

Synchronisation of linear continuous multi-agent systems with switching topology and communication delay

Article (Accepted Version)

Xiang, Ji, Li, Yanjun, Wei, Wei and Yang, Taicheng (2014) Synchronisation of linear continuous multi-agent systems with switching topology and communication delay. IET Control Theory and Applications, 8 (14). pp. 1415-1420. ISSN 1751-8644

This version is available from Sussex Research Online: <http://sro.sussex.ac.uk/id/eprint/73625/>

This document is made available in accordance with publisher policies and may differ from the published version or from the version of record. If you wish to cite this item you are advised to consult the publisher's version. Please see the URL above for details on accessing the published version.

Copyright and reuse:

Sussex Research Online is a digital repository of the research output of the University.

Copyright and all moral rights to the version of the paper presented here belong to the individual author(s) and/or other copyright owners. To the extent reasonable and practicable, the material made available in SRO has been checked for eligibility before being made available.

Copies of full text items generally can be reproduced, displayed or performed and given to third parties in any format or medium for personal research or study, educational, or not-for-profit purposes without prior permission or charge, provided that the authors, title and full bibliographic details are credited, a hyperlink and/or URL is given for the original metadata page and the content is not changed in any way.

Synchronization of linear continuous multi-agent systems with switching topology and communication delay

Ji Xiang, Yanjun Li, Wei Wei, and Taicheng Yang

Abstract

A distributed dynamic output feedback control is designed by Scardovi and Sepulchre for the synchronization of a network of identical linear systems, known as agents in literature. The design is based on some mild conditions allowing switching topology. But it assumes that there is no time delay in signal transfer between the neighbouring agents. In this paper we extend their work to include known time delay in communications. Furthermore, our design has some special features: (a) the delay can be arbitrary and only need to be uniformly bounded by a constant, (b) the conditions that time delay should be the same and sufficiently small in some literature are not required here, and (c) no local buffer is required to store past data due to time-delay effect.

Index Terms

Synchronization, time-delay, switching topology, linear system

This research was supported by a grant from the National 863 Program of China (2011AA050204), the National Natural Science Foundation of China (61104149, 61374174), the Zhejiang Province Natural Science Fund (LY13F030001), and the Program for New Century Excellent Talents in University (NCET-11-0459).

J. Xiang and W. Wei are with the Department of System Science and Engineering, College of Electrical Engineering, Zhejiang University, P. R. China.

Y. Li is with the School of Information and Electrical Engineering, Zhejiang University City College, Hangzhou 310027, P. R. China.

T. Yang is with the Department of Engineering and Design University of Sussex, Brighton BN1 9QT, England.

I. INTRODUCTION

Consensus and cooperation in networked multi-agent systems has attracted a great deal of research interests. In recent years, more research has focused on the synchronization of high-order multi agent systems. Whereas consensus is seeking common “positions” or “formations”, synchronization is much more than that – it is seeking common time-varying “velocity vectors” before some formations can be achieved. An early work of Ren and Atkins [1] studied synchronization of a network of double-integrator agents with static state feedback. It was shown that a spanning tree is only a necessary rather than a sufficient condition for the synchronization of a multi-agent system, even when the communication graph is fixed. Further, Yu, Chen and Cao [2] presented an eigenvalue-dependent condition. Wang and Wu [3] studied synchronization of second-order nonlinear multi-agent systems with fixed and switching topology. Seo, Shim and Back [4] proposed a dynamical output feedback controller for the synchronization of high-order systems with fixed communication topology. Li, et al., proposed an observer-like dynamical output feedback controller in [5] for fixed topology. One of major differences between [4] and [5] is that an exchange of agents’ controller states is required in [5] but not in [4].

It is worth pointing out that although there are considerable researches on consensus problems with time delays in communications - to name a few see [6]–[9], on the synchronization of high order linear agents, it appears that little work has been reported when communication delay is considered. Recently, a dynamic output feedback control is proposed for the asymptotical synchronization of a network of identical n -th order linear state-space models [10]. The design is based on some mild assumptions on the dynamics of the model, and the time-varying topology of the communication graph. It also assumes that there is no time delay in signal transfer over the communication graph. In this note, we extend the work of [10] to include known time delay in communications. The delay is arbitrary and uniformly bounded by a constant. However, the conditions that time delay should be the same and sufficiently small in some literature [2], [11]–[13] are not required here. More recently, Yang et al. [14] proposed a dynamic controller for output synchronization of discrete-time agents with arbitrary bounded communication delays. But the communication delays are assumed to be uniform and constant, and moreover agents are restricted to be introspective and right-invertible. Under similar mild assumptions as those stated in [10], a new dynamic controller is proposed in this paper for the synchronization when there is

communication delay. In the controller designed in this paper no local buffer is required to store past data due to time-delay effect and therefore it is easy to implement. The proposed design is applicable to a class of applications where GPS-timing signals are available to all agents and a time-stamp is included in a data packet.

This paper is organised as follows. Some preliminaries are given in Section II. Our main result, Theorem 1, for continuous-time systems is presented in Section III. This is extended to discrete-time systems in Section IV. A simulation example and a brief conclusion are presented in Sections V and VI respectively.

II. PRELIMINARIES

A. Problem formulation

Consider a multi-agent system,

$$\begin{cases} \dot{x}_i = Ax_i + Bu_i, \\ y_i = Cx_i \end{cases}, \quad i = 1, 2, \dots, N, \quad (\text{II.1})$$

where $x_i \in \mathcal{R}^n$ is the state vector, $u_i \in \mathcal{R}^m$ the input vector, $y_i \in \mathcal{R}^q$ the measurement output vector, A, B, C are constant matrices with compatible dimensions. Agents communicate between them through a network. Specifically, the information transmitted by agent j is denoted by ψ_j , whose details will be explained later. In general ψ_j is a vector of system and controller states of agent j .

The information exchanges among agents are described by a switching graph $\mathcal{G}_{\sigma(t)} : \mathcal{R} \mapsto \Xi$, where $\Xi = \{\mathcal{G}_p\}$ with $p \in \mathcal{P} = \{1, 2, \dots, P\}$ denotes a finite set consisting of P directed graphs. Each graph is denoted by $\mathcal{G}_p = (\mathcal{N}, \mathcal{E}_p)$ where $\mathcal{N} = \{1, 2, \dots, N\}$ is a nonempty finite set of nodes and $\mathcal{E}_p = \{\mathcal{N} \times \mathcal{N}\}$ is a set of edges satisfying that an edge $(i, j) \in \mathcal{E}_p$ if and only if there is an information channel from node j to node i . The switching between the graphs is described by a function $\sigma : \mathcal{R}^+ \mapsto \mathcal{P}$. This is a piecewise constant function from the right. The time instance when σ switches is denoted by an increasing sequence t_k , $k = 0, 1, 2, \dots$, with $t_0 = 0$. We assume that any two consecutive switching instants are separated by a dwell-time D_t , i.e., $t_k - t_{k-1} \geq D_t$. This guarantees that the switching graph is non-chattering and zero behavior cannot occur. A union graph $\mathcal{G}_{[t_1, t_2)}$ over an interval time $[t_1, t_2)$ is defined by $\mathcal{G}_{[t_1, t_2)} \triangleq (\mathcal{N}, \bigcup_{t \in [t_1, t_2)} \mathcal{E}_{\sigma(t)})$ corresponding to a graph consisting of all nodes in \mathcal{N} and all edges at any time $t \in [t_1, t_2)$.

For each \mathcal{G}_p with $p \in \mathcal{P}$, the adjacency matrix is denoted by $\mathcal{A}_p = [a_{ij}^p]_{N \times N}$, which is defined by $a_{ij}^p > 0$ if $(j, i) \in \mathcal{E}_p$; otherwise $a_{ij}^p = 0$. $a_{ii} = 0$ due to the absence of self-loop. A directed path of digraph is a sequence of edges $(i_1, i_2), (i_2, i_3), \dots$, where $i_j \in \mathcal{N}$ for all $j = 1, 2, \dots$. A tree $\mathcal{G}_t = (\mathcal{N}_t, \mathcal{E}_t)$ is a graph where every node has exactly one parent node except for one node: root node, which has no parent node but has a directed path to every other node. The graph $\mathcal{G}_s = (\mathcal{N}_s, \mathcal{E}_s)$ is a subgraph of \mathcal{G} if $\mathcal{N}_s \subseteq \mathcal{N}$ and $\mathcal{E}_s = \mathcal{E} \cap (\mathcal{N}_s \times \mathcal{N}_s)$. The tree \mathcal{G}_t is a spanning tree of graph \mathcal{G} iff \mathcal{G}_t is a subgraph of \mathcal{G} with $|\mathcal{N}_t| = |\mathcal{N}|$, where $|S|$ denotes the size of set S .

Denote by $\tau_{ij}(t)$ the communication delay from agent j to i at time t . All delays among agents are denoted by $\Omega(t) = [\tau_{ij}(t)]_{N \times N}$. Then the information of agent j received by node i at time t is $\psi_j(t - \tau_{ij}(t))$. Without loss of generality, we assume that $\psi_i(t - \tau) = \psi_i(0)$ if $t \in [0, \tau]$, for some positive scalar τ and for all i .

We recall a specific definition for a switching graph $\mathcal{G}_{\sigma(t)}$ ¹.

Definition 1 (Uniformly Quasi-Strongly Connected [7]): A switching graph $\mathcal{G}_{\sigma(t)}$ is *uniformly quasi-strongly connected* if there is a $T > 0$ such that for all $t \geq 0$, the union graph $\mathcal{G}_{[t, t+T]}$ is quasi-strongly connected, i.e., $\mathcal{G}_{[t, t+T]}$ contains a spanning tree.

Define the maximal real part of all eigenvalues of A by

$$\alpha(A) = \max_i \operatorname{Re}(\lambda_i(A)), \quad i = 1, 2, \dots, n,$$

where, and henceforth, $\lambda(A)$ denotes the eigenvalue of A . The value $\alpha(A)$ denotes the divergence rate of an isolated agent.

Following assumptions are made in this paper:

- A1)** The triplet (A, B, C) is stabilizable and detectable,
- A2)** The switching graph $\mathcal{G}_{\sigma(t)}$ associated with (II.1) is uniformly quasi-strongly connected,
- A3)** The time delay $\tau_{ij}(t)$ is arbitrary but uniformly bounded by a constant τ , i.e., $\tau_{ij}(t) \leq \tau$ for all $i, j \in \mathcal{N}$, and
- A4)** $\alpha(A) \geq 0$.

The synchronization problem studied in this paper is defined as follows:

¹Here for the consistence with the right-continuous switching signal the time interval is altered correspondingly.

Definition 2 (Synchronization with known time delay): Given a multi-agent system (II.1) with the above four assumptions, find a distributed control law u_i based on the output $y_i(t)$ and the delayed information of neighboring agents $\psi_j(t - \tau_{ij})$, where j satisfying $(i, j) \in \mathcal{E}_{\sigma(t)}$ and τ_{ij} is known for node i , such that the solutions of (II.1) asymptotically synchronize to a nontrivial solution of an isolated agent, that is, there is a $\xi_0 \in \mathcal{R}^n$ such that $x_i(t) \rightarrow e^{At}\xi_0$, as $t \rightarrow \infty$.

Assumption A2) suggests that a bounded time interval exists in such a way that during the time interval starting any time there must be a node, probably different for different starting time, whose information will be transmitted node by node to all remaining nodes. This assumption is a very weak condition for the consensus seeking of a switching digraph; it maybe hold even none of \mathcal{G}_p with $p \in \mathcal{P}$ has a spanning tree embedded.

Assumption A4) is to exclude a trivial case that all modes of the isolated agent are decaying. Thus the synchronization can be achieved without interactions between agents. Similarly, "non-trivial solution" means that the steady state will demonstrate the dynamics of the isolated agent rather than be silence in the origin unless the initial conditions and communication graph happen to meet $\xi_0 = 0$.

B. Consensus with communication delays

While the focus of this paper is on synchronization, the result of convergence rate of consensus protocols with time delay is first reviewed in this subsection. This naturally leads to Lemma 1 that will be used in the proof of our main result in Section III.

Consider a consensus protocol for a first-order integrator multi-agent system with nonuniform time varying delays:

$$\dot{w}_i(t) = \sum_j a_{ij}^{\sigma(t)} (w_j(t - \tau_{ij}(t)) - w_i(t)). \quad (\text{II.2})$$

where $w_i \in \mathcal{R}$ is the state, $a_{ij}^{\sigma(t)}$ is the element of adjacency matrix $\mathcal{A}_{\sigma(t)}$ associated with $\mathcal{G}_{\sigma(t)}$. The element $\tau_{ij}(t)$ of matrix $\Omega(t)$ denotes the transfer delay of signal between every pair of nodes. In [7], it has been shown that system (II.2) can achieve consensus if the communication graph satisfies Assumption A2) and the delay matrix $\Omega(t)$ satisfies Assumption A3). The consensus convergence rate is influenced by $\Omega(t)$. In this consideration, we denote by $\mu(\mathcal{G}_{\sigma(t)}, \Omega(t))$ the consensus convergence rate associated with system (II.2). Similar to that in [15], $\mu(\mathcal{G}_{\sigma(t)}, \Omega(t))$ is

defined as

$$\mu(\mathcal{G}_{\sigma(t)}, \Omega(t)) = \sup_{w(0) \neq \mathbf{1}\bar{w}} \lim_{t \rightarrow \infty} \ln \left(\frac{\|w(t) - \mathbf{1}\bar{w}\|}{\|w(0) - \mathbf{1}\bar{w}\|} \right)^{1/t}, \quad (\text{II.3})$$

where $w \in \mathcal{R}^n$ is the concatenated vector of w_i s and $\bar{w} \in \mathcal{R}$ denotes the converged value. From the above, one can easily obtain:

Lemma 1: Given a system (II.2) with a switching digraph $\mathcal{G}_{\sigma(t)}$ subject to communication delay $\Omega(t)$ satisfying Assumptions A2) and A3), then $\mu(\mathcal{G}_{\sigma(t)}, \Omega(t)) < 0$ holds.

Proof: From (II.3), $\|w(t) - \mathbf{1}\bar{w}\| \rightarrow \|w(0) - \mathbf{1}\bar{w}\|e^{\mu t}$ as $t \rightarrow \infty$. If $\mu(\mathcal{G}_{\sigma(t)}, \Omega(t)) \geq 0$, then $w(t) - \mathbf{1}\bar{w}$ will not converge to zero. This is contractive to the fact that system (II.2) will achieve consensus with Assumption A2) and A3). ■

Remark 1: In the simplest case when the digraph is fixed and there is no time delay, then μ is just the algebraic connectivity of digraph that can be calculated directly [16]. In the general case when both switching graph and communication delay exist, it is almost impossible to find the exact value of μ . For an estimation of μ in such a general case, the interested readers can refer to [9] [15] (continuous-time systems) and [8] (discrete-time systems). As shown below, μ is a qualitative measure of the effect of information flow, which is characterised by both network topology and communication delay. However, if $\alpha(A) = 0$, then the value of μ is irrelevant; only $\mu < 0$ is required.

III. SYNCHRONIZATION CONTROLLER

Theorem 1: Given a multi-agent system (II.1) satisfying Assumptions A1)~A4). The following distributed dynamic controller

$$\begin{cases} \dot{\eta}_i(t) = (A + BK)\eta_i(t) - \sum_j a_{ij}^{\sigma(t)} (e^{A\tau_{ij}(t)}\psi_j(t - \tau_{ij}(t)) - \psi_i(t)) \\ \quad + H(C\hat{x}_i(t) - y_i(t)) \\ \dot{\hat{x}}_i(t) = A\hat{x}_i(t) + Bu_i(t) + H(C\hat{x}_i(t) - y_i(t)) \\ u_i(t) = K\eta_i(t) \end{cases}, \quad i \in \mathcal{N}, \quad (\text{III.1})$$

will exponentially synchronize all the solutions x_i s to an isolated agent system $\dot{\xi} = A\xi$, if the consensus convergence rate of diagraph $\mathcal{G}_{\sigma(t)}$ subject to communication delay $\Omega(t)$ satisfies

$$\alpha(A) + \mu(\mathcal{G}_{\sigma(t)}, \Omega(t)) < 0, \quad (\text{III.2})$$

where $\hat{x}_i \in \mathcal{R}^n$ is the local observer state, η_i is the controller state, $\psi_j = \hat{x}_j - \eta_j$ is the information transmitted by node j , and K and H are such that both $A + BK$ and $A + HC$ are Hurwitz.

Proof: The dynamics of ψ_i has the form

$$\dot{\psi}_i = A\psi_i + \sum_j a_{ij}^{\sigma(t)} (e^{A\tau_{ij}(t)}\psi_j(t - \tau_{ij}(t)) - \psi_i(t)), \quad i \in \mathcal{V}.$$

Introduce a time-varying coordinates transformation

$$z_i(t) = e^{-At}\psi_i(t), \quad i \in \mathcal{N}. \quad (\text{III.3})$$

It follows that $z_i(t - \tau_{ij}(t)) = e^{-A(t-\tau_{ij}(t))}\psi_i(t - \tau_{ij}(t))$. Taking time derivative of both side of (III.3) yields

$$\begin{aligned} \dot{z}_i(t) &= -e^{-At}A\psi_i(t) + e^{-At}\dot{\psi}_i(t) = e^{-At} \sum_j a_{ij}^{\sigma(t)} (e^{A\tau_{ij}(t)}\psi_j(t - \tau_{ij}(t)) - \psi_i(t)) \\ &= \sum_j a_{ij}^{\sigma(t)} (e^{-A(t-\tau_{ij}(t))}\psi_j(t - \tau_{ij}(t)) - e^{-At}\psi_i(t)) \\ &= \sum_j a_{ij}^{\sigma(t)} (z_j(t - \tau_{ij}(t)) - z_i(t)). \end{aligned} \quad (\text{III.4})$$

By Lemma 1, the solutions $z_i(t)$, $i \in \mathcal{N}$ will converge to a common value $z_0 \in \mathcal{R}^n$ when $t \rightarrow \infty$ with a convergence rate μ . According to the definition of μ , it follows that for any given positive scalar ε_1 , there is a constant scalar k_1 dependent on ε_1 such that

$$\|z_i(t) - z_0\| \leq k_1(\varepsilon_1)e^{(\mu+\varepsilon_1)t}\|z_i(0) - z_0\|, \quad \forall t > 0, i \in \mathcal{N}.$$

In the original coordinates, it becomes

$$\|\psi_i(t) - e^{At}z_0\| \leq \|e^{At}\| \|z_i(t) - z_0\| \leq k_1(\varepsilon_1)\|e^{At}\|e^{(\mu+\varepsilon_1)t}\|z_i(0) - z_0\|.$$

Since $\|e^{At}\| \leq k_2(\varepsilon_2)e^{(\alpha+\varepsilon_2)t}$ for any arbitrary positive scalar ε_2 with k_2 being some constant scalar dependent on ε_2 , there exists a constant scalar k_3 such that

$$\|\psi_i(t) - e^{At}z_0\| \leq k_3e^{(\alpha+\mu+\varepsilon_1+\varepsilon_2)t}, \quad \forall t > 0, i \in \mathcal{N}.$$

Given inequality (III.2), there are three sufficiently small scalar ε_i , ε_2 and ε_3 satisfying

$$\alpha + \mu + \varepsilon_1 + \varepsilon_2 < -\varepsilon_3,$$

and therefore, for all $t > 0$,

$$\|\psi_i(t) - e^{At}z_0\| \leq k_3 e^{-\varepsilon_3 t}, \quad i \in \mathcal{N}. \quad (\text{III.5})$$

Now, recall the dynamics of η_i ,

$$\dot{\eta}_i(t) = (A + BK)\eta_i(t) - \sum_j a_{ij}^{\sigma(t)} (e^{A\tau_{ij}(t)}\psi_j(t - \tau_{ij}(t)) - \psi_i(t)) + H(C\hat{x}_i(t) - y_i(t)). \quad (\text{III.6})$$

From (III.5), when $t \rightarrow \infty$, $\psi_j(t - \tau_{ij}(t)) \rightarrow e^{A(t - \tau_{ij}(t))}z_0$ for all i , and subsequently $e^{A\tau_{ij}(t)}\psi_j(t - \tau_{ij}(t)) - \psi_i(t) \rightarrow 0$ exponentially. On the other hand, consider the observer error $e_i = \hat{x}_i - x_i$. Its dynamics is

$$\dot{e}_i = (A + HC)e_i.$$

Since $A + HC$ is Hurwitz, it follows that $e_i \rightarrow 0$ exponentially for all $i \in \mathcal{V}$. Therefore $C\hat{x}_i(t) - y_i(t)$ exponentially converge to zero.

The exponential convergence to zero of the last two terms in (III.6), together with that $A + BK$ is Hurwitz, implies that $\eta_i \rightarrow 0$ exponentially. Combining $\hat{x}_i - \eta_i \rightarrow e^{At}z_0$, $x_i - \hat{x}_i \rightarrow 0$ and $\eta_i \rightarrow 0$ yields $x_i \rightarrow e^{At}z_0$, which means that all the solutions x_i s under controller (III.1) will synchronize on a trajectory of $\dot{\xi} = A\xi$ with $\xi(0) = z_0$. ■

Four remarks are in order.

Firstly, it is always difficult to handle heterogenous delays. One of our contributions is a novel approach to deal with them. We convert the original synchronization problem, via a time-delay dependent gain, to the consensus problem and the consensus state is time-independent.

Secondly, although the technique used is similar to that in [10], the proposed controller (III.1) has a different structure from the controller in [10], which has the form of

$$\begin{aligned} \dot{\eta}_i &= (A + BK)\eta_i - \sum_j a_{ij}^{\sigma(t)} (\hat{x}_j - \eta_j - (\hat{x}_i - \eta_i)) \\ \dot{\hat{x}}_i &= A\hat{x}_i + Bu_i + H(C\hat{x}_i - y_i) \\ u_i &= K\eta_i \end{aligned} \quad (\text{III.7})$$

The first equation of the controller (III.1) in this paper, reduces to

$$\dot{\eta}_i = (A + BK)\eta_i - \sum_j a_{ij}^{\sigma(t)} (\hat{x}_j - \eta_j - (\hat{x}_i - \eta_i)) + H(C\hat{x}_i - y_i),$$

when $\tau_{ij} = 0$. Compared with (III.7), it has an additional term $H(C\hat{x}_i - y_i)$, which decouples observer error from other signals. This is particularly important when both switching graphs and communication delays exist.

Thirdly, to deal with communication delay, in some literature [11] [17], *delay matching* controllers are proposed. For example, in [11] the following system

$$\dot{\psi}_i = A\psi_i + \sum_{j=1}^N a_{ij}(t)(\psi_j(t - \tau_{ij}) - \psi_i(t - \tau_{ij})),$$

has been studied with an assumption that all τ_{ij} s are same and sufficiently small. To achieve the match, all τ_{ij} s are also required to be known and extra data storage is required. Compare with the methods in [11], [17] and [2], the approach used in this paper has some advantages.

Finally, the feedback gain and observer gain matrices K and H are required to be the same for all nodes. These matrices can be predefined and send to the all nodes in the initialisation of the distributed controls [10] [5].

IV. EXTENSION TO DISCRETE-TIME MULTI-AGENT SYSTEM

Consider a discrete-time system

$$x_i(k+1) = Ax_i(k) + Bu_i(k), \quad y_i = Cx_i(k), \quad i \in \mathcal{N}. \quad (\text{IV.1})$$

Define the maximal eigenvalue modulus by

$$m(A) = \max_i |\lambda_i(A)|, \quad i = 1, 2, \dots, n.$$

To exclude the trivial case, Assumption A4) is replaced by,

A5) $m(A) \geq 1$.

Similarly, a first-order integrator discrete-time system over digraph $\mathcal{G}_{\sigma(k)}$ subject to communication delay $\Omega(k)$ is:

$$w_i(k+1) = w_i(k) + \epsilon_i(k) \sum_j a_{ij}(k)(w_j(k - \tau_{ij}(k)) - w_i(k)), \quad i \in \mathcal{N}, \quad (\text{IV.2})$$

where $w_i(k) \in \mathcal{R}$, and $\epsilon_i(k) = \frac{1}{d_i(k)+1}$, where $d_i(k) = \sum_j a_{ij}^{\sigma(k)}$. The consensus convergence rate of (IV.2) is defined as follows [18],

$$\mu(\mathcal{G}_{\sigma(k)}, \Omega(k)) = \sup_{w(0) \neq \mathbf{1}\bar{w}} \lim_{t \rightarrow \infty} \left(\frac{\|w(t) - \mathbf{1}\bar{w}\|}{\|w(0) - \mathbf{1}\bar{w}\|} \right)^{1/t}.$$

An estimation of $\mu(\mathcal{G}_{\sigma(k),\Omega(k)})$ is given in [8], where Assumptions A2) and A3) lead to $\mu(\mathcal{G}_{\sigma(k),\Omega(k)}) < 1$.

As a counterpart of Theorem 1:

Theorem 2: For a discrete-time multi-agent system (IV.1) with assumptions A1), A2), A3) and A5). the following distributed dynamic controller

$$\begin{cases} \eta_i(k+1) = A\eta_i(k) + Bu(k) - \epsilon_i(k)A \sum_j a_{ij}(k) (A^{\tau_{ij}(k)}\psi_j(k - \tau_{ij}(k)) - \psi_i(k)) \\ \quad + H(C\hat{x}_i(k) - y_i(k)) \\ \hat{x}_i(k+1) = A\hat{x}_i(k) + Bu_i(k) + H(C\hat{x}_i(k) - y_i(k)) \\ u_i(k) = K\eta_i(k) \end{cases}, \quad i \in \mathcal{N}, \quad (\text{IV.3})$$

will exponentially synchronize all the solutions x_i s to an isolated agent system $\xi(k+1) = A\xi(k)$, if (a) both $A + BK$ and $A + HC$ are Schur stable, and (b) the consensus convergence rate of diagraph $\mathcal{G}_{\sigma(k)}$ subject to communication delays $\Omega(k)$ satisfies

$$\mu(\mathcal{G}_{\sigma(k),\Omega(k)})m(A) < 1, \quad (\text{IV.4})$$

where $\hat{x}_i \in \mathcal{R}^n$ is the observer state, $\eta_i \in \mathcal{R}^n$ is the controller state, and $\psi_i(k) = \hat{x}_i(k) - \eta_i(k)$ is the information sent by agent i .

Proof: A brief proof is given here, following the line of the proof of Theorem 1. The transmitted signal $\psi_i(k)$ has the following dynamics

$$\psi_i(k+1) = A\psi_i(k) + \epsilon_i(k)A \sum_j a_{ij}(k) (A^{\tau_{ij}(k)}\psi_j(k - \tau_{ij}(k)) - \psi_i(k)).$$

Introduce a time-varying coordinates transformation $z_i(k) = A^{-k}\psi_i(k)$, since $z_i(k - \tau_{ij}(k)) = A^{-(k-\tau_{ij}(k))}\psi_i(k - \tau_{ij}(k))$, then

$$z_i(k+1) = z_i(k) + \epsilon_i(k)A \sum_j a_{ij}(k) (z_j(k - \tau_{ij}(k)) - z_i(k)).$$

Under Assumptions A2) and A3), it will achieve consensus, that is, $z_i(k) \rightarrow z_0$ for some vector z_0 . Given (IV.4), there are sufficient small ϵ_1, ϵ_2 such that there is a positive scalar ϵ_3 satisfying

$$(m + \epsilon_1)(\mu + \epsilon_2) < \epsilon_3 < 1,$$

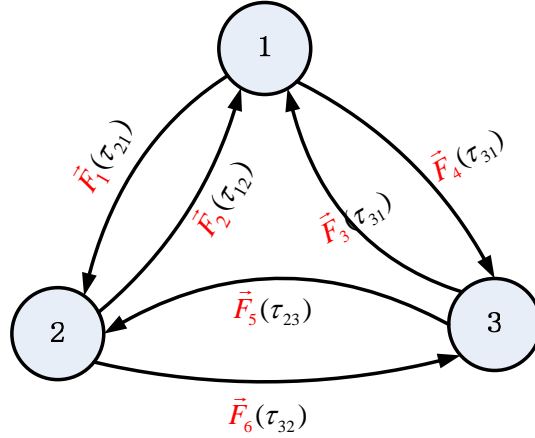


Fig. 1. Communication digraph

and subsequently $\|\psi_i(k) - A^k z_0\| \leq k_3(\varepsilon_3)^k$ for some positive scalar k_3 . On the other hand, since $A + HC$ is Schur stable, $e_i = \hat{x}_i - x_i \rightarrow 0$. Then since $A + BK$ is also Schur stable, $\eta_i \rightarrow 0$. Therefore $x_i \rightarrow A^k z_0$. This completes the proof. \blacksquare

Remark 2: Synchronization of a general multi-agent system depends on both the consensus convergence rate of the underlying communication digraph and the divergence rate of the dynamics of the isolated agent. If the former dominates the latter, then synchronization can be achieved. If the isolated agent has a critical divergence rate, namely $\alpha(A) = 0$ (or $m(A) = 1$), then the inequality condition (III.2) (or (IV.4)) is implied by Assumptions A2) and A3) (see Lemma 1) and therefore, is no longer required.

V. SIMULATION EXAMPLES

For simplicity, consider a discrete-time multi-agent system with $N = 3$:

$$x_i(k+1) = Ax_i(k) + Bu_i(k), \quad y(k) = Cx_i(k), \quad i = 1, 2, 3,$$

where

$$A = \begin{bmatrix} 0.995 & 0.1 \\ -0.1 & 0.995 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

satisfy Assumption A1). The divergence rate of isolated agent is $m(A) = 1$.

The information graph $\mathcal{G} = \{\mathcal{N}, \mathcal{E}\}$ is shown in Fig. 1, where all six possible communication channels are denoted by \vec{F}_i , $i = 1, \dots, 6$. Nonuniform delay times τ_{ij} associated with these

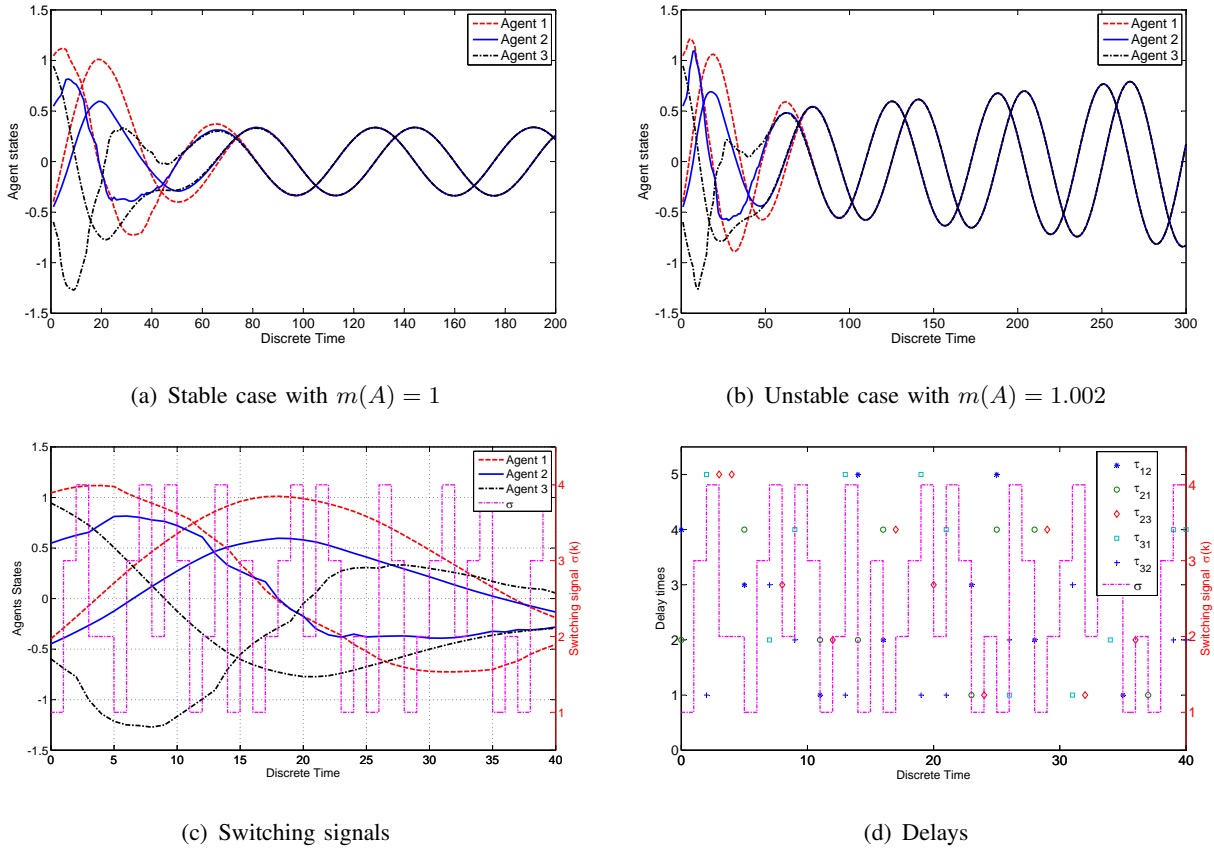


Fig. 2. Simulation results under controller (IV.3). (a) Trajectories of three agents with a stable system matrix; (b)trajectories of three agents with a slightly unstable system matrix; (c) enlarged plots of stable case with switching signals in the transition phase of $[0, 40]$; (d) communication delays between the nodes in the transition phase of $[0, 40]$.

channels belong to a finite set $\Delta = \{1, 2, \dots, 5\}$. The digraph collection Ξ consists of four simple digraphs,

$$\mathcal{G}_1 = \{\mathcal{N}, \{\vec{F}_1, \vec{F}_2\}\}, \mathcal{G}_2 = \{\mathcal{N}, \{\vec{F}_5\}\}, \mathcal{G}_3 = \{\mathcal{N}, \{\}\} \mathcal{G}_4 = \{\mathcal{N}, \{\vec{F}_4, \vec{F}_6\}\}.$$

Each of them does not have a spanning tree, and \mathcal{G}_3 is even completely isolated. Switching signal $\sigma(k)$ is randomly selected at time k but transverses the set $\{1, 2, 3, 4\}$. For each active communication channel, the delay time τ_{ij} is randomly selected from Δ . In this example, when $T > 4$, the switching digraph $\mathcal{G}_{\sigma(k)}$ satisfies assumptions A2) and A3).

The output feedback synchronization controller (IV.3) is designed with $a_{ij}(k) = 1$ if $(j, i) \in \mathcal{E}_{\sigma(k)}$, $K = [-0.5, -0.5]$, and $H = [-0.6, -0.4]^T$. Simulation results are given in Fig. 2. The non-smooth trajectories in Fig. 2(c) is due to the topology change. It can be seen that after the

transition phase all the agents have their states synchronize to an isolated agent, despite that the communication graph is switching and the communication delays are non-uniform and time-varying, as shown in Fig. 2(d).

According to Proposition 1 in [8], the consensus convergence rate $\mu(\mathcal{G}_{\sigma(k)}, \Omega(k))$ is less than $(1 - 0.5^{20})^{1/20} \triangleq \bar{\mu}$. Therefore in theory for any Schur unstable matrix A satisfying $m(A) < 1/\bar{\mu}$, the synchronization will be achieved. However such an estimation is too conservative. For example, Fig. 2(b) shows a synchronization process when A becomes $A + 0.002I_2$ for which $m(A) \approx 1.002$.

VI. CONCLUSION

The synchronization controllers with the gain matrix depending on communication delays are proposed for the linear multi-agent systems having switching topology and communication delays. It is shown that the synchronization can be achieved if the divergence rate of the isolated agent is less than the consensus convergence rate of the graph with communication delays. When the agent has no unstable modes, the uniform quadratic-strongly connected condition is sufficient for the synchronization in the presence of communication delays that are uniformly bounded but can be time-varying and non-uniform.

REFERENCES

- [1] Ren W., Atkins E.: ‘Distributed multi-vehicle coordinated control via local information exchange’, *International Journal of Robust and Nonlinear Control*, 2007, 17, (10-11), pp. 1002-1033
- [2] Yu W., Chen G., Cao M.: ‘Some necessary and sufficient conditions for second-order consensus in multi-agent dynamical systems’, *Automatica*, 2010, 46, (6), pp. 1089-1095
- [3] Wang J. H., Wu H. N.: ‘Leader-following formation control of multi-agent systems under fixed and switching topologies’, *International Journal of Control*, 2012, 85, (6), pp. 695-705
- [4] Seo J. H., Shim H., Back J.: ‘Consensus of high-order linear systems using dynamic output feedback compensator: Low gain approach’, *Automatica*, 2009, 45, (11), pp. 2659-2664
- [5] Li Z., Duan Z., Chen G., *et. al.*: ‘Consensus of Multiagent Systems and Synchronization of Complex Networks: A Unified Viewpoint’, *IEEE Transactions on Circuits and Systems I: Regular Papers*, 2010, 57, (1), pp. 213-224
- [6] Lin P., Jia Y.: ‘Consensus of a class of second-order multi-agent systems with time-delay and jointly-connected topologies’, *IEEE Transactions on Automatic Control*, 2010, 55, (3), pp. 778-784
- [7] Münz U., Papachristodoulou A., Allgöwer F.: ‘Consensus in multi-agent systems with coupling delays and switching topology’, *IEEE Transactions on Automatic Control*, 2011, 56, (12), pp. 2976-2982
- [8] Nedić A., Ozdaglar A.: ‘Convergence rate for consensus with delays’, *Journal of Global Optimization*, 2010, 47, (3), pp.437-456

- [9] Münz U., Papachristodoulou A., Allgöwer F.: ‘Delay robustness in consensus problems’, *Automatica*, 2010, 46, (8), pp. 1252-1265
- [10] Scardovi L., Sepulchre R.: ‘Synchronization in networks of identical linear systems’, *Automatica*, 2009, 45, (11), pp. 2557-2562
- [11] Hu J., Hong Y.: ‘Leader-following coordination of multi-agent systems with coupling time delays’, *Physica A: Statistical Mechanics and its Applications*, 2007, 374, (2), pp. 853-863
- [12] Wang J. L., Yang Z. C., Huang T., *et. al.*: ‘Local and global exponential synchronization of complex delayed dynamical networks with general topology’, *Discrete and Continuous Dynamical Systems - Series B*, 2011, 16, (1), pp. 393-408
- [13] Wang J. L., Wu H. N.: ‘Local and global exponential output synchronization of complex delayed dynamical networks’, *Nonlinear Dynamics*, 2012, 67, pp. 497-504
- [14] Yang T., Wang X., Saberi A., *et. al.*: ‘Output synchronization for heterogeneous networks of discrete-time introspective right-invertible agents with uniform constant communication delay’, In *American Control Conference (ACC)*, pages 516–521, 2013.
- [15] Xiang J., Wei W.: ‘On local synchronisability of nonlinear networked systems with a unit inner-coupling matrix and switching topology’, *International Journal of Control*, 2011, 84, (11), pp. 1769-1778
- [16] Wu C. W.: ‘Algebraic connectivity of directed graphs’, *Linear and Multilinear Algebra*, 2005, 53, (3), pp. 203-223
- [17] Zhou J., Chen T.: ‘Synchronization in general complex delayed dynamical networks’, *IEEE Transactions on Circuits and Systems I: Regular Papers*, 2006, 53, (3), pp. 733-744
- [18] Xiao L., Boyd S.: ‘Fast linear iterations for distributed averaging’, *Systems & Control Letters*, 2004, 53, (1), pp. 65-78