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Whirl Mappings on Generalised Annuli and the Incompressible Symmetric Equilibria of the Dirichlet Energy

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Abstract

In this paper we show a striking contrast in the symmetries of equilibria and extremisers of the total elastic energy of a hyperelastic incompressible annulus subject to pure displacement boundary conditions. Indeed upon considering the equilibrium equations, here, the nonlinear second order elliptic system formulated for the deformation $u = (u_1, ..., u_N)$:

$$\mathbb{EL}[u, \mathbf{X}] = \begin{cases} \Delta u = div\ (\mathscr{P}(x) \mathrm{cof}\, \nabla u) & \text{in } \mathbf{X}, \\ \det \nabla u = 1 & \text{in } \mathbf{X}, \\ u \equiv \varphi & \text{on } \partial \mathbf{X}, \end{cases}$$

where $\mathbf{X}$ is a finite, open, symmetric N-annulus (with $N \geq 2$), $\mathscr{P} = \mathscr{P}(x)$ is an unknown hydrostatic pressure field and φ is the identity mapping, we prove that, despite the inherent rotational symmetry in the system, when $N = 3$, the problem possesses no non-trivial symmetric equilibria whereas in sharp contrast, when $N = 2$, the problem possesses an infinite family of symmetric and topologically distinct equilibria. We extend and prove the counterparts of these results in higher dimensions by way of showing that a similar dichotomy persists between all odd *vs.* even dimensions $N \geq 4$ and discuss a number of closely related issues.

1 Introduction

A problem of major interest and significance in nonlinear elasticity is the understanding of qualitative features and symmetries of the energy minimisers and equilibria under the so-called incompressibility constraint (*see*, e.g., [1]-[6], [19] [20, 21]). Motivated by the above and the earlier works [19, 20], [25]-[27], in this paper we make an interesting contribution towards certain aspects of this problem by considering the geometric setup where $\mathbf{X} \subset \mathbb{R}^N$ $(N \geq 2)$ is a finite, symmetric, open annulus and the total elastic energy is given by an integral

$$\mathbb{E}[u; \mathbf{X}] = \int_{\mathbf{X}} W(\nabla u(x))\, dx. \tag{1.1}$$

The stored energy function $W = W(\mathbf{F}) : \mathbb{R}^{N\times N} \to \mathbb{R} \cup \{\infty\}$ is taken to be isotropic, frame indifferent and polyconvex (*see* below) while the deformation u is restricted to the class of admissible incompressible Sobolev mappings

$$\mathcal{A}(\mathbf{X}) = \left\{ u \in W^{1,2}(\mathbf{X}, \mathbb{R}^N) : \det \nabla u = 1 \ a.e. \text{ in } \mathbf{X},\ u \equiv \varphi \text{ on } \partial \mathbf{X} \right\}, \tag{1.2}$$

with the last condition in (1.2) asserting that u agrees with the identity mapping on $\partial\mathbf{X}$ in the sense of traces. To prevent interpenetration of matter, $W(\mathbf{F}) \equiv \infty$ for $\det \mathbf{F} \le 0$ and so as far as the elastic deformations and their energetics are concerned it is the open set $\mathbb{R}^{N\times N}_{+} = \{\mathbf{F} \in \mathbb{R}^{N\times N} : \det \mathbf{F} > 0\}$ that is of interest. Now recall that the stored energy function $W = W(\mathbf{F})$ is said to be polyconvex (*see* [2, 15] and [6, 7]) *iff* it can be expressed as a convex function of the minors (or equivalently sub-determinants) of $\mathbf{F}$, that is, *iff* $W(\mathbf{F}) = \phi(\mathbf{F}, \mathrm{adj}_2\mathbf{F}, \cdots, \mathrm{adj}_N\mathbf{F})$ for some convex function $\phi : \mathbb{R}^{\tau(N)} \to \mathbb{R} \cup \{\infty\}$. Here $\mathrm{adj}_s\mathbf{F}$ (with $1 \le s \le N$) stands for the matrix of all $s \times s$ minors of $\mathbf{F}$ and $\tau(N) = \sum (N!)^2/[s!(N-s)!]^2$ (summation over $s = 1, ..., N$). Additionally the stored energy function is said to be frame indifferent and isotropic *iff* $W(\mathbf{QF}) = W(\mathbf{F})$ and $W(\mathbf{FQ}) = W(\mathbf{F})$ for all $\mathbf{Q} \in \mathbf{SO}(N)$ respectively, that is, it is invariant under the left and right actions of the rotation group $\mathbf{SO}(N)$. For more discussion on the implications of these assumptions to the representation of $W = W(\mathbf{F})$ in terms of the right- and left- Cauchy-Green tensors or the singular values of $\mathbf{F}$ *see* [1, 2] or [6, 7]. A central example of a frame indifferent, isotropic and polyconvex stored energy function is given by

$$W(\mathbf{F}) = tr\{\mathbf{F}^t\mathbf{F}\} + h(\det \mathbf{F}) = \sum_{j=1}^{N} v_j^2 + h(\prod_{j=1}^{N} v_j), \tag{1.3}$$

where h is any convex function on the line, $v_1, ..., v_N$ are the singular values of $\mathbf{F}$, that is, the eigenvalues of $\sqrt{\mathbf{F}^t\mathbf{F}}$ and the second equality assumes $\det \mathbf{F} > 0$ (*see*, [1]-[2] and [6, 7] for more).

Now subject to the differentiability of the stored energy function, by invoking the Lagrange multipliers method (*cf.* [3, 6, 26]) it can be seen that the Euler-Lagrange system associated with the elastic energy $\mathbb{E}[\cdot; \mathbf{X}]$ over the class of admissible deformations $\mathcal{A}(\mathbf{X})$ takes the form

$$\mathbb{EL}[u, \mathbf{X}] = \begin{cases} div\, \mathfrak{S}[x, \nabla u(x)] = 0 & \text{in } \mathbf{X}, \\ \det \nabla u = 1 & \text{in } \mathbf{X}, \\ u \equiv \varphi & \text{on } \partial\mathbf{X}, \end{cases} \tag{1.4}$$

where the divergence operator is understood to act row-wise while the tensor field $\mathfrak{S} = \mathfrak{S}[x, \nabla u]$ – the Piola-Kirchhoff stress tensor – is given by

$$\mathfrak{S}[x, \mathbf{F}] = \frac{\partial W}{\partial \mathbf{F}}(\mathbf{F}) - \mathscr{P}(x)\mathbf{F}^{-t} = \frac{\partial W}{\partial \mathbf{F}}(\mathbf{F}) - \mathscr{P}(x)\mathrm{cof}\, \mathbf{F}. \tag{1.5}$$

Here the function $\mathscr{P}$ is an *a priori* unknown Lagrange multiplier associated with the incompressibility constraint often called the hydrostatic pressure.

In this paper we shall introduce and study a class of mappings called *whirl* mappings (or *whirls* for simplicity). These are certain continuous self-mappings of the N-dimensional annulus $\overline{\mathbf{X}}$ onto itself – for definiteness we hereafter write $\mathbf{X} = \mathbf{X}[a,b] = \{(x_1, ..., x_N) : a < |x| < b\}$ with $0 < a < b < \infty$ – that agree with the identity mapping on the boundary $\partial\mathbf{X}$ and admit the representation

$$u : x \mapsto \mathbf{Q}(\rho_1, \dots, \rho_n)\, x, \qquad x \in \overline{\mathbf{X}}. \tag{1.6}$$

In this representation $\mathbf{Q}$ is a continuous $\mathbf{SO}(N)$-valued matrix field depending on the variable x through the 2-plane radial variables $\varrho = (\rho_1, \dots, \rho_n)$ given by

$$\rho_j = \sqrt{x_{2j-1}^2 + x_{2j}^2}, \qquad 1 \le j \le n-1, \tag{1.7}$$

and

$$\rho_n = \begin{cases} \sqrt{x_{2n-1}^2 + x_{2n}^2} & \text{if } N = 2n, \\ x_{2n-1} & \text{if } N = 2n-1, \end{cases} \tag{1.8}$$

where $\varrho = (\rho_1, \dots, \rho_n) \in U_N \subset \mathbb{R}^n$, $U_N = \{\varrho \in \mathbb{R}^n_+ : a < |\varrho| < b\}$ when $N = 2n$ and $U_N = \{\varrho \in \mathbb{R}^{n-1}_+ \times \mathbb{R} : a < |\varrho| < b\}$ when $N = 2n-1$. Next for the purpose of symmetry considerations described below we demand $\mathbf{Q}$ to take values on a fixed *maximal torus* $\mathbb{T} \subset \mathbf{SO}(N)$ that here we set to be the subgroup of all 2×2 block diagonal rotation matrices. As a result

$$\mathbf{Q}(\varrho) = \mathbf{Q}(\rho_1, \dots, \rho_n) = \mathrm{diag}(\mathbf{R}[g_1], \dots, \mathbf{R}[g_n]), \qquad N = 2n, \tag{1.9}$$

and

$$\mathbf{Q}(\varrho) = \mathbf{Q}(\rho_1, \dots, \rho_n) = \mathrm{diag}(\mathbf{R}[g_1], \dots, \mathbf{R}[g_{n-1}], 1), \qquad N = 2n-1, \tag{1.10}$$

where each block $\mathbf{R} = \mathbf{R}[g] \in \mathbf{SO}(2)$ is determined by an *angle of rotation* or *whirl* function $g = g_j(\rho_1, \dots, \rho_n)$. Now any such whirl mapping u is invariant under the action of the maximal torus $\mathbb{T}$ since firstly each ρ_j with $1 \le j \le n$ remains fixed under the action of $\mathbb{T}$ and secondly,

$$[\mathbf{P}u \circ \mathbf{P}^t](x) = \mathbf{P}u(\mathbf{P}^t x) = \mathbf{P}\mathbf{Q}(\rho_1, \dots, \rho_n)\mathbf{P}^t x = \mathbf{Q}(\rho_1, \dots, \rho_n)x = u(x),$$

for all $\mathbf{P} \in \mathbb{T}$ and $x \in \overline{\mathbf{X}}$. (Note that the second to last equality above follows from the commutativity of $\mathbb{T}$.) Thus in this sense the whirl mappings as defined above are rotationally symmetric with respect to $\mathbb{T} \subset \mathbf{SO}(N)$.

Specialising momentarily to $N = 3$ the $\mathbf{SO}(3)$-valued matrix field $\mathbf{Q}$ in (1.6) has the form $\mathbf{Q} = \mathrm{diag}(\mathbf{R}[g], 1)$ where $g = g(\rho_1, x_3)$, $\rho_1 = \sqrt{x_1^2 + x_2^2}$ and $\rho_2 = x_3$ [*cf.* (2.4)]. Thus subject to g being differentiable, the deformation gradient (with $\rho = \rho_1$, $\dot{\mathbf{Q}} = \partial_g \mathbf{Q}$) is seen to be $\nabla u = \mathbf{Q}(\rho, x_3) + \dot{\mathbf{Q}}(\rho, x_3)x \otimes \nabla g$ where ∇g denotes the gradient in the x variables, i.e., $\nabla g = (g_\rho\, x_1/\rho, g_\rho\, x_2/\rho, g_{x_3})$. Setting now for each $x \in \mathbf{X}$ fixed, $\mathbf{P}_1 = \mathbf{P}\mathbf{R}\mathbf{Q}^t$, $\mathbf{P}_2 = \mathbf{R}^t\mathbf{P}^t$ where $\mathbf{P}, \mathbf{R} \in \mathbf{SO}(3)$ are the matrices of rotation about the axes $e_1 = (1, 0, 0)$ and $e_3 = (0, 0, 1)$ respectively

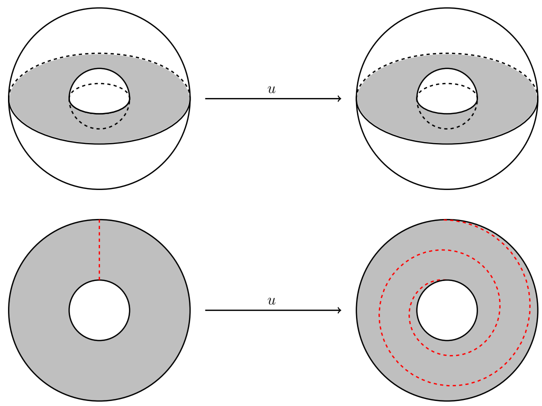


Figure 1: The figure here shows how a *whirl* mapping u acts on $\mathbf{X}$ when $N = 3$. In particular it shows how the *whirl* u wraps a radial line around the centre in the (x_1, x_2)-plane.

such that $\mathbf{R}\mathbf{Q}^t\dot{\mathbf{Q}}x = (\sqrt{x_1^2 + x_2^2}, 0, 0)$ and $\mathbf{P}\mathbf{R}\nabla g = (0, 0, |\nabla g|)$, upon noting $\langle \mathbf{Q}^t\dot{\mathbf{Q}}x, \nabla g\rangle = 0$, a straightforward calculation gives

$$\mathbf{P}_1 \nabla u \, \mathbf{P}_2 = \begin{bmatrix} 1 & 0 & \rho|\nabla g| \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} =: \mathbb{B}[\rho|\nabla g|]. \tag{1.11}$$

Hence in view of the assumptions on W the elastic energy $\mathbb{E}$ when restricted to these whirl mappings in three dimensions can be expressed as,

$$\mathbb{E}[u; \mathbf{X}] = \int_{\mathbf{X}} W(\nabla u) \, dx = \int_{\mathbf{X}} W(\mathbb{B}[\rho|\nabla g|]) \, dx = 2\pi \int_U f(\rho|\nabla g|)\rho \, d\rho dx_3.$$

Here $U = U_3$ as defined earlier is a half annulus (see also Section 2) and $f(t) = W(\mathbb{B}[t])$ is the restriction of the stored energy function to the rank-one line $\mathbb{B}[t] = \mathbf{I}_3 + te_1 \otimes e_3$ with $-\infty < t < \infty$.

For the sake of this paper we shall confine to invariant stored energy functions of the form (1.3) that by virtue of the incompressibility constraint can be taken without loss of generality to be $W(\mathbf{F}) = tr\{\mathbf{F}^t\mathbf{F}\}/2$ with the resulting elastic energy $\mathbb{E}$ in (1.1) being the Dirichlet energy. The main goal then is to highlight a curious difference in the existence and multiplicity of symmetric equilibria of the elastic energy (1.1) or equivalently solutions to the nonlinear system (1.4) in the form of whirl mappings between the cases $N = 2$ and $N = 3$. Indeed in the latter case we show that there are no non-trivial equilibria in the class of whirl mappings whereas in the former case they do exist and quite in abundance – in fact, here, we show that there are infinitely (countably) many equilibria in the form of whirl mappings, each of a distinct topological type. In the final sections we establish the counterpart of these results in higher dimensions and prove a similar dichotomy between all odd *vs.* even dimensions $N \geq 4$.

2 Existence and non-existence results in two and three dimensions

The aim of this section is to showcase a curious and sharp contrast in the nature and form of the symmetries of the equilibria and extremisers of the elastic energy between the cases $N = 2$ and $N = 3$. Specifically here we show that when $N = 3$ the only whirl solution to the system (2.1) below is the identity mapping whereas when $N = 2$ there is an infinite (countably many) scale of such solutions to (2.1), each of a different topological type. To this end let us recall that the Euler-Lagrange system associated with the Dirichlet energy ($W(\mathbf{F}) = tr\,(\mathbf{F}^t\mathbf{F})/2$) over $\mathcal{A}(\mathbf{X})$ (for any $N \geq 2$) is given by the nonlinear system [*cf.* (1.4)- (1.5)]

$$\mathbb{EL}[u, \mathbf{X}] = \begin{cases} div\ \mathfrak{S}[x, \nabla u(x)] = 0 & \text{in } \mathbf{X}, \\ \det \nabla u = 1 & \text{in } \mathbf{X}, \\ u \equiv \varphi & \text{on } \partial\mathbf{X}, \end{cases} \tag{2.1}$$

where the stress field $\mathfrak{S}$ in this case is given by $\mathfrak{S}[x, \mathbf{F}] = \mathbf{F} - \mathscr{P}(x)\mathbf{F}^{-t}$. Therefore, elaborating further, the divergence term on the left in the first line of the above system can be written as

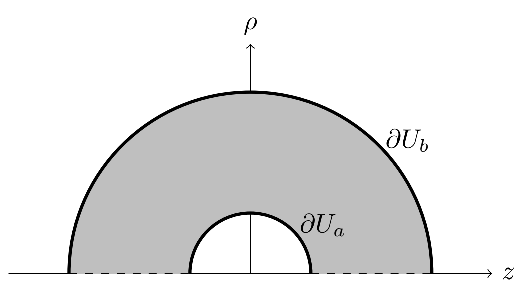
$$div\ \mathfrak{S}[x, \nabla u(x)] = \Delta u - \mathrm{div}(\mathscr{P}(\nabla u)^{-t}) = \Delta u - (\nabla u)^{-t}\nabla\mathscr{P}, \tag{2.2}$$

where the last identity subject to sufficient regularity of u results from an application of the so-called Piola identity (*see*, e.g., [2, 15, 28]). Hence the first equation in the system (2.1) reads as

$$div\ \mathfrak{S}[x, \nabla u(x)] = 0 \iff \nabla\mathscr{P} = (\mathrm{cof}\,\nabla u)^{-1}\Delta u = (\nabla u)^t\Delta u. \tag{2.3}$$

By a classical solution we mean a pair $(u, \mathscr{P})$ where u is admissible, that is, $u \in \mathcal{A}(\mathbf{X})$, $(u, \mathscr{P})$ is regular, i.e., $u \in \mathbf{C}(\overline{\mathbf{X}}, \mathbb{R}^3) \cap \mathbf{C}^2(\mathbf{X}, \mathbb{R}^3)$, $\mathscr{P} \in \mathbf{C}(\overline{\mathbf{X}}) \cap \mathbf{C}^1(\mathbf{X})$ and (2.1), or the equivalent formulation of the first equation, (2.3) holds. Now being prompted by considerations of symmetry we continue by seeking classical solutions to the system (2.1) from the class of whirl mappings. [1]

The case $N = 3$. Let $U = U[a, b] = \{(\rho, z) \in \mathbb{R}^2 : \rho > 0, a < \sqrt{\rho^2 + z^2} < b\}$, i.e., the *half* vertical open annulus whose closure upon a 2π rotation about the z-axis gives $\overline{\mathbf{X}}$.



[1]We consider only classical solutions in this paper. Less regular solutions or other possible pathological solutions are neither discussed not studied here.

Then by definition any whirl mapping u on $\mathbf{X}$ is represented as $u(x) = \mathbf{Q}(\rho, z)x$ where $x = (x_1, x_2, x_3)$, $\rho = \sqrt{x_1^2 + x_2^2}$ and for brevity $z = x_3$ where for each $(\rho, z) \in U$ the matrix field $\mathbf{Q}$ has the specific block diagonal form

$$\mathbf{Q} = \mathbf{Q}[g](\rho, z) = \mathrm{diag}(\mathbf{R}[g], 1) = \begin{bmatrix} \mathbf{R}[g] & 0 \\ 0 & 1 \end{bmatrix}. \tag{2.4}$$

Here $\mathbf{R} = \mathbf{R}[g]$ is a planar rotation matrix, that is, an element of the special orthogonal group $\mathbf{SO}(2)$ [*see* (3.7)] with $g = g(\rho, z)$ representing the angle of *rotation* or *whirl* function associated with $\mathbf{Q}$ (or u respectively). As a matter of fact here g is assumed to lie in one of the spaces

$$\mathscr{G}_k(U) := \left\{ g \in W^{1,2}(U) : g \equiv 0 \text{ on } \partial U_a \text{ and } g \equiv 2\pi k \text{ on } \partial U_b \right\}, \qquad k \in \mathbb{Z}, \tag{2.5}$$

with $\partial U_a = \{(\rho, z) \in \partial U : \sqrt{\rho^2 + z^2} = a\}$, $\partial U_b = \{(\rho, z) \in \partial U : \sqrt{\rho^2 + z^2} = b\}$. Note that the boundary conditions on g here ensure that upon passing to the associated whirl mapping we have $u \equiv x$ on $\partial \mathbf{X}$. A set of straightforward and direct calculations based on the representation of u and $\mathbf{Q}$ now give

$$\begin{aligned} \nabla u &= [\partial u_i / \partial x_j : 1 \le i, j \le 3] \\ &= \begin{bmatrix} \cos g - x_1(x_1 \sin g + x_2 \cos g) g_\rho/\rho & -\sin g - x_2(x_1 \sin g + x_2 \cos g) g_\rho/\rho & -(x_1 \sin g + x_2 \cos g) g_z \\ \sin g + x_1(x_1 \cos g - x_2 \sin g) g_\rho/\rho & +\cos g + x_2(x_1 \cos g - x_2 \sin g) g_\rho/\rho & +(x_1 \cos g - x_2 \sin g) g_z \\ 0 & 0 & 1 \end{bmatrix}, \\ &= \begin{bmatrix} \mathbf{R}[g](\rho, z) & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -(x_1 \sin g + x_2 \cos g) \\ x_1 \cos g - x_2 \sin g \\ 0 \end{bmatrix} \otimes \begin{bmatrix} x_1 g_\rho/\rho \\ x_2 g_\rho/\rho \\ g_z \end{bmatrix}. \end{aligned} \tag{2.6}$$

The last equation in particular gives $\det \nabla u = 1$ in $\mathbf{X}$. (Note that for any invertible matrix $\mathbf{A}$ and vectors a, b: $\det(\mathbf{A} + a \otimes b) = \det \mathbf{A} \times \det(\mathrm{I} + \mathbf{A}^{-1} a \otimes b) = \det \mathbf{A} + \langle \mathbf{A}^{-1} a, b \rangle \det \mathbf{A}$ as a consequence of the determinant being quasiaffine.) Therefore the whirl mapping u satisfies the boundary condition $u = \varphi$ on $\partial \mathbf{X}$ and the incompressibility constraint $\det \nabla u = 1$ in $\mathbf{X}$ and so to have the inclusion $u \in \mathcal{A}(\mathbf{X})$ all that remains is to justify the L^2-integrability of ∇u. Towards this end a straightforward calculation gives

$$|\nabla u|^2 = \mathrm{tr}\{[\nabla u]^t [\nabla u]\} = 3 + \rho^2 (g_\rho^2 + g_z^2), \tag{2.7}$$

and so it is evident that $||(g_\rho, g_z)||_{L^2(U, \mathbb{R}^2)} < \infty \implies ||\nabla u||_{L^2(\mathbf{X}, \mathbb{R}^{3\times 3})} < \infty$. As a result by combining all the above ingredients we have

$$g \in \mathscr{G}(U) = \bigcup_{k \in \mathbb{Z}} \mathscr{G}_k(U) \implies u = \mathbf{Q}[g](\rho, z)x \in \mathcal{A}(\mathbf{X}). \tag{2.8}$$

Next upon integrating (2.7) and changing variables it is seen that the elastic energy of u can be expressed as an associated restricted energy of the angle of rotation function g through

$$
\begin{aligned}
2\mathbb{E}[u;\mathbf{X}] &= \int_{\mathbf{X}} |\nabla u|^2\,dx = \int_{\mathbf{X}} [3+\rho^2(g_\rho^2+g_z^2)]\,dx \\
&= 2\pi\int_U [3+\rho^2(g_\rho^2+g_z^2)]\rho\,d\rho dz =: 3|\mathbf{X}| + 2\pi\mathbb{H}[g;U]. \qquad (2.9)
\end{aligned}
$$

Hence aiming to resolve the system (2.3) and obtaining multiple solutions in the form of whirl mappings we proceed on to extremising the energy $\mathbb{E}[u;\mathbf{X}]$ over the subclass of whirl mappings in $\mathcal{A}(\mathbf{X})$. In fact in view of the formulation (2.9) above we initially set ourselves the task of extremising the restricted energy $\mathbb{H}[g;U]$ over the grand class $\mathscr{G}(U)$ of all admissible angle of rotation functions; specifically:

$$
\begin{aligned}
&\mathbb{H}[g;U] = \int_U \rho^3(g_\rho^2+g_z^2)\,d\rho dz, \\
&g \in \mathscr{G}(U) = \bigcup_{k\in\mathbb{Z}} \mathscr{G}_k(U). \qquad (2.10)
\end{aligned}
$$

Evidently the Euler-Lagrange equation associated with $\mathbb{H}[\cdot;U]$ over each $\mathscr{G}_k(U)$ is seen to take the divergence form

$$
\mathbb{EL}[g,U] = \begin{cases} \partial_\rho(\rho^3 g_\rho) + \partial_z(\rho^3 g_z) = 0 & \text{in } U, \\ g = 0 & \text{on } \partial U_a, \\ g = 2\pi k & \text{on } \partial U_b, \\ \rho^3\partial_\nu g = 0 & \text{on } \partial U\backslash[\partial U_a \cup \partial U_b]. \end{cases} \qquad (2.11)
$$

Notice that the horizontal part of the boundary $\partial U\backslash[\partial U_a \cup \partial U_b] = \{(\rho,z) \in \partial U : \rho = 0\}$ is left free accounting for the natural boundary condition in the last line. Any solution to (2.11) defines a corresponding whirl mapping u (as outlined earlier in the introduction) which is a possible candidate for a solution to (2.1). The following proposition establishes the existence of a unique solution to the restricted Euler-Lagrange equation (2.11) for each fixed $k \in \mathbb{Z}$.

Proposition 2.1. *For every $k \in \mathbb{Z}$ the restricted Euler-Lagrange equation* (2.11) *has a unique solution $g = g(\rho,z;k)$ in $\mathscr{G}_k(U)$ given explicitly by*

$$
g(\rho,z;k) = \frac{2\pi k a^3 b^3}{b^3-a^3}\left[\frac{1}{a^3} - \frac{1}{(\rho^2+z^2)^{3/2}}\right], \qquad (\rho,z) \in \overline{U}. \qquad (2.12)
$$

Proof. First a straightforward calculation shows that g satisfies the required boundary conditions in (2.11): For the segment $U\backslash[\partial U_a \cup \partial U_b]$ this is a result of $\rho \equiv 0$ while for ∂U_a and ∂U_b this follows as a result $\rho^2+z^2 = a^2$ and $\rho^2+z^2 = b^2$ respectively. Now from the explicit formulation of g we can compute the gradient vector $\nabla_U g = (\partial_\rho g, \partial_z g)$ as

$$
\partial_\rho g = \frac{6\pi k a^3 b^3}{b^3-a^3}\frac{\rho}{(\rho^2+z^2)^{5/2}}, \quad \partial_z g = \frac{6\pi k a^3 b^3}{b^3-a^3}\frac{z}{(\rho^2+z^2)^{5/2}}. \qquad (2.13)
$$

In particular upon recalling $(\rho, z) \in U \implies a < \sqrt{\rho^2+z^2} < b$ it follows at once that $g \in W^{1,2}(U)$. Next referring to the first equation in (2.11) and upon writing $div_U(X_1, X_2) = \partial_\rho X_1 + \partial_z X_2$ it follows using the above description of $\nabla_U g$ that

$$\begin{aligned} div_U\left(\rho^3 \nabla_U g\right) =& \partial_\rho(\rho^3 \partial_\rho g) + \partial_z(\rho^3 \partial_z g) = \frac{6\pi k a^3 b^3}{b^3-a^3} \times \\ & \times \left[\partial_\rho\{\rho^4(\rho^2+z^2)^{-5/2}\} + \partial_z\{\rho^3 z(\rho^2+z^2)^{-5/2}\}\right] = 0. \quad (2.14) \end{aligned}$$

(In other words the vector field $\rho^3(\rho^2+z^2)^{-5/2}(\rho, z)$ is divergence free in U.) Thus g solves (2.11). The proof of uniqueness is now standard: Indeed assume that g^1, g^2 are solutions to (2.11) and set $g = g^1 - g^2$. Then g solves (2.11) with $g \equiv 0$ on $\partial U_a \cup \partial U_b$ and so an application of the divergence theorem along with the above gives

$$\int_U \rho^3 |\nabla_U g|^2 \, d\rho dz = \int_U div_U\left(\rho^3 g \nabla_U g\right) \, d\rho dz = \int_{\partial U} \rho^3 g \partial_\nu g \, d\sigma = 0. \quad (2.15)$$

However as $\rho > 0$ in U and $|\nabla_U g|^2 \ge 0$ we must have $g \equiv c$ for some constant c that as a result of the zero boundary conditions gives $g \equiv 0$, i.e., $g^1 \equiv g^2$. This therefore completes the proof. $\square$

We are now in a position to prove that when $N = 3$ the whirl mapping u with the associated angle of rotation g is *not* a solution to the system (2.1) except for when $k = 0$.

Theorem 2.1. ($N = 3$) *Let* $\mathbf{X} = \mathbf{X}[a, b] \subset \mathbb{R}^3$ *be an open annulus and consider the elastic energy* (1.1) *with* $W(\mathbf{F}) = tr(\mathbf{F}^t\mathbf{F})/2$ *over the space of incompressible admissible mappings* $\mathcal{A}(\mathbf{X})$. *Then there are no non-trivial equilibria in the form of a whirl mapping.*

Proof. The idea is to examine the Euler-Lagrange system associated with the Dirichlet energy against the whirl mapping $u \in \mathcal{A}(\mathbf{X})$ corresponding to the whirl function $g \in \mathscr{G}_k(U)$ described in the previous proposition. Towards this end we first note that

$$\nabla u(x) = \mathbf{Q}[g] + \dot{\mathbf{Q}}[g]x \otimes \nabla g, \quad (2.16)$$

$$\Delta u(x) = \Delta g \dot{\mathbf{Q}}[g]x + 2\dot{\mathbf{Q}}[g]\nabla g + |\nabla g|^2 \ddot{\mathbf{Q}}x. \quad (2.17)$$

Now referring to the PDE (2.3) an explicit calculation using the above gives

$$\begin{aligned} (\nabla u)^t \Delta u = & 2\mathbf{Q}^t\dot{\mathbf{Q}}\nabla g + \Delta g \mathbf{Q}^t\dot{\mathbf{Q}}x + |\nabla g|^2 \mathbf{Q}^t\ddot{\mathbf{Q}}x \\ & + 2\langle \dot{\mathbf{Q}}x, \dot{\mathbf{Q}}\nabla g\rangle \nabla g + \Delta g |\dot{\mathbf{Q}}x|^2 \nabla g \\ & + |\nabla g|^2 \langle \dot{\mathbf{Q}}x, \ddot{\mathbf{Q}}x\rangle \nabla g. \quad (2.18) \end{aligned}$$

This upon noting the identities $\langle \dot{\mathbf{Q}}x, \ddot{\mathbf{Q}}x\rangle = 0$, $|\dot{\mathbf{Q}}x|^2 = \rho^2$ and $\langle \dot{\mathbf{Q}}x, \dot{\mathbf{Q}}\nabla g\rangle = \rho g_\rho$ is then seen to simplify to

$$(\nabla u)^t \Delta u = 2\mathbf{Q}^t\dot{\mathbf{Q}}\nabla g + \Delta g \mathbf{Q}^t\dot{\mathbf{Q}}x + |\nabla g|^2 \mathbf{Q}^t\ddot{\mathbf{Q}}x + \left(2\rho g_\rho + \rho^2 \Delta g\right) \nabla g. \quad (2.19)$$

Now referring to the reduced Euler-Lagrange system for the whirl function g and using the identity $\Delta g = \rho^{-1} g_\rho + g_{\rho\rho} + g_{zz}$ it follows that $\Delta g + 2g_\rho/\rho = 0$. Hence plugging the latter into (2.19) results in the formulation

$$(\nabla u)^t \Delta u = 2\mathbf{Q}^t \dot{\mathbf{Q}} \nabla g + \Delta g \mathbf{Q}^t \dot{\mathbf{Q}} x + |\nabla g|^2 \mathbf{Q}^t \ddot{\mathbf{Q}} x, \tag{2.20}$$

or in components

$$\begin{aligned} (\nabla u)^t \Delta u &= \begin{bmatrix} -\rho^{-1} x_2 (3g_\rho + \rho g_{\rho\rho} + \rho g_{zz}) + 2x_1 g_\rho^2 + \rho x_1 g_\rho (g_{\rho\rho} + g_{zz}) - x_1 g_z^2 \\ +\rho^{-1} x_1 (3g_\rho + \rho g_{\rho\rho} + \rho g_{zz}) + 2x_2 g_\rho^2 + \rho x_2 g_\rho (g_{\rho\rho} + g_{zz}) - x_2 g_z^2 \\ \rho g_z (3g_\rho + \rho g_{\rho\rho} + \rho g_{zz}) \end{bmatrix} \\ &= \begin{bmatrix} -\rho^{-1} x_2 (2g_\rho + \rho \Delta g) - x_1 (g_\rho^2 + g_z^2) \\ +\rho^{-1} x_1 (2g_\rho + \rho \Delta g) - x_2 (g_\rho^2 + g_z^2) \\ 0 \end{bmatrix}. \end{aligned} \tag{2.21}$$

A further application of $\Delta g + 2g_\rho/\rho = 0$ to the above components finally results in the identity

$$\nabla \mathscr{P} = (\nabla u)^t \Delta u = \begin{bmatrix} -x_1 (g_\rho^2 + g_z^2) \\ -x_2 (g_\rho^2 + g_z^2) \\ 0 \end{bmatrix}. \tag{2.22}$$

Here (2.22) is the formulation of the Euler-Lagrange system for the candidate whirl mapping u in terms of its associated whirl function g. Before we use the explicit form of the solution g to the restricted Euler-Lagrange equation (2.11) we note that a necessary condition for the solvability of the above system for $\mathscr{P}$ is that the vector field on the *right* is *curl-free*. Thus,

$$\begin{aligned} \mathbf{curl}(\nabla \mathscr{P}) = \nabla \times \left[(\nabla u)^t \Delta u \right] &= 0 \\ &\Longleftrightarrow \\ \frac{\partial}{\partial z} \left[(g_\rho^2 + g_z^2) \right] &= 0, \\ x_1 \frac{\partial}{\partial x_2} \left[(g_\rho^2 + g_z^2) \right] - x_2 \frac{\partial}{\partial x_1} \left[(g_\rho^2 + g_z^2) \right] &= 0. \end{aligned} \tag{2.23}$$

Now using the explicit description of the whirl function g in (2.12), for each $k \in \mathbb{Z}$, a basic calculation gives $g_\rho^2 + g_z^2 = (6\pi k a^3 b^3)^2/(b^3 - a^3)^2 (\rho^2 + z^2)^{-4}$. Thus, in particular, when $k \neq 0$, this results in

$$\frac{\partial}{\partial z} (g_\rho^2 + g_z^2) = -\left(\frac{6\pi k a^3 b^3}{b^3 - a^3} \right)^2 \frac{8z}{(\rho^2 + z^2)^5} \neq 0, \tag{2.24}$$

that clearly contradicts (2.23). It therefore follows as claimed that here there are no non-trivial whirl mappings serving as equilibria of the elastic energy. □

The case $N = 2$. Let us next contrast the non-existence result for (non-trivial) whirl mappings in three dimensions with an interesting multiplicity result in two dimensions.

Theorem 2.2. $(N = 2)$ *Consider the whirl mapping* $u_k = u(x; k) = \mathbf{Q}[g(r; k)]x$ *with* $k \in \mathbb{Z}$ *whose angle of rotation function is given by*

$$g = g(r; k) = \frac{2\pi a^2 b^2 k}{b^2 - a^2} \left(\frac{1}{a^2} - \frac{1}{r^2} \right) \qquad a \le r \le b. \tag{2.25}$$

Then $(u_k : k \in \mathbb{Z})$ *are equilibria of the energy* (1.1) *with* $W(\mathbf{F}) = tr(\mathbf{F}^t\mathbf{F})/2$ *over* $\mathcal{A}(\mathbf{X})$*; specifically, for a suitable hydrostatic pressure* $\mathscr{P}_k$*, the pair* $(u_k, \mathscr{P}_k)$ *is a smooth solution to the system* (2.1)*.*

Proof. Fix k and for notational convenience put $u = u_k$ and $g = g(r; k)$. Then upon noting that for $N = 2$ we have $\rho = \sqrt{x_1^2 + x_2^2} = r$ we can write

$$\nabla u(x) = \mathbf{Q}[g](r) + \frac{\dot{g}(r)}{r} \dot{\mathbf{Q}}[g](r) x \otimes x, \tag{2.26}$$

$$\Delta u(x) = \left(\frac{r\ddot{g} + 3\dot{g}}{r} \right) \dot{\mathbf{Q}}[g](r) x + \dot{g}^2 \ddot{\mathbf{Q}}[g](r) x. \tag{2.27}$$

Hence referring to (2.3) and using the calculations above we can write

$$\begin{aligned} \nabla \mathscr{P} = (\nabla u)^t \Delta u = & \left\{ \mathbf{Q}^t[g](r) + x \otimes \frac{\dot{g}(r)}{r} \dot{\mathbf{Q}}[g](r) x \right\} \times \\ & \times \left\{ \left(\frac{r\ddot{g} + 3\dot{g}}{r} \right) \dot{\mathbf{Q}}[g](r) x + \dot{g}^2 \ddot{\mathbf{Q}}[g](r) x \right\} \\ = & \left\{ (r\ddot{g} + 3\dot{g}) \left[\dot{g} \mathbf{I}_2 + \mathbf{Q}^t \dot{\mathbf{Q}} \right] + \dot{g}^2 \mathbf{Q}^t \ddot{\mathbf{Q}} + \frac{\dot{g}^3}{r} \langle \dot{\mathbf{Q}} x, \ddot{\mathbf{Q}} x \rangle \right\} x, \end{aligned} \tag{2.28}$$

that by virtue of the orthogonality relation $\langle \dot{\mathbf{Q}} x, \ddot{\mathbf{Q}} x \rangle = 0$ results in the equation

$$\nabla \mathscr{P} = (\nabla u)^t \Delta u = \begin{bmatrix} -r\dot{g}^2 \cos\theta + (3\dot{g} + r\ddot{g})(r\dot{g}\cos\theta - \sin\theta) \\ -r\dot{g}^2 \sin\theta + (3\dot{g} + r\ddot{g})(r\dot{g}\sin\theta + \cos\theta) \end{bmatrix}. \tag{2.29}$$

Now referring to the explicit form of the angle of rotation function g as given by (2.25) we have $\dot{g}(r) = 4\pi a^2 b^2 k/(b^2 - a^2) r^{-3}$ and $\ddot{g}(r) = -12\pi a^2 b^2 k/(b^2 - a^2) r^{-4}$ and so by a straightforward calculation $3\dot{g} + r\ddot{g} = 0$. As a consequence the PDE (2.29) simplifies further and gives

$$\begin{aligned} (\nabla u)^t \Delta u &= \left\{ (r\ddot{g} + 3\dot{g}) \left[\dot{g} \mathbf{I}_2 + \mathbf{Q}^t \dot{\mathbf{Q}} \right] + \dot{g}^2 \mathbf{Q}^t \ddot{\mathbf{Q}} \right\} x \\ &= \begin{bmatrix} -r\dot{g}^2 \cos\theta + (3\dot{g} + r\ddot{g})(r\dot{g}\cos\theta - \sin\theta) \\ -r\dot{g}^2 \sin\theta + (3\dot{g} + r\ddot{g})(r\dot{g}\sin\theta + \cos\theta) \end{bmatrix} \\ &= \begin{bmatrix} -r\dot{g}^2 \cos\theta \\ -r\dot{g}^2 \sin\theta \end{bmatrix} = \frac{4\pi^2 a^4 b^4 k^2}{(b^2 - a^2)^2} \nabla \frac{1}{|x|^4} = \nabla \mathscr{P}. \end{aligned} \tag{2.30}$$

Therefore the whirl mapping u is a solution to the system (2.1) for a suitable choice of the hydrostatic pressure $\mathscr{P}$. It is interesting to note that here the whirl mapping u is *totally* rotationally symmetric in that for every $\mathbf{R} \in \mathbf{SO}(2)$ we have

$[\mathbf{R}u\circ\mathbf{R}^t](x) = \mathbf{R}u(\mathbf{R}^t x) = \mathbf{R}\mathbf{Q}[g](r)\mathbf{R}^t x = \mathbf{R}\mathbf{R}^t\mathbf{Q}[g](r)x = \mathbf{Q}[g](r)x = u(x)$ in view of the rotation group $\mathbf{SO}(2)$ being commutative. Additionally the energy of the whirl mapping $u = u_k$ here is seen to be

$$\mathbb{E}[u;\mathbf{X}] - |\mathbf{X}| = \int_{\mathbf{X}} \frac{|\nabla u|^2 - 2}{2}\,dx = \frac{16\pi^3 a^4 b^4 k^2}{(b^2-a^2)^2}\int_a^b \frac{dr}{r^3} = \frac{8\pi^3 a^2 b^2 k^2}{b^2-a^2}, \quad (2.31)$$

showing that the elastic energy of u_k diverges to infinity like k^2 as $|k| \nearrow \infty$. □

Remark 2.1. The space of those continuous self-mappings of an N-annulus $\mathbf{X}$ ($N \geq 2$) onto itself that agree with the identity mapping on the boundary $\partial\mathbf{X}$ has a rich and interesting topology. To discuss this further and the link it bears to the extremising whirls consider

$$\mathscr{C}(\mathbf{X}) = \left\{ u \in \mathbf{C}(\overline{\mathbf{X}}, \mathbb{R}^N) : u(\overline{\mathbf{X}}) \subset \overline{\mathbf{X}},\, u \equiv x \text{ on } \partial\mathbf{X} \right\}, \quad (2.32)$$

equipped with the compact open topology. Then as a result of the isomorphisms (*cf.* [25, 27])

$$\begin{aligned} \pi_0(\mathscr{C}(\mathbf{X})) &\cong \pi_1[\mathscr{C}_\phi(\mathbb{S}^{N-1}, \mathbb{S}^{N-1})] \\ &\cong \pi_1[\mathbf{SO}(N)] \cong \left\{ \begin{array}{ll} \mathbb{Z} & \text{in } N = 2, \\ \mathbb{Z}_2 & \text{on } N \geq 3, \end{array} \right. \end{aligned} \quad (2.33)$$

it is seen that the space $\mathscr{C}(\mathbf{X})$ has an infinite number of connected components when $N = 2$ and only two when $N \geq 3$. Here $\mathscr{C}_\phi(\mathbb{S}^{N-1}, \mathbb{S}^{N-1})$ is the component of the space of continuous self-mappings of the sphere onto itself containing the identity or equivalently the component containing mappings with Brouwer-Hopf degree $+1$, π_1 stands for the first homotopy group (or the fundamental group) functor and $\pi_0(\mathscr{C}(\mathbf{X}))$ is the set of connected components of $\mathscr{C}(\mathbf{X})$.

Now when $N = 2$ any admissible $u \in \mathcal{A}(\mathbf{X})$ has a continuous representative (also denoted u) satisfying $u \in \mathscr{C}(\mathbf{X})$. To see this observe firstly that $\det \nabla u = 1$ *a.e.* combined with a Lebesgue-type monotonicity argument as, e.g., in [30] (*see* also [11, 15, 23]) implies that u has a continuous representative $u \in \mathbf{C}(\overline{\mathbf{X}}; \mathbb{R}^2)$. Next the identity boundary condition on u and a degree theoretical argument gives that $u^{-1}(p)$ is non-empty for every $p \in \overline{\mathbf{X}}$ and so $\overline{\mathbf{X}} \subset u(\overline{\mathbf{X}})$. [2] Finally to justify $u(\overline{\mathbf{X}}) \subset \overline{\mathbf{X}}$ one argues by contradiction: Suppose that there exists $x \in \mathbf{X}$ such that $u(x) \notin \mathbf{X}$; then the continuity of u and Lemma 2.4 in [10] contradicts $d(u, \mathbf{X}, p) = 0$ for $p \notin \overline{\mathbf{X}}$. Note that by the same degree theory discussion, for any $u \in \mathscr{C}(\mathbf{X})$ we have $u(\overline{\mathbf{X}}) = \overline{\mathbf{X}}$. Now returning to the connected components, for $N = 2$ we can enumerate these and accordingly partition $\mathscr{C}(\mathbf{X})$ as,

$$\mathscr{C}(\mathbf{X}) = \bigcup_{k\in\mathbb{Z}} \mathscr{C}_k(\mathbf{X}), \qquad \mathscr{C}_k(\mathbf{X}) = \left\{ u \in \mathscr{C}(\mathbf{X}) : \mathbf{Ind}(u) = k \right\}, \quad (2.34)$$

[2] Due to $u \equiv x$ on $\partial\mathbf{X}$ we have $d(u, \mathbf{X}, p) = 1$ for $p \in \mathbf{X}$ and $d(u, \mathbf{X}, p) = 0$ for $p \in \mathbb{R}^2 \backslash \overline{\mathbf{X}}$ where d here stands for the Brouwer degree of $u \in \mathbf{C}(\overline{\mathbf{X}}; \mathbb{R}^2)$.

where the integer $\mathbf{Ind}(u)$ is the index or Brouwer-Hopf degree of the mapping $u|u|^{-1} \in \mathscr{C}(\mathbb{S}^1, \mathbb{S}^1)$ through the identification $\mathbb{S}^1 \cong [a, b]/\{a, b\}$. Using this we can now define $\mathscr{A}_k(\mathbf{X})$ as the class of all those admissible mappings u whose continuous representative lies in $\mathscr{C}_k(\mathbf{X})$, that is, with a slight abuse of notation, and upon using (2.34),

$$\mathcal{A}(\mathbf{X}) = \bigcup_{k\in\mathbb{Z}} \mathscr{A}_k(\mathbf{X}), \qquad \mathscr{A}_k(\mathbf{X}) := \Big\{ u \in \mathcal{A}(\mathbf{X}) : \mathbf{Ind}(u) = k \Big\}, \tag{2.35}$$

where $\mathbf{Ind}(u)$ is now the index of the continuous representative of $u \in \mathcal{A}(\mathbf{X})$. By going back to Theorem 2.2 it is seen that the equilibrium whirl mapping $u_k = \mathbf{Q}[g_k](r)x$ with the whirl function $g = g_k$ as in (2.25) lies in $\mathscr{A}_k(\mathbf{X})$ as a result of:

$$\mathbf{Ind}(u_k) = \frac{1}{2\pi} \int_0^{2\pi} \frac{u_k \times (u_k)_{,r}}{|u_k|^2}\, dr = \frac{1}{2\pi} \int_0^{2\pi} \dot{g}_k(r)\, dr = k. \tag{2.36}$$

In particular the mappings u_k are all of different topological types. Furthermore it is easily seen that u_k is the unique minimiser of $\mathbb{E}$ amongst all whirl mappings in $\mathscr{A}_k(\mathbf{X})$. A much stronger statement and plausible conjecture is that u_k is the unique minimiser of the elastic energy $\mathbb{E}$ over the full component $\mathscr{A}_k(\mathbf{X})$ $\forall k \in \mathbb{Z}$. (*See* [26, 27] for more and [16, 17] for related results. See also [13, 14, 18, 24].)

3 Structure of whirl mappings in higher dimensions

In the remainder of this paper we show how the concepts of whirl mappings and their symmetries can be extended to higher dimensions and investigate whether this class of mappings provides equilibria for the Dirichlet energy over the space $\mathcal{A}(\mathbf{X})$. Towards this end let us start by defining a generalised whirl mapping u of an annulus $\mathbf{X} = \mathbf{X}[a, b] \subset \mathbb{R}^N$ as a continuous self-mapping of $\overline{\mathbf{X}}$ onto itself agreeing with the identity mapping on the boundary $\partial\mathbf{X}$ and having the specific representation

$$u : x \mapsto u(x) = \begin{cases} \mathbf{Q}(\rho_1, \dots, \rho_{n-1}, \rho_n)\, x, & \text{if } N = 2n, \\ \mathbf{Q}(\rho_1, \dots, \rho_{n-1}, x_N)\, x, & \text{if } N = 2n - 1. \end{cases} \tag{3.1}$$

Here $x = (x_1, ..., x_N) \in \mathbf{X}$ and for $1 \le j \le n$ when $N = 2n$ and $1 \le j \le n-1$ when $N = 2n - 1$ we set $\rho_j = (x_{2j-1}^2 + x_{2j}^2)^{1/2}$ while $\mathbf{Q}$ is a suitable mapping (*see* below) taking values in the special orthogonal group $\mathbf{SO}(N)$. For the sake of brevity we agree to set, when $N = 2n - 1$, $\rho_n = x_N$, so that as a result we can write u in (3.1), regardless of N being even or odd, as

$$u : x \mapsto u(x) = \mathbf{Q}(\rho_1, \dots, \rho_n)x, \qquad x \in \overline{\mathbf{X}}. \tag{3.2}$$

With this notation in place we now require $\mathbf{Q}$ to lie in $\mathbf{C}(\overline{U}_N, \mathbb{T})$ where $\mathbb{T}$ is the fixed maximal torus in $\mathbf{SO}(N)$ of all block diagonal 2×2 rotation matrices (with an additional unit block at the end for N odd – see below) and $U_N \subset \mathbb{R}^n$ is the semi-annular region defined according to whether the spatial dimension N is even or odd by

$$U_N = \Big\{ \varrho = (\rho_1, \dots, \rho_n) \in \mathbb{R}^n_+ : a < |\varrho| < b \Big\}, \qquad \text{if } N = 2n, \tag{3.3}$$

and

$$U_N = \Big\{ \varrho = (\rho_1, \dots, \rho_n) \in \mathbb{R}^{n-1}_+ \times \mathbb{R} : a < |\varrho| < b \Big\}, \qquad \text{if } N = 2n-1, \tag{3.4}$$

respectively. Note that here and in what follows we write $|\varrho| = \sqrt{\rho_1^2 + ... + \rho_n^2}$. As a result of $\mathbf{Q}$ taking values on the maximal torus $\mathbb{T} \subset \mathbf{SO}(N)$ we can write $\mathbf{Q}(\varrho) = \mathrm{diag}(\mathbf{R}[g_1](\varrho), \dots, \mathbf{R}[g_n](\varrho), 1)$ when $N = 2n-1$, or more specifically,

$$\mathbf{Q}(\varrho) = \mathbf{Q}(\rho_1, \cdots, \rho_n) = \begin{bmatrix} \mathbf{R}[g_1] & 0 & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \cdots & \mathbf{R}[g_{n-1}] & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \tag{3.5}$$

and $\mathbf{Q}(\varrho) = \mathrm{diag}(\mathbf{R}[g_1](\varrho), \dots, \mathbf{R}[g_n](\varrho))$ when $N = 2n$, that is,

$$\mathbf{Q}(\varrho) = \mathbf{Q}(\rho_1, \cdots \rho_n) = \begin{bmatrix} \mathbf{R}[g_1] & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \\ 0 & \cdots & \mathbf{R}[g_{n-1}] & 0 \\ 0 & 0 & \cdots & \mathbf{R}[g_n] \end{bmatrix}. \tag{3.6}$$

In either case the angle of rotation or whirl function $g_j = g_j(\varrho)$ lies in $\mathbf{C}(\overline{U}_N, \mathbb{R})$ for $1 \le j \le n$ while each 2×2 rotation block in (3.5), (3.6) is the usual $\mathbf{SO}(2)$ matrix of the from

$$\mathbf{R}[g] = \exp\{g\mathbf{J}\} = \begin{bmatrix} \cos g & -\sin g \\ \sin g & \cos g \end{bmatrix}, \qquad \mathbf{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{3.7}$$

Let us now, by assuming sufficient differentiability of the whirl functions g_j, proceed with the calculation of some quantities that will be needed later on. Firstly by a straightforward differentiation it is seen that

$$\nabla u = \mathbf{Q} + \sum_{j=1}^n \mathbf{Q}_{,j} x \otimes \nabla \rho_j. \tag{3.8}$$

Hence upon calculating the square of the Hilbert-Schmidt matrix norm we have

$$\begin{aligned} |\nabla u|^2 &= tr \Big\{ (\mathbf{Q} + \sum_{j=1}^n \mathbf{Q}_{,j} x \otimes \nabla \rho_j)(\mathbf{Q} + \sum_{j=1}^n \mathbf{Q}_{,j} x \otimes \nabla \rho_j)^t \Big\} \\ &= tr \Big\{ \mathrm{I}_N + \sum_{j=1}^n [\mathbf{Q}_{,j} x \otimes \mathbf{Q} \nabla \rho_j + \mathbf{Q} \nabla \rho_j \otimes \mathbf{Q}_{,j} x] + \sum_{j=1}^n \mathbf{Q}_{,j} x \otimes \mathbf{Q}_{,j} x \Big\}. \end{aligned} \tag{3.9}$$

Here $\mathbf{Q}_{,j}$ denotes the partial derivative of $\mathbf{Q}$ with respect to the ρ_j variable. Next we show, upon utilising (3.8), that whirl mappings satisfy the incompressibility constraint. Indeed by the above calculation we have

$$\det \nabla u = \det(\mathbf{Q} + \sum_{j=1}^{n} \mathbf{Q}_{,j} x \otimes \nabla \rho_j) = \det(\mathrm{I}_N + \sum_{j=1}^{n} \mathbf{Q}^t \mathbf{Q}_{,j} x \otimes \nabla \rho_j). \quad (3.10)$$

Now, upon going further into the structure of $\mathbf{Q}$, we note that,

$$\mathbf{Q}^t \mathbf{Q}_{,j} = \begin{cases} \mathrm{diag}(\partial_j g_1 \mathbf{J}, \dots, \partial_j g_{n-1} \mathbf{J}, 0) & \text{if } N = 2n-1, \\ \mathrm{diag}(\partial_j g_1 \mathbf{J}, \dots, \partial_j g_{n-1} \mathbf{J}, \partial_j g_n \mathbf{J}) & \text{if } N = 2n, \end{cases} \quad (3.11)$$

where in the above for short we have set $\partial_j g_i = \partial_{\rho_j} g_i$ while $\mathbf{J}$ is as in (3.7). Next let $y_j = (x_{2j-1}, x_{2j})$ for $1 \le j \le n$ if $N = 2n$ and $y_n = x_{2n-1}$ when $N = 2n-1$. Then it is easily seen that

$$\mathbf{Q}^t \mathbf{Q}_{,j} x = \begin{cases} (\partial_j g_1 \mathbf{J} y_1, \dots, \partial_j g_{n-1} \mathbf{J} y_{n-1}, 0)^t & \text{if } N = 2n-1, \\ (\partial_j g_1 \mathbf{J} y_1, \dots, \partial_j g_{n-1} \mathbf{J} y_{n-1}, \partial_j g_n \mathbf{J} y_n)^t & \text{if } N = 2n. \end{cases} \quad (3.12)$$

Furthermore from the definition of ρ_j it is clear that $\nabla \rho_j = (0, \dots, y_j/\rho_j, \dots, 0)$ and therefore in view of $\mathbf{J}$ being skew-symmetric it is plain that

$$\langle \mathbf{Q}^t \mathbf{Q}_{,j} x, \nabla \rho_i \rangle = \langle \frac{\partial g_i}{\partial \rho_j} \mathbf{J} y_i, \frac{y_i}{\rho_i} \rangle = 0, \quad (3.13)$$

for all $1 \le i, j \le n$. With this observation in mind we now state a lemma which will later assist us in proving that whirl mappings are incompressible mappings.

Lemma 3.1. *Suppose that $a_1, \dots, a_k$ and $b_1, \dots, b_k$ are two sequences of vectors in $\mathbb{R}^d$ satisfying the orthogonality relations $\langle a_i, b_j \rangle = 0$ for each $1 \le i, j \le k$. Then*

$$\det(\mathrm{I}_d + \sum_{j=1}^{k} a_j \otimes b_j) = 1. \quad (3.14)$$

Here as usual $a \otimes b$ stands for the rank-one matrix $[a_i b_j : 1 \le i, j \le d]$.

Proof. This is by induction on k. Firstly when $k = 1$ for the rank-one perturbation of I_d we have $\det(\mathrm{I}_d + a_1 \otimes b_1) = 1 + \langle a_1, b_1 \rangle = 1$. Before proceeding to the case of an arbitrary $k \ge 2$ it is instructive to see how the case $k = 2$ works. Towards this end let $A_1 = \mathrm{I}_d + a_1 \otimes b_1$ and note that $A_1^{-1} = \mathrm{I}_d - a_1 \otimes b_1$. Then $\det(\mathrm{I}_d + a_1 \otimes b_1 + a_2 \otimes b_2) = \big(1 + \langle b_2, A_1^{-1} a_2 \rangle\big) \det A_1$. However as we have $\det A_1 = 1$ and $A_1^{-1} a_2 = a_2 - a_1 \langle a_2, b_1 \rangle = a_2$ it follows easily upon substitution that $\det(\mathrm{I}_d + a_1 \otimes b_1 + a_2 \otimes b_2) = (1 + \langle b_2, a_2 \rangle) = 1$. Thus we have shown that (3.14) holds in the case that $k = 2$.

Let us now assume that (3.14) holds for a fixed k. Put $A_k = \mathrm{I}_d + \sum a_j \otimes b_j$ which then gives $A_k^{-1} = \mathrm{I}_d - \sum a_j \otimes b_j$ (both summations over $1 \le j \le k$). Therefore

$$\det(\mathrm{I}_d + \sum_{j=1}^{k+1} a_j \otimes b_j) = \big(1 + \langle b_{k+1}, A_k^{-1} a_{k+1} \rangle\big) \det A_k. \tag{3.15}$$

Next by the inductive hypothesis we have $\det A_k = 1$ and by invoking the assumption on the vectors a_j and b_j we have $A_k^{-1} a_{k+1} = a_{k+1}$. Therefore this results in $(1 + \langle b_{k+1}, A_k^{-1} a_{k+1} \rangle) \det A_k = (1 + \langle b_{k+1}, a_{k+1} \rangle) = 1$ which then gives the required conclusion for $k+1$. The proof is thus complete. □

Now taking $a_j = \mathbf{Q}^t \mathbf{Q}_{,j} x$ and $b_j = \nabla \rho_j$ combined with the orthogonality relations (3.13), it follows from Lemma 3.1 that,

$$\det \nabla u = \det(\mathrm{I}_N + \sum_{j=1}^{n} \mathbf{Q}^t \mathbf{Q}_{,j} x \otimes \nabla \rho_j) = 1. \tag{3.16}$$

Therefore whirl mappings are incompressible. Now for a whirl mapping u to be admissible, i.e., $u \in \mathcal{A}(\mathbf{X})$, we need $\mathbf{Q}(\varrho) \equiv \mathrm{I}_N$ for $\varrho \in (\partial U_N)_a \cup (\partial U_N)_b$. Recall that $(\partial U_N)_a = \{\varrho \in \partial U_N : |\varrho| = a\}$ and $(\partial U_N)_b = \{\varrho \in \partial U_N : |\varrho| = b\}$. From this point onwards we shall impose this boundary condition on $\mathbf{Q}$ and by virtue of the direct dependence of $\mathbf{Q}$ on the angle of rotation functions g_j do so by requiring each $g_j \equiv 0$ on $(\partial U_N)_a$ and $g_j \equiv 2\pi k_j$ on $(\partial U_N)_b$ for $k_j \in \mathbb{Z}$ [*see* (3.24)]. Now recalling (3.9) we obtain

$$\begin{aligned} |\nabla u|^2 &= tr\left[(\nabla u)(\nabla u)^t\right] \\ &= tr(\mathrm{I}_N + \sum_{j=1}^{n} [\mathbf{Q}_{,j} x \otimes \mathbf{Q} \nabla \rho_j + \mathbf{Q} \nabla \rho_j \otimes \mathbf{Q}_{,j} x] + \sum_{j=1}^{n} \mathbf{Q}_{,j} x \otimes \mathbf{Q}_{,j} x) \\ &= N + \sum_{j=1}^{n} |\mathbf{Q}_{,j} x|^2 + \sum_{j=1}^{n} tr\left(\mathbf{Q}_{,j} x \otimes \mathbf{Q} \nabla \rho_j + \mathbf{Q} \nabla \rho_j \otimes \mathbf{Q}_{,j} x\right). \end{aligned} \tag{3.17}$$

However since as a result of the orthogonality relations described earlier we have the identities

$$\begin{aligned} tr\left(\mathbf{Q}_{,j} x \otimes \mathbf{Q} \nabla \rho_j\right) &= tr\left(\mathbf{Q} \nabla \rho_j \otimes \mathbf{Q}_{,j} x\right) \\ &= \langle \mathbf{Q} \nabla \rho_j, \mathbf{Q}_{,j} x \rangle \\ &= \langle \nabla \rho_j, \mathbf{Q}^t \mathbf{Q}_{,j} x \rangle = 0 \end{aligned} \tag{3.18}$$

it follows that the third term in (3.17) vanishes and therefore the square of the Hilbert-Schmidt norm of the deformation gradient can be simplified and expressed as

$$|\nabla u|^2 = N + \sum_{j=1}^{n} |\mathbf{Q}_{,j} x|^2. \tag{3.19}$$

Consequently the total elastic energy $\mathbb{E}$ of a whirl mapping u results from integrating (3.19) and can be written as

$$\mathbb{E}[u;\mathbf{X}] = \frac{1}{2}\int_{\mathbf{X}} |\nabla u|^2 = \frac{N}{2}|\mathbf{X}| + \frac{1}{2}\int_{\mathbf{X}} \sum_{j=1}^{n} |\mathbf{Q}_{,j} x|^2 \, dx. \tag{3.20}$$

Now referring to the explicit representation of $\mathbf{Q}$ in terms of g as expressed at the beginning of the section it is easily seen that

$$\sum_{j=1}^{n} |\mathbf{Q}_{,j} x|^2 = \sum_{j=1}^{n}\sum_{l=1}^{s} (g_{l,j})^2 \rho_l^2 = \sum_{l=1}^{s} |\nabla g_l|^2 \rho_l^2, \tag{3.21}$$

where we have set $s = n$ if $N = 2n$ or $s = n - 1$ if $N = 2n - 1$. Therefore returning to (3.20), a change of variables gives,

$$\begin{aligned} \mathbb{E}[u;\mathbf{X}] - \frac{N}{2}|\mathbf{X}| &= \frac{1}{2}\int_{\mathbf{X}} \sum_{l=1}^{s} |\nabla g_l|^2 \rho_l^2 \, dx \\ &= \frac{1}{2}\int_{U_N} \sum_{l=1}^{s} (2\pi)^s |\nabla g_l|^2 \rho_l^2 \prod_{j=1}^{s} \rho_j \, d\rho \\ &= \frac{(2\pi)^s}{2} \mathbb{H}[\mathbf{g}; U_N]. \end{aligned} \tag{3.22}$$

A close inspection of the integral above shows that the energy functional $\mathbb{H}$ is a sum of independent energies $\mathbb{H}_l$ (with $1 \le l \le s$), defined for $\mathbf{g} = (g_1, ..., g_s)$, respectively by,

$$\mathbb{H}[\mathbf{g}; U_N] = \sum_{l=1}^{s} \mathbb{H}_l[g_l; U_N], \qquad \mathbb{H}_l[g; U_N] = \int_{U_N} |\nabla g|^2 \rho_l^2 \prod_{j=1}^{s} \rho_j \, d\rho. \tag{3.23}$$

Now as $\mathbb{H}$ is a sum of $\mathbb{H}_l$'s and each $\mathbb{H}_l$ depends only on the angle of rotation function $g = g_l$ it follows that the resulting Euler-Lagrange equations *decouple* and so this means that we can analyse each energy $\mathbb{H}_l$ and its associated Euler-Lagrange equation separately and independently of the rest. Towards this end we introduce the admissible class of mappings for each $\mathbb{H}_l$ as the grand class $\mathscr{G}(U_N)$ defined through

$$\mathscr{G}(U_N) = \bigcup_{k \in \mathbb{Z}} \mathscr{G}_k(U_N) \tag{3.24}$$

where the components in the union on the right for each fixed $k \in \mathbb{Z}$ are in turn given by

$$\mathscr{G}_k(U_N) = \left\{ g \in W^{1,2}(U_N) : \ g \equiv 0 \text{ on } (\partial U_N)_a \text{ and } g \equiv 2\pi k \text{ on } (\partial U_N)_b \right\}.$$

Now for fixed $1 \leq l \leq s$ and $k \in \mathbb{Z}$ the Euler-Lagrange equation associated with the energy $\mathbb{H}_l$ in (3.23) over the space $\mathscr{G}_k(U_N)$ is easily seen to take the form

$$\mathbb{EL}[g, U_N] = \begin{cases} div\left(\rho_l^2 \nabla g \prod_{j=1}^s \rho_j\right) = 0 & \varrho \in U_N, \\ g = 0 & \varrho \in (\partial U_N)_a, \\ g = 2\pi k & \varrho \in (\partial U_N)_b, \\ \rho_l^2 \partial_\nu g \prod_{j=1}^s \rho_j = 0 & \varrho \in \partial U_N \backslash [(\partial U_N)_a \cup (\partial U_N)_b]. \end{cases} \tag{3.25}$$

A quick comment on notation: here and in the remainder of the section all differential operators are understood in reference to the $(\rho_1, ..., \rho_n)$ variables. To avoid any confusion with those in reference to the $(x_1, ..., x_N)$ variables, when necessary, a subscript $U = U_N$ is used. Thus in particular here $div(X_1, ..., X_n) = div_{U_N}(X_1, ..., X_n) = \partial_{\rho_1} X_1 + ... + \partial_{\rho_n} X_n$ and $\nabla f = \nabla_{U_N} f = (\partial_{\rho_1} f, ..., \partial_{\rho_n} f)$. The next result states that for each $k \in \mathbb{Z}$ the boundary value problem (3.25) has a unique solution that can be described in explicit terms.

Proposition 3.1. *For every $k \in \mathbb{Z}$ the Euler-Lagrange system* (3.25) *associated with the energy $\mathbb{H}_l$ over $\mathscr{G}_k(U_N)$ has a unique solution $g = g(\varrho) = g(\rho_1, ..., \rho_n)$ given explicitly by,*

$$g(\rho_1, ..., \rho_n; k) = \frac{2\pi a^N b^N k}{b^N - a^N} \left(\frac{1}{a^N} - \frac{1}{\left(\sqrt{\sum_{i=1}^n \rho_i^2}\right)^N} \right), \qquad \rho \in \overline{U}_N. \tag{3.26}$$

Proof. First we attend to the uniqueness part. To this end suppose that g_1, g_2 are two solutions to (3.25) and put $g = g_1 - g_2$. Then g is a solution to (3.25) but now with all boundary conditions being zero. By an application of the divergence theorem it then follows that

$$\int_{U_N} |\nabla g|^2 \rho_l^2 \prod_{j=1}^s \rho_j \, d\rho = \int_{\partial U_N} g \partial_\nu g \rho_l^2 \prod_{j=1}^s \rho_j \, d\sigma = 0, \tag{3.27}$$

as $g(\varrho) = 0$ for $\varrho \in (\partial U_N)_a$ and $\varrho \in (\partial U_N)_b$. Now in view of $\rho_j > 0$ in U_N for $1 \leq j \leq s$ it follows that $|\nabla g|^2 \equiv 0$ and therefore $g \equiv 0$ in $\overline{U}_N$ as a result of the zero boundary conditions on g. Hence $g_1 = g_2$ in $\overline{U}_N$.

Hence it remains to show that for each $k \in \mathbb{Z}$ the given whirl function $g(\cdot\,; k)$ is a solution to (3.25). To this end we first note that g satisfies all the required boundary conditions as a consequence of $|\rho| = a$ on $(\partial U_N)_a$, $|\rho| = b$ on $(\partial U_N)_b$ and $\rho_1 \times ... \times \rho_s \equiv 0$ on $\partial U_N \backslash [(\partial U_N)_a \cup (\partial U_N)_b]$. Next by a straightforward differentiation it is seen that

$$\partial_{\rho_l} g(\varrho\,; k) = \frac{2\pi a^N b^N k}{b^N - a^N} \frac{N \rho_l}{\left(\sqrt{\sum_{i=1}^n \rho_i^2}\right)^{N+2}}. \tag{3.28}$$

For the sake of convenience and clarity let us now proceed by considering the even and odd cases of N separately. In the former case $N = 2n$ upon using

(3.28) we obtain that

$$
\begin{aligned}
div\left(\rho_l^2\nabla g\prod_{j=1}^n\rho_j\right) =&\frac{2\pi N a^N b^N k}{b^N-a^N}\sum_{r=1}^n\frac{\partial}{\partial\rho_r}\left(\rho_l^2\prod_{\substack{j=1\\ j\neq r}}^n\rho_j\frac{\rho_r^2}{\left(\sum_{i=1}^n\rho_i^2\right)^{n+1}}\right)\\
=&\frac{2\pi N a^N b^N k}{b^N-a^N}\left\{\sum_{\substack{r=1\\ r\neq l}}^n\rho_l^2\prod_{\substack{j=1\\ j\neq r}}^n\rho_j\frac{\partial}{\partial\rho_r}\left(\frac{\rho_r^2}{\left(\sum_{i=1}^n\rho_i^2\right)^{n+1}}\right)+\right.\\
&\left.+\left(\prod_{\substack{j=1\\ j\neq l}}^n\rho_j\right)\frac{\partial}{\partial\rho_l}\left(\frac{\rho_l^4}{\left(\sum_{i=1}^n\rho_i^2\right)^{n+1}}\right)\right\}\\
=&\frac{2\pi N a^N b^N k}{b^N-a^N}\left\{\sum_{\substack{r=1\\ r\neq l}}^n\frac{\rho_l^2\prod_{j=1,j\neq r}^n\rho_j}{\left(\sum_{i=1}^n\rho_i^2\right)^{n+1}}\left(2\rho_r-\frac{(2n+2)\rho_r^3}{\left(\sum_{i=1}^n\rho_i^2\right)}\right)+\right.\\
&\left.+\frac{\prod_{j=1,j\neq l}^n\rho_j}{\left(\sum_{i=1}^n\rho_i^2\right)^{n+1}}\left(4\rho_l^3-\frac{(2n+2)\rho_l^5}{\left(\sum_{i=1}^n\rho_i^2\right)}\right)\right\} \qquad (3.29)
\end{aligned}
$$

and consequently

$$
\begin{aligned}
div\left(\rho_l^2\nabla g\prod_{j=1}^n\rho_j\right) =&\frac{2\pi N a^N b^N k}{b^N-a^N}\left\{\sum_{r=1}^n\frac{\rho_l^2\prod_{j=1}^n\rho_j}{\left(\sum_{i=1}^n\rho_i^2\right)^{n+1}}\left(2-\frac{(2n+2)\rho_r^2}{\left(\sum_{i=1}^n\rho_i^2\right)}\right)+\right.\\
&\left.+2\frac{\rho_l^2\prod_{j=1}^n\rho_j}{\left(\sum_{i=1}^n\rho_i^2\right)^{n+1}}\right\}\\
=&\frac{2\pi N a^N b^N k}{b^N-a^N}\left\{\frac{\rho_l^2\prod_{j=1}^n\rho_j}{\left(\sum_{i=1}^n\rho_i^2\right)^{n+1}}(2n-(2n+2))+\right.\\
&\left.+2\frac{\rho_l^2\prod_{j=1}^n\rho_j}{\left(\sum_{i=1}^n\rho_i^2\right)^{n+1}}\right\}=0. \qquad (3.30)
\end{aligned}
$$

Thus we have that the angle of rotation function g given by (3.26) solves (3.25) and therefore in the even dimensional case $N=2n$ is the unique solution. Now for when $N=2n-1$ the calculations proceed in a similar fashion but with a careful change accounting for $\rho_n=x_N$. Indeed proceeding with the divergence and the first equation in (3.25) here we have

$$div\left(\rho_l^2\nabla g\prod_{j=1}^{n-1}\rho_j\right)=\frac{2\pi Na^Nb^Nk}{b^N-a^N}\left\{\sum_{r=1}^{n-1}\frac{\partial}{\partial\rho_r}\left(\rho_l^2\prod_{\substack{j=1\\j\neq r}}^{n-1}\rho_j\frac{\rho_r^2}{\left(\sum_{i=1}^n\rho_i^2\right)^{n+1/2}}\right)+\right.$$
$$\left.+\rho_l^2\prod_{j=1}^{n-1}\rho_j\frac{\partial}{\partial\rho_n}\left(\frac{\rho_n}{\left(\sum_{i=1}^n\rho_i^2\right)^{n+1/2}}\right)\right\}$$
$$=\frac{2\pi Na^Nb^Nk}{b^N-a^N}\times(\mathbf{I}+\mathbf{II}). \tag{3.31}$$

Now proceeding directly and referring to (3.31) we can write

$$\mathbf{I}=\frac{1}{\left(\sum_{i=1}^n\rho_i^2\right)^{1/2}}\sum_{r=1}^{n-1}\frac{\partial}{\partial\rho_r}\left(\rho_l^2\prod_{\substack{j=1\\j\neq r}}^{n-1}\rho_j\frac{\rho_r^2}{\left(\sum_{i=1}^n\rho_i^2\right)^{n}}\right)+$$
$$+\sum_{r=1}^{n-1}\frac{\partial}{\partial\rho_r}\left(\frac{1}{\left(\sum_{i=1}^n\rho_i^2\right)^{1/2}}\right)\left(\rho_l^2\prod_{\substack{j=1\\j\neq r}}^{n-1}\rho_j\frac{\rho_r^2}{\left(\sum_{i=1}^n\rho_i^2\right)^{n}}\right)=\quad\mathbf{I}_1+\mathbf{I}_2, \tag{3.32}$$

where we have

$$\begin{aligned}\mathbf{I}_1&=\sum_{r=1}^{n-1}\frac{\rho_l^2\prod_{j=1}^{n-1}\rho_j}{\left(\sum_{i=1}^n\rho_i^2\right)^{n+1/2}}\left(2-\frac{2n\rho_r^2}{\left(\sum_{i=1}^n\rho_i^2\right)}\right)+2\frac{\rho_l^2\prod_{j=1}^{n-1}\rho_j}{\left(\sum_{i=1}^n\rho_i^2\right)^{n+1/2}}\\&=\frac{\rho_l^2\prod_{j=1}^{n-1}\rho_j}{\left(\sum_{i=1}^n\rho_i^2\right)^{n+1/2}}\left(2n-2n\frac{\sum_{i=1}^n\rho_i^2-\rho_n^2}{\sum_{i=1}^n\rho_i^2}\right)\\&=2n\frac{\rho_l^2\rho_n^2\prod_{j=1}^{n-1}\rho_j}{\left(\sum_{i=1}^n\rho_i^2\right)^{n+3/2}},\end{aligned} \tag{3.33}$$

and

$$\mathbf{I}_2=-\sum_{r=1}^{n-1}\rho_l^2\prod_{j=1}^{n-1}\rho_j\frac{\rho_r^2}{\left(\sum_{i=1}^n\rho_i^2\right)^{n+3/2}}=-\frac{\rho_l^2\prod_{j=1}^{n-1}\rho_j}{\left(\sum_{i=1}^n\rho_i^2\right)^{n+3/2}}\left(\sum_{i=1}^{n-1}\rho_i^2\right).$$

Likewise a straightforward calculation gives

$$\mathbf{II}=\frac{\rho_l^2\prod_{j=1}^{n-1}\rho_j}{\left(\sum_{i=1}^n\rho_i^2\right)^{n+1/2}}\left(1-\frac{(2n+1)\rho_n^2}{\sum_{i=1}^n\rho_i^2}\right). \tag{3.34}$$

Hence by putting the various fragments of the above calculations and derivations together it is seen at once that the full divergence term in the above set of equations can be written as

$$
\begin{aligned}
div\left(\rho_l^2\nabla g\prod_{j=1}^{n}\rho_j\right) =& \frac{2\pi N a^N b^N k}{b^N-a^N}\times(\mathbf{I}+\mathbf{II})\\
=& \frac{2\pi N a^N b^N k}{b^N-a^N}\frac{\rho_l^2\prod_{j=1}^{n-1}\rho_j}{(\sum_{i=1}^n\rho_i^2)^{n+1/2}}\times\\
&\times\left(1-\frac{(2n+1)\rho_n^2-2n\rho_n^2+\sum_{r=1}^{n-1}\rho_r^2}{\sum_{i=1}^n\rho_i^2}\right)=0. \qquad (3.35)
\end{aligned}
$$

Therefore we have shown that (3.26) is also the unique solution to (3.25) in the odd dimensional case $N=2n-1$. □

Remark 3.1. In view of Proposition 3.1 the whirl function $g=g(\rho_1,\dots,\rho_n;k)$ is solely a function of $|\varrho|=\sqrt{\rho_1^2+...+\rho_n^2}=\sqrt{x_1^2+...+x_N^2}=r$ [*cf.* (3.26)]. Moreover as a solution to the boundary value problem (3.25), the dependence of $g(\varrho;k)$ on $k\in\mathbb{Z}$ is linear, that is, we have $g(\varrho;k)=kg(\varrho;1)$. These observations put together prompt us to write the solution g to (3.25), with a slight abuse of notation, as

$$
g(r\,;k)=d(k)-c(k)r^{-N},\qquad a\le r\le b,\quad k\in\mathbb{Z}, \qquad (3.36)
$$

with the choice of coefficients

$$
c(k)=\frac{2\pi a^N b^N k}{b^N-a^N},\qquad d(k)=\frac{2\pi b^N k}{b^N-a^N}. \qquad (3.37)
$$

Thus any solution $\mathbf{g}=(g_1,...,g_s)$ to the Euler-Lagrange equation associated with the restricted energy $\mathbb{H}[\mathbf{g};U_N]$, in turn, depends solely on the radial variable r and as a matter of fact, in vector notation $\mathbf{g}(r)=\mathbf{d}-\mathbf{c}/r^N$ with $\mathbf{c}=\mathbf{c}(k)=(c_1,\dots,c_s)$ and $\mathbf{d}=\mathbf{d}(k)=(d_1,\dots,d_s)$. In particular the corresponding whirl mapping u is of the form $u(x)=\mathbf{Q}(r)x$ for a suitable $\mathbf{Q}\in\mathbf{C}^\infty([a,b],\mathbf{SO}(N))$.

4 Whirl mappings as solutions to the nonlinear system (2.1) in higher dimensions $N\ge 4$

In this final section of the paper we show that in higher dimensions $N\ge 4$, the non-trivial whirl mappings obtained as extremisers (equivalently critical points) of the restricted energy in the previous section can only go on to satisfy the full Euler-Lagrange system (2.1) associated with the Dirichlet energy $\mathbb{E}$, i.e., with $W(\mathbf{F})=tr(\mathbf{F}^t\mathbf{F})/2$, when $N=2n$. In contrast, when $N=2n-1$ the only whirl solution to (2.1) will be shown to be the identity mapping. The conclusion is therefore similar in spirit to the cases $N=2$ *vs.* $N=3$ discussed earlier in the paper but with further and natural complications.

Towards this end recall that the Euler-Lagrange equation associated with the Dirichlet energy over $\mathcal{A}(\mathbf{X})$ takes the form [*cf.* (1.4), (2.1)]

$$\begin{aligned} div\, \mathfrak{S}[x, \nabla u(x)] &= \Delta u - div(\mathscr{P}(x) \operatorname{cof} \nabla u) \\ &= \Delta u - (\operatorname{cof} \nabla u) \nabla \mathscr{P}(x) \\ &= 0. \end{aligned} \tag{4.1}$$

Subsequently by invoking the incompressibility constraint $\det \nabla u = 1$ and basic identities, upon rearranging terms, it must be that

$$(\operatorname{cof} \nabla u)^{-1} \Delta u = (\nabla u)^t \Delta u = \nabla \mathscr{P}. \tag{4.2}$$

Again, by a classical solution we mean a pair $(u, \mathscr{P})$ where u is admissible, that is, $u \in \mathcal{A}(\mathbf{X})$, $(u, \mathscr{P})$ is regular, i.e., $u \in \mathbf{C}(\overline{\mathbf{X}}, \mathbb{R}^N) \cap \mathbf{C}^2(\mathbf{X}, \mathbb{R}^N)$, $\mathscr{P} \in \mathbf{C}(\overline{\mathbf{X}}) \cap \mathbf{C}^1(\mathbf{X})$ and (1.4) or the equivalent formulation of the first equation (4.2) holds. Now straightforward calculations using the notation introduced in the previous section lead to the identities (here *dots* denote derivatives of $\mathbf{Q}$ with respect to $r = \sqrt{\rho_1^2 + \cdots + \rho_n^2} = \sqrt{x_1^2 + \cdots + x_N^2}$ in light of Remark 3.1),

$$\nabla u = \mathbf{Q} + r^{-1} \dot{\mathbf{Q}} x \otimes x, \tag{4.3}$$

$$\Delta u = r^{-1} \left[(N+1) \dot{\mathbf{Q}} + r \ddot{\mathbf{Q}} \right] x. \tag{4.4}$$

Next referring to the previous section and by using the explicit form of $\mathbf{Q}$ and g as given by (3.36) we can easily verify that

$$\dot{\mathbf{Q}} = N r^{-(N+1)} \mathbf{Q} \mathbf{C}, \tag{4.5}$$

$$\ddot{\mathbf{Q}} = -N(N+1) r^{-(N+2)} \mathbf{Q} \mathbf{C} + N^2 r^{-(2N+2)} \mathbf{Q} \mathbf{C}^2. \tag{4.6}$$

Here $\mathbf{C}$ is the $N \times N$ skew-symmetric block diagonal matrix given by,

$$\mathbf{C} = \begin{cases} \operatorname{diag}(c_1 \mathbf{J}, \ldots, c_{n-1} \mathbf{J}, 0) & \text{if } N = 2n-1, \\ \operatorname{diag}(c_1 \mathbf{J}, \ldots, c_{n-1} \mathbf{J}, c_n \mathbf{J}) & \text{if } N = 2n, \end{cases} \tag{4.7}$$

and as seen before $\mathbf{J}$ is the 2×2 skew-symmetric square root of $-\mathbf{I}_2$ [*see* (3.7)]. Furthermore the boundary conditions $\mathbf{Q}(a) = \mathbf{I}_N$ and $\mathbf{Q}(b) = \mathbf{I}_N$ force the block entries of $\mathbf{C}$ to take on the values [*cf.* (3.37)]

$$c_j \mathbf{J} = c_j(k_j) \mathbf{J} = \frac{2\pi a^N b^N}{b^N - a^N} k_j \mathbf{J}, \qquad k_j \in \mathbb{Z}. \tag{4.8}$$

Therefore returning to the PDE (4.2) it is not difficult to verify that firstly, the Laplacian of u can be computed and simplified to,

$$\begin{aligned} \Delta u =& r^{-1} \left[(N+1) \dot{\mathbf{Q}} + r \ddot{\mathbf{Q}} \right] x \\ =& r^{-1} \Big\{ (N+1) N r^{-(N+1)} \mathbf{Q} \mathbf{C} + r \Big[- N(N+1) r^{-(N+2)} \mathbf{Q} \mathbf{C} + \\ &+ N^2 r^{-(2N+2)} \mathbf{Q} \mathbf{C}^2 \Big] \Big\} = \frac{N^2}{r^{2N+2}} \mathbf{Q} \mathbf{C}^2 x, \end{aligned} \tag{4.9}$$

and secondly, and as a consequence, that (4.2) results in the gradient of the pressure field $\mathscr{P}$ being

$$\begin{aligned}\nabla\mathscr{P} &= (\operatorname{cof}\nabla u)^{-1}\Delta u = (\nabla u)^t\Delta u\\ &= \frac{N^2}{r^{2N+2}}\left(\mathbf{Q}^t + \frac{N}{r^{N+2}}x\otimes\mathbf{QC}x\right)\mathbf{QC}^2x = \frac{N^2}{r^{2N+2}}\mathbf{C}^2x.\end{aligned} \tag{4.10}$$

Note that here $\mathbf{C}^2$ is the $N\times N$ symmetric block-diagonal matrix [*cf.* (4.7)]

$$\mathbf{C}^2 = \begin{cases}\operatorname{diag}(-c_1^2\mathbf{I}_2,\dots,-c_{n-1}^2\mathbf{I}_2,0) & \text{if } N=2n-1,\\ \operatorname{diag}(-c_1^2\mathbf{I}_2,\dots,-c_{n-1}^2\mathbf{I}_2,-c_n^2\mathbf{I}_2) & \text{if } N=2n.\end{cases} \tag{4.11}$$

Now as the vector field on the RHS in (4.10) must necessarily be *curl-free* in $\mathbf{X}$ (as a result of being a gradient field) we proceed by computing for $1\le i,j\le N$,

$$\begin{aligned}\left\{\mathbf{curl}\left[(\nabla u)^t\Delta u\right]\right\}_{ij}=0 &\iff \frac{\partial}{\partial x_j}\left[\sum_{\ell=1}^N u_{\ell,i}\Delta u_\ell\right] - \frac{\partial}{\partial x_i}\left[\sum_{\ell=1}^N u_{\ell,j}\Delta u_\ell\right]=0\\ &\iff \left[(\nabla u)^t\Delta\nabla u\right]_{ij} - \left[(\nabla u)^t\Delta\nabla u\right]_{ji}=0\\ &\iff \frac{\partial}{\partial x_j}\frac{c^2_{[(i+1)/2]}x_i}{r^{2N+2}} - \frac{\partial}{\partial x_i}\frac{c^2_{[(j+1)/2]}x_j}{r^{2N+2}}=0\\ &\iff (2N+2)\left(c^2_{[(i+1)/2]} - c^2_{[(j+1)/2]}\right)\frac{x_ix_j}{r^{2N+4}}=0,\end{aligned} \tag{4.12}$$

for all $x\in\mathbf{X}$. Therefore the whirl mapping u is a solution to the Euler-Lagrange system (2.1) *iff* we have $|c_1|^2=\cdots=|c_n|^2\equiv c^2$. However in odd dimensions due to the presence of a zero entry in the last block of $\mathbf{C}$ [*cf.* (4.7)] this gives $c=0$ and so $|c_1|=\cdots=|c_n|=0$. Therefore $\mathbf{Q}^t\dot{\mathbf{Q}}=0$ and this in turn gives

$$\mathbf{Q}^t\dot{\mathbf{Q}}=0 \iff \dot{\mathbf{Q}}=0 \iff \mathbf{Q}\equiv \mathrm{I}_N, \tag{4.13}$$

as $\mathbf{Q}(a)=\mathbf{Q}(b)=\mathrm{I}_N$. As a result for N odd the only whirl mapping satisfying the Euler-Lagrange system (2.1) is the identity mapping $u\equiv x$. In contrast for N even we have $|c_1|=\cdots=|c_n|$ and so (2.1) admits an infinite family of solutions in the form $u_k=\mathbf{Q}(\rho_1,\dots,\rho_n;k)x$ (with $k\in\mathbb{Z}$) where $\mathbf{Q}\in\mathbf{C}^\infty(\overline{U}_N;\mathbf{SO}(N))$ is block diagonal as in (3.6) and the whirl functions $g_j=g(\rho_1,\dots,\rho_n;k_j)$ (with $1\le j\le n$) are given explicitly by (3.26) subject to $|k_1|=\dots|k_n|=|k|$, that is, $g_1,\dots,g_n\in\{\pm g\}$ where $g=g(\rho;k)$ is as in (3.26). We have therefore proved the following result.

Theorem 4.1. *Let* $\mathbf{X}=\mathbf{X}[a,b]\subset\mathbb{R}^N$ $(N\ge 2)$ *and consider the elastic energy* $\mathbb{E}[u;\mathbf{X}]$ *with* $W(\mathbf{F})=tr(\mathbf{F}^t\mathbf{F})/2$ *over the space of incompressible admissible mappings* $\mathcal{A}(\mathbf{X})$ *along with the system of Euler-Lagrange equations* (2.1). *Then the following hold:*

- (N *even*) *The nonlinear system* (2.1) *admits an infinite family of solutions in the form of whirl mappings, specifically,* $u_k=\mathbf{Q}(\rho_1,\dots\rho_n;k)x$ *with* $k\in\mathbb{Z}$, $\mathbf{Q}$ *as in* (3.6), $g_1,\dots,g_{N/2}\in\{\pm g\}$ *and* $g=g(\rho;k)$ *as in* (3.26).

- (N *odd*) *The only solution to* (2.1) *in the form of a whirl mapping is the trivial one, that is, the identity mapping* $u \equiv x$.

Remark 4.1. In the case N even we can explicitly calculate the elastic energy of the whirl solutions given by Theorem 4.1. Indeed it is seen that the Dirichlet energy of $u = u_k$ (with $k \in \mathbb{Z}$) can be expressed as

$$\begin{aligned} \mathbb{E}[u; \mathbf{X}] &= \frac{N}{2}|\mathbf{X}| + 2N^2\omega_N \int_a^b \left(\frac{\pi a^N b^N k}{b^N - a^N} \right)^2 r^{-N-1}\, dr \\ &= \frac{N}{2}|\mathbf{X}| + \frac{2N^2\omega_N \pi^2 a^N b^N}{b^N - a^N} k^2. \end{aligned} \tag{4.14}$$

Hence similar to what was seen earlier in the planar case the energy of the whirl solutions diverge to infinity quadratically in k. Likewise the gradient of the hydrostatic pressure in (4.10) is given by $\nabla \mathscr{P} = Nc^2 \nabla |x|^{-2N}/2$ where $-c^2$ as indicated earlier is the common value of the diagonal entries of the symmetric matrix $\mathbf{C}^2$ in (4.11).

Remark 4.2. For $N \geq 4$ even the extremising whirls u_k in Theorem 4.1 have the following topological parity. Indeed referring to Remark 2.1 the space $\mathscr{C}(\mathbf{X})$ here has only two connected components. Although for $N \geq 3$ the admissible mappings $u \in \mathcal{A}(\mathbf{X})$ do not in general have continuous representatives it is clear however that the whirls u_k do lie in $\mathscr{C}(\mathbf{X})$. It turns out that when $N/2$ is even the whirls u_k are all in one component (the same component as the identity). In contrast when $N/2$ is odd the whirls u_k with $k \in \mathbb{Z}$ even all lie in one component and the whirls u_k with $k \in \mathbb{Z}$ odd all lie in the other component. This follows by a direct calculation of their associated spin degree (*cf.* [27]).

References

[1] S.S. Antman, *Nonlinear Problems of Elasticity*, Applied Mathematical Sciences, Vol. **107**, 2nd Edition, Springer, 2005.

[2] J.M. Ball, Convexity conditions and existence theorems in nonlinear elasticity, *Arch. Ration. Mech. Anal.*, Vol. **63**, pp. 337-403, 1977.

[3] J.M. Ball, D.G. Schaeffer, Bifurcation and stability of homogeneous equilibrium configurations of an elastic body under dead-load traction, *Math. Proc. Camb. Phil. Soc.*, Vol. **94**, 1983, pp. 315-339.

[4] J.M. Ball, Differentiability properties of symmetric and isotropic functions, *Duke Math. J.*, Vol. **51**, No. **3**, 1984, pp. 699-728.

[5] J.M. Ball, Some open problems in elasticity, In: *Geometry, mechanics and dynamics*, Springer, New York, pp. 3-59, 2002.

[6] P. Ciarlet, *Mathematical Elasticity: Three dimensional elasticity*, Vol. **1**, Elsevier, 1988.

[7] B. Dacorogna, *Direct methods in the Calculus of Variations*, Vol. **78**, Springer, 2007.

[8] S. Day, A. Taheri, Geodesics on $\mathbf{SO}(n)$ and a class of spherically symmetric maps as solutions to a nonlinear generalised harmonic map problem, *Topol. Meth. Nonlin. Anal.*, To appear 2018.

[9] L.C. Evans, R.F. Gariepy, On the partial regularity of energy minimizing area preserving maps, *Calc. Var. & PDEs*, Vol. **63**, 1999, pp. 357-372.

[10] I. Fonseca, W Gangbo, Local invertibility of Sobolev functions, *SIAM J. Math. Anal.*, Vol. **26**, 1995, pp. 280-304.

[11] V.M. Gol'dshtein, Yu.G. Reshetnyak, *Quasiconformal mappings and Sobolev spaces*, Kluwer Academic Publishers, 1990.

[12] T. Iwaniec, J. Onninen, n-harmonic mappings between annuli: the art of integrating free Lagrangians, *Mem. Amer. Math. Soc.*, Vol. **218**, No. **1023**, viii+105 pp., 2012.

[13] F. John, Remarks on the non-linear theory of elasticity, *Semin. Ist. Naz. Alta Mat.*, Ediz. Cremonese, Rome, 1962/1963, pp. 474-482.

[14] F. John, Uniqueness of non-linear elastic equilibrium for prescribed boundary displacements and sufficiently small strains, *Comm. Pure Appl. Math.*, Vol. **25**, 1972, pp. 617?634.

[15] C.B. Morrey. *Multiple Integrals in the Calculus of Variations*, Classics in Mathematics, Vol. **130**, Springer, 1966.

[16] C. Morris, A. Taheri, Twist mappings as energy minimisers in homotopy classes: symmetrisation and the coarea formula, *Nonlinear Anal.*, Vol. **152**, 2017, pp. 250-275.

[17] C. Morris, A. Taheri, On the Uniqueness and Monotonicity of Energy Minimisers in the Homotopy Classes of Incompressible Mappings and Related Problems, Submitted, 2017.

[18] K. Post, J. Sivaloganathan, On homotopy conditions and the existence of multiple equilibria in finite elasticity, *Proc. Roy. Soc. Edin. A*, **127**, 1997, pp. 595-614.

[19] M.S. Shahrokhi-Dehkordi, A.Taheri, Generalised twists, stationary loops and the Dirichlet energy over a space of measure preserving maps, *Calc. Var. & PDEs*, Vol. **35**, 2009, pp. 191-213.

[20] M.S. Shahrokhi-Dehkordi, A. Taheri, Quasiconvexity and uniqueness of stationary points on a space of measure preserving maps, *J. Convex Anal.*, Vol. **17**, No. **1**, 2010, pp. 69-79.

[21] J. Sivaloganathan, S.J. Spector. On the symmetry of energy-minimising deformations in nonlinear elasticity, I. Incompressible materials, *Arch. Ration. Mech. Anal.*, Vol. **196**, 2010, pp. 363-394.

[22] J. Sivaloganathan, S.J. Spector. On the symmetry of energy-minimising deformations in nonlinear elasticity, II. Compressible materials, *Arch. Ration. Mech. Anal.*, Vol. **196**, 2010, pp. 395-431.

[23] V. Sverak, Regularity properties of deformations with finite energy, *Arch. Rational Mech. Anal.*, Vol. **100**, 1988, pp. 105-127.

[24] A. Taheri, On Artin's braid group and polyconvexity in the calculus of variations, *J. London Math. Soc.*, Vol. **67**, No. **3**, 2003, pp. 752-768.

[25] A. Taheri, Local minimizers and quasiconvexity - the impact of Topology, *Arch. Rational Mech. Anal.*, Vol. **176**, 2005, pp. 363-414.

[26] A. Taheri, Minimizing the Dirichlet energy over a space of measure preserving maps, *Top. Meth. Nonlinear Anal.*, Vol. **33**, 2009, pp. 179-204.

[27] A. Taheri, Homotopy classes of self-maps of annuli, generalised twists and spin degree, *Arch. Rational Mech. Anal.*, Vol. **197**, 2010, pp. 239-270.

[28] A. Taheri, *Function Spaces and Partial Differential Equations*, Vol. **I** & **II**, Oxford Lecture Series in Mathematics and its Applications, OUP, 2015.

[29] N.J. Vilenkin, *Special Functions and the Theory of Group Representations*, Translations of Mathematical Monographs, Vol. **22**, AMS, 1968.

[30] S.K. Vodopyanov, V.M. Gol'dshtein. Quasiconformal mappings and spaces of functions with generalized first derivatives, *Siberian Math. J.*, Vol. **17**, 1977, pp. 515-531.