10xy^2z+6xyz^2+2xz^3+4y^4+2y^3z+6yz^3$	0	16	29
f_{103}	$7x^3y+6x^2y^2+6x^2yz+6x^2z^2+4xy^3+4xy^2z+9xyz^2+11xz^3+y^4+4y^3z+7y^2z^2+8yz^3+3z^4$	0	15	25
f_{104}	$x^4+2x^3y+2x^3z+3x^2y^2+3x^2yz+2x^2z^2+10xy^3+10xy^2z+6xyz^2+8xz^3+8y^3z+3yz^3+9z^4$	0	14	23
f_{105}	$x^4+3x^3y-x^3z-x^2y^2+8x^2yz+3x^2z^2+8xy^3+7xy^2z+3xyz^2+xz^3+2y^4+2y^3z+6y^2z^2+6z^4$	1	14	25
f_{106}	$x^4+3x^3y-x^3z-x^2y^2+8x^2yz+3x^2z^2+8xy^3+7xy^2z+3xyz^2+xz^3+2y^4+2y^3z+6y^2z^2+6z^4$	1	14	25
f_{107}	$11x^3y+3x^3z+2x^2y^2+4x^2yz+8x^2z^2-xy^3+10xy^2z+6xyz^2-xz^3+y^4+3y^3z+11y^2z^2+9yz^3+6z^4$	1	15	26
f_{108}	$x^4+10x^3z-x^2y^2+9x^2yz+4x^2z^2+xy^3+xy^2z+6xyz^2+9xz^3+9y^4+4y^3z+6y^2z^2+8yz^3+11z^4$	0	16	26
f_{109}	$x^4+10x^3z-x^2y^2+9x^2yz+4x^2z^2+xy^3+xy^2z+6xyz^2+9xz^3+9y^4+4y^3z+6y^2z^2+8yz^3+11z^4$	0	16	26

f_{110}	$8x^3y+10x^3z+6x^2y^2+10x^2yz+6x^2z^2+11xy^3+11xy^2z+9xyz^2+xz^3+y^4+11y^3z+2y^2z^2+5yz^3+4z^4$	0	15	20
f_{111}	$x^4+11x^3y+2x^3z+11x^2y^2+4x^2yz+7x^2z^2-xy^3+2xy^2z-xyz^2+xz^3+10y^4+6y^3z+9y^2z^2+8yz^3+z^4$	0	14	27
f_{112}	$x^4+9x^3y-x^3z+x^2yz+3x^2z^2+8xy^3+2xy^2z+3xyz^2-xz^3+y^4+y^3z+2y^2z^2+4yz^3+6z^4$	1	17	27
f_{113}	$x^4+2x^3z+11x^2y^2-x^2yz+3x^2z^2-xy^3+10xy^2z+7xyz^2+3xz^3+2y^4+2y^3z+9y^2z^2+3yz^3+8z^4$	0	16	26
f_{114}	$x^4+2x^3z+11x^2y^2-x^2yz+3x^2z^2-xy^3+10xy^2z+7xyz^2+3xz^3+2y^4+2y^3z+9y^2z^2+3yz^3+8z^4$	0	16	26
f_{115}	$x^4-x^3y+11x^3z-x^2y^2+9x^2yz-x^2z^2+5xy^3+xy^2z+3xyz^2+xz^3+7y^4+4y^3z+7y^2z^2+11yz^3+7z^4$	0	15	24
f_{116}	$x^4+9x^3y-x^3z+x^2yz+3x^2z^2+8xy^3+2xy^2z+3xyz^2-xz^3+y^4+y^3z+2y^2z^2+4yz^3+6z^4$	1	16	27
f_{117}	$x^4+4x^3y+2x^3z+3x^2y^2+9x^2yz+5x^2z^2+2xy^3+9xy^2z+4xyz^2+8xz^3+y^4+11y^3z+10y^2z^2+4z^4$	2	15	22
f_{118}	$x^4+2x^3y+8x^2y^2+5x^2yz+8x^2z^2+11xy^3+7xy^2z+4xyz^2+9xz^3+6y^4+4y^3z+9y^2z^2+7yz^3+4z^4$	1	15	26
f_{119}	$x^4+11x^3y+11x^3z+3x^2y^2+3x^2yz+x^2z^2+3xy^3+6xy^2z+5xz^3+4y^4+6y^3z+y^2z^2+3yz^3+7z^4$	1	15	28
f_{120}	$x^4+3x^3y+9x^3z+10x^2yz+9x^2z^2+5xy^3+8xy^2z+9xyz^2+3xz^3+7y^3z+6y^2z^2+4yz^3+2z^4$	1	15	24
f_{121}	$x^4+5x^3y+10x^3z+11x^2y^2+11x^2yz+7x^2z^2+11xy^3+7xy^2z+6xyz^2+8xz^3+5y^4+y^2z^2+2yz^3+7z^4$	1	14	26
f_{122}	$x^4+8x^3y+6x^3z-x^2y^2+3x^2yz+3x^2z^2+7xy^3+5xy^2z+7xyz^2+3xz^3+4y^4-y^3z+9y^2z^2+10yz^3+6z^4$	1	15	27
f_{123}	$3x^3y+x^3z+3x^2y^2+7x^2yz+5x^2z^2+10xy^3+6xy^2z+11xyz^2+4xz^3+y^4+6y^3z+2y^2z^2+5yz^3+3z^4$	0	16	29

5.10 Quartic curves for complete $(32;4)$-arcs

From Table 5.9, the number of quartic curves is 119. The time of these curves is 2560 msc. Among them the number of associated quartic curves of the complete $(32;4)$-arcs that do not have inflexion points is 54. There are 65 quartic curves have between 1 and 3 inflexions. Also, the number $|\mathscr{C}\cap\mathscr{K}|$ for the 119 quartic curves on the corresponding arc ranges between 14 and 19 points and the number N_1 is between 19 and 30 points. It took 9412 msec.

Table 5.9: **Quartic curves for complete (32;4)-arcs**

Symbol	$\mathscr{C}(f_i)$	I	$\lvert\mathscr{C}\cap\mathscr{K}\rvert$	N_1
f_1	$x^4+7x^3z+5x^2y^2+6x^2yz+9x^2z^2+9xy^3+6xy^2z+9xyz^2+8xz^3+11y^4+y^3z+6y^2z^2+11yz^3+2z^4$	0	16	30
f_2	$x^4+10x^3y-x^3z+5x^2y^2-x^2yz+8x^2z^2+9xy^3+10xy^2z+9xyz^2+9xz^3+3y^4+9y^2z^2+5yz^3+3z^4$	0	15	19
f_3	$x^4+9x^3y+4x^3z+9x^2yz+6x^2z^2+8xy^3+xy^2z+3xyz^2+3xz^3+3y^4+8y^3z+4y^2z^2+2yz^3+8z^4$	0	16	28
f_4	$x^4+3x^3y+10x^3z+8x^2y^2+7x^2yz+5x^2z^2+7xy^3+2xy^2z+9xyz^2-xz^3-y^4+3y^3z+9y^2z^2+4yz^3+11z^4$	0	15	27
f_5	$x^4+x^3y+8x^2y^2+4x^2yz+2x^2z^2+10xy^3+4xy^2z+11xyz^2+3xz^3+5y^4+10y^3z-y^2z^2-yz^3+z^4$	0	14	24
f_6	$x^4+10x^3y-x^3z-x^2y^2+8x^2yz+6x^2z^2+9xy^3+11xy^2z+9xyz^2+9xz^3+3y^4+6y^3z+2y^2z^2+yz^3+8z^4$	1	15	23
f_7	$x^4+5x^3y+10x^3z+10x^2y^2+11x^2yz+3x^2z^2+3xy^3+8xy^2z+5xyz^2+xz^3+3y^4+7y^3z+8y^2z^2+6yz^3+9z^4$	1	14	23
f_8	$x^4+8x^3y+5x^3z+4x^2y^2+11x^2yz+10xy^3+7xy^2z+9xyz^2+7xz^3-y^4+6y^3z+8y^2z^2+7yz^3+10z^4$	1	16	24
f_9	$x^4+6x^3y+3x^2y^2+11x^2yz+2x^2z^2+9xy^3-xy^2z+6xyz^2-y^4+2y^3z+9y^2z^2+yz^3+5z^4$	1	16	28
f_{10}	$x^4+5x^3y+10x^3z+9x^2y^2+5x^2yz+5x^2z^2-xy^3+7xy^2z+4xyz^2+11xz^3+6y^4+5y^3z+2y^2z^2+yz^3+7z^4$	1	17	27
f_{11}	$x^4+4x^3y+11x^3z+2x^2y^2+x^2yz+6x^2z^2+3xy^2z+4xyz^2+5xz^3+4y^4+10y^3z-y^2z^2+9yz^3+8z^4$	1	14	22
f_{12}	$x^4+3x^3y+10x^3z+2x^2y^2+11x^2yz+x^2z^2+7xy^3+4xy^2z+6xyz^2+5xz^3+10y^3z+7y^2z^2+6yz^3$	1	18	28

f_{13}	$x^4+7x^3y+6x^3z+10x^2y^2+7x^2yz+8x^2z^2+7xy^3+9xy^2z+xyz^2+11xz^3+8y^4+10y^3z+5yz^3+3z^4$	2	15	28
f_{14}	$x^4+7x^3y+10x^3z+4x^2y^2+8x^2yz+4x^2z^2+11xy^3+8xy^2z-xyz^2+xz^3+9y^4+y^3z-y^2z^2+10yz^3$	1	16	23
f_{15}	$x^4+9x^3y+4x^3z+8x^2yz+6x^2z^2+4xy^3+xy^2z+5xyz^2+3xz^3+2y^4+3y^3z+4y^2z^2+7yz^3+8z^4$	3	15	27
f_{16}	$x^4+2x^3y+2x^3z+9x^2y^2+2x^2yz+2x^2z^2+10xy^3-xy^2z+2xyz^2+9xz^3+9y^4+4y^2z^2+2yz^3+8z^4$	0	15	29
f_{17}	$x^4+7x^3y+6x^3z+10x^2y^2+7x^2yz+8x^2z^2+7xy^3+9xy^2z+xyz^2+11xz^3+8y^4+10y^3z+5yz^3+3z^4$	2	15	28
f_{18}	$x^4+4x^3y+11x^3z+x^2y^2+10x^2yz+7x^2z^2+xy^3+4xy^2z+9xyz^2+2y^4+10y^3z+8y^2z^2+7yz^3+7z^4$	2	15	22
f_{19}	$x^4+3x^3y+6x^3z-x^2y^2-x^2z^2+4xy^3+5xy^2z+4xyz^2+6xz^3+11y^4+2y^3z+5y^2z^2+11yz^3+4z^4$	0	14	25
f_{20}	$x^4+7x^3y+8x^3z+7x^2y^2+3x^2yz-x^2z^2+4xy^3+6xy^2z-xyz^2+5xz^3+6y^4+8y^3z+7y^2z^2+8yz^3+3z^4$	1	14	25
f_{21}	$x^4+8x^3y+5x^3z+5x^2y^2+6x^2yz+9x^2z^2+7xy^3+10xy^2z+xyz^2+7xz^3+7y^4+6y^3z+4y^2z^2+2yz^3+7z^4$	1	14	26
f_{22}	$x^4+3x^3y-x^3z+3x^2y^2+2x^2yz+4x^2z^2+11xy^2z+9xz^3-y^4+6y^3z+7y^2z^2+5yz^3$	0	16	25
f_{23}	$x^4+2x^3y+8x^3z-x^2y^2+5x^2yz+7xy^3-xy^2z+8xyz^2+6xz^3+2y^4+11y^3z-y^2z^2+3yz^3+10z^4$	0	14	24
f_{24}	$x^4+3x^3y+11x^2y^2+11x^2z^2+3xy^3+10xy^2z+6xyz^2+4y^4+5y^3z+6y^2z^2+8yz^3+2z^4$	0	16	27
f_{25}	$x^4+10x^3y+10x^3z+11x^2y^2+11x^2yz+3xy^3+xy^2z+2xyz^2+xz^3+11y^4+10y^3z+10y^2z^2+2yz^3+10z^4$	2	15	24
f_{26}	$x^4+x^3z+3x^2y^2-x^2yz+7x^2z^2+2xy^3+2xy^2z+xyz^2+7xz^3+3y^4+6y^3z+5yz^3+8z^4$	0	16	27
f_{27}	$x^4-x^3y+8x^3z+3x^2y^2-x^2yz+3x^2z^2+4xy^3+11xy^2z+7xyz^2+10xz^3+3y^4+10y^3z+y^2z^2+10yz^3+2z^4$	0	16	27
f_{28}	$x^4+5x^3y+8x^3z+3x^2y^2+2x^2yz+11x^2z^2+4xy^3+5xy^2z+5xyz^2+6xz^3-y^4+y^3z+3y^2z^2+4yz^3+2z^4$	3	14	24
f_{29}	$x^4+11x^3y+6x^3z+5x^2y^2-x^2yz+11x^2z^2+7xy^3+10xy^2z+xyz^2+9xz^3+11y^4+8y^3z+8y^2z^2+5yz^3+9z^4$	3	16	28
f_{30}	$3x^3y+3x^3z+11x^2yz+4xy^3+4xy^2z+5xyz^2-xz^3+y^4+7y^3z-y^2z^2$	0	16	27

f_{31}	$x^4+9x^3y-x^3z+8x^2y^2+2x^2yz-x^2z^2+10xy^3+xy^2z+11xyz^2+9xz^3+4y^4+3y^3z+8y^2z^2+8yz^3+6z^4$	3	16	21
f_{32}	$x^4+5x^3y+11x^3z+8x^2y^2+6x^2yz+8x^2z^2-xy^3+8xy^2z+2xz^3+7y^4+5y^3z+4y^2z^2+4yz^3+9z^4$	0	16	26
f_{33}	$x^4+5x^3y+11x^3z+8x^2y^2+6x^2yz+8x^2z^2-xy^3+8xy^2z+2xz^3+7y^4+5y^3z+4y^2z^2+4yz^3+9z^4$	0	16	26
f_{34}	$x^4+7x^3y-x^3z+8x^2y^2+5x^2yz+2x^2z^2+10xy^3-xy^2z+5xyz^2+4xz^3+4y^4-y^3z+10y^2z^2+5yz^3+3z^4$	1	16	26
f_{35}	$x^4-x^3y+6x^2yz+7x^2z^2+7xy^3+11xy^2z+7xyz^2+7y^4+5y^3z+9y^2z^2+6yz^3-z^4$	1	16	24
f_{36}	$x^4+4x^3y+7x^2y^2+3x^2yz+7x^2z^2+5xy^3+5xy^2z+9xyz^2+9xz^3+8y^4+6y^2z^2$	2	16	29
f_{37}	$x^4+11x^3y+6x^3z+5x^2y^2-x^2yz+11x^2z^2+7xy^3+10xy^2z+xyz^2+9xz^3+11y^4+8y^3z+8y^2z^2+5yz^3+9z^4$	3	16	28
f_{38}	$x^4+x^3y+11x^3z+5x^2y^2+x^2yz+3x^2z^2+2xy^3+3xy^2z+xyz^2+11xz^3+3y^4+10y^3z+7y^2z^2+11yz^3+z^4$	1	14	24
f_{39}	$x^4+6x^3y+3x^3z+2x^2y^2+9x^2yz+5xy^2z+8xz^3+7y^4+3y^3z+7yz^3+9z^4$	0	15	21
f_{40}	$x^4+8x^3y-x^3z-x^2y^2+4x^2yz+8x^2z^2+9xy^3+xy^2z+5xyz^2+4y^4+6y^3z+8y^2z^2+6yz^3+5z^4$	0	14	21
f_{41}	$x^4+10x^3y+6x^3z+x^2y^2+2x^2yz+10x^2z^2+xy^3+2xy^2z+11xyz^2-xz^3+11y^4+11y^3z+3y^2z^2+8yz^3+10z^4$	2	15	20
f_{42}	$x^4-x^3z+9x^2y^2+3x^2yz+9x^2z^2-xy^3+11xyz^2+xz^3+5y^4+2y^3z+4y^2z^2+11yz^3+11z^4$	0	16	25
f_{43}	$x^4+4x^3y+10x^3z+11x^2yz+7x^2z^2+10xy^3+2xy^2z+6xyz^2+3xz^3+11y^4+2y^3z+y^2z^2+2yz^3+5z^4$	0	15	30
f_{44}	$x^4+4x^3y+10x^3z+11x^2yz+7x^2z^2+10xy^3+2xy^2z+6xyz^2+3xz^3+11y^4+2y^3z+y^2z^2+2yz^3+5z^4$	0	15	30
f_{45}	$x^4+x^3y+2x^3z+10x^2y^2+11x^2yz+x^2z^2-xy^3+xy^2z+3xyz^2+3xz^3-y^4+6y^3z+8y^2z^2+3yz^3+10z^4$	0	16	23
f_{46}	$x^4+2x^3y+2x^3z+4x^2y^2-x^2yz+2x^2z^2+5xy^2z+5xz^3-y^4+y^2z^2+11yz^3+6z^4$	1	15	27

f_{47}	$x^4+5x^3z+9x^2y^2-x^2yz-x^2z^2+4xy^3+6xy^2z+10xyz^2+10xz^3+10y^4+4y^3z+3y^2z^2+11yz^3+11z^4$	2	16	23
f_{48}	$x^4+3x^3y+11x^3z+7x^2y^2+3x^2yz+10x^2z^2+10xy^3+3xy^2z+4xz^3+11y^4+11y^3z-y^2z^2+7yz^3+8z^4$	0	14	27
f_{49}	$x^4+5x^3y+11x^3z+7x^2y^2+5x^2yz+2x^2z^2+4xy^3+7xy^2z+3xyz^2-xz^3+y^4+8y^3z+6y^2z^2-yz^3$	0	15	30
f_{50}	$x^4+4x^3y+10x^2y^2+7x^2yz+10x^2z^2+11xy^3+9xy^2z+9xyz^2+6xz^3+10y^4+5y^3z+5y^2z^2+9yz^3+10z^4$	0	14	20
f_{51}	$x^4+5x^3z+9x^2y^2-x^2yz-x^2z^2+4xy^3+6xy^2z+10xyz^2+10xz^3+10y^4+4y^3z+3y^2z^2+11yz^3+11z^4$	2	16	23
f_{52}	$x^4+x^3y+11x^3z+11x^2y^2+9x^2yz-x^2z^2+6xy^3+6xy^2z+8xz^3+11y^4+y^3z+6y^2z^2+9yz^3-z^4$	1	15	25
f_{53}	$x^4+4x^3y+10x^2y^2+9x^2z^2+11xy^3+7xy^2z+5xyz^2+5xz^3+2y^4+11y^3z+8y^2z^2+8yz^3+9z^4$	3	14	30
f_{54}	$x^4+4x^3y+6x^3z+9x^2y^2+4x^2yz+x^2z^2+11xy^3+6xy^2z+xyz^2+10xz^3-y^4+10y^3z+2y^2z^2+yz^3+11z^4$	0	15	23
f_{55}	$x^4+3x^3y-x^3z+9x^2y^2+10x^2yz+4x^2z^2+2xy^3+10xy^2z+8xz^3+8y^4+y^3z+10y^2z^2+5yz^3+10z^4$	3	14	24
f_{56}	$x^4+2x^3y+9x^3z-x^2y^2+2x^2yz+6x^2z^2+xy^3+xy^2z+9xyz^2+5xz^3+8y^4+6y^3z+6y^2z^2+5yz^3+6z^4$	0	15	25
f_{57}	$x^4+9x^3y+11x^3z+x^2y^2-x^2yz+11x^2z^2+xy^3+2xyz^2+5xz^3+6y^4+9y^3z+9y^2z^2+3yz^3+2z^4$	1	15	24
f_{58}	$x^4+9x^3y+11x^3z+x^2y^2-x^2yz+11x^2z^2+xy^3+2xyz^2+5xz^3+6y^4+9y^3z+9y^2z^2+3yz^3+2z^4$	1	15	24
f_{59}	$x^4+3x^3y+7x^3z+10x^2y^2+x^2yz+6x^2z^2-xy^3+3xy^2z+7xyz^2+5xz^3+5y^4+3y^3z+yz^3+4z^4$	0	15	27
f_{60}	$x^4+8x^3y+6x^3z-x^2y^2+3x^2yz+3x^2z^2+7xy^3+5xy^2z+7xyz^2+3xz^3+4y^4-y^3z+9y^2z^2+10yz^3+6z^4$	1	14	27
f_{61}	$4x^3y+7x^3z+6x^2y^2+2x^2yz+x^2z^2+5xy^3+8xy^2z+11xyz^2+7xz^3+y^4-y^3z+6y^2z^2+7yz^3+6z^4$	0	18	24
f_{62}	$x^4+11x^3y+3x^3z+6x^2y^2+11x^2yz+7x^2z^2+10xy^3+6xy^2z+8xyz^2+4xz^3+9y^4+4y^3z+11y^2z^2+10yz^3+z^4$	0	14	22

f_{63}	$x^4+8x^3y+8x^3z+5x^2y^2+9x^2yz+x^2z^2+10xy^3+5xy^2z+7xyz^2+7xz^3+y^4+11y^3z+6y^2z^2+2yz^3+6z^4$	0	17	29
f_{64}	$x^4+6x^3y+11x^2y^2+4x^2yz+2x^2z^2+11xy^3+3xy^2z+6xyz^2+11xz^3+5y^4+2y^3z+y^2z^2+8yz^3+11z^4$	1	16	26
f_{65}	$x^4+3x^3y+7x^3z+6x^2y^2-x^2yz+x^2z^2+11xy^2z-xyz^2+7xz^3+8y^4+4y^3z+2y^2z^2+6yz^3-z^4$	1	14	23
f_{66}	$x^4+2x^3y+5x^2y^2+x^2yz-x^2z^2+4xy^3+7xy^2z+4xyz^2+5xz^3-y^4+11y^3z+4y^2z^2+4yz^3+5z^4$	1	16	25
f_{67}	$x^4+5x^3y+7x^3z+7x^2y^2-x^2yz+3x^2z^2-xy^3+xy^2z+4xyz^2+xz^3+8y^4+11y^3z+5y^2z^2+4yz^3+6z^4$	1	15	23
f_{68}	$x^4+9x^3y+5x^3z+5x^2y^2+11x^2yz+3x^2z^2-xy^2z+11xyz^2+xz^3+8y^4+6y^3z+6y^2z^2+3yz^3+8z^4$	1	14	24
f_{69}	$x^4+7x^3y+7x^3z-x^2y^2+9x^2yz+5x^2z^2-xy^3+10xy^2z-xz^3+8y^4+6y^3z+10y^2z^2-yz^3$	1	16	27
f_{70}	$x^4+2x^3y-x^2y^2+7x^2yz+11x^2z^2+5xy^3+4xy^2z+7xz^3+4y^4+5y^3z+4y^2z^2+11yz^3$	0	15	28
f_{71}	$x^4+11x^3y+10x^3z-x^2y^2-x^2yz+8x^2z^2+8xy^3+3xyz^2+6xz^3+10y^4+5y^3z+3y^2z^2-yz^3+z^4$	3	14	25
f_{72}	$x^4+x^3y+5x^3z+x^2y^2+4x^2yz+5x^2z^2+4xy^3+7xy^2z+3xyz^2+5y^4+6y^3z+11y^2z^2+5yz^3-z^4$	2	14	24
f_{73}	$x^4+11x^3y+3x^3z+5x^2y^2+3x^2yz+9x^2z^2-xy^3+7xy^2z+9xyz^2+10xz^3+11y^3z+2z^4$	0	17	29
f_{74}	$x^4+2x^3y+5x^2y^2+x^2yz-x^2z^2+4xy^3+7xy^2z+4xyz^2+5xz^3-y^4+11y^3z+4y^2z^2+4yz^3+5z^4$	1	16	25
f_{75}	$x^4+4x^3y+10x^3z+3x^2y^2+5x^2yz+2x^2z^2+xy^3+4xy^2z+7xyz^2+8xz^3+8y^3z+2y^2z^2+11yz^3-z^4$	0	17	26
f_{76}	$x^4+8x^3y+4x^3z+4x^2yz+2x^2z^2+11xy^3+3xy^2z-xyz^2+2xz^3+7y^4+2y^3z+yz^3+2z^4$	1	14	26
f_{77}	$x^4+8x^3y+4x^3z+4x^2yz+2x^2z^2+11xy^3+3xy^2z-xyz^2+2xz^3+7y^4+2y^3z+yz^3+2z^4$	1	14	26
f_{78}	$x^4+6x^3y+2x^3z+5x^2y^2+11x^2yz+8x^2z^2+5xy^3+xy^2z+5xyz^2+10xz^3+4y^4+5y^3z-y^2z^2+yz^3$	1	16	26
f_{79}	$x^4+9x^3y+10x^3z+10x^2y^2+x^2z^2+6xy^3+3xy^2z+6xyz^2+xz^3-y^4-y^3z+2y^2z^2+5yz^3+11z^4$	0	14	26
f_{80}	$x^4+7x^3y+2x^3z+3x^2y^2+7x^2z^2+3xy^3+11xy^2z+4xyz^2+xz^3+4y^4+8y^3z+y^2z^2+6yz^3+4z^4$	1	14	24

f_{81}	$x^4+11x^3y+3x^3z+6x^2y^2+11x^2yz+7x^2z^2+10xy^3+6xy^2z+8xyz^2+4xz^3+9y^4+4y^3z+11y^2z^2+10yz^3+z^4$	0	14	22
f_{82}	$x^4+x^3y+9x^3z+5x^2y^2+4x^2yz-x^2z^2+9xy^3+xy^2z+2xyz^2+4xz^3+6y^4+5y^3z+y^2z^2+9yz^3+11z^4$	0	15	24
f_{83}	$x^4+11x^3y+10x^3z+10x^2y^2+2x^2yz+9x^2z^2+3xy^3+6xy^2z+7xyz^2+5xz^3-y^4+5y^3z+y^2z^2+4yz^3+8z^4$	1	14	21
f_{84}	$x^4+11x^3y+10x^3z+7x^2y^2+6x^2yz+11x^2z^2-xy^3+9xy^2z+xyz^2+7xz^3+y^4+y^3z+8y^2z^2+yz^3$	0	15	24
f_{85}	$x^4+2x^3y+9x^3z+3x^2y^2+10x^2yz+4x^2z^2+10xy^3+4xy^2z+11xz^3+y^4+3y^3z+11y^2z^2+3yz^3$	2	16	25
f_{86}	$x^4-x^3y+11x^3z-x^2yz+7x^2z^2-xy^3+6xy^2z+3xyz^2+4xz^3+2y^3z-y^2z^2+7yz^3+10z^4$	1	15	25
f_{87}	$x^4+9x^3y-x^3z+2x^2y^2+5x^2yz+11x^2z^2+2xy^3+xy^2z+10xyz^2+3xz^3+11y^4+y^3z+6y^2z^2+7yz^3+z^4$	1	15	27
f_{88}	$x^4+7x^3z+5x^2y^2+6x^2yz+9x^2z^2+9xy^3+6xy^2z+9xyz^2+8xz^3+11y^4+y^3z+6y^2z^2+11yz^3+2z^4$	0	15	30
f_{89}	$x^4+10x^3y-x^3z+5x^2y^2-x^2yz+8x^2z^2+9xy^3+10xy^2z+9xyz^2+9xz^3+3y^4+9y^2z^2+5yz^3+3z^4$	0	14	19
f_{90}	$x^4+3x^3y+10x^3z+8x^2y^2+7x^2yz+5x^2z^2+7xy^3+2xy^2z+9xyz^2-xz^3-y^4+3y^3z+9y^2z^2+4yz^3+11z^4$	0	15	27
f_{91}	$x^4+3x^3y+10x^3z+8x^2y^2+7x^2yz+5x^2z^2+7xy^3+2xy^2z+9xyz^2-xz^3-y^4+3y^3z+9y^2z^2+4yz^3+11z^4$	0	14	27
f_{92}	$x^4+3x^3y+10x^3z+8x^2y^2+7x^2yz+5x^2z^2+7xy^3+2xy^2z+9xyz^2-xz^3-y^4+3y^3z+9y^2z^2+4yz^3+11z^4$	0	14	27
f_{93}	$x^4+2x^3y+2x^3z+9x^2y^2+2x^2yz+2x^2z^2+10xy^3-xy^2z+2xyz^2+9xz^3+9y^4+4y^2z^2+2yz^3+8z^4$	0	14	29
f_{94}	$x^4+8x^3y+5x^3z+4x^2y^2+11x^2yz+10xy^3+7xy^2z+9xyz^2+7xz^3-y^4+6y^3z+8y^2z^2+7yz^3+10z^4$	0	15	24
f_{95}	$x^4+8x^3y+5x^3z+4x^2y^2+11x^2yz+10xy^3+7xy^2z+9xyz^2+7xz^3-y^4+6y^3z+8y^2z^2+7yz^3+10z^4$	0	15	24
f_{96}	$x^4+6x^3y+3x^2y^2+11x^2yz+2x^2z^2+9xy^3-xy^2z+6xyz^2-y^4+2y^3z+9y^2z^2+yz^3+5z^4$	1	16	28

f_{97}	$x^4+5x^3y+10x^3z+9x^2y^2+5x^2yz+5x^2z^2-xy^3+7xy^2z+4xyz^2+11xz^3+6y^4+5y^3z+2y^2z^2+yz^3+7z^4$	1	16	27
f_{98}	$x^4+3x^3y+10x^3z+2x^2y^2+11x^2yz+x^2z^2+7xy^3+4xy^2z+6xyz^2+5xz^3+10y^3z+7y^2z^2+6yz^3$	1	17	28
f_{99}	$x^4+7x^3y+6x^3z+10x^2y^2+7x^2yz+8x^2z^2+7xy^3+9xy^2z+xyz^2+11xz^3+8y^4+10y^3z+5yz^3+3z^4$	2	14	28
f_{100}	$x^4+4x^3y+11x^3z+x^2y^2+10x^2yz+7x^2z^2+xy^3+4xy^2z+9xyz^2+2y^4+10y^3z+8y^2z^2+7yz^3+7z^4$	2	14	22
f_{101}	$x^4-x^3y+x^3z+3x^2y^2+6x^2yz+7x^2z^2+4xy^3+7xyz^2+11y^4+3y^3z-y^2z^2+7yz^3$	3	19	30
f_{102}	$x^4-x^3y+x^3z+3x^2y^2+6x^2yz+7x^2z^2+4xy^3+7xyz^2+11y^4+3y^3z-y^2z^2+7yz^3$	3	19	30
f_{103}	$x^4+10x^3y+10x^3z+11x^2y^2+11x^2yz+3xy^3+xy^2z+2xyz^2+xz^3+11y^4+10y^3z+10y^2z^2+2yz^3+10z^4$	2	14	24
f_{104}	$x^4-x^3y+8x^3z+3x^2y^2-x^2yz+3x^2z^2+4xy^3+11xy^2z+7xyz^2+10xz^3+3y^4+10y^3z+y^2z^2+10yz^3+2z^4$	0	15	27
f_{105}	$3x^3y+3x^3z+11x^2yz+4xy^3+4xy^2z+5xyz^2-xz^3+y^4+7y^3z-y^2z^2$	0	15	27
f_{106}	$x^4+9x^3y-x^3z+8x^2y^2+2x^2yz-x^2z^2+10xy^3+xy^2z+11xyz^2+9xz^3+4y^4+3y^3z+8y^2z^2+8yz^3+6z^4$	3	15	21
f_{107}	$x^4+5x^3y+11x^3z+8x^2y^2+6x^2yz+8x^2z^2-xy^3+8xy^2z+2xz^3+7y^4+5y^3z+4y^2z^2+4yz^3+9z^4$	0	16	26
f_{108}	$6x^3y+10x^3z+8x^2yz+11x^2z^2+3xy^3+10xy^2z+5xyz^2+2xz^3+y^4+9y^3z+9y^2z^2+2yz^3+8z^4$	1	19	29
f_{109}	$x^4+6x^3y+3x^3z+2x^2y^2+9x^2yz+5xy^2z+8xz^3+7y^4+3y^3z+7yz^3+9z^4$	0	14	21
f_{110}	$x^4+5x^3z+9x^2y^2-x^2yz-x^2z^2+4xy^3+6xy^2z+10xyz^2+10xz^3+10y^4+4y^3z+3y^2z^2+11yz^3+11z^4$	2	16	23
f_{111}	$x^4+10x^3y+6x^3z+x^2y^2+2x^2yz+10x^2z^2+xy^3+2xy^2z+11xyz^2-xz^3+11y^4+11y^3z+3y^2z^2+8yz^3+10z^4$	2	14	20
f_{112}	$x^4+x^3y+2x^3z+10x^2y^2+11x^2yz+x^2z^2-xy^3+xy^2z+3xyz^2+3xz^3-y^4+6y^3z+8y^2z^2+3yz^3+10z^4$	0	15	23

f_{113}	$x^4+3x^3y+11x^3z+7x^2y^2+3x^2yz+10x^2z^2+10xy^3+3xy^2z+4xz^3+11y^4+11y^3z-y^2z^2+7yz^3+8z^4$	0	15	27
f_{114}	$x^4+3x^3y+11x^3z+7x^2y^2+3x^2yz+10x^2z^2+10xy^3+3xy^2z+4xz^3+11y^4+11y^3z-y^2z^2+7yz^3+8z^4$	0	14	27
f_{115}	$x^4+x^3y+11x^3z+11x^2y^2+9x^2yz-x^2z^2+6xy^3+6xy^2z+8xz^3+11y^4+y^3z+6y^2z^2+9yz^3-z^4$	1	16	25
f_{116}	$x^4+9x^3y+11x^3z+x^2y^2-x^2yz+11x^2z^2+xy^3+2xyz^2+5xz^3+6y^4+9y^3z+9y^2z^2+3yz^3+2z^4$	1	14	24
f_{117}	$4x^3y+7x^3z+6x^2y^2+2x^2yz+x^2z^2+5xy^3+8xy^2z+11xyz^2+7xz^3+y^4-y^3z+6y^2z^2+7yz^3+6z^4$	0	17	24
f_{118}	$x^4+6x^3y+11x^2y^2+4x^2yz+2x^2z^2+11xy^3+3xy^2z+6xyz^2+11xz^3+5y^4+2y^3z+y^2z^2+8yz^3+11z^4$	1	16	26
f_{119}	$x^4+x^3y+9x^3z+5x^2y^2+4x^2yz-x^2z^2+9xy^3+xy^2z+2xyz^2+4xz^3+6y^4+5y^3z+y^2z^2+9yz^3+11z^4$	0	15	24

5.11 Quartic curves for complete $(33;4)$-arcs

There are 77 associated quartic curves of the complete $(33;4)$-arcs. It took 2395 msc. These curves are divided into 34 curves that do not have inflexion points and 43 curves whose number I of inflexions is $1,2$ or 3. Here, the number $|\mathscr{C}\cap\mathscr{K}|$ for the 77 quartic curves ranges between 14 and 19 points. Also, the number N_1 for these curves is between 21 and 30. It took 4289 msec. The statistics of the quartic curves are given in Table 5.10.

Table 5.10: **Quartic curves for complete (33;4)-arcs**

Symbol	$\mathscr{C}(f_i)$	I	$\lvert\mathscr{C}\cap\mathscr{K}\rvert$	N_1
f_1	$x^4+5x^3y+8x^3z-x^2y^2+11x^2yz+x^2z^2+7xy^3+2xyz^2+6xz^3+4y^4+2y^3z+10y^2z^2-yz^3+z^4$	0	14	23
f_2	$x^4+11x^3y+9x^2y^2+7x^2yz+2x^2z^2+6xy^3+6xy^2z+2xyz^2+6y^4+5y^3z+9y^2z^2+7yz^3+5z^4$	0	16	25
f_3	$x^4+6x^3y-x^3z+x^2y^2-x^2z^2-xy^2z+xyz^2+9xz^3+7y^4+y^3z+3y^2z^2+9yz^3+6z^4$	0	16	26
f_4	$x^4+9x^3y+7x^2y^2+3x^2yz+4x^2z^2+xy^3+9xyz^2+5xz^3+10y^4+7y^3z+3y^2z^2+2z^4$	0	17	28
f_5	$x^4+5x^3y+4x^3z+11x^2y^2+7x^2yz+x^2z^2+6xy^3+4xy^2z+xyz^2+3xz^3+7y^4+5y^3z+8y^2z^2+8yz^3+z^4$	2	15	25

f_6	$x^4+4x^3y+x^3z+9x^2y^2-x^2yz+3x^2z^2+2xy^3+7xy^2z+5xyz^2+4xz^3+7y^4+4y^3z+y^2z^2+3yz^3+9z^4$	0	15	23
f_7	$x^4+11x^3z+2x^2yz+9x^2z^2+10xy^3+4xy^2z+4xyz^2+5xz^3+6y^4-y^3z+9y^2z^2+yz^3+7z^4$	2	14	25
f_8	$x^4+3x^3y+2x^2y^2+10x^2yz+10xy^3+10xy^2z+3xyz^2+7xz^3-y^4+y^3z+4y^2z^2+8yz^3+5z^4$	2	14	25
f_9	$x^4+6x^3z-x^2y^2+4x^2yz+7xy^3+5xy^2z+4xyz^2+11xz^3+10y^4-y^3z+11y^2z^2-yz^3+10z^4$	2	14	23
f_{10}	$x^4+6x^3y+2x^3z+x^2y^2+7x^2yz+7x^2z^2+3xy^2z+5xyz^2+8xz^3+6y^4+6y^3z+7y^2z^2+3yz^3-z^4$	0	15	25
f_{11}	$x^4+5x^3y+7x^3z+9x^2y^2-x^2yz+x^2z^2+11xy^3+7xy^2z+3xyz^2+8xz^3+9y^4+5y^3z+9y^2z^2+11yz^3+4z^4$	1	17	25
f_{12}	$x^4+11x^3y+8x^3z+3x^2y^2+11x^2yz+2x^2z^2+xy^3+3xy^2z+9xyz^2+6xz^3+y^4+6y^3z+y^2z^2+8yz^3+5z^4$	2	15	21
f_{13}	$x^4+5x^3y+5x^3z+6x^2y^2+10x^2yz-x^2z^2+5xy^3+8xy^2z+9xyz^2+7xz^3+4y^4+4y^3z+9yz^3+6z^4$	2	16	26
f_{14}	$x^4+6x^3y-x^3z+5x^2y^2+8x^2yz+9x^2z^2+6xy^3+10xy^2z+3xyz^2+5xz^3+y^4+y^3z+9y^2z^2+8z^4$	0	14	28
f_{15}	$x^4+6x^3y+10x^3z+4x^2z^2+10xy^3+9xy^2z+xz^3+y^4-y^3z+2y^2z^2+3yz^3$	0	16	29
f_{16}	$x^4+8x^3y+8x^3z+9x^2y^2+3x^2yz+7x^2z^2+6xy^3+8xyz^2+6xz^3+3y^4+5y^2z^2+9yz^3-z^4$	0	15	21
f_{17}	$x^4+2x^3y+5x^3z+9x^2y^2-x^2yz+7x^2z^2+10xy^3+11xy^2z+5xyz^2+7xz^3+4y^4+4y^3z+8y^2z^2+9yz^3-z^4$	0	14	24
f_{18}	$x^4+5x^3y+5x^3z+6x^2y^2+10x^2yz-x^2z^2+5xy^3+8xy^2z+9xyz^2+7xz^3+4y^4+4y^3z+9yz^3+6z^4$	2	16	26
f_{19}	$x^4-x^3y+10x^3z+11x^2y^2+7x^2yz+7x^2z^2+10xy^3-xy^2z+3xyz^2+xz^3+2y^4+7y^3z+8y^2z^2+6yz^3-z^4$	0	15	23
f_{20}	$x^4+3x^3y+10x^3z+2x^2y^2+11x^2yz+x^2z^2+7xy^3+4xy^2z+6xyz^2+5xz^3+10y^3z+7y^2z^2+6yz^3$	1	19	28
f_{21}	$x^4+3x^3y+7x^3z+11x^2y^2+x^2yz+7x^2z^2+2xy^3+xy^2z+7xyz^2+8xz^3-y^4+11y^3z+8y^2z^2-yz^3+4z^4$	2	15	28
f_{22}	$x^4+2x^3y+2x^3z+9x^2y^2+2x^2yz+2x^2z^2+10xy^3-xy^2z+2xyz^2+9xz^3+9y^4+4y^2z^2+2yz^3+8z^4$	0	14	29
f_{23}	$x^4+11x^3y+8x^3z+3x^2y^2+11x^2yz+2x^2z^2+xy^3+3xy^2z+9xyz^2+6xz^3+y^4+6y^3z+y^2z^2+8yz^3+5z^4$	2	15	21

f_{24}	$x^4+11x^3y-x^3z+4x^2y^2+8x^2yz+7x^2z^2+8xy^3+4xy^2z+11xyz^2+6xz^3+11y^4+4y^3z+9y^2z^2+3yz^3+3z^4$	0	14	24
f_{25}	$x^4+7x^3y+11x^3z+x^2y^2-x^2yz+2x^2z^2+6xy^3+xy^2z+5xz^3+4y^4+8y^3z+5y^2z^2+5yz^3+5z^4$	0	14	24
f_{26}	$x^4+11x^3y-x^3z+5x^2y^2+7x^2yz+3x^2z^2+2xy^3+7xy^2z+9xyz^2+8xz^3+10y^4+2y^3z+2y^2z^2+6z^4$	0	15	24
f_{27}	$x^4+11x^3y-x^3z+4x^2y^2+8x^2yz+7x^2z^2+8xy^3+4xy^2z+11xyz^2+6xz^3+11y^4+4y^3z+9y^2z^2+3yz^3+3z^4$	0	14	24
f_{28}	$x^4+x^3z+3x^2y^2-x^2yz+7x^2z^2+2xy^3+2xy^2z+xyz^2+7xz^3+3y^4+6y^3z+5yz^3+8z^4$	0	16	27
f_{29}	$x^4+11x^3y+10x^3z+6x^2y^2+4x^2yz+xy^3+2xy^2z+2xyz^2+5xz^3+5y^4+8y^2z^2+7z^4$	3	15	25
f_{30}	$x^4+7x^3y+11x^3z+x^2y^2-x^2yz+2x^2z^2+6xy^3+xy^2z+5xz^3+4y^4+8y^3z+5y^2z^2+5yz^3+5z^4$	0	14	24
f_{31}	$x^4+11x^3y-x^3z+5x^2y^2+7x^2yz+3x^2z^2+2xy^3+7xy^2z+9xyz^2+8xz^3+10y^4+2y^3z+2y^2z^2+6z^4$	1	15	24
f_{32}	$9x^3y+5x^3z+9x^2yz+6x^2z^2+2xy^3+11xy^2z+6xyz^2+y^4+4y^3z+11y^2z^2-yz^3+3z^4$	0	15	22
f_{33}	$x^4+6x^3y+3x^3z+8x^2yz+9x^2z^2+2xy^3+10xy^2z-xyz^2-xz^3+y^4+9y^3z+10y^2z^2+7z^4$	1	14	27
f_{34}	$x^4+6x^3y+3x^3z+8x^2yz+9x^2z^2+2xy^3+10xy^2z-xyz^2-xz^3+y^4+9y^3z+10y^2z^2+7z^4$	1	14	27
f_{35}	$x^4+11x^3y+10x^3z+6x^2y^2+4x^2yz+xy^3+2xy^2z+2xyz^2+5xz^3+5y^4+8y^2z^2+7z^4$	3	15	25
f_{36}	$x^4+11x^3y+10x^3z+6x^2y^2+4x^2yz+xy^3+2xy^2z+2xyz^2+5xz^3+5y^4+8y^2z^2+7z^4$	3	15	25
f_{37}	$x^4+x^3y+6x^3z+10x^2y^2+10x^2yz+9x^2z^2+xy^2z+10xyz^2+8xz^3+y^4+11y^3z+6y^2z^2+6yz^3+11z^4$	0	15	22
f_{38}	$x^4+3x^3y+6x^3z+8x^2y^2+x^2yz+11x^2z^2+11xy^3+4xy^2z+6xyz^2+6xz^3+9y^4+2y^2z^2+2yz^3$	2	16	25
f_{39}	$x^4-x^3y+5x^3z+10x^2y^2+4x^2yz+5x^2z^2+7xy^3+9xy^2z+2xyz^2+5xz^3+3y^4+3y^3z+4y^2z^2+11yz^3+5z^4$	3	17	27
f_{40}	$x^4+11x^3y+10x^3z+6x^2y^2+4x^2yz+xy^3+2xy^2z+2xyz^2+5xz^3+5y^4+8y^2z^2+7z^4$	3	15	25
f_{41}	$x^4-x^3y+5x^3z+10x^2y^2+4x^2yz+5x^2z^2+7xy^3+9xy^2z+2xyz^2+5xz^3+3y^4+3y^3z+4y^2z^2+11yz^3+5z^4$	3	17	27

f_{42}	$x^4-x^3y+8x^3z+4x^2y^2+4x^2yz+2xy^3+xy^2z+8xyz^2+6xz^3+9y^4+3y^3z+7yz^3+10z^4$	0	15	22
f_{43}	$x^4+6x^3y+10x^3z+9x^2y^2+11x^2yz-x^2z^2+7xy^3+8xy^2z+10xyz^2+3xz^3+3y^4+6y^3z+6y^2z^2+10yz^3-z^4$	0	16	26
f_{44}	$-x^3y+8x^3z+6x^2y^2-x^2yz-x^2z^2+9xy^3+7xy^2z+8xyz^2+7xz^3+y^4+7y^3z+5y^2z^2+9yz^3-z^4$	2	15	23
f_{45}	$x^4+2x^3y+3x^3z+10x^2y^2+8x^2yz+11x^2z^2+5xy^3-xy^2z+7xyz^2+7xz^3+11y^4+7y^3z+2yz^3+6z^4$	1	14	24
f_{46}	$x^4+11x^3y+4x^3z+4x^2y^2+6x^2yz+7x^2z^2+5xy^3+6xy^2z+8xyz^2+xz^3+8y^4+3y^3z+8y^2z^2+3yz^3+10z^4$	1	15	25
f_{47}	$7x^3y+3x^3z+11x^2y^2+4x^2yz+8x^2z^2+6xy^3+2xy^2z+5xyz^2+y^4+2y^3z+7y^2z^2-yz^3+9z^4$	1	18	30
f_{48}	$x^4+x^3y+9x^3z+4x^2y^2-x^2yz+9x^2z^2+3xy^3-xy^2z+11xyz^2+xz^3+6y^4-y^3z+4yz^3+6z^4$	1	14	23
f_{49}	$x^4+8x^3y+11x^3z+7x^2y^2+3x^2yz-x^2z^2+8xy^2z-xyz^2+8xz^3+3y^4+y^3z+3y^2z^2-yz^3$	1	15	22
f_{50}	$x^4+3x^3y+4x^3z+5x^2z^2+10xy^3+9xy^2z+2xyz^2+11xz^3-y^4+10y^3z+10y^2z^2+4yz^3+6z^4$	2	16	27
f_{51}	$x^4+2x^3y+2x^3z+9x^2y^2+2x^2yz+2x^2z^2+10xy^3-xy^2z+2xyz^2+9xz^3+9y^4+4y^2z^2+2yz^3+8z^4$	0	14	29
f_{52}	$x^4-x^3y+5x^3z+7x^2y^2+x^2yz+x^2z^2+2xy^3+8xy^2z+2xyz^2+8xz^3+7y^4+4y^3z+7y^2z^2+11yz^3+5z^4$	1	15	27
f_{53}	$x^4+3x^3y-x^3z+2x^2y^2+9x^2yz+8x^2z^2+10xy^3+2xy^2z+5xyz^2+10xz^3+6y^4+9y^3z+11y^2z^2+3yz^3$	0	19	28
f_{54}	$7x^3y+3x^3z+11x^2y^2+4x^2yz+8x^2z^2+6xy^3+2xy^2z+5xyz^2+y^4+2y^3z+7y^2z^2-yz^3+9z^4$	1	18	30
f_{55}	$x^4+x^3y+5x^3z+11x^2y^2+x^2yz+10x^2z^2+9xy^3+2xy^2z+10xyz^2+3xz^3+2y^4+9y^3z+9y^2z^2+yz^3-z^4$	2	14	26
f_{56}	$x^4+2x^3y+4x^3z+4x^2y^2+4x^2yz+3xy^3+xy^2z+11xyz^2+3xz^3+11y^4+9y^3z+2y^2z^2+10z^4$	2	17	25
f_{57}	$x^4+10x^3y+6x^3z+5x^2y^2+9x^2yz+8x^2z^2+7xy^3+9xy^2z+2xyz^2+8xz^3-y^4+6y^3z+5y^2z^2-yz^3+z^4$	2	15	26
f_{58}	$x^4+x^3y+7x^3z+9x^2yz+4x^2z^2+9xy^3+10xy^2z+6xyz^2+2xz^3+4y^4+2y^3z+6yz^3$	0	15	29
f_{59}	$x^4+3x^3y-x^3z+2x^2y^2+9x^2yz+8x^2z^2+10xy^3+2xy^2z+5xyz^2+10xz^3+6y^4+9y^3z+11y^2z^2+3yz^3$	0	19	28

f_{60}	$x^4+7x^3y-x^3z+6x^2y^2-x^2yz+4x^2z^2+2xy^3+6xy^2z+11xyz^2+9xz^3+4y^4+4yz^3$	1	17	30
f_{61}	$x^4+3x^3y-x^3z+2x^2y^2+9x^2yz+8x^2z^2+10xy^3+2xy^2z+5xyz^2+10xz^3+6y^4+9y^3z+11y^2z^2+3yz^3$	0	19	28
f_{62}	$x^4+6x^3y-x^3z+x^2y^2-x^2z^2-xy^2z+xyz^2+9xz^3+7y^4+y^3z+3y^2z^2+9yz^3+6z^4$	0	15	26
f_{63}	$x^4+6x^3y-x^3z+x^2y^2-x^2z^2-xy^2z+xyz^2+9xz^3+7y^4+y^3z+3y^2z^2+9yz^3+6z^4$	0	15	26
f_{64}	$x^4+6x^3y-x^3z+x^2y^2-x^2z^2-xy^2z+xyz^2+9xz^3+7y^4+y^3z+3y^2z^2+9yz^3+6z^4$	0	15	26
f_{65}	$x^4+9x^3y+7x^2y^2+3x^2yz+4x^2z^2+xy^3+9xyz^2+5xz^3+10y^4+7y^3z+3y^2z^2+2z^4$	0	16	28
f_{66}	$x^4+5x^3y+4x^3z+11x^2y^2+7x^2yz+x^2z^2+6xy^3+4xy^2z+xyz^2+3xz^3+7y^4+5y^3z+8y^2z^2+8yz^3+z^4$	2	14	25
f_{67}	$x^4+4x^3y+x^3z+9x^2y^2-x^2yz+3x^2z^2+2xy^3+7xy^2z+5xyz^2+4xz^3+7y^4+4y^3z+y^2z^2+3yz^3+9z^4$	0	14	23
f_{68}	$x^4+3x^3y+2x^2y^2+10x^2yz+10xy^3+10xy^2z+3xyz^2+7xz^3-y^4+y^3z+4y^2z^2+8yz^3+5z^4$	2	14	25
f_{69}	$x^4+11x^3y+8x^3z+3x^2y^2+11x^2yz+2x^2z^2+xy^3+3xy^2z+9xyz^2+6xz^3+y^4+6y^3z+y^2z^2+8yz^3+5z^4$	2	14	21
f_{70}	$x^4+8x^3y+8x^3z+9x^2y^2+3x^2yz+7x^2z^2+6xy^3+8xyz^2+6xz^3+3y^4+5y^2z^2+9yz^3-z^4$	0	14	21
f_{71}	$x^4+5x^3y+5x^3z+6x^2y^2+10x^2yz-x^2z^2+5xy^3+8xy^2z+9xyz^2+7xz^3+4y^4+4y^3z+9yz^3+6z^4$	2	15	26
f_{72}	$x^4+3x^3y+7x^3z+11x^2y^2+x^2yz+7x^2z^2+2xy^3+xy^2z+7xyz^2+8xz^3-y^4+11y^3z+8y^2z^2-yz^3+4z^4$	2	14	28
f_{73}	$x^4+11x^3y+8x^3z+3x^2y^2+11x^2yz+2x^2z^2+xy^3+3xy^2z+9xyz^2+6xz^3+y^4+6y^3z+y^2z^2+8yz^3+5z^4$	2	14	21
f_{74}	$x^4+6x^3y-x^3z+5x^2y^2+8x^2yz+9x^2z^2+6xy^3+10xy^2z+3xyz^2+5xz^3+y^4+y^3z+9y^2z^2+8z^4$	0	14	28
f_{75}	$x^4+11x^3y-x^3z+5x^2y^2+7x^2yz+3x^2z^2+2xy^3+7xy^2z+9xyz^2+8xz^3+10y^4+2y^3z+2y^2z^2+6z^4$	1	14	24
f_{76}	$x^4+11x^3y+10x^3z+6x^2y^2+4x^2yz+xy^3+2xy^2z+2xyz^2+5xz^3+5y^4+8y^2z^2+7z^4$	3	15	25
f_{77}	$x^4+3x^3y+6x^3z+8x^2y^2+x^2yz+11x^2z^2+11xy^3+4xy^2z+6xyz^2+6xz^3+9y^4+2y^2z^2+2yz^3$	2	16	25

5.12 Quartic curves for complete $(34;4)$-arcs

From Table 5.11, there are 24 quartic curves; this took 2221 msc. These curves divide into 15 quartic curves that do not have inflexion points and 9 quartic curves that have 1 or 2 inflexion points. In addition, the number $|\mathscr{C} \cap \mathscr{K}|$ of rational points of each of the 24 curves on the corresponding arc ranges between 14 and 17 points. Furthermore, the number N_1 of rational points of the curves ranges between 21 and 30 points. It took 3100 msec.

Table 5.11: **Quartic curves for complete (34;4)-arcs**

Symbol	$\mathscr{C}(f_i)$	I	$\lvert\mathscr{C} \cap \mathscr{K}\rvert$	N_1
f_1	$x^4+11x^3y+11x^3z+x^2yz+9x^2z^2+5xy^3+2xy^2z+8xyz^2+2xz^3+3y^4+4y^3z+6y^2z^2-yz^3+3z^4$	0	15	27
f_2	$x^4+5x^3y+9x^3z+3x^2y^2+4x^2yz+x^2z^2+10xy^3+2xy^2z+3xyz^2+2xz^3+3y^4+8y^3z+10y^2z^2+yz^3+3z^4$	0	14	22
f_3	$x^4+x^3y+5x^3z+5x^2y^2-x^2yz+4x^2z^2+9xy^3+11xy^2z+8xyz^2+5xz^3+3y^4+3y^3z+y^2z^2+3yz^3+7z^4$	0	15	25
f_4	$x^4+x^3y+5x^2y^2+9x^2yz+10x^2z^2+xy^3+3xy^2z+3xyz^2+y^4+10y^2z^2+11z^4$	2	15	24
f_5	$x^4-x^3y+5x^3z+9x^2y^2+x^2yz+x^2z^2+7xy^3+5xy^2z+11xyz^2-xz^3+7y^4+7y^3z+10yz^3+3z^4$	0	14	23
f_6	$x^4+2x^3y+4x^3z+8x^2y^2+9x^2yz+6x^2z^2+3xy^3+3xy^2z+3xyz^2+3xz^3+6y^4+7y^3z+9y^2z^2+3yz^3+8z^4$	1	14	21
f_7	$x^4+2x^3y+2x^3z+9x^2y^2+2x^2yz+2x^2z^2+10xy^3-xy^2z+2xyz^2+9xz^3+9y^4+4y^2z^2+2yz^3+8z^4$	0	14	29
f_8	$x^4+3x^3y+9x^3z-x^2yz+11x^2z^2+5xy^3+xy^2z+xyz^2+10xz^3+6y^4+y^3z+5y^2z^2+4yz^3+2z^4$	0	16	26
f_9	$x^4-x^3y+3x^3z+9x^2y^2+9x^2yz+5x^2z^2-xy^3+5xy^2z+9xyz^2+5xz^3+2y^4+4y^3z+3y^2z^2+4yz^3+9z^4$	0	16	28
f_{10}	$x^3y+10x^3z+6x^2y^2+5x^2yz+4x^2z^2+10xy^3+7xy^2z+5xyz^2+2xz^3+4y^3z+11yz^3$	1	14	21
f_{11}	$x^4+10x^3y+4x^3z+4x^2y^2+x^2yz+3x^2z^2+3xy^3+3xy^2z+6xyz^2-y^4+11y^3z+4y^2z^2+3yz^3$	0	15	24
f_{12}	$2x^3y+8x^3z+3x^2y^2+4x^2yz+5x^2z^2+5xy^3+xy^2z+xyz^2+8xz^3+y^4+9y^2z^2+3yz^3+6z^4$	0	17	24

f_{13}	$x^4-x^3y+6x^3z+2x^2y^2+4x^2yz+6x^2z^2+9xy^3+5xy^2z+9xyz^2+11xz^3+10y^4+y^3z+3y^2z^2+4yz^3+8z^4$	1	16	25
f_{14}	$x^4+10x^3y+11x^3z+x^2y^2+3x^2yz+5x^2z^2+2xyz^2+5xz^3+4y^4+9y^3z+4y^2z^2+10yz^3+4z^4$	2	16	26
f_{15}	$x^4+2x^3y+x^3z+7x^2y^2+5x^2yz+6x^2z^2+6xy^3+2xy^2z+11xyz^2+8xz^3-y^4+2y^3z+6y^2z^2+10yz^3+7z^4$	2	16	27
f_{16}	$x^4+2x^3y+4x^3z+3x^2y^2+11x^2yz+2x^2z^2+11xy^3+2xy^2z+9xyz^2+3y^3z+10y^2z^2+9yz^3+2z^4$	2	15	26
f_{17}	$x^4+11x^3y+11x^3z+x^2yz+9x^2z^2+5xy^3+2xy^2z+8xyz^2+2xz^3+3y^4+4y^3z+6y^2z^2-yz^3+3z^4$	0	15	27
f_{18}	$x^4+x^3y+5x^3z+5x^2y^2-x^2yz+4x^2z^2+9xy^3+11xy^2z+8xyz^2+5xz^3+3y^4+3y^3z+y^2z^2+3yz^3+7z^4$	0	16	25
f_{19}	$x^4+10x^3y+10x^3z+10x^2y^2+7x^2yz+4x^2z^2+9xy^3-xy^2z+2xyz^2+xz^3+3y^4+3y^3z+10y^2z^2+6yz^3$	2	16	30
f_{20}	$x^4+10x^3y+10x^3z+10x^2y^2+7x^2yz+4x^2z^2+9xy^3-xy^2z+2xyz^2+xz^3+3y^4+3y^3z+10y^2z^2+6yz^3$	2	16	30
f_{21}	$x^4+2x^3y+2x^3z+9x^2y^2+2x^2yz+2x^2z^2+10xy^3-xy^2z+2xyz^2+9xz^3+9y^4+4y^2z^2+2yz^3+8z^4$	0	14	29
f_{22}	$x^4+3x^3y+9x^3z-x^2yz+11x^2z^2+5xy^3+xy^2z+xyz^2+10xz^3+6y^4+y^3z+5y^2z^2+4yz^3+2z^4$	0	15	26
f_{23}	$x^4-x^3y+3x^3z+9x^2y^2+9x^2yz+5x^2z^2-xy^3+5xy^2z+9xyz^2+5xz^3+2y^4+4y^3z+3y^2z^2+4yz^3+9z^4$	0	17	28
f_{24}	$2x^3y+8x^3z+3x^2y^2+4x^2yz+5x^2z^2+5xy^3+xy^2z+xyz^2+8xz^3+y^4+9y^2z^2+3yz^3+6z^4$	0	16	24

5.13 Quartic curves for complete $(35;4)$-arcs

There are 20 quartic curves associated to complete (35;4)-arcs. These curves took 2081 msc. Among them the number of associated quartic curves of the complete (35;4)-arcs that do not have inflexion points is 7. These curves are $\mathscr{C}(f_4)$, $\mathscr{C}(f_{11})$, $\mathscr{C}(f_{12})$, $\mathscr{C}(f_{13})$, $\mathscr{C}(f_{14})$, $\mathscr{C}(f_{17})$ and $\mathscr{C}(f_{20})$. Also, there are 13 quartic curves that have 1, 2 or 3 inflexion points. The number $|\mathscr{C}\cap\mathscr{K}|$ of rational points of each of the 20 quartic curves on the corresponding arc is ranges between 14 and 16. Furthermore, the number N_1 of rational points of the 20 curves is between 23 and 29. It took 3026 msec. The detailed statistics are given in Table 5.12.

Table 5.12: **Quartic curves for complete (35;4)-arcs**

Symbol	$\mathscr{C}(f_i)$	I	$\lvert\mathscr{C}\cap\mathscr{K}\rvert$	N_1
f_1	$x^4+7x^3y+6x^3z+3x^2y^2-x^2yz+9x^2z^2+3xy^3+9xy^2z+4xyz^2+11xz^3+10y^4+y^3z+6y^2z^2+4yz^3+7z^4$	2	14	23
f_2	$x^3y+9x^3z+3x^2y^2+7x^2yz+2x^2z^2+3xy^3+3xy^2z+4xz^3+5y^3z+4y^2z^2+11yz^3$	2	16	27
f_3	$x^4+3x^3y+7x^3z+3x^2y^2-x^2yz+5xy^3+9xy^2z+6xyz^2+2xz^3+2y^4+6y^3z+3y^2z^2+8yz^3+10z^4$	3	14	27
f_4	$x^4+2x^3y+2x^3z+9x^2y^2+2x^2yz+2x^2z^2+10xy^3-xy^2z+2xyz^2+9xz^3+9y^4+4y^2z^2+2yz^3+8z^4$	0	14	29
f_5	$x^4+3x^3y+2x^2y^2+10x^2yz+10xy^3+10xy^2z+3xyz^2+7xz^3-y^4+y^3z+4y^2z^2+8yz^3+5z^4$	2	14	25
f_6	$x^4+3x^3y+7x^3z+3x^2y^2-x^2yz+5xy^3+9xy^2z+6xyz^2+2xz^3+2y^4+6y^3z+3y^2z^2+8yz^3+10z^4$	3	14	27
f_7	$x^4+3x^3y+6x^3z+6x^2y^2-x^2yz+2xy^3+11xy^2z+7xyz^2+10xz^3+10y^4+7y^3z+y^2z^2+2yz^3$	1	16	27
f_8	$x^4+5x^3y+8x^3z+x^2y^2+10x^2z^2+3xy^3+3xy^2z+5xyz^2-xz^3+y^4+11y^3z+7y^2z^2+2yz^3+3z^4$	1	16	25
f_9	$x^4+5x^3y+8x^3z+x^2y^2+10x^2z^2+3xy^3+3xy^2z+5xyz^2-xz^3+y^4+11y^3z+7y^2z^2+2yz^3+3z^4$	1	16	25
f_{10}	$x^4-x^3y+11x^3z+5x^2y^2+6x^2yz+x^2z^2+2xy^3+3xy^2z+10xyz^2+11xz^3+10y^4+3y^3z-y^2z^2+2yz^3+9z^4$	2	15	27
f_{11}	$x^3y+9x^3z+6x^2y^2+6x^2yz+3x^2z^2+10xy^3+9xy^2z-xz^3+7y^3z+6y^2z^2+9yz^3$	0	15	24
f_{12}	$x^4-x^3y+3x^3z+6x^2y^2+x^2yz+x^2z^2+xy^3+xy^2z+8xyz^2-xz^3+9y^4+2y^3z+4y^2z^2-yz^3+z^4$	0	15	26
f_{13}	$x^4-x^3y+3x^3z+9x^2y^2+9x^2yz+5x^2z^2-xy^3+5xy^2z+9xyz^2+5xz^3+2y^4+4y^3z+3y^2z^2+4yz^3+9z^4$	0	16	28
f_{14}	$x^4-x^3y+3x^3z+9x^2y^2+9x^2yz+5x^2z^2-xy^3+5xy^2z+9xyz^2+5xz^3+2y^4+4y^3z+3y^2z^2+4yz^3+9z^4$	0	16	28
f_{15}	$x^3y+9x^3z+3x^2y^2+7x^2yz+2x^2z^2+3xy^3+3xy^2z+4xz^3+5y^3z+4y^2z^2+11yz^3$	2	15	27
f_{16}	$x^4+3x^3y+7x^3z+3x^2y^2-x^2yz+5xy^3+9xy^2z+6xyz^2+2xz^3+2y^4+6y^3z+3y^2z^2+8yz^3+10z^4$	3	14	27
f_{17}	$x^4+2x^3y+2x^3z+9x^2y^2+2x^2yz+2x^2z^2+10xy^3-xy^2z+2xyz^2+9xz^3+9y^4+4y^2z^2+2yz^3+8z^4$	0	14	29
f_{18}	$x^4+3x^3y+7x^3z+3x^2y^2-x^2yz+5xy^3+9xy^2z+6xyz^2+2xz^3+2y^4+6y^3z+3y^2z^2+8yz^3+10z^4$	3	14	27
f_{19}	$x^4+3x^3y+6x^3z+6x^2y^2-x^2yz+2xy^3+11xy^2z+7xyz^2+10xz^3+10y^4+7y^3z+y^2z^2+2yz^3$	1	16	27
f_{20}	$x^4-x^3y+3x^3z+6x^2y^2+x^2yz+x^2z^2+xy^3+xy^2z+8xyz^2-xz^3+9y^4+2y^3z+4y^2z^2-yz^3+z^4$	0	15	26

5.14 The algebraic properties of the quartic curve for the complete $(36;4)$-arc

In Chapter 2, Section 2.37, a new complete $(k;4)$-arc for $k=36$ is investigated. Here, the second largest size of a complete $(k;4)$-arc in PG$(2,13)$ is $k=36$. The quartic curve $\mathscr{C}(f)$ of the complete $(36;4)$-arc that took 1929 msc has polynomial

$$f = x^4+9x^3z+10x^2y^2+8x^2yz+x^2z^2+11xy^2z+9xyz^2+9y^4+5y^3z-y^2z^2+3yz^3.$$

This quartic curve does not have inflexion points. Also, the number $|\mathscr{C}\cap\mathscr{K}|$ of rational points of $\mathscr{C}(f)$ on the associated arc is 14 points, while the number N_1 of rational points of $\mathscr{C}(f)$ is 30. It took 1771 msec. In addition, the group of $\mathscr{C}(f)$ is Z_2. It took 2256 msec. This group partitions the 30 rational points of $\mathscr{C}(f)$ into 16 orbits as follows:

$$\{3,63\},\{5,88\},\{7\},\{9,115\},\{11,34\},\{21,83\},\{22,108\},\{46,176\},\{55,67\},$$
$$\{85,101\},\{95,120\},\{96,159\},\{109,168\},\{124,134\},\{148\},\{151,154\}.$$

The statistics of the quartic curve $\mathscr{C}(f)$ of the complete $(36;4)$-arc are given in Table 5.13.

Table 5.13: **Quartic curve for the complete $(36;4)$-arc**

Symbol	$\mathscr{C}(f)$	I	$\lvert\mathscr{C}\cap\mathscr{K}\rvert$	N_1	S
f	$x^4+9x^3z+10x^2y^2+8x^2yz+x^2z^2+11xy^2z+9xyz^2+9y^4+5y^3z-y^2z^2+3yz^3$	0	14	30	Z_2

5.15 The algebraic properties of the quartic curve for the complete $(38;4)$-arc

The largest size of complete $(k;4)$-arc in PG$(2,13)$ is at least 38 as shown in Chapter 2, Section 2.34. The corresponding quartic curve $\mathscr{C}(g)$ of the complete $(38;4)$-arc that took 1937 msc is given by the following polynomial:

$$g = x^4 + x^3y + 6x^3z + 2x^2y^2 + 10x^2yz + x^2z^2 + 9xy^3 + 7xy^2z + 5xyz^2 + 11y^4 + 6y^3z - y^2z^2 + 11yz^3 + z^4.$$

Here, $\mathscr{C}(g)$ contains one inflexion point. Also the number $|\mathscr{C} \cap \mathscr{K}|$ of rational points on the corresponding arc is 14. In addition, the number N_1 of rational points of $\mathscr{C}(g)$ is 24 points. This number took 2106 msec. The group S of $\mathscr{C}(g)$ is the identity group. It took 2717 msec. This group splits the 24 points into a number of orbits as follows:

$$\{17\}, \{24\}, \{32\}, \{48\}, \{52\}, \{55\}, \{82\}, \{86\}, \{94\}, \{98\}, \{100\}, \{101\}, \{102\}, \{136\}, \{141\}, \{148\}, \{152\}, \{154\}, \{155\}, \{158\}, \{164\}, \{166\}, \{168\}, \{175\}.$$

The statistics of the quartic curve $\mathscr{C}(g)$ of the complete $(38;4)$-arc are given in the following table:

Table 5.14: **Quartic curve for the complete** $(38;4)$**-arc**

Symbol	$\mathscr{C}(g)$	I	$\lvert\mathscr{C} \cap \mathscr{K}\rvert$	N_1	S
g	$x^4 + x^3y + 6x^3z + 2x^2y^2 + 10x^2yz + x^2z^2 + 9xy^3 + 7xy^2z + 5xyz^2 + 11y^4 + 6y^3z - y^2z^2 + 11yz^3 + z^4$	1	14	24	I

5.16 Quartic curves and the Hasse–Weil–Serre bound

In Chapter 1, Section 1.7.8, the Hasse–Weil–Serre bound states that the number N_1 of rational points of the algebraic curve $\mathscr{C}$ over $\mathbf{F}_q$ satisfies the following inequality:

$$|N_1 - (q+1)| \leq g\lfloor 2\sqrt{q} \rfloor ,$$

where g is the genus of the curve $\mathscr{C}$.

This upper bound over the finite field $\mathbf{F}_{13}$ is 32. In this section, there are two examples of algebraic quartic curves attaining the Hasse–Weil–Serre bound. These curves are associated to the complete

$(32;4)$-arcs $\mathscr{K}_1'$ and $\mathscr{K}_2'$. These complete arcs are given as follows:

(1) $\mathscr{K}_1' = \{1,2,3,88,9,21,12,38,42,64,65,74,76,79,95,97,102,110,118,121,125,127,129,$
$141,145,147,149,151,157,165,175,180\}$;

(2) $\mathscr{K}_2' = \{1,2,3,88,22,145,4,5,7,8,28,40,47,48,58,61,75,78,86,97,100,108,111,127,137,$
$156,163,166,167,169,178,179\}$.

The corresponding quartic curves of $\mathscr{K}_1'$ and $\mathscr{K}_2'$ that took 1953 msc are as follows:

(1) $g_1' = x^3y+x^3z+6x^2y^2+8x^2yz+10xy^3+8xy^2z-xyz^2+4xz^3+2y^3z+11y^2z^2+2yz^3$;

(2) $g_2' = x^3y+6x^3z-x^2z^2+5xy^3+6xy^2z+7xyz^2-xz^3+10y^3z+y^2z^2+5yz^3$.

Both the curves $\mathscr{C}(g_1')$ and $\mathscr{C}(g_2')$ have no inflexions. The number of rational points $|V(g_1')\cap\mathscr{K}_1'|$ and $|V(g_2')\cap\mathscr{K}_2'|$ that lie on $\mathscr{K}_1'$ and $\mathscr{K}_2'$ is 32. Also, the number N_1 of rational points on $\mathscr{C}(g_1')$ and $\mathscr{C}(g_2')$ is 32. It took 2098 msec. Furthermore, $\mathscr{K}_1'$ and $\mathscr{K}_2'$ have *sd*-inequivalent classes N_c of secant distribution $\{t_4,t_3,t_2,t_1,t_0\}$ as follows:

$$\{72,0,64,32,15\},\{65,20,46,36,16\}.$$

In addition, the stabiliser group S of $\mathscr{C}(g_1')$ is $((Z_4\times Z_4)\rtimes Z_3)\rtimes Z_2$. It took 2763 msec. This group permutes the 32 rational points of $\mathscr{C}(g_1')$ in one orbit:

$\mathscr{O}rb_1(\mathscr{C}(g_1')) = \{1,2,129,38,147,42,3,110,64,127,74,125,97,149,157,141,76,9,65,79,121,$
$175,151,118,21,95,12,165,102,145,88,180\}$.

Also, the group S of $\mathscr{C}(g_2')$ is D_4. It took 2211 msec. This group partitions the 32 points of $\mathscr{C}(g_2')$ into 5 orbits. These orbits are the following:

$\mathscr{O}rb_1(g_2') = \{1,8,48,100,179,111,178,108\}$,

$\mathscr{O}rb_2(g_2') = \{2,137,3,47,86,7,88,127\}$,

$\mathscr{O}rb_3(g_2') = \{4,40,22,97\}$,

$\mathscr{O}rb_4(g_2') = \{5,145,166,169\}$,

$\mathscr{O}rb_5(g'_2) = \{28, 61, 163, 156, 58, 75, 78, 167\}$.

The statistics of the quartic curves $\mathscr{C}(g'_1)$ and $\mathscr{C}(g'_2)$ are given in Table 5.15.

Table 5.15: **Algebraic statistics of $\mathscr{C}(g'_1)$ and $\mathscr{C}(g'_2)$**

Symbol	$\mathscr{C}(g'_i)$	I	$\lvert\mathscr{C}\cap\mathscr{K}\rvert$	N_1	S
g'_1	$x^3y + x^3z + 6x^2y^2 + 8x^2yz + 10xy^3 + 8xy^2z - xyz^2 + 4xz^3 + 2y^3z + 11y^2z^2 + 2yz^3$	0	32	32	$((Z_4 \times Z_4) \rtimes Z_3) \rtimes Z_2$
g'_2	$x^3y + 6x^3z - x^2z^2 + 5xy^3 + 6xy^2z + 7xyz^2 - xz^3 + 10y^3z + y^2z^2 + 5yz^3$	0	32	32	D_4

Chapter 6

Classification of $(k;4)$-arcs up to projective inequivalence, for $k < 10$

6.1 Introduction

In this chapter, the classification of $(k;4)$-arcs up to projective inequivalence for $k < 10$ in $\mathrm{PG}(2,13)$ is established. The strategy is to start from the projective line $\mathrm{PG}(1,13)$ where there are three projectively inequivalent tetrads. Then this classification is continued to discover if there is a new size of *sd*-inequivalent $(k;4)$-arcs for $k > 38$. The approach here is done by extending each of the 36 *sd*-inequivalent $(10;4)$-arcs found to a desired length.

6.2 Projectively inequivalent $(4;4)$-arcs

In this classification, the number of tetrads is constructed by fixing a triad, $\mathscr{U}_1 = \{1,2,9\}$. There are eleven tetrads containing $\mathscr{U}_1$. The lexicographically least sets in the G-orbits of tetrads, where $G = PGL(2,13)$ took 2104 msec. Then among these canonical sets there are three projectively inequivalent tetrads; this took 1699 msec. Also, the three tetrads have *sd*-equivalent secant distribution. It took 1734. The statistics are shown in Table 6.1.

Table 6.1: **Projectively inequivalent tetrads**

Number	Tetrad	$\{t_4,t_3,t_2,t_1,t_0\}$
1	$\{1,2,9,21\}$	$\{1,0,0,52,130\}$
2	$\{1,2,9,83\}$	$\{1,0,0,52,130\}$
3	$\{1,2,9,115\}$	$\{1,0,0,52,130\}$

Theorem 6.1. *In* PG$(1,13)$*, there are exactly three projectively inequivalent tetrads.*

Remark

Theoretically, there are three inequivalent types of tetrad in PG$(1,13)$. The three types are the harmonic tetrad denoted by $\mathbb{H}$, the equianharmonic tetrad denoted by $\mathbb{E}$, and the tetrad that is neither harmonic nor equianharmonic tetrad denoted by $\mathbb{N}$. These tetrads are determined according to their cross ratios, where for the harmonic tetrad the cross ratio is $-1,2$, or -6. Also, it is -3 or 4 for the equianharmonic tetrad. Then, for the tetrad that is neither harmonic nor equianharmonic it is one of the values $-2,3,-4,5,-5,6$.

6.3 Projectively inequivalent $(5;4)$-arcs

The $(5;4)$-arcs are constructed by adding all the points from the plane, PG$(2,13)$, which are not on the line to each inequivalent tetrad given in Table 6.1. So, the constructed number of $(5;4)$-arcs is 507. The lexicographically least set images of the 507 $(5;4)$-arcs are computed. This shows that the number Φ_4 of projectively inequivalent $(5;4)$-arcs is three. The three $(5;4)$-arcs all have the same secant distribution, that is, $\{1,0,4,58,120\}$. In addition, the stabiliser of each of the three projectively inequivalent $(5;4)$-arcs is $Z_3\times((Z_4\times Z_4)\rtimes Z_2)$, $Z_3\times(Z_8\rtimes Z_2)$, $Z_3\times(\mathrm{SL}(2,3)\rtimes Z_2)$. The statistics are given in the following tables:

Table 6.2: **Projectively inequivalent** $(5;4)$**-arcs**

Number	Φ_4	Stabiliser	$\{t_4,t_3,t_2,t_1,t_0\}$
1	$\{1,2,9,83,3\}$	$Z_3\times((Z_4\times Z_4)\rtimes Z_2)$	$\{1,0,4,58,120\}$
2	$\{1,2,9,21,3\}$	$Z_3\times(Z_8\rtimes Z_2)$	$\{1,0,4,58,120\}$
3	$\{1,2,9,115,3\}$	$Z_3\times(\mathrm{SL}(2,3)\rtimes Z_2)$	$\{1,0,4,58,120\}$

Theorem 6.2. *In* PG$(2,13)$*, there are exactly three projectively inequivalent* $(5;4)$*-arcs.*

Table 6.3: **Points added**

Tetrad	Points added
1	3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 84, 85, 86, 87, 88, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 116, 117, 118, 120, 121, 122, 123, 124, 125, 126, 127, 129, 130, 131, 132, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 145, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 162, 163, 164, 165, 166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 183
2	3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 84, 85, 86, 87, 88, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 116, 117, 118, 120, 121, 122, 123, 124, 125, 126, 127, 129, 130, 131, 132, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 145, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 162, 163, 164, 165, 166, 167,168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 183
3	3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 37, 38, 39,40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 84, 85, 86, 87, 88,90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 116, 117, 118, 120, 121, 122, 123, 124, 125, 126, 127, 129, 130, 131, 132, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 145, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 162, 163, 164, 165, 166, 167,168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 183

Remark

The stabiliser groups in Table 6.2 split the associated projectively inequivalent $(5;4)$-arcs into 2 orbits. They are given as follows.

(1) The group $Z_3 \times ((Z_4 \times Z_4) \rtimes Z_2)$ partitions the $(5;4)$-arc $\{1,2,9,83,3\}$ into 2 orbits $\{1,9,2,83\}$, $\{3\}$.

(2) The group $Z_3 \times (Z_8 \rtimes Z_2)$ splits the $(5;4)$-arc $\{1,2,9,21,3\}$ into 2 orbits $\{1,9,2,21\},\{3\}$.

(3) The group $Z_3 \times (\mathrm{SL}(2,3) \rtimes Z_2)$ divides the $(5;4)$-arc $\{1,2,9,115,3\}$ into 2 orbits $\{1,2,115,9\}$, $\{3\}$.

6.4 Projectively inequivalent $(6;4)$-arcs

In Table 6.2, for each projectively inequivalent $(5;4)$-arc the points from the plane which are not on any 4-secant are added to construct the $(6;4)$-arcs. Therefore, the number of $(6;4)$-arcs that constructed is 504. Among the 504 $(6;4)$-arcs the lexicographically least set image and the stabiliser are calculated. So, the number Φ_4 of projectively inequivalent $(6;4)$-arcs is 10. Also, the secant distribution $\{t_4,t_3,t_2,t_1,t_0\}$ for each of the 10 projectively inequivalent $(6;4)$-arcs is computed. It shows that there are only two *sd*-inequivalent classes N_c of secant distributions. The statistics of the 10 projectively inequivalent $(6;4)$-arcs are given in the following tables:

Table 6.4: **Projectively inequivalent $(6;4)$-arcs**

Number	Φ_4	Stabiliser	Orbits
1	$\{1,2,9,83,3,4\}$	$Z_2 \times Z_2$	$\{1\},\{2\},\{3,4\},\{9,83\}$
2	$\{1,2,9,21,3,4\}$	Z_2	$\{1\},\{2\},\{3,4\},\{9\},\{21\}$
3	$\{1,2,9,115,3,4\}$	Z_6	$\{1\},\{2,115,9\},\{3,4\}$
4	$\{1,2,9,83,3,8\}$	$Z_4 \times Z_2$	$\{1,9,2,83\},\{3,8\}$
5	$\{1,2,9,21,3,5\}$	Z_2	$\{1\},\{2\},\{3,5\},\{9\},\{21\}$
6	$\{1,2,9,21,3,12\}$	Z_2	$\{1\},\{2\},\{3,12\},\{9\},\{21\}$
7	$\{1,2,9,21,3,14\}$	$Z_2 \times Z_2$	$\{1,2\},\{3,14\},\{9,21\}$
8	$\{1,2,9,83,3,5\}$	Z_2	$\{1\},\{2\},\{3,5\},\{9\},\{83\}$
9	$\{1,2,9,115,3,7\}$	$Z_2 \times Z_2$	$\{1,115\},\{2,9\},\{3,7\}$
10	$\{1,2,9,115,3,5\}$	Z_6	$\{1\},\{2,115,9\},\{3,5\}$

Table 6.5: N_c **of** $\{t_4,t_3,t_2,t_1,t_0\}$ **of projectively inequivalent $(6;4)$-arcs**

Number	N_c	Number of N_c
1	{ 1, 1, 6, 65, 110 }	3
2	{ 1, 0, 9, 62, 111 }	7

Theorem 6.3. *In* PG$(2,13)$*, there are exactly ten projectively inequivalent* $(6;4)$*-arcs.*

6.5 Projectively inequivalent $(7;4)$-arcs

In this process, the constructed number of $(7;4)$-arcs is 1670. According to their lexicographically least set images, the number of projectively inequivalent $(7;4)$-arcs is 207. Among the 207 arcs, there are eleven types of the stabiliser groups. In addition, the secant distribution $\{t_4,t_3,t_2,t_1,t_0\}$

of each of the $(7;4)$-arcs is also computed. It shows that there are five *sd*-inequivalent classes of secant distributions. The statistics are given in Tables 6.6 and 6.7.

Table 6.6: **Projectively inequivalent $(7;4)$-arcs**

Number	Φ_4	Stabiliser
1	$\{1,2,9,83,3,4,57\}$	I
2	$\{1,2,9,83,3,4,5\}$	I
3	$\{1,2,9,21,3,4,20\}$	I
4	$\{1,2,9,21,3,4,5\}$	I
5	$\{1,2,9,21,3,4,22\}$	I
6	$\{1,2,9,21,3,4,32\}$	I
7	$\{1,2,9,21,3,4,37\}$	I
8	$\{1,2,9,21,3,4,58\}$	I
9	$\{1,2,9,115,3,4,22\}$	Z_2
10	$\{1,2,9,115,3,4,5\}$	I
11	$\{1,2,9,83,3,4,51\}$	I
12	$\{1,2,9,83,3,4,6\}$	I
13	$\{1,2,9,83,3,4,19\}$	I
14	$\{1,2,9,21,3,4,13\}$	I
15	$\{1,2,9,21,3,4,19\}$	I
16	$\{1,2,9,21,3,4,96\}$	I
17	$\{1,2,9,21,3,4,27\}$	I
18	$\{1,2,9,21,3,4,28\}$	I
19	$\{1,2,9,21,3,4,56\}$	I
20	$\{1,2,9,21,3,4,149\}$	I
21	$\{1,2,9,21,3,4,122\}$	I
22	$\{1,2,9,21,3,4,6\}$	I
23	$\{1,2,9,83,3,4,30\}$	I
24	$\{1,2,9,115,3,4,50\}$	I
25	$\{1,2,9,115,3,4,15\}$	I
26	$\{1,2,9,83,3,4,27\}$	I
27	$\{1,2,9,83,3,4,33\}$	I
28	$\{1,2,9,115,3,4,10\}$	I
29	$\{1,2,9,115,3,4,30\}$	I
30	$\{1,2,9,21,3,4,101\}$	I
31	$\{1,2,9,21,3,4,30\}$	I
32	$\{1,2,9,83,3,4,47\}$	I
33	$\{1,2,9,21,3,4,40\}$	I
34	$\{1,2,9,21,3,4,75\}$	I
35	$\{1,2,9,21,3,4,127\}$	I

36	$\{1,2,9,21,3,4,100\}$	I
37	$\{1,2,9,21,3,4,14\}$	I
38	$\{1,2,9,21,3,4,111\}$	I
39	$\{1,2,9,21,3,4,12\}$	I
40	$\{1,2,9,83,3,4,15\}$	I
41	$\{1,2,9,83,3,4,16\}$	I
42	$\{1,2,9,115,3,4,103\}$	I
43	$\{1,2,9,115,3,4,6\}$	I
44	$\{1,2,9,83,3,4,8\}$	I
45	$\{1,2,9,83,3,4,43\}$	I
46	$\{1,2,9,115,3,4,7\}$	I
47	$\{1,2,9,115,3,4,20\}$	I
48	$\{1,2,9,21,3,4,18\}$	I
49	$\{1,2,9,21,3,4,65\}$	I
50	$\{1,2,9,83,3,4,20\}$	I
51	$\{1,2,9,83,3,4,92\}$	I
52	$\{1,2,9,21,3,4,136\}$	Z_2
53	$\{1,2,9,21,3,4,95\}$	I
54	$\{1,2,9,21,3,4,49\}$	I
55	$\{1,2,9,21,3,4,44\}$	I
56	$\{1,2,9,115,3,4,18\}$	I
57	$\{1,2,9,83,3,4,11\}$	D_4
58	$\{1,2,9,83,3,4,31\}$	I
59	$\{1,2,9,83,3,4,10\}$	Z_2
60	$\{1,2,9,83,3,4,17\}$	I
61	$\{1,2,9,83,3,4,49\}$	I
62	$\{1,2,9,83,3,4,23\}$	Z_2
63	$\{1,2,9,83,3,4,28\}$	I
64	$\{1,2,9,83,3,4,54\}$	I
65	$\{1,2,9,83,3,4,13\}$	I
66	$\{1,2,9,83,3,4,37\}$	Z_2
67	$\{1,2,9,83,3,4,40\}$	I
68	$\{1,2,9,83,3,4,26\}$	I
69	$\{1,2,9,83,3,4,76\}$	Z_2
70	$\{1,2,9,83,3,4,25\}$	I
71	$\{1,2,9,83,3,4,7\}$	I
72	$\{1,2,9,83,3,4,82\}$	Z_2
73	$\{1,2,9,83,3,4,71\}$	I
74	$\{1,2,9,83,3,4,108\}$	Z_2
75	$\{1,2,9,83,3,4,126\}$	Z_2
76	$\{1,2,9,83,3,4,14\}$	Z_2

77	$\{1,2,9,83,3,4,130\}$	Z_6
78	$\{1,2,9,83,3,4,100\}$	I
79	$\{1,2,9,21,3,4,35\}$	I
80	$\{1,2,9,115,3,4,24\}$	Z_2
81	$\{1,2,9,21,3,4,16\}$	I
82	$\{1,2,9,21,3,4,43\}$	I
83	$\{1,2,9,21,3,4,46\}$	I
84	$\{1,2,9,21,3,4,51\}$	I
85	$\{1,2,9,115,3,4,8\}$	I
86	$\{1,2,9,115,3,4,16\}$	I
87	$\{1,2,9,21,3,4,50\}$	I
88	$\{1,2,9,21,3,4,82\}$	I
89	$\{1,2,9,21,3,4,17\}$	I
90	$\{1,2,9,21,3,4,152\}$	I
91	$\{1,2,9,21,3,4,76\}$	I
92	$\{1,2,9,21,3,4,55\}$	I
93	$\{1,2,9,21,3,4,94\}$	I
94	$\{1,2,9,115,3,4,17\}$	I
95	$\{1,2,9,115,3,4,33\}$	I
96	$\{1,2,9,115,3,4,34\}$	I
97	$\{1,2,9,21,3,4,8\}$	I
98	$\{1,2,9,21,3,4,57\}$	I
99	$\{1,2,9,21,3,4,103\}$	I
100	$\{1,2,9,21,3,4,47\}$	I
101	$\{1,2,9,21,3,4,48\}$	I
102	$\{1,2,9,21,3,4,34\}$	I
103	$\{1,2,9,21,3,4,26\}$	I
104	$\{1,2,9,21,3,4,108\}$	I
105	$\{1,2,9,21,3,4,25\}$	I
106	$\{1,2,9,115,3,4,74\}$	I
107	$\{1,2,9,115,3,4,26\}$	I
108	$\{1,2,9,21,3,4,15\}$	I
109	$\{1,2,9,21,3,4,7\}$	I
110	$\{1,2,9,21,3,4,118\}$	Z_2
111	$\{1,2,9,115,3,4,35\}$	Z_3
112	$\{1,2,9,21,3,4,23\}$	Z_2
113	$\{1,2,9,21,3,4,71\}$	I
114	$\{1,2,9,21,3,4,110\}$	I
115	$\{1,2,9,21,3,4,74\}$	I
116	$\{1,2,9,21,3,4,54\}$	I

117	$\{1,2,9,21,3,4,31\}$	I
118	$\{1,2,9,21,3,4,33\}$	I
119	$\{1,2,9,21,3,4,66\}$	I
120	$\{1,2,9,21,3,4,130\}$	Z_3
121	$\{1,2,9,21,3,4,77\}$	I
122	$\{1,2,9,115,3,4,32\}$	I
123	$\{1,2,9,115,3,4,14\}$	I
124	$\{1,2,9,115,3,4,130\}$	$Z_3 \times S_3$
125	$\{1,2,9,83,3,8,17\}$	Z_2
126	$\{1,2,9,21,3,5,56\}$	I
127	$\{1,2,9,21,3,12,18\}$	I
128	$\{1,2,9,83,3,5,7\}$	I
129	$\{1,2,9,115,3,7,12\}$	Z_2
130	$\{1,2,9,115,3,7,6\}$	Z_2
131	$\{1,2,9,21,3,5,31\}$	I
132	$\{1,2,9,83,3,8,60\}$	$Z_4 \times Z_2$
133	$\{1,2,9,83,3,8,7\}$	I
134	$\{1,2,9,83,3,8,18\}$	I
135	$\{1,2,9,83,3,8,57\}$	Z_{12}
136	$\{1,2,9,83,3,8,40\}$	Z_2
137	$\{1,2,9,83,3,8,24\}$	I
138	$\{1,2,9,83,3,8,62\}$	I
139	$\{1,2,9,83,3,8,26\}$	I
140	$\{1,2,9,83,3,8,5\}$	I
141	$\{1,2,9,83,3,8,19\}$	Z_4
142	$\{1,2,9,83,3,8,12\}$	I
143	$\{1,2,9,21,3,12,17\}$	I
144	$\{1,2,9,21,3,5,13\}$	Z_2
145	$\{1,2,9,115,3,7,16\}$	Z_2
146	$\{1,2,9,83,3,5,6\}$	I
147	$\{1,2,9,21,3,12,68\}$	Z_2
148	$\{1,2,9,21,3,5,6\}$	I
149	$\{1,2,9,21,3,14,52\}$	$Z_2 \times Z_2$
150	$\{1,2,9,83,3,5,13\}$	Z_2
151	$\{1,2,9,115,3,7,49\}$	$Z_2 \times Z_2$
152	$\{1,2,9,115,3,5,13\}$	Z_6
153	$\{1,2,9,21,3,5,111\}$	I
154	$\{1,2,9,21,3,5,79\}$	I
155	$\{1,2,9,21,3,5,50\}$	I
156	$\{1,2,9,21,3,12,30\}$	I
157	$\{1,2,9,83,3,5,44\}$	I

158	$\{1,2,9,115,3,7,19\}$	Z_2
159	$\{1,2,9,115,3,7,41\}$	I
160	$\{1,2,9,21,3,5,76\}$	I
161	$\{1,2,9,21,3,5,106\}$	I
162	$\{1,2,9,21,3,5,95\}$	I
163	$\{1,2,9,21,3,12,58\}$	I
164	$\{1,2,9,83,3,5,17\}$	I
165	$\{1,2,9,21,3,5,65\}$	I
166	$\{1,2,9,21,3,5,66\}$	I
167	$\{1,2,9,21,3,5,99\}$	I
168	$\{1,2,9,21,3,5,45\}$	I
169	$\{1,2,9,21,3,5,42\}$	I
170	$\{1,2,9,83,3,5,16\}$	I
171	$\{1,2,9,83,3,5,32\}$	I
172	$\{1,2,9,21,3,5,40\}$	Z_3
173	$\{1,2,9,21,3,14,55\}$	Z_2
174	$\{1,2,9,21,3,12,66\}$	Z_3
175	$\{1,2,9,115,3,5,6\}$	Z_3
176	$\{1,2,9,21,3,14,31\}$	Z_6
177	$\{1,2,9,83,3,5,40\}$	Z_3
178	$\{1,2,9,115,3,7,92\}$	Z_6
179	$\{1,2,9,115,3,5,40\}$	$Z_3 \times Z_3$
180	$\{1,2,9,21,3,5,28\}$	I
181	$\{1,2,9,21,3,5,26\}$	I
182	$\{1,2,9,21,3,5,8\}$	Z_2
183	$\{1,2,9,21,3,5,41\}$	I
184	$\{1,2,9,21,3,5,27\}$	I
185	$\{1,2,9,21,3,5,20\}$	I
186	$\{1,2,9,21,3,5,126\}$	I
187	$\{1,2,9,21,3,5,17\}$	I
188	$\{1,2,9,21,3,5,100\}$	I
189	$\{1,2,9,21,3,5,29\}$	Z_2
190	$\{1,2,9,21,3,5,43\}$	I
191	$\{1,2,9,21,3,5,167\}$	I
192	$\{1,2,9,21,3,5,15\}$	I
193	$\{1,2,9,115,3,7,8\}$	I
194	$\{1,2,9,115,3,7,26\}$	I
195	$\{1,2,9,115,3,7,15\}$	I
196	$\{1,2,9,83,3,5,42\}$	I
197	$\{1,2,9,115,3,5,42\}$	Z_3
198	$\{1,2,9,115,3,7,13\}$	I

199	$\{1,2,9,21,3,12,14\}$	Z_2
200	$\{1,2,9,115,3,7,45\}$	Z_3
201	$\{1,2,9,83,3,5,27\}$	I
202	$\{1,2,9,115,3,7,25\}$	I
203	$\{1,2,9,115,3,7,5\}$	I
204	$\{1,2,9,115,3,7,20\}$	Z_3
205	$\{1,2,9,115,3,7,52\}$	Z_2
206	$\{1,2,9,21,3,12,96\}$	Z_2
207	$\{1,2,9,21,3,12,15\}$	I

Table 6.7: N_c **of** $\{t_4,t_3,t_2,t_1,t_0\}$ **of projectively inequivalent** $(7;4)$**-arcs**

Number	N_c	Number of N_c
1	{ 1, 0, 15, 64, 103 }	62
2	{ 1, 1, 12, 67, 102 }	106
3	{ 1, 2, 9, 70, 101 }	30
4	{ 1, 3, 6, 73, 100 }	3
5	{ 2, 0, 9, 72, 100 }	6

Theorem 6.4. *In* PG$(2,13)$*, there are exactly* 207 *projectively inequivalent* $(7;4)$*-arcs.*

Remark

In Table 6.6, there are 11 types of the stabiliser groups as follows:

$$I,\ Z_2,\ Z_3,\ Z_4,\ Z_6,\ D_4,\ Z_3\times S_3,\ Z_4\times Z_2,\ Z_{12},\ Z_2\times Z_2,\ Z_3\times Z_3.$$

These stabiliser groups of order at least two divide their corresponding projectively inequivalent $(7;4)$-arcs into a number of orbits. All orbits of these groups are listed in Table 6.8.

Table 6.8: **Group orbits of projectively inequivalent** $(7;4)$**-arcs in** PG$(2,13)$

Φ_4	Stabiliser	Orbits
$\{1,2,9,115,3,4,22\}$	Z_2	$\{1,2\},\{3\},\{4,22\},\{9,115\}$
$\{1,2,9,21,3,4,136\}$	Z_2	$\{1,2\},\{3,136\},\{4\},\{9,21\}$
$\{1,2,9,83,3,4,11\}$	D_4	$\{1\},\{2,3\},\{4,11,83,9\}$
$\{1,2,9,83,3,4,10\}$	Z_2	$\{1,2\},\{3\},\{4,10\},\{9,83\}$
$\{1,2,9,83,3,4,23\}$	Z_2	$\{1\},\{2\},\{3\},\{4\},\{9,83\},\{23\}$
$\{1,2,9,83,3,4,37\}$	Z_2	$\{1,2\},\{3\},\{4,37\},\{9\},\{83\}$
$\{1,2,9,83,3,4,76\}$	Z_2	$\{1\},\{2\},\{3,4\},\{9,83\},\{76\}$

{1,2,9,83,3,4,82}	Z_2	{1},{2},{3,4},{9,83},{82}
{1,2,9,83,3,4,108}	Z_2	{1},{2},{3,4},{9,83},{108}
{1,2,9,83,3,4,126}	Z_2	{1},{2},{3,4},{9,83},{126}
{1,2,9,83,3,4,14}	Z_2	{1},{2},{3,4},{9,83},{14}
{1,2,9,83,3,4,130}	Z_6	{1},{2},{3,4,130},{9,83}
{1,2,9,115,3,4,24}	Z_2	{1,2},{3,24},{4},{9,115}
{1,2,9,21,3,4,118}	Z_2	{1,2},{3,118},{4},{9,21}
{1,2,9,115,3,4,35}	Z_3	{1,9,115},{2},{3,35,4}
{1,2,9,21,3,4,23}	Z_2	{1},{2,3},{4,21},{9,23}
{1,2,9,21,3,4,130}	Z_3	{1},{2},{3,4,130},{9},{21}
{1,2,9,115,3,4,130}	$Z_3 \times S_3$	{1},{2,3,9,115,130,4}
{1,2,9,83,3,8,17}	Z_2	{1,2},{3,8},{9,83},{17}
{1,2,9,115,3,7,12}	Z_2	{1,9},{2,115},{3,12},{7}
{1,2,9,115,3,7,6}	Z_2	{1,115},{2,9},{3,7},{6}
{1,2,9,83,3,8,60}	$Z_4 \times Z_2$	{1,9,2,83},{3,8},{60}
{1,2,9,83,3,8,57}	Z_{12}	{1,9,2,83},{3,8,57}
{1,2,9,83,3,8,40}	Z_2	{1,2},{3,8},{9,83},{40}
{1,2,9,83,3,8,19}	Z_4	{1,9,2,83},{3},{8},{19}
{1,2,9,21,3,5,13}	Z_2	{1},{2},{3,13},{5},{9},{21}
{1,2,9,115,3,7,16}	Z_2	{1,115},{2,9},{3},{7},{16}
{1,2,9,21,3,12,68}	Z_2	{1},{2},{3},{9},{12,68},{21}
{1,2,9,21,3,14,52}	$Z_2 \times Z_2$	{1,2},{3},{9,21},{14,52}
{1,2,9,83,3,5,13}	Z_2	{1},{2},{3,13},{5},{9},{83}
{1,2,9,115,3,7,49}	$Z_2 \times Z_2$	{1,115},{2,9},{3},{7,49}
{1,2,9,115,3,5,13}	Z_6	{1},{2,115,9},{3,13},{5}
{1,2,9,115,3,7,19}	Z_2	{1,115},{2,9},{3,7},{19}
{1,2,9,21,3,5,40}	Z_3	{1},{2},{3,5,40},{9},{21}
{1,2,9,21,3,14,55}	Z_2	{1,2},{3},{9,21},{14},{55}
{1,2,9,21,3,12,66}	Z_3	{1},{2},{3,12,66},{9},{21}
{1,2,9,115,3,5,6}	Z_3	{1},{2,9,115},{3},{5},{6}
{1,2,9,21,3,14,31}	Z_6	{1,2},{3,14,31},{9,21}
{1,2,9,83,3,5,40}	Z_3	{1},{2},{3,5,40},{9},{83}
{1,2,9,115,3,7,92}	Z_6	{1,115},{2,9},{3,7,92}
{1,2,9,115,3,5,40}	$Z_3 \times Z_3$	{1},{2,9,115},{3,5,40}
{1,2,9,21,3,5,8}	Z_2	{1,2},{3},{5,8},{9,21}
{1,2,9,21,3,5,29}	Z_2	{1,2},{3,29},{5},{9,21}
{1,2,9,115,3,5,42}	Z_3	{1,9,115},{2},{3,5,42}
{1,2,9,21,3,12,14}	Z_2	{1,2},{3,14},{9,21},{12}
{1,2,9,115,3,7,45}	Z_3	{1,9,115},{2},{3,45,7}
{1,2,9,115,3,7,20}	Z_3	{1,2,9},{3,7,20},{115}
{1,2,9,115,3,7,52}	Z_2	{1,115},{2,9},{3,7},{52}
{1,2,9,21,3,12,96}	Z_2	{1,2},{3,96},{9,21},{12}

6.6 Projectively inequivalent $(8;4)$-arcs

In PG$(2,13)$, the number of projectively inequivalent $(8;4)$-arcs is 7399. The stabliser groups of 7399 projectively inequivalent $(8;4)$-arcs are as follows:

$$I,\ Z_2,\ Z_3,\ Z_4,\ Z_6,\ Z_{12},\ Z_2\times Z_2,\ Z_4\times Z_2,\ (Z_4\times Z_4)\rtimes Z_2,\ Z_3\times S_3,\ D_4.$$

The number of these groups is listed in Table 6.9. Also, the 7399 projectively inequivalent $(8;4)$-arcs have eleven *sd*-inequivalent classes of secant distributions as shown in Table 6.10.

Table 6.9: **Group statistics of the projectively inequivalent $(8;4)$-arcs**

Number	Stabiliser	Number of stabiliser
1	I	6895
2	Z_2	443
3	Z_3	12
4	Z_4	15
5	Z_6	4
6	Z_{12}	1
7	$Z_2\times Z_2$	18
8	$Z_4\times Z_2$	2
9	$(Z_4\times Z_4)\rtimes Z_2$	1
10	$Z_3\times S_3$	1
11	D_4	7

Note that the groups of order at least eight are as follows:

$$Z_4\times Z_2,\ Z_{12},\ (Z_4\times Z_4)\rtimes Z_2,\ Z_3\times S_3.$$

These groups partition the associated projectively inequivalent $(8;4)$-arcs into a number of orbits as shown below.

(1) The group Z_{12} splits the $(8;4)$-arc $\{1,2,9,83,3,8,57,19\}$ into 3 orbits of sizes $4,3,1$. They are $\{1,9,2,83\},\{3,8,57\},\{19\}$.

(2) The group $Z_4\times Z_2$ partitions the $(8;4)$-arcs $\{1,2,9,83,3,8,60,19\}$ and $\{1,2,9,83,3,8,57,59\}$ into 3 orbits. They are $\{\{1,9,2,83\},\{3,60\},\{8,19\}\}$ and $\{\{1,9,2,83\},\{3,59\},\{8,57\}\}$.

(3) The group $(Z_4\times Z_4)\rtimes Z_2$ divides the $(8;4)$-arc $\{1,2,9,83,3,8,19,59\}$ into one orbit, that is, $\{1,2,3,19,8,83,59,9\}$.

(4) The group $Z_3 \times S_3$ separates the $(8;4)$-arc $\{1,2,9,115,3,5,6,132\}$ into two orbits of sizes $2,6$. Thay are $\{\{1,5\},\{2,6,9,115,132,3\}\}$.

Table 6.10: N_c **of** $\{t_4,t_3,t_2,t_1,t_0\}$ **of projectively inequivalent** $(8;4)$**-arcs**

Number	N_c	Number of N_c
1	{ 1, 0, 22, 64, 96 }	534
2	{ 1, 1, 19, 67, 95 }	2272
3	{ 1, 2, 16, 70, 94 }	2905
4	{ 1, 3, 13, 73, 93 }	1188
5	{ 2, 0, 16, 72, 93 }	146
6	{ 1, 4, 10, 76, 92 }	182
7	{ 2, 1, 13, 75, 92 }	128
8	{ 1, 5, 7, 79, 91 }	10
9	{ 2, 2, 10, 78, 91 }	30
10	{ 1, 6, 4, 82, 90 }	1
11	{ 2, 3, 7, 81, 90 }	3

Theorem 6.5. *In* PG$(2,13)$*, there are exactly* 7399 *projectively inequivalent* $(8;4)$*-arcs.*

6.7 Projectively inequivalent $(9;4)$-arcs

In PG$(2,13)$, the number of projectively inequivalent $(9;4)$-arcs is 222536 according to the inequivalent lexicographically least set in the G-orbit of each $(9;4)$-arc. These arcs have one of the groups I, Z_2, Z_3, Z_4, Z_6, $Z_2 \times Z_2$, $Z_4 \times Z_2$, D_4, S_3, S_4, A_4. In addition, the secant distribution of each of the 222536 projectively inequivalent arcs is calculated. There are 21 *sd*-inequivalent classes of secant distributions of the projectively inequivalent $(9;4)$-arcs. The statistics are given in Tables 6.11, 6.12, and 6.13.

Table 6.11: **Group statistics of the projectively inequivalent** $(9;4)$**-arcs**

Number	Stabiliser	Number of stabiliser
1	I	220719
2	Z_2	1702
3	Z_3	59
4	Z_4	14
5	Z_6	8
6	$Z_2 \times Z_2$	22
7	$Z_4 \times Z_2$	1
8	S_3	4
9	S_4	2
10	D_4	3
11	A_4	2

Remark

In Table 6.11, the large groups of order at least 4 are Z_4, Z_6, $Z_4 \times Z_2$, S_3, S_4, D_4, A_4. The action of these groups is shown in the following table:

Table 6.12: **Group orbits of projectively inequivalent** $(9;4)$**-arcs in** $\mathrm{PG}(2,13)$

Φ_4	Stabiliser	Orbits
$\{1,2,9,83,3,4,57,99,105\}$	Z_4	$\{1,9,2,83\},\{3,105,57,4\},\{99\}$
$\{1,2,9,83,3,4,5,24,135\}$	Z_4	$\{1\},\{2,4\},\{3,83,135,9\},\{5,24\}$
$\{1,2,9,21,3,4,22,24,108\}$	Z_4	$\{1,2\},\{3,22,24,4\},\{9,21\},\{108\}$
$\{1,2,9,115,3,4,18,151,159\}$	Z_4	$\{1,115\},\{2,9\},\{3,18,159,4\},\{151\}$
$\{1,2,9,83,3,4,30,84,124\}$	Z_4	$\{1,9,2,83\},\{3\},\{4,124,84,30\}$
$\{1,2,9,83,3,4,92,135,118\}$	Z_4	$\{1\},\{2,4\},\{3,83,135,9\},\{92,118\}$
$\{1,2,9,83,3,5,13,49,101\}$	Z_4	$\{1\},\{2\},\{3,49,13,101\},\{5\},\{9,83\}$
$\{1,2,9,21,3,12,68,56,151\}$	Z_4	$\{1,2\},\{3\},\{9,21\},\{12,56,68,151\}$
$\{1,2,9,83,3,5,13,16,33\}$	Z_4	$\{1,2\},\{3,16,13,33\},\{5\},\{9,83\}$
$\{1,2,9,83,3,5,13,58,97\}$	Z_4	$\{1,2\},\{3,58,13,97\},\{5\},\{9\},\{83\}$
$\{1,2,9,83,3,8,17,32,61\}$	Z_4	$\{1,9,2,83\},\{3,32,8,61\},\{17\}$
$\{1,2,9,83,3,8,17,79,147\}$	Z_4	$\{1,9,2,83\},\{3,79,8,147\},\{17\}$
$\{1,2,9,115,3,7,6,154,160\}$	Z_4	$\{1,154,115,160\},\{2,3,9,7\},\{6\}$
$\{1,2,9,21,3,14,31,8,74\}$	Z_4	$\{1,14,2,74\},\{3,21,31,9\},\{8\}$
$\{1,2,9,115,3,4,5,25,148\}$	Z_6	$\{1,3\},\{2,25,115,9,148,5\},\{4\}$
$\{1,2,9,115,3,4,30,43,59\}$	Z_6	$\{1,4\},\{2,43,9,115,59,30\},\{3\}$
$\{1,2,9,115,3,4,18,35,39\}$	Z_6	$\{1,3,2,115,39,35\},\{4,18\},\{9\}$
$\{1,2,9,115,3,4,8,51,130\}$	Z_6	$\{1\},\{2,4,9,3,115,130\},\{8,51\}$
$\{1,2,9,115,3,4,16,37,145\}$	Z_6	$\{1,9,3,115,37,145\},\{2\},\{4\},\{16\}$
$\{1,2,9,115,3,4,32,31,130\}$	Z_6	$\{1\},\{2,115,130,9,3,4\},\{31,32\}$
$\{1,2,9,115,3,4,32,29,130\}$	Z_6	$\{1\},\{2,3,115,4,9,130\},\{29,32\}$
$\{1,2,9,115,3,4,32,130,149\}$	Z_6	$\{1\},\{2,4,115,9,130,3\},\{32,149\}$
$\{1,2,9,83,3,4,57,60,147\}$	$Z_4 \times Z_2$	$\{1,9,3,2,60,83,147,57\},\{4\}$
$\{1,2,9,21,3,4,58,7,80\}$	S_3	$\{1,2,3,7,58,4\},\{9,21,80\}$
$\{1,2,9,115,3,4,5,130,131\}$	S_3	$\{1,2,4\},\{3,9,131,115,5,130\}$
$\{1,2,9,21,3,4,96,163,166\}$	S_3	$\{1,2,163\},\{3,9,21,96,4,166\}$
$\{1,2,9,115,3,4,15,130,45\}$	S_3	$\{1,9,130\},\{2,3,45,115,15,4\}$
$\{1,2,9,83,3,4,11,10,84\}$	S_4	$\{1,2,3\},\{4,9,83,84,11,10\}$
$\{1,2,9,83,3,4,11,37,129\}$	S_4	$\{1,2,3\},\{4,83,11,9,37,129\}$
$\{1,2,9,83,3,4,10,82,86\}$	D_4	$\{1,2\},\{3,86,10,4\},\{9,83\},\{82\}$
$\{1,2,9,115,3,7,12,77,76\}$	D_4	$\{1,2,9,115\},\{3,12,7,77\},\{76\}$
$\{1,2,9,115,3,7,12,70,177\}$	D_4	$\{1,2,9,115\},\{3,70,12,177\},\{7\}$
$\{1,2,9,83,3,4,5,12,135\}$	A_4	$\{1,2,4\},\{3,83,135,5,9,12\}$
$\{1,2,9,83,3,4,92,135,164\}$	A_4	$\{1,2,4\},\{3,83,135,164,9,92\}$

Table 6.13: N_c **of** $\{t_4,t_3,t_2,t_1,t_0\}$ **of projectively inequivalent** $(9;4)$**-arcs**

Number	N_c	Number of N_c
1	{ 1, 0, 30, 62, 90 }	1199
2	{ 1, 1, 27, 65, 89 }	13688
3	{ 1, 2, 24, 68, 88 }	50341
4	{ 1, 3, 21, 71, 87 }	74174
5	{ 2, 0, 24, 70, 87 }	1776
6	{ 1, 4, 18, 74, 86 }	47139
7	{ 2, 1, 21, 73, 86 }	7227
8	{ 2, 2, 18, 76, 85 }	8259
9	{ 1, 5, 15, 77, 85 }	12848
10	{ 1, 6, 12, 80, 84 }	1487
11	{ 2, 3, 15, 79, 84 }	3388
12	{ 3, 0, 18, 78, 84 }	182
13	{ 1, 7, 9, 83, 83 }	68
14	{ 2, 4, 12, 82, 83 }	518
15	{ 3, 1, 15, 81, 83 }	151
16	{ 1, 8, 6, 86, 82 }	2
17	{ 2, 5, 9, 85, 82 }	39
18	{ 3, 2, 12, 84, 82 }	42
19	{ 2, 6, 6, 88, 81 }	2
20	{ 3, 3, 9, 87, 81 }	5
21	{ 3, 4, 6, 90, 80 }	1

Theorem 6.6. *In* $\mathrm{PG}(2,13)$*, there are exactly* 222536 *projectively inequivalent* $(9;4)$*-arcs.*

6.8 Projectively inequivalent $(10;4)$-arcs

The number of $(10;4)$-arcs is paralleled into 5 processes; each took $6:22:54:11$, $4:15:36:77$, $5:09:28:12$, $5:12:40:46$, $3:21:52:13$ of CPU time respectively for the construction. Then according to the canonical images of the $(10;4)$-arcs found from 4 processes, there are at least 5268378 projectively inequivalent $(10;4)$-arcs. This took 2403232618 msc. The 5268378 arcs have 36 sd-inequivalent classes N_c of i-secant distributions as listed in Table 6.14. The total time is 1726578 msc where it was computed in six processes. Then according to the number of N_c there are 36 sd-inequivalent $(10;4)$-arcs, which have five types of stabilisers I, $Z_2\times Z_2$, Z_2, S_3, $Z_3\times S_3$. The timing of these groups was 3633 msec. The statistics of the sd-inequivalent $(10;4)$-arcs are given in Table 6.15.

Table 6.14: N_c **of** $\{t_4, t_3, t_2, t_1, t_0\}$ **of projectively inequivalent** $(10;4)$**-arcs**

Number	N_c	Number of N_c
1	{1, 7, 18, 79, 78 }	192599
2	{1, 0, 39, 58, 85 }	661
3	{1, 1, 36, 61, 84 }	15664
4	{1, 2, 33, 64, 83 }	145027
5	{1, 3, 30, 67, 82 }	592731
6	{1, 4, 27, 70, 81 }	1227187
7	{1, 5, 24, 73, 80 }	1322219
8	{1, 6, 21, 76, 79 }	719144
9	{1, 8, 15, 82, 77 }	24434
10	{1, 9, 12, 85, 76 }	1399
11	{1, 10, 9, 88, 75 }	31
12	{1, 11, 6, 91, 74 }	3
13	{2, 0, 33, 66, 82 }	4572
14	{2, 1, 30, 69, 81 }	52934
15	{2, 2, 27, 72, 80 }	207496
16	{2, 3, 24, 75, 79 }	344994
17	{2, 4, 21, 78, 78 }	255989
18	{2, 5, 18, 81, 77 }	87359
19	{2, 6, 15, 84, 76 }	13784
20	{2, 7, 12, 87, 75 }	954
21	{2, 8, 9, 90, 74 }	38
22	{2, 10, 3, 96, 72 }	1
23	{3, 0, 27, 74, 79 }	3944
24	{3, 1, 24, 77, 78 }	17244
25	{3, 2, 21, 80, 77 }	22990
26	{3, 3, 18, 83, 76 }	11598
27	{3, 4, 15, 86, 75 }	2477
28	{3, 5, 12, 89, 74 }	257
29	{3, 6, 9, 92, 73 }	12
30	{3, 7, 6, 95, 72 }	2
31	{4, 0, 21, 82, 76 }	222
32	{4, 1, 18, 85, 75 }	297
33	{4, 2, 15, 88, 74 }	97
34	{4, 3, 12, 91, 73 }	13
35	{4, 4, 9, 94, 72 }	2
36	{5, 0, 15, 90, 73 }	3

Theorem 6.7. *In* PG$(2,13)$*, there are at least* 5268378 *projectively inequivalent* $(10;4)$*-arcs.*

Table 6.15: *sd*-**inequivalent** (10;4)-**arcs**

Symbol	(10;4)-arc	$\{t_4,t_3,t_2,t_1,t_0\}$	Stabiliser
$\mathscr{K}_1'$	$\{1,2,9,83,3,4,57,6,166,8\}$	$\{1,7,18,79,78\}$	I
$\mathscr{K}_2'$	$\{1,2,9,83,3,8,17,40,72,78\}$	$\{1,0,39,58,85\}$	I
$\mathscr{K}_3'$	$\{1,2,9,83,3,4,6,50,67,63\}$	$\{1,1,36,61,84\}$	I
$\mathscr{K}_4'$	$\{1,2,9,83,3,4,57,166,99,40\}$	$\{1,2,33,64,83\}$	I
$\mathscr{K}_5'$	$\{1,2,9,83,3,4,57,6,107,18\}$	$\{1,3,30,67,82\}$	I
$\mathscr{K}_6'$	$\{1,2,9,83,3,4,57,6,166,33\}$	$\{1,4,27,70,81\}$	I
$\mathscr{K}_7'$	$\{1,2,9,83,3,4,57,6,166,16\}$	$\{1,5,24,73,80\}$	I
$\mathscr{K}_8'$	$\{1,2,9,115,3,4,5,6,7,8\}$	$\{1,6,21,76,79\}$	I
$\mathscr{K}_9'$	$\{1,2,9,115,3,4,5,6,7,90\}$	$\{1,8,15,82,77\}$	I
$\mathscr{K}_{10}'$	$\{1,2,9,83,3,4,5,129,137,178\}$	$\{1,9,12,85,76\}$	I
$\mathscr{K}_{11}'$	$\{1,2,9,83,3,4,5,129,178,104\}$	$\{1,10,9,88,75\}$	I
$\mathscr{K}_{12}'$	$\{1,2,9,83,3,4,5,30,37,51\}$	$\{1,11,6,91,74\}$	$Z_2\times Z_2$
$\mathscr{K}_{13}'$	$\{1,2,9,83,3,4,6,11,167,33\}$	$\{2,0,33,66,82\}$	I
$\mathscr{K}_{14}'$	$\{1,2,9,83,3,4,57,166,11,33\}$	$\{2,1,30,69,81\}$	I
$\mathscr{K}_{15}'$	$\{1,2,9,83,3,4,57,6,11,18\}$	$\{2,2,27,72,80\}$	I
$\mathscr{K}_{16}'$	$\{1,2,9,83,3,4,57,6,166,7\}$	$\{2,3,24,75,79\}$	I
$\mathscr{K}_{17}'$	$\{1,2,9,83,3,4,57,6,166,17\}$	$\{2,4,21,78,78\}$	I
$\mathscr{K}_{18}'$	$\{1,2,9,83,3,4,57,6,166,87\}$	$\{2,5,18,81,77\}$	I
$\mathscr{K}_{19}'$	$\{1,2,9,83,3,4,57,6,166,163\}$	$\{2,6,15,84,76\}$	I
$\mathscr{K}_{20}'$	$\{1,2,9,83,3,4,5,129,137,37\}$	$\{2,7,12,87,75\}$	I
$\mathscr{K}_{21}'$	$\{1,2,9,83,3,4,5,129,68,11\}$	$\{2,8,9,90,74\}$	Z_2
$\mathscr{K}_{22}'$	$\{1,2,9,115,3,4,18,183,35,39\}$	$\{2,10,3,96,72\}$	$Z_3\times S_3$
$\mathscr{K}_{23}'$	$\{1,2,9,83,3,4,57,166,11,51\}$	$\{3,0,27,74,79\}$	I
$\mathscr{K}_{24}'$	$\{1,2,9,83,3,4,57,6,11,17\}$	$\{3,1,24,77,78\}$	Z_2
$\mathscr{K}_{25}'$	$\{1,2,9,83,3,4,57,6,113,77\}$	$\{3,2,21,80,77\}$	I
$\mathscr{K}_{26}'$	$\{1,2,9,83,3,4,5,129,137,87\}$	$\{3,3,18,83,76\}$	I
$\mathscr{K}_{27}'$	$\{1,2,9,83,3,4,57,142,131,163\}$	$\{3,4,15,86,75\}$	I
$\mathscr{K}_{28}'$	$\{1,2,9,83,3,4,57,6,95,163\}$	$\{3,5,12,89,74\}$	I
$\mathscr{K}_{29}'$	$\{1,2,9,83,3,4,5,129,51,37\}$	$\{3,6,9,92,73\}$	I
$\mathscr{K}_{30}'$	$\{1,2,9,83,3,4,5,51,37,122\}$	$\{3,7,6,95,72\}$	S_3
$\mathscr{K}_{31}'$	$\{1,2,9,83,3,4,57,166,38,160\}$	$\{4,0,21,82,76\}$	I
$\mathscr{K}_{32}'$	$\{1,2,9,83,3,4,57,6,153,91\}$	$\{4,1,18,85,75\}$	I
$\mathscr{K}_{33}'$	$\{1,2,9,83,3,4,57,142,163,96\}$	$\{4,2,15,88,74\}$	I
$\mathscr{K}_{34}'$	$\{1,2,9,83,3,4,5,129,112,39\}$	$\{4,3,12,91,73\}$	I
$\mathscr{K}_{35}'$	$\{1,2,9,83,3,4,5,129,37,11\}$	$\{4,4,9,94,72\}$	Z_2
$\mathscr{K}_{36}'$	$\{1,2,9,21,3,4,37,91,90,178\}$	$\{5,0,15,90,73\}$	Z_2

6.9 Complete $(38;4)$-arcs from the sd-inequivalent $(10;4)$-arcs

In Table 6.15, there are 36 sd-inequivalent $(10;4)$-arcs together with the corresponding sd-inequivalent classes of the i-secant distributions. Among these 36 sd-inequivalent arcs, there are two new sd-inequivalent classes of secant distributions given as follows:

$$\{2,10,3,96,72\},\{5,0,15,90,73\}.$$

These sd-inequivalent classes did not appear in the process of sd-inequivalent (10;4)-arcs in Chapter 2, Section 2.6, where in Table 2.11 the number of sd-inequivalent classes of secant distributions $\{t_4,t_3,t_2,t_1,t_0\}$ is 34. Therefore, at this stage of the classification the 36-arcs of Table 6.15 have been extended. The aim of this process is to discover if there is a new complete $(k;4)$-arc in $\mathrm{PG}(2,13)$ for $k>38$. The result of this process is a complete $(38;4)$-arc $\mathscr{K}'$. This complete arc is comes from the sd-inequivalent $(10;4)$-arc $\mathscr{K}_8'$. The complete $(38;4)$-arc is as follows:

$\mathscr{K}'=\{1,2,9,115,3,4,5,6,7,8,10,19,25,60,74,98,107,78,130,27,106,69,116,46,63,126,99,$ $51,81,65,52,176,88,92,53,181,169,178\}$. The properties of $\mathscr{K}'$ are given in Table 6.16.

Table 6.16: **Complete** $(38;4)$**-arc in** $\mathrm{PG}(2,13)$

Symbol	Complete (38;4)-arc	Stabiliser	$\{t_4,t_3,t_2,t_1,t_0\}$
$\mathscr{K}'$	$\{1,2,9,115,3,4,5,6,7,8,10,19,25,60,74,$ 98, 107, 78, 130, 27, 106, 69, 116, 46, 63, 126, 99, 51, $81,65,52,176,88,92,53,181,169,178\}$	D_{12}	$\{102,24,19,14,24\}$

Remark

In Table 6.17, The classification timings of the projectively inquivalent $(k;4)$-arcs for $k=5,\dots,9$ are given.

Table 6.17: **Timing (msec) of projectively inequivalent** $(k;4)$**-arcs for** $k=5,\dots,9$

$(k;4)$-arcs	Construction	Lexicographically least sets	$\{t_4,t_3,t_2,t_1,t_0\}$	Stabilisers
$(5;4)$-arcs	2011	2134	2193	2181
$(6;4)$-arcs	2138	2168	2329	2230
$(7;4)$-arcs	2516	2201	2999	3615
$(8;4)$-arcs	26606	711630	19554	80338
$(9;4)$-arcs	22729912	32126643	176130	3848131

Chapter 7

The classification of certain *sd*-inequivalent $(k;6)$-arcs in $\mathrm{PG}(2,13)$

In this chapter, some results on the classification of certain *sd*-inequivalent $(k;6)$-arcs in PG(2,13) for the values of $k = 9,\ldots,25$ are given. The programming tool used in this computation is GAP. The approach that has been used to establish certain *sd*-inequivalent $(k;6)$-arcs for the values of $k = 9,\ldots,25$ is the method used in Chapter 2, Section 2.3. The approach to the classification starts by fixing a certain $(8;6)$-arc of eight points where an arc in this set contains six collinear points from a particular line of the 183 lines in $\mathrm{PG}(2,13)$. This set is $S_1 = \{1,2,3,88,9,21,83,89\}$. The *i*-secant distribution of this $(8;6)$-arc is as follows:

$$\{t_6,t_5,t_4,t_3,t_2,t_1,t_0\} = \{1,0,0,0,13,80,89\}.$$

Also, the stabiliser group of S_1 is Z_2.

7.1 *sd*-inequivalent $(9;6)$-arcs

The $(9;6)$-arcs are constructed by adding all the points from the plane, $\mathrm{PG}(2,13)$, which do not lie on any 6-secant to S_1. The constructed number of $(9;6)$-arcs is 167 and the number of *sd*-inequivalent (9;6)-arcs is three, each having the identity group. The statistics are given in Table

7.1.

Table 7.1: sd**-inequivalent** $(9;6)$**-arcs**

Symbol	sd-inequivalent (9;6)-arc	$\{t_6,t_5,t_4,t_3,t_2,t_1,t_0\}$	Stabiliser
$\mathscr{A}_1$	$\{1,2,3,88,9,21,83,89,4\}$	$\{1,0,0,1,18,81,82\}$	I
$\mathscr{A}_2$	$\{1,2,3,88,9,21,83,89,5\}$	$\{1,0,0,0,21,78,83\}$	I
$\mathscr{A}_3$	$\{1,2,3,88,9,21,83,89,23\}$	$\{1,0,0,2,15,84,81\}$	I

Theorem 7.1. *In* $\mathrm{PG}(2,13)$*, there are at least three sd-inequivalent* $(9;6)$*-arcs.*

7.2 sd-inequivalent $(10;6)$-arcs

For each $(9;6)$-arc constructed, the points from the plane which are not on any 6-secant to each of the $(9;6)$-arcs $\mathscr{A}_1,\mathscr{A}_2$, and $\mathscr{A}_3$ are added. The constructed number of $(10;6)$-arcs is 498 while the number of sd-inequivalent (10;6)-arcs is 9. Also, the values of t_i of the secant distributions of the sd-inequivalent $(10;6)$-arcs are as follows:

$$t_6=1, t_5=0, t_4\in\{0,1\}, t_3\in\{0,\ldots,5\}, t_2\in\{15,\ldots,30\},$$
$$t_1\in\{74,\ldots,89\}, t_0\in\{73,\ldots,78\}.$$

The notation S in Table 7.2 stands for the stabiliser groups of sd-inequivalent $(10;6)$-arcs while w stands for the number of these groups and similarly in each associated statistics table for $k=11,12,\ldots,25$ respectively.

Table 7.2: sd**-inequivalent** $(10;6)$**-arcs**

sd-inequivalent (10;6)-arcs	$S:w$
9	$I:9$

Theorem 7.2. *In* $\mathrm{PG}(2,13)$*, there are at least nine sd-inequivalent* $(10;6)$*-arcs.*

7.3 *sd*-inequivalent $(11;6)$-arcs

In this process, there are 1485 (11;6)-arcs found including 24 *sd*-inequivalent (11;6)-arcs. Also, the values of t_i of the secant distributions of the *sd*-inequivalent $(11;6)$-arcs are as follows:

$$t_6 = 1, t_5 \in \{0,1\}, t_4 \in \{0,\dots,2\}, t_3 \in \{0,\dots,8\},$$
$$t_2 \in \{16,\dots,40\}, t_1 \in \{68,\dots,94\}, t_0 \in \{65,\dots,74\}.$$

Table 7.3: *sd*-**inequivalent** $(11;6)$**-arcs**

sd-inequivalent (11;6)-arcs	$S:w$
24	$I:24$

Theorem 7.3. *In* $\mathrm{PG}(2,13)$*, there are at least* 24 *sd-inequivalent* $(11;6)$*-arcs.*

7.4 *sd*-inequivalent $(12;6)$-arcs

In this process, there are 3936 $(12;6)$-arcs found. Among them the number of *sd*-inequivalent $(12;6)$-arcs is 54. Also, the values of t_i of the secant distributions of the *sd*-inequivalent $(12;6)$-arcs are as follows:

$$t_6 \in \{1,2\}, t_5 \in \{0,1\}, t_4 \in \{0,\dots,3\}, t_3 \in \{0,\dots,11\},$$
$$t_2 \in \{18,\dots,48\}, t_1 \in \{63,\dots,99\}, t_0 \in \{57,\dots,70\}.$$

The statistics are given in Table 7.4.

Table 7.4: *sd*-**inequivalent** $(12;6)$**-arcs**

sd-inequivalent (12;6)-arcs	$S:w$
54	$I:53$, $Z_2:1$

Theorem 7.4. *In* $\mathrm{PG}(2,13)$*, there are at least* 54 *sd-inequivalent* $(12;6)$*-arcs.*

7.5 sd-inequivalent $(13;6)$-arcs

In this process, among the 8762 $(13;6)$-arcs found, there are 114 sd-inequivalent $(13;6)$-arcs; all have the identity group. Also, the values of t_i of the secant distributions of the sd-inequivalent $(13;6)$-arcs are as follows:

$$t_6 \in \{1,2\}, t_5 \in \{0,\ldots,2\}, t_4 \in \{0,\ldots,5\}, t_3 \in \{0,\ldots,15\},$$
$$t_2 \in \{18,\ldots,54\}, t_1 \in \{59,\ldots,103\}, t_0 \in \{50,\ldots,66\}.$$

The statistics are given in Table 7.5.

Table 7.5: sd-**inequivalent** $(13;6)$-**arcs**

sd-inequivalent (13;6)-arcs	$S:w$
114	$I:114$

Theorem 7.5. *In* $\mathrm{PG}(2,13)$*, there are at least* 114 *sd-inequivalent* $(13;6)$*-arcs.*

7.6 sd-inequivalent $(14;6)$-arcs

There are 18344 $(14;6)$-arcs found including 228 sd-inequivalent $(14;6)$-arcs. The stabiliser groups of the 228 arcs are I and Z_2. Also, the values of t_i of the secant distributions of the sd-inequivalent $(14;6)$-arcs are as follows:

$$t_6 \in \{1,2\}, t_5 \in \{0,\ldots,2\}, t_4 \in \{0,\ldots,7\}, t_3 \in \{0,\ldots,19\},$$
$$t_2 \in \{16,\ldots,61\}, t_1 \in \{53,\ldots,107\}, t_0 \in \{43,\ldots,63\}.$$

The statistics are given in Table 7.6.

Table 7.6: sd-**inequivalent** $(14;6)$-**arcs**

sd-inequivalent (14;6)-arcs	$S:w$
228	$I:224$, $Z_2:4$

Theorem 7.6. *In* $\mathrm{PG}(2,13)$*, there are at least* 228 *sd-inequivalent* $(14;6)$*-arcs.*

7.7 *sd*-inequivalent $(15;6)$-arcs

In this process, there are 36325 $(15;6)$-arcs found including 428 *sd*-inequivalent $(15;6)$-arcs. The stabiliser groups of these arcs are I and Z_2. In addition, the values of t_i of the secant distributions of these arcs are as follows:

$$t_6 \in \{1,\dots,3\}, t_5 \in \{0,\dots,3\}, t_4 \in \{0,\dots,8\}, t_3 \in \{0,\dots,23\},$$
$$t_2 \in \{15,\dots,68\}, t_1 \in \{48,\dots,109\}, t_0 \in \{36,\dots,60\}.$$

The statistics are given in Table 7.7.

Table 7.7: *sd*-**inequivalent** (15;6)**-arcs**

sd-inequivalent (15;6)-arcs	$S:w$
428	$I:425$, $Z_2:3$

Theorem 7.7. *In* $\mathrm{PG}(2,13)$*, there are at least* 428 *sd-inequivalent* $(15;6)$*-arcs.*

7.8 *sd*-inequivalent $(16;6)$-arcs

In this process, among the 67445 $(16;6)$-arcs found, the number of *sd*-inequivalent $(16;6)$-arcs is 783. These arcs have two types of the stabiliser groups; they are: I and Z_2. In addition, the values of t_i of the secant distributions of these arcs are as follows:

$$t_6 \in \{1,\dots,3\}, t_5 \in \{0,\dots,4\}, t_4 \in \{0,\dots,11\}, t_3 \in \{0,\dots,27\},$$
$$t_2 \in \{15,\dots,76\}, t_1 \in \{43,\dots,112\}, t_0 \in \{30,\dots,57\}.$$

The statistics are given in Table 7.8.

Table 7.8: sd**-inequivalent** $(16;6)$**-arcs**

sd-inequivalent (16;6)-arcs	$S:w$
783	$I:775$, $Z_2:8$

Theorem 7.8. *In* $\mathrm{PG}(2,13)$*, there are at least* 783 *sd-inequivalent* $(16;6)$*-arcs.*

7.9 sd-inequivalent $(17;6)$-arcs

There are 1318 sd-inequivalent $(17;6)$-arcs. In addition, the values of t_i of the secant distributions of the sd-inequivalent $(17;6)$-arcs are as follows:

$$t_6 \in \{1,\dots,3\}, t_5 \in \{0,\dots,5\}, t_4 \in \{0,\dots,13\}, t_3 \in \{0,\dots,32\},$$
$$t_2 \in \{15,\dots,81\}, t_1 \in \{38,\dots,115\}, t_0 \in \{25,\dots,57\}.$$

The statistics are given in Table 7.9.

Table 7.9: sd**-inequivalent** $(17;6)$**-arcs**

sd-inequivalent (17;6)-arcs	$S:w$
1318	$I:1306$, $Z_2:12$

Theorem 7.9. *In* $\mathrm{PG}(2,13)$*, there are at least* 1318 *sd-inequivalent* $(17;6)$*-arcs.*

7.10 sd-inequivalent $(18;6)$-arcs

The number of sd-inequivalent $(18;6)$-arcs is 2165. Here, there are five types of the stabiliser groups they are I, Z_2, $Z_2 \times Z_2$, Z_3, Z_4. The statistics are shown in Table 7.10.

Table 7.10: sd**-inequivalent** $(18;6)$**-arcs**

sd-inequivalent (18;6)-arcs	$S:w$
2165	$I:2147$, $Z_2:14$, $Z_2 \times Z_2:2$, $Z_3:1, Z_4:1$

Theorem 7.10. *In* $\mathrm{PG}(2,13)$*, there are at least* 2165 *sd-inequivalent* $(18;6)$*-arcs.*

7.11 *sd*-inequivalent $(19;6)$-arcs

Here, the number of *sd*-inequivalent $(19;6)$-arcs is 3391. The statistics are shown in Table 7.11.

Table 7.11: *sd*-**inequivalent** $(19;6)$-**arcs**

sd-inequivalent (19;6)-arcs	$S:w$
3391	$I:3369$, $Z_2:21$, $D_4:1$

Theorem 7.11. *In* $\mathrm{PG}(2,13)$*, there are at least* 3391 *sd-inequivalent* $(19;6)$*-arcs.*

7.12 *sd*-inequivalent $(20;6)$-arcs

In this process, the number of *sd*-inequivalent $(20;6)$-arcs is 5157. The stabilisers of these arcs are I, Z_2, $Z_2 \times Z_2$. The statistics are shown in Table 7.12.

Table 7.12: *sd*-**inequivalent** $(20;6)$-**arcs**

sd-inequivalent (20;6)-arcs	$S:w$
5157	$I:5137$, $Z_2:19$, $Z_2 \times Z_2:1$

Theorem 7.12. *In* $\mathrm{PG}(2,13)$*, there are at least* 5157 *sd-inequivalent* $(20;6)$*-arcs.*

7.13 *sd*-inequivalent $(21;6)$-arcs

The number of *sd*-inequivalent $(21;6)$-arcs is 7580. The statistics are shown in Table 7.13.

Table 7.13: *sd*-**inequivalent** $(21;6)$-**arcs**

sd-inequivalent (21;6)-arcs	$S:w$
7580	$I:7545$, $Z_2:28$, $Z_2 \times Z_2:2$, $Z_4:1$, $Z_3:3$, $D_4:1$

Theorem 7.13. *In* $\mathrm{PG}(2,13)$*, there are at least* 7580 *sd-inequivalent* $(21;6)$*-arcs.*

7.14 *sd*-inequivalent $(22;6)$-arcs

Here, the number of *sd*-inequivalent $(22;6)$-arcs is 10743. The statistics are shown in Table 7.14.

Table 7.14: *sd*-**inequivalent** $(22;6)$-**arcs**

sd-inequivalent (22;6)-arcs	$S:w$
10743	$I:10706,\ Z_2:35,\ Z_3:1,\ Z_4:1$

Theorem 7.14. *In* $\mathrm{PG}(2,13)$*, there are at least* 10743 *sd-inequivalent* $(22;6)$*-arcs.*

7.15 *sd*-inequivalent $(23;6)$-arcs

In this process, the number of *sd*-inequivalent $(23;6)$-arcs is 14777. The statistics are given in Table 7.15.

Table 7.15: *sd*-**inequivalent** $(23;6)$-**arcs**

sd-inequivalent (23;6)-arcs	$S:w$
14777	$I:14721,\ Z_2:53,\ Z_2\times Z_2:3$

Theorem 7.15. *In* $\mathrm{PG}(2,13)$*, there are at least* 14777 *sd-inequivalent* $(23;6)$*-arcs.*

7.16 *sd*-inequivalent $(24;6)$-arcs

There are 19941 *sd*-inequivalent $(24;6)$-arcs. The statistics are given in Table 7.16.

Table 7.16: *sd*-**inequivalent** $(24;6)$-**arcs**

sd-inequivalent (24;6)-arcs	$S:w$
19941	$I:19899,\ Z_2:38,\ Z_2\times Z_2:1,\ Z_3:2, S_3:1$

Theorem 7.16. *In* $\mathrm{PG}(2,13)$*, there are at least* 19941 *sd-inequivalent* $(24;6)$*-arcs.*

7.17 *sd*-inequivalent $(25;6)$-arcs

The number of *sd*-inequivalent $(25;6)$-arcs is 26148. Table 7.17 presents the related statistics.

Table 7.17: *sd*-**inequivalent** $(25;6)$-**arcs**

sd-inequivalent (25;6)-arcs	$S : w$
26148	$I : 26068, Z_2 : 69, Z_3 : 2, Z_2 \times Z_2 : 4, Z_4 : 1, S_3 : 3, D_6 : 1$

Theorem 7.17. *In* $\mathrm{PG}(2,13)$*, there are at least* 26148 *sd-inequivalent* $(25;6)$*-arcs.*

Remark

The timings in (msec) of this classification for the construction, the *i*-secant distributions of $(k;6)$-arcs, and the stabilisers of *sd*-inequivalent $(k;6)$-arcs are given in Table 7.18. The CPU timings for $(25;6)$-arcs are $4:21:58:34$, $01:35:57$, $01:21:42$.

Table 7.18: **Timing (msec) of** $(k;6)$**-arcs for** $k = 9, \ldots, 24$

$(k;6)$-arcs	Construction	$\{t_6, t_5, t_4, t_3, t_2, t_1, t_0\}$	Stabilisers
$(9;6)$-arcs	2182	2157	2164
$(10;6)$-arcs	2281	2295	2145
$(11;6)$-arcs	2668	2956	2331
$(12;6)$-arcs	3833	4684	2795
$(13;6)$-arcs	6337	9147	3943
$(14;6)$-arcs	14795	19080	6020
$(15;6)$-arcs	31815	32912	48862
$(16;6)$-arcs	109736	72366	131783
$(17;6)$-arcs	451940	114838	223694
$(18;6)$-arcs	1317723	225384	495909
$(19;6)$-arcs	3267376	334899	821086
$(20;6)$-arcs	7451239	541622	1328093
$(21;6)$-arcs	16863648	1010704	2014101
$(22;6)$-arcs	36277978	1531558	2932407
$(23;6)$-arcs	69236502	2340801	4034725
$(24;6)$-arcs	132275399	3218687	5233797

Appendix A

The following tables represent the points and the lines of $\mathrm{PG}(2,13)$.

Table 7.19: **The points of** $\mathrm{PG}(2,13)$

1	1 0 0	38	3 0 1	75	6 12 1	112	10 5 1	149	6 11 1
2	0 1 0	39	7 10 1	76	10 10 1	113	8 9 1	150	11 11 1
3	0 0 1	40	5 8 1	77	5 4 1	114	4 1 1	151	11 5 1
4	7 0 1	41	12 10 1	78	2 6 1	115	8 1 0	152	8 6 1
5	7 6 1	42	5 10 1	79	9 3 1	116	0 8 1	153	9 12 1
6	9 4 1	43	5 2 1	80	3 11 1	117	12 0 1	154	10 2 1
7	2 3 1	44	6 5 1	81	11 12 1	118	7 1 1	155	6 10 1
8	3 1 1	45	8 8 1	82	10 1 1	119	12 1 0	156	5 5 1
9	2 1 0	46	12 3 1	83	11 1 0	120	0 12 1	157	8 11 1
10	0 2 1	47	3 6 1	84	0 11 1	121	10 0 1	158	11 6 1
11	6 0 1	48	9 11 1	85	11 0 1	122	7 3 1	159	9 10 1
12	7 7 1	49	11 10 1	86	7 2 1	123	3 10 1	160	5 1 1
13	1 12 1	50	5 7 1	87	6 7 1	124	5 9 1	161	9 1 0
14	10 6 1	51	1 3 1	88	1 1 1	125	4 12 1	162	0 9 1
15	9 2 1	52	3 7 1	89	6 1 0	126	10 11 1	163	4 0 1
16	6 9 1	53	1 7 1	90	0 6 1	127	11 1 1	164	7 9 1
17	4 4 1	54	1 11 1	91	9 0 1	128	10 1 0	165	4 9 1
18	2 10 1	55	11 4 1	92	7 4 1	129	0 10 1	166	4 7 1
19	5 6 1	56	2 8 1	93	2 11 1	130	5 0 1	167	1 5 1
20	9 1 1	57	12 4 1	94	11 8 1	131	7 8 1	168	8 10 1
21	5 1 0	58	2 4 1	95	12 9 1	132	12 1 1	169	5 11 1
22	0 5 1	59	2 5 1	96	4 8 1	133	7 1 0	170	11 7 1
23	8 0 1	60	8 7 1	97	12 8 1	134	0 7 1	171	1 4 1
24	7 5 1	61	1 10 1	98	12 11 1	135	1 0 1	172	2 9 1
25	8 5 1	62	5 3 1	99	11 9 1	136	7 12 1	173	4 10 1
26	8 2 1	63	3 9 1	100	4 3 1	137	10 3 1	174	5 12 1
27	6 8 1	64	4 2 1	101	3 2 1	138	3 5 1	175	10 4 1
28	12 12 1	65	6 4 1	102	6 3 1	139	8 4 1	176	2 12 1
29	10 7 1	66	2 2 1	103	3 3 1	140	2 7 1	177	10 12 1
30	1 6 1	67	6 2 1	104	3 8 1	141	1 9 1	178	10 8 1
31	9 8 1	68	6 6 1	105	12 6 1	142	4 5 1	179	12 7 1
32	12 5 1	69	9 9 1	106	9 5 1	143	8 1 1	180	1 2 1
33	8 3 1	70	4 6 1	107	8 12 1	144	4 1 0	181	6 1 1
34	3 4 1	71	9 6 1	108	10 9 1	145	0 4 1	182	1 1 0
35	2 1 1	72	9 7 1	109	4 11 1	146	2 0 1	183	0 1 1
36	3 1 0	73	1 8 1	110	11 3 1	147	7 11 1		
37	0 3 1	74	12 2 1	111	3 12 1	148	11 2 1		

Table 7.20: **The lines of** $\mathrm{PG}(2,13)$

l_1	1	2	9	21	36	83	89	115	119	128	133	144	161	182
l_2	2	3	10	22	37	84	90	116	120	129	134	145	162	183
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
l_{183}	183	1	8	20	35	82	88	114	118	127	132	143	160	181

Appendix B

The following table gives the groups that occur in this work.

Table 7.21: **Groups**

Group	Description of group
Z_n	Cyclic group of order n
S_n	Symmetric group of degree n
D_n	Dihedral group of order $2n$
A_n	Alternating group of degree n
$Z_n \times Z_m$	Direct product of Z_n and Z_m
$Z_n \times S_n$	Direct product of Z_n and S_n
$(Z_n \times Z_m) \rtimes Z_r$	Semi-direct product of $Z_n \times Z_m$ and Z_r
$\mathrm{SL}(n,q) \rtimes Z_r$	Semi-direct product of special linear group and Z_r

Appendix C

Algorithms

The following algorithms are used in this thesis.

Algorithm (1) Compute the sd-inequivalent $(k;4)$-arcs

Step (1): Input $(k;4)$-arcs.

Step (2): For each $(k;4)$-arc do the following:

Step (3): Compute the 4-secants of $(k;4)$-arc.

Step (4): Compute the points lying on the 4-secants of $(k;4)$-arc.

Step (5): Compute the set $V = \{\mathrm{PG}(2,13) \setminus 4-\text{secants}\}$.

Step (6): If $V \neq \phi$, then add each point $P \in V$ separately to $(k;4)$-arc to form $(k+1;4)$-arcs.

Step (7): Otherwise, $V = \phi$ and then $(k;4)$-arc is complete.

Step (8): End

The Output is $(k+1;4)$-arcs.

Step (9): Input $(k+1;4)$-arcs.

Step (10): Compute the i-secant distributions $\{t_4,t_3,t_2,t_1,t_0\}$ for $(k+1;4)$-arcs.

Step (11): Compute the sd-inequivalent classes N_c of secant distributions for $(k+1;4)$-arcs.

Step (12): End.

Note

In Chapter 2, algorithm (1) used the projectively inequivalent $(6;4)$-arcs as input to establish the sd-inequivalent $(k;4)$-arcs for $k=7$ only. However, to establish the sd-inequivalent $(k;4)$ for $k>7$, the input will be the sd-inequivalent $(k;4)$-arcs for $k=7,\ldots,37$. Also, In chapter 6, for $k \leq 10$, the input is the projectively inequivalent $(k;4)$-arcs. In addition, the stabiliser of each

$(k;4)$-arc is computed via Gap function Stabilizer(G,acts,OnSets).

Algorithm (2) Compute the quartic curve

(1) Consider the finite field $\mathbf{F}_{13}$.

(2) Consider the homogeneous polynomial f for a quartic curve over the finite field $\mathbf{F}_{13}$.

(3) Given a $(k;4)$-arc $\mathscr{K}$ where $k \geq 24$, substitute a particular set of 14 points of $\mathscr{K}$ in f.

(4) Solve the system of homogeneous equations to give the curve $\mathscr{C}$.

Algorithm (3) Identify the rational points of $\mathscr{K}$ on the curve $\mathscr{C}$ over $\mathrm{PG}(2,13)$

(1) Consider the points P_i for $i = 1,\dots,183$ in $\mathrm{PG}(2,13)$.

(2) Substitute the points from $\mathrm{PG}(2,13)$ in a polynomial quartic curve.

(3) For an $(k;4)$-arc $\mathscr{K}$, let $\mathscr{K}(\mathscr{C}) = \{P_i \mid f(P_i) = 0\}$; that is, $\mathscr{K}(\mathscr{C})$ is the subset of $\mathscr{K}$ of points lying on $\mathscr{C}$.

Appendix D

The following code establishes the $(k;n)$-arcs.

```
vvv:=[[]];;
lines:= Set(lines);;
k:=[];; D:=[];; C:=[];;
for x in [1..Length(vvv)] do
v:=vvv[x];
for y in [1..q^2+q+1] do
l:=lines[y];
if Size(Intersection(v,l)) = n then
Append(C,l); Add(D,l);
fi;
od;
Ep:=[];; H:=[];;
e:=Difference([1..q^2+q+1],Set(C));
h:=Difference(lines,D);
Add(Ep,e); Add(H,h);
od;
for i in [1..1] do
v:=vvv[i];
m:=Ep[i];; m1:=H[i];;
mm:=[];; nn:=[];;
for cc in m1 do
if Size(Intersection(vvvv,cc)) =n then
Append(mm,cc); Add(nn,cc);
fi; od;
M:=Difference(m,Set(mm));
N:=Difference(m1,nn);
if Size(M)=ϕ then continue;
```

```
else Print(vvvv, "\ n");
fi; od; od;
```

The following code computes the secant distribution.

```
v:=[];;
lines:= Set(lines);;
for x in v do
u0:=0;u1:=0;u2:=0;u3:=0;u4:=0;u5:=0;u6:=0;
for i in [1..q^2+q+1] do
l:=lines[i];
SI :=Size(Intersection(Set(x),Set(l)));
if SI = 0 then u0:=u0+1;
elif SI = 1 then u1:=u1+1;
elif SI = 2 then u2:=u2+1;
elif SI = 3 then u3:=u3+1;
elif SI = 4 then u4:=u4+1;
elif SI = 5 then u5:=u5+1;
elif SI = 6 then u6:=u6+1;
fi; od;
SD:=[];
Add(SD,[u6,u5,u4,u3,u2,u1,u0]);
od;
```

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