The Geometry of The Plane of Order Nineteen and its Application to Error-Correcting Codes

by

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Declaration

I hereby declare that the work presented in this thesis is entirely my own, unless otherwise stated, and has been not presented for examination, in whole or in part, to this or any other university.

Signed:

Date:

To The Soul of My Father

Abstract

In the projective space $PG(k-1,q)$ over $\mathbf{F}_q$, the finite field of order q, an $(n;r)$-arc $\mathcal{K}$ is a set of n points with at most r on a hyperplane and there is some hyperplane meeting $\mathcal{K}$ in exactly r points. An arc is complete if it is maximal with respect to inclusion. The arc $\mathcal{K}$ corresponds to a projective $[n,k,n-r]_q$-code of length n, dimension k, and minimum distance $n-r$; if $\mathcal{K}$ is a complete arc, then the corresponding projective code cannot be extended.

In this thesis, the n-sets in $PG(1,19)$ up to $n=10$ and the n-arcs in $PG(2,19)$ for $4 \le n \le 20$ in both the complete and incomplete cases are classified.

The set of rational points of a non-singular, plane cubic curve can be considered as an arc of degree three. Over $\mathbf{F}_{19}$, these curves are classified, and the maximum size of the complete arc of degree three that can be constructed from each such incomplete arc is given.

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Preface

In recent years there has been an increasing interest in finite projective spaces, and important applications to practical topics such as coding theory, cryptography and design of experiments have made the field even more attractive.

A problem, first studied in statistics by Fisher [20, 21], has proved to be equivalent to a problem in geometry [13]. The statistician Bose [11], in his investigations on graph theory, design theory and finite projective spaces, generalized this application of finite projective geometry for the design of experiments and called it the *packing problem*. mainly using purely combinatorial arguments in combination with some linear algebra. He also presented, in 1961, connections between the design of experiments and coding theory [12, 13].

After initial consideration by Bose and his followers as a statistical problem, the topic was taken up by Segre [45, 46] applied to finite projective spaces. Using geometric methods, he showed that, in the projective plane $PG(2, q)$ over the Galois field $\mathbf{GF}(q)$ with q odd, every set of $q + 1$ points, no three of which are collinear, is a conic.

Coding theory provides a second motivation for these problems, which have equivalent formulations in finite projective spaces and coding theory. This amounts in coding theory to studying the row space of a generator matrix of a code and in Galois geometry to studying the column space. The classical example, that is, the equivalence of linear maximum distance separable MDS codes and arcs in projective spaces, has been stressed in many books on Galois geometries and coding theory. In [36], MacWilliams and Sloane introduce the chapter on MDS codes in their standard work on coding theory *as one of the most fascinating chapters in all of coding theory*.

Applications of curves over finite fields to coding theory have been given by Goppa [22]. These stimulated the enthusiasm of many geometers to work on the problems that arise from these relations amongst finite projective spaces, coding theory and statistics. Some of these problems in the view of finite projective spaces and coding theory are as follows:

(I) to classify the $(n; r)$-arcs in $PG(k - 1, q)$ which correspond to projective $[n, k, n - r]_q$-codes;

(II) to classify non-singular plane cubic curves in $PG(k - 1, q)$ and find the number of rational inflexion points on each curve.

These two problems has been studied by Hirschfeld theoretically and also for $q = 2, 3, 4, 5, 7, 9, 11, 13$; see [28]. Hirschfeld and Storm have given a historical survey on the problem (I); see [31].

In this thesis the case $q = 19$ is studied. Since

$$19 \equiv 1 \pmod 3$$

this property affects the geometry considerably. For example, there exists a tetrad of equianharmonic type in $PG(1, 19)$ and there exists a non-singular plane cubic curve with nine rational inflexion points.

The principle themes of this thesis are the following.

(1) Classify the subsets of the projective line $PG(1, 19)$ up to size 10.

(2) Classify arcs in the projective plane $PG(2, 19)$ up to size 20.

(3) Classify those arcs which are contained in a conic.

(4) Classify non-singular plane cubic curves in $PG(2, 19)$ according to the number of rational inflexion points and the number of rational inflexion triangles; determine which of them are complete as $(k; 3)$-arcs and, for each incomplete $(k; 3)$-arc, find the largest complete $(k; 3)$-arc which contain it.

(5) Give the corresponding error-correcting projective $[n, k, n-r]_{19}$-codes to these arcs.

The main computing tool that was used in this thesis is the mathematical programming language GAP [24]. A windows machine and a cluster computer have been used to execute the programs.

The material developed in each chapter is as follows.

Chapter 1 is devoted to basic definitions and some background material on the theory of finite fields, projective spaces, their arcs, conics, plane cubics and coding theory. The relation between coding theory and projective spaces is explained. The main reference used in this chapter is [28] beside other important references that are related to the subjects in this work referred to throughout.

Chapter 2 deals with the projective line $PG(1, 19)$ of order nineteen. The classification of all n-sets for $4 \le n \le 10$ is given. Partitions of $PG(1, 19)$ into two decads and into five disjoint tetrads are found. Links with MDS codes of dimension two are also given.

Chapter 3 introduces the background to the projective plane $PG(2, q)$. Important properties of ovals, conics and complete arcs in the projective plane are introduced and their relation to each other are explained. Also, some of the known results on complete arcs are stated and some conditions for a k-arc $\mathcal{K}$ to be complete or incomplete are given. Some bounds for the size of complete arcs are also given. The algorithms to calculate the matrix transformation between any two 4-arcs and construction of a complete arc are given.

Chapter 4 deals with classification in the projective plane $PG(2, 19)$ of complete and incomplete n-arcs for $4 \leq n \leq 20$. Some other configurations are also given. Links with MDS codes of dimension three are described.

Chapter 5 is devoted to non-singular, plane cubic curves over $\mathbf{F}_{19}$. The canonical form, the number of rational points and the stabilizer group for each non-singular plane cubic curve are found. The number of non-singular plane cubic curves is determined in both the complete and incomplete cases, and the maximum size of a complete arc of degree three that can be constructed from each incomplete arc are given. Links with AMDS codes of dimension three are also explained.

Finally, in the appendix there is a table of the points of $PG(2, 19)$.

Chapter 1

Introduction

1.1 Finite Fields

A *field* is a non-empty set K with two binary operations, usually called addition and multiplication, with the property that K is an additive group with identity 0 and $K\backslash\{0\}$ is a multiplicative group and distributive law holds.

A *Galois field* is a finite field with $q = p^h$ elements, where p is a prime number and h is a natural number. This field is denoted by $\mathbf{GF}(q)$ or $\mathbf{F}_q$. Here p is called the *characteristic* of this field and is the smallest prime such that

$$px = \underbrace{x + \cdots + x}_{p} = 0, \text{ for all } x \in \mathbf{F}_q.$$

If $f(x)$ is an irreducible polynomial of degree h over $\mathbf{F}_p$, then

$$\begin{aligned} \mathbf{F}_{p^h} &= \mathbf{F}_p[x]/(f(x)) \\ &= \{a_0 + a_1 x + a_2 x^2 + \cdots + a_{h-1}x^{h-1} \mid a_i \in \mathbf{F}_p, f(x) = 0\}, \end{aligned}$$

is a field of characteristic p satisfying the following properties.

(1) The elements x of $\mathbf{F}_q$ satisfy $x^q - x = 0$.

(2) Let $x, y \in \mathbf{F}_q$. then

$$(x + y)^q = x^q + y^q.$$

(3) There exists ϵ in $\mathbf{F}_q\backslash\{0\}$ such that

$$\mathbf{F}_q = \{0, 1, \epsilon, \ldots, \epsilon^{q-2} \mid \epsilon^{q-1} = 1\};$$

such an ϵ is called a *primitive element* or *primitive root* of $\mathbf{F}_q$.

(4) The additive structure of $\mathbf{F}_q$ is given by the group isomorphism

$$\mathbf{F}_q \cong \underbrace{\mathbf{Z}_p \times \cdots \times \mathbf{Z}_p}_{h}.$$

(5) The multiplicative structure of $\mathbf{F}_q$ is given by the group isomorphism

$$\mathbf{F}_q \backslash \{0\} \cong \mathbf{Z}_{q-1}.$$

(6) $\mathbf{F}_{p^h}$ is a vector space of dimension h over $\mathbf{F}_p$.

(7) (**Uniqueness**): Any finite field K of q elements is isomorphic to $\mathbf{F}_q$.

Throughout this work the linear space of n-tuples over the finite field $\mathbf{F}_q$ is denoted by $\mathbf{F}_q^n$.

For an extensive introduction to finite fields see [35].

1.2 Primitive and subprimitive polynomials

Let $f(x) = x^n - a_{n-1}x^{n-1} - \cdots - a_0$ be a monic polynomial of degree $n \geq 1$ over $\mathbf{F}_q$.

(I) Its *companion matrix* $\mathbf{C}(f)$ is given by the $n \times n$ matrix

$$\mathbf{C}(f) = \begin{bmatrix} 0 & & & \\ \vdots & & I_{n-1} & \\ 0 & & & \\ a_0 & a_1 & \cdots & a_{n-1} \end{bmatrix}.$$

(II) Let f be irreducible over $\mathbf{F}_q$ and $\alpha \in \mathbf{F}_{q^n}$ be a root of f.

- It is called *primitive* if the smallest power s of α such that $\alpha^s = 1$ is $(q^n - 1)$; that is, α is a primitive root over $\mathbf{F}_{q^n}$.
- It is called *subprimitive* if the smallest power s of α such that $\alpha^s \in \mathbf{F}_q$ is $\theta(n-1, q) = (q^n - 1)/(q - 1)$.

Lemma 1.2.1. *The monic polynomial $f(x)$ of degree $n \geq 1$ over $\mathbf{F}_q$ is a primitive polynomial over $\mathbf{F}_q$ if and only if $(-1)^n f(0)$ is a primitive element of $\mathbf{F}_q$ and the least positive integer r for which $f(x)$ divides $(x^r - (-1)^n f(0))$ is $\theta(n-1, q)$.*

Proof. See [35, Theorem 3.18]. □

Corollary 1.2.2. *According to Lemma 1.2.1, every primitive polynomial over* $\mathbf{F}_q$ *is also subprimitive.*

1.3 Roots

To solve the equation $x^n = c$ in $\mathbf{F}_q$ with $q = p^h$, let $d = (n, q-1)$, $e = (q-1)/d$ and let s be a primitive element of $\mathbf{F}_q$.

(i) $x^n = 1$ has d solutions in $\mathbf{F}_q$, namely $x = 1, s^e, s^{2e}, \ldots, s^{(d-1)e}$.

(ii) $x^n = 1$ has the unique solution $x = 1$ when $d = 1$.

(iii) $x^n = 1$ has n solutions when $n|(q-1)$; these are $x = 1$, $s^{(q-1)/n}, \ldots, s^{(n-1)(q-1)/n}$.

(iv) $x^n = c$ has a unique solution when $d = 1$; this is $x = c^r$ where $r, r' \in \mathbf{Z}$ and $rn + r'(q-1) = 1$.

(v) $x^n = c$ has n solutions when $n|(q-1)$ and $c^{(q-1)/n} = 1$.

(vi) When $p > 2$, the equation $x^2 = c$ has two solutions for exactly half the non-zero values of c and no solutions for the other half.

(vii) The following are equivalent:

   (a) $(q-1, 3) = 3$;

   (b) $q \equiv 1 \pmod 3$;

   (c) $x^2 + x + 1 = 0$ has two distinct roots in $\mathbf{F}_q$;

   (d) $x^3 = 1$ has exactly three solutions in $\mathbf{F}_q$.

(viii) The following are equivalent:

   (a) $q \equiv 0 \pmod 3$;

   (b) $x^2 + x + 1 = 0$ has exactly one root in $\mathbf{F}_q$;

   (c) $x^3 = 1$ has exactly one solution in $\mathbf{F}_q$ and in $\mathbf{F}_{q^2}$;

   (d) $q = 3^h$.

1.4 Group Theory and Group Actions

A group G acts on a set K if there is a map $\varphi : K \times G \longrightarrow K$ such that if e is the identity and $g,\ g'$ are elements in G; then, for any $x \in K$,

(i) $\varphi(x, e) = x$;

(ii) $\varphi(\varphi(x, g), g') = \varphi(x, gg')$.

The *orbit* of a set S is $SG = \{\varphi(S, g) \mid g \in G\}$, a subset of K; the *stabilizer* of S is $G_S = \{g \in G \mid \varphi(S, g) = S\}$, a subgroup of G.

The action of G on K is *transitive* if there is only one orbit; that is, given $x,\ y \in K$ there exists $g \in G$ such that $y = \varphi(x, g)$. The action is *sharply transitive* if it is transitive and if $G_x = \{e\}$ for all $x \in K$.

The action of G on K is k-transitive if there is some element of G transforming any ordered k-tuple of distinct elements of K to any other such k-tuple.

Lemma 1.4.1. *Let the group G act on the set K.*

(1) *If $y = \varphi(x, g)$, for $x, y \in K,\ g \in G$, then*

  (a) $yG = xG$;

  (b) $G_y = g^{-1} G_x g$.

(2) $|G_x| = |G|/|xG|$; *that is, the order of the stabilizer group of x is the order of G divided by the length of the orbit of x.*

Some groups that occur in this work are listed below.

$\mathbf{Z}$	=	group of integers;
$\mathbf{Z}_n$	=	cyclic group of order n;
$\mathbf{V}_4$	=	Klein 4-group which is the direct product of two copies of the cyclic group of order 2;
$\mathbf{S}_n$	=	symmetric group of degree n;
$\mathbf{A}_n$	=	alternating group of degree n;
$\mathbf{D}_n$	=	dihedral group of order $2n = \langle r, s \mid r^n = s^2 = (rs)^2 = 1 \rangle$;
$G \times H$	=	the direct product of G and H;
$G \rtimes H$	=	a semi-direct product of G with H, where G is a normal subgroup.

1.5 Projective Space over a Finite Field

Let $V = V(n+1,q)$ be an $(n+1)$-dimensional vector space over the field $\mathbf{F}_q$ with zero element 0 which can be regarded as $\mathbf{F}_q^{n+1}$. Consider the equivalence relation on the elements of $V_0 = V\backslash\{0\}$ whose equivalence classes are the one-dimensional subspaces of V with zero removed. Thus, if $X, Y \in V_0$, then X is equivalent to Y if $Y = tX$ for some t in $\mathbf{F}_q\backslash\{0\}$; that is, $y_i = tx_i$ for all i. Then the set of equivalence classes is the *n-dimensional projective space over* $\mathbf{F}_q$ and is denoted by $PG(n,q)$. The elements of $PG(n,q)$ are called *points*; the equivalence class of the vector X is the point $\mathbf{P}(X)$. It will also be said that X is a *coordinate vector* for $\mathbf{P}(X)$ or that X is a *vector representing* $\mathbf{P}(X)$. In this case, tX with t in $\mathbf{F}_q\backslash\{0\}$ also represents $\mathbf{P}(X)$; that is, by definition, $\mathbf{P}(tX) = \mathbf{P}(X)$. So, the points of $PG(n,q)$ can be described in terms of coordinates as in Table 1.1,

TABLE 1.1: Type of elements of $PG(n,q)$

Type of elements	No. of elements
$\mathbf{P}(x_0,\dots,x_{n-1},1)$	q^n
$\mathbf{P}(x_0,\dots,x_{n-2},1,0)$	q^{n-1}
$\vdots$	$\vdots$
$\mathbf{P}(x_0,1,0,\dots,0)$	q
$\mathbf{P}(1,0,\dots,0)$	1
	$\theta(n,q)$

where $x_0, x_1, \dots, x_{n-1} \in \mathbf{F}_q$. So

$$|PG(n,q)| = \theta(n,q) = (q^{n+1}-1)/(q-1).$$

The points $\mathbf{P}(X_1),\dots,\mathbf{P}(X_r)$ are *linearly independent* if a set of vectors $X_1,\dots,X_r$ representing them is linearly independent.

For any $m = -1, 0, 1, 2, \dots, n$, a *subspace of dimension* m, or m*-space*, of $PG(n,q)$ is a set of points all of whose representing vectors form, together with the zero, a subspace of dimension $m+1$ of $V = V(n+1,q)$; it is denoted by Π_m. A subspace Π_0 of dimension zero has already been called a *point*; a subspace of dimension -1 is the empty set. Subspaces Π_1 of dimension one, Π_2 of dimension two are respectively a *line*, a *plane*. A subspace Π_{n-1} of dimension $n-1$ is a *hyperplane*. A hyperplane is the set of points $\mathbf{P}(X)$ whose vectors $X = (x_0,\dots,x_n)$ satisfy a linear equation

$$u_0x_0 + u_1x_1 + \cdots + u_nx_n = 0$$

with $U = (u_0, \ldots, u_n)$ in $\mathbf{F}_q^{n+1}\backslash\{(0,\ldots,0)\}$. An m-space Π_m is the set of points whose representing vectors $X = (x_0, \ldots, x_n)$ satisfy the equations $XA = 0$, where A is an $(n+1)\times(n-m)$ matrix of rank $n-m$ with coefficients in $\mathbf{F}_q$.

Throughout this work a line through two distinct points P_i and P_j is denoted by P_iP_j.

1.6 Projectivities

Let Ω_1 and Ω_2 be two projective spaces of dimension n.

A *projectivity* $\mathfrak{T} : \Omega_1 \longrightarrow \Omega_2$ is a bijection given by a non-singular $(n+1)\times(n+1)$ matrix A such that $\mathbf{P}(X') = \mathbf{P}(X)\mathfrak{T}$ if and only if $tX' = XA$, where $t \in \mathbf{F}_q\backslash\{0\}$. Write $\mathfrak{T} = \mathbf{M}(A)$; then $\mathfrak{T} = \mathbf{M}(\lambda A)$ for any λ in $\mathbf{F}_q\backslash\{0\}$.

The two projective spaces Ω_1 and Ω_2 are *projectively equivalent* if there is a projectivity between them; this is denoted by $\Omega_1 \cong \Omega_2$.

A projectivity $\mathfrak{T}$ which permutes the $\theta(n,q)$ points of $PG(n,q)$ in a single cycle is called a *cyclic projectivity.*

Lemma 1.6.1. *A projectivity $\mathfrak{T}$ of $PG(n,q)$ is cyclic if and only if the characteristic polynomial of an associated matrix is subprimitive.*

Proof. See [28, Theorem 4.2]. □

Remark 1.6.2. In Lemma 1.6.1, $\mathfrak{T}$ is not unique. See [28, Section 1.6(ix)].

The *projective general linear group* $PGL(n+1,q)$ is the group of projectivities of $PG(n,q)$. The *general linear group* $GL(n+1,q)$ is the group of non-singular linear transformations of $V(n+1,q)$. It is isomorphic to the multiplicative group of $(n+1)\times(n+1)$ non-singular matrices whose entries come from $\mathbf{F}_q$. The order of $GL(n+1,q)$ is

$$(q^{n+1}-1)(q^{n+1}-q)\cdots(q^{n+1}-q^n).$$

Since each projectivity $\mathfrak{T}$ of $PG(n,q)$ is given by $q-1$ matrices $\lambda A, \lambda \in \mathbf{F}_q\backslash\{0\}$, then the order of $PGL(n+1,q)$ is

$$|GL(n+1,q)|/(q-1).$$

1.7 The Fundamental Theorem of Projective Geometry

(i) If $\{P_0, \ldots, P_{n+1}\}$ and $\{P'_0, \ldots, P'_{n+1}\}$ are both subsets of $PG(n, q)$ of cardinality $n+2$ such that no $n+1$ points chosen from the same set lie in a hyperplane, then there exists a unique projectivity $\mathfrak{T}$ such that $P'_i = P_i\mathfrak{T}$, for $i = 0, 1, \ldots, n+1$.

(ii) For $n = 1$, (i) simplifies: there is a unique projectivity of $PG(1, q)$ transforming any three distinct points on a line to any other three.

(iii) For $n = 2$, (i) simplifies: there is a unique projectivity of $PG(2, q)$ transforming any four distinct points no three on a line to any other four points no three on a line.

1.8 The Principle of Duality

For any space $\Omega = PG(n, q)$, there is a dual space Ω^*, whose points and hyperplanes are respectively the hyperplanes and points of Ω. So, for any projective result established using points and hyperplanes, a symmetrical result holds in which the conditions on hyperplanes and points are interchanged: points become hyperplanes, the points lying on a hyperplane become the hyperplanes through a point, non-collinear points become non-concurrent hyperplanes. Hence the dual of an r-space in Ω is an $(n-r-1)$-space in Ω^*. In particular, in $PG(2, q)$, point and line are dual.

1.9 Coordinate Frames

The Fundamental Theorem of Projective Geometry emphasizes a basic difference between $V(n+1, q)$ and $PG(n, q)$. In the former, linear transformations are determined by the images of $n+1$ points; in the latter, projectivities are determined by the images of $n+2$ points. Let $\{P_0, \ldots, P_{n+1}\}$ be any set of $n+2$ points in $PG(n, q)$, no $n+1$ in a hyperplane. If P is any other point of the space, then a coordinate vector for P is determined in the following manner. Let P_i be represented by the vector X_i for some vector X_i in $V(n+1, q)$. Since X_{n+1} is linearly dependent on $X_0, \ldots, X_n$, for any given t in $\mathbf{F}_q\backslash\{0\}$ there exist a_i in $\mathbf{F}_q\backslash\{0\}$ for $i = 0, 1, \ldots, n$ such that

$$tX_{n+1} = a_0X_0 + \cdots + a_nX_n.$$

So, for variable t, the ratios a_i/a_j remain fixed. Thus, if P is any point with $P = \mathbf{P}(X)$, then

$$X = t_0a_0X_0 + \cdots + t_na_nX_n.$$

So, with respect to $\{P_0, \ldots, P_{n+1}\}$, the point P is given by $(t_0, \ldots, t_n)$ where the t_i are determined up to a common factor. Then $\{P_0, \ldots, P_n\}$ is the *simplex of reference* and P_{n+1} the *unit point*. Together the $n+2$ points form a (*coordinate*) *frame*. In particular, let $E_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ be the vector with 1 in the $(i+1)$-th place and zeros elsewhere, and let $E = (1, \ldots, 1)$. Write

$$\mathbf{U}_i = \mathbf{P}(E_i),\ \mathbf{U} = \mathbf{P}(E).$$

Then $\{\mathbf{U}_0, \ldots, \mathbf{U}_n\}$ is the simplex of reference and $\mathbf{U}$ the unit point forming a frame $\{\mathbf{U}_0, \ldots, \mathbf{U}_n, \mathbf{U}\}$, which is called the *standard frame*.

Thus, in $V(n+1, q)$, a basis is a set of $n+1$ linearly independent points and, in $PG(n, q)$, a frame is a set of $n+2$ points, no $n+1$ in a hyperplane; that is, every subset of $n+1$ points is linearly independent. Dually, a coordinate frame is determined by $n+2$ hyperplanes no $n+1$ of which have a point in common. The faces of the simplex of reference are written $\mathbf{u}_0, \ldots, \mathbf{u}_n$ and the unit hyperplane $\mathbf{u}$. So $\mathbf{u}_i$ has equation $x_i = 0$ and $\mathbf{u}$ has equation $\sum_{i=0}^{n} x_i = 0$.

Again from the Fundamental Theorem, if two coordinate frames are given by the vectors $X = (x_0, \ldots, x_n)$ and $Y = (y_0, \ldots, y_n)$, then a change from one frame to the other is given by $Y = XA$, where A is an $(n+1) \times (n+1)$ non-singular matrix.

1.10 Arcs in a Projective Space

An $(n; r)$-*arc* or *arc of degree* r in $PG(k, q)$ with $n \geq r+1$ is a set of n points $\mathcal{K}$ with property that every hyperplane meets $\mathcal{K}$ in at most r points of $\mathcal{K}$ and there is some hyperplane meeting $\mathcal{K}$ in exactly r points. An $(n; 2)$-arc is also called an n-arc. An $(n; r)$-arc $\mathcal{K}$ is *complete* if it is maximal with respect to inclusion; that is, it is not contained in an $(n+1; r)$-arc. The maximum value n for an $(n; r)$-arc is denoted by $m_r(k, q)$.

A line ℓ of $PG(k, q)$, $k > 1$ is an *i-secant* of an $(n; r)$-arc $\mathcal{K}$ if $|\ell \cap K| = i$. A 2-secant is called a *bisecant*, a 1-secant a *unisecant (tangent)* and a 0-secant is an *external line*. Define τ_i as the number of i-secants to $\mathcal{K}$.

From Lemma 1.4.1(2), the number of $(n;r)$-arcs projectively equivalent to an $(n;r)$-arc $\mathcal{K}$ with stabilizer group G in $PG(k,q)$ is

$$|PGL(k+1,q)|/|G|.$$

More details are given in Chapter 3.

1.11 Projective Plane Curves

A homogeneous polynomial F in the three indeterminate variables X_0, X_1, X_2 over $\mathbf{F}_q$ is called a *form*. Let $X = (x_0, x_1, x_2)$ and $F(X) = F(x_0, x_1, x_2)$. A *projective plane curve* $\mathcal{F}$ or *plane curve* for short is the set

$$\mathcal{F} = \mathbf{v}(F) = \{\mathbf{P}(X) \in PG(2,q) \mid F(X) = 0\}.$$

A point $\mathbf{P}(X)$ of F is a *rational point* of $\mathcal{F}$.

A plane curve $\mathcal{F}$ is *irreducible* if F is irreducible over $\mathbf{F}_q$. The *order* or *degree* of $\mathcal{F}$ is the degree of F.

Let $P = \mathbf{P}(A)$ be a point of the irreducible plane curve $\mathcal{F} = \mathbf{v}(F)$ of degree d and let $\ell = \mathbf{P}(A)\mathbf{P}(B)$. Then

$$f(t) = F(A + tB) = F^{(0)} + F^{(1)}t + \cdots + F^{(d)}t^d.$$

Since $P(A) \in \mathbf{v}(F)$, so $F^{(0)} = F(A) = 0$; also $F^{(d)} = F(B)$. Suppose ℓ is not on $\mathcal{F} = \mathbf{v}(F)$; that is, not all the $F^{(i)}$ are zero. The *intersection multiplicity* of ℓ and $\mathcal{F}$ at $P(A)$, denoted $m_P(\ell, \mathcal{F})$, is the multiplicity of the root $t = 0$ of $f(t)$; that is, it is the highest power of t in the factorization of $f(t)$.

The *multiplicity* of P on $\mathcal{F}$, denoted $m_P(\mathcal{F})$, is the minimum of $m_P(\ell, \mathcal{F})$ for all lines ℓ through P. Then P is a *singular* or *multiple* point of $\mathcal{F}$ if $m_P(\mathcal{F}) \geq 2$ and a *simple* or *non-singular* point of $\mathcal{F}$ if $m_P(\mathcal{F}) = 1$. The curve $\mathcal{F}$ is called *singular* or *non-singular* according as $\mathcal{F}$ does or does not have a singular point. A line ℓ is a *tangent* line to $\mathcal{F}$ at P if $m_P(\ell, \mathcal{F}) > m_P(\mathcal{F})$ and then ℓ denote by ℓ_P.

If $m_P(\mathcal{F}) = 1$, then P has a unique tangent ℓ_P. If $m_P(\mathcal{F}) = 2$, then P is a *double point* of $\mathcal{F}$. A double point P with two distinct tangents to $\mathcal{F}$ at P is called a *node*, and with only one tangent to $\mathcal{F}$ at P is a *cusp*. If P is a double point with two distinct tangents, neither of them defined over $\mathbf{F}_q$, then P is an *isolated double point over* $\mathbf{F}_q$. If $m_P(\mathcal{F}) = 3$, then P is a *triple point* of $\mathcal{F}$.

A non-singular rational point P of $\mathcal{F}$ is a *point of inflexion* of $\mathcal{F}$ if

$$m_P(\ell_P, \mathcal{F}) \geq 3.$$

Here, P is also called a *rational inflexion*; the tangent line ℓ_P at P is the *inflexion tangent.*

If F is a form of degree one; that is,

$$F = \sum_{i=0}^{2} a_i X_i,$$

with not all $a_i = 0$ in $\mathbf{F}_q$, then $\mathbf{v}(F)$ is a line.

If F is a form of degree two; that is,

$$F = \sum_{0 \leq i \leq j \leq 2} a_{ij} X_i X_j,$$

with not all $a_{ij} = 0$ in $\mathbf{F}_q$, then $\mathbf{v}(F)$ is called a *plane quadric.* The *discriminant* of a plane quadric $\mathcal{Q} = \mathbf{v}(F)$ is the determinant

$$\Delta = \det \begin{vmatrix} 2a_{00} & a_{01} & a_{02} \\ a_{01} & 2a_{11} & a_{12} \\ a_{02} & a_{12} & 2a_{22} \end{vmatrix}.$$

Put

$$\delta = \begin{cases} 4a_{00}a_{11}a_{22} + a_{01}a_{02}a_{12} - a_{00}a_{12}^2 - a_{11}a_{02}^2 - a_{22}a_{01}^2 & \text{for } q \text{ odd,} \\ a_{01}a_{02}a_{12} - a_{00}a_{12}^2 - a_{11}a_{02}^2 - a_{22}a_{01}^2 & \text{for } q \text{ even.} \end{cases}$$

So, $\Delta = 2\delta$ for q odd.

Lemma 1.11.1. *On $PG(2,q)$, $q = p^h$, a plane quadric $\mathcal{Q}$ is non-singular if and only if $\delta \neq 0$.*

Proof. See [28, Theorem 7.16]. □

A non-singular plane quadric $\mathcal{Q}$ is called a *conic.*

More details about conics and their relation to arcs are discussed in Chapter 3.

If F is a form of degree three, that is

$$F = \sum_{0\le i\le j\le k\le 2} a_{ijk} X_i X_j X_k \ ,$$

with not all $a_{ijk} = 0$ in $\mathbf{F}_q$, then $\mathbf{v}(F)$ is called a *cubic*.

More details about cubics and their relation to arcs are discussed in Chapter 5.

Example 1.11.2. The conic has no singular point and no inflexion point. □

Remark 1.11.3. Let $\mathcal{F}$ be an irreducible plane curve of degree d that has k rational points. Then $\mathcal{F}$ can be regarded as a $(k;r)$-arc with $r \le d$.

Lemma 1.11.4. (i) *With F homogeneous, a point $P = \mathbf{P}(x_0, x_1, x_2)$ of $\mathcal{F} = \mathbf{v}(F)$ is singular if and only if*

$$\frac{\partial F}{\partial X_0}(P) = \frac{\partial F}{\partial X_1}(P) = \frac{\partial F}{\partial X_2}(P) = 0.$$

(ii) *An irreducible plane curve of degree d has at most $\binom{d-1}{2}$ singularities.*

Proof. (i) See [34, Theorem 6.8].
(ii) See [34, Corollary 7.16]. □

Let $\mathcal{F} = \mathbf{v}(F)$ be a projective plane curve of degree d. Write

$$F_{X_i} = \frac{\partial F}{\partial X_i}, \ F_{X_iX_j} = \frac{\partial^2 F}{\partial X_i \partial X_j}.$$

If the determinant

$$\hat{H}(X_0, X_1, X_2) = \begin{vmatrix} F_{X_0X_0} & F_{X_0X_1} & F_{X_0X_2} \\ F_{X_0X_1} & F_{X_1X_1} & F_{X_1X_2} \\ F_{X_0X_2} & F_{X_1X_2} & F_{X_2X_2} \end{vmatrix}$$

is not vanishing, then the projective curve $\hat{\mathcal{H}} = \mathbf{v}(\hat{H}(X_0, X_1, X_2))$ is the *Hessian curve* of $\mathcal{F}$; it has degree $3(d-2)$.

Lemma 1.11.5. *Let $\mathcal{F} = \mathbf{v}(F)$ be a projective plane curve of degree d such that $2(d-1)$ is invertible in $\mathbf{F}_q$.*

(i) *A non-singular point P of $\mathcal{F}$ is an inflexion point of $\mathcal{F}$ if and only if it is a common point of $\mathcal{F}$ and $\hat{\mathcal{H}}$.*

(ii) *Every singular point of $\mathcal{F}$ lies on $\hat{\mathcal{H}}$.*

Proof. See [30, Theorem 1.35]. □

Remark 1.11.6. (1) If $2(d-1)$ is not invertible in $\mathbf{F}_q$ then $\hat{\mathcal{H}}$ is identically zero.

(2) Let $\mathcal{F} = \mathbf{v}(F)$ be a non-singular plane cubic curve. Then its Hessian is also a plane cubic.

1.12 Coding Theory

1.12.1 Basic Definitions and Results

Let $V(n,q) = \mathbf{F}_q^n$ be the n-dimensional vector space over $\mathbf{F}_q$. In general any subset C of $V(n,q)$ is a *q-ary code*. A *linear* q-ary $[n,k,d]$ *code* or an $[n,k,d]_q$*-code* C is a subspace of $V(n,q)$, where the dimension of C is

$$\dim C = k,$$

and the *minimum distance* is

$$d(C) = d = \min\{w(x) \mid x \in C\backslash\{0\}\} = \min\{d(x,y) \mid x \neq y\}.$$

Here, with $x = (x_1,\ldots,x_n) = x_1\ldots x_n$ and $y = (y_1,\ldots,y_n) = y_1\ldots y_n$,

$$w(x) = |\{i \mid x_i \neq 0\}|$$

is the *weight* of the word x and

$$d(x,y) = |\{i \mid x_i \neq y_i\}|$$

is the *(Hamming) distance* between the words x and y. If d is not specified, then the term $[n,k]_q$-code is used. The vectors $v \in C$ are called the codewords.

A central problem in coding theory is that of optimizing one of the parameters n, k and d for given values of the other two and q fixed.

A code C with minimum distance at least $2e+1$ can correct up to e errors. So, if a received codeword is distorted in at most e entries, then it can correctly deduced which codeword was sent. This type of code is called an *e-error correcting code.*

Lemma 1.12.1. *If a code C has minimum distance d, then it can correct $e = \lfloor (d-1)/2 \rfloor$ errors, where $\lfloor m \rfloor$ denotes the integer part of m.*

A *generator matrix* $\mathbb{G}$ of an $[n,k,d]_q$-code C is a $k \times n$ matrix whose rows form a basis of C; thus, if $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbf{F}_q^k \backslash \{0\}$ and $c_1, \dots, c_n$ are the columns of $\mathbb{G}$, then

$$x \in C \Longleftrightarrow x = (\sum_{i=1}^{k} \lambda_i c_1, \dots, \sum_{i=1}^{k} \lambda_i c_n).$$

A linear code for which any two columns of a generator matrix are linearly independent is called a *projective code.*

For $u = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$, let

$$x \cdot y = \sum_{i=1}^{n} x_i y_i$$

be the *standard scalar (inner) product* of x and y. The *dual code* $C^\perp$ of an $[n,k,d]_q$-code C is

$$C^\perp = \{x \in \mathbf{F}_q^n \mid x \cdot y = 0, \text{ for all } y \in C\},$$

which is an $[n, n-k, d']_q$-code. A *parity-check matrix* $\mathbb{H}$ for C is an $(n-k) \times n$ matrix that is a generator matrix for the dual code $C^\perp$; thus, if $x = (x_1, \dots, x_n) \in \mathbf{F}_q^n$ and $c_1, \dots, c_n$ are the columns of $\mathbb{H}$, then

$$x \in C \Longleftrightarrow x\mathbb{H}^\top = 0 \text{ or equivalently } x_1 c_1 + \dots + x_n c_n = 0.$$

Two codes C_1 and C_2 are *equivalent* if C_2 can be obtained from C_1 by permuting coordinates and by multiplying coordinates by non-zero elements of $\mathbf{F}_q$. There is a code C' equivalent to C for which the generator matrix has the *standard* form $\mathbb{G} = [I_k A]$, where I_k is the $k \times k$ identity matrix and A is a $k \times (n-k)$ matrix; in this case, a parity-check matrix for C' is $\mathbb{H} = [-A^\top I_{n-k}]$.

The minimum distance d of C can be calculated from the next result.

Lemma 1.12.2. *If C is an $[n,k]_q$-code with corresponding generator matrix $\mathbb{G}$ and parity-check matrix $\mathbb{H}$, then the following are equivalent:*

(i) *$d(C) = d$;*

(ii) *every $d-1$ columns of the parity-check matrix $\mathbb{H}$ are linearly independent but some d columns are dependent;*

(iii) *at most* $n-d$ *columns of the generator matrix* $\mathbb{G}$ *lie in any hyperplane of the projective space* $PG(k-1,q)$*;*

Corollary 1.12.3. *(Singleton bound) For an* $[n,k,d]_q$*-code,*

$$d \le n-k+1.$$

For an extensive introduction to this subject see [27], [36], [37].

1.12.2 MDS Codes and AMDS Codes

Let C be an $[n,k,d]_q$-code.

(I) C is *maximum distance separable* (MDS) if

$$d = d(C) = n-k+1;$$

that is, d achieves the upper limit in the Singleton bound.

(II) C is *almost*-MDS (AMDS) if

$$d = d(C) = n-k.$$

Theorem 1.12.4. *If* C *is an* $[n,k,d]_q$*-code with corresponding generator matrix* $\mathbb{G}$ *and parity-check matrix* $\mathbb{H}$*, then the following are equivalent:*

(i) C *is an* MDS *code;*

(ii) *any* $n-k$ *columns of the parity-check matrix* $\mathbb{H}$ *are linearly independent;*

(iii) *at most* $k-1$ *columns of the generator matrix* $\mathbb{G}$ *lie in a hyperplane of* $PG(k-1,q)$*; that is, every* k *columns are linearly independent;*

(iv) *the dual code* $C^{\perp}$ *is an* MDS *code.*

Remark 1.12.5. The dual code of an AMDS code need not to be AMDS as illustrated in the following example.

Let C $[n,k,n-k+1]_q$-code be an MDS code with parity check matrix

$$\mathbb{H} = \begin{bmatrix} h_1 \\ \vdots \\ h_{n-k} \end{bmatrix}_{(n-k)\times n},$$

where h_i is the ith row of $\mathbb{H}$. Choose $h \in V(n,q)$ which is not a linear combination of rows of $\mathbb{H}$ and which is of weight less than $k-1$. Consider

$$\mathbb{H}_1 = \begin{bmatrix} h_1 \\ \vdots \\ h_{n-k} \\ h \end{bmatrix}_{(n-k+1)\times n}$$

as a parity check matrix of the $[n, k-1]_q$-code C_1. Then C_1 is an AMDS code but the dual is not.

For further background on linear MDS and AMDS codes, see [9], [10], [19] and [31].

1.13 The Relationship Between Coding Theory and Finite Projective Spaces

The projective geometries over finite fields have been introduced and it was seen that linear codes come from finite fields. These two different ideas are linked by their underlying vector spaces. The following explanation and result are presented in many references, for instance [4].

Let $v_1, v_2, \ldots, v_k$ be the rows of a generator matrix $\mathbb{G}$ for a projective $[n,k,d]_q$-code and for $i = 1, 2, \ldots, n$ define vectors u_i of $V(k,q)$, by the rule

$$(u_i)_j = (v_j)_i.$$

In other words, the jth coordinate of u_i is the ith coordinate of v_j; that is, u_i is column vector of $\mathbb{G}$. For all $a \in \mathbf{F}_q^k \backslash \{0\}$ the vector $\sum_{j=1}^{k} a_j v_j$ has at most $n-d$ zero coordinates and so, for $i = 1, 2, \ldots, n$,

$$\sum_{j=1}^{k} a_j (v_j)_i = 0$$

has at most $n-d$ solutions. Hence

$$\sum_{j=1}^{k} a_j (u_i)_j = 0$$

has at most $n - d$ solutions, or in other words there are at most $n - d$ of the n vectors u_i on the hyperplane with equation

$$\sum_{j=1}^{k} a_j X_j = 0.$$

So, this gives the following fundamental theorem.

Theorem 1.13.1. *There exists a projective* $[n, k, d]_q$*-code if and only if there exists an* $(n; n - d)$*-arc in* $PG(k - 1, q)$*.*

Finally, as a conclusion from this chapter, the geometrical objects considered in this work can be viewed as a linear codes defined over a finite field. Hence, all results on their geometry can be translated to results on coding theory as is shown in the next chapters.

Chapter 2

The Projective Line of Order Nineteen

2.1 Introduction

The main reference for this section is [28, Chapter 6].

In general, the $q+1$ points of $PG(1,q)$ are $\mathbf{P}(t_0,t_1)$, $t_i \in \mathbf{F}_q$. So,

$$PG(1,q) = \{\mathbf{P}(t,1) \mid t \in \mathbf{F}_q\} \cup \{\mathbf{P}(1,0)\}.$$

Each point $\mathbf{P}(t_0,t_1)$ with $t_1 \neq 0$ is determined by the non-homogeneous coordinate t_0/t_1; the coordinate for $\mathbf{P}(1,0)$ is ∞. So, the points of $PG(1,q)$ can be represented by the set

$$\mathbf{F}_q \cup \{\infty\} = \{\infty, \lambda_1, \lambda_2, \dots, \lambda_q \mid \lambda_i \in \mathbf{F}_q\}.$$

A projectivity $\mathfrak{T} = \mathbf{M}(A)$ of $PG(1,q)$ is given by

$$Y = XA, \text{where } X = (x_0,x_1), Y = (y_0,y_1) \text{ and } A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$

Let $s = y_0/y_1$ and $t = x_0/x_1$. Its *projective equation* is

$$s = (at+b)/(ct+d).$$

If $Q_i = P_i\mathfrak{T}$ for $i = 2,3,4$ and P_i and Q_i have the respective coordinates t_i and s_i, then $\mathfrak{T}$ is given by

$$\frac{(s-s_3)(s_2-s_4)}{(s-s_4)(s_2-s_3)} = \frac{(t-t_3)(t_2-t_4)}{(t-t_4)(t_2-t_3)}. \tag{2.1}$$

2.2 The Cross-Ratio and Stabilizer Group of a Tetrad

The cross-ratio $\lambda = \{P_1, P_2; P_3, P_4\}$ of four ordered points $P_1, P_2, P_3, P_4 \in PG(1, q)$ with coordinates t_1, t_2, t_3, t_4 is

$$\lambda = \{P_1, P_2; P_3, P_4\} = \{t_1, t_2; t_3, t_4\} = \frac{(t_1 - t_3)(t_2 - t_4)}{(t_1 - t_4)(t_2 - t_3)}.$$

The cross-ratio has the property that

(1) $\lambda = \{t_1, t_2; t_3, t_4\} = \{t_2, t_1; t_4, t_3\} = \{t_3, t_4; t_1, t_2\} = \{t_4, t_3; t_2, t_1\}$. So, $\{P_1, P_2; P_3, P_4\}$ is invariant under a projective group of order four, given by

$$\{I, (P_1P_2)(P_3P_4), (P_1P_3)(P_2P_4), (P_1P_4)(P_2P_3)\} \cong \mathbf{V}_4.$$

Thus, under all 24 permutations of $\{P_1, P_2, P_3, P_4\}$, the cross-ratio takes just the six values

$$\lambda,\ 1/\lambda,\ 1 - \lambda,\ 1/(1 - \lambda),\ (\lambda - 1)/\lambda,\ \lambda/(\lambda - 1).$$

(2) $\lambda = \{t_1, t_2; t_3, t_4\}$ takes the values $\infty, 0$ or 1 if and only if two of the t_i are equal.

A projectivity is determined by the images of three points, by (2.1). Therefore there exists a projectivity $\mathfrak{T} = \mathbf{M}(A)$ such that $Q_i = P_iA,\ i = 1, 2, 3, 4$ if and only if the cross-ratios of the two sets of four points in the corresponding order are equal. Also the order of $PGL(2, q)$ is $q(q^2 - 1)$, which is the number of ordered sets of three points in $PG(1, q)$.

Remark 2.2.1. The action of $PGL(2, q)$ on $PG(1, q)$ is sharply 3-transitive.

An unordered set of four distinct points is called a *tetrad*. Let λ be the cross-ratio of a given order, the tetrad is called

(I) *harmonic*, denoted by H, if $\lambda = 1/\lambda$ or $\lambda = \lambda/(\lambda - 1)$ or $\lambda = 1 - \lambda$;

(II) *equianharmonic*, denoted by E, if $\lambda = 1/(1-\lambda)$ or, equivalently, $\lambda = (\lambda-1)/\lambda$;

(III) *neither harmonic nor equianharmonic*, denoted by N, if the cross-ratio is another value .

The cross-ratio of any harmonic tetrad has the values −1, 2, 1/2.

Let $q = p^h$, where p is a prime. When $p = 2$, there are no harmonic tetrads and when $p = 3$, then $\lambda = -1$ is the unique solution.

The cross-ratio of a tetrad of type E satisfies the equation

$$\lambda^2 - \lambda + 1 = 0. \tag{2.2}$$

So, equianharmonic tetrads exist if and only if $\lambda^3 + 1 = 0$ has three solutions in $\mathbf{F}_q$ or $\lambda = -1$ is a unique solution of (2.2) in $\mathbf{F}_q$. Therefore equianharmonic tetrads exist if $q \equiv 1$ or $0 \pmod 3$; so if $p = 3$, harmonic and equianharmonic are the same. In particular, a tetrad of type E exists when $q = 19$. Since the values ∞ , 0 and 1 cannot appear as the cross ratio of a tetrad whose four points are distinct and since every three distinct points in $PG(1,q)$ are projectively equivalent, so we choose the tetrad where three of the points are ∞, 0 and 1. As the cross-ratio $\lambda = \{\infty, 0; 1, t\} = t$, it is only necessary to consider the elements $t \in \mathbf{F}_q \backslash \{0,1\}$ and the corresponding tetrads $\{\infty, 0, 1, t\}$. Hence there are three classes of tetrads:

$$\begin{aligned}
\mathcal{X}_1 &= \{\text{tetrads of type } H\},\\
\mathcal{X}_2 &= \{\text{tetrads of type } E\},\\
\mathcal{X}_3 &= \{\text{tetrads of type } N\}.
\end{aligned}$$

Now the question is: *Which subgroup of* $\mathbf{S}_4$ *fixes the tetrad in each class?*

Let T = $\{P_1, P_2, P_3, P_4\}$ be a tetrad in classes $\mathcal{X}_i$ with cross-ratio $\lambda = \{P_1, P_2; P_3, P_4\}$.

(I) If $i = 1$ (harmonic case), then $\lambda = 1/\lambda$, whence for $p > 3$ there are eight permutations of T amongst the 24 permutations which are projectively equivalent as follows:

$$\begin{aligned}
&I, (P_1P_2)(P_3P_4), (P_1P_3)(P_2P_4), (P_1P_4)(P_2P_3),\\
&(P_3P_4), (P_1P_2), (P_1P_3P_2P_4), (P_1P_4P_2P_3).
\end{aligned}$$

These permutations form a group isomorphic to the dihedral group $\mathbf{D}_4$ of order eight.

(II) If $i = 2$ (equianharmonic case), then $\lambda = 1/(1-\lambda) = (\lambda - 1)/\lambda$, whence for $q \equiv 1 \pmod 3$ there are 12 projectively equivalent permutations of T amongst the

24 permutations which are projectively equivalent as follows:

$$I, (P_1P_2)(P_3P_4), (P_1P_3)(P_2P_4), (P_1P_4)(P_2P_3), (P_2P_4P_3), (P_1P_2P_3),$$
$$(P_1P_3P_4), (P_1P_4P_2), (P_2P_3P_4), (P_1P_2P_4), (P_1P_3P_2), (P_1P_4P_3).$$

These permutations form a group isomorphic to the alternating group $\mathbf{A}_4$ of degree four.

(III) If $i = 3$, then from the definition of a tetrad of type N there only four permutations of T amongst the 24 permutations which are projectively equivalent as follows:

$$I, (P_1P_2)(P_3P_4), (P_1P_3)(P_2P_4), (P_1P_4)(P_2P_3).$$

These permutations form a group isomorphic to the Klein 4-group $\mathbf{V}_4$.

So, following lemma is obtained.

Lemma 2.2.2. *On* $PG(1, q)$, $q = p^h$,

(i) *the number of harmonic tetrads* n_H *and the stabilizer group* G *of each one are as in the following table:*

	n_H	G
$p = 3$	$q(q^2-1)/24$	$\mathbf{S}_4$
$p > 3$	$q(q^2-1)/8$	$\mathbf{D}_4$

(ii) *the number of equianharmonic tetrads* n_E *and the stabilizer group* G *of each one are as in the following table:*

	n_E	G
$p = 3$	$q(q^2-1)/24$	$\mathbf{S}_4$
$q \equiv 1 \pmod 3$	$q(q^2-1)/12$	$\mathbf{A}_4$

Proof. See [28, Lemma 6.1]. □

On $PG(1, q)$, a $(k; 1)$-arc is just an unordered set of k distinct points simply called a *k-set*. A 3-set is called a *triad*, a 4-set a *tetrad*, a 5-set a *pentad*, a 6-set a *hexad*, a 7-set a *heptad*, an 8-set a *octad*, a 9-set a *nonad*, a 10-set a *decad*.

The question arises here: *How many projectively inequivalent k-sets in $PG(1, q)$ are there and what is the stabilizer group of each one*?

2.3 The Algorithm for Classification of the k-Sets in $PG(1,q)$

On $PG(1,q)$, a k-set can be constructed by adding to any $(k-1)$-set one point from the other $q-k+2$ points. According to the Fundamental Theorem of Projective Geometry, Section 1.7(ii), any three distinct points on a line are projectively equivalent; so choose a fixed triad $\mathfrak{R}$. A 4-set is formed by adding to $\mathfrak{R}$ one point from the other $q-2$ points on $PG(1,q)$; that is, from $PG(1,q)\backslash\mathfrak{R} = \mathfrak{R}^c$. It is shown in Section 2.2 that there is a unique tetrad of type H and unique tetrad of type E but the tetrad of type N might be divided into subclasses. A 5-set is formed by adding to any tetrad T in $\mathcal{X}_i$ one point from the other $q-3$ points on $PG(1,q)$. The group G_{T} fixes T and splits the other $q-3$ points into a number of orbits; so, different 5-sets are formed by adding one point from each different orbit. The procedure can be extended to construct $6,7,8,9,\ldots,(\frac{q+1}{2})$-sets in $PG(1,q)$. The $(n-1)$-subsets of an n-set are classified according to their projective type.

Let K and K$'$ be two pentads. To check they are equivalent the following steps are used.

(1) Classify tetrads in both pentads.

(2) If the classifications of K and K$'$ are different then they are projectively inequivalent.

(3) If the classifications of K and K$'$ are the same, then transformation matrices A_α are constructed from a tetrad T with highest recurrence in the algebraic structure of K to tetrads T_α in K$'$ with same types of T.

(4) If the action of one A_α on the remaining points of T are equal to the remaining points of T$'$ then K and K$'$ are projectively equivalent. If not, it means they are projectively inequivalent.

This procedure can be extended to check the equivalence between k-sets, $k = 6,7,\ldots,(\frac{q+1}{2})$, and also can be used to calculate the stabilizer group of each k-set.

2.4 Preliminary to $PG(1,19)$

On $PG(1,19)$, the projective line over Galois field of order 19, there are 20 points. The points of $PG(1,19)$ are the elements of the set

$$\mathbf{F}_{19} \cup \{\infty\} = \{\infty, 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 7, \pm 8, \pm 9\}.$$

The order of the projective group $PGL(2,19)$ is $20 \cdot 19 \cdot 18 = 6840$. This is the number of ordered sets of three points.

In the following sections in this chapter, the k-sets in $PG(1,19)$, $k = 4, \ldots, 10$, are classified by giving the projectively inequivalent k-sets with their stabilizer groups.

2.5 The Tetrads

Let $\mathcal{S}$ be the set of all different tetrads in $PG(1,19)$. Then the order of $\mathcal{S}$ is

$$|\mathcal{S}| = \binom{20}{4} = 4845.$$

As mentioned in Section 2.2, to consider the action of $PGL(2,19)$ on $\mathcal{S}$, it is only necessary to consider the tetrads $\{\infty, 0, 1, t\}, t \in \mathbf{F}_{19} \backslash \{0,1\}$. A tetrad is of type H if the cross-ratio is $-1, 2$ or $1/2 = -9$. It is of type E if the cross-ratio is -7 or 8, and it is of type N if the cross-ratio is $-2, 3, -3, 4, -4, 5, -5, 6, -6, 7, -8$ or 9. As a tetrad of type N has six possible values of its cross-ratios so, there are two tetrads of type N, one with cross-ratios $-2, 3, -6, 7, -8, 9$ denoted by N_1 and the other with $-3, 4, -4, 5, -5, 6$ denoted by N_2. Hence there are four classes of tetrads:

$$\mathcal{C}_1 = \{\text{the class of } H \text{ tetrads}\} \ni \{\infty, 0, 1, a\} \text{ for } a = -1, 2, -9;$$
$$\mathcal{C}_2 = \{\text{the class of } E \text{ tetrads}\} \ni \{\infty, 0, 1, b\} \text{ for } b = -7, 8;$$
$$\mathcal{C}_3 = \{\text{the class of } N_1 \text{ tetrads}\} \ni \{\infty, 0, 1, c\} \text{ for } c = -2, 3, -6, 7, -8, 9;$$
$$\mathcal{C}_4 = \{\text{the class of } N_2 \text{ tetrads}\} \ni \{\infty, 0, 1, d\} \text{ for } d = -3, 4, -4, 5, -5, 6.$$

From Lemma 2.2.2 $|\mathcal{C}_1| = 855$, $|\mathcal{C}_2| = 570$ and therefore $|\mathcal{C}_3| = |\mathcal{C}_4| = 1710$. As mentioned in Section 2.2 any two tetrads with the same cross-ratio are projectively equivalent; so each class $\mathcal{C}_i$, $i = 1, 2, 3, 4$, is projectively unique. Then there are only four projectively distinct tetrads.

Case(1): The tetrad $H = \{\infty, 0, 1, -1\}$ chosen from $\mathcal{C}_1$.
Case(2): The tetrad $E = \{\infty, 0, 1, -7\}$ chosen from $\mathcal{C}_2$.
Case(3): The tetrad $N_1 = \{\infty, 0, 1, -2\}$ chosen from $\mathcal{C}_3$.
Case(4): The tetrad $N_2 = \{\infty, 0, 1, -3\}$ chosen from $\mathcal{C}_4$.

A. Ali [2] in 1993 classified the tetrads, pentads and hexads on $PG(1,19)$. The results are rechecked and rewritten to make the research continuous. For a chosen tetrad from each class $\mathcal{C}_i$ its stabilizer group in $PGL(2,19)$ identified it as a subgroup of $\mathbf{S}_4$ using the projective equation. As mentioned in Section 2.2 about tetrads and Lemma 2.2.2, the following are satisfied.

Case(1): Let the tetrad $H = \{\infty, 0, 1, -1\}$ be chosen from the class $\mathcal{C}_1$. The stabilizer group G_H of H consists of the following eight permutations:

$$I, (P_1P_2)(P_3P_4), (P_1P_3)(P_2P_4), (P_1P_4)(P_2P_3),$$
$$(P_3P_4), (P_1P_2), (P_1P_3P_2P_4), (P_1P_4P_2P_3).$$

The eight permutations of H and the respective projectivities are given in Table 2.1.

TABLE 2.1: Projectivities fixing an H tetrad

No.	H tetrad	Projectivity
1	$\{\infty, 0, 1, -1\}$	t
2	$\{0, \infty, -1, 1\}$	$-1/t$
3	$\{1, -1, \infty, 0\}$	$(t+1)/(t-1)$
4	$\{-1, 1, 0, \infty\}$	$(1-t)/(1+t)$
5	$\{\infty, 0, -1, 1\}$	$-t$
6	$\{0, \infty, 1, -1\}$	$1/t$
7	$\{-1, 1, \infty, 0\}$	$(1+t)/(1-t)$
8	$\{1, -1, 0, \infty\}$	$(t-1)/(t+1)$

The group

$$G_H = \{t, -1/t, (t+1)/(t-1), (1-t)/(1+t), -t, 1/t, (1+t)/(1-t), (t-1)/(t+1)\},$$

fixes the tetrad H; it is isomorphic to

$$\mathbf{D}_4 = \langle (1+t)/(1-t),\ (t+1)/(t-1) \rangle.$$

Case(2): Let the tetrad $E = \{\infty, 0, 1, -7\}$ be chosen from the class $\mathcal{C}_2$. The stabilizer group G_E of E consists of the following twelve permutations:

$$I, (P_1P_2)(P_3P_4), (P_1P_3)(P_2P_4), (P_1P_4)(P_2P_3), (P_2P_4P_3), (P_1P_2P_3),$$
$$(P_1P_3P_4), (P_1P_4P_2), (P_2P_3P_4), (P_1P_2P_4), (P_1P_3P_2), (P_1P_4P_3).$$

The twelve permutations of H and the respective projectivities are given in Table 2.2.

TABLE 2.2: Projectivities fixing an E tetrad

No.	E tetrad	Projectivity
1	$\{\infty, 0, 1, -7\}$	t
2	$\{0, \infty, -7, 1\}$	$1/(8t)$
3	$\{1, -7, \infty, 0\}$	$(t+7)/(t-1)$
4	$\{-7, 1, 0, \infty\}$	$(t-1)/(8t-1)$
5	$\{\infty, 1, -7, 0\}$	$(1-8t)$
6	$\{1, \infty, 0, -7\}$	$(t-1)/t$
7	$\{-7, 0, \infty, 1\}$	$t/(8t-8)$
8	$\{0, -7, 1, \infty\}$	$7/(8t-1)$
9	$\{\infty, -7, 0, 1\}$	$(7t-7)$
10	$\{-7, \infty, 1, 0\}$	$(t+7)/(8t)$
11	$\{0, 1, \infty, -7\}$	$1/(1-t)$
12	$\{1, 0, -7, \infty\}$	$t/(t+7)$

The group

$$G_E = \{t, 1/(8t), (t+7)/(t-1), (t-1)/(8t-1), (1-8t), (t-1)/t, t/(8t-8), (7t-7), (t+7)/(8t), 1/(1-t), t/(t+7)\},$$

fixes E; it is isomorphic to

$$\mathbf{A}_4 = \langle (t+7)/(8t),\ 1/(8t) \rangle.$$

Case(3): Let the tetrad $N_1 = \{\infty, 0, 1, -2\}$ be chosen from the class $\mathcal{C}_3$. The stabilizer group G_{N_1} of N_1 consists of the following four permutations:

$$I, (P_1P_2)(P_3P_4), (P_1P_3)(P_2P_4), (P_1P_4)(P_2P_3).$$

The four permutations of N_1 and the respective projectivities are given in Table 2.3.

Table 2.3: Projectivities fixing an N_1 tetrad

No.	N_1 tetrad	Projectivity
1	$\{\infty, 0, 1, -2\}$	t
2	$\{0, \infty, -2, 1\}$	$-2/t$
3	$\{1, -2, \infty, 0\}$	$(t+2)/(t-1)$
4	$\{-2, 1, 0, \infty\}$	$(t-1)/(9t-1)$

The group

$$G_{N_1} = \{t, -2/t, (t+2)/(t-1), (t-1)/(9t-1)\},$$

fixes N_1; it is isomorphic to

$$\mathbf{V}_4 = \langle -2/t,\ (t-1)/(9t-1) \rangle.$$

Case(4): Let the tetrad $N_2 = \{\infty, 0, 1, -3\}$ be chosen from the class $\mathcal{C}_4$. The stabilizer group G_{N_2} of N_2 consists of the following four permutations:

$$I, (P_1P_2)(P_3P_4), (P_1P_3)(P_2P_4), (P_1P_4)(P_2P_3).$$

The four permutations of the elements of N_2 and the respective projectivities are given in Table 2.4.

TABLE 2.4: Projectivities fixing an N_2 tetrad

No.	N_2 tetrad	Projectivity
1	$\{\infty, 0, 1, -3\}$	t
2	$\{0, \infty, -3, 1\}$	$-3/t$
3	$\{1, -3, \infty, 0\}$	$(t+3)/(t-1)$
4	$\{-3, 1, 0, \infty\}$	$(t-1)/(6t-1)$

The group

$$G_{N_2} = \{t, -3/t, (t+3)/(t-1), (t-1)/(6t-1)\},$$

fixes N_2; it is isomorphic to

$$\mathbf{V}_4 = \langle -3/t,\ (t-1)/(6t-1)\rangle.$$

From the cases 1,2,3 and 4, the following conclusion is obtained.

Theorem 2.5.1. *On* $PG(1, 19)$*, there are precisely four projectively distinct tetrads given with their stabilizer groups in Table* 2.5.

TABLE 2.5: Distinct tetrads on $PG(1, 19)$

Type	The tetrad	Stabilizer
H	$\{\infty, 0, 1, -1\}$	$\mathbf{D}_4 = \langle (1+t)/(1-t),\ (t+1)/(t-1)\rangle$
E	$\{\infty, 0, 1, -7\}$	$\mathbf{A}_4 = \langle (t+7)/(8t),\ 1/(8t)\rangle$
N_1	$\{\infty, 0, 1, -2\}$	$\mathbf{V}_4 = \langle -2/t,\ (t-1)/(9t-1)\rangle$
N_2	$\{\infty, 0, 1, -3\}$	$\mathbf{V}_4 = \langle -3/t,\ (t-1)/(6t-1)\rangle$

2.6 The Pentads

To construct the pentad in $PG(1, 19)$, as mentioned in Section 2.3, it is enough to add one point from each orbit that comes from the action of the projective group of the tetrad G_{T} on the complement of T, where $\mathrm{T} = H, E, N_1, N_2$. All orbits of the tetrads in Table 2.5 are given in Table 2.6.

TABLE 2.6: Partition of $PG(1, 19)$ by the projectivities of tetrads

T	Partition of T^c
H	(1) $\{2, -2, 3, -3, 6, -6, 9, -9\}$ (2) $\{4, -4, 5, -5, 7, -7, 8, -8\}$
E	(1) $\{-1, 2, 3, -3, 4, -4, 5, -5, 6, 7, 9, -9\}$ (2) $\{-2, -6, 8, -8\}$
N_1	(1) $\{-1, 2, 4, 9\}$ (2) $\{3, -7\}$ (3) $\{-3, 5, 7, -8\}$ (4) $\{-4, -5, 8, -9\}$ (5) $\{6, -6\}$
N_2	(1) $\{-1, 3\}$ (2) $\{-2, 6, -8, 9\}$ (3) $\{2, 5, 7, 8\}$ (4) $\{4, -4\}$ (5) $\{-5, -6, -7, -9\}$

According to Table 2.6, there are fourteen pentads constructed by adding one point from each orbit to the corresponding tetrad. Each pentad contains five tetrads. In Table 2.7, for each pentad $\mathcal{P} = \{a_1, a_2, a_3, a_4, a_5\}$ the classification of its tetrads in the order

$$\{a_1, a_2, a_3, a_4\}, \{a_1, a_2, a_3, a_5\}, \{a_1, a_2, a_4, a_5\}, \{a_1, a_3, a_4, a_5\}, \{a_2, a_3, a_4, a_5\}$$

is given. Also the stabilizer group of each pentad is given.

Table 2.7: Pentads on $PG(1,19)$

Symbol	The pentad	Types of tetrads	Stabilizer
$\mathcal{P}''_1$	$\{\infty, 0, 1, -1, 2\}$	$H\ H\ N_1\ N_1\ N_2$	$\mathbf{Z}_2 = \langle 1-t \rangle$
$\mathcal{P}''_2$	$\{\infty, 0, 1, -1, 4\}$	$H\ N_2\ N_2\ E\ N_1$	$I = \langle t \rangle$
$\mathcal{P}''_3$	$\{\infty, 0, 1, -7, -1\}$	$E\ H\ N_1\ N_2\ N_2$	$I = \langle t \rangle$
$\mathcal{P}''_4$	$\{\infty, 0, 1, -7, -2\}$	$E\ N_1\ N_1\ N_1\ E$	$\mathbf{S}_3 = \langle (1-8t),\ t/(9t-1) \rangle$
$\mathcal{P}''_5$	$\{\infty, 0, 1, -2, -1\}$	$N_1\ H\ H\ N_1\ N_2$	$\mathbf{Z}_2 = \langle -(t+1) \rangle$
$\mathcal{P}''_6$	$\{\infty, 0, 1, -2, 3\}$	$N_1\ N_1\ E\ E\ N_1$	$\mathbf{S}_3 = \langle t/(7-6t),\ (t-1)/(9t-1) \rangle$
$\mathcal{P}''_7$	$\{\infty, 0, 1, -2, -3\}$	$N_1\ N_2\ N_1\ N_2\ N_1$	$\mathbf{Z}_2 = \langle -(t+2) \rangle$
$\mathcal{P}''_8$	$\{\infty, 0, 1, -2, -4\}$	$N_1\ N_2\ H\ E\ N_2$	$I = \langle t \rangle$
$\mathcal{P}''_9$	$\{\infty, 0, 1, -2, 6\}$	$N_1\ N_2\ N_2\ N_1\ N_1$	$\mathbf{Z}_2 = \langle 1/(9t) \rangle$
$\mathcal{P}''_{10}$	$\{\infty, 0, 1, -3, -1\}$	$N_2\ H\ N_1\ H\ N_1$	$\mathbf{Z}_2 = \langle (t+3)/(t-1) \rangle$
$\mathcal{P}''_{11}$	$\{\infty, 0, 1, -3, 2\}$	$N_2\ H\ E\ N_2\ N_1$	$I = \langle t \rangle$
$\mathcal{P}''_{12}$	$\{\infty, 0, 1, -3, -2\}$	$N_2\ N_1\ N_1\ N_2\ N_1$	$\mathbf{Z}_2 = \langle -(t+2) \rangle$
$\mathcal{P}''_{13}$	$\{\infty, 0, 1, -3, 4\}$	$N_2\ N_2\ N_2\ N_2\ N_2$	$\mathbf{D}_5 = \langle (t-4)/(5t-4),\ (t+3)/(5t-1) \rangle$
$\mathcal{P}''_{14}$	$\{\infty, 0, 1, -3, -5\}$	$N_2\ N_2\ E\ N_1\ H$	$I = \langle t \rangle$

For those pentads with an equivalent sets of tetrads, Table 2.8 gives the projectivities between them.

Table 2.8: The equivalence of pentads

No.	Equivalent pentads	Projective equation
1	$\mathcal{P}''_1 \longrightarrow \mathcal{P}''_5$	$-t$
2	$\mathcal{P}''_1 \longrightarrow \mathcal{P}''_{10}$	$(2-t)/t$
3	$\mathcal{P}''_2 \longrightarrow \mathcal{P}''_3$	$(t-1)/(t+1)$
4	$\mathcal{P}''_2 \longrightarrow \mathcal{P}''_8$	$4t/(1-t)$
5	$\mathcal{P}''_2 \longrightarrow \mathcal{P}''_{11}$	$(1-t)$
6	$\mathcal{P}''_2 \longrightarrow \mathcal{P}''_{14}$	$1/(1-5t)$
7	$\mathcal{P}''_4 \longrightarrow \mathcal{P}''_6$	$-2/t$
8	$\mathcal{P}''_7 \longrightarrow \mathcal{P}''_9$	$(6t-1)/(t-1)$
9	$\mathcal{P}''_7 \longrightarrow \mathcal{P}''_{12}$	t

Table 2.8 gives the following conclusion.

Theorem 2.6.1. *On $PG(1,19)$, there are precisely five projectively distinct pentads given with their stabilizer groups in Table 2.9.*

TABLE 2.9: Inequivalent pentads on $PG(1,19)$

Type	The pentad	Stabilizer
$\mathcal{P}_1$	$\{\infty, 0, 1, -1, 2\}$	$\mathbf{Z}_2 = \langle 1-t \rangle$
$\mathcal{P}_2$	$\{\infty, 0, 1, -1, 4\}$	$I = \langle t \rangle$
$\mathcal{P}_3$	$\{\infty, 0, 1, -7, -2\}$	$\mathbf{S}_3 = \langle (1-8t),\ t/(9t-1) \rangle$
$\mathcal{P}_4$	$\{\infty, 0, 1, -2, -3\}$	$\mathbf{Z}_2 = \langle -(t+2) \rangle$
$\mathcal{P}_5$	$\{\infty, 0, 1, -3, 4\}$	$\mathbf{D}_5 = \langle (t-4)/(5t-4),\ (t+3)/(5t-1) \rangle$

2.7 The Hexads

The projective group $G_{\mathcal{P}_i}$ splits $\mathcal{P}_i^c$, $i = 1,2,3,4,5$, into a number of orbits. The hexads are constructed by adding one point from each orbit to the corresponding pentad. All orbits are listed in Table 2.10.

TABLE 2.10: Partition of $PG(1,19)$ by the projectivities of pentads

$\mathcal{P}_i$	Partition of $\mathcal{P}_i^c$
$\mathcal{P}_1$	(1) $\{-2,3\}$ (2) $\{-3,4\}$ (3) $\{-4,5\}$ (4) $\{-5,6\}$ (5) $\{-6,7\}$ (6) $\{-7,8\}$ (7) $\{-8,9\}$ (8) $\{-9\}$
$\mathcal{P}_2$	$G_{\mathcal{P}_2}$ splits $\mathcal{P}_2^c$ into 15 orbits of single points
$\mathcal{P}_3$	(1) $\{-1,2,4,5,7,9\}$ (2) $\{3,-4,-5\}$ (3) $\{-3,6,-6,8,-8,9\}$
$\mathcal{P}_4$	(1) $\{-1\}$ (2) $\{2,-4\}$ (3) $\{3,-5\}$ (4) $\{4,-6\}$ (5) $\{5,-7\}$ (6) $\{6,-8\}$ (7) $\{7,-9\}$ (8) $\{8,9\}$
$\mathcal{P}_5$	(1) $\{-1,2,-2,3,-5,6,-7,8,-8,9\}$ (2) $\{-4,5,-5,-6,-9\}$

The total numbers of all orbits is 36; therefore 36 hexads can be constructed in $PG(1,19)$. In Table 2.11 all equivalent hexads with their projective equations are listed.

Table 2.11: The equivalence of hexads

No.	Equivalent hexads	Projective equation
1	$\mathcal{P}_1 \cup \{-2\} \longrightarrow \mathcal{P}_4 \cup \{-1\}$	$(t-1)$
2	$\mathcal{P}_1 \cup \{-3\} \longrightarrow \mathcal{P}_1 \cup \{-6\}$	$(t-1)/(t+1)$
3	$\mathcal{P}_1 \cup \{-3\} \longrightarrow \mathcal{P}_2 \cup \{2\}$	$(1-t)$
4	$\mathcal{P}_1 \cup \{-3\} \longrightarrow \mathcal{P}_2 \cup \{6\}$	$(t-1)/(t+3)$
5	$\mathcal{P}_1 \cup \{-3\} \longrightarrow \mathcal{P}_3 \cup \{-1\}$	$-2/(t+1)$
6	$\mathcal{P}_1 \cup \{-3\} \longrightarrow \mathcal{P}_4 \cup \{3\}$	$3t/(t-2)$
7	$\mathcal{P}_1 \cup \{-4\} \longrightarrow \mathcal{P}_2 \cup \{-2\}$	$(-t)$
8	$\mathcal{P}_1 \cup \{-4\} \longrightarrow \mathcal{P}_2 \cup \{-9\}$	$(2-t)/(4+t)$
9	$\mathcal{P}_1 \cup \{-5\} \longrightarrow \mathcal{P}_2 \cup \{9\}$	$-1/t$
10	$\mathcal{P}_1 \cup \{-5\} \longrightarrow \mathcal{P}_4 \cup \{8\}$	$t/(6t+6)$
11	$\mathcal{P}_1 \cup \{-7\} \longrightarrow \mathcal{P}_2 \cup \{3\}$	$(t-2)/t$
12	$\mathcal{P}_1 \cup \{-7\} \longrightarrow \mathcal{P}_2 \cup \{-3\}$	$(1+t)/(1-t)$
13	$\mathcal{P}_1 \cup \{-7\} \longrightarrow \mathcal{P}_4 \cup \{2\}$	$-1/(9t)$
14	$\mathcal{P}_1 \cup \{-7\} \longrightarrow \mathcal{P}_4 \cup \{3\}$	$(t-2)/(9t+9)$
15	$\mathcal{P}_1 \cup \{-7\} \longrightarrow \mathcal{P}_5 \cup \{-1\}$	$(1+t)/(1-t)$
16	$\mathcal{P}_1 \cup \{-8\} \longrightarrow \mathcal{P}_2 \cup \{-6\}$	$(t-1)/(t+1)$
17	$\mathcal{P}_1 \cup \{-8\} \longrightarrow \mathcal{P}_2 \cup \{8\}$	$(t-2)/(5t+2)$
18	$\mathcal{P}_2 \cup \{-4\} \longrightarrow \mathcal{P}_3 \cup \{3\}$	$(t-4)/(6-6t)$
19	$\mathcal{P}_2 \cup \{5\} \longrightarrow \mathcal{P}_5 \cup \{-4\}$	$1/(5t)$
20	$\mathcal{P}_2 \cup \{-5\} \longrightarrow \mathcal{P}_3 \cup \{-3\}$	$-(4t+3)$
21	$\mathcal{P}_2 \cup \{-5\} \longrightarrow \mathcal{P}_4 \cup \{5\}$	$(t-4)/t$
22	$\mathcal{P}_2 \cup \{-7\} \longrightarrow \mathcal{P}_2 \cup \{-8\}$	$(7t-8)$
23	$\mathcal{P}_2 \cup \{-7\} \longrightarrow \mathcal{P}_4 \cup \{7\}$	$(t+7)/(6t+6)$

Table 2.11 gives the following conclusion.

Theorem 2.7.1. *On* $PG(1,19)$*, there are precisely* 13 *projectively distinct hexads summarized in Table 2.12.*

Table 2.12: Inequivalent hexads on $PG(1,19)$

Type	The hexad	Types of pentads	Stabilizer
$\mathcal{H}_1$	$\{\infty,0,1,-1,2,-2\}$	$\mathcal{P}_1\mathcal{P}_1\mathcal{P}_1\mathcal{P}_1\mathcal{P}_4\mathcal{P}_4$	$\mathbf{V}_4 = \langle -t,\ -2/t\rangle$
$\mathcal{H}_2$	$\{\infty,0,1,-1,2,-3\}$	$\mathcal{P}_1\mathcal{P}_1\mathcal{P}_2\mathcal{P}_3\mathcal{P}_2\mathcal{P}_4$	$I = \langle t\rangle$
$\mathcal{H}_3$	$\{\infty,0,1,-1,2,-4\}$	$\mathcal{P}_1\mathcal{P}_2\mathcal{P}_2\mathcal{P}_1\mathcal{P}_2\mathcal{P}_2$	$\mathbf{Z}_2 = \langle (2t+2)/(t-2)\rangle$
$\mathcal{H}_4$	$\{\infty,0,1,-1,2,-5\}$	$\mathcal{P}_1\mathcal{P}_2\mathcal{P}_1\mathcal{P}_4\mathcal{P}_4\mathcal{P}_2$	$\mathbf{Z}_2 = \langle (1-t)/(1+9t)\rangle$
$\mathcal{H}_5$	$\{\infty,0,1,-1,2,-7\}$	$\mathcal{P}_1\mathcal{P}_2\mathcal{P}_2\mathcal{P}_4\mathcal{P}_4\mathcal{P}_5$	$I = \langle t\rangle$
$\mathcal{H}_6$	$\{\infty,0,1,-1,2,-8\}$	$\mathcal{P}_1\mathcal{P}_2\mathcal{P}_1\mathcal{P}_2\mathcal{P}_2\mathcal{P}_2$	$\mathbf{Z}_2 = \langle (t-2)/(t-1)\rangle$
$\mathcal{H}_7$	$\{\infty,0,1,-1,2,-9\}$	$\mathcal{P}_1\mathcal{P}_1\mathcal{P}_1\mathcal{P}_1\mathcal{P}_1\mathcal{P}_1$	$\mathbf{D}_6 = \langle (1+t)/(2-t),\ (2t-1)/(t-2)\rangle$
$\mathcal{H}_8$	$\{\infty,0,1,-1,4,-4\}$	$\mathcal{P}_2\mathcal{P}_2\mathcal{P}_2\mathcal{P}_2\mathcal{P}_3\mathcal{P}_3$	$\mathbf{V}_4 = \langle -t,\ 4/t\rangle$
$\mathcal{H}_9$	$\{\infty,0,1,-1,4,5\}$	$\mathcal{P}_2\mathcal{P}_2\mathcal{P}_5\mathcal{P}_5\mathcal{P}_2\mathcal{P}_2$	$\mathbf{V}_4 = \langle (t-4)/(4t-1),\ (t-5)/(5t-1)\rangle$
$\mathcal{H}_{10}$	$\{\infty,0,1,-1,4,-5\}$	$\mathcal{P}_2\mathcal{P}_2\mathcal{P}_4\mathcal{P}_4\mathcal{P}_3\mathcal{P}_3$	$\mathbf{Z}_2 = \langle -1/t\rangle$
$\mathcal{H}_{11}$	$\{\infty,0,1,-1,4,7\}$	$\mathcal{P}_2\mathcal{P}_2\mathcal{P}_2\mathcal{P}_2\mathcal{P}_2\mathcal{P}_2$	$\mathbf{S}_3 = \langle (4t+3)/t,\ -(t+1)/(8t+1)\rangle$
$\mathcal{H}_{12}$	$\{\infty,0,1,-1,4,-7\}$	$\mathcal{P}_2\mathcal{P}_2\mathcal{P}_2\mathcal{P}_4\mathcal{P}_2\mathcal{P}_4$	$\mathbf{Z}_2 = \langle (t-4)/(t-1)\rangle$
$\mathcal{H}_{13}$	$\{\infty,0,1,-2,-3,6\}$	$\mathcal{P}_4\mathcal{P}_4\mathcal{P}_4\mathcal{P}_4\mathcal{P}_4\mathcal{P}_4$	$\mathbf{S}_3 = \langle (t+2)/(6t+2),\ (t+3)/(t-1)\rangle$

Remark 2.7.2. Note that $\mathcal{H}_3$ and $\mathcal{H}_6$ both have the same structure of pentads and the same type of stabilizer group but they are projectively inequivalent.

2.8 The Heptads

The projective group G_{H_i} splits $\mathcal{H}_i^c$, $i = 1, \ldots, 13$, into a number of orbits. The heptads are constructed by adding one point from each orbit to the corresponding hexad. All orbits are listed in Table 2.13.

TABLE 2.13: Partition of $PG(1,19)$ by the projectivities of hexads

$\mathcal{H}_i$	Partition of $\mathcal{H}_i^c$
$\mathcal{H}_1$	(1) $\{3,-3,7,-7\}$ (2) $\{4,-4,9,-9\}$ (3) $\{5,-5,8,-8\}$ (4) $\{6,-6\}$
$\mathcal{H}_2$	G_{H_2} splits $\mathcal{H}_2^c$ into 14 orbits of single points
$\mathcal{H}_3$	(1) $\{-2,-9\}$ (2) $\{3,8\}$ (3) $\{-3\}$ (4) $\{4,5\}$ (5) $\{-5,-7\}$ (6) $\{6,-6\}$ (7) $\{7\}$ (8) $\{-8,9\}$
$\mathcal{H}_4$	(1) $\{-2,-8\}$ (2) $\{3,4\}$ (3) $\{-3,-6\}$ (4) $\{-4,8\}$ (5) $\{5,9\}$ (6) $\{6,-7\}$ (7) $\{7,-9\}$
$\mathcal{H}_5$	G_{H_5} splits $\mathcal{H}_5^c$ into 14 orbits of single points
$\mathcal{H}_6$	(1) $\{-2,-5\}$ (2) $\{3,-9\}$ (3) $\{-3,6\}$ (4) $\{4,7\}$ (5) $\{-4,5\}$ (6) $\{-6,-7\}$ (7) $\{8,9\}$
$\mathcal{H}_7$	(1) $\{-2,3,-3,4,-4,5,-5,6,-6,7,-8,9\}$ (2) $\{-7,8\}$
$\mathcal{H}_8$	(1) $\{2,-2\}$ (2) $\{3,-3,5,-5\}$ (3) $\{6,-6,7,-7\}$ (4) $\{8,-8,9,-9\}$
$\mathcal{H}_9^c$	(1) $\{2,-3,6,-9\}$ (2) $\{-2,7,-8,9\}$ (3) $\{-4,-5\}$ (4) $\{3,-6,-7,8\}$
$\mathcal{H}_{10}$	(1) $\{2,9\}$ (2) $\{-2,-9\}$ (3) $\{3,6\}$ (4) $\{-3,-6\}$ (5) $\{-4,5\}$ (6) $\{7,8\}$ (7) $\{-7,-8\}$
$\mathcal{H}_{11}$	(1) $\{2,-4,-5,6,8,-8\}$ (2) $\{-2,3,-3,5,-7,9\}$ (3) $\{-6,-9\}$
$\mathcal{H}_{12}$	(1) $\{2,-2\}$ (2) $\{3,9\}$ (3) $\{-3\}$ (4) $\{-4,-6\}$ (5) $\{5\}$ (6) $\{-5,-8\}$ (7) $\{7,-9\}$ (8) $\{6,8\}$
$\mathcal{H}_{13}$	(1) $\{-1,-5,-6\}$ (2) $\{2,3,5\}$ (3) $\{4,-4,7,8,-8,9\}$ (4) $\{-7,-9\}$

There are 86 different orbits; therefore 86 heptads can be constructed in $PG(1,19)$. The projectively distinct heptads with their types of hexads and the stabilizer groups are given in the following theorem.

Theorem 2.8.1. *On* $PG(1,19)$, *there are* 18 *projectively distinct heptads, as summarized in Table* 2.14.

Table 2.14: Inequivalent heptads on $PG(1,19)$

Type	The heptad	Types of hexads	Stabilizer
$\mathcal{T}_1$	$\{\infty,0,1,-1,2,-2,-3\}$	$\mathcal{H}_1\mathcal{H}_2\mathcal{H}_1\mathcal{H}_5\mathcal{H}_2\mathcal{H}_5\mathcal{H}_{13}$	$\mathbf{Z}_2=\langle -(t+1)\rangle$
$\mathcal{T}_2$	$\{\infty,0,1,-1,2,-2,6\}$	$\mathcal{H}_1\mathcal{H}_4\mathcal{H}_2\mathcal{H}_2\mathcal{H}_4\mathcal{H}_{10}\mathcal{H}_{10}$	$\mathbf{Z}_2=\langle -2/t\rangle$
$\mathcal{T}_3$	$\{\infty,0,1,-1,2,-2,-4\}$	$\mathcal{H}_1\mathcal{H}_3\mathcal{H}_2\mathcal{H}_6\mathcal{H}_7\mathcal{H}_2\mathcal{H}_4$	$I=\langle t\rangle$
$\mathcal{T}_4$	$\{\infty,0,1,-1,2,-2,-5\}$	$\mathcal{H}_1\mathcal{H}_4\mathcal{H}_3\mathcal{H}_6\mathcal{H}_5\mathcal{H}_5\mathcal{H}_{12}$	$I=\langle t\rangle$
$\mathcal{T}_5$	$\{\infty,0,1,-1,2,-3,9\}$	$\mathcal{H}_2\mathcal{H}_6\mathcal{H}_2\mathcal{H}_{12}\mathcal{H}_8\mathcal{H}_{11}\mathcal{H}_{10}$	$I=\langle t\rangle$
$\mathcal{T}_6$	$\{\infty,0,1,-1,2,-3,8\}$	$\mathcal{H}_2\mathcal{H}_5\mathcal{H}_3\mathcal{H}_{12}\mathcal{H}_2\mathcal{H}_{12}\mathcal{H}_5$	$\mathbf{Z}_2=\langle (t+1)/(6t-1)\rangle$
$\mathcal{T}_7$	$\{\infty,0,1,-1,2,-3,4\}$	$\mathcal{H}_2\mathcal{H}_2\mathcal{H}_5\mathcal{H}_5\mathcal{H}_{10}\mathcal{H}_{10}\mathcal{H}_{13}$	$\mathbf{Z}_2=\langle 1-t\rangle$
$\mathcal{T}_8$	$\{\infty,0,1,-1,2,-3,5\}$	$\mathcal{H}_2\mathcal{H}_3\mathcal{H}_2\mathcal{H}_8\mathcal{H}_{10}\mathcal{H}_3\mathcal{H}_{10}$	$\mathbf{Z}_2=\langle (t+3)/(t-1)\rangle$
$\mathcal{T}_9$	$\{\infty,0,1,-1,2,-3,7\}$	$\mathcal{H}_2\mathcal{H}_2\mathcal{H}_3\mathcal{H}_6\mathcal{H}_8\mathcal{H}_{12}\mathcal{H}_4$	$I=\langle t\rangle$
$\mathcal{T}_{10}$	$\{\infty,0,1,-1,2,-3,-7\}$	$\mathcal{H}_2\mathcal{H}_5\mathcal{H}_4\mathcal{H}_6\mathcal{H}_{10}\mathcal{H}_2\mathcal{H}_5$	$I=\langle t\rangle$
$\mathcal{T}_{11}$	$\{\infty,0,1,-1,2,-3,-4\}$	$\mathcal{H}_2\mathcal{H}_3\mathcal{H}_5\mathcal{H}_9\mathcal{H}_2\mathcal{H}_9\mathcal{H}_5$	$\mathbf{Z}_2=\langle -(t+1)/(9t+1)\rangle$
$\mathcal{T}_{12}$	$\{\infty,0,1,-1,2,-4,-8\}$	$\mathcal{H}_3\mathcal{H}_6\mathcal{H}_6\mathcal{H}_3\mathcal{H}_6\mathcal{H}_{11}\mathcal{H}_3$	$\mathbf{Z}_3=\langle t/(7t+7)\rangle$
$\mathcal{T}_{13}$	$\{\infty,0,1,-1,2,-4,-5\}$	$\mathcal{H}_3\mathcal{H}_4\mathcal{H}_9\mathcal{H}_5\mathcal{H}_5\mathcal{H}_{12}\mathcal{H}_{11}$	$I=\langle t\rangle$
$\mathcal{T}_{14}$	$\{\infty,0,1,-1,2,-5,-7\}$	$\mathcal{H}_4\mathcal{H}_5\mathcal{H}_{12}\mathcal{H}_4\mathcal{H}_{13}\mathcal{H}_{13}\mathcal{H}_5$	$\mathbf{Z}_2=\langle (1-t)/(4t+1)\rangle$
$\mathcal{T}_{15}$	$\{\infty,0,1,-1,2,-7,9\}$	$\mathcal{H}_5\mathcal{H}_6\mathcal{H}_6\mathcal{H}_9\mathcal{H}_5\mathcal{H}_{12}\mathcal{H}_9$	$\mathbf{Z}_2=\langle -t/(t+1)\rangle$
$\mathcal{T}_{16}$	$\{\infty,0,1,-1,2,-7,8\}$	$\mathcal{H}_5\mathcal{H}_5\mathcal{H}_8\mathcal{H}_8\mathcal{H}_{10}\mathcal{H}_{10}\mathcal{H}_9$	$\mathbf{Z}_2=\langle 1-t\rangle$
$\mathcal{T}_{17}$	$\{\infty,0,1,-1,2,-7,-9\}$	$\mathcal{H}_5\mathcal{H}_7\mathcal{H}_5\mathcal{H}_5\mathcal{H}_5\mathcal{H}_5\mathcal{H}_5$	$\mathbf{Z}_6=\langle (2t-1)/(t+1)\rangle$
$\mathcal{T}_{18}$	$\{\infty,0,1,-1,4,-5,-7\}$	$\mathcal{H}_{10}\mathcal{H}_{12}\mathcal{H}_{12}\mathcal{H}_{12}\mathcal{H}_{13}\mathcal{H}_{10}\mathcal{H}_{10}$	$\mathbf{Z}_3=\langle 8/(t+7)\rangle$

2.9 The Octads

The projective group $G_{\mathcal{T}_i}$ splits $\mathcal{T}_i^c$, $i = 1, \ldots, 18$, into a number of orbits. The octads are constructed by adding one point from each orbit to the corresponding heptad . All orbits are listed in Table 2.15.

TABLE 2.15: Partition of $PG(1,19)$ by the projectivities of heptads

$\mathcal{T}_i$	Partition of $\mathcal{T}_i^c$
$\mathcal{T}_1$	(1) $\{3,-4\}$(2) $\{4,-5\}$(3) $\{5,-6\}$ (4) $\{6,-7\}$ (5) $\{7,-8\}$ (6) $\{8,-9\}$ (7) $\{9\}$
$\mathcal{T}_2$	(1) $\{3,-7\}$ (2) $\{-3,7\}$ (3) $\{4,9\}$ (4) $\{-4,-9\}$ (5) $\{5,-8\}$ (6) $\{-5,8\}$ (7) $\{-6\}$
$\mathcal{T}_3$	$G_{\mathcal{T}_3}$ splits $\mathcal{T}_3^c$ into 13 orbits of single points
$\mathcal{T}_4$	$G_{\mathcal{T}_4}$ splits $\mathcal{T}_4^c$ into 13 orbits of single points
$\mathcal{T}_5$	$G_{\mathcal{T}_5}$ splits $\mathcal{T}_5^c$ into 13 orbits of single points
$\mathcal{T}_6$	(1) $\{3,-2\}$ (2) $\{4,6\}$ (3) $\{-4,-9\}$ (4) $\{5,-7\}$ (5) $\{-5,-6\}$ (6) $\{7,9\}$ (7) $\{-8\}$
$\mathcal{T}_7$	(1) $\{-2,3\}$ (2) $\{-4,5\}$ (3) $\{-5,6\}$ (4) $\{-6,7\}$ (5) $\{-7,8\}$ (6) $\{-8,9\}$ (7) $\{-9\}$
$\mathcal{T}_8$	(1) $\{-2,6\}$ (2) $\{3\}$ (3) $\{4,-4\}$ (4) $\{-5,-6\}$ (5) $\{7,8\}$ (6) $\{-7,-9\}$ (7) $\{-8,9\}$
$\mathcal{T}_9$	$G_{\mathcal{T}_9}$ splits $\mathcal{T}_9^c$ into 13 orbits of single points
$\mathcal{T}_{10}$	$G_{\mathcal{T}_{10}}$ splits $\mathcal{T}_{10}^c$ into 13 orbits of single points
$\mathcal{T}_{11}$	(1) $\{-2,-9\}$ (2) $\{3,8\}$ (3) $\{4,5\}$ (4) $\{-5,-7\}$ (5) $\{6,-6\}$ (6) $\{7\}$ (7) $\{-8,9\}$
$\mathcal{T}_{12}$	(1) $\{-2,3,-6\}$ (2) $\{-3,7,-7\}$ (3) $\{4,5,6\}$ (4) $\{-5,8,9\}$ (5) $\{-9\}$
$\mathcal{T}_{13}$	$G_{\mathcal{T}_{13}}$ splits $\mathcal{T}_{13}^c$ into 13 orbits of single points
$\mathcal{T}_{14}$	(1) $\{-2,5\}$ (2) $\{3,-6\}$ (3) $\{-3,-9\}$ (4) $\{4,-8\}$ (5) $\{-4,6\}$ (6) $\{7\}$ (7) $\{8,9\}$
$\mathcal{T}_{15}$	(1) $\{-2\}$ (2) $\{3,4\}$ (3) $\{-3,8\}$ (4) $\{-4,5\}$ (5) $\{-5,-6\}$ (6) $\{6,-9\}$ (7) $\{7,-8\}$
$\mathcal{T}_{16}$	(1) $\{-2,3\}$ (2) $\{-3,4\}$ (3) $\{-4,5\}$ (4) $\{-5,6\}$ (5) $\{-6,7\}$ (6) $\{-8,9\}$ (7) $\{-9\}$
$\mathcal{T}_{17}$	(1) $\{-2,-3,5,-5,-6,-8\}$ (2) $\{3,4,-4,6,7,9\}$ (3) $\{8\}$
$\mathcal{T}_{18}$	(1) $\{-2,3,-3\}$ (2) $\{-2,-6,8\}$ (3) $\{-4,9,-9\}$ (4) $\{5,6,7\}$ (5) $\{-8\}$

There are 154 different orbits; therefore 154 octads can be constructed in $PG(1,19)$. The projectively distinct octads with their types of heptads and the stabilizer groups are given in the following theorem.

Theorem 2.9.1. *On $PG(1,19)$, there are 31 projectively distinct octads, as summarized in Table 2.16.*

TABLE 2.16: Inequivalent octads on $PG(1,19)$

Type	The octad	Types of heptads	Stabilizer
$\mathcal{O}_1$	{∞,0,1,−1,2,−2,−3,−4}	$\mathcal{T}_1\mathcal{T}_3\mathcal{T}_{11}\mathcal{T}_1\mathcal{T}_{15}\mathcal{T}_3\mathcal{T}_{11}\mathcal{T}_{14}$	$\mathbf{Z}_2=\langle -(t+2)\rangle$
$\mathcal{O}_2$	{∞,0,1,−1,2,−2,−3,4}	$\mathcal{T}_1\mathcal{T}_3\mathcal{T}_7\mathcal{T}_4\mathcal{T}_{17}\mathcal{T}_{10}\mathcal{T}_{10}\mathcal{T}_{14}$	$I=\langle t\rangle$
$\mathcal{O}_3$	{∞,0,1,−1,2,−2,−3,5}	$\mathcal{T}_1\mathcal{T}_4\mathcal{T}_8\mathcal{T}_2\mathcal{T}_{16}\mathcal{T}_5\mathcal{T}_{13}\mathcal{T}_{18}$	$I=\langle t\rangle$
$\mathcal{O}_4$	{∞,0,1,−1,2,−2,−3,6}	$\mathcal{T}_1\mathcal{T}_2\mathcal{T}_2\mathcal{T}_1\mathcal{T}_7\mathcal{T}_{10}\mathcal{T}_{10}\mathcal{T}_7$	$\mathbf{Z}_2=\langle (1-t)/(1+t)\rangle$
$\mathcal{O}_5$	{∞,0,1,−1,2,−2,−3,7}	$\mathcal{T}_1\mathcal{T}_1\mathcal{T}_9\mathcal{T}_4\mathcal{T}_4\mathcal{T}_9\mathcal{T}_{14}\mathcal{T}_{14}$	$\mathbf{Z}_2=\langle -2/t\rangle$
$\mathcal{O}_6$	{∞,0,1,−1,2,−2,−3,8}	$\mathcal{T}_1\mathcal{T}_4\mathcal{T}_6\mathcal{T}_3\mathcal{T}_4\mathcal{T}_3\mathcal{T}_6\mathcal{T}_1$	$\mathbf{Z}_2=\langle -(t+2)/(7t+1)\rangle$
$\mathcal{O}_7$	{∞,0,1,−1,2,−2,−3,9}	$\mathcal{T}_1\mathcal{T}_3\mathcal{T}_5\mathcal{T}_3\mathcal{T}_{13}\mathcal{T}_5\mathcal{T}_{13}\mathcal{T}_7$	$\mathbf{Z}_2=\langle -(t+1)\rangle$
$\mathcal{O}_8$	{∞,0,1,−1,2,−2,6,−4}	$\mathcal{T}_2\mathcal{T}_3\mathcal{T}_9\mathcal{T}_9\mathcal{T}_5\mathcal{T}_3\mathcal{T}_5\mathcal{T}_2$	$\mathbf{Z}_2=\langle -(t+1)/(5t+1)\rangle$
$\mathcal{O}_9$	{∞,0,1,−1,2,−2,6,4}	$\mathcal{T}_2\mathcal{T}_3\mathcal{T}_{10}\mathcal{T}_9\mathcal{T}_3\mathcal{T}_4\mathcal{T}_8\mathcal{T}_{10}$	$I=\langle t\rangle$
$\mathcal{O}_{10}$	{∞,0,1,−1,2,−2,6,5}	$\mathcal{T}_2\mathcal{T}_4\mathcal{T}_{13}\mathcal{T}_{10}\mathcal{T}_{11}\mathcal{T}_9\mathcal{T}_{16}\mathcal{T}_5$	$I=\langle t\rangle$
$\mathcal{O}_{11}$	{∞,0,1,−1,2,−2,6,−5}	$\mathcal{T}_2\mathcal{T}_4\mathcal{T}_{14}\mathcal{T}_6\mathcal{T}_{10}\mathcal{T}_{14}\mathcal{T}_7\mathcal{T}_{18}$	$I=\langle t\rangle$
$\mathcal{O}_{12}$	{∞,0,1,−1,2,−2,6,−6}	$\mathcal{T}_2\mathcal{T}_2\mathcal{T}_3\mathcal{T}_3\mathcal{T}_3\mathcal{T}_3\mathcal{T}_8\mathcal{T}_8$	$\mathbf{V}_4=\langle 2/t,\ -2/t\rangle$
$\mathcal{O}_{13}$	{∞,0,1,−1,2,−2,−4,−5}	$\mathcal{T}_3\mathcal{T}_4\mathcal{T}_{13}\mathcal{T}_{11}\mathcal{T}_{15}\mathcal{T}_{17}\mathcal{T}_6\mathcal{T}_{13}$	$I=\langle t\rangle$
$\mathcal{O}_{14}$	{∞,0,1,−1,2,−2,−4,8}	$\mathcal{T}_3\mathcal{T}_4\mathcal{T}_4\mathcal{T}_5\mathcal{T}_{12}\mathcal{T}_3\mathcal{T}_9\mathcal{T}_{10}$	$I=\langle t\rangle$
$\mathcal{O}_{15}$	{∞,0,1,−1,2,−2,−4,9}	$\mathcal{T}_3\mathcal{T}_3\mathcal{T}_{12}\mathcal{T}_3\mathcal{T}_{12}\mathcal{T}_3\mathcal{T}_3\mathcal{T}_3$	$\mathbf{S}_3=\langle (t-2)/(t+4),\ (2/t)\rangle$
$\mathcal{O}_{16}$	{∞,0,1,−1,2,−2,−4,−9}	$\mathcal{T}_3\mathcal{T}_3\mathcal{T}_3\mathcal{T}_9\mathcal{T}_9\mathcal{T}_3\mathcal{T}_9\mathcal{T}_9$	$\mathbf{V}_4=\langle -2/t,\ (2-t)/(1+t)\rangle$
$\mathcal{O}_{17}$	{∞,0,1,−1,2,−2,−5,5}	$\mathcal{T}_4\mathcal{T}_4\mathcal{T}_9\mathcal{T}_9\mathcal{T}_{15}\mathcal{T}_{15}\mathcal{T}_{11}\mathcal{T}_6$	$\mathbf{Z}_2=\langle -t\rangle$
$\mathcal{O}_{18}$	{∞,0,1,−1,2,−2,−5,8}	$\mathcal{T}_4\mathcal{T}_4\mathcal{T}_{13}\mathcal{T}_{12}\mathcal{T}_{12}\mathcal{T}_{13}\mathcal{T}_{15}\mathcal{T}_{15}$	$\mathbf{Z}_2=\langle -2/t\rangle$
$\mathcal{O}_{19}$	{∞,0,1,−1,2,−2,−5,−8}	$\mathcal{T}_4\mathcal{T}_4\mathcal{T}_4\mathcal{T}_{13}\mathcal{T}_4\mathcal{T}_{13}\mathcal{T}_{13}\mathcal{T}_{13}$	$\mathbf{V}_4=\langle (t-2)/(t-1),\ (2t-2)/(t-2)\rangle$
$\mathcal{O}_{20}$	{∞,0,1,−1,2,−3,9,4}	$\mathcal{T}_5\mathcal{T}_7\mathcal{T}_9\mathcal{T}_{10}\mathcal{T}_6\mathcal{T}_8\mathcal{T}_5\mathcal{T}_{18}$	$I=\langle t\rangle$
$\mathcal{O}_{21}$	{∞,0,1,−1,2,−3,9,−4}	$\mathcal{T}_5\mathcal{T}_{11}\mathcal{T}_{12}\mathcal{T}_6\mathcal{T}_{13}\mathcal{T}_9\mathcal{T}_{13}\mathcal{T}_{10}$	$I=\langle t\rangle$

$\mathcal{O}_{22}$	{∞,0,1,−1,2,−3,9,5}	$\mathcal{T}_5\mathcal{T}_8\mathcal{T}_{12}\mathcal{T}_9\mathcal{T}_9\mathcal{T}_5\mathcal{T}_{12}\mathcal{T}_8$	$\mathbf{Z}_2$=⟨−(t+3)/(2t+1)⟩
$\mathcal{O}_{23}$	{∞,0,1,−1,2,−3,9,−5}	$\mathcal{T}_5\mathcal{T}_{10}\mathcal{T}_9\mathcal{T}_9\mathcal{T}_6\mathcal{T}_8\mathcal{T}_{13}\mathcal{T}_{16}$	I=⟨t⟩
$\mathcal{O}_{24}$	{∞,0,1,−1,2,−3,9,−6}	$\mathcal{T}_5\mathcal{T}_5\mathcal{T}_5\mathcal{T}_5\mathcal{T}_5\mathcal{T}_5\mathcal{T}_5\mathcal{T}_5$	$\mathbf{D}_4$=⟨(t−1)/(t+1),(2t+1)/(t−2)⟩
$\mathcal{O}_{25}$	{∞,0,1,−1,2,−3,9,−7}	$\mathcal{T}_5\mathcal{T}_{10}\mathcal{T}_{15}\mathcal{T}_{10}\mathcal{T}_{15}\mathcal{T}_{16}\mathcal{T}_5\mathcal{T}_{16}$	$\mathbf{Z}_2$=⟨(t+7)/(t−1)⟩
$\mathcal{O}_{26}$	{∞,0,1,−1,2,−3,9,−8}	$\mathcal{T}_5\mathcal{T}_9\mathcal{T}_{15}\mathcal{T}_7\mathcal{T}_{14}\mathcal{T}_{16}\mathcal{T}_{13}\mathcal{T}_{18}$	I=⟨t⟩
$\mathcal{O}_{27}$	{∞,0,1,−1,2,−3,4,5}	$\mathcal{T}_7\mathcal{T}_8\mathcal{T}_{11}\mathcal{T}_{11}\mathcal{T}_{16}\mathcal{T}_{16}\mathcal{T}_8\mathcal{T}_7$	$\mathbf{Z}_2$=⟨(t−4)/(4t−1)⟩
$\mathcal{O}_{28}$	{∞,0,1,−1,2,−3,−7,−4}	$\mathcal{T}_{10}\mathcal{T}_{11}\mathcal{T}_{13}\mathcal{T}_{13}\mathcal{T}_{15}\mathcal{T}_{10}\mathcal{T}_{11}\mathcal{T}_{15}$	$\mathbf{Z}_2$=⟨−(t+7)/(t+1)⟩
$\mathcal{O}_{29}$	{∞,0,1,−1,2,−4,−5,−7}	$\mathcal{T}_{13}\mathcal{T}_{13}\mathcal{T}_{14}\mathcal{T}_{13}\mathcal{T}_{14}\mathcal{T}_{14}\mathcal{T}_{14}\mathcal{T}_{13}$	$\mathbf{V}_4$=⟨(t−1)/(8t−1),(2t+2)/(t−2)⟩
$\mathcal{O}_{30}$	{∞,0,1,−1,2,−7,8,−9}	$\mathcal{T}_{16}\mathcal{T}_{17}\mathcal{T}_{17}\mathcal{T}_{16}\mathcal{T}_{16}\mathcal{T}_{16}\mathcal{T}_{16}\mathcal{T}_{16}$	$\mathbf{D}_6$=⟨(1+t)/(2−t), t/(t−1)⟩
$\mathcal{O}_{31}$	{∞,0,1,−1,4,−5,−7,−8}	$\mathcal{T}_{18}\mathcal{T}_{18}\mathcal{T}_{18}\mathcal{T}_{18}\mathcal{T}_{18}\mathcal{T}_{18}\mathcal{T}_{18}\mathcal{T}_{18}$	$\mathbf{S}_4$=⟨(1+t)/(1−t), (t−4)/(t−1)⟩

2.10 The Nonads

The 31 projectivities of the octads $G_{\mathcal{O}_i}$, $i = 1,\ldots,31$, split $\mathcal{O}_i^c$ into a number of orbits. The nonads are constructed by adding one point from each orbit to the corresponding octad. All orbits are listed in Table 2.17.

TABLE 2.17: Partition of $PG(1,19)$ by the projectivities of octads

$\mathcal{O}_i$	Partition of $\mathcal{O}_i^c$
$\mathcal{O}_1$	(1) {3, −5} (2) {4, −6} (3) {5, −7} (4) {6, −8} (5) {7, −9} (6) {8, 9}
$\mathcal{O}_2$	$G_{\mathcal{O}_2}$ splits $\mathcal{O}_2^c$ into 12 orbits of single points
$\mathcal{O}_3$	$G_{\mathcal{O}_3}$ splits $\mathcal{O}_3^c$ into 12 orbits of single points
$\mathcal{O}_4$	(1) {3, 9} (2) {4, 7} (3) {−4, −8} (4) {5, −7} (5) {−5, 8} (6) {−6, −9}
$\mathcal{O}_5$	(1) {3, −7} (2) {4, 9} (3) {−4, −9} (4) {5, −8} (5) {−5, −8} (6) {6} (7) {−6}
$\mathcal{O}_6$	(1) {3, −8} (2) {4, 7} (3) {−4, −5} (4) {5, −6} (5) {6} (6) {−7, 9} (7) {−9}
$\mathcal{O}_7$	(1) {3, −4} (2) {4, −5} (3) {5, −6} (4) {6, −7} (5) {7, −8} (6) {8, −9}

$\mathcal{O}_8$	(1) $\{3,-5\}$ (2) $\{-3,8\}$ (3) $\{4,7\}$ (4) $\{5,-9\}$ (5) $\{-6,9\}$ (6) $\{-7,-8\}$
$\mathcal{O}_9$	$G_{\mathcal{O}_9}$ splits $\mathcal{O}_9^c$ into 12 orbits of single points
$\mathcal{O}_{10}$	$G_{\mathcal{O}_{10}}$ splits $\mathcal{O}_{10}^c$ into 12 orbits of single points
$\mathcal{O}_{11}$	$G_{\mathcal{O}_{11}}$ splits $\mathcal{O}_{11}^c$ into 12 orbits of single points
$\mathcal{O}_{12}$	(1) $\{3,-3,7,-7\}$ (2) $\{4,-4,9,-9\}$ (3) $\{5,-5,8,-8\}$
$\mathcal{O}_{13}$	$G_{\mathcal{O}_{13}}$ splits $\mathcal{O}_{13}^c$ into 12 orbits of single points
$\mathcal{O}_{14}$	$G_{\mathcal{O}_{14}}$ splits $\mathcal{O}_{14}^c$ into 12 orbits of single points
$\mathcal{O}_{15}$	(1) $\{3,-3,-5,7,-7,-8\}$ (2) $\{4,5,6,-6,8,-9\}$
$\mathcal{O}_{16}$	(1) $\{3,-5,-7,8\}$ (2) $\{-3,7\}$ (3) $\{4,5,-8,9\}$ (4) $\{6,-6\}$
$\mathcal{O}_{17}$	(1) $\{3,-3\}$ (2) $\{4,-4\}$ (3) $\{6,-6\}$ (4) $\{7,-7\}$ (5) $\{8,-8\}$ (6) $\{9,-9\}$
$\mathcal{O}_{18}$	(1) $\{3,-7\}$ (2) $\{-3,7\}$ (3) $\{4,9\}$ (4) $\{-4,-9\}$ (5) $\{5,-8\}$ (6) $\{6\}$ (7) $\{-6\}$
$\mathcal{O}_{19}$	(1) $\{3,4,7,-9\}$ (2) $\{-3,6,-6,-7\}$ (3) $\{-4,5,8,9\}$
$\mathcal{O}_{20}$	$G_{\mathcal{O}_{20}}$ splits $\mathcal{O}_{20}^c$ into 12 orbits of single points
$\mathcal{O}_{21}$	$G_{\mathcal{O}_{21}}$ splits $\mathcal{O}_{21}^c$ into 12 orbits of single points
$\mathcal{O}_{22}$	(1) $\{-2,-6\}$ (2) $\{3,-9\}$ (3) $\{4,-5\}$ (4) $\{-4,8\}$ (5) $\{6,-8\}$ (6) $\{7,-7\}$
$\mathcal{O}_{23}$	$G_{\mathcal{O}_{23}}$ splits $\mathcal{O}_{23}^c$ into 12 orbits of single points
$\mathcal{O}_{24}$	(1) $\{-2,3,-4,5,6,7,-8,-9\}$ (2) $\{4,-5,-7,8\}$
$\mathcal{O}_{25}$	(1) $\{-2,-8\}$ (2) $\{3,5\}$ (3) $\{4,-9\}$ (4) $\{-4,7\}$ (5) $\{-5,6\}$ (6) $\{-6,8\}$
$\mathcal{O}_{26}$	$G_{\mathcal{O}_{26}}$ splits $\mathcal{O}_{26}^c$ into 12 orbits of single points
$\mathcal{O}_{27}$	(1) $\{-2,7\}$ (2) $\{3,-7\}$ (3) $\{-4\}$ (4) $\{-5\}$ (5) $\{6,-9\}$ (6) $\{-6,8\}$ (7) $\{-8,9\}$
$\mathcal{O}_{28}$	(1) $\{-2,5\}$ (2) $\{3,57\}$ (3) $\{4,-6\}$ (4) $\{-5,-9\}$ (5) $\{6,9\}$ (6) $\{8,-8\}$
$\mathcal{O}_{29}$	(1) $\{-2,3,8,-9\}$ (2) $\{-3,7\}$ (3) $\{4,5\}$ (4) $\{6,-6,-8,9\}$
$\mathcal{O}_{30}$	(1) $\{-2,3,-3,4,-4,5,-5,6,-6,7,-8,9\}$
$\mathcal{O}_{31}$	(1) $\{-2,-2,3,-3,-4,5,6,-6,7,8,9,-9\}$

There are 228 different orbits; therefore 228 nonads can be constructed in $PG(1,19)$. The projectively distinct nonads with their types of octads and the stabilizer groups are given in the following result.

Theorem 2.10.1. *On $PG(1,19)$, there are 33 projectively distinct nonads, as summarized in Table 2.18.*

TABLE 2.18: Inequivalent nonads on $PG(1,19)$

Type	The nonad	Types of octads	Stabilizer
$\mathcal{N}_1$	{∞,0,1,−1,2,−2,−3,−4,3}	$\mathcal{O}_1\mathcal{O}_1\mathcal{O}_2\mathcal{O}_{13}\mathcal{O}_2\mathcal{O}_{28}\mathcal{O}_{13}\mathcal{O}_{28}\mathcal{O}_{29}$	$\mathbf{Z}_2=\langle -(t+1)\rangle$
$\mathcal{N}_2$	{∞,0,1,−1,2,−2,−3,−4,4}	$\mathcal{O}_1\mathcal{O}_2\mathcal{O}_{12}\mathcal{O}_{27}\mathcal{O}_3\mathcal{O}_{13}\mathcal{O}_9\mathcal{O}_{10}\mathcal{O}_{11}$	$I=\langle t\rangle$
$\mathcal{N}_3$	{∞,0,1,−1,2,−2,−3,−4,5}	$\mathcal{O}_1\mathcal{O}_3\mathcal{O}_9\mathcal{O}_{27}\mathcal{O}_4\mathcal{O}_{25}\mathcal{O}_7\mathcal{O}_{28}\mathcal{O}_{26}$	$I=\langle t\rangle$
$\mathcal{N}_4$	{∞,0,1,−1,2,−2,−3,−4,6}	$\mathcal{O}_1\mathcal{O}_4\mathcal{O}_8\mathcal{O}_{10}\mathcal{O}_5\mathcal{O}_{26}\mathcal{O}_{14}\mathcal{O}_{21}\mathcal{O}_{11}$	$I=\langle t\rangle$
$\mathcal{N}_5$	{∞,0,1,−1,2,−2,−3,−4,7}	$\mathcal{O}_1\mathcal{O}_5\mathcal{O}_6\mathcal{O}_{17}\mathcal{O}_6\mathcal{O}_{17}\mathcal{O}_{16}\mathcal{O}_1\mathcal{O}_5$	$\mathbf{Z}_2=\langle (7-t)/(1+5t)\rangle$
$\mathcal{N}_6$	{∞,0,1,−1,2,−2,−3,−4,8}	$\mathcal{O}_1\mathcal{O}_6\mathcal{O}_{14}\mathcal{O}_{13}\mathcal{O}_7\mathcal{O}_{18}\mathcal{O}_{15}\mathcal{O}_{21}\mathcal{O}_2$	$I=\langle t\rangle$
$\mathcal{N}_7$	{∞,0,1,−1,2,−2,−3,4,5}	$\mathcal{O}_2\mathcal{O}_3\mathcal{O}_{13}\mathcal{O}_{27}\mathcal{O}_{10}\mathcal{O}_{30}\mathcal{O}_{25}\mathcal{O}_{23}\mathcal{O}_{26}$	$I=\langle t\rangle$
$\mathcal{N}_8$	{∞,0,1,−1,2,−2,−3,4,−5}	$\mathcal{O}_2\mathcal{O}_2\mathcal{O}_9\mathcal{O}_4\mathcal{O}_9\mathcal{O}_2\mathcal{O}_4\mathcal{O}_2\mathcal{O}_5$	$\mathbf{Z}_2=\langle -(t+1)\rangle$
$\mathcal{N}_9$	{∞,0,1,−1,2,−2,−3,4,−6}	$\mathcal{O}_2\mathcal{O}_3\mathcal{O}_8\mathcal{O}_{26}\mathcal{O}_{17}\mathcal{O}_{13}\mathcal{O}_{20}\mathcal{O}_{25}\mathcal{O}_{11}$	$I=\langle t\rangle$
$\mathcal{N}_{10}$	{∞,0,1,−1,2,−2,−3,4,7}	$\mathcal{O}_2\mathcal{O}_5\mathcal{O}_7\mathcal{O}_{26}\mathcal{O}_{19}\mathcal{O}_{13}\mathcal{O}_{10}\mathcal{O}_{11}\mathcal{O}_{29}$	$I=\langle t\rangle$
$\mathcal{N}_{11}$	{∞,0,1,−1,2,−2,−3,4,−7}	$\mathcal{O}_2\mathcal{O}_4\mathcal{O}_6\mathcal{O}_{11}\mathcal{O}_{11}\mathcal{O}_2\mathcal{O}_{11}\mathcal{O}_4\mathcal{O}_{11}$	$\mathbf{Z}_2=\langle (t-4)/(t-1)\rangle$
$\mathcal{N}_{12}$	{∞,0,1,−1,2,−2,−3,4,8}	$\mathcal{O}_2\mathcal{O}_6\mathcal{O}_{14}\mathcal{O}_{11}\mathcal{O}_{14}\mathcal{O}_2\mathcal{O}_9\mathcal{O}_{20}\mathcal{O}_5$	$I=\langle t\rangle$
$\mathcal{N}_{13}$	{∞,0,1,−1,2,−2,−3,4,9}	$\mathcal{O}_2\mathcal{O}_7\mathcal{O}_{16}\mathcal{O}_{20}\mathcal{O}_{14}\mathcal{O}_{13}\mathcal{O}_{23}\mathcal{O}_{21}\mathcal{O}_{26}$	$I=\langle t\rangle$
$\mathcal{N}_{14}$	{∞,0,1,−1,2,−2,−3,5,6}	$\mathcal{O}_3\mathcal{O}_4\mathcal{O}_{10}\mathcal{O}_3\mathcal{O}_4\mathcal{O}_{27}\mathcal{O}_{20}\mathcal{O}_{10}\mathcal{O}_{20}$	$\mathbf{Z}_2=\langle (t+3)/(t-1)\rangle$
$\mathcal{N}_{15}$	{∞,0,1,−1,2,−2,−3,5,−6}	$\mathcal{O}_3\mathcal{O}_3\mathcal{O}_{11}\mathcal{O}_{20}\mathcal{O}_{11}\mathcal{O}_{26}\mathcal{O}_{20}\mathcal{O}_{26}\mathcal{O}_{31}$	$\mathbf{Z}_2=\langle -(t+1)\rangle$
$\mathcal{N}_{16}$	{∞,0,1,−1,2,−2,−3,5,7}	$\mathcal{O}_3\mathcal{O}_5\mathcal{O}_5\mathcal{O}_{23}\mathcal{O}_{11}\mathcal{O}_3\mathcal{O}_{23}\mathcal{O}_{29}\mathcal{O}_{11}$	$\mathbf{Z}_2=\langle (2t)/(t-2)\rangle$
$\mathcal{N}_{17}$	{∞,0,1,−1,2,−2,−3,5,−7}	$\mathcal{O}_3\mathcal{O}_4\mathcal{O}_6\mathcal{O}_9\mathcal{O}_{12}\mathcal{O}_{23}\mathcal{O}_8\mathcal{O}_7\mathcal{O}_{20}$	$I=\langle t\rangle$
$\mathcal{N}_{18}$	{∞,0,1,−1,2,−2,−3,5,8}	$\mathcal{O}_3\mathcal{O}_6\mathcal{O}_{19}\mathcal{O}_{23}\mathcal{O}_9\mathcal{O}_{10}\mathcal{O}_{14}\mathcal{O}_{21}\mathcal{O}_3$	$I=\langle t\rangle$
$\mathcal{N}_{19}$	{∞,0,1,−1,2,−2,−3,5,−8}	$\mathcal{O}_3\mathcal{O}_5\mathcal{O}_{18}\mathcal{O}_{22}\mathcal{O}_3\mathcal{O}_{26}\mathcal{O}_{22}\mathcal{O}_{18}\mathcal{O}_{26}$	$\mathbf{Z}_2=\langle -(t+3)/(t+1)\rangle$
$\mathcal{N}_{20}$	{∞,0,1,−1,2,−2,−3,5,9}	$\mathcal{O}_3\mathcal{O}_7\mathcal{O}_{14}\mathcal{O}_{22}\mathcal{O}_8\mathcal{O}_{10}\mathcal{O}_{24}\mathcal{O}_{21}\mathcal{O}_{20}$	$I=\langle t\rangle$
$\mathcal{N}_{21}$	{∞,0,1,−1,2,−2,−3,8,−9}	$\mathcal{O}_6\mathcal{O}_6\mathcal{O}_{13}\mathcal{O}_{13}\mathcal{O}_{13}\mathcal{O}_{13}\mathcal{O}_{13}\mathcal{O}_{13}\mathcal{O}_6$	$\mathbf{S}_3=\langle -(t+1),\ (t+3)/(2t+3)\rangle$

$\mathcal{N}_{22}$	{∞,0,1,−1,2,−2,6,−4,5}	$\mathcal{O}_8\mathcal{O}_{10}\mathcal{O}_9\mathcal{O}_{23}\mathcal{O}_9\mathcal{O}_{10}\mathcal{O}_{16}\mathcal{O}_{23}\mathcal{O}_8$	$\mathbf{Z}_2$=⟨(t+2)/(4t−1)⟩
$\mathcal{N}_{23}$	{∞,0,1,−1,2,−2,6,−4,−6}	$\mathcal{O}_8\mathcal{O}_{12}\mathcal{O}_9\mathcal{O}_{16}\mathcal{O}_{14}\mathcal{O}_{14}\mathcal{O}_{15}\mathcal{O}_{22}\mathcal{O}_9$	I=⟨t⟩
$\mathcal{N}_{24}$	{∞,0,1,−1,2,−2,6,4,5}	$\mathcal{O}_9\mathcal{O}_{10}\mathcal{O}_{13}\mathcal{O}_{28}\mathcal{O}_{21}\mathcal{O}_{13}\mathcal{O}_{17}\mathcal{O}_{23}\mathcal{O}_{21}$	I=⟨t⟩
$\mathcal{N}_{25}$	{∞,0,1,−1,2,−2,6,4,−5}	$\mathcal{O}_9\mathcal{O}_{11}\mathcal{O}_9\mathcal{O}_{11}\mathcal{O}_{20}\mathcal{O}_9\mathcal{O}_{11}\mathcal{O}_{20}\mathcal{O}_{20}$	$\mathbf{Z}_3$=⟨4/(2−t)⟩
$\mathcal{N}_{26}$	{∞,0,1,−1,2,−2,6,4,−8}	$\mathcal{O}_9\mathcal{O}_{10}\mathcal{O}_{14}\mathcal{O}_{23}\mathcal{O}_{17}\mathcal{O}_{14}\mathcal{O}_{18}\mathcal{O}_{22}\mathcal{O}_{25}$	I=⟨t⟩
$\mathcal{N}_{27}$	{∞,0,1,−1,2,−2,6,5,−5}	$\mathcal{O}_{10}\mathcal{O}_{11}\mathcal{O}_{17}\mathcal{O}_{26}\mathcal{O}_{23}\mathcal{O}_{28}\mathcal{O}_{26}\mathcal{O}_{27}\mathcal{O}_{20}$	I=⟨t⟩
$\mathcal{N}_{28}$	{∞,0,1,−1,2,−2,6,5,−8}	$\mathcal{O}_{10}\mathcal{O}_{10}\mathcal{O}_{18}\mathcal{O}_{21}\mathcal{O}_{28}\mathcal{O}_{28}\mathcal{O}_{21}\mathcal{O}_{25}\mathcal{O}_{25}$	$\mathbf{Z}_2$=⟨1/(9t)⟩
$\mathcal{N}_{29}$	{∞,0,1,−1,2,−2,6,−5,8}	$\mathcal{O}_{11}\mathcal{O}_{11}\mathcal{O}_{18}\mathcal{O}_{29}\mathcal{O}_{21}\mathcal{O}_{21}\mathcal{O}_{29}\mathcal{O}_{26}\mathcal{O}_{26}$	$\mathbf{Z}_2$=⟨1/(9t)⟩
$\mathcal{N}_{30}$	{∞,0,1,−1,2,−2,−4,−5,8}	$\mathcal{O}_{13}\mathcal{O}_{14}\mathcal{O}_{18}\mathcal{O}_{19}\mathcal{O}_{21}\mathcal{O}_{18}\mathcal{O}_{13}\mathcal{O}_{17}\mathcal{O}_{28}$	I=⟨t⟩
$\mathcal{N}_{31}$	{∞,0,1,−1,2,−3,9,4,−4}	$\mathcal{O}_{20}\mathcal{O}_{21}\mathcal{O}_{27}\mathcal{O}_{22}\mathcal{O}_{23}\mathcal{O}_{21}\mathcal{O}_{22}\mathcal{O}_{23}\mathcal{O}_{20}$	I=⟨t⟩
$\mathcal{N}_{32}$	{∞,0,1,−1,2,−3,9,4,−6}	$\mathcal{O}_{20}\mathcal{O}_{24}\mathcal{O}_{26}\mathcal{O}_{23}\mathcal{O}_{25}\mathcal{O}_{23}\mathcal{O}_{20}\mathcal{O}_{25}\mathcal{O}_{26}$	$\mathbf{Z}_2$=⟨(2−t)/(1+2t)⟩
$\mathcal{N}_{33}$	{∞,0,1,−1,2,−3,4,5,−4}	$\mathcal{O}_{27}\mathcal{O}_{27}\mathcal{O}_{27}\mathcal{O}_{27}\mathcal{O}_{27}\mathcal{O}_{27}\mathcal{O}_{27}\mathcal{O}_{27}\mathcal{O}_{27}$	$\mathbf{D}_9$=⟨(4+t)/(2−t), (1−t)⟩

2.11 The Decads

The 33 projectivities of the nonads $G_{\mathcal{N}_i}$, $i = 1,\dots,33$, split $\mathcal{N}_i^c$ into a number of orbits. The decads are constructed by adding one point from each orbit to the corresponding nonad. All orbits are listed in Table 2.19.

TABLE 2.19: Partition of $PG(1,19)$ by the projective group of nonads

$\mathcal{N}_i$	Partition of $\mathcal{N}_i^c$
$\mathcal{N}_1$	(1) {4, −5} (2) {5, −6} (3) {6, −7} (4) {7, −8} (5) {8, −9} (6) {9}
$\mathcal{N}_2$	$G_{\mathcal{N}_2}$ splits $\mathcal{N}_2^c$ into 11 orbits of single points
$\mathcal{N}_3$	$G_{\mathcal{N}_3}$ splits $\mathcal{N}_3^c$ into 11 orbits of single points
$\mathcal{N}_4$	$G_{\mathcal{N}_4}$ splits $\mathcal{N}_4^c$ into 11 orbits of single points
$\mathcal{N}_5$	(1) {3, 5} (2) {4, −8} (3) {−5, 9} (4) {6, 8} (5) {−6, −7} (6) {−9}
$\mathcal{N}_6$	$G_{\mathcal{N}_6}$ splits $\mathcal{N}_6^c$ into 11 orbits of single points

$\mathcal{N}_7$	$G_{\mathcal{N}_7}$ splits $\mathcal{N}_7^c$ into 11 orbits of single points
$\mathcal{N}_8$	(1) $\{3,-4\}$ (2) $\{5,-6\}$ (3) $\{6,-7\}$ (4) $\{7,-8\}$ (5) $\{8,-9\}$ (6) $\{9\}$
$\mathcal{N}_9$	$G_{\mathcal{N}_9}$ splits $\mathcal{N}_9^c$ into 11 orbits of single points
$\mathcal{N}_{10}$	$G_{\mathcal{N}_{10}}$ splits $\mathcal{N}_{10}^c$ into 11 orbits of single points
$\mathcal{N}_{11}$	(1) $\{3,9\}$ (2) $\{-4,-6\}$ (3) $\{5\}$ (4) $\{-5,-8\}$ (5) $\{6,8\}$ (6) $\{7,-9\}$
$\mathcal{N}_{12}$	$G_{\mathcal{N}_{12}}$ splits $\mathcal{N}_{12}^c$ into 11 orbits of single points
$\mathcal{N}_{13}$	$G_{\mathcal{N}_{13}}$ splits $\mathcal{N}_{13}^c$ into 11 orbits of single points
$\mathcal{N}_{14}$	(1) $\{3\}$ (2) $\{4,-4\}$ (3) $\{-5,-6\}$ (4) $\{7,8\}$ (5) $\{-7,-9\}$ (6) $\{-8,9\}$
$\mathcal{N}_{15}$	(1) $\{3,-4\}$ (2) $\{4,-5\}$ (3) $\{6,-7\}$ (4) $\{7,-8\}$ (5) $\{8,-9\}$ (6) $\{9\}$
$\mathcal{N}_{16}$	(1) $\{3,6\}$ (2) $\{4\}$ (3) $\{-4,-5\}$ (4) $\{-6,-8\}$ (5) $\{-7,-9\}$ (6) $\{8,9\}$
$\mathcal{N}_{17}$	$G_{\mathcal{N}_{17}}$ splits $\mathcal{N}_{17}^c$ into 11 orbits of single points
$\mathcal{N}_{18}$	$G_{\mathcal{N}_{18}}$ splits $\mathcal{N}_{18}^c$ into 11 orbits of single points
$\mathcal{N}_{19}$	(1) $\{3,8\}$ (2) $\{4,-9\}$ (3) $\{-4,6\}$ (4) $\{-5,9\}$ (5) $\{-6,7\}$ (6) $\{-7\}$
$\mathcal{N}_{20}$	$G_{\mathcal{N}_{20}}$ splits $\mathcal{N}_{20}^c$ into 11 orbits of single points
$\mathcal{N}_{21}$	(1) $\{3,4,-4,-5,7,-8\}$ (2) $\{5,-6\}$ (3) $\{6,-7,9\}$
$\mathcal{N}_{22}$	(1) $\{3,-3\}$ (2) $\{4,8\}$ (3) $\{-5,-8\}$ (4) $\{-6,7\}$ (5) $\{-7,-9\}$ (6) $\{9\}$
$\mathcal{N}_{23}$	$G_{\mathcal{N}_{23}}$ splits $\mathcal{N}_{23}^c$ into 11 orbits of single points
$\mathcal{N}_{24}$	$G_{\mathcal{N}_{24}}$ splits $\mathcal{N}_{24}^c$ into 11 orbits of single points
$\mathcal{N}_{25}$	(1) $\{3,-4,7\}$ (2) $\{-3\}$ (3) $\{5\}$ (4) $\{-6,9,-9\}$ (5) $\{-7,8,-8\}$
$\mathcal{N}_{26}$	$G_{\mathcal{N}_{26}}$ splits $\mathcal{N}_{26}^c$ into 11 orbits of single points
$\mathcal{N}_{27}$	$G_{\mathcal{N}_{27}}$ splits $\mathcal{N}_{27}^c$ into 11 orbits of single points
$\mathcal{N}_{28}$	(1) $\{3,-7\}$ (2) $\{-3,7\}$ (3) $\{4,9\}$ (4) $\{-4,-9\}$ (5) $\{-5,8\}$ (6) $\{-6\}$
$\mathcal{N}_{29}$	(1) $\{3,-7\}$ (2) $\{-3,7\}$ (3) $\{4,9\}$ (4) $\{-4,-9\}$ (5) $\{5,-8\}$ (6) $\{-6\}$
$\mathcal{N}_{30}$	$G_{\mathcal{N}_{30}}$ splits $\mathcal{N}_{30}^c$ into 11 orbits of single points
$\mathcal{N}_{31}$	$G_{\mathcal{N}_{31}}$ splits $\mathcal{N}_{31}^c$ into 11 orbits of single points
$\mathcal{N}_{32}$	(1) $\{-2,5\}$ (2) $\{3,8\}$ (3) $\{-4,-9\}$ (4) $\{-5\}$ (5) $\{6,7\}$ (6) $\{-7,-8\}$
$\mathcal{N}_{33}$	(1) $\{-2,3,-5,6,-6,7,-7,8,-9\}$ (2) $\{-8,9\}$

There are 280 different orbits; therefore 280 decads can be constructed in $PG(1,19)$. The projectively distinct decads with their types of nonads and the stabilizer groups are given in the following theorem.

Theorem 2.11.1. *On* $PG(1,19)$*, there are* 44 *projectively distinct decads, as summarized in Table 2.20.*

Table 2.20: Inequivalent decads on $PG(1,19)$

Type	The decad	Type of Nonads	Stabilizer
$\mathcal{D}_1$	$\{\infty,0,1,-1,2,-2,-3,-4,3,4\}$	$\mathcal{N}_1\mathcal{N}_2\mathcal{N}_1\mathcal{N}_2\mathcal{N}_7\mathcal{N}_7\mathcal{N}_{24}\mathcal{N}_{24}\mathcal{N}_{28}\mathcal{N}_{29}$	$\mathbf{Z}_2=\langle -t\rangle$
$\mathcal{D}_2$	$\{\infty,0,1,-1,2,-2,-3,-4,3,5\}$	$\mathcal{N}_1\mathcal{N}_3\mathcal{N}_2\mathcal{N}_8\mathcal{N}_2\mathcal{N}_8\mathcal{N}_3\mathcal{N}_{10}\mathcal{N}_1\mathcal{N}_{10}$	$\mathbf{Z}_2=\langle (t-5)/(t-1)\rangle$
$\mathcal{D}_3$	$\{\infty,0,1,-1,2,-2,-3,-4,3,6\}$	$\mathcal{N}_1\mathcal{N}_4\mathcal{N}_3\mathcal{N}_9\mathcal{N}_{24}\mathcal{N}_{10}\mathcal{N}_{27}\mathcal{N}_{30}\mathcal{N}_{28}\mathcal{N}_{29}$	$I=\langle t\rangle$
$\mathcal{D}_4$	$\{\infty,0,1,-1,2,-2,-3,-4,3,7\}$	$\mathcal{N}_1\mathcal{N}_5\mathcal{N}_4\mathcal{N}_{11}\mathcal{N}_9\mathcal{N}_{12}\mathcal{N}_{27}\mathcal{N}_{13}\mathcal{N}_3\mathcal{N}_{16}$	$I=\langle t\rangle$
$\mathcal{D}_5$	$\{\infty,0,1,-1,2,-2,-3,-4,3,8\}$	$\mathcal{N}_1\mathcal{N}_6\mathcal{N}_5\mathcal{N}_{12}\mathcal{N}_{21}\mathcal{N}_{13}\mathcal{N}_{30}\mathcal{N}_6\mathcal{N}_{24}\mathcal{N}_{10}$	$I=\langle t\rangle$
$\mathcal{D}_6$	$\{\infty,0,1,-1,2,-2,-3,-4,3,9\}$	$\mathcal{N}_1\mathcal{N}_6\mathcal{N}_6\mathcal{N}_6\mathcal{N}_{30}\mathcal{N}_6\mathcal{N}_{30}\mathcal{N}_{30}\mathcal{N}_{30}\mathcal{N}_1$	$\mathbf{V}_4=\langle -(t+1),\ (1-t)/(1+2t)\rangle$
$\mathcal{D}_7$	$\{\infty,0,1,-1,2,-2,-3,-4,4,5\}$	$\mathcal{N}_2\mathcal{N}_3\mathcal{N}_7\mathcal{N}_2\mathcal{N}_{33}\mathcal{N}_{14}\mathcal{N}_7\mathcal{N}_3\mathcal{N}_{27}\mathcal{N}_{27}$	$\mathbf{Z}_2=\langle 4/t\rangle$
$\mathcal{D}_8$	$\{\infty,0,1,-1,2,-2,-3,-4,4,6\}$	$\mathcal{N}_2\mathcal{N}_4\mathcal{N}_8\mathcal{N}_{17}\mathcal{N}_{14}\mathcal{N}_{16}\mathcal{N}_{10}\mathcal{N}_{12}\mathcal{N}_{18}\mathcal{N}_{11}$	$I=\langle t\rangle$
$\mathcal{D}_9$	$\{\infty,0,1,-1,2,-2,-3,-4,4,-6\}$	$\mathcal{N}_2\mathcal{N}_2\mathcal{N}_9\mathcal{N}_{17}\mathcal{N}_7\mathcal{N}_9\mathcal{N}_{21}\mathcal{N}_{17}\mathcal{N}_7\mathcal{N}_{11}$	$\mathbf{Z}_2=\langle -(t+2)\rangle$
$\mathcal{D}_{10}$	$\{\infty,0,1,-1,2,-2,-3,-4,4,7\}$	$\mathcal{N}_2\mathcal{N}_5\mathcal{N}_{10}\mathcal{N}_{17}\mathcal{N}_{27}\mathcal{N}_{18}\mathcal{N}_{24}\mathcal{N}_{22}\mathcal{N}_4\mathcal{N}_{16}$	$I=\langle t\rangle$
$\mathcal{D}_{11}$	$\{\infty,0,1,-1,2,-2,-3,-4,4,-7\}$	$\mathcal{N}_2\mathcal{N}_3\mathcal{N}_{11}\mathcal{N}_{17}\mathcal{N}_{27}\mathcal{N}_{15}\mathcal{N}_9\mathcal{N}_{25}\mathcal{N}_{14}\mathcal{N}_{15}$	$I=\langle t\rangle$
$\mathcal{D}_{12}$	$\{\infty,0,1,-1,2,-2,-3,-4,4,8\}$	$\mathcal{N}_2\mathcal{N}_6\mathcal{N}_{12}\mathcal{N}_{23}\mathcal{N}_2\mathcal{N}_{20}\mathcal{N}_6\mathcal{N}_{23}\mathcal{N}_{20}\mathcal{N}_{12}$	$\mathbf{Z}_2=\langle (2-t)/(1+5t)\rangle$
$\mathcal{D}_{13}$	$\{\infty,0,1,-1,2,-2,-3,-4,4,-8\}$	$\mathcal{N}_2\mathcal{N}_4\mathcal{N}_{12}\mathcal{N}_{23}\mathcal{N}_{31}\mathcal{N}_{17}\mathcal{N}_{13}\mathcal{N}_{23}\mathcal{N}_{22}\mathcal{N}_{25}$	$I=\langle t\rangle$
$\mathcal{D}_{14}$	$\{\infty,0,1,-1,2,-2,-3,-4,4,9\}$	$\mathcal{N}_2\mathcal{N}_6\mathcal{N}_{13}\mathcal{N}_{23}\mathcal{N}_{31}\mathcal{N}_{18}\mathcal{N}_{30}\mathcal{N}_{26}\mathcal{N}_{24}\mathcal{N}_9$	$I=\langle t\rangle$
$\mathcal{D}_{15}$	$\{\infty,0,1,-1,2,-2,-3,-4,4,-9\}$	$\mathcal{N}_2\mathcal{N}_5\mathcal{N}_6\mathcal{N}_{23}\mathcal{N}_3\mathcal{N}_{19}\mathcal{N}_{30}\mathcal{N}_{18}\mathcal{N}_{26}\mathcal{N}_4$	$I=\langle t\rangle$
$\mathcal{D}_{16}$	$\{\infty,0,1,-1,2,-2,-3,-4,5,6\}$	$\mathcal{N}_3\mathcal{N}_4\mathcal{N}_{14}\mathcal{N}_{22}\mathcal{N}_7\mathcal{N}_8\mathcal{N}_7\mathcal{N}_{13}\mathcal{N}_{24}\mathcal{N}_9$	$I=\langle t\rangle$
$\mathcal{D}_{17}$	$\{\infty,0,1,-1,2,-2,-3,-4,5,-7\}$	$\mathcal{N}_3\mathcal{N}_3\mathcal{N}_3\mathcal{N}_3\mathcal{N}_{17}\mathcal{N}_{17}\mathcal{N}_{17}\mathcal{N}_{17}\mathcal{N}_{32}\mathcal{N}_{32}$	$\mathbf{V}_4=\langle -(t+2),\ (5-t)/(1+t)\rangle$
$\mathcal{D}_{18}$	$\{\infty,0,1,-1,2,-2,-3,-4,5,8\}$	$\mathcal{N}_3\mathcal{N}_6\mathcal{N}_{18}\mathcal{N}_{18}\mathcal{N}_7\mathcal{N}_3\mathcal{N}_{28}\mathcal{N}_6\mathcal{N}_{28}\mathcal{N}_7$	$\mathbf{Z}_2=\langle (t+1)/(9t-1)\rangle$
$\mathcal{D}_{19}$	$\{\infty,0,1,-1,2,-2,-3,-4,5,-8\}$	$\mathcal{N}_3\mathcal{N}_4\mathcal{N}_{19}\mathcal{N}_{26}\mathcal{N}_{31}\mathcal{N}_{14}\mathcal{N}_{32}\mathcal{N}_{20}\mathcal{N}_{28}\mathcal{N}_{27}$	$I=\langle t\rangle$

$\mathcal{D}_{20}$	{∞,0,1,−1,2,−2,−3,−4,5,9}	$\mathcal{N}_3\mathcal{N}_6\mathcal{N}_{20}\mathcal{N}_{23}\mathcal{N}_{31}\mathcal{N}_{17}\mathcal{N}_{26}\mathcal{N}_{20}\mathcal{N}_{24}\mathcal{N}_{13}$	$I=\langle t\rangle$
$\mathcal{D}_{21}$	{∞,0,1,−1,2,−2,−3,−4,6,7}	$\mathcal{N}_4\mathcal{N}_5\mathcal{N}_8\mathcal{N}_{17}\mathcal{N}_{26}\mathcal{N}_{12}\mathcal{N}_9\mathcal{N}_{23}\mathcal{N}_6\mathcal{N}_{12}$	$I=\langle t\rangle$
$\mathcal{D}_{22}$	{∞,0,1,−1,2,−2,−3,−4,6,8}	$\mathcal{N}_4\mathcal{N}_6\mathcal{N}_{11}\mathcal{N}_4\mathcal{N}_{10}\mathcal{N}_{10}\mathcal{N}_{29}\mathcal{N}_6\mathcal{N}_{29}\mathcal{N}_{11}$	$\mathbf{Z}_2=\langle -(t+1)/(5t+1)\rangle$
$\mathcal{D}_{23}$	{∞,0,1,−1,2,−2,−3,−4,6,−8}	$\mathcal{N}_4\mathcal{N}_4\mathcal{N}_4\mathcal{N}_{20}\mathcal{N}_{20}\mathcal{N}_4\mathcal{N}_{15}\mathcal{N}_{20}\mathcal{N}_{20}\mathcal{N}_{15}$	$\mathbf{V}_4=\langle -(t+2),\ (1-t)/(1+t)\rangle$
$\mathcal{D}_{24}$	{∞,0,1,−1,2,−2,−3,−4,7,−9}	$\mathcal{N}_5\mathcal{N}_5\mathcal{N}_5\mathcal{N}_5\mathcal{N}_5\mathcal{N}_5\mathcal{N}_5\mathcal{N}_5\mathcal{N}_5\mathcal{N}_5$	$\mathbf{D}_{10}=\langle 2/(t+2),\ -2/t\rangle$
$\mathcal{D}_{25}$	{∞,0,1,−1,2,−2,−3,−4,8,9}	$\mathcal{N}_6\mathcal{N}_6\mathcal{N}_{17}\mathcal{N}_{23}\mathcal{N}_{13}\mathcal{N}_{17}\mathcal{N}_{19}\mathcal{N}_{23}\mathcal{N}_{13}\mathcal{N}_8$	$\mathbf{Z}_2=\langle -(t+2)\ \rangle$
$\mathcal{D}_{26}$	{∞,0,1,−1,2,−2,−3,4,5,−6}	$\mathcal{N}_7\mathcal{N}_9\mathcal{N}_{15}\mathcal{N}_9\mathcal{N}_{27}\mathcal{N}_{27}\mathcal{N}_7\mathcal{N}_{32}\mathcal{N}_{32}\mathcal{N}_{15}$	$\mathbf{Z}_2=\langle (t-1)/(5t-1)\ \rangle$
$\mathcal{D}_{27}$	{∞,0,1,−1,2,−2,−3,4,5,7}	$\mathcal{N}_7\mathcal{N}_{10}\mathcal{N}_{16}\mathcal{N}_{10}\mathcal{N}_7\mathcal{N}_{10}\mathcal{N}_7\mathcal{N}_7\mathcal{N}_{16}\mathcal{N}_{10}$	$\mathbf{V}_4=\langle (t-4)/(4t-1),\ 2t/(t-2)\rangle$
$\mathcal{D}_{28}$	{∞,0,1,−1,2,−2,−3,4,5,8}	$\mathcal{N}_7\mathcal{N}_{12}\mathcal{N}_{18}\mathcal{N}_{30}\mathcal{N}_{27}\mathcal{N}_{26}\mathcal{N}_7\mathcal{N}_{26}\mathcal{N}_{31}\mathcal{N}_{19}$	$I=\langle t\rangle$
$\mathcal{D}_{29}$	{∞,0,1,−1,2,−2,−3,4,5,9}	$\mathcal{N}_7\mathcal{N}_{13}\mathcal{N}_{20}\mathcal{N}_{13}\mathcal{N}_{31}\mathcal{N}_{20}\mathcal{N}_7\mathcal{N}_{32}\mathcal{N}_{31}\mathcal{N}_{32}$	$\mathbf{Z}_2=\langle (t+2)/(t-1)\rangle$
$\mathcal{D}_{30}$	{∞,0,1,−1,2,−2,−3,4,−5,−7}	$\mathcal{N}_8\mathcal{N}_{11}\mathcal{N}_8\mathcal{N}_{12}\mathcal{N}_{11}\mathcal{N}_{25}\mathcal{N}_{12}\mathcal{N}_{11}\mathcal{N}_8\mathcal{N}_{12}$	$\mathbf{Z}_3=\langle (t-4)/(6t+6)\rangle$
$\mathcal{D}_{31}$	{∞,0,1,−1,2,−2,−3,4,−6,7}	$\mathcal{N}_9\mathcal{N}_{10}\mathcal{N}_{19}\mathcal{N}_{20}\mathcal{N}_{19}\mathcal{N}_{30}\mathcal{N}_{30}\mathcal{N}_{20}\mathcal{N}_9\mathcal{N}_{10}$	$\mathbf{Z}_2=\langle (t-7)/(3t-1)\rangle$
$\mathcal{D}_{32}$	{∞,0,1,−1,2,−2,−3,4,−6,8}	$\mathcal{N}_9\mathcal{N}_{12}\mathcal{N}_{18}\mathcal{N}_{20}\mathcal{N}_{29}\mathcal{N}_{26}\mathcal{N}_{10}\mathcal{N}_{25}\mathcal{N}_{32}\mathcal{N}_{16}$	$I=\langle t\rangle$
$\mathcal{D}_{33}$	{∞,0,1,−1,2,−2,−3,4,−6,9}	$\mathcal{N}_9\mathcal{N}_{13}\mathcal{N}_{20}\mathcal{N}_{22}\mathcal{N}_{32}\mathcal{N}_{26}\mathcal{N}_{24}\mathcal{N}_{31}\mathcal{N}_{28}\mathcal{N}_{27}$	$I=\langle t\rangle$
$\mathcal{D}_{34}$	{∞,0,1,−1,2,−2,−3,4,7,8}	$\mathcal{N}_{10}\mathcal{N}_{12}\mathcal{N}_{12}\mathcal{N}_{13}\mathcal{N}_{15}\mathcal{N}_{18}\mathcal{N}_{13}\mathcal{N}_{18}\mathcal{N}_{15}\mathcal{N}_{10}$	$\mathbf{Z}_2=\langle -(t+2)/(7t+1)\rangle$
$\mathcal{D}_{35}$	{∞,0,1,−1,2,−2,−3,4,7,9}	$\mathcal{N}_{10}\mathcal{N}_{13}\mathcal{N}_{10}\mathcal{N}_{13}\mathcal{N}_{27}\mathcal{N}_{30}\mathcal{N}_{30}\mathcal{N}_{27}\mathcal{N}_{29}\mathcal{N}_{29}$	$\mathbf{Z}_2=\langle -2/t\rangle$
$\mathcal{D}_{36}$	{∞,0,1,−1,2,−2,−3,5,6,−7}	$\mathcal{N}_{14}\mathcal{N}_{17}\mathcal{N}_{14}\mathcal{N}_{18}\mathcal{N}_{18}\mathcal{N}_{17}\mathcal{N}_{31}\mathcal{N}_{20}\mathcal{N}_{20}\mathcal{N}_{31}$	$\mathbf{Z}_2=\langle (1-t)/(1+t)\rangle$
$\mathcal{D}_{37}$	{∞,0,1,−1,2,−2,−3,5,−6,7}	$\mathcal{N}_{15}\mathcal{N}_{16}\mathcal{N}_{19}\mathcal{N}_{16}\mathcal{N}_{31}\mathcal{N}_{29}\mathcal{N}_{19}\mathcal{N}_{31}\mathcal{N}_{29}\mathcal{N}_{15}$	$\mathbf{Z}_2=\langle (2-t)/(1+8t)\rangle$
$\mathcal{D}_{38}$	{∞,0,1,−1,2,−2,−3,5,−7,8}	$\mathcal{N}_{17}\mathcal{N}_{18}\mathcal{N}_{17}\mathcal{N}_{18}\mathcal{N}_{22}\mathcal{N}_{23}\mathcal{N}_{22}\mathcal{N}_{23}\mathcal{N}_{20}\mathcal{N}_{20}$	$\mathbf{Z}_2=\langle 2/t\rangle$
$\mathcal{D}_{39}$	{∞,0,1,−1,2,−2,−3,5,8,−9}	$\mathcal{N}_{18}\mathcal{N}_{18}\mathcal{N}_{21}\mathcal{N}_{30}\mathcal{N}_{24}\mathcal{N}_{24}\mathcal{N}_{24}\mathcal{N}_{30}\mathcal{N}_{30}\mathcal{N}_{18}$	$\mathbf{Z}_3=\langle (t+3)/(2t+3)\rangle$
$\mathcal{D}_{40}$	{∞,0,1,−1,2,−2,6,−4,5,9}	$\mathcal{N}_{22}\mathcal{N}_{23}\mathcal{N}_{26}\mathcal{N}_{23}\mathcal{N}_{26}\mathcal{N}_{23}\mathcal{N}_{26}\mathcal{N}_{22}\mathcal{N}_{26}\mathcal{N}_{23}$	$\mathbf{V}_4=\langle (t+2)/(4t-1),\ -(t+4)/(9t+1)\rangle$
$\mathcal{D}_{41}$	{∞,0,1,−1,2,−2,6,−4,−6,9}	$\mathcal{N}_{23}\mathcal{N}_{23}\mathcal{N}_{23}\mathcal{N}_{23}\mathcal{N}_{23}\mathcal{N}_{23}\mathcal{N}_{23}\mathcal{N}_{23}\mathcal{N}_{23}\mathcal{N}_{23}$	$\mathbf{D}_5=\langle (2+t)/(9-t),\ (t-6)/(t-1)\rangle$
$\mathcal{D}_{42}$	{∞,0,1,−1,2,−2,6,4,5,−5}	$\mathcal{N}_{24}\mathcal{N}_{25}\mathcal{N}_{27}\mathcal{N}_{24}\mathcal{N}_{27}\mathcal{N}_{31}\mathcal{N}_{24}\mathcal{N}_{27}\mathcal{N}_{31}\mathcal{N}_{31}$	$\mathbf{Z}_3=\langle 4/(2-t)\rangle$
$\mathcal{D}_{43}$	{∞,0,1,−1,2,−2,6,4,5,−8}	$\mathcal{N}_{24}\mathcal{N}_{26}\mathcal{N}_{28}\mathcal{N}_{30}\mathcal{N}_{24}\mathcal{N}_{30}\mathcal{N}_{30}\mathcal{N}_{30}\mathcal{N}_{26}\mathcal{N}_{28}$	$\mathbf{Z}_2=\langle (t-5)/(5t-1)\rangle$
$\mathcal{D}_{44}$	{∞,0,1,−1,2,−3,9,4,−4,5}	$\mathcal{N}_{31}\mathcal{N}_{31}\mathcal{N}_{31}\mathcal{N}_{33}\mathcal{N}_{31}\mathcal{N}_{31}\mathcal{N}_{31}\mathcal{N}_{31}\mathcal{N}_{31}\mathcal{N}_{31}$	$\mathbf{Z}_9=\langle (4+t)/(2-t)\rangle$

2.12 The partition of $PG(1,19)$

Each decad $\mathcal{D}_i$, and its complement $\mathcal{D}_i^c$ partition $PG(1,19)$. The stabilizer $G_{\mathcal{D}_i}$ of the decad $\mathcal{D}_i$ also fixes the complement $\mathcal{D}_i^c$. Since there are 44 projectively distinct decads in $PG(1,19)$, so the question arises: *Are $\mathcal{D}_i^c$ and $\mathcal{D}_i$ equivalent? What is the group of projectivities of $PG(1,19)$ of the partition?*

In Table 2.21, all $\mathcal{D}_i^c$ are listed with their types of the nonads. Also the projective equation from each $\mathcal{D}_j$ to its equivalent decad $\mathcal{D}_i^c$ is given.

TABLE 2.21: Classification of the complements of the decads in $PG(1,19)$

$\mathcal{D}_i^c$	Types of nonads	$\mathcal{D}_j$	Projective equation
$\mathcal{D}_1^c$	$\mathcal{N}_{30}\mathcal{N}_{30}\mathcal{N}_{13}\mathcal{N}_{13}\mathcal{N}_{27}\mathcal{N}_{27}\mathcal{N}_{10}\mathcal{N}_{10}\mathcal{N}_{29}\mathcal{N}_{29}$	$\mathcal{D}_{35}$	$(t+6)/(4t-5)$
$\mathcal{D}_2^c$	$\mathcal{N}_{6}\mathcal{N}_{6}\mathcal{N}_{4}\mathcal{N}_{10}\mathcal{N}_{4}\mathcal{N}_{11}\mathcal{N}_{11}\mathcal{N}_{29}\mathcal{N}_{10}\mathcal{N}_{29}$	$\mathcal{D}_{22}$	$(t+5)/(3t+4)$
$\mathcal{D}_3^c$	$\mathcal{N}_{24}\mathcal{N}_{30}\mathcal{N}_{3}\mathcal{N}_{1}\mathcal{N}_{28}\mathcal{N}_{9}\mathcal{N}_{4}\mathcal{N}_{27}\mathcal{N}_{29}\mathcal{N}_{10}$	$\mathcal{D}_{3}$	$(t+7)/(4t-1)$
$\mathcal{D}_4^c$	$\mathcal{N}_{5}\mathcal{N}_{1}\mathcal{N}_{16}\mathcal{N}_{12}\mathcal{N}_{3}\mathcal{N}_{4}\mathcal{N}_{9}\mathcal{N}_{13}\mathcal{N}_{11}\mathcal{N}_{27}$	$\mathcal{D}_{4}$	$(t+7)/(7t-1)$
$\mathcal{D}_5^c$	$\mathcal{N}_{21}\mathcal{N}_{6}\mathcal{N}_{5}\mathcal{N}_{24}\mathcal{N}_{12}\mathcal{N}_{6}\mathcal{N}_{1}\mathcal{N}_{30}\mathcal{N}_{10}\mathcal{N}_{13}$	$\mathcal{D}_{5}$	$(t+7)/(4t-1)$
$\mathcal{D}_6^c$	$\mathcal{N}_{6}\mathcal{N}_{1}\mathcal{N}_{6}\mathcal{N}_{30}\mathcal{N}_{1}\mathcal{N}_{6}\mathcal{N}_{30}\mathcal{N}_{30}\mathcal{N}_{6}\mathcal{N}_{30}$	$\mathcal{D}_{6}$	$(t+7)/(7t-1)$
$\mathcal{D}_7^c$	$\mathcal{N}_{19}\mathcal{N}_{31}\mathcal{N}_{31}\mathcal{N}_{19}\mathcal{N}_{15}\mathcal{N}_{16}\mathcal{N}_{15}\mathcal{N}_{16}\mathcal{N}_{29}\mathcal{N}_{29}$	$\mathcal{D}_{37}$	$(t+7)/(3t-4)$
$\mathcal{D}_8^c$	$\mathcal{N}_{2}\mathcal{N}_{18}\mathcal{N}_{17}\mathcal{N}_{8}\mathcal{N}_{14}\mathcal{N}_{4}\mathcal{N}_{12}\mathcal{N}_{11}\mathcal{N}_{16}\mathcal{N}_{10}$	$\mathcal{D}_{8}$	$(9-t)/(1+4t)$
$\mathcal{D}_9^c$	$\mathcal{N}_{18}\mathcal{N}_{13}\mathcal{N}_{12}\mathcal{N}_{13}\mathcal{N}_{15}\mathcal{N}_{18}\mathcal{N}_{12}\mathcal{N}_{10}\mathcal{N}_{15}\mathcal{N}_{10}$	$\mathcal{D}_{34}$	$(5-t)/(2+6t)$
$\mathcal{D}_{10}^c$	$\mathcal{N}_{5}\mathcal{N}_{24}\mathcal{N}_{22}\mathcal{N}_{17}\mathcal{N}_{2}\mathcal{N}_{18}\mathcal{N}_{4}\mathcal{N}_{10}\mathcal{N}_{16}\mathcal{N}_{27}$	$\mathcal{D}_{10}$	$(t-6)/(4t-1)$
$\mathcal{D}_{11}^c$	$\mathcal{N}_{3}\mathcal{N}_{9}\mathcal{N}_{25}\mathcal{N}_{17}\mathcal{N}_{2}\mathcal{N}_{15}\mathcal{N}_{14}\mathcal{N}_{11}\mathcal{N}_{15}\mathcal{N}_{27}$	$\mathcal{D}_{11}$	$(t-1)/(4t-1)$
$\mathcal{D}_{12}^c$	$\mathcal{N}_{6}\mathcal{N}_{23}\mathcal{N}_{23}\mathcal{N}_{17}\mathcal{N}_{17}\mathcal{N}_{13}\mathcal{N}_{8}\mathcal{N}_{6}\mathcal{N}_{19}\mathcal{N}_{13}$	$\mathcal{D}_{25}$	$(7+t)/(2-3t)$
$\mathcal{D}_{13}^c$	$\mathcal{N}_{23}\mathcal{N}_{2}\mathcal{N}_{23}\mathcal{N}_{25}\mathcal{N}_{22}\mathcal{N}_{12}\mathcal{N}_{17}\mathcal{N}_{4}\mathcal{N}_{31}\mathcal{N}_{13}$	$\mathcal{D}_{13}$	$(t-1)/(8t-1)$
$\mathcal{D}_{14}^c$	$\mathcal{N}_{26}\mathcal{N}_{2}\mathcal{N}_{23}\mathcal{N}_{9}\mathcal{N}_{24}\mathcal{N}_{13}\mathcal{N}_{18}\mathcal{N}_{6}\mathcal{N}_{31}\mathcal{N}_{30}$	$\mathcal{D}_{14}$	$(t-1)/(8t-1)$
$\mathcal{D}_{15}^c$	$\mathcal{N}_{26}\mathcal{N}_{23}\mathcal{N}_{6}\mathcal{N}_{3}\mathcal{N}_{5}\mathcal{N}_{18}\mathcal{N}_{2}\mathcal{N}_{4}\mathcal{N}_{19}\mathcal{N}_{30}$	$\mathcal{D}_{15}$	$(9-t)/(1+4t)$
$\mathcal{D}_{16}^c$	$\mathcal{N}_{18}\mathcal{N}_{26}\mathcal{N}_{32}\mathcal{N}_{10}\mathcal{N}_{20}\mathcal{N}_{12}\mathcal{N}_{25}\mathcal{N}_{9}\mathcal{N}_{16}\mathcal{N}_{29}$	$\mathcal{D}_{32}$	$(t+9)/(5t+6)$
$\mathcal{D}_{17}^c$	$\mathcal{N}_{4}\mathcal{N}_{20}\mathcal{N}_{20}\mathcal{N}_{20}\mathcal{N}_{4}\mathcal{N}_{15}\mathcal{N}_{20}\mathcal{N}_{4}\mathcal{N}_{15}\mathcal{N}_{4}$	$\mathcal{D}_{23}$	$(t-8)/(5t-3)$

$\mathcal{D}_{18}^c$	$\mathcal{N}_{30}\mathcal{N}_{20}\mathcal{N}_{19}\mathcal{N}_{20}\mathcal{N}_{9}\mathcal{N}_{9}\mathcal{N}_{10}\mathcal{N}_{30}\mathcal{N}_{19}\mathcal{N}_{10}$	$\mathcal{D}_{31}$	$(3-t)/(9+6t)$
$\mathcal{D}_{19}^c$	$\mathcal{N}_{26}\mathcal{N}_{3}\mathcal{N}_{19}\mathcal{N}_{20}\mathcal{N}_{27}\mathcal{N}_{14}\mathcal{N}_{32}\mathcal{N}_{28}\mathcal{N}_{31}\mathcal{N}_{4}$	$\mathcal{D}_{19}$	$-(t+5)/(8t+1)$
$\mathcal{D}_{20}^c$	$\mathcal{N}_{23}\mathcal{N}_{3}\mathcal{N}_{20}\mathcal{N}_{20}\mathcal{N}_{13}\mathcal{N}_{17}\mathcal{N}_{26}\mathcal{N}_{24}\mathcal{N}_{31}\mathcal{N}_{6}$	$\mathcal{D}_{20}$	$-(t+5)/(8t+1)$
$\mathcal{D}_{21}^c$	$\mathcal{N}_{5}\mathcal{N}_{6}\mathcal{N}_{17}\mathcal{N}_{8}\mathcal{N}_{26}\mathcal{N}_{23}\mathcal{N}_{12}\mathcal{N}_{12}\mathcal{N}_{4}\mathcal{N}_{9}$	$\mathcal{D}_{21}$	$(9-t)/(1+4t)$
$\mathcal{D}_{22}^c$	$\mathcal{N}_{10}\mathcal{N}_{2}\mathcal{N}_{3}\mathcal{N}_{3}\mathcal{N}_{8}\mathcal{N}_{2}\mathcal{N}_{1}\mathcal{N}_{10}\mathcal{N}_{8}\mathcal{N}_{1}$	$\mathcal{D}_{2}$	$(t-6)/(4t+5)$
$\mathcal{D}_{23}^c$	$\mathcal{N}_{17}\mathcal{N}_{3}\mathcal{N}_{3}\mathcal{N}_{32}\mathcal{N}_{17}\mathcal{N}_{17}\mathcal{N}_{3}\mathcal{N}_{32}\mathcal{N}_{17}\mathcal{N}_{3}$	$\mathcal{D}_{17}$	$(t-9)/(8t+6)$
$\mathcal{D}_{24}^c$	$\mathcal{N}_{5}\mathcal{N}_{5}\mathcal{N}_{5}\mathcal{N}_{5}\mathcal{N}_{5}\mathcal{N}_{5}\mathcal{N}_{5}\mathcal{N}_{5}\mathcal{N}_{5}\mathcal{N}_{5}$	$\mathcal{D}_{24}$	$(4-t)/(3+2t)$
$\mathcal{D}_{25}^c$	$\mathcal{N}_{12}\mathcal{N}_{2}\mathcal{N}_{20}\mathcal{N}_{12}\mathcal{N}_{23}\mathcal{N}_{2}\mathcal{N}_{6}\mathcal{N}_{20}\mathcal{N}_{23}\mathcal{N}_{6}$	$\mathcal{D}_{12}$	$(t-3)/(2t+7)$
$\mathcal{D}_{26}^c$	$\mathcal{N}_{9}\mathcal{N}_{32}\mathcal{N}_{27}\mathcal{N}_{7}\mathcal{N}_{15}\mathcal{N}_{7}\mathcal{N}_{9}\mathcal{N}_{32}\mathcal{N}_{15}\mathcal{N}_{27}$	$\mathcal{D}_{26}$	$(t+5)/(8t-1)$
$\mathcal{D}_{27}^c$	$\mathcal{N}_{10}\mathcal{N}_{7}\mathcal{N}_{7}\mathcal{N}_{7}\mathcal{N}_{10}\mathcal{N}_{7}\mathcal{N}_{10}\mathcal{N}_{16}\mathcal{N}_{16}\mathcal{N}_{10}$	$\mathcal{D}_{27}$	$(t+5)/(2t-6)$
$\mathcal{D}_{28}^c$	$\mathcal{N}_{30}\mathcal{N}_{31}\mathcal{N}_{26}\mathcal{N}_{18}\mathcal{N}_{7}\mathcal{N}_{7}\mathcal{N}_{12}\mathcal{N}_{26}\mathcal{N}_{19}\mathcal{N}_{27}$	$\mathcal{D}_{28}$	$(9-t)/(1+5t)$
$\mathcal{D}_{29}^c$	$\mathcal{N}_{20}\mathcal{N}_{7}\mathcal{N}_{31}\mathcal{N}_{13}\mathcal{N}_{7}\mathcal{N}_{32}\mathcal{N}_{32}\mathcal{N}_{20}\mathcal{N}_{31}\mathcal{N}_{13}$	$\mathcal{D}_{29}$	$(t-8)/(3t-1)$
$\mathcal{D}_{30}^c$	$\mathcal{N}_{8}\mathcal{N}_{12}\mathcal{N}_{11}\mathcal{N}_{12}\mathcal{N}_{8}\mathcal{N}_{25}\mathcal{N}_{8}\mathcal{N}_{12}\mathcal{N}_{11}\mathcal{N}_{11}$	$\mathcal{D}_{30}$	$(t-6)/(4t-1)$
$\mathcal{D}_{31}^c$	$\mathcal{N}_{6}\mathcal{N}_{28}\mathcal{N}_{3}\mathcal{N}_{7}\mathcal{N}_{3}\mathcal{N}_{6}\mathcal{N}_{18}\mathcal{N}_{7}\mathcal{N}_{18}\mathcal{N}_{28}$	$\mathcal{D}_{18}$	$(t-7)/(4t-5)$
$\mathcal{D}_{32}^c$	$\mathcal{N}_{24}\mathcal{N}_{22}\mathcal{N}_{4}\mathcal{N}_{14}\mathcal{N}_{7}\mathcal{N}_{8}\mathcal{N}_{9}\mathcal{N}_{7}\mathcal{N}_{13}\mathcal{N}_{3}$	$\mathcal{D}_{16}$	$-(t+8)/(4t+3)$
$\mathcal{D}_{33}^c$	$\mathcal{N}_{31}\mathcal{N}_{9}\mathcal{N}_{22}\mathcal{N}_{27}\mathcal{N}_{28}\mathcal{N}_{26}\mathcal{N}_{20}\mathcal{N}_{32}\mathcal{N}_{13}\mathcal{N}_{24}$	$\mathcal{D}_{33}$	$(t-7)/(8t-1)$
$\mathcal{D}_{34}^c$	$\mathcal{N}_{21}\mathcal{N}_{2}\mathcal{N}_{9}\mathcal{N}_{2}\mathcal{N}_{7}\mathcal{N}_{11}\mathcal{N}_{17}\mathcal{N}_{7}\mathcal{N}_{17}\mathcal{N}_{9}$	$\mathcal{D}_{9}$	$(7+t)/(9-3t)$
$\mathcal{D}_{35}^c$	$\mathcal{N}_{24}\mathcal{N}_{7}\mathcal{N}_{2}\mathcal{N}_{1}\mathcal{N}_{28}\mathcal{N}_{29}\mathcal{N}_{2}\mathcal{N}_{7}\mathcal{N}_{24}\mathcal{N}_{1}$	$\mathcal{D}_{1}$	$-(t+5)/(3t+4)$
$\mathcal{D}_{36}^c$	$\mathcal{N}_{14}\mathcal{N}_{31}\mathcal{N}_{20}\mathcal{N}_{17}\mathcal{N}_{18}\mathcal{N}_{14}\mathcal{N}_{17}\mathcal{N}_{20}\mathcal{N}_{18}\mathcal{N}_{31}$	$\mathcal{D}_{36}$	$-(t+4)/(2t+1)$
$\mathcal{D}_{37}^c$	$\mathcal{N}_{2}\mathcal{N}_{3}\mathcal{N}_{33}\mathcal{N}_{7}\mathcal{N}_{27}\mathcal{N}_{2}\mathcal{N}_{27}\mathcal{N}_{3}\mathcal{N}_{7}\mathcal{N}_{14}$	$\mathcal{D}_{7}$	$(t+5)/(8t-1)$
$\mathcal{D}_{38}^c$	$\mathcal{N}_{18}\mathcal{N}_{20}\mathcal{N}_{23}\mathcal{N}_{22}\mathcal{N}_{17}\mathcal{N}_{17}\mathcal{N}_{23}\mathcal{N}_{20}\mathcal{N}_{18}\mathcal{N}_{22}$	$\mathcal{D}_{38}$	$-(t+4)/(2t+1)$
$\mathcal{D}_{39}^c$	$\mathcal{N}_{18}\mathcal{N}_{30}\mathcal{N}_{18}\mathcal{N}_{24}\mathcal{N}_{21}\mathcal{N}_{18}\mathcal{N}_{24}\mathcal{N}_{30}\mathcal{N}_{30}\mathcal{N}_{24}$	$\mathcal{D}_{39}$	$(t+8)/(8t-1)$
$\mathcal{D}_{40}^c$	$\mathcal{N}_{23}\mathcal{N}_{26}\mathcal{N}_{22}\mathcal{N}_{23}\mathcal{N}_{23}\mathcal{N}_{23}\mathcal{N}_{26}\mathcal{N}_{22}\mathcal{N}_{26}\mathcal{N}_{26}$	$\mathcal{D}_{40}$	$(t+8)/(8t-1)$
$\mathcal{D}_{41}^c$	$\mathcal{N}_{23}\mathcal{N}_{23}\mathcal{N}_{23}\mathcal{N}_{23}\mathcal{N}_{23}\mathcal{N}_{23}\mathcal{N}_{23}\mathcal{N}_{23}\mathcal{N}_{23}\mathcal{N}_{23}$	$\mathcal{D}_{41}$	$(t-8)/(4t+7)$
$\mathcal{D}_{42}^c$	$\mathcal{N}_{31}\mathcal{N}_{31}\mathcal{N}_{24}\mathcal{N}_{24}\mathcal{N}_{24}\mathcal{N}_{27}\mathcal{N}_{31}\mathcal{N}_{27}\mathcal{N}_{25}\mathcal{N}_{27}$	$\mathcal{D}_{42}$	$(t+6)/(2t-1)$
$\mathcal{D}_{43}^c$	$\mathcal{N}_{26}\mathcal{N}_{28}\mathcal{N}_{30}\mathcal{N}_{30}\mathcal{N}_{28}\mathcal{N}_{30}\mathcal{N}_{24}\mathcal{N}_{24}\mathcal{N}_{26}\mathcal{N}_{30}$	$\mathcal{D}_{43}$	$-(t+9)/(2t+1)$
$\mathcal{D}_{44}^c$	$\mathcal{N}_{31}\mathcal{N}_{33}\mathcal{N}_{31}\mathcal{N}_{31}\mathcal{N}_{31}\mathcal{N}_{31}\mathcal{N}_{31}\mathcal{N}_{31}\mathcal{N}_{31}\mathcal{N}_{31}$	$\mathcal{D}_{44}$	$(t-7)/(8t-1)$

Amongst the 44 decads $\mathcal{D}_i$ there are 16 of them which are not equivalent to their complements as shown in Table 2.21.

Theorem 2.12.1. *The projective line* $PG(1,19)$ *has*

(i) 28 *projectively distinct partitions into two equivalent decads;*

(ii) 16 *projectively distinct partitions into two inequivalent decads.*

They are given in Table 2.22 and Table 2.23 with their stabilizer groups in $PGL(2,19)$ *and the number of partitions of that type.*

TABLE 2.22: Partition of $PG(1,19)$ into two inequivalent decads

$\{\mathcal{D}_i;\mathcal{D}_i^c\}$	Stabilizer of the partition	Number
$\{\mathcal{D}_1;\mathcal{D}_1^c\}$	$\mathbf{Z}_2=\langle -t\rangle$	3420
$\{\mathcal{D}_2;\mathcal{D}_2^c\}$	$\mathbf{Z}_2=\langle (t-5)/(t-1)\rangle$	3420
$\{\mathcal{D}_7;\mathcal{D}_7^c\}$	$\mathbf{Z}_2=\langle 4/t\rangle$	3420
$\{\mathcal{D}_9;\mathcal{D}_9^c\}$	$\mathbf{Z}_2=\langle -(t+2)\rangle$	3420
$\{\mathcal{D}_{12};\mathcal{D}_{12}^c\}$	$\mathbf{Z}_2=\langle (2-t)/(1+5t)\rangle$	3420
$\{\mathcal{D}_{16};\mathcal{D}_{16}^c\}$	$I=\langle t\rangle$	6840
$\{\mathcal{D}_{17};\mathcal{D}_{17}^c\}$	$\mathbf{V}_4=\langle -(t+2),\ (5-t)/(1+t)\rangle$	1710
$\{\mathcal{D}_{18};\mathcal{D}_{18}^c\}$	$\mathbf{Z}_2=\langle (t+1)/(9t-1)\rangle$	3420
$\{\mathcal{D}_{22};\mathcal{D}_{22}^c\}$	$\mathbf{Z}_2=\langle -(t+1)/(5t+1)\rangle$	3420
$\{\mathcal{D}_{23};\mathcal{D}_{23}^c\}$	$\mathbf{V}_4=\langle -(t+2),\ (1-t)/(1+t)\rangle$	1710
$\{\mathcal{D}_{25};\mathcal{D}_{25}^c\}$	$\mathbf{Z}_2=\langle -(t+2)\rangle$	3420
$\{\mathcal{D}_{31};\mathcal{D}_{31}^c\}$	$\mathbf{Z}_2=\langle (t-7)/(3t-1)\rangle$	3420
$\{\mathcal{D}_{32};\mathcal{D}_{32}^c\}$	$I=\langle t\rangle$	6840
$\{\mathcal{D}_{34};\mathcal{D}_{34}^c\}$	$\mathbf{Z}_2=\langle -(t+2)/(7t+1)\rangle$	3420
$\{\mathcal{D}_{35};\mathcal{D}_{35}^c\}$	$\mathbf{Z}_2=\langle -2/t\rangle$	3420
$\{\mathcal{D}_{37};\mathcal{D}_{37}^c\}$	$\mathbf{Z}_2=\langle (2-t)/(1+8t)\rangle$	3420

TABLE 2.23: Partition of $PG(1,19)$ into two equivalent decads

$\{\mathcal{D}_i;\mathcal{D}_i^c\}$	Stabilizer of the partition	Number
$\{\mathcal{D}_3;\mathcal{D}_3^c\}$	$\mathbf{Z}_2 = \langle (t+7)/(4t-1) \rangle$	3420
$\{\mathcal{D}_4;\mathcal{D}_4^c\}$	$\mathbf{Z}_2 = \langle (t+7)/(7t-1) \rangle$	3420
$\{\mathcal{D}_5;\mathcal{D}_5^c\}$	$\mathbf{Z}_2 = \langle (t+7)/(4t-1) \rangle$	3420
$\{\mathcal{D}_6;\mathcal{D}_6^c\}$	$\mathbf{D}_4 = \langle (t+7)/(7t-1), -(t+1) \rangle$	855
$\{\mathcal{D}_8;\mathcal{D}_8^c\}$	$\mathbf{Z}_2 = \langle (9-t)/(1+4t) \rangle$	3420
$\{\mathcal{D}_{10};\mathcal{D}_{10}^c\}$	$\mathbf{Z}_2 = \langle (t-6)/(4t-1) \rangle$	3420
$\{\mathcal{D}_{11};\mathcal{D}_{11}^c\}$	$\mathbf{Z}_2 = \langle (t-1)/(4t-1) \rangle$	3420
$\{\mathcal{D}_{13};\mathcal{D}_{13}^c\}$	$\mathbf{Z}_2 = \langle (t-1)/(8t-1) \rangle$	3420
$\{\mathcal{D}_{14};\mathcal{D}_{14}^c\}$	$\mathbf{Z}_2 = \langle (t-1)/(8t-1) \rangle$	3420
$\{\mathcal{D}_{15};\mathcal{D}_{15}^c\}$	$\mathbf{Z}_2 = \langle (9-t)/(1+4t) \rangle$	3420
$\{\mathcal{D}_{19};\mathcal{D}_{19}^c\}$	$\mathbf{Z}_2 = \langle -(t+5)/(8t+1) \rangle$	3420
$\{\mathcal{D}_{20};\mathcal{D}_{20}^c\}$	$\mathbf{Z}_2 = \langle -(t+5)/(8t+1) \rangle$	3420
$\{\mathcal{D}_{21};\mathcal{D}_{21}^c\}$	$\mathbf{Z}_2 = \langle (9-t)/(1+4t) \rangle$	3420
$\{\mathcal{D}_{24};\mathcal{D}_{24}^c\}$	$\mathbf{D}_{20} = \langle (4-t)/(3+2t), -2/t \rangle$	171
$\{\mathcal{D}_{26};\mathcal{D}_{26}^c\}$	$\mathbf{V}_4 = \langle (t+5)/(8t-1),\ (t-1)/(5t-1) \rangle$	1710
$\{\mathcal{D}_{27};\mathcal{D}_{27}^c\}$	$\mathbf{D}_4 = \langle (t+5)/(2t-6),\ 2t/(t-2) \rangle$	855
$\{\mathcal{D}_{28};\mathcal{D}_{28}^c\}$	$\mathbf{Z}_2 = \langle (9-t)/(1+5t) \rangle$	3420
$\{\mathcal{D}_{29};\mathcal{D}_{29}^c\}$	$\mathbf{V}_4 = \langle (t-8)/(3t-1),\ (t+2)/(t-1) \rangle$	1710
$\{\mathcal{D}_{30};\mathcal{D}_{30}^c\}$	$\mathbf{S}_3 = \langle (t-6)/(4t-1),\ (t-6)/(6t+6) \rangle$	1140
$\{\mathcal{D}_{33};\mathcal{D}_{33}^c\}$	$\mathbf{Z}_2 = \langle (t-7)/(8t-1) \rangle$	3420
$\{\mathcal{D}_{36};\mathcal{D}_{36}^c\}$	$\mathbf{V}_4 = \langle -(t+4)/(2t+1),\ (1-t)/(1+t) \rangle$	1710
$\{\mathcal{D}_{38};\mathcal{D}_{38}^c\}$	$\mathbf{V}_4 = \langle -(t+4)/(2t+1),\ 2/t \rangle$	1710
$\{\mathcal{D}_{39};\mathcal{D}_{39}^c\}$	$\mathbf{S}_3 = \langle (t+8)/(8t-1),\ (t+3)/(2t+3) \rangle$	1140
$\{\mathcal{D}_{40};\mathcal{D}_{40}^c\}$	$\mathbf{D}_4 = \langle (t+5)/(3t+8),\ (t+2)/(4t-1) \rangle$	855
$\{\mathcal{D}_{41};\mathcal{D}_{41}^c\}$	$\mathbf{D}_{10} = \langle (t-8)/(4t+7),\ (t-6)/(t-1) \rangle$	342
$\{\mathcal{D}_{42};\mathcal{D}_{42}^c\}$	$\mathbf{S}_3 = \langle (t+6)/(2t-1),\ 4/(2-t) \rangle$	1140
$\{\mathcal{D}_{43};\mathcal{D}_{43}^c\}$	$\mathbf{V}_4 = \langle -(t+9)/(2t+1),\ (t-5)/(5t-1) \rangle$	1710
$\{\mathcal{D}_{44};\mathcal{D}_{44}^c\}$	$\mathbf{D}_9 = \langle (t-7)/(8t-1),\ (4+t)/(2-t) \rangle$	380

Remark 2.12.2. In Table 2.23, when the stabilizer group of the partition is generated by two elements, the first generator transforms the decad $\mathcal{D}_i$ to $\mathcal{D}_i^c$, while the second fixes $\mathcal{D}_i$.

2.13 Splitting $PG(1,19)$ into Five Disjoint Tetrads

There are four types of tetrads H, E, N_1, N_2 on $PG(1,19)$. The question arises here: *Does the projective line $PG(1,19)$ split into five disjoint harmonic tetrads, five equianharmonic tetrads, five tetrads of type N_1 or five tetrads of type N_2?*

The answer is yes for each type as given below. Here the symbol $CR(a_1)$ refers to the cross-ratio of the set a_i.

(i) Harmonic

$a_1 = \{\infty, 0, 1, -1\}$, $CR(a_1) = -1$;
$a_2 = \{2, -2, 3, -5\}$, $CR(a_2) = -1$;
$a_3 = \{-3, 4, -4, -6\}$, $CR(a_3) = 2$;
$a_4 = \{5, 7, 8, -8\}$, $CR(a_4) = 2$;
$a_5 = \{6, -7, 9, -9\}$, $CR(a_5) = -9$.

(ii) Equianharmonic

$a_1 = \{\infty, 0, 1, -7\}$, $CR(a_1) = -7$;
$a_2 = \{-1, 2, -2, 5\}$, $CR(a_2) = -7$;
$a_3 = \{3, -3, -4, 7\}$, $CR(a_3) = 8$;
$a_4 = \{4, 8, 9, -9\}$, $CR(a_4) = 8$;
$a_5 = \{-5, 6, -6, -8\}$, $CR(a_5) = -7$.

(iii) Tetrads of Type N_1

$a_1 = \{\infty, 0, 1, -2\}$, $CR(a_1) = -2$;
$a_2 = \{-1, 2, 3, 4\}$, $CR(a_2) = -6$;
$a_3 = \{-3, -4, 5, -5\}$, $CR(a_3) = -8$;
$a_4 = \{6, -6, 8, -8\}$, $CR(a_4) = 7$;
$a_5 = \{7, -7, 9, -9\}$, $CR(a_5) = -8$.

(iv) Tetrads of Type N_2

$a_1 = \{\infty, 0, 1, -3\}$, $CR(a_1) = -3$;
$a_2 = \{-1, 2, -2, 3\}$, $CR(a_2) = 6$;
$a_3 = \{4, -4, 5, -5\}$, $CR(a_3) = 4$;
$a_4 = \{6, -6, 8, -9\}$, $CR(a_4) = 6$;
$a_5 = \{7, -7, -8, 9\}$, $CR(a_5) = 6$.

Remark 2.13.1. These partitions are not unique. There are 519156 partitions of five disjoint harmonic tetrads, 67944 of five disjoint equianharmonic tetrads, and more than 100000 of five disjoint tetrads of type N_1 and N_2.

2.14 Summary

The information about $PG(1,19)$ is summarized in the following two tables. Table 2.24 presents the full details about the number of k-sets in $PG(1,19)$; $5 \le k \le 10$, containing a certain $(k-1)$-set K and inequivalent under the stabilizer group of K, as well as the type of their stabilizer groups. Table 2.25 presents the number of all inequivalent k-sets in $PG(1,19)$, $4 \le k \le 10$, and the type of the their stabilizer groups. A cell $m : G$ of the tables means that m is the number of k-sets stabilized by the group G; n_k means the number of k-sets and $\bar{n}_k$ means the number of inequivalent k-sets.

TABLE 2.24: The classification of k-sets in $PG(1,19)$, $5 \le k \le 10$

k-set	n_k	$m : G$						
Pentad	14	$5 : I$	$6 : \mathbf{Z}_2$	$2 : \mathbf{S}_3$	$1 : \mathbf{D}_5$			
Hexad	36	$12 : I$	$15 : \mathbf{Z}_2$	$6 : \mathbf{V}_4$	$2 : \mathbf{S}_3$	$1 : \mathbf{D}_6$		
Heptad	86	$42 : I$	$36 : \mathbf{Z}_2$	$6 : \mathbf{Z}_3$	$1 : \mathbf{Z}_6$			
Octad	154	$88 : I$	$51 : \mathbf{Z}_2$	$9 : \mathbf{V}_4$	$1 : \mathbf{S}_3$	$1 : \mathbf{D}_4$	$2 : \mathbf{D}_6$	$2 : \mathbf{S}_4$
Nonad	228	$162 : I$	$60 : \mathbf{Z}_2$	$3 : \mathbf{Z}_3$	$2 : \mathbf{S}_3$	$1 : \mathbf{D}_9$		
Decad	280	$160 : I$	$89 : \mathbf{Z}_2$	$12 : \mathbf{Z}_3$	$15 : \mathbf{V}_4$	$2 : \mathbf{Z}_9$	$1 : \mathbf{D}_5$	$1 : \mathbf{D}_{10}$

TABLE 2.25: The classification of inequivalent k-sets in $PG(1,19)$, $4 \le k \le 10$

k-set	$\bar{n}_k$	$m : G$						
Tetrad	4	$1 : \mathbf{V}_4$	$1 : \mathbf{V}_4$	$1 : \mathbf{A}_4$	$1 : \mathbf{D}_4$			
Pentad	5	$1 : I$	$2 : \mathbf{Z}_2$	$1 : \mathbf{S}_3$	$1 : \mathbf{D}_5$			
Hexad	13	$1 : I$	$5 : \mathbf{Z}_2$	$3 : \mathbf{V}_4$	$2 : \mathbf{S}_3$	$2 : \mathbf{D}_6$		
Heptad	18	$6 : I$	$9 : \mathbf{Z}_2$	$2 : \mathbf{Z}_3$	$1 : \mathbf{Z}_6$			
Octad	31	$11 : I$	$12 : \mathbf{Z}_2$	$4 : \mathbf{V}_4$	$1 : \mathbf{S}_3$	$1 : \mathbf{D}_4$	$1 : \mathbf{D}_6$	$1 : \mathbf{S}_4$
Nonad	33	$18 : I$	$12 : \mathbf{Z}_2$	$1 : \mathbf{Z}_3$	$1 : \mathbf{S}_3$	$1 : \mathbf{D}_9$		
Decad	44	$16 : I$	$17 : \mathbf{Z}_2$	$3 : \mathbf{Z}_3$	$5 : \mathbf{V}_4$	$1 : \mathbf{Z}_9$	$1 : \mathbf{D}_5$	$1 : \mathbf{D}_{10}$

2.15 MDS Codes of Dimension Two

As in Theorem 1.13.1, an $(n; n-d)$-arc in $PG(k-1, q)$ is equivalent to a projective $[n, k, d]_q$-code. So, if $k = 2$ and $n - d = 1$, then there is a one-to-one correspondence between n-sets in $PG(1, 19)$ and projective $[n, 2, n-1]_{19}$-codes C. Since $d(C)$ of the code C is equal to $n - k + 1$, thus the projective code C is MDS.

In Table 2.26, the MDS codes corresponding to the n-sets in $PG(1, 19)$ and the parameter e of errors corrected are given.

TABLE 2.26: MDS code over $PG(1, 19)$

n-Set	MDS code	e
Tetrad	$[4, 2, 3]_{19}$	1
Pentad	$[5, 2, 4]_{19}$	1
Hexad	$[6, 2, 5]_{19}$	2
Heptad	$[7, 2, 6]_{19}$	2
Octad	$[8, 2, 7]_{19}$	3
Nonad	$[9, 2, 8]_{19}$	3
Decad	$[10, 2, 9]_{19}$	4

Chapter 3

The Projective Plane

3.1 Introduction

The projective plane $PG(2,q)$ is a 2-dimensional projective space over $\mathbf{F}_q$. In a plane, each point P is joined to the remaining points by a *pencil* which consists of $q+1$ lines; each of these lines contains P and q other points. Hence the plane contains

$$q(q+1)+1 = q^2+q+1 = \theta(2,q)$$

points and by duality a plane contains q^2+q+1 lines. The integer q is called the *order of the plane*.

Throughout, $\Upsilon = \{\mathbf{U}_0, \mathbf{U}_1, \mathbf{U}_2, \mathbf{U}\}$ denotes the standard frame in $PG(2,q)$.

3.2 Construction of $PG(2,q)$

In this section three ways to represent the points of $PG(2,q)$ are given.

(1) Let $f(x) = x^3 - a_2x^2 - a_1x - a_0$ be a cubic subprimitive polynomial over $\mathbf{F}_q$. An example of a cyclic projectivity $\mathfrak{T} = \mathbf{M}(A)$ occurs when $A = \mathbf{C}(f)$; that is,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_0 & a_1 & a_2 \end{bmatrix},$$

since $|xI_3 - \mathbf{C}(f)| = f(x)$, see [17, Theorem 7.1].

So, $\mathbf{P}(x_0, x_1, x_2)\mathfrak{T} = \mathbf{P}(x_0', x_1', x_2')$, where

$$x_0' = a_0 x_2,$$
$$x_1' = x_0 + a_1 x_2,$$
$$x_2' = x_1 + a_2 x_2.$$

If α is a root of $f(x)$ in $\mathbf{F}_{q^3}$, then

$$\alpha^3 = a_2\alpha^2 + a_1\alpha + a_0. \tag{3.1}$$

Let $P(0) = \mathbf{U}_0$ and define $P(i) = P(0)\mathfrak{T}^i$. For $i = 0, 1, \ldots, q^2 + q$, there exist $y_0^{(i)}, y_1^{(i)}, y_2^{(i)}$ in $\mathbf{F}_q$ such that

$$\alpha^i = y_0^{(i)} + y_1^{(i)}\alpha + y_2^{(i)}\alpha^2.$$

Then

$$\alpha^{i+1} = \alpha \cdot \alpha^i = y_0^{(i)}\alpha + y_1^{(i)}\alpha^2 + y_2^{(i)}\alpha^3. \tag{3.2}$$

By substituting (3.1) in (3.2),

$$\alpha^{i+1} = a_0 y_2^{(i)} + (y_0^{(i)} + a_1 y_2^{(i)})\alpha + (y_1^{(i)} + a_2 y_2^{(i)})\alpha^2.$$

But also

$$\alpha^{i+1} = y_0^{(i+1)} + y_1^{(i+1)}\alpha + y_2^{(i+1)}\alpha^2.$$

Thus

$$(y_0^{(i+1)}, y_1^{(i+1)}, y_2^{(i+1)}) = (y_0^{(i)}, y_1^{(i)}, y_2^{(i)})A.$$

So, $P(i) = \mathbf{P}(y_0^{(i)}, y_1^{(i)}, y_2^{(i)})$ or more generally $P(i)\mathfrak{T}^{j-i} = P(j),\ 0 \le i < j \le q^2 + q$. The order of the projectivity $\mathfrak{T}$ is $\theta(2, q)$, and

$$PG(2, q) = \{\mathbf{U}_0\mathfrak{T}^i \mid i = 0, 1, \ldots, q^2 + q\}.$$

Note that $P(0) = \mathbf{U}_0,\ P(1) = \mathbf{U}_1,\ P(2) = \mathbf{U}_2$. Since $\mathfrak{T}$ acts cyclically on the points of $PG(2, q)$, then dually it acts cyclically on the lines of $PG(2, q)$.

The existence of a cyclic projectivity gives an attractive representation of the points and lines of $PG(2, q)$ as illustrated in (2).

(2) The plane can be represented by a *regular array*; that is, each row is a cyclic permutation of the previous one.

Suppose that the points collinear with $P(0)$ and $P(1)$ are those $P(i)$ with indices $i = d_2, d_3, \ldots, d_q$. Write d_0 for 0 and d_1 for 1, and consider the array $\mathcal{M}$:

$$\begin{array}{ccccc} d_0 & d_0+1 & d_0+2 & \ldots & d_0+q^2+q \\ d_1 & d_1+1 & d_1+2 & \ldots & d_1+q^2+q \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ d_q & d_q+1 & d_q+2 & \ldots & d_q+q^2+q \end{array},$$

where each entry has been reduced modulo q^2+q+1. The rows of $\mathcal{M}$ represents the points of $PG(2,q)$ and the columns represents the lines of $PG(2,q)$.

According to Berman [7, Theorem 2.1], the integers $d_0, d_1, \ldots, d_q$ form a *perfect difference set*; that is, q^2+q integers $d_i - d_j$ with $i \neq j$ are all distinct modulo q^2+q+1.

These types of representations for $PG(2,q)$ can be extended to higher dimensions. See [28, Section 4.2].

(3) The points of $PG(2,q)$ as shown in Table 1.1 can be represented by vectors of three coordinate over $\mathbf{F}_q$ as in Table 3.1.

TABLE 3.1: Type of elements of $PG(2,q)$

Type of elements	No. of elements
$\mathbf{P}(x_0, x_1, 1)$	q^2
$\mathbf{P}(x_0, 1, 0)$	q
$\mathbf{P}(1, 0, 0)$	1
	$\theta(2,q)$

Throughout, the numeral form is used to refer to the points of $PG(2,q)$; that is, $1 = P(0), 2 = P(1), \ldots, q^2+q+1 = P(q^2+q)$.

3.3 Arcs in a Plane

As in Section 1.10, a $(k;r)$-arc in $PG(2,q)$ is a set of k points no $r+1$ of them collinear but some r collinear. In the terms of τ_i this becomes the following: a $(k;r)$-arc is a set of k points of $PG(2,q)$ for which $\tau_i \geq 0$ for $i < r$, $\tau_r > 0$ and $\tau_i = 0$ when $i > r$. In terms of i-secants the definition of a complete $(k;r)$-arc becomes the following: a $(k;r)$-arc $\mathcal{K}$ is complete if every point lies on some r-secant of

$\mathcal{K}$. In $PG(2,q)$, the number of $(k;r)$-arcs projectively equivalent to a $(k;r)$-arc $\mathcal{K}$ with stabilizer group G is

$$q^3(q^3-1)(q^2-1)/|G|.$$

3.4 Some Basic Equations

Let P be a point of k-arc $\mathcal{K}$ and let $t(P)$ be the number of unisecants through P. Then through P there are $k-1$ bisecants of $\mathcal{K}$; hence

$$t(P) = (q+1) - (k-1) = q+2-k = t.$$

So,

$$t+k = q+2 \qquad \text{and} \quad t \geq 0; \tag{3.3}$$

hence $k \leq q+2$ and t is independent of the point P. Therefore a k-arc $\mathcal{K}$ can have at most $q+2$ points. There are exactly $\frac{1}{2}k(k-1)$ bisecants, kt unisecants and

$$q^2+q+1-\frac{1}{2}k(k-1)-kt = \frac{1}{2}q(q-1)+\frac{1}{2}t(t-1)$$

external lines. That is,

$$\tau_2 = \binom{k}{2}, \quad \tau_1 = kt, \quad \tau_0 = \binom{q}{2}+\binom{t}{2}.$$

Let Q be a point of $PG(2,q)$ not on the k-arc $\mathcal{K}$. Let $\sigma_i(Q)$ be the number of i-secants through Q. The number $\sigma_2(Q)$ of bisecants is called the *index of Q with respect to $\mathcal{K}$*. Then $\sigma_0(Q)+\sigma_1(Q)+\sigma_2(Q) = q+1$ and $\sigma_1(Q)+2\sigma_2(Q) = k$. So, $\sigma_1(Q) \equiv k \pmod 2$ and from (3.3), the following holds:

$$t \equiv k \pmod 2 \text{ if } q \text{ is even;}$$
$$t \not\equiv k \pmod 2 \text{ if } q \text{ is odd.}$$

Example 3.4.1. Let $\mathcal{K}$ be a k-arc in $PG(2,q)$.

(i) If $k = q$, then

  (1) $\sigma_1(Q)+2\sigma_2(Q) = q$;

  (2) if q even, then $\sigma_1(Q)$ is even;

(3) if q odd, then $\sigma_1(Q)$ is odd;

(4) $t = 2$, $\tau_2 = \frac{1}{2}q(q-1)$, $\tau_1 = 2q$ and $\tau_0 = \frac{1}{2}q(q-1) + 1$.

(ii) If $k = q+1$, then

(1) $\sigma_1(Q) + 2\sigma_2(Q) = q+1$;

(2) if q even, then $\sigma_1(Q)$ is odd;

(3) if q odd, then $\sigma_1(Q)$ is even;

(4) $t = 1$, $\tau_2 = \frac{1}{2}q(q+1)$, $\tau_1 = q+1$ and $\tau_0 = \frac{1}{2}q(q-1)$. □

Let c_i be the number of points of $PG(2,q)\backslash\mathcal{K}$ of index exactly i. So, c_0 is the number of points through which no bisecant of $\mathcal{K}$ passes. Then $\mathcal{K}$ is complete if and only if $c_0 = 0$; that is, $\sigma_2(Q) \neq 0$ for all Q off $\mathcal{K}$. Also, c_3 is the number of points where three bisecants meet. The maximum possible index of a point, with respect to a given k-arc, is $k' = \lfloor \frac{1}{2}k \rfloor$.

If two k-arcs are projectively equivalent then it is necessary that both have the constant c_i for each i. However, the converse is not true as shown in next chapter.

Lemma 3.4.2. *The constants c_i of a k-arc $\mathcal{K}$ in $PG(2,q)$ satisfy the following equations:*

$$\sum_{i=0}^{k'} c_i = q^2 + q + 1 - k; \tag{3.4}$$

$$\sum_{i=1}^{k'} ic_i = k(k-1)(q-1)/2; \tag{3.5}$$

$$\sum_{i=2}^{k'} i(i-1)c_i = k(k-1)(k-2)(k-3)/8. \tag{3.6}$$

Proof.

$$\begin{aligned}
\sum_{i=0}^{k'} c_i &= |\{Q \mid Q \in PG(2,q)\backslash\mathcal{K}\}|;\\
\sum_{i=1}^{k'} ic_i &= |\{(Q,\ell) \mid Q \in \ell\backslash\mathcal{K}; \ell \text{ a bisecant of } \mathcal{K}\}|;\\
\sum_{i=2}^{k'} i(i-1)c_i &= |\{(Q,\{\ell,\ell'\}) \mid Q \in (\ell\cap\ell')\backslash\mathcal{K}; \ell,\ell' \text{ bisecants of } \mathcal{K}\}|.
\end{aligned}$$

□

The values of the constant c_i for a k-arc with $2 \le k \le 7$ are given in Table 3.2.

TABLE 3.2: Constants for small arcs

k	c_0	c_1	c_2
2	q^2	$q-1$	
3	$(q-1)^2$	$3(q-1)$	
4	$(q-2)(q-3)$	$6(q-2)$	3
5	$(q-4)(q-5)+1$	$10(q-4)$	15
6	$(q-7)^2+6-c_3$	$3\{5(q-7)+c_3\}$	$3(15-c_3)$
7	$(q-10)^2+20-c_3$	$3\{7(q-11)+c_3\}$	$3(35-c_3)$

It is clear from Table 3.2 that a complete 4-arc exists only for $q = 2$ and 3. A 5-arc is never complete.

3.5 n-Stigms

An *n-stigm* in $PG(2,q)$ is a set of n points, no three of which are collinear, together with the $\frac{1}{2}n(n-1)$ lines that are joins of pairs of the points. The points and lines are called *vertices* and *sides* of the n-stigm. The vertices form an n-arc. A 3-stigm is also called a *triangle*, a 4-stigm a *tetrastigm*, a 5-stigm a *pentastigm* and a 6-stigm a *hexastigm*.

Let $l(n,q)$ be the number of points on the sides of an n-stigm, and

$$l^*(n,q) = q^2 + q + 1 - l(n,q).$$

The *diagonal points* of an n-stigm are the intersections of two sides which do not pass through the same vertex.

For $n \le 5$, The number of points on the sides of an n-stigm is

$$l(n,q) = \binom{n}{2}(q-1) + n - \frac{1}{2}\binom{n}{2}\binom{n-2}{2}.$$

Table 3.3 gives the values of $l(n,q)$ and $l^*(n,q)$. See [28, Lemma 7.1(i)].

TABLE 3.3: The number of points on the sides of an n-stigm

n	2	3	4	5
$l(n,q)$	$q+1$	$3q$	$6q-5$	$10q-20$
$l^*(n,q)$	q^2	$(q-1)^2$	$(q-2)(q-3)$	$(q-4)(q-5)+1$

Note that $l^*(n,q) = c_0$ for $2 \le n \le 5$.

3.6 Conics

Let

$$F = a_{00}X_0^2 + a_{11}X_1^2 + a_{22}X_2^2 + a_{01}X_0X_1 + a_{02}X_0X_2 + a_{12}X_1X_2,$$

be a form of degree 2 and $\mathcal{Q} = \mathbf{v}(F)$ be a plane quadric . As mentioned in Section 1.11, a conic is a non-singular plane quadric. Then the following properties hold.

(1) A conic is an irreducible plane quadric.

(2) Any plane quadric through a 5-arc is non-singular. See [28, Theorem 7.4].

(3) Every conic is determined by the ratios of the coefficients $(a_{00}, a_{11}, a_{22}, a_{01}, a_{02}, a_{12})$. So, it is determined by five of its points.

(4) In $PG(2,q)$ with $q \ge 4$, there is a unique conic through a 5-arc. See [28, Corollary 7.5].

(5) In $PG(2,q)$, number of conics is $q^5 - q^2$. See [28, Theorem 7.4].

(6) If a conic contains one rational point, then it contains exactly $q+1$. See [28, Lemma 7.6].

(7) Let $\mathcal{C}$ be a conic and $\mathbf{P}(A) \in \mathcal{C}$. Then the $q+1$ lines of the plane through $\mathbf{P}(A)$ comprise the tangent and q bisecants.

(8) Every conic in $PG(2,q)$ is a $(q+1)$-arc. So, its unisecants are its tangents. See [28, Lemma 7.7 and Corollary 8.3].

(9) In $PG(2,q)$ for q even, the $q+1$ tangents to a conic $\mathcal{C}$ are concurrent. The point of intersection of these tangents is called the *nucleus*. See [28, Corollary 7.11]. Note that the nucleus is not on any bisecants of $\mathcal{C}$.

(10) In $PG(2,q)$ for q odd, every point off a conic $\mathcal{C}$ lies on exactly two or no tangents of $\mathcal{C}$. See [28, Lemma 8.10].

A point of $PG(2,q)$ is *external* or *internal* to the conic $\mathcal{C}$ according as it lies on two or no tangents of $\mathcal{C}$. Hence, with respect to $\mathcal{C}$, the q^2+q+1 points of $PG(2,q)$ are partitioned into three classes:

(a) $q+1$ points on $\mathcal{C}$;

(b) $q(q+1)/2$ external points;

(c) $q(q-1)/2$ internal points.

Similarly, the q^2+q+1 lines of $PG(2,q)$ are partitioned into three classes with respect to $\mathcal{C}$:

(a) $q+1$ unisecants;

(b) $q(q+1)/2$ bisecants;

(c) $q(q-1)/2$ external lines.

(11) Let $\mathcal{C} = \mathbf{v}(F)$ be a conic and $PGO(3,q)$ denote the projective group of the conic. Then

(i) $\mathcal{C} \cong \mathcal{C}^* = \mathbf{v}(X_1^2 - X_0X_2) = \{\mathbf{P}(t^2,t,1) \mid t \in \mathbf{F}_q \cup \{\infty\}\}$.

(ii) Let $\mathfrak{T} = \mathbf{M}(A)$ be an element of $PGL(2,q)$ given by $\mathbf{P}(t,1) \mapsto \mathbf{P}(t,1)\mathfrak{T}$, where

$$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$

The matrix A has the following effect on $\mathcal{C}^*$:

$$\mathbf{P}(t^2,t,1) \mapsto \mathbf{P}(t^2,t,1)\mathfrak{T}',$$

where $\mathfrak{T}' = \mathbf{M}(A')$ and

$$A' = \begin{bmatrix} a^2 & ac & c^2 \\ 2ab & ad+bc & 2cd \\ b^2 & bd & d^2 \end{bmatrix}.$$

Thus

$$\varphi : PGL(2,q) \longrightarrow PGO(3,q)$$

given by $\mathfrak{T}\varphi = \mathfrak{T}'$ is an group isomorphism. Therefore,

$$PGL(2,q) \cong PGO(3,q).$$

For more details see [28, Corollary 7.14].

(12) There is a one-to-one correspondence between $PG(1,q)$ and a conic in $PG(2,q)$. Therefore, there is a one-to-one correspondence between a set of points on $PG(1,q)$ and an arc of the same size on a conic in $PG(2,q)$.

3.7 Ovals

A k-arc in $PG(2,q)$ with maximum number of points is an *oval*. The maximum value of k for a k-arc is denoted by $m_2(2,q)$ and was determined by Bose [11] in the following theorem.

Theorem 3.7.1. *In* $PG(2,q)$

$$m_2(2,q) = \begin{cases} q+2 & \text{for } q \text{ even}, \\ q+1 & \text{for } q \text{ odd}. \end{cases}$$

Proof. As mentioned in Section 3.4, $k \le q+2$ because $t(P) = q+2-k = t \ge 0$ for any k-arc. When q is even, the union of a conic $\mathcal{C}$ with its nucleus N is a $(q+2)$-arc, whence $m_2(2,q) = q+2$ for q even.

If q is odd and $t = 0$, then $\sigma_1(Q) = 0$ for all Q off the k-arc $\mathcal{K}$. Thus $2\sigma_2(Q) = k$ is even and $q+2-t = k$ is odd, which is a contradiction. So $t \ne 0$; that is, $k \le q+1$. As a conic is a $(q+1)$-arc, so $m_2(2,q) = q+1$ for q odd. □

Theorem 3.7.2. (Segre's Theorem)

In $PG(2,q)$, with q odd, every oval is a conic.

Proof. See [28, Theorem 8.14]. □

For more results about $m_r(2,q)$, $r > (q+3)/2$ see [18].

3.8 Complete k-Arcs

A k-arc $\mathcal{K}$ is complete if and only if the points on the bisecants of $\mathcal{K}$ cover the whole plane. An example of a complete arc is the oval for q even and the conic for q odd. There is no complete arc other than a conic lying on the conic.

The questions arise here: *Are there complete arcs in $PG(2,q)$ other than ovals? What is the lower bound for the minimum integer k for complete k-arcs in $PG(2,q)$?*

Any q-arc is incomplete for q odd as given in the following lemma.

Lemma 3.8.1. *In $PG(2,q)$ for q odd, a q-arc $\mathcal{K}$ lies on a conic; the number of such conics is one or four as $q \neq 3$ or $q = 3$.*

Proof. See [28, Theorem 10.28]. □

Lemma 3.8.2. *If a k-arc $\mathcal{K}$ in $PG(2,q)$ is complete, then $q \leq \frac{1}{2}(k-1)(k-2)$. If q is not a prime, the equality cannot hold.*

Proof. See [6, Theorem 2.5.1]. □

Corollary 3.8.3. *In $PG(2,q)$, if $q \geq 16$, there is no complete 6-arc and 7-arc.*

Lemma 3.8.4. *In $PG(2,q)$ with q odd, let the k-arc $\mathcal{K}$, where $k = \frac{1}{2}(q+5)$, have its points arbitrarily chosen from a conic $\mathcal{C}$. Then the bisecants of $\mathcal{K}$ contain all the points in the plane other than those of $\mathcal{C}\backslash\mathcal{K}$.*

Proof. See [28, Lemma 9.27]. □

Corollary 3.8.5. *In $PG(2,q)$ with q odd, if $\mathcal{K}$ is a k-arc not contained in a conic, then $\mathcal{K}$ has at most $k = \frac{1}{2}(q+3)$ points in common with a conic.*

Proof. If there is a conic $\mathcal{C}$ containing $\frac{1}{2}(q+5)$ points of $\mathcal{K}$ and if $P \in \mathcal{K}\backslash\mathcal{C}$, then by Lemma 3.8.4 there is a bisecant of $\mathcal{K}$ through P; that is, $\mathcal{K}$ has 3-secant, a contradiction. □

Lemma 3.8.6. *In $PG(2,q)$, q odd and $q \geq 7$, there exists a complete k-arc not on a conic.*

Proof. Outline: Let Q be an external point of the conic $\mathcal{C}$. The $q+1$ lines through Q consist of two tangents to $\mathcal{C}$, $\frac{1}{2}(q-1)$ bisecants to $\mathcal{C}$ and $\frac{1}{2}(q-1)$ external lines to $\mathcal{C}$. Let the k-arc $\mathcal{K}^*$ consist of Q, the two points P, P' of contact of the tangents to $\mathcal{C}$ through Q, and one of the two points of $\mathcal{C}$ on each of the $\frac{1}{2}(q-1)$ bisecants of $\mathcal{C}$ through Q. Then $k = \frac{1}{2}(q+5)$. Now $\mathcal{K}^*$ can be completed to $\mathcal{K}^{*\prime}$ not on a conic. See [28, Lemma 9.29]. □

Lemma 3.8.7. *In $PG(2,q)$, $q \equiv -1 \pmod 4$, there exists a complete k-arc with $k = \frac{1}{2}(q+5)$.*

Proof. Outline: The set $\mathcal{K}^*$ of the previous lemma can be chosen so that $\mathcal{K}^* = \mathcal{K}^{*\prime}$. Let $\mathcal{C} = \mathbf{v}(X_1^2 - X_0X_2)$ and let $Q = \mathbf{U}_1$; then $\{P, P'\} = \{\mathbf{U}_0, \mathbf{U}_2\}$. Let α be a primitive element of $\mathbf{F}_q$. Then the $q-1$ points of $\mathcal{C}\backslash\{\mathbf{U}_0, \mathbf{U}_2\}$ fall into two branches S^* and N^*, where

$$S^* = \{\mathbf{P}(\alpha^{2i}, 1, \alpha^{-2i}) \mid i = 1, \ldots, (q-1)/2\}, \text{ the branch of the squares,}$$
$$N^* = \{\mathbf{P}(\alpha^{2i-1}, 1, \alpha^{-2i+1}) \mid i = 1, \ldots, (q-1)/2\}, \text{ the branch of the non-squares.}$$

Let $\mathcal{K}_1^* = S^* \cup \{\mathbf{U}_0, \mathbf{U}_1, \mathbf{U}_2\}$ and $\mathcal{K}_2^* = N^* \cup \{\mathbf{U}_0, \mathbf{U}_1, \mathbf{U}_2\}$. Then $\mathcal{K}_1^*$ and $\mathcal{K}_2^*$ are complete k-arcs with $k = \frac{1}{2}(q+5)$. See [28, Theorem 9.30]. □

Lemma 3.8.8. *If $\mathcal{K}$ is a k-arc in $PG(2,q)$, q odd, and $k > \frac{2}{3}(q+2)$, then there is a unique complete arc containing $\mathcal{K}$.*

Proof. See [28, Theorem 10.23]. □

3.9 The Algorithms

The calculation of a matrix transformations between two 4-arcs as well as methods of constructions of inequivalent and complete arcs are illustrated in the following algorithm.

3.9.1 Projectivity Between Two 4-Arcs

In general, a projectivity $\mathfrak{T} = \mathbf{M}(A)$ in $PG(2,q)$ is given by the equation

$$tY = XA,$$

where $Y = (y_0, y_1, y_2)$, $X = (x_0, x_1, x_2)$, $A = (t_{ij})$, $t \in \mathbf{F}_q\backslash\{0\}$; that is,

$$x_0t_{00} + x_1t_{10} + x_2t_{20} = ty_0, \tag{3.7}$$
$$x_0t_{01} + x_1t_{11} + x_2t_{21} = ty_1, \tag{3.8}$$
$$x_0t_{02} + x_1t_{12} + x_2t_{22} = ty_2. \tag{3.9}$$

By the Fundamental Theorem of Projective Geometry, Section 1.7(iii), $\mathfrak{T}$ is uniquely determined when the points P_i, $i = 1, 2, 3, 4$, of a 4-arc and their images $P_i\mathfrak{T}$, also points of a 4-arc, are given. The mapping $\mathfrak{T}$ is in fact determined by eight conditions. This can be seen in two ways. The matrix A has nine entries but $\mathfrak{T}$ is determined by their ratios and so by eight conditions; alternatively, given the four points P_i, each image $P_i\mathfrak{T}$ is determined by two conditions as follows.

Eliminating t from equations (3.7) and (3.8), and from (3.8) and (3.9) give the following two homogeneous equations:

$$y_1(x_0t_{00} + x_1t_{10} + x_2t_{20}) - y_0(x_0t_{01} + x_1t_{11} + x_2t_{21}) = 0,$$
$$y_2(x_0t_{01} + x_1t_{11} + x_2t_{21}) - y_1(x_0t_{02} + x_1t_{12} + x_2t_{22}) = 0.$$

So, $\mathfrak{T}$ is determined by $4 \times 2 = 8$ conditions.

Alternatively, to find a projectivity between any two arcs the following procedure can be used. Let

$$\mathcal{K} = \{\mathbf{P}(a_0, a_1, a_2), \mathbf{P}(b_0, b_1, b_2), \mathbf{P}(c_0, c_1, c_2), \mathbf{P}(d_0, d_1, d_2)\},$$
$$\mathcal{K}' = \{\mathbf{P}(a_0', a_1', a_2'), \mathbf{P}(b_0', b_1', b_2'), \mathbf{P}(c_0', c_1', c_2'), \mathbf{P}(d_0', d_1', d_2')\},$$

be two 4-arcs and $\Upsilon = \{\mathbf{P}(1,0,0), \mathbf{P}(0,1,0), \mathbf{P}(0,0,1), \mathbf{P}(1,1,1)\}$ be the standard frame. If A is a matrix which transforms Υ to $\mathcal{K}$ and B is a matrix which transforms Υ to $\mathcal{K}'$, then the matrix $A^{-1}B$ transforms $\mathcal{K}$ to $\mathcal{K}'$.

The procedure to find the projective transformation $\mathfrak{T} = \mathbf{M}(A)$ which maps

$$\begin{aligned} \mathbf{P}(1,0,0) &\quad \text{to} \quad \mathbf{P}(a_0, a_1, a_2), \\ \mathbf{P}(0,1,0) &\quad \text{to} \quad \mathbf{P}(b_0, b_1, b_2), \\ \mathbf{P}(0,0,1) &\quad \text{to} \quad \mathbf{P}(c_0, c_1, c_2), \\ \mathbf{P}(1,1,1) &\quad \text{to} \quad \mathbf{P}(d_0, d_1, d_2), \end{aligned}$$

is as follows. Let $\alpha, \beta, \gamma \in \mathbf{F}_q \backslash \{0\}$ and

$$(1,0,0)A = \alpha(a_0, a_1, a_2),$$
$$(0,1,0)A = \beta(b_0, b_1, b_2),$$
$$(0,0,1)A = \gamma(c_0, c_1, c_2).$$

Then,

$$A = \begin{bmatrix} \alpha a_0 & \alpha a_1 & \alpha a_2 \\ \beta b_0 & \beta b_1 & \beta b_2 \\ \gamma c_0 & \gamma c_1 & \gamma c_2 \end{bmatrix}.$$

Also there is $\nu \in \mathbf{F}_q \backslash \{0\}$ such that $(1,1,1)A = \nu(d_0, d_1, d_2)$; so the following non-homogeneous system is obtained:

$$\begin{bmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} \nu d_0 \\ \nu d_1 \\ \nu d_2 \end{bmatrix}.$$

This system has a unique solution given by

$$\frac{\alpha}{\begin{vmatrix} d_0 & b_0 & c_0 \\ d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \end{vmatrix}} = \frac{\beta}{\begin{vmatrix} a_0 & d_0 & c_0 \\ a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \end{vmatrix}} = \frac{\gamma}{\begin{vmatrix} a_0 & b_0 & d_0 \\ a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \end{vmatrix}} = \frac{\nu}{\begin{vmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}$$

or

$$\frac{\alpha}{D_1} = \frac{\beta}{D_2} = \frac{\gamma}{D_3} = \frac{\nu}{D_4},$$

where $D_1 D_2 D_3 D_4 \neq 0$. Therefore,

$$\frac{D_4}{\nu} A = \begin{bmatrix} D_1 a_0 & D_1 a_1 & D_1 a_2 \\ D_2 b_0 & D_2 b_1 & D_2 b_2 \\ D_3 c_0 & D_3 c_1 & D_3 c_2 \end{bmatrix},$$

and

$$\mathfrak{T} = \mathbf{M}(A) = \mathbf{M}\left(\frac{D_4}{\nu} A\right).$$

3.9.2 Construction of Inequivalent k-Arcs

In this section, the algorithm used to classify the k-arcs that contain the standard frame is described.

Let $\mathcal{K}$ be a $(k-1)$-arc, $k \geq 5$, containing the standard frame Υ.

(1) Define C_0^{k-1} to be a set of points not on the bisecants of $\mathcal{K}$; that is, points of index zero. Here $|C_0^{k-1}| = c_0$.

(2) If C_0^{k-1} is not empty, that is, $\mathcal{K}$ is not complete, then C_0^{k-1} is separated into orbits by the stabilizer group $G_{\mathcal{K}}$ of $\mathcal{K}$.

(3) A k-arc is constructed by adding one point to $\mathcal{K}$ from an orbit.

(4) Let $\lfloor \frac{k}{2} \rfloor = n$. Then the values of the constants $c_0, c_1, \ldots, c_n$, are calculated for each k-arc.

(5) Let M^k be the set of all different k-arcs that are constructed from $(k-1)$-arcs in $PG(2,q)$. Then M^k is partitioned into classes $\{M_i^k\}_{i\in\Lambda}$ according to $c_0,\ldots,c_n$.

(6) In general, two k-arcs, $\mathcal{K}$ and $\mathcal{K}'$ are equivalent if there is a projective transformation $\mathfrak{T}$ which transforms the frame Υ to any permutation of four points in $\mathcal{K}'$ such that $\mathfrak{T}$ transforms $\mathcal{K}\backslash\Upsilon$ to any permutation of the other $k-4$ points in $\mathcal{K}'$. Accordingly, any two k-arcs in the same class M_i^k are equivalent if there is a projective transformation between them.

Remark 3.9.1. (1) If $C_0^{k-1} \neq \phi$, then $C_0^k \subset C_0^{k-1}$.

(2) If $\mathcal{K}$ and $\mathcal{K}'$ are two equivalent arcs, then it is necessary that they have the same type of projective group. It is shown in the next chapter that the converse is not true. So, the k-arcs in each class M_i^k can be partitioned according to their projective groups.

3.9.3 Construction of Complete k-Arcs

To find the complete arcs of size s, first try to calculate all inequivalent arcs of size equal to a fixed threshold n, $n < s$. Then try to complete each inequivalent n-arc by extending it until it reaches the desired length s. An s-arc is complete if $C_0^s = \phi$. In doing the extension, the repeated arcs are skipped and the information furnished by C_0^h, $n < h < s$, is exploited to save time. Inequivalent complete arcs are checked as in Section 3.9.2, steps 5 and 6.

Chapter 4

The Projective Plane of Order Nineteen

4.1 Introduction

According to Section 3.1, the projective plane of order nineteen, $PG(2,19)$, has 381 points and lines, 20 points on each line and 20 lines passing through each point.

Let $\ell_1 = \mathbf{v}(X_2)$; that is, ℓ_1 is the line passing through points $\mathbf{P}(X_0, X_1, X_2)$ with third coordinate equal to zero. Then ℓ_1 forms the following difference set.

1 2 26 46 80 86 112 183 216 220 238 251 259 266 289 308 318 366 371 380

From Section 3.2, the points and the lines ℓ_i of $PG(2,19)$ can be represented by the following array. See Appendix A.

TABLE 4.1: The points and the lines of $PG(2,19)$

ℓ_1	ℓ_2	ℓ_3	ℓ_4	$\dots$	ℓ_{q^2+q+1}
1	2	3	4	...	381
2	3	4	5	...	1
26	27	28	29	...	25
46	47	48	49	...	45
80	81	82	83	...	79
86	87	88	89	...	85
112	113	114	115	...	111
183	184	185	186	...	182
216	217	218	219	...	215
220	221	222	223	...	219
238	239	240	241	...	237
251	252	253	254	...	250
259	260	261	262	...	258
266	267	268	269	...	265
289	290	291	292	...	288
308	309	310	311	...	307
318	319	320	321	...	317
366	367	368	369	...	365
371	372	373	374	...	370
380	381	1	2	...	379

A vector representation of the points in $PG(2,19)$ by three coordinates over $\mathbf{F}_q$ is as follows.

TABLE 4.2: Type of elements of $PG(2,19)$

Type of elements	No. of elements
$\mathbf{P}(x_0,x_1,1)$	361
$\mathbf{P}(x_0,1,0)$	19
$\mathbf{P}(1,0,0)$	1
	$\theta(2,19)$

4.2 The Unique 4-Arc

From the Fundamental Theorem of Projective Geometry applied to the projective plane, Section 1.7(iii), the frame Υ is projectively the unique 4-arc in $PG(2,19)$. The frame points in $PG(2,19)$ are the points 1,2,3,263 in numeral form. The stabilizer group of Υ is $\mathbf{S}_4$, which can be found by transforming Υ to its 24 permutations. The matrix determining each element of $\mathbf{S}_4$ for each permutation $(ijkl)$ of Υ is given by the rows of Table 4.3. The two matrices marked by g_1, g_2 are generators of $\mathbf{S}_4$.

TABLE 4.3: The stabilizer of the standard frame in $PG(2,19)$

$(ijkl)$	Matrix transformation									
(1234)	1	0	0	0	1	0	0	0	1	
(1243)	−1	0	0	0	−1	0	1	1	1	
(1324)	−1	0	0	0	0	−1	0	−1	0	
(1342)	1	0	0	0	0	1	−1	−1	−1	
(1423)	1	0	0	−1	−1	−1	0	1	0	
(1432)	−1	0	0	1	1	1	0	0	−1	
(2134)	0	−1	0	−1	0	0	0	0	−1	g_1

(2143)	0	1	0	1	0	0	−1	−1	−1	
(2314)	0	1	0	0	0	1	1	0	0	
(2341)	0	−1	0	0	0	−1	1	1	1	g_2
(2413)	0	−1	0	1	1	1	−1	0	0	
(2431)	0	1	0	−1	−1	−1	0	0	1	
(3124)	0	0	1	1	0	0	0	1	0	
(3142)	0	0	−1	−1	0	0	1	1	1	
(3214)	0	0	−1	0	−1	0	−1	0	0	
(3241)	0	0	1	0	1	0	−1	−1	−1	
(3412)	0	0	1	−1	−1	−1	1	0	0	
(3421)	0	0	−1	1	1	1	0	−1	1	
(4123)	1	1	1	−1	0	0	0	−1	0	
(4132)	−1	−1	−1	1	0	0	0	0	1	
(4213)	−1	−1	−1	0	1	0	1	0	0	
(4231)	1	1	1	0	−1	0	0	0	−1	
(4312)	1	1	1	0	0	−1	−1	0	0	
(4321)	−1	−1	−1	0	0	1	0	1	0	

Remark 4.2.1. (1) From Table 3.2, the values of the constants c_i for any 4-arc are

$$c_0 = 272, \quad c_1 = 102, \quad c_2 = 3.$$

(2) The three diagonal points of the frame are

$$\mathbf{U}_0\mathbf{U}_1 \cap \mathbf{U}_2\mathbf{U} = \mathbf{P}(1,1,0), \quad \mathbf{U}_0\mathbf{U} \cap \mathbf{U}_1\mathbf{U}_2 = \mathbf{P}(0,1,1), \quad \mathbf{U}_0\mathbf{U}_2 \cap \mathbf{U}_1\mathbf{U} = \mathbf{P}(1,0,1).$$

These points are not collinear.

(3) The diagonal points are exactly the three points of index two. The set of diagonal points is fixed by $\mathbf{S}_4$, the stabilizer group of the frame.

4.3 5-Arcs

The number of points on the sides of a tetrastigm is $l(4,19) = 109$. Hence the number of points not on the sides of tetrastigm is $l^*(4,19) = 381 - 109 = 272$. The projective group $\mathbf{S}_4$ of the standard frame Υ splits the 272 points not on the bisecants of Υ into 14 disjoint orbits as follows.

(1) { 7, 155, 94, 15, 33, 64, 65, 108, 103, 115, 12, 269, 51, 141, 274, 139, 342, 135, 198, 145, 298, 277, 343, 327 }.

(2) {8, 55, 258, 348, 194, 180, 297, 77, 39, 166, 323, 333, 42, 71, 328, 19, 313, 356, 376, 117, 349, 68, 307, 241 }.

(3) { 9, 118, 56, 345, 61, 128, 142, 37, 49, 225, 134, 165, 95, 182, 160, 197, 226, 16, 247, 235, 85, 275, 248, 59 }.

(4) {10, 317, 369, 44, 256, 207, 377, 233, 341, 364, 281, 304, 287, 339, 276, 292, 72, 74, 122, 181, 119, 111, 264, 130 }.

(5) {17, 45, 121, 229, 363, 303, 286, 60, 283, 223, 227, 105, 36, 208, 210, 20, 101, 144, 131, 230, 107, 215, 234, 314}.

(6) { 18, 350, 344, 53, 294, 189, 70, 353, 168, 62, 196, 254, 152, 32, 150, 100, 162, 280, 73, 285, 136, 57, 31, 360 }.

(7) {21, 34, 54, 243, 25, 133, 202, 213, 340, 305, 40, 265 }.

(8) { 22, 99, 324, 169, 306, 249, 316, 110, 75, 359, 106, 199, 147, 123, 29, 257, 167, 78, 93, 126, 148, 311, 332, 63 }.

(9) {23, 83, 246, 186, 140, 188, 200, 358, 178, 302, 365, 237, 346, 352, 282, 351, 52, 300, 354, 132, 149, 125, 143, 91 }.

(10) {24, 236, 120, 201, 176, 153, 161, 159}.

(11) {43, 109, 173, 191, 69, 288, 79, 158, 212, 211, 379, 219, 355, 315, 157, 347, 102, 163, 357, 177, 154, 329, 203, 190 }.

(12) {66, 209, 231, 179, 299, 192, 170, 89, 96, 279, 127,325 }.

(13) {76, 137, 104, 334, 164, 284, 172, 336, 295, 204, 174, 245 }.

(14) { 97, 272, 331, 361, 129, 296, 151, 205, 374, 146, 262, 156 }.

Hence, fourteen 5-arcs are constructed by adding one point from each orbit to Υ. They are listed with their stabilizer groups in Table 4.4.

TABLE 4.4: 5-arcs in $PG(2, 19)$

No.	The 5-arc	Stabilizer	The generator
1	$\{1, 2, 3, 263, 7\}$	I	1 0 0 0 1 0 0 0 1
2	$\{1, 2, 3, 263, 8\}$	I	1 0 0 0 1 0 0 0 1
3	$\{1, 2, 3, 263, 9\}$	$\mathbf{Z}_2$	7 2 1 0 0 6 0 5 0
4	$\{1, 2, 3, 263, 10\}$	I	1 0 0 0 1 0 0 0 1
5	$\{1, 2, 3, 263, 17\}$	I	1 0 0 0 1 0 0 0 1
6	$\{1, 2, 3, 263, 18\}$	$\mathbf{Z}_2$	1 1 1 −3 −1 −7 0 0 6
7	$\{1, 2, 3, 263, 21\}$	$\mathbf{Z}_2$	0 0 −1 1 1 1 −1 0 0
8	$\{1, 2, 3, 263, 22\}$	$\mathbf{Z}_2$	0 0 1 0 6 0 −2 0 0
9	$\{1, 2, 3, 263, 23\}$	I	1 0 0 0 1 0 0 0 1
10	$\{1, 2, 3, 263, 24\}$	$\mathbf{S}_3$	0 1 0 0 0 1 1 0 0 0 −7 0 8 0 0 0 0 −1
11	$\{1, 2, 3, 263, 43\}$	$\mathbf{Z}_2$	−1 9 4 0 9 0 1 1 1
12	$\{1, 2, 3, 263, 66\}$	$\mathbf{Z}_2$	1 1 1 0 0 −1 0 −1 0
13	$\{1, 2, 3, 263, 76\}$	$\mathbf{S}_3$	0 0 8 0 1 0 7 6 −1 0 0 −1 1 1 1 −1 0 0
14	$\{1, 2, 3, 263, 97\}$	$\mathbf{D}_5$	5 4 −1 −6 0 0 1 1 1 0 0 −1 1 1 1 −1 0 0

The stabilizer column in Table 4.4 shows that at least four of the fourteen 5-arcs are projectively distinct.

Theorem 4.3.1. *In $PG(2, 19)$, there are precisely five projectively distinct 5-arcs, as summarized in Table 4.5.*

Table 4.5: Inequivalent 5-arcs in $PG(2,19)$

Symbol	The 5-arc	Stabilizer
$\mathcal{A}_1$	$\{1,2,3,263,7\}$	I
$\mathcal{A}_2$	$\{1,2,3,263,9\}$	$\mathbf{Z}_2$
$\mathcal{A}_3$	$\{1,2,3,263,18\}$	$\mathbf{Z}_2$
$\mathcal{A}_4$	$\{1,2,3,263,24\}$	$\mathbf{S}_3$
$\mathcal{A}_5$	$\{1,2,3,263,97\}$	$\mathbf{D}_5$

Remark 4.3.2. (1) From Table 3.2, the values of the constants c_i for any 5-arc are

$$c_0 = 211, \quad c_1 = 150, \quad c_2 = 15.$$

(2) The 5-arcs $\mathcal{A}_2$ and $\mathcal{A}_3$ have the same constants c_i and isomorphic stabilizer groups but they are inequivalent.

(3) Because of the one-to-one correspondence between $PG(1,19)$ and a conic, Theorem 4.3.1 can be deduced as follows.

Let

$$\mathcal{C}^* = \mathbf{v}(X_1^2 - X_0X_2) = \{\mathbf{P}(t^2,t,1) \mid t \in \mathbf{F}_{19} \cup \{\infty\}\}$$

be a conic. Then the five pentads $\mathcal{P}_i$ as given in Table 2.9 correspond to inequivalent five 5-arcs $\mathcal{P}'_i$ on the conic $\mathcal{C}^*$. Each 5-arc $\mathcal{P}'_i$, $i = 1,\ldots,5$, is equivalent to one of $\mathcal{A}_j$, $j = 1,\ldots,5$. These equivalences and the matrix transformations are given in Table 4.6.

Table 4.6: Transforming $\mathcal{P}'_i$ to $\mathcal{A}_j$

$\mathcal{P}'_i \cong \mathcal{A}_j$	Matrix transformation								
$\mathcal{P}'_1 = \{1,3,263,250,177\} \cong \mathcal{A}_2$	−9	−9	−9	9	−9	0	0	0	9
$\mathcal{P}'_2 = \{1,3,263,250,374\} \cong \mathcal{A}_1$	0	0	9	−9	1	−9	9	0	0
$\mathcal{P}'_3 = \{1,3,263,104,248\} \cong \mathcal{A}_4$	5	−2	−3	−8	5	3	3	0	0
$\mathcal{P}'_4 = \{1,3,263,248,93\} \cong \mathcal{A}_3$	0	0	8	6	−5	−8	0	5	0
$\mathcal{P}'_5 = \{1,3,263,93,374\} \cong \mathcal{A}_5$	7	0	0	−9	−2	8	0	8	0

The process of finding the projectivity matrices of the 5-arcs $\mathcal{P}'_i$ is illustrated in the following example.

Example 4.3.3. Let $\mathcal{P}_1 = \{\infty, 0, 1, -1, 2\}$ be the pentad with stabilizer group $G_{\mathcal{P}_1} = \langle 1-t\rangle \cong \mathbf{Z}_2$ as in Table 2.9. Using the parametrization of the conic $\mathcal{C}^*$, the projective transformation $\mathfrak{I} : PG(1,19) \longrightarrow PG(1,19)$ given by $t \mapsto 1-t$ has the following effect on $\mathcal{C}^*$:

$$\mathfrak{I}' : \mathbf{P}(t^2, t, 1) \mapsto \mathbf{P}((1-t)^2, (1-t), 1).$$

So

$$\begin{aligned} x'_0 &= x_0 - 2x_1 + x_2, \\ x'_1 &= -x_1 + x_2, \\ x'_2 &= x_2. \end{aligned}$$

Therefore, $\mathfrak{I}' = \mathbf{M}\left(\begin{bmatrix} 1 & 0 & 0 \\ -2 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\right)$ and $G_{\mathcal{P}'_1} = \langle \mathfrak{I}' \rangle$.

Alternatively, the projective transformation can be deduced using the matrix in Section 3.6(11)(ii). □

4.4 Collinearities of the Diagonal Points of Pentastigm

Let $P_0 = \mathbf{U}_0, P_1 = \mathbf{U}_1, P_2 = \mathbf{U}_2, P_3 = \mathbf{U}, P_4 = \mathbf{P}(a_0, a_1, a_2)$ be the vertices of a pentastigm $\mathcal{P}$. Since the vertices of $\mathcal{P}$ form a 5-arc then P_4 cannot be collinear with any pair of other vertices; so

$$a_0 a_1 a_2 (a_0 - a_1)(a_0 - a_2)(a_1 - a_2) \neq 0.$$

Write $ij \cdot kl$ for $P_iP_j \cap P_kP_l$; then the following fifteen points are the diagonal points of $\mathcal{P}$.

$01 \cdot 23 = \mathbf{P}(1, 1, 0)$,
$01 \cdot 24 = \mathbf{P}(a_0, a_1, 0)$,
$01 \cdot 34 = \mathbf{P}(a_2 - a_0, a_2 - a_1, 0)$,
$02 \cdot 13 = \mathbf{P}(1, 0, 1)$,
$02 \cdot 14 = \mathbf{P}(a_0, 0, a_2)$,
$02 \cdot 34 = \mathbf{P}(a_1 - a_0, 0, a_1 - a_2)$,
$03 \cdot 12 = \mathbf{P}(0, 1, 1)$,
$03 \cdot 14 = \mathbf{P}(a_0, a_2, a_2)$,
$03 \cdot 24 = \mathbf{P}(a_0, a_1, a_1)$,
$04 \cdot 12 = \mathbf{P}(0, a_1, a_2)$,
$04 \cdot 13 = \mathbf{P}(a_2, a_1, a_2)$,
$04 \cdot 23 = \mathbf{P}(a_1, a_1, a_2)$,
$12 \cdot 34 = \mathbf{P}(0, a_1 - a_0, a_2 - a_0)$,
$13 \cdot 24 = \mathbf{P}(a_0, a_1, a_0)$,
$14 \cdot 23 = \mathbf{P}(a_0, a_0, a_2)$.

Lemma 4.4.1. *The condition that five diagonal points of a pentastigm $\mathcal{P}$ are collinear in $PG(2, q)$ is that $x^2 = x + 1$ has a solution in $\mathbf{F}_q$.*

Proof. See [28, Lemma 7.3(i)]. □

Since in $\mathbf{F}_{19}$ the equation $x^2 = x + 1$ has two solutions $5, -4$, so there is a pentastigm with five collinear diagonal points in $PG(2, 19)$.

The pentastigm $\mathcal{P}$ which has the 5-arc $\mathcal{A}_5 = \{\mathbf{U}_0, \mathbf{U}_1, \mathbf{U}_2, \mathbf{U}, \mathbf{P}(-5, -4, 1)\}$ as vertices has five diagonal points which are collinear as shown below.

The fifteen diagonal points of $\mathcal{A}_5$ in coordinate and numeral form are

$01 \cdot 23 = \mathbf{P}(1, 1, 0) = 220$,
$01 \cdot 24 = \mathbf{P}(6, 1, 0) = 80$,
$01 \cdot 34 = \mathbf{P}(5, 1, 0) = 308$,
$02 \cdot 13 = \mathbf{P}(1, 0, 1) = 320$,
$02 \cdot 14 = \mathbf{P}(-5, 0, 1) = 268$,
$02 \cdot 34 = \mathbf{P}(-4, 0, 1) = 261$,
$03 \cdot 12 = \mathbf{P}(0, 1, 1) = 221$,
$03 \cdot 14 = \mathbf{P}(-5, 1, 1) = 335$,
$03 \cdot 24 = \mathbf{P}(6, 1, 1) = 175$,
$04 \cdot 12 = \mathbf{P}(0, -4, 1) = 319$,
$04 \cdot 13 = \mathbf{P}(1, -4, 1) = 13$,
$04 \cdot 23 = \mathbf{P}(-4, -4, 1) = 270$,
$12 \cdot 34 = \mathbf{P}(0, -3, 1) = 252$,
$13 \cdot 24 = \mathbf{P}(1, -3, 1) = 278$,
$14 \cdot 23 = \mathbf{P}(-5, -5, 1) = 255$.

Amongst these, the five diagonal points

$$
\begin{aligned}
03 \cdot 24 &= \mathbf{P}(6,1,1), \\
14 \cdot 23 &= \mathbf{P}(-5,-5,1), \\
01 \cdot 34 &= \mathbf{P}(5,1,0), \\
04 \cdot 12 &= \mathbf{P}(0,-4,1), \\
02 \cdot 13 &= \mathbf{P}(1,0,1),
\end{aligned}
$$

lie on the line $\mathbf{v}(-X_0 + 5X_1 + X_2)$.

Remark 4.4.2. (1) The ten sides of the pentastigm $\mathcal{P}$ are separated into five pairs such that no pair meets at vertex. Also the point $P_4 = \mathbf{P}(-5,-4,1)$ satisfies the equations

$$a_1 = a_0 + a_2, \qquad a_0^2 - a_2^2 = -a_0a_2,$$

which are the conditions for the above collinearities.

(2) The fifteen diagonals points of $\mathcal{A}_5$ are exactly the fifteen points of index two.

4.5 Conics Through the Inequivalent 5-Arcs

As mentioned in Section 3.6(4), there is a unique conic through each 5-arc. Let

$$F = a_0X_0^2 + a_1X_1^2 + a_2X_2^2 + a_3X_0X_1 + a_4X_0X_2 + a_5X_1X_2$$

be a form of degree two and $\mathcal{C} = \mathbf{v}(F)$ be a conic. Since all five 5-arcs $\mathcal{A}_i$ contain the points $\mathbf{U}_0, \mathbf{U}_1, \mathbf{U}_2$ then the form F reduces to

$$X_0X_1 + a_4'X_0X_2 + a_5'X_1X_2. \tag{4.1}$$

Therefore, by substituting $\mathbf{U}$ and the $5th$ point of each 5-arc $\mathcal{A}_i$ in (4.1) the following is deduced. Let $t \in \mathbf{F}_{19} \cup \{\infty\}$; then

$$
\begin{aligned}
\mathcal{C}_{\mathcal{A}_1} &= \mathbf{v}(X_0X_1 - 2X_0X_2 + X_1X_2) = \{\mathbf{P}(9(1-t), t, 9(t^2-t))\}; \\
\mathcal{C}_{\mathcal{A}_2} &= \mathbf{v}(X_0X_1 + 9X_0X_2 + 9X_1X_2) = \{\mathbf{P}(9(t-t^2), -9(t^2+t), 9(1-t^2))\}; \\
\mathcal{C}_{\mathcal{A}_3} &= \mathbf{v}(X_0X_1 + 3X_0X_2 - 4X_1X_2) = \{\mathbf{P}(6t, 5(1-t), 8(t^2-t))\}; \\
\mathcal{C}_{\mathcal{A}_4} &= \mathbf{v}(X_0X_1 - 8X_0X_2 + 7X_1X_2) = \{\mathbf{P}((t-1)(5t-3), (5t-2t^2), 3(t-t^2))\}; \\
\mathcal{C}_{\mathcal{A}_5} &= \mathbf{v}(X_0X_1 + 5X_0X_2 - 6X_1X_2) = \{\mathbf{P}(7(t^2-4t), 8(1-5t), 8t)\}.
\end{aligned}
$$

4.6 The Group Action of D_5 on the Pentad $\mathcal{A}_5$

From Table 4.5, the group $\mathbf{D}_5 = \langle h, g \mid g^2 = h^5 = I,\ hg = gh^{-1}\rangle$ where

$$g = \begin{bmatrix} 1 & 0 & 0 \\ -5 & -4 & 1 \\ 4 & 4 & 4 \end{bmatrix}, \quad h = \begin{bmatrix} 0 & -1 & 0 \\ 5 & 5 & 5 \\ -5 & -4 & 1 \end{bmatrix}$$

is the stabilizer group of the 5-arc $\mathcal{A}_5 = \{1, 2, 3, 263, 97\}$.

The Group $\mathbf{D}_5$ acts transitively on $\mathcal{A}_5$ as given below:

$$1 \stackrel{gh}{\longmapsto} 2, \qquad 1 \stackrel{gh^2}{\longmapsto} 263,$$
$$1 \stackrel{gh^3}{\longmapsto} 3, \qquad 1 \stackrel{h^4}{\longmapsto} 97.$$

Each of the five projectivities g, gh, gh^3, gh^2, gh^4 fixes 15 points amongst the 211 points of index zero by transforming each point to itself. Each of these 15 points lies on a line which is a unisecant to $\mathcal{A}_5$ and a bisecant of the conic

$$\mathcal{C}_{\mathcal{A}_5} = \mathbf{v}(X_0X_1 + 5X_0X_2 - 6X_1X_2).$$

These lines are

$$\ell_{167} = \mathbf{v}(X_1 + 3X_2); \qquad \ell_{272} = \mathbf{v}(X_0 + 4X_2);$$
$$\ell_{77} = \mathbf{v}(4X_0 + X_1); \qquad \ell_{279} = \mathbf{v}(X_0 - 3X_1 - X_2).$$
$$\ell_{220} = \mathbf{v}(X_0 - X_1 + X_2);$$

In Table 4.7, each row contains the projectivity f that fixes the set of 15 points which lies on the line ℓ_i.

Table 4.7: Projectivities fixing 15 points

f	Set of 15 points lies on ℓ_i fixed by f	ℓ_i
g	$\{23, 36, 44, 51, 74, 93, 151, 156, 165, 167, 168, 192, 212, 246, 349\}$	ℓ_{167}
gh	$\{61, 66, 75, 77, 78, 102, 122, 156, 162, 188, 292, 296, 314, 327, 365\}$	ℓ_{77}
gh^3	$\{21, 54, 76, 89, 104, 127, 146, 156, 204, 209, 245, 265, 299, 305, 331\}$	ℓ_{220}
gh^2	$\{73, 106, 110, 128, 149, 156, 179, 198, 208, 256, 272, 297, 317, 351, 357\}$	ℓ_{272}
gh^4	$\{9, 117, 135, 148, 156, 163, 186, 205, 215, 279, 280, 304, 324, 358, 364\}$	ℓ_{279}

The five lines $\ell_{167}, \ell_{77}, \ell_{220}, \ell_{272}, \ell_{279}$ are concurrent at an internal point $\mathbf{P}(-4, -3, 1) =$ 156 which is fixed by $\mathbf{D}_5$ as well.

4.7 6-Arcs

The number of points on the sides of pentastigm is $l(5,19) = 170$. Hence the number of points not on the sides of each pentastigm is $l^*(5,19) = 381 - 170 = 211$. So the total number of points not on the sides of the five pentastigms is 1055. The action of the stabilizer group of each inequivalent 5-arc on the corresponding set C_0^5 splits the 1055 points into 509 orbits. The details about the 509 orbits are given in Table 4.8.

A cell $n' : m'$ in Table 4.8 means that n' is the number of orbits of the 5-arc of length m'.

TABLE 4.8: Size of orbits of the 5-arcs

5-arc	$\mathcal{A}_1$	$\mathcal{A}_2$		$\mathcal{A}_3$		$\mathcal{A}_4$			$\mathcal{A}_5$		
Total number of orbits	211	113		113		43			29		
$n' : m'$	211:1	15:1	98:2	15:1	98:2	1:1	14:3	28:6	1:1	14:5	14:10

There are seven different classes of 6-arcs of type $[c_0, c_1, c_2, c_3]$ and eight different sizes of stabilizer groups. The details about them are given Table 4.9. A cell $n : |G|$ in Table 4.9 means that n is the number of 6-arcs stabilized by the group G of size m.

TABLE 4.9: Statistics of the constants c_i of 6-arcs

No.	$[c_0, c_1, c_2, c_3]$	$n : \lvert G \rvert$
1	$[140, 210, 15, 10]$	$1 : 60$
2	$[144, 198, 27, 6]$	$3 : 12, \quad 1 : 36$
3	$[146, 192, 33, 4]$	$10 : 6, \quad 1 : 12$
4	$[147, 189, 36, 3]$	$28 : 3, \quad 2 : 6$
5	$[148, 186, 39, 2]$	$76 : 2, \quad 12 : 4$
6	$[149, 183, 42, 1]$	$210 : 1, \quad 15 : 2$
7	$[150, 180, 45, 0]$	$150 : 1$

Note that the constants c_i in Table 4.9 satisfy the values in Table 3.2 for $k = 6$. From Table 4.9, a 6-arc have at most ten points of index three.

In Table 4.10, the 509 6-arcs are arranged according to the additional points to each 5-arc $\mathcal{A}_i, i = 1, 2, 3, 4, 5$, and the size of the stabilizer groups of the 6-arcs. Here $O(G)$ refers to size of the stabilizer groups of the 6-arcs.

Table 4.10: Points of index zero and order of the stabilizer group of 6-arcs

5-arc	The additional points	$O(G)$
$\mathcal{A}_1$	8, 9, 10, 20, 21, 23, 24, 29, 31, 32, 34, 36, 37, 39, 40, 42, 44, 52, 54, 55, 57, 59, 60, 61, 62, 66, 68, 69, 71, 75, 78, 83, 85, 89, 91, 93, 96, 99, 100, 101, 102, 108, 109, 110, 111, 115, 118, 119, 120, 121, 123, 125, 128, 130, 131, 132, 133, 134, 139, 140, 143, 147, 149, 150, 151, 152, 154, 157, 161, 162, 165, 168, 173, 174, 176, 177, 178, 179, 181, 182, 186, 189, 190, 191, 196, 201, 202, 203, 204, 205, 207, 208, 209, 211, 212, 213, 215, 219, 223, 226, 231, 234, 236, 241, 243, 245, 246, 247, 249, 258, 262, 264, 265, 272, 274, 275, 277, 281, 282, 284, 286, 288, 295, 297, 299, 302, 305, 307, 311, 314, 315, 316, 317, 324, 329, 332, 333, 336, 340, 343, 344, 345, 347, 349, 350, 352, 353, 354, 355, 357, 359, 360, 361, 363, 374, 376, 377, 379	(1)
	12, 15, 45, 49, 51, 64, 73, 94, 136, 142, 146, 148, 155, 156, 163, 164, 188, 197, 199, 235, 256, 257, 269, 285, 294, 298, 323, 334, 341, 342	(2)
	22, 72, 74, 77, 107, 117, 122, 127, 144, 225, 227, 230, 279, 313, 356, 358, 364,365	(3)
	56, 135, 145, 198	(4)
	327	(6)
$\mathcal{A}_2$	7, 8, 10, 15, 17, 18, 20, 21, 22, 23, 25, 29, 33, 36, 37, 40, 42, 45, 49, 52, 53, 57, 60, 66, 69, 70, 71, 72, 76, 77, 78, 79, 89, 91, 93, 94, 100, 105, 107, 110, 111, 115, 119, 120, 122, 129, 131, 134, 137, 146, 149, 151, 153, 155, 158, 165, 168, 169, 172, 181, 192, 194, 196, 198, 208, 209, 213, 225, 227, 229, 233, 245, 246, 248, 254, 262, 275, 276, 331, 332, 345, 353	(1)
	19, 34, 44, 55, 59, 62, 95, 97, 101, 103, 104, 167, 178, 179, 199, 223, 235, 257, 360	(2)
	31, 43, 54, 108, 202, 286	(3)
	12, 118	(4)
	162, 176	(6)
	85, 203	(12)

$\mathcal{A}_3$	8, 9, 10, 12, 20, 21, 23, 25, 29, 32, 33, 34, 36, 44, 45, 49, 51, 52, 55, 56, 57, 65, 66, 71, 72, 74, 76, 79, 85, 89, 93, 94, 95, 99, 101, 104, 109, 111, 119, 122, 133, 137, 139, 140, 143, 144, 146, 149, 150, 155, 157, 161, 164, 165, 167, 169, 180, 181, 188, 190, 196, 203, 207, 210, 215, 223, 234, 235, 241, 247, 249, 262, 274, 281, 284, 287, 294, 302, 333, 359	(1)
	31, 37, 39, 59, 83, 100, 105, 141, 145, 147, 153, 162, 166, 172, 192, 258, 280, 304, 313, 329	(2)
	75, 77, 127, 197	(3)
	54, 131, 156	(4)
	24, 73, 136, 152, 226	(6)
	148	(12)
$\mathcal{A}_4$	7, 8, 12, 19, 20, 21, 22, 33, 36, 39, 44, 55, 56, 59, 65, 66, 70, 76, 77, 78, 79, 93, 100, 132, 140, 156	(1)
	9, 17, 32, 42, 62, 63, 89, 91, 102, 120, 126, 243	(2)
	23	(4)
	18, 61	(6)
	204	(12)
	236	(36)
$\mathcal{A}_5$	8, 15, 17, 20, 24, 33, 39, 42, 52, 56, 60, 62, 68, 72	(1)
	9, 23, 36, 51, 54, 66, 74, 76, 78, 102	(2)
	73, 77	(4)
	75, 110	(6)
	156	(60)

For each of the eight different sizes of stabilizer groups in Table 4.10 and the type of parameters $[c_0, c_1, c_2, c_3]$, the inequivalence of the corresponding 6-arcs was checked.

Theorem 4.7.1. *In $PG(2,19)$, there are precisely 117 projectively distinct 6-arcs given with their stabilizer group types in Table 4.11.*

Table 4.11: Inequivalent 6-arcs

Symbol	The 6-arc	Stabilizer	Symbol	The 6-arc	Stabilizer
$\mathcal{B}_1$	$\{1,2,3,263,7,8\}$	I	$\mathcal{B}_2$	$\{1,2,3,263,7,9\}$	I
$\mathcal{B}_3$	$\{1,2,3,263,7,10\}$	I	$\mathcal{B}_4$	$\{1,2,3,263,7,20\}$	I
$\mathcal{B}_5$	$\{1,2,3,263,7,21\}$	I	$\mathcal{B}_6$	$\{1,2,3,263,7,23\}$	I
$\mathcal{B}_7$	$\{1,2,3,263,7,24\}$	I	$\mathcal{B}_8$	$\{1,2,3,263,7,29\}$	I
$\mathcal{B}_9$	$\{1,2,3,263,7,31\}$	I	$\mathcal{B}_{10}$	$\{1,2,3,263,7,32\}$	I
$\mathcal{B}_{11}$	$\{1,2,3,263,7,34\}$	I	$\mathcal{B}_{12}$	$\{1,2,3,263,7,36\}$	I
$\mathcal{B}_{13}$	$\{1,2,3,263,7,37\}$	I	$\mathcal{B}_{14}$	$\{1,2,3,263,7,39\}$	I
$\mathcal{B}_{15}$	$\{1,2,3,263,7,40\}$	I	$\mathcal{B}_{16}$	$\{1,2,3,263,7,42\}$	I
$\mathcal{B}_{17}$	$\{1,2,3,263,7,44\}$	I	$\mathcal{B}_{18}$	$\{1,2,3,263,7,54\}$	I
$\mathcal{B}_{19}$	$\{1,2,3,263,7,55\}$	I	$\mathcal{B}_{20}$	$\{1,2,3,263,7,57\}$	I
$\mathcal{B}_{21}$	$\{1,2,3,263,7,60\}$	I	$\mathcal{B}_{22}$	$\{1,2,3,263,7,61\}$	I
$\mathcal{B}_{23}$	$\{1,2,3,263,7,62\}$	I	$\mathcal{B}_{24}$	$\{1,2,3,263,7,66\}$	I
$\mathcal{B}_{25}$	$\{1,2,3,263,7,69\}$	I	$\mathcal{B}_{26}$	$\{1,2,3,263,7,71\}$	I
$\mathcal{B}_{27}$	$\{1,2,3,263,7,75\}$	I	$\mathcal{B}_{28}$	$\{1,2,3,263,7,78\}$	I
$\mathcal{B}_{29}$	$\{1,2,3,263,7,91\}$	I	$\mathcal{B}_{30}$	$\{1,2,3,263,7,96\}$	I
$\mathcal{B}_{31}$	$\{1,2,3,263,7,99\}$	I	$\mathcal{B}_{32}$	$\{1,2,3,263,7,100\}$	I
$\mathcal{B}_{33}$	$\{1,2,3,263,7,101\}$	I	$\mathcal{B}_{34}$	$\{1,2,3,263,7,102\}$	I
$\mathcal{B}_{35}$	$\{1,2,3,263,7,109\}$	I	$\mathcal{B}_{36}$	$\{1,2,3,263,7,118\}$	I
$\mathcal{B}_{37}$	$\{1,2,3,263,7,119\}$	I	$\mathcal{B}_{38}$	$\{1,2,3,263,7,121\}$	I
$\mathcal{B}_{39}$	$\{1,2,3,263,7,130\}$	I	$\mathcal{B}_{40}$	$\{1,2,3,263,7,134\}$	I
$\mathcal{B}_{41}$	$\{1,2,3,263,7,139\}$	I	$\mathcal{B}_{42}$	$\{1,2,3,263,7,147\}$	I
$\mathcal{B}_{43}$	$\{1,2,3,263,7,150\}$	I	$\mathcal{B}_{44}$	$\{1,2,3,263,7,151\}$	I
$\mathcal{B}_{45}$	$\{1,2,3,263,7,161\}$	I	$\mathcal{B}_{46}$	$\{1,2,3,263,7,168\}$	I
$\mathcal{B}_{47}$	$\{1,2,3,263,7,173\}$	I	$\mathcal{B}_{48}$	$\{1,2,3,263,7,176\}$	I
$\mathcal{B}_{49}$	$\{1,2,3,263,7,182\}$	I	$\mathcal{B}_{50}$	$\{1,2,3,263,7,189\}$	I
$\mathcal{B}_{51}$	$\{1,2,3,263,7,190\}$	I	$\mathcal{B}_{52}$	$\{1,2,3,263,7,213\}$	I
$\mathcal{B}_{53}$	$\{1,2,3,263,7,247\}$	I	$\mathcal{B}_{54}$	$\{1,2,3,263,7,272\}$	I
$\mathcal{B}_{55}$	$\{1,2,3,263,7,275\}$	I	$\mathcal{B}_{56}$	$\{1,2,3,263,7,282\}$	I
$\mathcal{B}_{57}$	$\{1,2,3,263,7,295\}$	I	$\mathcal{B}_{58}$	$\{1,2,3,263,7,317\}$	I
$\mathcal{B}_{59}$	$\{1,2,3,263,7,324\}$	I	$\mathcal{B}_{60}$	$\{1,2,3,263,7,374\}$	I
$\mathcal{B}_{61}$	$\{1,2,3,263,7,12\}$	$\mathbf{Z}_2$	$\mathcal{B}_{62}$	$\{1,2,3,263,7,15\}$	$\mathbf{Z}_2$
$\mathcal{B}_{63}$	$\{1,2,3,263,7,45\}$	$\mathbf{Z}_2$	$\mathcal{B}_{64}$	$\{1,2,3,263,7,49\}$	$\mathbf{Z}_2$
$\mathcal{B}_{65}$	$\{1,2,3,263,7,64\}$	$\mathbf{Z}_2$	$\mathcal{B}_{66}$	$\{1,2,3,263,7,73\}$	$\mathbf{Z}_2$
$\mathcal{B}_{67}$	$\{1,2,3,263,7,136\}$	$\mathbf{Z}_2$	$\mathcal{B}_{68}$	$\{1,2,3,263,7,142\}$	$\mathbf{Z}_2$
$\mathcal{B}_{69}$	$\{1,2,3,263,7,146\}$	$\mathbf{Z}_2$	$\mathcal{B}_{70}$	$\{1,2,3,263,7,148\}$	$\mathbf{Z}_2$
$\mathcal{B}_{71}$	$\{1,2,3,263,7,155\}$	$\mathbf{Z}_2$	$\mathcal{B}_{72}$	$\{1,2,3,263,7,156\}$	$\mathbf{Z}_2$
$\mathcal{B}_{73}$	$\{1,2,3,263,7,163\}$	$\mathbf{Z}_2$	$\mathcal{B}_{74}$	$\{1,2,3,263,7,164\}$	$\mathbf{Z}_2$
$\mathcal{B}_{75}$	$\{1,2,3,263,7,188\}$	$\mathbf{Z}_2$	$\mathcal{B}_{76}$	$\{1,2,3,263,7,197\}$	$\mathbf{Z}_2$
$\mathcal{B}_{77}$	$\{1,2,3,263,7,199\}$	$\mathbf{Z}_2$	$\mathcal{B}_{78}$	$\{1,2,3,263,7,257\}$	$\mathbf{Z}_2$
$\mathcal{B}_{79}$	$\{1,2,3,263,7,285\}$	$\mathbf{Z}_2$	$\mathcal{B}_{80}$	$\{1,2,3,263,7,294\}$	$\mathbf{Z}_2$
$\mathcal{B}_{81}$	$\{1,2,3,263,7,298\}$	$\mathbf{Z}_2$	$\mathcal{B}_{82}$	$\{1,2,3,263,7,334\}$	$\mathbf{Z}_2$
$\mathcal{B}_{83}$	$\{1,2,3,263,7,342\}$	$\mathbf{Z}_3$	$\mathcal{B}_{84}$	$\{1,2,3,263,7,22\}$	$\mathbf{Z}_3$
$\mathcal{B}_{85}$	$\{1,2,3,263,7,72\}$	$\mathbf{Z}_3$	$\mathcal{B}_{86}$	$\{1,2,3,263,7,74\}$	$\mathbf{Z}_3$

$\mathcal{B}_{87}$	$\{1,2,3,263,7,77\}$	$\mathbf{Z}_3$	$\mathcal{B}_{88}$	$\{1,2,3,263,7,107\}$	$\mathbf{Z}_3$
$\mathcal{B}_{89}$	$\{1,2,3,263,7,117\}$	$\mathbf{Z}_3$	$\mathcal{B}_{90}$	$\{1,2,3,263,7,127\}$	$\mathbf{Z}_3$
$\mathcal{B}_{91}$	$\{1,2,3,263,7,225\}$	$\mathbf{Z}_3$	$\mathcal{B}_{92}$	$\{1,2,3,263,7,227\}$	$\mathbf{Z}_3$
$\mathcal{B}_{93}$	$\{1,2,3,263,7,279\}$	$\mathbf{Z}_3$	$\mathcal{B}_{94}$	$\{1,2,3,263,7,356\}$	$\mathbf{Z}_3$
$\mathcal{B}_{95}$	$\{1,2,3,263,7,358\}$	$\mathbf{Z}_3$	$\mathcal{B}_{96}$	$\{1,2,3,263,7,364\}$	$\mathbf{Z}_3$
$\mathcal{B}_{97}$	$\{1,2,3,263,7,56\}$	$\mathbf{Z}_4$	$\mathcal{B}_{98}$	$\{1,2,3,263,7,135\}$	$\mathbf{V}_4$
$\mathcal{B}_{99}$	$\{1,2,3,263,7,145\}$	$\mathbf{Z}_4$	$\mathcal{B}_{100}$	$\{1,2,3,263,7,198\}$	$\mathbf{V}_4$
$\mathcal{B}_{101}$	$\{1,2,3,263,7,327\}$	$\mathbf{S}_3$			
$\mathcal{B}_{102}$	$\{1,2,3,263,9,62\}$	$\mathbf{Z}_2$	$\mathcal{B}_{103}$	$\{1,2,3,263,9,31\}$	$\mathbf{Z}_3$
$\mathcal{B}_{104}$	$\{1,2,3,263,9,118\}$	$\mathbf{V}_4$	$\mathcal{B}_{105}$	$\{1,2,3,263,9,162\}$	$\mathbf{S}_3$
$\mathcal{B}_{106}$	$\{1,2,3,263,9,176\}$	$\mathbf{S}_3$	$\mathcal{B}_{107}$	$\{1,2,3,263,9,85\}$	$\mathbf{D}_6$
$\mathcal{B}_{108}$	$\{1,2,3,263,9,203\}$	$\mathbf{A}_4$			
$\mathcal{B}_{109}$	$\{1,2,3,263,18,156\}$	$\mathbf{Z}_4$	$\mathcal{B}_{110}$	$\{1,2,3,263,18,24\}$	$\mathbf{S}_3$
$\mathcal{B}_{111}$	$\{1,2,3,263,18,73\}$	$\mathbf{S}_3$	$\mathcal{B}_{112}$	$\{1,2,3,263,18,136\}$	$\mathbf{S}_3$
$\mathcal{B}_{113}$	$\{1,2,3,263,18,152\}$	$\mathbf{S}_3$	$\mathcal{B}_{114}$	$\{1,2,3,263,18,148\}$	$\mathbf{A}_4$
$\mathcal{B}_{115}$	$\{1,2,3,263,24,204\}$	$\mathbf{A}_4$	$\mathcal{B}_{116}$	$\{1,2,3,263,24,236\}$	G_{36}
$\mathcal{B}_{117}$	$\{1,2,3,263,97,156\}$	$\mathbf{A}_5$			

The group G_{36} is a group of order 36 that has 9 elements of order 2, 8 elements of order 3 and 18 elements of order 4.

According to Table 4.9, the inequivalent 6-arcs fall into seven classes. The classes are given in Table 4.12. A cell $\mathcal{B}_{i'}\ldots\mathcal{B}_{j'} : n$ in Table 4.12 means that n of the 6-arcs have the parameters $[c_0, c_1, c_2, c_3]$.

Table 4.12: Statistics of the constants c_i of the inequivalent 6-arcs

$[c_0, c_1, c_2, c_3]$	$\mathcal{B}_{i'}\ldots\mathcal{B}_{j'} : n$
$[140, 210, 15, 10]$	$\mathcal{B}_{117} : 1$
$[144, 198, 27, 6]$	$\mathcal{B}_{108}$, $\mathcal{B}_{114}$, $\mathcal{B}_{115}$, $\mathcal{B}_{116}$: 4
$[146, 192, 33, 4]$	$\mathcal{B}_{105}$,$\mathcal{B}_{106}$,$\mathcal{B}_{107}$, $\mathcal{B}_{110}$,$\mathcal{B}_{111}$,$\mathcal{B}_{112}$: 6
$[147, 189, 36, 3]$	$\mathcal{B}_{84}$,$\mathcal{B}_{85}$,$\mathcal{B}_{86}$,$\mathcal{B}_{87}$,$\mathcal{B}_{88}$,$\mathcal{B}_{89}$,$\mathcal{B}_{90}$,$\mathcal{B}_{91}$,$\mathcal{B}_{92}$,$\mathcal{B}_{93}$,$\mathcal{B}_{94}$,$\mathcal{B}_{95}$,$\mathcal{B}_{96}$,$\mathcal{B}_{101}$,$\mathcal{B}_{103}$,$\mathcal{B}_{113}$: 16
$[148, 186, 39, 2]$	$\mathcal{B}_{62}$, $\mathcal{B}_{63}$, $\mathcal{B}_{66}$, $\mathcal{B}_{67}$, $\mathcal{B}_{69}$, $\mathcal{B}_{70}$, $\mathcal{B}_{71}$, $\mathcal{B}_{72}$, $\mathcal{B}_{73}$, $\mathcal{B}_{74}$, $\mathcal{B}_{75}$, $\mathcal{B}_{77}$, $\mathcal{B}_{78}$, $\mathcal{B}_{79}$, $\mathcal{B}_{80}$, $\mathcal{B}_{81}$, $\mathcal{B}_{82}$, $\mathcal{B}_{83}$, $\mathcal{B}_{97}$, $\mathcal{B}_{98}$, $\mathcal{B}_{99}$, $\mathcal{B}_{100}$, $\mathcal{B}_{102}$, $\mathcal{B}_{104}$, $\mathcal{B}_{109}$: 25
$[149, 183, 42, 1]$	$\mathcal{B}_{2}$, $\mathcal{B}_{6}$, $\mathcal{B}_{8}$, $\mathcal{B}_{9}$, $\mathcal{B}_{10}$, $\mathcal{B}_{11}$, $\mathcal{B}_{13}$, $\mathcal{B}_{14}$, $\mathcal{B}_{19}$, $\mathcal{B}_{21}$, $\mathcal{B}_{22}$, $\mathcal{B}_{23}$, $\mathcal{B}_{24}$, $\mathcal{B}_{25}$, $\mathcal{B}_{26}$, $\mathcal{B}_{27}$, $\mathcal{B}_{28}$, $\mathcal{B}_{29}$, $\mathcal{B}_{33}$, $\mathcal{B}_{34}$, $\mathcal{B}_{35}$, $\mathcal{B}_{37}$, $\mathcal{B}_{38}$, $\mathcal{B}_{39}$, $\mathcal{B}_{40}$, $\mathcal{B}_{41}$, $\mathcal{B}_{42}$, $\mathcal{B}_{43}$, $\mathcal{B}_{45}$, $\mathcal{B}_{50}$, $\mathcal{B}_{51}$, $\mathcal{B}_{54}$, $\mathcal{B}_{57}$, $\mathcal{B}_{58}$, $\mathcal{B}_{59}$, $\mathcal{B}_{61}$, $\mathcal{B}_{64}$, $\mathcal{B}_{65}$, $\mathcal{B}_{68}$, $\mathcal{B}_{76}$: 40
$[150, 180, 45, 0]$	$\mathcal{B}_{1}$, $\mathcal{B}_{3}$, $\mathcal{B}_{4}$, $\mathcal{B}_{5}$, $\mathcal{B}_{7}$, $\mathcal{B}_{12}$, $\mathcal{B}_{15}$, $\mathcal{B}_{16}$, $\mathcal{B}_{17}$, $\mathcal{B}_{18}$, $\mathcal{B}_{20}$, $\mathcal{B}_{30}$, $\mathcal{B}_{31}$, $\mathcal{B}_{32}$, $\mathcal{B}_{36}$, $\mathcal{B}_{44}$, $\mathcal{B}_{46}$, $\mathcal{B}_{47}$, $\mathcal{B}_{48}$, $\mathcal{B}_{49}$, $\mathcal{B}_{52}$, $\mathcal{B}_{53}$, $\mathcal{B}_{55}$, $\mathcal{B}_{56}$, $\mathcal{B}_{60}$: 25

The calculations shows that $c_0 \neq 0$ for all 6-arcs in $PG(2,19)$ as in Table 4.9. This result coincides with Corollary 3.8.3 that there is no complete 6-arc in $PG(2,19)$.

4.8 Properties of the 6-Arc $\mathcal{B}_{117}$

(1) Let

$$\begin{aligned}\mathcal{K} =& \{\mathbf{U}_0, \mathbf{U}_1, \mathbf{U}_2, \mathbf{U}, \mathbf{P}(a, b, 1), \mathbf{P}(c, d, 1)\}\\ =& \{P_1, P_2, P_3, P_4, P_5, P_6\}\end{aligned}$$

be a 6-arc. A point of index three is called a *Brianchon point* or B-point for short. Write $ij \cdot kl \cdot mn = P_iP_j \cap P_kP_l \cap P_mP_n$ for a B-point. There are fifteen ways of choosing three bisecants no two of which intersect on $\mathcal{K}$. The following are the conditions for these fifteen ways.

(1)	$12 \cdot 34 \cdot 56$	$a - c = b - d;$
(2)	$12 \cdot 35 \cdot 46$	$a(d-1) = b(c-1);$
(3)	$12 \cdot 36 \cdot 45$	$d(a-1) = c(b-1);$
(4)	$13 \cdot 24 \cdot 56$	$d(a-1) = b(c-1);$
(5)	$13 \cdot 25, 46$	$(a-1)(d-1) = (1-c);$
(6)	$13 \cdot 26 \cdot 45$	$1 - a = (b-1)(c-1);$
(7)	$14 \cdot 23 \cdot 56$	$a(d-1) = c(b-1);$
(8)	$14 \cdot 25 \cdot 36$	$c = ad;$
(9)	$14 \cdot 26 \cdot 35$	$a = bc;$
(10)	$15 \cdot 23 \cdot 46$	$1 - d = (b - 1(c-1);$
(11)	$15 \cdot 24 \cdot 36$	$d = bc;$
(12)	$15 \cdot 26 \cdot 34$	$b = c;$
(13)	$16 \cdot 23 \cdot 45$	$(a-1)(d-1) = (1-b);$
(14)	$16 \cdot 24 \cdot 35$	$b = ad;$
(15)	$16 \cdot 25 \cdot 34$	$a = d.$

The ten B-points of $\mathcal{B}_{117} = \{1, 2, 3, 263, 97, 156\}$ are

(1)	$12 \cdot 34 \cdot 56 = 220;$	(6)	$14 \cdot 25 \cdot 36 = 335;$
(2)	$12 \cdot 35 \cdot 46 = 80;$	(7)	$15 \cdot 24 \cdot 36 = 13;$
(3)	$13 \cdot 25 \cdot 46 = 268;$	(8)	$15 \cdot 26 \cdot 34 = 270;$
(4)	$13 \cdot 26 \cdot 45 = 261;$	(9)	$16 \cdot 23 \cdot 45 = 252;$
(5)	$14 \cdot 23 \cdot 56 = 221;$	(10)	$16 \cdot 24 \cdot 35 = 278.$

The set $\mathcal{K}_{10} = \{13, 80, 220, 221, 252, 261, 268, 270, 278, 335\}$ of B-points of $\mathcal{B}_{117}$ forms a 10-arc. More details about this 10-arc are given in Section 4.17.

The remaining five possibilities form triangles as shown below.

	The three lines			The vertices
(I)	$P_1P_2 = \mathbf{v}(X_2)$	$P_3P_6 = \mathbf{v}(3X_0 - 4X_1)$	$P_4P_5 = \mathbf{v}(5X_0 - 6X_1 + X_2)$	$\{259, 308, 342\}$
(II)	$P_1P_3 = \mathbf{v}(X_1)$	$P_2P_4 = \mathbf{v}(X_0 - X_2)$	$P_5P_6 = \mathbf{v}(X_0 - X_1 + X_2)$	$\{320, 218, 58\}$
(III)	$P_1P_4 = \mathbf{v}(X_1 - X_2)$	$P_2P_6 = \mathbf{v}(X_0 + 4X_2)$	$P_3P_5 = \mathbf{v}(X_0 - 6X_1)$	$\{273, 175, 141\}$
(IV)	$P_1P_5 = \mathbf{v}(X_1 + 4X_2)$	$P_2P_3 = \mathbf{v}(X_0)$	$P_4P_6 = \mathbf{v}(X_0 - 6X_1 + 5X_2)$	$\{319, 277, 113\}$
(V)	$P_1P_6 = \mathbf{v}(X_1 + 3X_2)$	$P_2P_5 = \mathbf{v}(X_0 + 5X_2)$	$P_3P_4 = \mathbf{v}(X_0 - X_1)$	$\{103, 5, 255\}$

Let $\mathcal{W} = \{\text{I,II,III,IV,V}\}$ be the set of five triangles. The stabilizer group $\mathbf{A}_5$ of $\mathcal{B}_{117}$ also fixes the set $\mathcal{W}$ of five triangles .

(2) L. Storme and V. Maldeghem [48] in Proposition 13 proved that, with $4t^2 - 2t - 1 = 0,\ t \in \mathbf{F}_q$, a 6-arc

$$K_6^* = \{(1, 0, 1 - 2t), (1, 0, 2t - 1), (1, 2t, 0), (1, -2t, 0), (0, 1, 2t), (0, 1, -2t)\}$$

in $PG(2, q)$ when $q \equiv \pm 1 \pmod{10}$ is the unique 6-arc with stabilizer group $\mathbf{A}_5$. In $\mathbf{F}_{19}$, the equation $4t^2 - 2t - 1 = 0$ has two solutions $-2, -7$. In $PG(2, 19)$, for $t = -2$, the 6-arc

$$K_6^* = \{373, 261, 259, 308, 260, 309\}$$

in numeral form is equivalent to the 6-arc $\mathcal{B}_{117}$ by the matrix transformation

$$A = \begin{bmatrix} 5 & 6 & 0 \\ 1 & -4 & 9 \\ 5 & -1 & 0 \end{bmatrix}.$$

4.9 6-Arcs on a Conic

The thirteen hexads $\mathcal{H}_i$ as given in Table 2.12 correspond to thirteen inequivalent 6-arcs $\mathcal{H}'_i$ on the conic $\mathcal{C}^*$. Each 6-arc $\mathcal{H}'_i$, $i = 1, \dots, 13$, is equivalent to one in Table 4.11. These equivalences and the matrix transformations are given in Table 4.13.

TABLE 4.13: Transforming $\mathcal{H}'_i$ to $\mathcal{B}_j$

$\mathcal{H}'_i \cong \mathcal{B}_j$	Matrix transformation								
$\mathcal{H}'_1 = \{1,3,263,250,177,248\} \cong \mathcal{B}_{104}$	9	9	9	9	−9	0	0	0	−9
$\mathcal{H}'_2 = \{1,3,263,250,177,93\} \cong \mathcal{B}_{55}$	0	6	0	7	−7	−7	2	1	7
$\mathcal{H}'_3 = \{1,3,263,250,177,262\} \cong \mathcal{B}_{68}$	0	0	−9	−9	1	−9	−9	0	0
$\mathcal{H}'_4 = \{1,3,263,250,177,296\} \cong \mathcal{B}_{76}$	9	0	0	9	−1	9	0	0	9
$\mathcal{H}'_5 = \{1,3,263,250,177,104\} \cong \mathcal{B}_{49}$	0	−9	0	9	−1	−9	0	0	−1
$\mathcal{H}'_6 = \{1,3,263,250,177,236\} \cong \mathcal{B}_{64}$	−4	0	0	−9	−4	−2	−6	6	2
$\mathcal{H}'_7 = \{1,3,263,250,177,191\} \cong \mathcal{B}_{107}$	−9	0	0	5	−4	−8	−5	−4	4
$\mathcal{H}'_8 = \{1,3,263,250,374,262\} \cong \mathcal{B}_{98}$	−4	0	0	−3	−6	−3	0	0	−7
$\mathcal{H}'_9 = \{1,3,263,250,374,205\} \cong \mathcal{B}_{100}$	−9	0	0	9	−1	9	0	0	−9
$\mathcal{H}'_{10} = \{1,3,263,250,374,296\} \cong \mathcal{B}_{65}$	9	0	0	9	−1	9	0	0	9
$\mathcal{H}'_{11} = \{1,3,263,250,374,24\} \cong \mathcal{B}_{101}$	6	0	0	−9	2	1	−3	−8	−7
$\mathcal{H}'_{12} = \{1,3,263,250,374,104\} \cong \mathcal{B}_{61}$	9	9	9	−9	0	9	0	−9	0
$\mathcal{H}'_{13} = \{1,3,263,248,93,22\} \cong \mathcal{B}_{113}$	−7	0	0	−8	7	4	−4	−4	−4

Alternatively, the 6-arcs on the conics are found by substituting the $6th$ point of each 6-arc in Table 4.11 in the corresponding conic form of $\mathcal{C}_{\mathcal{A}_i}$.

Theorem 4.9.1. *In $PG(2,19)$, there are precisely 13 projectively distinct 6-arcs on a conic, as summarized in Table 4.14.*

TABLE 4.14: Inequivalent 6-arcs on the conics

The conic	$\mathcal{B}_i : G$
$\mathcal{C}_{\mathcal{A}_1}$	$\mathcal{B}_{49}, \mathcal{B}_{55} : I$ $\mathcal{B}_{61}, \mathcal{B}_{64}, \mathcal{B}_{65}, \mathcal{B}_{68}, \mathcal{B}_{76} : \mathbf{Z}_2$ $\mathcal{B}_{98}, \mathcal{B}_{100} : \mathbf{V}_4$ $\mathcal{B}_{101} : \mathbf{S}_3$
$\mathcal{C}_{\mathcal{A}_2}$	$\mathcal{B}_{104} : \mathbf{V}_4$, $\mathcal{B}_{107} : \mathbf{D}_6$
$\mathcal{C}_{\mathcal{A}_3}$	$\mathcal{B}_{113} : \mathbf{S}_3$

Remark 4.9.2. Let $\mathcal{K} = \{\mathbf{U}_0, \mathbf{U}_1, \mathbf{U}_2, \mathbf{U}, \mathbf{P}(a,b,1), \mathbf{P}(c,d,1)\}$ be a 6-arc. The 6-arc $\mathcal{K}$ lies on the conic if and only if

$$ad(b-1)(c-1) - bc(a-1)(d-1) = 0.$$

4.10 7-Arcs

From Table 4.12, the total number of points not on the sides of the hexastigms is 17354. The action of the stabilizer group of each inequivalent 6-arc on the corresponding set C_0^6 splits the 17354 points into 11948 orbits. There are fourteen different classes of 7-arcs of type $[c_0, c_1, c_2, c_3]$ and four different sizes of stabilizer groups. A cell $n : |G|$ denote the number n of 7-arcs with stabilizer group size $|G|$.

TABLE 4.15: Statistics of the constants c_i of 7-arcs

No.	$[c_0, c_1, c_2, c_3]$	$n : \|G\|$
1	$[87, 210, 63, 14]$	$14 : 1$
2	$[88, 207, 66, 13]$	$28 : 1$
3	$[89, 204, 69, 12]$	$77 : 1,\ 12 : 2,\ 6 : 3$
4	$[90, 201, 72, 11]$	$252 : 1,\ 24 : 2$
5	$[91, 198, 75, 10]$	$644 : 1,\ 24 : 2,\ 18 : 3$
6	$[92, 195, 78, 9]$	$1358 : 1,\ 64 : 2,\ 15 : 3$
7	$[93, 192, 81, 8]$	$2044 : 1,\ 52 : 2$
8	$[94, 189, 84, 7]$	$2387 : 1,\ 84 : 2$
9	$[95, 186, 87, 6]$	$2121 : 1,\ 68 : 2$
10	$[96, 183, 90, 5]$	$1407 : 1,\ 80 : 2$
11	$[97, 180, 93, 4]$	$805 : 1,\ 16 : 2,\ 2 : 6$
12	$[98, 177, 96, 3]$	$245 : 1,\ 44 : 2,\ 3 : 3,\ 4 : 6$
13	$[99, 174, 99, 2]$	$35 : 1,\ 4 : 2$
14	$[100, 171, 102, 1]$	$7 : 1,\ 4 : 2$

Note that the constants c_i in Table 4.15 satisfy the values in Table 3.2 for $k = 7$.

Theorem 4.10.1. *In $PG(2, 19)$, there are precisely 1768 projectively distinct 7-arcs.*

The number n of inequivalent 7-arcs with stabilizer group of type G with respect to the constants c_i are given in Table 4.16.

TABLE 4.16: Statistics of the constants c_i of inequivalent 7-arcs

No.	$[c_0, c_1, c_2, c_3]$	$n : G$
1	$[87, 210, 63, 14]$	$2 : I$
2	$[88, 207, 66, 13]$	$4 : I$
3	$[89, 204, 69, 12]$	$11 : I,\ 3 : \mathbf{Z}_2,\ 2 : \mathbf{Z}_3$
4	$[90, 201, 72, 11]$	$36 : I,\ 6 : \mathbf{Z}_2$
5	$[91, 198, 75, 10]$	$92 : I,\ 6 : \mathbf{Z}_2,\ 6 : \mathbf{Z}_3$
6	$[92, 195, 78, 9]$	$194 : I,\ 16 : \mathbf{Z}_2,\ 5 : \mathbf{Z}_3$
7	$[93, 192, 81, 8]$	$292 : I,\ 13 : \mathbf{Z}_2$
8	$[94, 189, 84, 7]$	$341 : I,\ 21 : \mathbf{Z}_2$
9	$[95, 186, 87, 6]$	$303 : I,\ 17 : \mathbf{Z}_2$
10	$[96, 183, 90, 5]$	$201 : I,\ 20 : \mathbf{Z}_2$
11	$[97, 180, 93, 4]$	$115 : I,\ 4 : \mathbf{Z}_2,\ 1 : \mathbf{Z}_6$
12	$[98, 177, 96, 3]$	$35 : I,\ 11 : \mathbf{Z}_2,\ 1 : \mathbf{Z}_3,\ 2 : \mathbf{S}_3$
13	$[99, 174, 99, 2]$	$5 : I,\ 1 : \mathbf{Z}_2$
14	$[100, 171, 102, 1]$	$1 : I,\ 1 : \mathbf{Z}_2$

The constant $c_0 \neq 0$ for all 7-arcs in $PG(2, 19)$ as shown in Table 4.15. This result coincides with Corollary 3.8.3 that no complete 7-arcs exist in $PG(2, 19)$.

4.11 7-Arcs on a Conic

The eighteen heptads $\mathcal{T}_i$ as given in Table 2.14 correspond to eighteen inequivalent 7-arcs $\mathcal{T}_i'$ on the conic $\mathcal{C}^*$.

In Table 4.17, each row consists of a 7-arc $\mathcal{T}_i'$ and its projectively equivalent 7-arc $\mathcal{B}_j \cup \{P\}$, as well as the matrix transformation between them.

TABLE 4.17: Transforming $\mathcal{T}_i'$ to $\mathcal{B}_j \cup \{P\}$

$\mathcal{T}_1' = \{1,3,263,250,177,248,93\} \cong \mathcal{B}_{49} \cup \{345\}$	0 0 −9 −9 1 9 0 −1 0
$\mathcal{T}_2' = \{1,3,263,250,177,248,22\} \cong \mathcal{B}_{65} \cup \{197\}$	−1 −7 9 6 −7 −7 0 0 3
$\mathcal{T}_3' = \{1,3,263,250,177,248,262\} \cong \mathcal{B}_{64} \cup \{235\}$	9 0 0 −9 −1 9 0 1 0
$\mathcal{T}_4' = \{1,3,263,250,177,248,296\} \cong \mathcal{B}_{61} \cup \{49\}$	−9 9 3 −8 7 −6 −5 0 0
$\mathcal{T}_5' = \{1,3,263,250,177,93,225\} \cong \mathcal{B}_{61} \cup \{327\}$	−9 −9 −9 4 −9 −3 9 −3 −4
$\mathcal{T}_6' = \{1,3,263,250,177,93,204\} \cong \mathcal{B}_{61} \cup \{226\}$	−9 −9 −9 9 −1 8 0 0 1
$\mathcal{T}_7' = \{1,3,263,250,177,93,374\} \cong \mathcal{B}_{65} \cup \{182\}$	−6 −6 −6 0 7 −5 6 −1 1
$\mathcal{T}_8' = \{1,3,263,250,177,93,205\} \cong \mathcal{B}_{65} \cup \{142\}$	7 3 −5 0 −3 −4 −7 −6 9
$\mathcal{T}_9' = \{1,3,263,250,177,93,24\} \cong \mathcal{B}_{61} \cup \{135\}$	−6 −6 −6 −6 5 −3 −2 −7 −8
$\mathcal{T}_{10}' = \{1,3,263,250,177,93,104\} \cong \mathcal{B}_{64} \cup \{64\}$	−5 −7 1 4 8 −9 0 0 2
$\mathcal{T}_{11}' = \{1,3,263,250,177,93,262\} \cong \mathcal{B}_{68} \cup \{182\}$	0 0 −9 −9 1 −9 −9 0 0
$\mathcal{T}_{12}' = \{1,3,263,250,177,262,236\} \cong \mathcal{B}_{64} \cup \{51\}$	0 −9 0 9 0 −9 9 9 9
$\mathcal{T}_{13}' = \{1,3,263,250,177,262,296\} \cong \mathcal{B}_{61} \cup \{182\}$	6 0 0 −6 2 1 7 2 −2
$\mathcal{T}_{14}' = \{1,3,263,250,177,296,104\} \cong \mathcal{B}_{61} \cup \{197\}$	9 0 0 9 −1 9 0 0 9
$\mathcal{T}_{15}' = \{1,3,263,250,177,104,225\} \cong \mathcal{B}_{61} \cup \{51\}$	−9 0 0 3 8 4 6 −8 9
$\mathcal{T}_{16}' = \{1,3,263,250,177,104,204\} \cong \mathcal{B}_{65} \cup \{135\}$	6 −3 −5 −8 6 3 −8 0 0
$\mathcal{T}_{17}' = \{1,3,263,250,177,104,191\} \cong \mathcal{B}_{49} \cup \{226\}$	9 0 0 −5 8 4 5 −4 4
$\mathcal{T}_{18}' = \{1,3,263,250,374,296,104\} \cong \mathcal{B}_{61} \cup \{64\}$	9 0 0 9 −1 9 0 0 9

Note that each 7-arc $\mathcal{B}_i \cup \{P\}$ in Table 4.17 is on the conic $\mathcal{C}_{\mathcal{A}_1}$. The 7-arcs on the conics are also found by substituting the $6th$ and $7th$ points of each 7-arc in the conic form of $\mathcal{C}_{\mathcal{A}_1}$.

Theorem 4.11.1. *On $PG(2,19)$, there are precisely 18 projectively distinct 7-arcs on the conic summarized in Table 4.18.*

TABLE 4.18: Inequivalent 7-arcs on the conic

No.	The 7-arc	Stabilizer	$[c_0, c_1, c_2, c_3]$
1	$\mathcal{B}_{49} \cup \{345\}$	$\mathbf{Z}_2$	$[94, 189, 84, 7]$
2	$\mathcal{B}_{65} \cup \{197\}$	$\mathbf{Z}_2$	$[95, 186, 87, 6]$
3	$\mathcal{B}_{64} \cup \{235\}$	I	$[92, 195, 78, 9]$
4	$\mathcal{B}_{61} \cup \{49\}$	I	$[95, 186, 87, 6]$
5	$\mathcal{B}_{61} \cup \{327\}$	I	$[93, 192, 81, 8]$
6	$\mathcal{B}_{61} \cup \{226\}$	$\mathbf{Z}_2$	$[98, 177, 96, 3]$
7	$\mathcal{B}_{65} \cup \{182\}$	$\mathbf{Z}_2$	$[96, 183, 90, 5]$
8	$\mathcal{B}_{65} \cup \{142\}$	$\mathbf{Z}_2$	$[95, 186, 87, 6]$
9	$\mathcal{B}_{61} \cup \{135\}$	I	$[95, 186, 87, 6]$
10	$\mathcal{B}_{64} \cup \{64\}$	I	$[98, 177, 96, 3]$
11	$\mathcal{B}_{68} \cup \{182\}$	$\mathbf{Z}_2$	$[96, 183, 90, 5]$
12	$\mathcal{B}_{64} \cup \{51\}$	$\mathbf{Z}_3$	$[92, 195, 78, 9]$
13	$\mathcal{B}_{61} \cup \{182\}$	I	$[93, 192, 81, 8]$
14	$\mathcal{B}_{61} \cup \{197\}$	$\mathbf{Z}_2$	$[92, 195, 78, 9]$
15	$\mathcal{B}_{61} \cup \{51\}$	$\mathbf{Z}_2$	$[94, 189, 84, 7]$
16	$\mathcal{B}_{65} \cup \{135\}$	$\mathbf{Z}_2$	$[93, 192, 81, 8]$
17	$\mathcal{B}_{49} \cup \{226\}$	$\mathbf{Z}_6$	$[97, 180, 93, 4]$
18	$\mathcal{B}_{61} \cup \{64\}$	$\mathbf{Z}_3$	$[92, 195, 78, 9]$

4.12 8-Arcs

From Table 4.16 the total number of points not on the sides of the 7-stigms is 166219. The action of the stabilizer group of each inequivalent 7-arc on the corresponding set C_0^7 splits the 166219 points into 160164 orbits. There are 100 different classes of 8-arcs of type $[c_0, c_1, c_2, c_3, c_4]$. The minimum and maximum value of each constant c_i for all 8-arcs is as follows:

$$\begin{array}{rclrcl} 40 \le & c_0 & \le 61, & 40 \le & c_1 & \le 61, \\ 138 \le & c_2 & \le 200, & 96 \le & c_3 & \le 156, \\ 0 \le & c_4 & \le 37. \end{array}$$

Since $c_0 \neq 0$ for all 8-arcs so there is no complete 8-arc in $PG(2, 19)$. There are nine different sizes of stabilizer groups of the 8-arcs. The details are given in Table 4.19.

TABLE 4.19: Statistics of the stabilizer groups of 8-arcs

Number of 8-arcs	$\lvert G\rvert$	Number of 8-arcs	$\lvert G\rvert$
156376	1	7	8
3641	2	2	12
32	3	1	16
84	4	1	24
20	6		

Theorem 4.12.1. *In* $PG(2, 19)$*, there are precisely* 20361 *projectively distinct* 8*-arcs.*

In Table 4.20, the numbers of inequivalent 8-arcs are listed according to the stabilizer group types G.

TABLE 4.20: Statistics of the inequivalent 8-arcs

Number of 8-arcs	G	Number of 8-arcs	G
19547	I	7	$\mathbf{S}_3$
760	$\mathbf{Z}_2$	4	$\mathbf{D}_4$
8	$\mathbf{Z}_3$	1	$\mathbf{D}_6$
9	$\mathbf{Z}_4$	1	$\mathbf{Z}_8 \rtimes \mathbf{Z}_2$
23	$\mathbf{V}_4$	1	$\mathbf{S}_4$

4.13 8-arcs on a Conic

The 31 octads $\mathcal{O}_i$ as given in Table 2.16 correspond to 31 inequivalent 8-arcs $\mathcal{O}'_i$ on the conic $\mathcal{C}^*$. Each 8-arc $\mathcal{O}'_i$ is equivalent to an 8-arc of the form $\mathcal{B}_j \cup \{P_1, P_2\}$. The details are given in Table 4.21.

TABLE 4.21: Transforming $\mathcal{O}'_i$ to $\mathcal{B}_j \cup \{P_1, P_2\}$

$\mathcal{O}'_1 = \{1, 3, 263, 250, 177, 248, 93, 262\} \cong \mathcal{B}_{49} \cup \{345, 142\}$ 0 0 −9 −9 1 −9 −9 0 0
$\mathcal{O}'_2 = \{1, 3, 263, 250, 177, 248, 93, 374\} \cong \mathcal{B}_{49} \cup \{226, 51\}$ 6 0 0 8 2 1 5 −2 −4
$\mathcal{O}'_3 = \{1, 3, 263, 250, 177, 248, 93, 205\} \cong \mathcal{B}_{49} \cup \{345, 135\}$ 0 0 −9 −9 1 9 0 −1 0
$\mathcal{O}'_4 = \{1, 3, 263, 250, 177, 248, 93, 22\} \cong \mathcal{B}_{49} \cup \{345, 51\}$ 0 0 9 −9 1 8 −9 2 −1
$\mathcal{O}'_5 = \{1, 3, 263, 250, 177, 248, 93, 24\} \cong \mathcal{B}_{49} \cup \{345, 49\}$ 0 0 −9 −9 1 9 0 −1 0
$\mathcal{O}'_6 = \{1, 3, 263, 250, 177, 248, 93, 204\} \cong \mathcal{B}_{49} \cup \{345, 94\}$ 0 0 −9 −9 1 9 0 −1 0
$\mathcal{O}'_7 = \{1, 3, 263, 250, 177, 248, 93, 225\} \cong \mathcal{B}_{49} \cup \{345, 12\}$ 0 0 −9 −9 1 9 0 −1 0
$\mathcal{O}'_8 = \{1, 3, 263, 250, 177, 248, 22, 262\} \cong \mathcal{B}_{61} \cup \{135, 327\}$ 0 0 6 8 −3 −4 9 3 −2
$\mathcal{O}'_9 = \{1, 3, 263, 250, 177, 248, 22, 374\} \cong \mathcal{B}_{61} \cup \{49, 235\}$ −9 0 0 −9 −1 9 0 −1 0
$\mathcal{O}'_{10} = \{1, 3, 263, 250, 177, 248, 22, 205\} \cong \mathcal{B}_{61} \cup \{49, 345\}$ 9 −9 −3 −8 7 −6 5 0 0
$\mathcal{O}'_{11} = \{1, 3, 263, 250, 177, 248, 22, 296\} \cong \mathcal{B}_{61} \cup \{49, 64\}$ −9 9 3 −8 7 −6 −5 0 0
$\mathcal{O}'_{12} = \{1, 3, 263, 250, 177, 248, 22, 294\} \cong \mathcal{B}_{64} \cup \{235, 275\}$ 9 9 9 7 −6 0 3 −3 −1
$\mathcal{O}'_{13} = \{1, 3, 263, 250, 177, 248, 262, 296\} \cong \mathcal{B}_{49} \cup \{226, 12\}$ 6 0 0 −6 2 1 7 2 −2
$\mathcal{O}'_{14} = \{1, 3, 263, 250, 177, 248, 262, 204\} \cong \mathcal{B}_{61} \cup \{49, 135\}$ 4 0 0 −9 −4 −2 6 −6 −2

$\mathcal{O}'_{15} = \{1,3,263,250,177,248,262,225\} \cong \mathcal{B}_{64} \cup \{235,327\}$
9 0 0 −9 −1 9 0 1 0
$\mathcal{O}'_{16} = \{1,3,263,250,177,248,262,191\} \cong \mathcal{B}_{61} \cup \{135,235\}$
1 −6 4 −9 9 0 9 −3 −4
$\mathcal{O}'_{17} = \{1,3,263,250,177,248,296,205\} \cong \mathcal{B}_{61} \cup \{49,226\}$
−9 9 3 −8 7 −6 −5 0 0
$\mathcal{O}'_{18} = \{1,3,263,250,177,248,296,204\} \cong \mathcal{B}_{61} \cup \{49,51\}$
4 0 0 −9 −4 −2 6 −6 −2
$\mathcal{O}'_{19} = \{1,3,263,250,177,248,296,236\} \cong \mathcal{B}_{61} \cup \{49,182\}$
−4 0 0 −9 −4 −2 −6 6 2
$\mathcal{O}'_{20} = \{1,3,263,250,177,93,225,374\} \cong \mathcal{B}_{61} \cup \{135,94\}$
−1 6 −4 −9 9 0 −9 3 4
$\mathcal{O}'_{21} = \{1,3,263,250,177,93,225,262\} \cong \mathcal{B}_{61} \cup \{135,275\}$
−4 0 0 −6 −4 −2 −2 −2 −2
$\mathcal{O}'_{22} = \{1,3,263,250,177,93,225,205\} \cong \mathcal{B}_{61} \cup \{135,345\}$
0 0 −9 −9 1 9 0 −1 0
$\mathcal{O}'_{23} = \{1,3,263,250,177,93,225,296\} \cong \mathcal{B}_{61} \cup \{135,198\}$
−9 3 4 9 −9 0 −1 6 −4
$\mathcal{O}'_{24} = \{1,3,263,250,177,93,225,294\} \cong \mathcal{B}_{61} \cup \{327,345\}$
6 0 0 0 −5 7 −6 −5 −7
$\mathcal{O}'_{25} = \{1,3,263,250,177,93,225,104\} \cong \mathcal{B}_{61} \cup \{327,51\}$
7 6 −9 −1 9 9 −7 3 −1
$\mathcal{O}'_{26} = \{1,3,263,250,177,93,225,236\} \cong \mathcal{B}_{61} \cup \{135,64\}$
1 −6 4 8 2 −8 0 1 0
$\mathcal{O}'_{27} = \{1,3,263,250,177,93,374,205\} \cong \mathcal{B}_{65} \cup \{135,142\}$
−4 0 0 5 −6 −3 −1 6 −4
$\mathcal{O}'_{28} = \{1,3,263,250,177,93,104,262\} \cong \mathcal{B}_{61} \cup \{182,51\}$
0 −9 0 9 −1 −9 0 0 −1
$\mathcal{O}'_{29} = \{1,3,263,250,177,262,296,104\} \cong \mathcal{B}_{61} \cup \{182,197\}$
6 0 0 −6 2 1 7 2 −2
$\mathcal{O}'_{30} = \{1,3,263,250,177,104,204,191\} \cong \mathcal{B}_{49} \cup \{226,135\}$
9 0 0 −5 8 4 5 −4 4
$\mathcal{O}'_{31} = \{1,3,263,250,374,296,104,236\} \cong \mathcal{B}_{61} \cup \{64,94\}$
9 0 0 −3 8 4 −6 −1 −4

Note that each 8-arc $\mathcal{B}_i \cup \{P_1, P_2\}$ in Table 4.21 is on the conic $\mathcal{C}_{\mathcal{A}_1}$.

The 8-arcs on the conic are also found by substituting the $6th$, $7th$ and $8th$ points of each 8-arcs in the conic form of $\mathcal{C}_{\mathcal{A}_1}$.

Theorem 4.13.1. *In $PG(2,19)$, there are precisely 31 projectively distinct 8-arcs on a conic, as summarized in Table 4.22.*

Table 4.22: Inequivalent 8-arcs on the conic

No.	The 8-arc	Stabilizer	$[c_0, c_1, c_2, c_3, c_4]$
1	$\mathcal{B}_{49} \cup \{345, 142\}$	$\mathbf{Z}_2$	$[50, 171, 123, 29, 0]$
2	$\mathcal{B}_{49} \cup \{226, 51\}$	I	$[56, 153, 141, 23, 0]$
3	$\mathcal{B}_{49} \cup \{345, 135\}$	I	$[50, 171, 123, 29, 0]$
4	$\mathcal{B}_{49} \cup \{345, 51\}$	$\mathbf{Z}_2$	$[59, 143, 153, 17, 1]$
5	$\mathcal{B}_{49} \cup \{345, 49\}$	$\mathbf{Z}_2$	$[52, 164, 132, 24, 1]$
6	$\mathcal{B}_{49} \cup \{345, 94\}$	$\mathbf{Z}_2$	$[55, 155, 141, 21, 1]$
7	$\mathcal{B}_{49} \cup \{345, 12\}$	$\mathbf{Z}_2$	$[48, 177, 117, 31, 0]$
8	$\mathcal{B}_{61} \cup \{135, 327\}$	$\mathbf{Z}_2$	$[51, 167, 129, 25, 1]$
9	$\mathcal{B}_{61} \cup \{49, 235\}$	I	$[55, 156, 138, 24, 0]$
10	$\mathcal{B}_{61} \cup \{49, 345\}$	I	$[54, 159, 135, 25, 0]$
11	$\mathcal{B}_{61} \cup \{49, 64\}$	I	$[54, 159, 135, 25, 0]$
12	$\mathcal{B}_{64} \cup \{235, 275\}$	$\mathbf{V}_4$	$[50, 170, 126, 26, 1]$
13	$\mathcal{B}_{49} \cup \{226, 12\}$	I	$[54, 159, 135, 25, 0]$
14	$\mathcal{B}_{61} \cup \{49, 135\}$	I	$[51, 168, 126, 28, 0]$
15	$\mathcal{B}_{64} \cup \{235, 327\}$	$\mathbf{S}_3$	$[46, 180, 120, 24, 3]$
16	$\mathcal{B}_{61} \cup \{135, 235\}$	$\mathbf{V}_4$	$[52, 162, 138, 18, 3]$
17	$\mathcal{B}_{61} \cup \{49, 226\}$	$\mathbf{Z}_2$	$[56, 153, 141, 23, 0]$
18	$\mathcal{B}_{61} \cup \{49, 51\}$	$\mathbf{Z}_2$	$[50, 170, 126, 26, 1]$
19	$\mathcal{B}_{61} \cup \{49, 182\}$	$\mathbf{V}_4$	$[54, 156, 144, 16, 3]$
20	$\mathcal{B}_{61} \cup \{135, 94\}$	I	$[55, 156, 138, 24, 0]$
21	$\mathcal{B}_{61} \cup \{135, 275\}$	I	$[54, 159, 135, 25, 0]$
22	$\mathcal{B}_{61} \cup \{135, 345\}$	$\mathbf{Z}_2$	$[51, 167, 129, 25, 1]$
23	$\mathcal{B}_{61} \cup \{135, 198\}$	I	$[55, 156, 138, 24, 0]$
24	$\mathcal{B}_{61} \cup \{327, 345\}$	$\mathbf{D}_4$	$[52, 160, 144, 12, 5]$
25	$\mathcal{B}_{61} \cup \{327, 51\}$	$\mathbf{Z}_2$	$[54, 158, 138, 22, 1]$
26	$\mathcal{B}_{61} \cup \{135, 64\}$	I	$[49, 174, 120, 30, 0]$
27	$\mathcal{B}_{65} \cup \{135, 142\}$	$\mathbf{Z}_2$	$[56, 152, 144, 20, 1]$
28	$\mathcal{B}_{61} \cup \{182, 51\}$	$\mathbf{Z}_2$	$[57, 149, 147, 19, 1]$
29	$\mathcal{B}_{61} \cup \{182, 197\}$	$\mathbf{V}_4$	$[48, 174, 126, 22, 3]$
30	$\mathcal{B}_{49} \cup \{226, 135\}$	$\mathbf{D}_6$	$[54, 156, 144, 16, 3]$
31	$\mathcal{B}_{61} \cup \{64, 94\}$	$\mathbf{S}_4$	$[52, 156, 156, 0, 9]$

4.14 9-Arcs

The total number of points not on the sides of the 8-stigms is 1053996. The action of the stabilizer group of each inequivalent 8-arc on the corresponding set C_0^8 splits the 1053996 points into 1033587 orbits. There are 243 different classes of 9-arcs of type $[c_0, c_1, c_2, c_3, c_4]$. The minimum and maximum value of each constant c_i for all 9-arcs is as follows:

$$\begin{array}{rcl rcl} 9 \leq & c_0 & \leq 39, & 72 \leq & c_1 & \leq 162, \\ 126 \leq & c_2 & \leq 216, & 21 \leq & c_3 & \leq 84, \\ 0 \leq & c_4 & \leq 18. & & & \end{array}$$

Since $c_0 \neq 0$ for all 9-arcs so there is no complete 9-arc in $PG(2,19)$. There are six different sizes of stabilizer groups of the 9-arcs. The details are given in Table 4.23.

Table 4.23: Statistics of the stabilizer groups of 9-arcs

Number of 9-arcs	$\|G\|$	Number of 9-arcs	$\|G\|$
1027314	1	44	6
5670	2	7	9
550	3	2	18

Theorem 4.14.1. *In* $PG(2,19)$, *there are precisely* 115492 *projectively distinct* 9-*arcs.*

In Table 4.24, the numbers of inequivalent 9-arcs are listed according to the stabilizer group types G.

Table 4.24: Statistics of the inequivalent 9-arcs

Number of 9-arcs	G	Number of 9-arcs	G
114146	I	21	$\mathbf{S}_3$
1134	$\mathbf{Z}_2$	7	$\mathbf{Z}_3 \times \mathbf{Z}_3$
182	$\mathbf{Z}_3$	2	$(\mathbf{Z}_3 \times \mathbf{Z}_3) \rtimes \mathbf{Z}_2$

4.15 9-Arcs on a Conic

The 33 nonads $\mathcal{N}_i$ as given in Table 2.18 correspond to 33 inequivalent 9-arcs $\mathcal{N}_i'$ on the conic $\mathcal{C}^*$. Each 9-arc $\mathcal{N}_i'$ is equivalent to a 9-arc of the form $\mathcal{B}_j \cup \{P_1, P_2, P_3\}$. The details are given in Table 4.25.

TABLE 4.25: Transforming $\mathcal{N}_i'$ to $\mathcal{B}_j \cup \{P_1, P_2, P_3\}$

$\mathcal{N}_1' = \{1,3,263,250,177,248,93,262,353\} \cong \mathcal{B}_{49} \cup \{226,12,51\}$ −6 0 0 8 2 1 −5 2 4
$\mathcal{N}_2' = \{1,3,263,250,177,248,93,262,374\} \cong \mathcal{B}_{49} \cup \{226,12,94\}$ 4 −7 7 −8 9 −5 6 0 0
$\mathcal{N}_3' = \{1,3,263,250,177,248,93,262,205\} \cong \mathcal{B}_{49} \cup \{345,12,51\}$ 0 9 0 9 −3 −9 −1 −4 2
$\mathcal{N}_4' = \{1,3,263,250,177,248,93,262,22\} \cong \mathcal{B}_{49} \cup \{345,49,51\}$ 0 9 0 9 −2 −9 9 8 −8
$\mathcal{N}_5' = \{1,3,263,250,177,248,93,262,24\} \cong \mathcal{B}_{49} \cup \{345,49,198\}$ 0 0 −9 −9 1 9 0 −1 0
$\mathcal{N}_6' = \{1,3,263,250,177,248,93,262,204\} \cong \mathcal{B}_{49} \cup \{226,12,275\}$ −7 −4 1 −8 4 9 −1 8 −3
$\mathcal{N}_7' = \{1,3,263,250,177,248,93,374,205\} \cong \mathcal{B}_{49} \cup \{226,12,135\}$ −6 0 0 −6 2 1 −7 −2 2
$\mathcal{N}_8' = \{1,3,263,250,177,248,93,374,296\} \cong \mathcal{B}_{49} \cup \{226,51,94\}$ 4 −7 7 −8 9 −5 6 0 0
$\mathcal{N}_9' = \{1,3,263,250,177,248,93,374,294\} \cong \mathcal{B}_{49} \cup \{226,12,64\}$ 6 6 6 7 −5 2 0 0 −9
$\mathcal{N}_{10}' = \{1,3,263,250,177,248,93,374,24\} \cong \mathcal{B}_{49} \cup \{226,12,197\}$ −8 7 −5 −8 0 −4 −3 −9 9
$\mathcal{N}_{11}' = \{1,3,263,250,177,248,93,374,104\} \cong \mathcal{B}_{49} \cup \{226,51,345\}$ 6 0 0 8 2 1 5 −2 −4
$\mathcal{N}_{12}' = \{1,3,263,250,177,248,93,374,204\} \cong \mathcal{B}_{49} \cup \{226,51,64\}$ 6 0 0 8 2 1 5 −2 −4
$\mathcal{N}_{13}' = \{1,3,263,250,177,248,93,374,225\} \cong \mathcal{B}_{49} \cup \{226,12,345\}$ 0 0 −9 −9 1 9 0 −1 0
$\mathcal{N}_{14}' = \{1,3,263,250,177,248,93,205,22\} \cong \mathcal{B}_{49} \cup \{345,51,64\}$ −3 7 6 0 7 −6 3 0 0
$\mathcal{N}_{15}' = \{1,3,263,250,177,248,93,205,294\} \cong \mathcal{B}_{49} \cup \{345,135,327\}$ 0 0 −9 −9 1 9 0 −1 0

$\mathcal{N}'_{16} = \{1,3,263,250,177,248,93,205,24\} \cong \mathcal{B}_{49} \cup \{345,49,135\}$	0 0 −9 −9 1 9 0 −1 0
$\mathcal{N}'_{17} = \{1,3,263,250,177,248,93,205,104\} \cong \mathcal{B}_{49} \cup \{345,12,94\}$	4 −7 7 3 4 0 −1 3 −7
$\mathcal{N}'_{18} = \{1,3,263,250,177,248,93,205,204\} \cong \mathcal{B}_{49} \cup \{345,94,135\}$	0 0 −9 −9 1 9 0 −1 0
$\mathcal{N}'_{19} = \{1,3,263,250,177,248,93,205,236\} \cong \mathcal{B}_{49} \cup \{345,49,327\}$	0 −9 0 9 0 −9 9 9 9
$\mathcal{N}'_{20} = \{1,3,263,250,177,248,93,205,225\} \cong \mathcal{B}_{49} \cup \{345,12,135\}$	0 0 −9 −9 1 9 0 −1 0
$\mathcal{N}'_{21} = \{1,3,263,250,177,248,93,204,191\} \cong \mathcal{B}_{49} \cup \{226,12,142\}$	−5 4 −4 5 −8 −4 −9 0 0
$\mathcal{N}'_{22} = \{1,3,263,250,177,248,22,262,205\} \cong \mathcal{B}_{61} \cup \{49,235,345\}$	9 −9 −3 −8 7 −6 5 0 0
$\mathcal{N}'_{23} = \{1,3,263,250,177,248,22,262,294\} \cong \mathcal{B}_{61} \cup \{49,135,235\}$	9 0 0 −9 −1 9 0 1 0
$\mathcal{N}'_{24} = \{1,3,263,250,177,248,22,374,205\} \cong \mathcal{B}_{49} \cup \{226,12,235\}$	−4 7 −7 8 0 4 −6 −9 1
$\mathcal{N}'_{25} = \{1,3,263,250,177,248,22,374,296\} \cong \mathcal{B}_{61} \cup \{49,64,235\}$	−9 0 0 −9 −1 9 0 −1 0
$\mathcal{N}'_{26} = \{1,3,263,250,177,248,22,374,236\} \cong \mathcal{B}_{61} \cup \{49,51,235\}$	−9 0 0 −9 −1 9 0 −1 0
$\mathcal{N}'_{27} = \{1,3,263,250,177,248,22,205,296\} \cong \mathcal{B}_{61} \cup \{49,64,226\}$	−9 9 3 −8 7 −6 −5 0 0
$\mathcal{N}'_{28} = \{1,3,263,250,177,248,22,205,236\} \cong \mathcal{B}_{61} \cup \{49,51,345\}$	−4 0 0 −9 −4 −2 −6 6 2
$\mathcal{N}'_{29} = \{1,3,263,250,177,248,22,296,204\} \cong \mathcal{B}_{61} \cup \{49,51,64\}$	4 0 0 −9 −4 −2 6 −6 −2
$\mathcal{N}'_{30} = \{1,3,263,250,177,248,262,296,204\} \cong \mathcal{B}_{49} \cup \{226,12,49\}$	6 0 0 −6 2 1 7 2 −2
$\mathcal{N}'_{31} = \{1,3,263,250,177,93,225,374,262\} \cong \mathcal{B}_{61} \cup \{135,94,275\}$	−3 8 −1 −5 4 2 3 0 0
$\mathcal{N}'_{32} = \{1,3,263,250,177,93,225,374,294\} \cong \mathcal{B}_{61} \cup \{135,64,226\}$	4 6 −7 −4 8 5 9 5 2
$\mathcal{N}'_{33} = \{1,3,263,250,177,93,374,205,262\} \cong \mathcal{B}_{65} \cup \{135,142,182\}$	−4 0 0 5 −6 −3 −1 6 −4

Note that each 9-arc $\mathcal{B}_i \cup \{P_1, P_2, P_3\}$ in Table 4.25 is on the conic $\mathcal{C}_{\mathcal{A}_1}$. The 9-arcs on the conic are also found by substituting the $6th$,$7th$,$8th$ and $9th$ points of each 9-arcs in the conic form of $\mathcal{C}_{\mathcal{A}_1}$.

Theorem 4.15.1. *In $PG(2, 19)$, there are precisely 33 projectively distinct 9-arcs on a conic, as summarized in Table 4.26.*

Table 4.26: Inequivalent 9-arcs on the conic

No.	The 9-arc	Stabilizer	$[c_0, c_1, c_2, c_3, c_4]$
1	$\mathcal{B}_{49} \cup \{226, 12, 51\}$	$\mathbf{Z}_2$	$[29, 106, 174, 58, 5]$
2	$\mathcal{B}_{49} \cup \{226, 12, 94\}$	I	$[26, 118, 156, 70, 2]$
3	$\mathcal{B}_{49} \cup \{345, 12, 51\}$	I	$[27, 113, 165, 63, 4]$
4	$\mathcal{B}_{49} \cup \{345, 49, 51\}$	I	$[25, 120, 156, 68, 3]$
5	$\mathcal{B}_{49} \cup \{345, 49, 198\}$	$\mathbf{Z}_2$	$[29, 104, 180, 52, 7]$
6	$\mathcal{B}_{49} \cup \{226, 12, 275\}$	I	$[23, 124, 156, 64, 5]$
7	$\mathcal{B}_{49} \cup \{226, 12, 135\}$	I	$[29, 106, 174, 58, 5]$
8	$\mathcal{B}_{49} \cup \{226, 51, 94\}$	$\mathbf{Z}_2$	$[35, 90, 186, 58, 3]$
9	$\mathcal{B}_{49} \cup \{226, 12, 64\}$	I	$[26, 118, 156, 70, 2]$
10	$\mathcal{B}_{49} \cup \{226, 12, 197\}$	I	$[26, 113, 171, 55, 7]$
11	$\mathcal{B}_{49} \cup \{226, 51, 345\}$	$\mathbf{Z}_2$	$[34, 93, 183, 59, 3]$
12	$\mathcal{B}_{49} \cup \{226, 51, 64\}$	I	$[28, 112, 162, 68, 2]$
13	$\mathcal{B}_{49} \cup \{226, 12, 345\}$	I	$[25, 120, 156, 68, 3]$
14	$\mathcal{B}_{49} \cup \{345, 51, 64\}$	$\mathbf{Z}_2$	$[31, 102, 174, 62, 3]$
15	$\mathcal{B}_{49} \cup \{345, 135, 327\}$	$\mathbf{Z}_2$	$[27, 108, 180, 48, 9]$
16	$\mathcal{B}_{49} \cup \{345, 49, 135\}$	$\mathbf{Z}_2$	$[25, 118, 162, 62, 5]$
17	$\mathcal{B}_{49} \cup \{345, 12, 94\}$	I	$[27, 113, 165, 63, 4]$
18	$\mathcal{B}_{49} \cup \{345, 94, 135\}$	I	$[27, 113, 165, 63, 4]$
19	$\mathcal{B}_{49} \cup \{345, 49, 327\}$	$\mathbf{Z}_2$	$[19, 136, 144, 68, 5]$
20	$\mathcal{B}_{49} \cup \{345, 12, 135\}$	I	$[25, 116, 168, 56, 7]$
21	$\mathcal{B}_{49} \cup \{226, 12, 142\}$	$\mathbf{S}_3$	$[30, 105, 171, 63, 3]$
22	$\mathcal{B}_{61} \cup \{49, 235, 345\}$	$\mathbf{Z}_2$	$[29, 106, 174, 58, 5]$
23	$\mathcal{B}_{61} \cup \{49, 135, 235\}$	I	$[25, 114, 174, 50, 9]$
24	$\mathcal{B}_{49} \cup \{226, 12, 235\}$	I	$[30, 107, 165, 69, 1]$
25	$\mathcal{B}_{61} \cup \{49, 64, 235\}$	$\mathbf{Z}_3$	$[29, 111, 159, 73, 0]$
26	$\mathcal{B}_{61} \cup \{49, 51, 235\}$	I	$[26, 117, 159, 67, 3]$
27	$\mathcal{B}_{61} \cup \{49, 64, 226\}$	I	$[28, 112, 162, 68, 2]$
28	$\mathcal{B}_{61} \cup \{49, 51, 345\}$	$\mathbf{Z}_2$	$[31, 100, 180, 56, 5]$
29	$\mathcal{B}_{61} \cup \{49, 51, 64\}$	$\mathbf{Z}_2$	$[23, 122, 162, 58, 7]$
30	$\mathcal{B}_{49} \cup \{226, 12, 49\}$	I	$[29, 105, 177, 55, 6]$
31	$\mathcal{B}_{61} \cup \{135, 94, 275\}$	I	$[29, 108, 168, 64, 3]$
32	$\mathcal{B}_{61} \cup \{135, 64, 226\}$	$\mathbf{Z}_2$	$[29, 104, 180, 52, 7]$
33	$\mathcal{B}_{65} \cup \{135, 142, 182\}$	$\mathbf{D}_9$	$[39, 72, 216, 36, 9]$

4.16 10-Arcs

The total number of points not on the sides of the 9-stigms is 2798052. The action of the stabilizer group of each inequivalent 9-arc on the corresponding set C_0^9 splits the 2798052 points into 2783527 orbits. There are 1235 different classes of 10-arcs of type $[c_0, c_1, c_2, c_3, c_4, c_5]$. The minimum and maximum value of each constant c_i for all 10-arcs is as follows

$$\begin{array}{rclrcl} 0 \le & c_0 & \le 22, & 39 \le & c_1 & \le 102, \\ 120 \le & c_2 & \le 206, & 62 \le & c_3 & \le 151, \\ 5 \le & c_4 & \le 37, & 0 \le & c_5 & \le 9. \end{array}$$

Since $c_0 = 0$ for some 10-arcs so there is a complete 10-arc in $PG(2, 19)$. There are 11 different sizes of stabilizer groups of the 10-arcs. The details are given in Table 4.27.

Table 4.27: Statistics of the stabilizer groups of 10-arcs

Number of 10-arcs	$\|G\|$	Number of 10-arcs	$\|G\|$
2760500	1	9	10
22341	2	2	12
244	3	2	18
377	4	1	20
48	6	1	60
2	9		

Theorem 4.16.1. *In $PG(2, 19)$, there are precisely* 280104 *projectively distinct* 10*-arcs divided into* 280075 *incomplete arcs and* 29 *complete arcs.*

In Table 4.28, the numbers of inequivalent 10-arcs are listed according to the stabilizer group types G.

TABLE 4.28: Statistics of the inequivalent incomplete 10-arcs

Number of 10-arcs	G	Number of 10-arcs	G
276049	I	14	$\mathbf{S}_3$
3833	$\mathbf{Z}_2$	1	$\mathbf{Z}_9$
60	$\mathbf{Z}_3$	3	$\mathbf{D}_5$
36	$\mathbf{Z}_4$	1	$\mathbf{D}_9$
77	$\mathbf{V}_4$	1	$\mathbf{D}_{10}$

According to the stabilizer group types G, the numbers of 10-complete arcs are listed in Table 4.29.

TABLE 4.29: Statistics of the inequivalent complete 10-arcs

Number of 10-arcs	G	Number of 10-arcs	G
1	I	2	$\mathbf{S}_3$
18	$\mathbf{Z}_2$	2	$\mathbf{D}_5$
1	$\mathbf{Z}_3$	1	$\mathbf{A}_4$
1	$\mathbf{Z}_4$	1	$\mathbf{A}_5$
2	$\mathbf{V}_4$		

4.17 The Unique Complete 10-Arc with Stabilizer Group $\mathbf{A}_5$

(1) From Section 4.8, the set $K_{10} = \{13, 80, 220, 221, 252, 261, 268, 270, 278, 335\}$ of B-points of the 6-arc $\mathcal{B}_{117}$ forms a 10-arc. The class of type $[c_0, c_1, c_2, c_3, c_4, c_5]$ of K_{10} is

$$[0, 90, 150, 110, 15, 6].$$

Since $c_0 = 0$ then K_{10} is a complete 10-arc. The stabilizer group of K_{10} is $\mathbf{A}_5$.

(2) L. Storme and V. Maldeghem [48] in Proposition 13 also proved that with $4t^2 - 2t - 1 = 0,\ t \in \mathbf{F}_q$, a 10-arc

$$\begin{aligned} K_{10}^* = &\{(1,1,1),(1,1,-1),(1,-1,1),(1,-1,-1),(0,4t^2,1),\\ &(0,-4t^2,1),(-4t^2,1,0),(4t^2,1,0),(1,0,4t^2),(1,0,-4t^2)\} \end{aligned}$$

in $PG(2,q)$ when $q \equiv \pm 1 \pmod{10}$ is the unique 10-arc with stabilizer group $\mathbf{A}_5$. For $t = -2$, the 10-arc

$$K_{10}^{*} = \{263, 370, 250, 244, 252, 372, 371, 251, 28, 185\}$$

in numeral form is equivalent to the 10-arc K_{10} by the matrix transformation

$$A = \begin{bmatrix} 9 & -6 & 2 \\ -8 & -8 & 9 \\ 2 & 2 & -8 \end{bmatrix}.$$

4.18 10-Arcs on a Conic

The 44 decads $\mathcal{D}_i$ as given in Table 2.20 correspond to 44 inequivalent 10-arcs $\mathcal{D}'_i$ on the conic $\mathcal{C}^*$. Each 10-arc $\mathcal{D}'_i$ is equivalent to a 10-arc of the form $\mathcal{B}_j \cup \{P_1, P_2, P_3, P_4\}$. The details are given in Table 4.30.

TABLE 4.30: Transforming $\mathcal{D}'_i$ to $\mathcal{B}_j \cup \{P_1, P_2, P_3, P_4\}$

$\mathcal{D}'_1 = \{1,3,263,250,177,248,93,262,353,374\} \cong \mathcal{B}_{49} \cup \{226,12,51,135\}$
−6 0 0 1 2 1 0 0 3
$\mathcal{D}'_2 = \{1,3,263,250,177,248,93,262,353,205\} \cong \mathcal{B}_{49} \cup \{226,12,51,94\}$
6 0 0 1 2 1 0 0 −3
$\mathcal{D}'_3 = \{1,3,263,250,177,248,93,262,353,22\} \cong \mathcal{B}_{49} \cup \{226,12,49,64\}$
−6 −6 −6 7 −5 2 0 0 9
$\mathcal{D}'_4 = \{1,3,263,250,177,248,93,262,353,24\} \cong \mathcal{B}_{49} \cup \{226,12,51,64\}$
6 0 0 1 2 1 0 0 −3
$\mathcal{D}'_5 = \{1,3,263,250,177,248,93,262,353,204\} \cong \mathcal{B}_{49} \cup \{226,12,49,142\}$
6 9 −1 8 −4 −9 −2 7 3
$\mathcal{D}'_6 = \{1,3,263,250,177,248,93,262,353,225\} \cong \mathcal{B}_{49} \cup \{226,12,49,51\}$
−6 0 0 8 2 1 −5 2 4
$\mathcal{D}'_7 = \{1,3,263,250,177,248,93,262,374,205\} \cong \mathcal{B}_{49} \cup \{226,12,94,135\}$
−6 0 0 −6 2 1 −7 −2 2
$\mathcal{D}'_8 = \{1,3,263,250,177,248,93,262,374,22\} \cong \mathcal{B}_{49} \cup \{226,12,94,197\}$
4 −7 7 −8 9 −5 6 0 0
$\mathcal{D}'_9 = \{1,3,263,250,177,248,93,262,374,294\} \cong \mathcal{B}_{49} \cup \{226,12,64,142\}$
6 6 6 7 −5 2 0 0 −9
$\mathcal{D}'_{10} = \{1,3,263,250,177,248,93,262,374,24\} \cong \mathcal{B}_{49} \cup \{226,12,94,327\}$
4 −7 7 −8 9 −5 6 0 0

$\mathcal{D}'_{11} = \{1,3,263,250,177,248,93,262,374,104\} \cong \mathcal{B}_{49} \cup \{226,12,64,94\}$ 4 −7 7 −8 9 −5 6 0 0
$\mathcal{D}'_{12} = \{1,3,263,250,177,248,93,262,374,204\} \cong \mathcal{B}_{49} \cup \{226,12,94,275\}$ 4 −7 7 −8 9 −5 6 0 0
$\mathcal{D}'_{13} = \{1,3,263,250,177,248,93,262,374,236\} \cong \mathcal{B}_{49} \cup \{226,12,94,345\}$ 4 −7 7 −8 9 −5 6 0 0
$\mathcal{D}'_{14} = \{1,3,263,250,177,248,93,262,374,225\} \cong \mathcal{B}_{49} \cup \{226,12,49,94\}$ 4 −7 7 −8 9 −5 6 0 0
$\mathcal{D}'_{15} = \{1,3,263,250,177,248,93,262,374,191\} \cong \mathcal{B}_{49} \cup \{226,12,49,275\}$ 0 −6 0 7 0 −7 5 5 5
$\mathcal{D}'_{16} = \{1,3,263,250,177,248,93,262,205,22\} \cong \mathcal{B}_{49} \cup \{226,12,64,345\}$ −3 7 6 −6 2 6 0 −5 0
$\mathcal{D}'_{17} = \{1,3,263,250,177,248,93,262,205,104\} \cong \mathcal{B}_{49} \cup \{345,12,51,94\}$ 4 −7 7 3 4 0 −1 3 −7
$\mathcal{D}'_{18} = \{1,3,263,250,177,248,93,262,205,204\} \cong \mathcal{B}_{49} \cup \{226,12,135,275\}$ −7 −4 1 −8 4 9 −1 8 −3
$\mathcal{D}'_{19} = \{1,3,263,250,177,248,93,262,205,236\} \cong \mathcal{B}_{49} \cup \{345,12,51,327\}$ 0 9 0 9 −3 −9 −1 −4 2
$\mathcal{D}'_{20} = \{1,3,263,250,177,248,93,262,205,225\} \cong \mathcal{B}_{49} \cup \{226,12,235,345\}$ −9 −1 −5 −5 7 −5 −5 −1 −9
$\mathcal{D}'_{21} = \{1,3,263,250,177,248,93,262,22,24\} \cong \mathcal{B}_{49} \cup \{226,12,64,275\}$ −8 −5 5 −6 −6 4 1 8 5
$\mathcal{D}'_{22} = \{1,3,263,250,177,248,93,262,22,204\} \cong \mathcal{B}_{49} \cup \{226,12,197,275\}$ −7 −4 1 −8 4 9 −1 8 −3
$\mathcal{D}'_{23} = \{1,3,263,250,177,248,93,262,22,236\} \cong \mathcal{B}_{49} \cup \{345,12,135,327\}$ 9 −3 −4 −9 −3 8 1 0 0
$\mathcal{D}'_{24} = \{1,3,263,250,177,248,93,262,24,191\} \cong \mathcal{B}_{49} \cup \{345,49,198,235\}$ 9 0 0 −9 −1 9 0 1 0
$\mathcal{D}'_{25} = \{1,3,263,250,177,248,93,262,204,225\} \cong \mathcal{B}_{49} \cup \{226,12,275,345\}$ −7 −4 1 −8 4 9 −1 8 −3
$\mathcal{D}'_{26} = \{1,3,263,250,177,248,93,374,205,294\} \cong \mathcal{B}_{49} \cup \{226,12,64,135\}$ −6 0 0 −6 2 1 −7 −2 2
$\mathcal{D}'_{27} = \{1,3,263,250,177,248,93,374,205,24\} \cong \mathcal{B}_{49} \cup \{226,12,135,197\}$ −6 0 0 −6 2 1 −7 −2 2
$\mathcal{D}'_{28} = \{1,3,263,250,177,248,93,374,205,204\} \cong \mathcal{B}_{49} \cup \{226,12,49,135\}$ −6 −6 −6 −7 0 7 0 5 0

$\mathcal{D}'_{29} = \{1, 3, 263, 250, 177, 248, 93, 374, 205, 225\} \cong \mathcal{B}_{49} \cup \{226, 12, 135, 345\}$ −6 0 0 −6 2 1 −7 −2 2
$\mathcal{D}'_{30} = \{1, 3, 263, 250, 177, 248, 93, 374, 296, 104\} \cong \mathcal{B}_{49} \cup \{226, 51, 64, 94\}$ 4 −7 7 −8 9 −5 6 0 0
$\mathcal{D}'_{31} = \{1, 3, 263, 250, 177, 248, 93, 374, 294, 24\} \cong \mathcal{B}_{49} \cup \{226, 12, 49, 197\}$ 9 −9 −3 −9 8 7 6 −2 −9
$\mathcal{D}'_{32} = \{1, 3, 263, 250, 177, 248, 93, 374, 294, 204\} \cong \mathcal{B}_{49} \cup \{226, 12, 64, 197\}$ 6 6 6 7 −5 2 0 0 −9
$\mathcal{D}'_{33} = \{1, 3, 263, 250, 177, 248, 93, 374, 294, 225\} \cong \mathcal{B}_{49} \cup \{226, 12, 64, 235\}$ 6 6 6 7 −5 2 0 0 −9
$\mathcal{D}'_{34} = \{1, 3, 263, 250, 177, 248, 93, 374, 24, 204\} \cong \mathcal{B}_{49} \cup \{226, 12, 197, 345\}$ −3 7 6 6 −7 1 0 0 −7
$\mathcal{D}'_{35} = \{1, 3, 263, 250, 177, 248, 93, 374, 24, 225\} \cong \mathcal{B}_{49} \cup \{226, 12, 49, 345\}$ 0 0 −9 −9 1 9 0 −1 0
$\mathcal{D}'_{36} = \{1, 3, 263, 250, 177, 248, 93, 205, 22, 104\} \cong \mathcal{B}_{49} \cup \{345, 12, 94, 135\}$ 0 4 0 5 9 −5 −3 0 0
$\mathcal{D}'_{37} = \{1, 3, 263, 250, 177, 248, 93, 205, 294, 24\} \cong \mathcal{B}_{49} \cup \{345, 49, 135, 197\}$ −3 7 6 6 −7 1 0 0 −7
$\mathcal{D}'_{38} = \{1, 3, 263, 250, 177, 248, 93, 205, 104, 204\} \cong \mathcal{B}_{49} \cup \{345, 12, 94, 327\}$ 4 −7 7 3 4 0 −1 3 −7
$\mathcal{D}'_{39} = \{1, 3, 263, 250, 177, 248, 93, 205, 204, 191\} \cong \mathcal{B}_{49} \cup \{226, 12, 49, 235\}$ 0 9 0 9 0 −9 −9 −9 −9
$\mathcal{D}'_{40} = \{1, 3, 263, 250, 177, 248, 22, 262, 205, 225\} \cong \mathcal{B}_{61} \cup \{49, 51, 235, 327\}$ 0 9 0 9 1 −9 1 0 0
$\mathcal{D}'_{41} = \{1, 3, 263, 250, 177, 248, 22, 262, 294, 225\} \cong \mathcal{B}_{61} \cup \{49, 135, 235, 327\}$ 9 0 0 −9 −1 9 0 1 0
$\mathcal{D}'_{42} = \{1, 3, 263, 250, 177, 248, 22, 374, 205, 296\} \cong \mathcal{B}_{49} \cup \{226, 12, 235, 327\}$ −6 0 0 −6 2 1 −7 −2 2
$\mathcal{D}'_{43} = \{1, 3, 263, 250, 177, 248, 22, 374, 205, 236\} \cong \mathcal{B}_{49} \cup \{226, 12, 49, 198\}$ 9 0 0 −5 −1 9 9 5 2
$\mathcal{D}'_{44} = \{1, 3, 263, 250, 177, 93, 225, 374, 262, 205\} \cong \mathcal{B}_{61} \cup \{135, 94, 275, 345\}$ −4 0 0 −6 −4 −2 −2 −2 −2

Note that each 10-arc $\mathcal{B}_i \cup \{P_1, P_2, P_3, P_4\}$ in Table 4.30 is on the conic $\mathcal{C}_{\mathcal{A}_1}$. The 10-arcs on the conic are also found by substituting the $6th$, $7th$, $8th$, $9th$ and $10th$ points of each 10-arc in the conic form of $\mathcal{C}_{\mathcal{A}_1}$.

Theorem 4.18.1. *In $PG(2,19)$, there are precisely 44 projectively distinct 10-arcs on a conic, as summarized in Table 4.31.*

Table 4.31: Inequivalent 10-arcs on the conic

No.	The 10-arc	Stabilizer	$[c_0, c_1, c_2, c_3, c_4, c_5]$
1	$\mathcal{B}_{49} \cup \{226, 12, 51, 135\}$	$\mathbf{Z}_2$	$[16, 56, 162, 118, 19, 0]$
2	$\mathcal{B}_{49} \cup \{226, 12, 51, 94\}$	$\mathbf{Z}_2$	$[17, 53, 164, 120, 16, 1]$
3	$\mathcal{B}_{49} \cup \{226, 12, 49, 64\}$	I	$[15, 57, 165, 113, 21, 0]$
4	$\mathcal{B}_{49} \cup \{226, 12, 51, 64\}$	I	$[14, 63, 153, 123, 18, 0]$
5	$\mathcal{B}_{49} \cup \{226, 12, 49, 142\}$	I	$[15, 56, 168, 110, 22, 0]$
6	$\mathcal{B}_{49} \cup \{226, 12, 49, 51\}$	$\mathbf{V}_4$	$[14, 58, 166, 114, 17, 2]$
7	$\mathcal{B}_{49} \cup \{226, 12, 94, 135\}$	$\mathbf{Z}_2$	$[18, 50, 168, 116, 19, 0]$
8	$\mathcal{B}_{49} \cup \{226, 12, 94, 197\}$	I	$[16, 57, 159, 121, 18, 0]$
9	$\mathcal{B}_{49} \cup \{226, 12, 64, 142\}$	$\mathbf{Z}_2$	$[14, 65, 147, 129, 16, 0]$
10	$\mathcal{B}_{49} \cup \{226, 12, 94, 327\}$	I	$[14, 61, 159, 117, 20, 0]$
11	$\mathcal{B}_{49} \cup \{226, 12, 64, 94\}$	I	$[16, 56, 162, 118, 19, 0]$
12	$\mathcal{B}_{49} \cup \{226, 12, 94, 275\}$	$\mathbf{Z}_2$	$[11, 67, 158, 114, 20, 1]$
13	$\mathcal{B}_{49} \cup \{226, 12, 94, 345\}$	I	$[13, 64, 156, 118, 20, 0]$
14	$\mathcal{B}_{49} \cup \{226, 12, 49, 94\}$	I	$[12, 68, 150, 122, 19, 0]$
15	$\mathcal{B}_{49} \cup \{226, 12, 49, 275\}$	I	$[12, 63, 165, 107, 24, 0]$
16	$\mathcal{B}_{49} \cup \{226, 12, 64, 345\}$	I	$[16, 58, 156, 124, 17, 0]$
17	$\mathcal{B}_{49} \cup \{345, 12, 51, 94\}$	$\mathbf{V}_4$	$[14, 62, 154, 126, 13, 2]$
18	$\mathcal{B}_{49} \cup \{226, 12, 135, 275\}$	$\mathbf{Z}_2$	$[15, 57, 164, 116, 18, 1]$
19	$\mathcal{B}_{49} \cup \{345, 12, 51, 327\}$	I	$[14, 60, 162, 114, 21, 0]$
20	$\mathcal{B}_{49} \cup \{226, 12, 235, 345\}$	I	$[13, 61, 165, 109, 23, 0]$
21	$\mathcal{B}_{49} \cup \{226, 12, 64, 275\}$	I	$[14, 61, 159, 117, 20, 0]$
22	$\mathcal{B}_{49} \cup \{226, 12, 197, 275\}$	$\mathbf{Z}_2$	$[13, 61, 164, 112, 20, 1]$
23	$\mathcal{B}_{49} \cup \{345, 12, 135, 327\}$	$\mathbf{V}_4$	$[12, 62, 166, 110, 19, 2]$
24	$\mathcal{B}_{49} \cup \{345, 49, 198, 235\}$	$\mathbf{D}_{10}$	$[20, 40, 180, 120, 5, 6]$
25	$\mathcal{B}_{49} \cup \{226, 12, 275, 345\}$	$\mathbf{Z}_2$	$[12, 62, 168, 104, 25, 0]$
26	$\mathcal{B}_{49} \cup \{226, 12, 64, 135\}$	$\mathbf{Z}_2$	$[17, 49, 176, 108, 20, 1]$
27	$\mathcal{B}_{49} \cup \{226, 12, 135, 197\}$	$\mathbf{V}_4$	$[16, 50, 178, 106, 19, 2]$
28	$\mathcal{B}_{49} \cup \{226, 12, 49, 135\}$	I	$[13, 65, 153, 121, 19, 0]$
29	$\mathcal{B}_{49} \cup \{226, 12, 135, 345\}$	$\mathbf{Z}_2$	$[16, 52, 173, 109, 20, 1]$
30	$\mathcal{B}_{49} \cup \{226, 51, 64, 94\}$	$\mathbf{Z}_3$	$[22, 45, 159, 133, 12, 0]$
31	$\mathcal{B}_{49} \cup \{226, 12, 49, 197\}$	$\mathbf{Z}_2$	$[11, 65, 164, 108, 22, 1]$
32	$\mathcal{B}_{49} \cup \{226, 12, 64, 197\}$	I	$[13, 62, 162, 112, 22, 0]$
33	$\mathcal{B}_{49} \cup \{226, 12, 64, 235\}$	I	$[15, 59, 159, 119, 19, 0]$
34	$\mathcal{B}_{49} \cup \{226, 12, 197, 345\}$	$\mathbf{Z}_2$	$[14, 58, 167, 111, 20, 1]$
35	$\mathcal{B}_{49} \cup \{226, 12, 49, 345\}$	$\mathbf{Z}_2$	$[13, 61, 164, 112, 20, 1]$
36	$\mathcal{B}_{49} \cup \{345, 12, 94, 135\}$	$\mathbf{Z}_2$	$[15, 59, 158, 122, 16, 1]$
37	$\mathcal{B}_{49} \cup \{345, 49, 135, 197\}$	$\mathbf{Z}_2$	$[11, 63, 170, 102, 24, 1]$

38	$\mathcal{B}_{49} \cup \{345, 12, 94, 327\}$	$\mathbf{Z}_2$	$[16, 48, 185, 97, 24, 1]$
39	$\mathcal{B}_{49} \cup \{226, 12, 49, 235\}$	$\mathbf{Z}_3$	$[17, 54, 162, 120, 18, 0]$
40	$\mathcal{B}_{61} \cup \{49, 51, 235, 327\}$	$\mathbf{V}_4$	$[14, 56, 172, 108, 19, 2]$
41	$\mathcal{B}_{61} \cup \{49, 135, 235, 327\}$	$\mathbf{D}_5$	$[16, 40, 205, 85, 20, 5]$
42	$\mathcal{B}_{49} \cup \{226, 12, 235, 327\}$	$\mathbf{Z}_3$	$[13, 75, 123, 151, 9, 0]$
43	$\mathcal{B}_{49} \cup \{226, 12, 49, 198\}$	$\mathbf{Z}_2$	$[18, 50, 167, 119, 16, 1]$
44	$\mathcal{B}_{61} \cup \{135, 94, 275, 345\}$	$\mathbf{Z}_9$	$[20, 45, 171, 117, 18, 0]$

4.19 11-Arcs

The total number of points not on the sides of the 10-stigms is 2594630. The action of the stabilizer group of each inequivalent 10-arc on the corresponding set C_0^{10} splits the 2594630 points into 2578375 orbits. There are 1736 different classes of 11-arcs of type $[c_0, c_1, c_2, c_3, c_4, c_5]$. The minimum and maximum value of each constant c_i for all 11-arcs is as follows:

$$\begin{array}{rcl rcl} 0 \leq & c_0 & \leq 13, & 6 \leq & c_1 & \leq 48, \\ 85 \leq & c_2 & \leq 154, & 109 \leq & c_3 & \leq 185, \\ 33 \leq & c_4 & \leq 81, & 0 \leq & c_5 & \leq 19. \end{array}$$

Since $c_0 = 0$ for some 11-arcs so there is a complete 11-arc in $PG(2, 19)$. There are seven different sizes of stabilizer groups of the 11-arcs. The details are given in Table 4.32.

TABLE 4.32: Statistics of the stabilizer groups of 11-arcs

Number of 11-arcs	$\|G\|$	Number of 11-arcs	$\|G\|$
2566355	1	3	9
11862	2	2	10
115	3	2	18
36	6		

Theorem 4.19.1. *In* $PG(2, 19)$, *there are precisely* 235320 *projectively distinct* 11*-arcs divided into* 225779 *incomplete arcs and* 9541 *complete arcs.*

In Table 4.33, the numbers of inequivalent 11-arcs are listed according to the stabilizer group types G.

Table 4.33: Statistics of the inequivalent incomplete 11-arcs

Number of 11-arcs	G	Number of 11-arcs	G
223804	I	1	$\mathbf{Z}_9$
1941	$\mathbf{Z}_2$	1	$\mathbf{D}_5$
19	$\mathbf{Z}_3$	1	$\mathbf{D}_9$
12	$\mathbf{S}_3$		

According to the stabilizer group types G, the numbers of 11-complete arcs are listed in Table 4.34.

Table 4.34: Statistics of the inequivalent complete 11-arcs

Number of 11-arcs	G
9501	I
36	$\mathbf{Z}_2$
4	$\mathbf{Z}_3$

4.20 12-Arcs

The total number of points not on the sides of the 11-stigms is 656507. The action of the stabilizer group of each inequivalent 11-arc on the corresponding set C_0^{11} splits the 656507 points into 654654 orbits. There are 2787 different classes of 12-arcs of type $[c_0, c_1, c_2, c_3, c_4, c_5, c_6]$. The minimum and maximum value of each constant c_i for all 12-arcs is as follows:

$$\begin{array}{rcl rcl} 0 \leq & c_0 & \leq 8, & 0 \leq & c_1 & \leq 24, \\ 24 \leq & c_2 & \leq 93, & 99 \leq & c_3 & \leq 204, \\ 36 \leq & c_4 & \leq 162, & 0 \leq & c_5 & \leq 54, \\ 0 \leq & c_6 & \leq 14. & & & \end{array}$$

Since $c_0 = 0$ for some 12-arcs so there is a complete 12-arc in $PG(2, 19)$. There are eleven different sizes of stabilizer groups of the 12-arcs. The details are given in Table 4.35.

TABLE 4.35: Statistics of the stabilizer groups of 12-arcs

Number of 12-arcs	$\|G\|$	Number of 12-arcs	$\|G\|$
638436	1	4	9
15327	2	9	12
396	3	7	18
320	4	4	24
132	6	1	72
18	8		

Theorem 4.20.1. *In* $PG(2, 19)$*, there are precisely* 55708 *projectively distinct* 12*-arcs divided into* 25573 *incomplete arcs and* 30135 *complete arcs.*

In Table 4.36, the numbers of inequivalent 12-arcs are listed according to the stabilizer group types G.

TABLE 4.36: Statistics of the inequivalent incomplete 12-arcs

Number of 12-arcs	G	Number of 12-arcs	G
24902	I	8	$\mathbf{S}_3$
610	$\mathbf{Z}_2$	5	$\mathbf{D}_5$
17	$\mathbf{Z}_3$	1	$\mathbf{A}_4$
5	$\mathbf{Z}_4$	2	$\mathbf{D}_6$
22	$\mathbf{V}_4$	1	$\mathbf{S}_4$

According to the stabilizer group types G, the numbers of 12-complete arcs are listed in Table 4.37.

TABLE 4.37: Statistics of the inequivalent complete 12-arcs

Number of 12-arcs	G	Number of 12-arcs	G
28301	I	2	$\mathbf{Z}_3 \times \mathbf{Z}_3$
1640	$\mathbf{Z}_2$	3	$\mathbf{A}_4$
82	$\mathbf{Z}_3$	1	$\mathbf{D}_6$
11	$\mathbf{Z}_4$	2	$(\mathbf{Z}_3 \times \mathbf{Z}_3) \rtimes \mathbf{Z}_2$
47	$\mathbf{V}_4$	1	$\mathbf{D}_9$
37	$\mathbf{S}_3$	3	$\mathbf{S}_4$
4	$\mathbf{D}_4$	1	G_{72}

The group G_{72} of order 72 has 21 elements of order 2, 26 elements of order 3, 18 elements of order 4 and 6 elements of order 6.

4.21 13-Arcs

The total number of points not on the sides of the 12-stigms is 34679. The action of the stabilizer group of each inequivalent 12-arc on the corresponding set C_0^{12} splits the 34679 points into 34007 orbits. There are 957 different classes of 13-arcs of type $[c_0, c_1, c_2, c_3, c_4, c_5, c_6]$. The minimum and maximum value of each constant c_i for all 13-arcs is as follows:

$$\begin{array}{rclrcl} 0 \leq & c_0 & \leq 7, & 0 \leq & c_1 & \leq 9, \\ 7 \leq & c_2 & \leq 40, & 71 \leq & c_3 & \leq 127, \\ 122 \leq & c_4 & \leq 189, & 48 \leq & c_5 & \leq 95, \\ 1 \leq & c_6 & \leq 21. & & & \end{array}$$

Since the value of $c_0 = 0$ for some 13-arcs so there is a complete 13-arc in $PG(2, 19)$. There are four different sizes of stabilizer groups of the 13-arcs. The details are given in Table 4.38.

Table 4.38: Statistics of the stabilizer groups of 13-arcs

Number of 13-arcs	$\|G\|$	Number of 13-arcs	$\|G\|$
32305	1	45	3
1645	2	12	6

Theorem 4.21.1. *In $PG(2, 19)$, there are precisely 2733 projectively distinct 13-arcs divided into 501 incomplete arcs and 2232 complete arcs.*

In Table 4.39, the numbers of incomplete 13-arcs are listed according to their stabilizer group types.

Table 4.39: Statistics of the inequivalent incomplete 13-arcs

Number of 13-arcs	G	Number of 13-arcs	G
395	I	6	$\mathbf{Z}_3$
98	$\mathbf{Z}_2$	2	$\mathbf{Z}_6$

According to the stabilizer group types G, the numbers of 13-complete arcs are listed in Table 4.40.

Table 4.40: Statistics of the inequivalent complete 13-arcs

Number of 13-arcs	G	Number of 13-arcs	G
2090	I	3	$\mathbf{Z}_3$
137	$\mathbf{Z}_2$	2	$\mathbf{S}_3$

4.22 14-Arcs

The total number of points not on the sides of the 13-stigms is 626. The action of the stabilizer group of each inequivalent 13-arc on the corresponding set C_0^{13} splits the 626 points into 584 orbits. There are 77 different classes of 14-arcs of type $[c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7]$. The minimum and maximum value of each constant c_i for all 14-arcs is as follows:

$$\begin{array}{ccccccc} 0 \leq & c_0 & \leq 6, & & 0 \leq & c_1 & \leq 4, \\ 0 \leq & c_2 & \leq 15, & & 18 \leq & c_3 & \leq 78, \\ 84 \leq & c_4 & \leq 168, & & 114 \leq & c_5 & \leq 183, \\ 9 \leq & c_6 & \leq 53, & & 0 \leq & c_7 & \leq 18. \end{array}$$

Since the value of $c_0 = 0$ for some 14-arcs so there is a complete 14-arc in $PG(2, 19)$. There are five different sizes of stabilizer groups of the 14-arcs. The details are given in Table 4.41.

Table 4.41: Statistics of the stabilizer groups of 14-arcs

Number of 14-arcs	$\lvert G\rvert$	Number of 14-arcs	$\lvert G\rvert$
140	1	22	6
311	2	5	12
106	4		

Theorem 4.22.1. *In $PG(2, 19)$, there are precisely 83 projectively distinct 14-arcs divided into 13 incomplete arcs and 70 complete arcs.*

In Table 4.42, the numbers of incomplete 14-arcs are listed according to their stabilizer group types.

Table 4.42: Statistics of the inequivalent incomplete 14-arcs

Number of 14-arcs	G	Number of 14-arcs	G
2	I	2	$\mathbf{S}_3$
5	$\mathbf{Z}_2$	1	$\mathbf{D}_6$
3	$\mathbf{V}_4$		

According to the stabilizer group types G, the numbers of 14-complete arcs are listed in Table 4.43.

TABLE 4.43: Statistics of the inequivalent complete 14-arcs

Number of 14-arcs	G	Number of 14-arcs	G
8	I	14	$\mathbf{V}_4$
35	$\mathbf{Z}_2$	4	$\mathbf{S}_3$
8	$\mathbf{Z}_4$	1	$\mathbf{D}_6$

4.23 15-Arcs

The total number of points not on the sides of the 14-stigms is 78. The action of the stabilizer group of each inequivalent 14-arc on the corresponding set C_0^{14} splits the 78 points into 36 orbits. There are only five different classes of 15-arcs of type of $[c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7]$ as given below:

$$[5, 0, 0, 0, 46, 198, 103, 14],$$
$$[5, 0, 0, 0, 48, 192, 109, 12],$$
$$[5, 0, 0, 0, 49, 189, 112, 11],$$
$$[5, 0, 0, 0, 50, 186, 115, 10],$$
$$[5, 0, 0, 0, 54, 174, 127, 6].$$

Since $c_0 \neq 0$ for all 15-arcs so there is no complete 15-arc in $PG(2, 19)$. There are four different sizes of stabilizer groups of the 15-arcs. The details are given in Table 4.44.

TABLE 4.44: Statistics of the stabilizer groups of 15-arcs

Number of 15-arcs	$\|G\|$	Number of 15-arcs	$\|G\|$
15	1	3	6
16	2	2	10

Let $\mathcal{J} = \{1, 2, 3, 263, 7, 64, 135, 142, 182, 12, 49, 51\}$.

Theorem 4.23.1. *In $PG(2, 19)$, there are precisely five projectively distinct incomplete 15-arcs, as summarized in Table 4.45.*

TABLE 4.45: The inequivalent 15-arcs

The 15-arc	Stabilizer	$[c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7]$
$\mathcal{J} \cup \{94, 197, 198\}$	I	$[5, 0, 0, 0, 49, 189, 112, 11]$
$\mathcal{J} \cup \{94, 197, 235\}$	$\mathbf{Z}_2$	$[5, 0, 0, 0, 54, 174, 127, 6]$
$\mathcal{J} \cup \{94, 197, 275\}$	$\mathbf{Z}_2$	$[5, 0, 0, 0, 46, 198, 103, 14]$
$\mathcal{J} \cup \{94, 197, 226\}$	$\mathbf{S}_3$	$[5, 0, 0, 0, 48, 192, 109, 12]$
$\mathcal{J} \cup \{197, 235, 275\}$	$\mathbf{D}_5$	$[5, 0, 0, 0, 50, 186, 115, 10]$

4.24 16-Arcs

From the five different classes of type $[c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7]$ in Table 4.45, the total number of points not on the sides of the 15-stigms is 25. The action of the stabilizer group of each inequivalent 15-arc on the corresponding set C_0^{15} splits the 25 points into 14 orbits. There are only three different classes of 16-arcs of type $[c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8]$ as given below:

$$[4, 0, 0, 0, 0, 78, 214, 66, 3],$$
$$[4, 0, 0, 0, 0, 76, 220, 60, 5],$$
$$[4, 0, 0, 0, 0, 80, 208, 72, 1].$$

Since $c_0 \neq 0$ for all 16-arcs so there is no complete 16-arc in $PG(2, 19)$. There are three different sizes of stabilizer groups of the 16-arcs. The details are given in Table 4.46.

TABLE 4.46: Statistics of the stabilizer groups of 16-arcs

Number of 16-arcs	$\|G\|$	Number of 16-arcs	$\|G\|$
10	2	2	12
2	8		

Let $\mathcal{J}' = \{1, 2, 3, 263, 7, 64, 135, 142, 182, 12, 49, 51, 94, 197, 198\} = \mathcal{J} \cup \{94, 197, 198\}$.

Theorem 4.24.1. *In $PG(2, 19)$, there are precisely four projectively distinct incomplete 16-arcs, as summarized in Table 4.47.*

TABLE 4.47: The inequivalent 16-arcs

The 16-arc	Stabilizer	$[c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8]$
$\mathcal{J}' \cup \{235\}$	$\mathbf{V}_4$	$[4, 0, 0, 0, 0, 80, 208, 72, 1]$
$\mathcal{J}' \cup \{327\}$	$\mathbf{V}_4$	$[4, 0, 0, 0, 0, 80, 208, 72, 1]$
$\mathcal{J}' \cup \{275\}$	$\mathbf{D}_4$	$[4, 0, 0, 0, 0, 76, 220, 60, 5]$
$\mathcal{J}' \cup \{226\}$	$\mathbf{A}_4$	$[4, 0, 0, 0, 0, 78, 214, 66, 3]$

4.25 The Unique k-Arcs, $k = 17, 18, 19, 20$

(1) From the four classes of type $[c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8]$ in Table 4.47, the total number of points not on the sides of the 16-stigms is sixteen. The action of the stabilizer group of each inequivalent 16-arc on the corresponding set C_0^{16} splits the sixteen points into four orbits. There is only one class of 17-arcs of type $[c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8]$ as given below:

$$[3, 0, 0, 0, 0, 0, 112, 216, 33].$$

Since $c_0 \neq 0$ for the four 17-arcs so there is no complete 17-arc in $PG(2, 19)$.

Theorem 4.25.1. *In $PG(2, 19)$, there is precisely one projectively distinct incomplete 17-arc*

$$\mathcal{J}_{17} = \mathcal{J}' \cup \{226, 235\}.$$

It is stabilized by the group of type $\mathbf{S}_3$.

(2) From (1), there are only three points not on the sides of the projectively unique 17-stigm whose vertices are the points of the 17-arc $\mathcal{J}_{17}$. One orbit is constructed from these three points. The only class of 18-arc of type

$$[c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9]$$

is

$$[2, 0, 0, 0, 0, 0, 0, 0, 144, 207, 10].$$

Since $c_0 \neq 0$ so there is no complete 18-arc in $PG(2, 19)$.

Theorem 4.25.2. *In $PG(2,19)$, there is precisely one projectively distinct incomplete 18-arc*

$$\mathcal{J}_{18} = \mathcal{J}' \cup \{226, 235, 275\}.$$

It is stabilized by the group of type $\mathbf{D}_{18}$.

(3) From (2), there are only two points not on the sides of the projectively unique 18-stigm whose vertices are the points of the 18-arc $\mathcal{J}_{18}$. One orbit is constructed from these two points. The only class of 19-arc of type $[c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9]$ is

$$[1, 0, 0, 0, 0, 0, 0, 0, 171, 190].$$

Since $c_0 \neq 0$ so there is no complete 19-arc in $PG(2,19)$.

Theorem 4.25.3. *In $PG(2,19)$, there is precisely one projectively distinct incomplete 19-arc*

$$\mathcal{J}_{19} = \mathcal{J}' \cup \{226, 235, 275, 327\}.$$

It is stabilized by the group G of size 342.

(4) From (3), there is only one point 345 in numeral form not on the sides of the projectively unique 19-stigm whose vertices are the points of the 19-arc $\mathcal{J}_{19}$. So, only one 20-arc can be construct from $\mathcal{J}_{19}$ which is

$$\mathcal{J}_{20} = \mathcal{J}' \cup \{226, 235, 275, 327, 345\}.$$

The class $[c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, c_{10}]$ of $\mathcal{J}_{20}$ is

$$[0, 0, 0, 0, 0, 0, 0, 0, 0, 190, 171].$$

Since $c_0 = 0$ so $\mathcal{J}_{20}$ is complete arc. The 20-arc $\mathcal{J}_{20}$ is exactly the conic $\mathcal{C}_{A_1}$.

Remark 4.25.4. The value of the constant c_9 represents the number of external points and the value of the constant c_{10} represents the number of internal points of $\mathcal{C}_{A_1}$.

Theorem 4.25.5. *In $PG(2,19)$, the conic $\mathcal{C}_{A_1}$ is the projectively unique 20-arc. It is complete and stabilized by the group $PGL(2,19)$.*

Theorem 4.25.6. *In $PG(2,19)$, the conic $\mathcal{C}_{A_1}$ contains thirteen incomplete 14-arcs, five incomplete 15-arcs as in Table 4.45, four incomplete 16-arcs as in Table 4.47 and unique incomplete 17-arc $\mathcal{J}_{17}$, 18-arc $\mathcal{J}_{18}$ and 19-arc $\mathcal{J}_{19}$.*

Remark 4.25.7. (1) From Theorems 4.25.3 and 4.25.6, the 19-arc $\mathcal{J}_{19}$ is projectively unique and lies on the conic $\mathcal{C}_{A_1}$. This agrees with Lemma 3.8.1 that there is a projectively unique 19-arc lying on a conic.

(2) Theorem 4.25.6 coincides with Lemma 3.8.8 that there is a unique complete arc containing the k-arcs $k = 15, 16, 17, 18, 19$ for $q = 19$.

(3) The number of common points of an incomplete k-arc with a conic is at most 10, which happens when $k = 11, 12$. The number of common points of a complete k-arc with a conic is at most 11, which happens when $k = 12$. This agrees with Lemma 3.8.5.

Remark 4.25.8. The uniqueness of the 17-arc, 18-arc and 19-arc on a conic $\mathcal{C}$ in $PG(2, 19)$ can be proved theoretically as follows.

By Remark 2.2.1, $PGO(3, 19)$ acts sharply 3-transitively on $\mathcal{C}$. Therefore there is projectively a unique 17-arc on $\mathcal{C}$. As a special case, $PGO(3, 19)$ is 2-transitive and 1-transitive; so the 18-arcs and 19-arcs on $\mathcal{C}$ are projectively unique.

4.26 Summary of Complete k-Arcs for $k = 10, 11, 12, 13, 14$

One of the main themes of the previous calculations for arcs in $PG(2,19)$ is to find the size of arcs which are complete and the number of complete arcs of each size with their stabilizer group types. The following table summarizes the results on the complete arcs in this chapter by giving, in each column, the size k of the complete arc, the number M_k of the complete arcs of that size and finally the number M_k split according to the types of the stabilizer groups represented by the cell $n : G$.

TABLE 4.48: The classification of the complete k-arcs in $PG(2,19)$

$k = 10$	$k = 11$	$k = 12$	$k = 13$	$k = 14$	$k = 20$
$M_k = 29$	$M_k = 9541$	$M_k = 30135$	$M_k = 2232$	$M_k = 70$	$M_k = 1$
$1 : I$ $18 : \mathbf{Z}_2$ $1 : \mathbf{Z}_3$ $1 : \mathbf{Z}_4$ $2 : \mathbf{V}_4$ $2 : \mathbf{S}_3$ $2 : \mathbf{D}_5$ $1 : \mathbf{A}_4$ $1 : \mathbf{A}_5$	$9501 : I$ $36 : \mathbf{Z}_2$ $4 : \mathbf{Z}_3$	$28301 : I$ $1640 : \mathbf{Z}_2$ $82 : \mathbf{Z}_3$ $11 : \mathbf{Z}_4$ $47 : \mathbf{V}_4$ $37 : \mathbf{S}_3$ $4 : \mathbf{D}_4$ $2 : \mathbf{Z}_3 \times \mathbf{Z}_3$ $3 : \mathbf{A}_4$ $1 : \mathbf{D}_6$ $2 : (\mathbf{Z}_3 \times \mathbf{Z}_3) \rtimes \mathbf{Z}_2$ $1 : \mathbf{D}_9$ $3 : \mathbf{S}_4$ $1 : G_{72}$	$2090 : I$ $137 : \mathbf{Z}_2$ $3 : \mathbf{Z}_3$ $2 : \mathbf{S}_3$	$8 : I$ $35 : \mathbf{Z}_2$ $8 : \mathbf{Z}_4$ $14 : \mathbf{V}_4$ $4 : \mathbf{S}_3$ $1 : \mathbf{D}_6$	$1 : PGL(2,19)$

4.27 MDS Codes of Dimension Three

According to Theorem 1.13.1, an $(n; n-d)$-arc in $PG(k-1, q)$ is equivalent to a projective $[n, k, d]_q$-code. Now, if $k = 3, n-d = 2$, and $q = 19$, then there is a one-to-one correspondence between n-arcs in $PG(2, 19)$ and projective $[n, 3, n-2]_{19}$-codes C. Since $d(C)$ of the code C is equal to $n - k + 1$, thus the projective code C is MDS.

In Table 4.49, the MDS codes corresponding to the n-arcs in $PG(2, 19)$ and the parameter e of errors corrected are given.

Table 4.49: MDS code over $PG(2, 19)$

n-arc	MDS code	e	$n-$ arc	MDS code	e
4-arc	$[4, 3, 2]_{19}$	0	13-arc	$[13, 3, 11]_{19}$	5
5-arc	$[5, 3, 3]_{19}$	1	14-arc	$[14, 3, 12]_{19}$	5
6-arc	$[6, 3, 4]_{19}$	1	15-arc	$[15, 3, 13]_{19}$	6
7-arc	$[7, 3, 5]_{19}$	2	16-arc	$[16, 3, 14]_{19}$	6
8-arc	$[8, 3, 6]_{19}$	2	17-arc	$[17, 3, 15]_{19}$	7
9-arc	$[9, 3, 7]_{19}$	3	18-arc	$[18, 3, 16]_{19}$	7
10-arc	$[10, 3, 8]_{19}$	3	19-arc	$[19, 3, 17]_{19}$	8
11-arc	$[11, 3, 9]_{19}$	4	20-arc	$[20, 3, 18]_{19}$	8
12-arc	$[12, 3, 10]_{19}$	4			

Chapter 5

Classification of Non-Singular Plane Cubic Curves

5.1 Introduction

From Section 1.11, a rational inflexion of a cubic curve $\mathcal{F}$ is a non-singular (simple) point at which the tangent has three-point contact. A non-singular plane cubic curve with k rational points can be regarded as a $(k;3)$-arc.

A conic in $PG(2,q)$ is projectively a unique, irreducible, plane curve of degree two and also a complete $(q+1)$-arc for q odd but an incomplete $(q+1)$-arc for q even, which can be completed uniquely to a $(q+2)$-arc by its nucleus. Now, the question arises here: *Which non-singular plane cubic curves in $PG(2,q)$ are complete as arcs of degree three*?

In this chapter the answer to this question for $q = 19$ is given by the following method.

(1) Find the projectively distinct non-singular plane cubic curves in $PG(2,19)$.

(2) For each of these, write down the canonical form.

(3) Then list the rational points of each one.

(4) Now, the 3-secants are checked if they fill $PG(2,19)$ or not.

Also, the maximum values of k for $(k;3)$-arcs containing the curves are calculated.

Firstly, some definitions and results which are related to curves over the field $\mathbf{F}_{19}$ are given in next two sections.

5.2 Properties of Non-Singular Plane Cubic Curves

Let $\mathcal{F}$ be a plane cubic curve defined over $\mathbf{F}_q$. The *class* $\kappa = \kappa(\mathcal{F})$ of $\mathcal{F}$ is the number of distinct tangents to $\mathcal{F}$ through an arbitrary point of $PG(2, \overline{\mathbf{F}}_q)$. The class κ satisfies the following:

$$\kappa \leq 6, \;\; q \text{ odd};$$
$$\kappa \leq 3, \;\; q \text{ even}.$$

See [28, Lemma 11.14].

Lemma 5.2.1. *If a non-singular plane cubic curve $\mathcal{F}$ defined over $\mathbf{F}_q$ with q odd has class six, then there are four tangents to $\mathcal{F}$ from a point P of $\mathcal{F}$, other than the tangent at P, and the cross-ratio of the four tangents is constant.*

Proof. See [28, Lemma 11.15]. □

The non-singular plane cubic curve $\mathcal{F}$ in Lemma 5.2.1 called *harmonic* or *equianharmonic* if the four tangents through a point form a harmonic or equianharmonic set. A non-singular cubic curve which is not harmonic or equianharmonic is called *general*. In general, over $\overline{\mathbf{F}}_q$, $q \not\equiv 0 \pmod 3$, a non-singular plane cubic curve $\mathcal{F}$ has nine rational inflexions, [28, Theorem 11.43].

Let F be a cubic form over $\mathbf{F}_q$. A *rational inflexional triangle* is a set of three lines over $\mathbf{F}_q$ through the nine inflexions of $\mathcal{F} = \mathbf{v}(F)$ over $\overline{\mathbf{F}}_q$.

Lemma 5.2.2. (i) *The number of rational inflexions on a non-singular plane cubic curve over $\mathbf{F}_q$, $q \equiv 1 \pmod 3$ is zero, one, three, or nine. See [28, Lemma 11.42].*

(ii) *The possible numbers of rational inflexional triangles if $q \equiv 1 \pmod 3$ is zero, one or four. See [28, Corollary 11.44].*

A non-singular plane cubic curve $\mathcal{F}$ over $\mathbf{F}_q$, $q \not\equiv 0 \pmod 3$, is denoted by $\mathcal{F}_n^r$, where n is the number of rational inflexions and r is the number of rational inflexional triangles. Also, $\mathcal{F}_n^r = \mathcal{G}_n^r, \mathcal{E}_n^r, \mathcal{H}_n^r$ when $\mathcal{F}$ is respectively general, equianharmonic, harmonic.

Since $19 \equiv 1 \pmod 3$, then a non-singular plane cubic curve over $\mathbf{F}_{19}$ is one of the following types:

$$\mathcal{F}_9^4, \; \mathcal{F}_3^1, \; \mathcal{F}_1^4, \; \mathcal{F}_1^1, \; \mathcal{F}_1^0, \; \mathcal{F}_0^4, \; \mathcal{F}_0^1.$$

See [28, Theorem 11.46].

Lemma 5.2.3. (i) *There are* $(q-1,3)$ *projectively distinct plane cubic curves with three collinear rational inflexions such that the inflexional tangents are concurrent. The canonical forms are as follows:*

(a) $(q-1,3)=1$,

$$F = X_0X_1(X_0+X_1)+X_2^3;$$

(b) $(q-1,3)=3$,

$$F = X_0X_1(X_0+X_1)+X_2^3,$$
$$F' = X_0X_1(X_0+X_1)+\alpha X_2^3,$$
$$F'' = X_0X_1(X_0+X_1)+\alpha^2 X_2^3,$$

where α *is a primitive element of* $\mathbf{F}_q$.

(ii) *A non-singular plane cubic curve over* $\mathbf{F}_q$ *with three collinear rational inflexions and concurrent inflexional tangents has three or nine rational inflexions.*

(iii) *A non-singular plane cubic curve over* $\mathbf{F}_q$ *with three collinear rational inflexions and non-concurrent inflexional tangents has three or nine rational inflexions and canonical form*

$$\mathcal{F} = \mathbf{v}(X_0X_1X_2 + e(X_0+X_1+X_2)^3),$$

where $e \neq 0, 1/27$.

Proof. (i) See [28, Lemma 11.39].
(ii) See [28, Theorem 11.40].
(iii) See [28, Theorem 11.41]. □

Remark 5.2.4. In Lemma 5.2.3, in case (i), the inflexions are

$$\mathbf{P}(1,0,0),\ \mathbf{P}(0,1,0),\ \mathbf{P}(1,-1,0);$$

in case (iii), the inflexions are

$$\mathbf{P}(0,1,-1),\ \mathbf{P}(1,0,-1),\ \mathbf{P}(1,-1,0).$$

For $q=19$, the results in Lemma 5.2.3 are detailed in Sections 5.4 and 5.5.

5.3 Number of Non-Singular Plane Cubics and Their Rational Points

Let n_i for $i = 0, 1, 3, 9$ be the number of projective equivalence classes of non-singular plane cubic curve with exactly i rational inflexions. Let P_q be the total number of projective equivalence classes. Hence,

$$P_q = n_9 + n_3 + n_1 + n_0.$$

Theorem 5.3.1. $P_q = 3q + 2 + \left(\frac{-4}{q}\right) + \left(\frac{-3}{q}\right)^2 + 3\left(\frac{-3}{q}\right).$

Proof. See [28, Theorem 11.100(ii)]. □

Here the bracketed numbers are *Legendre–Jacobi* symbols taking the following values:

$$\left(\frac{-4}{c}\right) = \begin{cases} 1 & \text{if } c \equiv 1 \pmod 4, \\ 0 & \text{if } c \equiv 0 \pmod 2, \\ -1 & \text{if } c \equiv -1 \pmod 4; \end{cases}$$

$$\left(\frac{-3}{c}\right) = \begin{cases} 1 & \text{if } c \equiv 1 \pmod 3, \\ 0 & \text{if } c \equiv 0 \pmod 3, \\ -1 & \text{if } c \equiv -1 \pmod 3. \end{cases}$$

Corollary 5.3.2. *Over* $\mathbf{F}_{19}$, $P_{19} = 62$.

Let $N_q(1)$ denote the maximum number of rational points on any non-singular plane cubic curve over $\mathbf{F}_q$ and $L_q(1)$ the minimum number.

Lemma 5.3.3. *(Hasse-Weil Bound) Let* $\mathtt{N}_1$ *be the number of rational points of a non-singular plane cubic curve over* $\mathbf{F}_q$. *Then*

$$q + 1 - \lfloor 2\sqrt{q} \rfloor \leq \mathtt{N}_1 \leq q + 1 + \lfloor 2\sqrt{q} \rfloor.$$

Proof. See [28, Corollary 2.9]. □

Corollary 5.3.4. *Over* $\mathbf{F}_{19}$, $12 \leq \mathtt{N}_1 \leq 28$.

Lemma 5.3.5. *When q is prime, the number* $\mathtt{N}_1$ *takes every value between* $L_q(1)$ *and* $N_q(1)$.

Proof. See [28, Corollary 11.97]. □

For further details, related results, and proof of the results in this section and previous two sections see [29], [32], [44] and [47].

The 62 non-singular plane cubics over $\mathbf{F}_{19}$ are classified in the next four sections according to the types of $\mathcal{F}$ and the values of n and r. Also, the numbers of rational points on these curves and the maximum size of a complete arc of degree three can be constructed from each plane cubic curve are given.

The related results for the field of order nineteen are given in the next four sections.

Throughout the following sections, a primitive element 2 in $\mathbf{F}_{19}$ is chosen.

5.4 Non-Singular Plane Cubics with Nine Rational Inflexions

Lemma 5.4.1. *There exists a non-singular plane cubic curve over* $\mathbf{F}_q$ *with nine rational inflexions if and only if* $q \equiv 1 \pmod 3$. *Then* $\mathcal{F}(c) = \mathbf{v}(F(c))$ *has canonical form*

$$F(c) = X_0^3 + X_1^3 + X_2^3 - 3cX_0X_1X_2.$$

Proof. See [28, Lemma 11.36]. □

Corollary 5.4.2. *Over* $\mathbf{F}_{19}$, *there exists a non-singular plane cubic curve with nine rational inflexions.*

Lemma 5.4.3. *In* $PG(2,q)$, $q \equiv 1 \pmod 3$, *with* ω *a root of* $x^2 + x + 1$,

(i) $\mathcal{F}(c)$ *is equianharmonic for* $c = 0, 2, 2\omega, 2\omega^2$;

(ii) $\mathcal{F}(c)$ *is harmonic for* $c = 1 \pm \sqrt{3}, (1 \pm \sqrt{3})\omega, (1 \pm \sqrt{3})\omega^2$.

Proof. See [28, Lemma 11.47]. □

Remark 5.4.4. For $q = 19$, the equation $x^2 + x + 1$ has two distinct roots, $7, -8$.

Corollary 5.4.5. *In* $PG(2,19)$,

(i) $\mathcal{F}(c)$ *is equianharmonic for* $c = 0, 2, 3, -5$;

(ii) *there is no harmonic type of* $\mathcal{F}(c)$.

In Table 5.1, the columns give the symbol of each type of $\mathcal{F}_n^r$, the canonical form, the number of rational points $|\mathcal{F}_n^r|$, the description, the maximum value $M(\mathcal{F}_n^r)$ of k for a $(k;3)$-arc containing the curve, and the stabilizer group G.

Table 5.1: Non-singular plane cubic curves with nine rational inflexions

$\mathcal{F}_n^r$	Canonical form	$\lvert\mathcal{F}_n^r\rvert$	Description	$M(\mathcal{F}_n^r)$	G
$\mathcal{G}_9^4$	$X_0^3 + X_1^3 + X_2^3 + 7X_0X_1X_2$	18	Incomplete	21	$(\mathbf{Z}_3 \times \mathbf{Z}_3) \rtimes \mathbf{Z}_2$
$\mathcal{E}_9^4$	$X_0^3 + X_1^3 + X_2^3$	27	Complete	–	G_{54}

The group G_{54} has 9 elements of order 2, 26 elements of order 3, and 18 elements of order 6.

5.5 Non-Singular Plane Cubics with Three Rational Inflexions

From Lemma 5.2.3, the non-singular plane cubic curves $\mathcal{F}$ with exactly three rational inflexions have the following canonical forms:

$F = X_0X_1(X_0 + X_1) + eX_2^3$ if the three inflexional tangents are concurrent,

$F = X_0X_1X_2 + e(X_0 + X_1 + X_2)^3$ if the three inflexional tangents are not concurrent.

Lemma 5.5.1. *The cubic* $\mathcal{F} = \mathbf{v}(X_0X_1X_2 + e(X_0 + X_1 + X_2)^3)$ *is*

(i) *singular and irreducible if* $e = -1/27$*;*

(ii) *equianharmonic if* $e = -1/24$*;*

(iii) *harmonic if* $216e^2 + 36e + 1 = 0$*, which has two roots when* 3 *is a square.*

Proof. See [28, Lemma 11.52]. □

Corollary 5.5.2. *In* $PG(2, 19)$*,*

(i) *the cubic* $\mathbf{v}(X_0X_1X_2 + 7(X_0 + X_1 + X_2)^3)$ *is singular and irreducible;*

(ii) *if* $e = -4$*, then the cubic* $\mathbf{v}(X_0X_1X_2 - 4(X_0 + X_1 + X_2)^3)$ *is projectively equivalent to* $\mathcal{E}_9^4$ *in Table 5.1 by the matrix transformation*

$$A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix};$$

(iii) *there is no harmonic type of* $\mathcal{F}$*.*

Let $\mathcal{G}$ be the general type of curve when the inflexional tangents are not concurrent, and $\overline{\mathcal{E}}$ the type when they are concurrent.

From Lemma 5.5.1 and Corollary 5.5.2, Table 5.2 is deduced. This table gives the canonical form for the non-singular plane cubic curves with exactly one rational inflexion, number of rational points, description complete or incomplete, maximum size of a complete arc contain each curve, and the stabilizer groups.

TABLE 5.2: Non-singular plane cubic curves with exactly three rational inflexions

$\mathcal{F}_n^r$	No.	Canonical form	$\lvert\mathcal{F}_n^r\rvert$	Description	$M(\mathcal{F}_n^r)$	G
$\mathcal{G}_3^1$	1	$X_0X_1X_2 - 6(X_0 + X_1 + X_2)^3$	12	Incomplete	27	$\mathbf{S}_3$
	2	$X_0X_1X_2 + 3(X_0 + X_1 + X_2)^3$	15	Incomplete	27	$\mathbf{S}_3$
	3	$X_0X_1X_2 + 4(X_0 + X_1 + X_2)^3$	15	Incomplete	27	$\mathbf{S}_3$
	4	$X_0X_1X_2 + (X_0 + X_1 + X_2)^3$	18	Incomplete	21	$\mathbf{S}_3$
	5	$X_0X_1X_2 - 7(X_0 + X_1 + X_2)^3$	18	Incomplete	22	$\mathbf{S}_3$
	6	$X_0X_1X_2 + 2(X_0 + X_1 + X_2)^3$	21	Complete	–	$\mathbf{S}_3$
	7	$X_0X_1X_2 + 5(X_0 + X_1 + X_2)^3$	21	Complete	–	$\mathbf{S}_3$
	8	$X_0X_1X_2 + 9(X_0 + X_1 + X_2)^3$	24	Complete	–	$\mathbf{S}_3$
	9	$X_0X_1X_2 - 9(X_0 + X_1 + X_2)^3$	24	Complete	–	$\mathbf{S}_3$
	10	$X_0X_1X_2 - 3(X_0 + X_1 + X_2)^3$	24	Complete	–	$\mathbf{S}_3$
	11	$X_0X_1X_2 - 2(X_0 + X_1 + X_2)^3$	24	Complete	–	$\mathbf{S}_3$
	12	$X_0X_1X_2 - 8(X_0 + X_1 + X_2)^3$	27	Complete	–	$\mathbf{S}_3$
$\bar{\mathcal{E}}_3^1$	13	$X_0X_1(X_0 + X_1) + 2X_2^3$	12	Incomplete	27	$\mathbf{S}_3 \times \mathbf{Z}_3$
	14	$X_0X_1(X_0 + X_1) + 4X_2^3$	21	Complete	–	$\mathbf{S}_3 \times \mathbf{Z}_3$

5.6 Non-Singular Plane Cubics with One Rational Inflexion

Lemma 5.6.1. *Let $\mathcal{F}$ be a non-singular plane cubic curve defined over $\mathbf{F}_q$, $q = p^h$, $p \neq 2, 3$, with at least one inflexion. Then the following holds.*

(i) *$\mathcal{F}$ has the canonical form*

$$F = X_2^2X_1 + X_0^3 + cX_0X_1^2 + dX_1^3,$$

where $4c^3 + 27d^2 \neq 0$.

(ii) *The curve* $\mathcal{F}$ *is general when* $cd \neq 0$, *harmonic when* $c \neq 0$ *and* $d = 0$, *equianharmonic when* $c = 0$ *and* $d \neq 0$, *and singular when* $4c^3 + 27d^2 = 0$.

Proof. See [28, Theorem 11.54]. □

Lemma 5.6.2. *Write* $\mathcal{F}' = \mathbf{v}(F')$, $F' = X_2^2X_1 + X_0^3 + c'X_0X_1^2 + d'X_1^3$. *If* $\mathcal{F}$ *and* $\mathcal{F}'$ *are general, they are projectively equivalent if and only if* $c^3/d^2 = c'^3/d'^2$ *and* d/d' *is a square.*

Proof. See [28, Lemma 11.55]. □

Corollary 5.6.3. *In* $PG(2,19)$, *amongst the* 306 *ordered pairs* (c,d) *satisfying the equation* $4c^3 + 27d^2 \neq 0$, *there are* 34 *different classes:*

(i) *for* $(c,d) = (-9,-8)$, $\mathcal{F}$ *has nine inflexions;*

(ii) *for* $(c,d) = (-9,-9), (-9,-7), (-9,1), (-9,4), (-9,5), (-8,-8), (-8,-7), (-8,-6), (-8,-5), (-8,-2), (-8,-1), (-8,4)$, $\mathcal{F}$ *has exactly three inflexions;*

(iii) *for* $(c,d) = (-9,-6), (-9,-5), (-9,-4), (-9,-3), (-9,-2), (-9,-1), (-9,2), (-9,3), (-9,6), (-9,7), (-9,8), (-9,9), (-8,-9), (-8,-4), (-8,1), (-8,2), (-8,5), (-8,6), (-8,7), (-8,8), (-8,9)$, $\mathcal{F}$ *has exactly one inflexion.*

Using Lemma 5.6.1 and Corollary 5.6.3, in Table 5.3 the full details about non-singular plane cubic curves with exactly one inflexion are given.

Table 5.3: Non-singular plane cubic curves with exactly one rational inflexion

$\mathcal{F}_n^r$	No.	Canonical form	$\lvert\mathcal{F}_n^r\rvert$	Description	$M(\mathcal{F}_n^r)$	G
$\mathcal{G}_1^0$	1	$X_2^2X_1+X_0^3-9X_0X_1^2-3X_1^3$	14	Incomplete	27	$\mathbf{Z}_2$
	2	$X_2^2X_1+X_0^3-8X_0X_1^2+9X_1^3$	14	Incomplete	27	$\mathbf{Z}_2$
	3	$X_2^2X_1+X_0^3-9X_0X_1^2+6X_1^3$	17	Incomplete	23	$\mathbf{Z}_2$
	4	$X_2^2X_1+X_0^3-9X_0X_1^2-2X_1^3$	20	Incomplete	22	$\mathbf{Z}_2$
	5	$X_2^2X_1+X_0^3-9X_0X_1^2+2X_1^3$	20	Incomplete	22	$\mathbf{Z}_2$
	6	$X_2^2X_1+X_0^3-9X_0X_1^2-6X_1^3$	23	Complete	–	$\mathbf{Z}_2$
	7	$X_2^2X_1+X_0^3-9X_0X_1^2+3X_1^3$	26	Complete	–	$\mathbf{Z}_2$
	8	$X_2^2X_1+X_0^3-8X_0X_1^2-9X_1^3$	26	Complete	–	$\mathbf{Z}_2$
$\mathcal{H}_1^0$	9	$X_2^2X_1+X_0^3+X_0X_1^2$	20	Incomplete	21	$\mathbf{Z}_2$
	10	$X_2^2X_1+X_0^3+2X_0X_1^2$	20	Complete	–	$\mathbf{Z}_2$
$\mathcal{G}_1^1$	11	$X_2^2X_1+X_0^3-9X_0X_1^2+7X_1^3$	13	Incomplete	27	$\mathbf{Z}_2$
	12	$X_2^2X_1+X_0^3-9X_0X_1^2-5X_1^3$	16	Incomplete	26	$\mathbf{Z}_2$
	13	$X_2^2X_1+X_0^3-9X_0X_1^2-4X_1^3$	16	Incomplete	25	$\mathbf{Z}_2$
	14	$X_2^2X_1+X_0^3-9X_0X_1^2+9X_1^3$	16	Incomplete	25	$\mathbf{Z}_2$
	15	$X_2^2X_1+X_0^3-8X_0X_1^2+8X_1^3$	16	Incomplete	25	$\mathbf{Z}_2$
	16	$X_2^2X_1+X_0^3-8X_0X_1^2-4X_1^3$	19	Incomplete	22	$\mathbf{Z}_2$
	17	$X_2^2X_1+X_0^3-8X_0X_1^2+X_1^3$	19	Incomplete	21	$\mathbf{Z}_2$

	18	$X_2^2X_1 + X_0^3 - 8X_0X_1^2 + 6X_1^3$	22	Incomplete	23	$\mathbf{Z}_2$
	19	$X_2^2X_1 + X_0^3 - 8X_0X_1^2 + 7X_1^3$	22	Incomplete	23	$\mathbf{Z}_2$
	20	$X_2^2X_1 + X_0^3 - 8X_0X_1^2 + 5X_1^3$	25	Complete	–	$\mathbf{Z}_2$
	21	$X_2^2X_1 + X_0^3 - 9X_0X_1^2 - X_1^3$	25	Complete	–	$\mathbf{Z}_2$
	22	$X_2^2X_1 + X_0^3 - 8X_0X_1^2 + 2X_1^3$	28	Complete	–	$\mathbf{Z}_2$
$\mathcal{E}_1^1$	23	$X_2^2X_1 + X_0^3 + 6X_1^3$	19	Incomplete	21	$\mathbf{Z}_6$
	24	$X_2^2X_1 + X_0^3 - 8X_1^3$	28	Complete	–	$\mathbf{Z}_6$
$\mathcal{G}_1^4$	25	$X_2^2X_1 + X_0^3 - 9X_0X_1^2 + 8X_1^3$	22	Complete	–	$\mathbf{Z}_2$
$\mathcal{E}_1^4$	26	$X_2^2X_1 + X_0^3 - 2X_1^3$	13	Incomplete	28	$\mathbf{Z}_6$

5.7 Non-Singular Plane Cubics with no Rational Inflexions

As mentioned in Section 5.2, a non-singular plane cubic curve $\mathcal{F}$ with zero rational inflexions is an $\mathcal{F}_0^4$ or an $\mathcal{F}_0^1$.

Lemma 5.7.1. (i) *If $q \equiv 1 \pmod 3$, then every $\mathcal{F}_0^4$ has canonical form $\mathcal{F} = \mathbf{v}(F)$, where*

$$F = X_0^3 + \alpha X_1^3 + \alpha^2 X_2^3 - 3cX_0X_1X_2,$$

with α a primitive element of $\mathbf{F}_q$.

(ii) *With $\lambda^3 = 1$, the curve $\mathcal{F}$ in (i) is equianharmonic for $c = 0, -2\alpha\lambda$, harmonic for $c = (1 \pm \sqrt{3})\alpha\lambda$, and an inflexional triangle for $c = \alpha, \lambda$.*

Proof. See [28, Lemmas 11.89, 11.90]. □

When $c \neq 0$ and $\mathcal{F}$ is equianharmonic, write $\mathcal{F} = \mathcal{E}_0^4$; when $c = 0$ and $\mathcal{F}$ is equianharmonic, write $\mathcal{F} = \bar{\mathcal{E}}_0^4$.

Remark 5.7.2. Over $\mathbf{F}_{19}$, the cubic equation $\lambda^3 = 1$ in Lemma 5.7.1(ii) has three solutions $1, 7, -8$.

Corollary 5.7.3. *In $PG(2, 19)$,*

(i) *the curve $\mathcal{F}$ is equianharmonic if $c = 0, -4$;*

(ii) *there is no harmonic type of $\mathcal{F}_0^4$.*

Lemma 5.7.4. *If $q \equiv 1 \pmod 3$, then $\mathcal{F}_0^1$ has canonical form $\mathcal{F} = \mathbf{v}(F)$, where*

$$F = X_0X_1^2 + X_0^2X_2 + eX_1X_2^2 - c(X_0^3 + eX_1^3 + e^2X_2^3 - 3eX_0X_1X_2),$$

with α a primitive element of $\mathbf{F}_q$ and $e = \alpha, \alpha^2$.

Proof. See [28, Lemma 11.91]. □

The curve $\mathcal{F}$ in Lemma 5.7.4 is equianharmonic for $c = 0$.

Corollary 5.7.5. *In $PG(2, 19)$, the curve $\mathcal{F}$ is of type $\mathcal{F}_0^1$ if $e = 2, 4$.*

From the details in the above, Table 5.4 is deduced.

Table 5.4: Non-singular plane cubic curves with zero rational inflexions

$\mathcal{F}_n^r$	No.	Canonical form	$\lvert\mathcal{F}_n^r\rvert$	Description	$M(\mathcal{F}_n^r)$	G
$\mathcal{G}_0^1$	1	$X_0X_1^2+X_0^2X_2+4X_1X_2^2-5(X_0^3+4X_1^3-3X_2^3+7X_0X_1X_2)$	12	Incomplete	27	$\mathbf{Z}_3$
	2	$X_0X_1^2+X_0^2X_2+4X_1X_2^2+9(X_0^3+4X_1^3-3X_2^3+7X_0X_1X_2)$	15	Incomplete	26	$\mathbf{Z}_3$
	3	$X_0X_1^2+X_0^2X_2+2X_1X_2^2-4(X_0^3+2X_1^3+4X_2^3-6X_0X_1X_2)$	15	Incomplete	26	$\mathbf{Z}_3$
	4	$X_0X_1^2+X_0^2X_2+2X_1X_2^2-5(X_0^3+2X_1^3+4X_2^3-6X_0X_1X_2)$	18	Incomplete	21	$\mathbf{Z}_3$
	5	$X_0X_1^2+X_0^2X_2+4X_1X_2^2-2(X_0^3+4X_1^3-3X_2^3+7X_0X_1X_2)$	18	Incomplete	21	$\mathbf{Z}_3$
	6	$X_0X_1^2+X_0^2X_2+2X_1X_2^2-(X_0^3+2X_1^3+4X_2^3-6X_0X_1X_2)$	21	Complete	–	$\mathbf{Z}_3$
	7	$X_0X_1^2+X_0^2X_2+2X_1X_2^2-2(X_0^3+2X_1^3+4X_2^3-6X_0X_1X_2)$	21	Complete	–	$\mathbf{Z}_3$
	8	$X_0X_1^2+X_0^2X_2+2X_1X_2^2-8(X_0^3+2X_1^3+4X_2^3-6X_0X_1X_2)$	24	Complete	–	$\mathbf{Z}_3$
	9	$X_0X_1^2+X_0^2X_2+2X_1X_2^2+9(X_0^3+2X_1^3+4X_2^3-6X_0X_1X_2)$	24	Complete	–	$\mathbf{Z}_3$
	10	$X_0X_1^2+X_0^2X_2+4X_1X_2^2-(X_0^3+4X_1^3-3X_2^3+7X_0X_1X_2)$	24	Complete	–	$\mathbf{Z}_3$
	11	$X_0X_1^2+X_0^2X_2+4X_1X_2^2-8(X_0^3+4X_1^3-3X_2^3+7X_0X_1X_2)$	24	Complete	–	$\mathbf{Z}_3$
	12	$X_0X_1^2+X_0^2X_2+4X_1X_2^2-4(X_0^3+4X_1^3-3X_2^3+7X_0X_1X_2)$	27	Complete	–	$\mathbf{Z}_3$
$\mathcal{E}_0^1$	13	$X_0X_1^2+X_0^2X_2+2X_1X_2^2$	12	Incomplete	27	$\mathbf{Z}_3\times\mathbf{Z}_3$
	14	$X_0X_1^2+X_0^2X_2+4X_1X_2^2$	21	Complete	–	$\mathbf{Z}_3\times\mathbf{Z}_3$
$\mathcal{G}_0^4$	15	$X_0^3+2X_1^3+4X_2^3-3X_0X_1X_2$	18	Complete	–	$\mathbf{Z}_3\times\mathbf{Z}_3$
	16	$X_0^3+2X_1^3+4X_2^3+7X_0X_1X_2$	18	Incomplete	24	$\mathbf{Z}_3\times\mathbf{Z}_3$
	17	$X_0^3+2X_1^3+4X_2^3+4X_0X_1X_2$	18	Complete	–	$\mathbf{Z}_3\times\mathbf{Z}_3$
	18	$X_0^3+2X_1^3+4X_2^3-5X_0X_1X_2$	18	Incomplete	21	$\mathbf{Z}_3\times\mathbf{Z}_3$
$\mathcal{E}_0^4$	19	$X_0^3+2X_1^3+4X_2^3-7X_0X_1X_2$	27	Complete	–	$\mathbf{Z}_3\times\mathbf{Z}_3$
$\bar{\mathcal{E}}_0^4$	20	$X_0^3+2X_1^3+4X_2^3$	27	Complete	–	$\mathbf{Z}_3\times\mathbf{Z}_3\times\mathbf{Z}_3$

5.8 Summary

From Tables 5.1, 5.2, 5.3 and 5.4 the following theorem is established.

Theorem 5.8.1. *In* $PG(2,19)$*, the* 62 *inequivalent non-singular plane cubic curves are divided into* 30 *complete and* 32 *non-complete arcs of degree three.*

Table 5.5 lists the number of each type of stabilizer group of complete and incomplete projectively distinct non-singular cubic curves.

TABLE 5.5: Groups of non-singular plane cubic curves

Complete	G	$\mathbf{Z}_2$	$\mathbf{Z}_3$	$\mathbf{Z}_6$	$\mathbf{S}_3$	$\mathbf{Z}_3\times\mathbf{Z}_3$	$\mathbf{S}_3\times\mathbf{Z}_3$	$\mathbf{Z}_3\times\mathbf{Z}_3\times\mathbf{Z}_3$	G_{54}
	No.	8	7	1	7	4	1	1	1
Incomplete	G	$\mathbf{Z}_2$	$\mathbf{Z}_3$	$\mathbf{Z}_6$	$\mathbf{S}_3$	$\mathbf{Z}_3\times\mathbf{Z}_3$	$\mathbf{S}_3\times\mathbf{Z}_3$	$(\mathbf{Z}_3\times\mathbf{Z}_3)\rtimes\mathbf{Z}_2$	
	No.	15	5	2	5	3	1	1	

In Table 5.6, a cell $n:m$ means that n is the number of points on the curve and m is the number of such distinct curves.

TABLE 5.6: Numbers of distinct non-singular plane cubic curves

9 inflexions	18 : 1	27 : 1									
3 inflexions	12 : 2	15 : 2	18 : 2	21 : 3	24 : 4	27 : 1					
1 inflexion	13 : 2	14 : 2	16 : 4	17 : 1	19 : 3	20 : 4	22 : 3	23 : 1	25 : 2	26 : 2	28 : 2
0 inflexion	12 : 2	15 : 2	18 : 6	21 : 3	24 : 4	27 : 3					

Also, from these tables the following statistics are deduced.

1. In $PG(2,19)$, $n_0 = 20$, $n_1 = 26$, $n_3 = 14$, and $n_9 = 2$. So, $P_{19} = 62$, which agrees with Corollary 5.3.2.

2. $L_{19}(1) = 12$ and $N_{19}(1) = 28$ and the number $\mathtt{N}_1$ takes every value between $L_{19}(1)$ and $N_{19}(1)$. This agrees with Corollary 5.3.4 and Lemma 5.3.5.

3. In $PG(2,19)$, a non-singular plane cubic curve with k points is a complete $(k,3)$-arc when k has the following values:

$$18, 20, 21, 22, 23, 24, 25, 26, 27, 28.$$

5.9 AMDS Codes of Dimension Three

According to Theorem 1.13.1, an $(n; n-d)$-arc in $PG(k-1, q)$ is equivalent to a projective $[n, k, d]_q$-code. Now, if $k = 3, n-d = 3$, and $q = 19$, then there is a one-to-one correspondence between $(n; 3)$-arcs in $PG(2, 19)$ and projective $[n, 3, n-3]_{19}$-codes C. Since $d(C)$ of the code C is equal to $n-k$, thus the projective code C is AMDS.

In Table 5.7, the AMDS codes corresponding to the $(n; 3)$-arcs for $12 \leq n \leq 28$ in $PG(2, 19)$ and the parameter e of errors corrected are given.

TABLE 5.7: AMDS code over $PG(2, 19)$

$(n;3)$-arc	AMDS code	e	$(n;3)-$arc	AMDS code	e
$(12;3)$-arc	$[12,3,9]_{19}$	4	$(21;3)$-arc	$[21,3,18]_{19}$	8
$(13;3)$-arc	$[13,3,10]_{19}$	4	$(22;3)$-arc	$[22,3,19]_{19}$	9
$(14;3)$-arc	$[14,3,11]_{19}$	5	$(23;3)$-arc	$[23,3,20]_{19}$	9
$(15;3)$-arc	$[15,3,12]_{19}$	5	$(24;3)$-arc	$[24,3,21]_{19}$	10
$(16;3)$-arc	$[16,3,13]_{19}$	6	$(25;3)$-arc	$[25,3,22]_{19}$	10
$(17;3)$-arc	$[17,3,14]_{19}$	6	$(26;3)$-arc	$[26,3,23]_{19}$	11
$(18;3)$-arc	$[18,3,15]_{19}$	7	$(27;3)$-arc	$[27,3,24]_{19}$	11
$(19;3)$-arc	$[19,3,16]_{19}$	7	$(28;3)$-arc	$[28,3,25]_{19}$	12
$(20;3)$-arc	$[20,3,17]_{19}$	8			

Appendix A

Points of $PG(2, 19)$

Points of $PG(2, 19)$ generated by (0 1 0   0 0 1   −3 0 1)

1	1	0	0	2	0	1	0	3	0	0	1	4	−3	0	1
5	−3	−3	1	6	−8	−8	1	7	−5	−7	1	8	−9	4	1
9	7	2	1	10	−1	−4	1	11	1	−6	1	12	−7	−4	1
13	1	−4	1	14	1	6	1	15	5	−8	1	16	−5	2	1
17	−1	−8	1	18	−5	−8	1	19	−5	−2	1	20	3	5	1
21	9	−9	1	22	−2	6	1	23	5	−3	1	24	−8	7	1
25	2	−1	1	26	8	1	0	27	0	8	1	28	6	0	1
29	−3	6	1	30	5	5	1	31	9	4	1	32	7	−2	1
33	3	−7	1	34	−9	9	1	35	−6	1	1	36	8	−3	1
37	−8	−4	1	38	1	9	1	39	−6	2	1	40	−1	−2	1
41	3	1	1	42	8	−8	1	43	−5	7	1	44	2	−3	1
45	−8	−1	1	46	−2	1	0	47	0	−2	1	48	3	0	1
49	−3	3	1	50	4	4	1	51	7	−3	1	52	−8	6	1
53	5	7	1	54	2	3	1	55	4	−9	1	56	−2	9	1
57	−6	−4	1	58	1	2	1	59	−1	−6	1	60	−7	4	1
61	7	−9	1	62	−2	−8	1	63	−5	3	1	64	4	−6	1
65	−7	3	1	66	4	3	1	67	4	1	1	68	8	2	1

69	−1	9	1	70	−6	−2	1	71	3	6	1	72	5	−5	1
73	−4	−6	1	74	−7	−3	1	75	−8	−6	1	76	−7	−6	1
77	−7	9	1	78	−6	5	1	79	9	−1	1	80	6	1	0
81	0	6	1	82	5	0	1	83	−3	5	1	84	9	9	1
85	−6	−1	1	86	−9	1	0	87	0	−9	1	88	−2	0	1
89	−3	−2	1	90	3	3	1	91	4	−4	1	92	1	5	1
93	9	−3	1	94	−8	5	1	95	9	5	1	96	9	−8	1
97	−5	−4	1	98	1	8	1	99	6	−2	1	100	3	−6	1
101	−7	7	1	102	2	−8	1	103	−5	−3	1	104	−8	−7	1
105	−9	−5	1	106	−4	7	1	107	2	9	1	108	−6	4	1
109	7	−5	1	110	−4	3	1	111	4	−1	1	112	4	1	0
113	0	4	1	114	7	0	1	115	−3	7	1	116	2	2	1
117	−1	7	1	118	2	7	1	119	2	5	1	120	9	−6	1
121	−7	2	1	122	−1	4	1	123	7	−4	1	124	1	4	1
125	7	4	1	126	7	9	1	127	−6	−5	1	128	−4	−8	1
129	−5	6	1	130	5	2	1	131	−1	8	1	132	6	2	1
133	−1	2	1	134	−1	6	1	135	5	8	1	136	6	9	1
137	−6	−7	1	138	−9	1	1	139	8	5	1	140	9	−5	1
141	−4	−7	1	142	−9	7	1	143	2	6	1	144	5	3	1
145	4	6	1	146	5	6	1	147	5	−2	1	148	3	−5	1
149	−4	4	1	150	7	3	1	151	4	−3	1	152	−8	−2	1
153	3	8	1	154	6	−6	1	155	−7	−5	1	156	−4	−3	1
157	−8	2	1	158	−1	−9	1	159	−2	−7	1	160	−9	−6	1
161	−7	−2	1	162	3	7	1	163	2	−2	1	164	3	−2	1
165	3	−3	1	166	−8	8	1	167	6	−3	1	168	−8	−3	1
169	−8	4	1	170	7	6	1	171	5	1	1	172	8	−7	1
173	−9	5	1	174	9	8	1	175	6	1	1	176	8	3	1
177	4	2	1	178	−1	−5	1	179	−4	5	1	180	9	−7	1

181	−9	8	1	182	6	−1	1	183	9	1	0	184	0	9	1
185	−6	0	1	186	−3	−6	1	187	−7	−7	1	188	−9	−2	1
189	3	9	1	190	−6	6	1	191	5	−9	1	192	−2	−3	1
193	−8	1	1	194	8	−4	1	195	1	−9	1	196	−2	7	1
197	2	−5	1	198	−4	9	1	199	−6	−8	1	200	−5	9	1
201	−6	9	1	202	−6	7	1	203	2	4	1	204	7	8	1
205	6	5	1	206	9	1	1	207	8	−5	1	208	−4	−2	1
209	3	4	1	210	7	−7	1	211	−9	2	1	212	−1	−3	1
213	−8	−9	1	214	−2	1	1	215	8	−1	1	216	2	1	0
217	0	2	1	218	−1	0	1	219	−3	−1	1	220	1	1	0
221	0	1	1	222	8	0	1	223	−3	8	1	224	6	6	1
225	5	9	1	226	−6	−9	1	227	−2	−4	1	228	1	7	1
229	2	−7	1	230	−9	6	1	231	5	−4	1	232	1	−8	1
233	−5	8	1	234	6	−9	1	235	−2	4	1	236	7	−8	1
237	−5	−1	1	238	−7	1	0	239	0	−7	1	240	−9	0	1
241	−3	−9	1	242	−2	−2	1	243	3	2	1	244	−1	1	1
245	8	9	1	246	−6	−3	1	247	−8	3	1	248	4	−2	1
249	3	−4	1	250	1	−1	1	251	−3	1	0	252	0	−3	1
253	−8	0	1	254	−3	−8	1	255	−5	−5	1	256	−4	6	1
257	5	−6	1	258	−7	−1	1	259	−5	1	0	260	0	−5	1
261	−4	0	1	262	−3	−4	1	263	1	1	1	264	8	−9	1
265	−2	−1	1	266	−8	1	0	267	0	−8	1	268	−5	0	1
269	−3	−5	1	270	−4	−4	1	271	1	−5	1	272	−4	−5	1
273	−4	1	1	274	8	−2	1	275	3	−8	1	276	−5	5	1
277	9	−4	1	278	1	−3	1	279	−8	9	1	280	−6	3	1
281	4	8	1	282	6	−8	1	283	−5	−9	1	284	−2	3	1
285	4	9	1	286	−6	8	1	287	6	−7	1	288	−9	−1	1

289	−6	1	0	290	0	−6	1	291	−7	0	1	292	−3	−7	1
293	−9	−9	1	294	−2	−6	1	295	−7	8	1	296	6	−5	1
297	−4	8	1	298	6	8	1	299	6	7	1	300	2	−4	1
301	1	−7	1	302	−9	3	1	303	4	−7	1	304	−9	−7	1
305	−9	−8	1	306	−5	4	1	307	7	−1	1	308	5	1	0
309	0	5	1	310	9	0	1	311	−3	9	1	312	−6	−6	1
313	−7	5	1	314	9	2	1	315	−1	3	1	316	4	−5	1
317	−4	−1	1	318	−4	1	0	319	0	−4	1	320	1	0	1
321	−3	1	1	322	8	8	1	323	6	3	1	324	4	−8	1
325	−5	−6	1	326	−7	1	1	327	8	6	1	328	5	−7	1
329	−9	−4	1	330	1	3	1	331	4	5	1	332	9	7	1
333	2	−6	1	334	−7	−8	1	335	−5	1	1	336	8	7	1
337	2	1	1	338	8	1	1	339	8	4	1	340	7	−6	1
341	−7	−9	1	342	−2	8	1	343	6	4	1	344	7	5	1
345	9	−2	1	346	3	−9	1	347	−2	2	1	348	−1	−7	1
349	−9	−3	1	350	−8	−5	1	351	−4	2	1	352	−1	5	1
353	9	3	1	354	4	7	1	355	2	−9	1	356	−2	−5	1
357	−4	−9	1	358	−2	−9	1	359	−2	5	1	360	9	6	1
361	5	4	1	362	7	1	1	363	8	−6	1	364	−7	6	1
365	5	−1	1	366	7	1	0	367	0	7	1	368	2	0	1
369	−3	2	1	370	−1	−1	1	371	3	1	0	372	0	3	1
373	4	0	1	374	−3	4	1	375	7	7	1	376	2	8	1
377	6	−4	1	378	1	−2	1	379	3	−1	1	380	−1	1	0
381	0	−1	1												

Appendix B

Notation

$\|X\|$	number of elements in the set X
$X \backslash Y$	the set of elements of X not in Y
ϕ	the empty set
$G \cong H$	the group G and H are isomorphic
$\mathcal{K} \cong \mathcal{K}'$	the arc $\mathcal{K}$ and $\mathcal{K}'$ are projectively equivalent
$G \times H$	the direct product of the groups G and H
$N \rtimes H$	a semi-direct product of N and H with N a normal subgroup of $N \rtimes H$
$\langle g_1, \ldots, g_n \rangle$	the group generated by $g_1, \ldots, g_n$
$\theta(n, q)$	$(q^{n+1} - 1)/(q - 1)$
$\binom{n}{r}$	$n(n-1)\cdots(n-r+1)/r!$
$\lfloor x \rfloor$	integer n where $n \leq x < n + 1$
(n, m)	the greatest common divisor of n and m
I_k	the $k \times k$ identity matrix
$A^\top$	transpose matrix of the matrix A
$\mathbf{M}(A)$	projectivity with matrix A
$\mathbf{C}(f)$	companion matrix of polynomial f
$P_i \overset{g}{\longmapsto} P_j$	transform the point P_i to P_j by the projectivity g
$\mathbf{Z}_n$	cyclic group of order n
$\mathbf{S}_n$	symmetric group of degree n

$\mathbf{A}_n$	alternating group of degree n
$\mathbf{D}_n$	dihedral group of order $2n$
$G_{\mathcal{K}}$	group fixing a set $\mathcal{K}$
G_i	group of order i
$\mathbf{F}_q, \mathbf{GF}(q)$	the Galois field of $q = p^h$ elements
$\overline{\mathbf{F}}_q$	algebraic closure of $\mathbf{F}_q$
$\mathbf{F}_q^n$	linear space of n-tuples over $\mathbf{F}_q$
$V(n,q)$	n-dimensional vector space over $\mathbf{F}_q$
$PG(n,q)$	n-dimensional projective space over $\mathbf{F}_q$
$GL(n,q)$	group of non-singular linear transformations of $V(n,q)$
$PGL(n,q)$	group of projectivities of $PG(n-1,q)$
$PGO(3,q)$	group of projectivities fixing the plane conic
$\mathbf{P}(X), \mathbf{P}(x_0,\dots,x_n)$	point of $PG(n,q)$ with vector $X = (x_0,\dots,x_n)$
$\mathbf{U}_i$	$\mathbf{P}(0,\dots,0,1,0,\dots,0)$ with 1 in the $(i+1)$-th place
$\mathbf{U}$	$\mathbf{P}(1,1,\dots,1)$
$\mathbf{u}_i$	hyperplane whose points satisfy the equation $x_i = 0$
$\mathbf{u}$	hyperplane whose points satisfy the equation $\sum_{i=0}^n x_i = 0$
$(n;r)$-arc	set of n points with at most r points on a hyperplane
n-arc	set of n points with at most two points on a line
Υ	standard frame $\{\mathbf{U}_0, \mathbf{U}_1, \dots, \mathbf{U}_n, \mathbf{U}\}$
C_0^n	set of points not on the bisecants of an n-arc
$l(n,q)$	number of points on the sides of an n-stigm
$l^*(n,q)$	$q^2 + q + 1 - l(n,q)$
ℓ_i	the lines of $PG(2,19)$ in Table 4.1
ℓ_P	the tangent line at the point P
P_iP_j	line passing through the points P_i and P_j
$t(P)$	number of unisecants of k-arc $\mathcal{K}$ through P
τ_i	number of i-secants to the arc $\mathcal{K}$
$\sigma_i(Q)$	number of i-secants to the arc $\mathcal{K}$ through $Q \notin \mathcal{K}$

c_i	$\mid \{Q \in PG(2,q) \backslash \mathcal{K} \mid \sigma_2(Q) = i\} \mid$
$\{\mathcal{D}_i; \mathcal{D}_i^c\}$	partition of $PG(1,19)$ into two decads
$\{P_1, P_2; P_3, P_4\}$	cross-ratio of the points P_1, P_2, P_3, P_4
$\{t_1, t_2; t_3, t_4\}$	cross-ratio of the parameters t_1, t_2, t_3, t_4
$CR(X)$	cross-ratio of the tetrad X
$\mathcal{F} = \mathbf{v}(F)$	$\{\mathbf{P}(X) \in PG(2,q) \mid F(X) = 0\}$
$m_P(\mathcal{F})$	multiplicity of P on $\mathcal{F}$
$m_P(\ell, \mathcal{F})$	intersection multiplicity of the line ℓ and $\mathcal{F}$ at P
$\mathtt{N}_1$	number of rational points of non-singular plane cubic curve over $\mathbf{F}_q$
$L_q(1)$	the minimum number of rational points on any non-singular plane cubic over $\mathbf{F}_q$
$N_q(1)$	the maximum number of rational points on any non-singular plane cubic over $\mathbf{F}_q$
$\mathcal{F}_n^r$	non-singular plane cubic curve with n rational inflexions and r rational inflexional triangles
$[n,k,d]_q$-code	code with length n, dimension k and minimum distance d over the field $\mathbf{F}_q$
$C^\perp$	dual code to the code C

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