# Almost sure stability of switching Markov Jump Linear Systems

Article (Published Version)

# Almost Sure Stability of Switching Markov Jump Linear Systems

Yang Song, Jie Yang, Taicheng Yang, Minrui Fei

*Abstract*—**Recently a special hybrid system called Switching Markov Jump Linear System (SMJLS) is studied. A SMJLS is subject to a deterministic switching and a stochastic Markovain switching. To extend the results already obtained and to investigate some new aspects of such systems, our main contributions in this paper are: (i) Transient analysis of Markov process, i.e. the expectations of the sojourn time, the activation number of any mode, and the number of switchings between any two modes; and (ii) Two sufficient conditions of the exponential almost sure stability for a general SMJLS. Different from previous work, which is a special case of our study, the transition rate matrix for the random Markov process in our study is not fixed, but varies when a deterministic switching takes place.**

*Index Terms*—**Almost sure stability, deterministic switching, switching Markov jump linear system, stochastic switching.**

## I. Introduction

A Markov Jump Linear System (MJLS) has a few linear subsystems and the switching between them is governed by a finite state Markov process. The basic properties of Markov process have been investigated extensively in previous works, see e.g., [1]. MJLSs are widely used to model the systems with abrupt random changes in structures and/or parameters. MJLS models are used in the study of fault tolerant systems [2], manufacturing systems [3], networked control systems [4], aerospace systems [5], etc.

Over the past decades, significant research is reported in the literature mainly on system stability. This includes δ-moment stability, Mean Square (MS) stability, and Almost Sure (AS) stability [6-9]. The δ-moment stability means that the expectation of the δ-th moment of the state norm converges to zero asymptotically. MS stability is a special case of the δ-moment stability if $\delta = 2$. AS-stability requires the convergence of the state trajectory with probability one. For MJLSs, the sufficient conditions of the MS-stability are in the form of coupled Lyapunov equations [7, 10], which can be solved by linear matrix inequality (LMI). For the AS-stability, the sufficient conditions presented in [7, 11, 12] are all based on the ergodic law of large numbers. Among them, the conditions in [11] are less restrictive than those in [7, 12] but are non-deterministic. An approach is developed based on the Monte Carlo algorithm to test the conditions presented in [11]. Overall, there is a general view that [7], for applications the AS-stability is more useful than the MS-stability and the δ-moment stability, since for the AS-stability one only needs to prove the convergence of the state trajectory with probability one.

This work was supported in part by the National Natural Science Funds of China (61573237,61533010) and Shanghai Natural Science Foundation (13ZR1416300).

Y. Song (corresponding author), J. Yang and M. R. Fei are with the Department of Automation, School of Mechatronic Engineering and Automation, Shanghai University, and with the Shanghai Key Laboratory of Power Station Automation Technology, 200072, Shanghai, China (e-mail: y_song@shu.edu.cn; yangjielanz@163.com; mrfei@staff.shu.edu.cn).

T. C. Yang is with Department of Engineering and Design, University of Sussex, Brighton, BN1 9QT, UK (e-mail: t.c.yang@sussex.ac.uk).

Recently, a new hybrid system subject to both deterministic and stochastic Markovain switchings is studied [13-15]. This system is called a Switching Markov Jump Linear System (SMJLS) [13, 14]. The concept of SMJLS can be illustrated by an example of a variable structure control system as in Fig. 1. The controller is of variable structure and the switching is by an operator or automatically based on certain conditions. Therefore, the controller switching is deterministic. On the other hand, the changes of plant modes are often trigged by uncertain stochastic factors and also may be affected by controller switching (e.g., an aggressive controller may increase the risk of fault occurrence [14]). As a result, the plant switching can be modeled as a finite state Markov process, and its transition rate can be fixed or variable. This appears to be one of the challenging topics in ever-increased complicated systems. For example, in renewable power generation industry, the performance of a wind turbines can be described by a Markov model considering the component reliability and the effect of wind speed [16]. In this area, various complicated switching controls are essential for stable and optimised operation [17].

Furthermore, SMJLSs can be divided into F-SMJLS and V-SMJLS. In a V-SMJLS, the transition rate matrix of Markov process is not fixed and is determined by the current position of deterministic switching. However, in the case of F-SMJLS the transition rate matrix remains the same despite the deterministic switching. For the stability of SMJLSs, based on our knowledge some results achieved are: (1) Sufficient conditions for the AS stability of a F-SMJLS based on the ergodicity of the Markov process [15], and (2) Sufficient conditions for the MS stability of a V-SMJLS by analyzing the time evolution of the second-order moment of the state [13, 14]. For the AS stability of a V-SMJLS, however, since the transition rate is variable and the ergodicity is no longer valid, the stability conditions are still an open challenge.

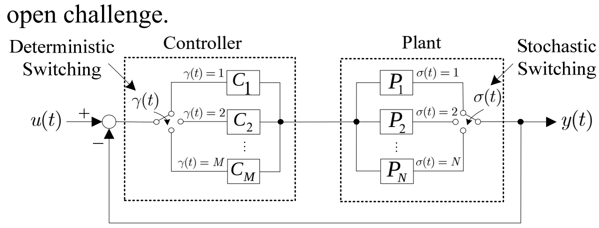


Fig. 1. An example of Switching MJLS.

The main contributions of this paper are: (i) Transient analysis of Markov process, i.e., the expectations of the sojourn time, the activation number of any mode, and the switching number between any two modes (Section III); and (ii) Two sufficient conditions of the exponential almost sure stability for V-SMJLSs (Section IV). These are obtained from the results of (i) and by applying the dwell time and average dwell time techniques. The rest of this paper is organized as follows.

The problem formulation and the preliminaries are presented in Section 2. After Sections 3 and 4 of our main results, two numerical examples are shown in Section 5, followed by conclusions in Section 6.

*Notations:* $\mathbf{0}_{N\times N}$ (or $\mathbf{1}_{N\times N}$) is an $N \times N$ dimension matrix with all the elements being 0 (or 1). $\|x\|$ and $\|A\|$ stand for the Euclidean norm of $x$ and the corresponding induced matrix norm of matrix $A$, respectively.

## II. PROBLEM FORMULATION

Consider a continuous-time V-SMJLS:

$$\dot{x}(t) = A_{\sigma(t,\gamma(t))}^{[\gamma(t)]}x(t),\ t \geq 0 \quad (1)$$

where state $x(t) \in R^n$, deterministic switching $\gamma(t) \in \mathcal{M} := \{1,2,\ldots,M\}$ is a exogenous piecewise constant function with $M$ level, $\sigma(t,\gamma(t)) \in \mathcal{N} := \{1,2,\ldots,N\}$ denotes a stochastic switching function which is governed by a $N$-mode Markov process. The structure of a V-SMJLS is illustrated by Fig. 2.

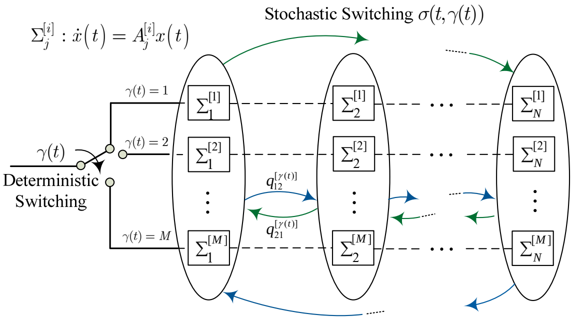


Fig. 2. The structure of a V-SMJLS.

The transition probability of the Markov process $\sigma(t,\gamma(t))$ is

$$p_{rs}^{[\gamma(t)]}(dt) = \Pr\left\{\sigma^{[\gamma(t)]}(t+dt) = s \mid \sigma^{[\gamma(t)]}(t) = r\right\}$$
$$= \begin{cases} q_{rs}^{[\gamma(t)]}dt + o(dt), & r \neq s \\ 1 + q_{rr}^{[\gamma(t)]}dt + o(dt), & r = s \end{cases},\ r,\ s \in \mathcal{N} \quad (2)$$

where $dt > 0$, $\lim_{dt\to 0} o(dt)/dt = 0$, $q_{rs}^{[\gamma(t)]}$ is the transition rate from mode $r$ at the instant $t$ to mode $s$ at the instant $t + dt$, and satisfy:

$$q_{rs}^{[\gamma(t)]} \geq 0,\ r \neq s,\ \sum\nolimits_{s=1,\,s\neq r}^{N} q_{rs}^{[\gamma(t)]} = -q_{rr}^{[\gamma(t)]} \quad (3)$$

Matrix $Q^{[\gamma(t)]} := [q_{rs}^{[\gamma(t)]}]_{N\times N}$ is called the transition rate matrix of $\sigma(t,\gamma(t))$. In this paper, $\forall j \in \mathcal{M}$, Markov process $\sigma(t,j)$ are assumed to be irreducible, therefore it is ergodic and has a unique invariant distribution $\boldsymbol{\pi}^{[j]} := [\pi_1^{[j]} \cdots \pi_N^{[j]}]$, which satisfies

$$\boldsymbol{\pi}^{[j]}Q^{[j]} = 0,\ \sum\nolimits_{i=1}^{N}\pi_i^{[j]} = 1 \quad (4)$$

The initial conditions of V-SMJLS (1) are: initial state $x_0$, initial deterministic switching position $\gamma_0$ and initial probability distribution $\boldsymbol{f}^{[\gamma_0]} := [f_1^{[\gamma_0]} \cdots f_N^{[\gamma_0]}]$, where $f_i^{[\gamma_0]} := \Pr\{\sigma^{[\gamma_0]}(0) = i\}$, $i \in \mathcal{N}$. Let $\overline{p}_i^{[j]}(t) = \Pr\{\sigma^{[j]}(t) = i\}$ denote the probability that the subsystem $\dot{x}(t) = A_i^{[j]}x(t)$ is active at the instant $t$. Let $D(t'',t')$ denote the number of deterministic switching in the interval $[t',t'')$. The deterministic switching $\gamma(t)$ is said to have an average dwell time (ADT) constrained with $\tau_a$ in $[t',t'')$, if $D(t'',t') \leq (t''-t')/\tau_a$. The stochastic system $\dot{x}(t) = A_{\sigma(t,j)}^{[j]}x(t)$ is called the $j$-th sub-MJLS of (1). Denote $T_k^{[j]}$ the interval that the $j$-th sub-MJLS operates continuously for the $k$-th time as shown in Fig. 3, where $t_m, t_{m+1}, \cdots$ are the deterministic switching instants. $T_{D\min}^{[j]}$ is said to be the minimal dwell-time of the $j$-th sub-MJLS, if $T_k^{[j]} \geq T_{D\min}^{[j]}$, $k = 1,2,\cdots$, $\forall j \in \mathcal{M}$.

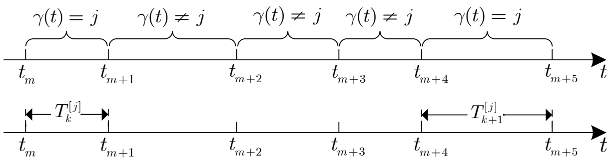


Fig. 3. The interval that the $j$-th sub-MJLS operates continuously.

*Definition 1: [15]*: The V-SMJLS (1) is said to be exponentially almost sure stable (EAS-stable), if there exist $\rho > 0$ such that, for any initial condition, it results that

$$\Pr\left\{\lim_{t\to\infty}\sup\frac{1}{t}\ln\left\|x(t)\right\| \leq -\rho\right\} = 1 \quad (5)$$

where $\rho$ is called the stability margin.

*Lemma 1: [11]:* The V-SMJLS (1) is EAS-stable if and only if $E[\lambda] < 0$, where the top Lyapunov exponent $\lambda$ is defined as

$$\lambda = \lim_{t\to\infty}\sup\frac{1}{t}\ln\left\|\Phi(t,0)\right\| \quad (6)$$

## III. TRANSIENT ANALYSIS OF MARKOV PROCESS

In this section, a new Lemma on the transient characteristic of a Markov process is given. This lemma will be used to derive the results in the next section.

For an $N$-mode Markov process $\sigma(t)$, the transition probability matrix $P(t) := [p_{ij}(t)]_{N\times N}$ and the transition rate matrix $Q := [q_{ij}]_{N\times N}$ satisfy the Kolmogorov differential equation [18]

$$\dot{P}(t) = P(t)Q,\ P(0) = I \quad (7)$$

Then by [18], this leads to

$$P(t) = e^{Qt} = \sum_{k=0}^{\infty}\frac{Q^k t^k}{k!},\ \text{where } Q^0 = I \quad (8)$$

*Lemma 2:* Consider an $N$-mode Markov process $\sigma(t)$ in the interval $[0,t)$: $T_j(t,0)$ is the accumulated time sojourns in mode $j$; $N_{ij}(t,0)$ is the total number of the switching from mode $i$ to mode $j$; $\bar{N}_j(t,0)$ is the number of the activations of the mode $j$. For all modes $i$, $j$,

$$E_{\boldsymbol{f}}\left[T_j(t,0)\right] = \pi_j t + h_j(t) \quad (9)$$

$$E_{\boldsymbol{f}}\left[N_{ij}(t,0)\right] = q_{ij}\pi_i t + h_i(t)q_{ij},\ i \neq j \quad (10)$$

$$E_{\boldsymbol{f}}\left[\bar{N}_j(t,0)\right] = -q_{jj}\pi_j t - h_j(t)q_{jj} + \boldsymbol{f}\left[P(t)\right]_j \quad (11)$$

where $h_i(t) = \boldsymbol{f}(P(t)-I)\left[(Q+\mathbf{1}_{N\times N})^{-1}\right]_i$, $[\cdot]_j$ denotes the $j$-th column of a matrix, unique invariant distribution $\boldsymbol{\pi} = [\pi_1 \cdots \pi_N]$, $\boldsymbol{f} = [f_1 \cdots f_N]$ is the initial probability distribution of each mode, $E_{\boldsymbol{f}}[\cdot]$ denotes the expectation operator with respect to the initial distribution $\boldsymbol{f}$. □

*Proof:* Clearly,

$$\overline{p}_j(t) = \sum_{i=1}^{N} f_i p_{ij}(t) = \boldsymbol{f}\left[P(t)\right]_j \tag{12}$$

$$E_{\boldsymbol{f}}\left[T_j(t,0)\right] = \int_0^t \overline{p}_j(\tau)d\tau = \boldsymbol{f}\left[\int_0^t P(\tau)d\tau\right]_j \tag{13}$$

Then integrating (7) from $0$ to $t$ and noticing (8), we have

$$P(t) - I = \left(\int_0^t P(\tau)d\tau\right)Q = \sum_{k=0}^{\infty}\frac{Q^k t^{k+1}}{(k+1)!}Q \tag{14}$$

From the definition of the transition rate matrix $Q$, it follows $Q\mathbf{1}_{N\times N} = \mathbf{0}_{N\times N}$. Then it can be seen from (14) that

$$\begin{aligned} P(t) - I &= \sum_{k=0}^{\infty}\frac{Q^k t^{k+1}}{(k+1)!}(Q + \mathbf{1}_{N\times N} - \mathbf{1}_{N\times N}) \\ &= \sum_{k=0}^{\infty}\frac{Q^k t^{k+1}}{(k+1)!}(Q + \mathbf{1}_{N\times N}) - t\mathbf{1}_{N\times N} \end{aligned} \tag{15}$$

From Proposition 5.7.1 in [19], $Q + \mathbf{1}_{N\times N}$ is invertible and $\mathbf{1}_{1\times N}(Q + \mathbf{1}_{N\times N})^{-1} = \boldsymbol{\pi}$. It follows from (15) that

$$\begin{aligned} \int_0^t P(\tau)d\tau &= \sum_{k=0}^{\infty}\frac{Q^k t^{k+1}}{(k+1)!} \\ &= (P(t) - I + t\mathbf{1}_{N\times N})(Q + \mathbf{1}_{N\times N})^{-1} \\ &= (P(t) - I)(Q + \mathbf{1}_{N\times N})^{-1} + t\mathbf{1}_{N\times 1}\mathbf{1}_{1\times N}(Q + \mathbf{1}_{N\times N})^{-1} \\ &= (P(t) - I)(Q + \mathbf{1}_{N\times N})^{-1} + t\mathbf{1}_{N\times 1}\boldsymbol{\pi} \end{aligned} \tag{16}$$

Substituting (16) into (13) and noticing $\boldsymbol{f}\mathbf{1}_{N\times 1}=1$, it yields

$$\begin{aligned} E_{\boldsymbol{f}}\left[T_j(t,0)\right] &= \boldsymbol{f}\left[(P(t) - I)(Q + \mathbf{1}_{N\times N})^{-1} + t\mathbf{1}_{N\times 1}\boldsymbol{\pi}\right]_j \\ &= h_j(t) + t\pi_j \end{aligned} \tag{17}$$

where $h_j(t) = \boldsymbol{f}(P(t) - I)\left[(Q + \mathbf{1}_{N\times N})^{-1}\right]_j$. Then the total number of switching from mode $i$ to mode $j$ in $[0,t)$ is

$$\begin{aligned} E_{\boldsymbol{f}}\left[N_{ij}(t,0)\right] &= \int_0^t q_{ij}\overline{p}_i(\tau)d\tau = q_{ij}\boldsymbol{f}\left[\int_0^t P(\tau)d\tau\right]_i \\ &= q_{ij}\pi_i t + h_i(t)q_{ij},\ i \neq j \end{aligned} \tag{18}$$

Denote $T_j^k$ the $k$-th successive sojourn time of the mode $j$, $(k = 1,2,\ldots)$. $T_j^k$ are independent and identically distributed and has an exponential distribution with mean $-1/q_{jj}$ [18]. Hence,

$$E_{\boldsymbol{f}}\left[\overline{N}_j(t,0) \mid T_j(t,0) = \overline{T}, \sigma(t) \neq j\right] = -q_{jj}\overline{T} \tag{19}$$

$$E_{\boldsymbol{f}}\left[\overline{N}_j(t,0) \mid T_j(t,0) = \overline{T}, \sigma(t) = j\right] = 1 - q_{jj}\overline{T} \tag{20}$$

From (19) and (20), it follows that

$$\begin{aligned} &E_{\boldsymbol{f}}\left[\overline{N}_j(t,0) \mid T_j(t,0) = \overline{T}\right] \\ &\quad = (1 - \overline{p}_j(t))E_{\boldsymbol{f}}\left[\overline{N}_j(t,0) \mid T_j(t,0) = \overline{T}, \sigma(t) \neq j\right] \\ &\qquad + \overline{p}_j(t)E_{\boldsymbol{f}}\left[\overline{N}_j(t,0) \mid T_j(t,0) = \overline{T}, \sigma(t) = j\right] \\ &\quad = -q_{jj}\overline{T} + \overline{p}_j(t) \end{aligned} \tag{21}$$

Given that $E_{\boldsymbol{f}}[X] = E[E_{\boldsymbol{f}}[X \mid Y]]$, (21) implies

$$\begin{aligned} E_{\boldsymbol{f}}\left[\overline{N}_j(t,0)\right] &= E\left[E_{\boldsymbol{f}}\left[\overline{N}_j(t,0) \mid T_j(t,0)\right]\right] \\ &= -q_{jj}E_{\boldsymbol{f}}\left[T_j(t,0)\right] + \overline{p}_j(t) \end{aligned} \tag{22}$$

Substituting (12) and (17) into (22):

$$E_{\boldsymbol{f}}\left[\overline{N}_j(t,0)\right] = -q_{jj}\pi_j t - h_j(t)q_{jj} + \boldsymbol{f}\left[P(t)\right]_j \tag{23}$$

This completes the proof. □

*Remark 1:* This lemma concerns with the transient statistical properties of a Markov process. When the Markov process reaches the steady state, or equivalently the initial distribution $\boldsymbol{f}$ equals the unique invariant distribution $\boldsymbol{\pi}$, it follows $\boldsymbol{\pi}P(t) = \boldsymbol{\pi}$. Then (9)-(11) are reduced to $E_{\boldsymbol{\pi}}[T_j(t,0)] = \pi_j t$, $E_{\boldsymbol{\pi}}[N_{ij}(t,0)] = q_{ij}\pi_i t$ and $E_{\boldsymbol{\pi}}[\overline{N}_j(t,0)] = -q_{jj}\pi_j t + \pi_j$. These are the same as the formulas (7a), (7b) and (7c) in [15]. However, Lemma 2 here is applicable to more general cases.

*Remark 2:* If $\boldsymbol{f}$ is uncertain, then the bounds of $E_{\boldsymbol{f}}[T_j(t,0)]$, $E_{\boldsymbol{f}}[N_{ij}(t,0)]$ and $E_{\boldsymbol{f}}[\overline{N}_j(t,0)]$ can be obtained as below:

For arbitrary $\boldsymbol{f}$ and $t$, it can be seen that $0 < \boldsymbol{f}\left[P(t)\right]_j \leq 1$. Let $[c_1\ c_2\ \cdots\ c_N] := \boldsymbol{f}(P(t) - I)$ and $[b_{1j}\ b_{2j}\ \cdots\ b_{Nj}]^T := [(Q + \mathbf{1}_{N\times N})^{-1}]_j$. Clearly, $\sum_{i=1}^N c_i = 0$, $-1 \leq c_i \leq 1$ and $h_j(t) = \boldsymbol{f}(P(t) - I)[(Q + \mathbf{1}_{N\times N})^{-1}]_j = \sum_{i=1}^N c_i b_{ij}$. Given that $\boldsymbol{f}$ is arbitrary, subject to the above the values of $c_1\ c_2\ \cdots\ c_N$ are also arbitrary. $b_{ij}$ is known when transition rate matrix $Q$ is given. Rearrange $b_{1j}, b_{2j}, \ldots, b_{Nj}$ into $\tilde{b}_{1j} \geq \tilde{b}_{2j} \geq \cdots \geq \tilde{b}_{Nj}$, then,

$$\begin{aligned} \delta_j &:= \max_{c_1,c_2,\cdots,c_N} \sum_{i=1}^N c_i b_{ij} = -\min_{c_1,c_2,\cdots,c_N} \sum_{i=1}^N c_i b_{ij} \\ &= \begin{cases} \sum_{i=1}^{N/2}\tilde{b}_{ij} - \sum_{i=1+N/2}^{N}\tilde{b}_{ij}, & \text{If } N \text{ is even number} \\ \sum_{i=1}^{(N-1)/2}\tilde{b}_{ij} - \sum_{i=2+(N-1)/2}^{N}\tilde{b}_{ij}, & \text{If } N \text{ is odd number} \end{cases} \end{aligned}$$

Then for arbitrary $\boldsymbol{f}$ and $t$, it follows that $-\delta_j \leq h_j(t) \leq \delta_j$. This leads to

$$\pi_j t - \delta_j \leq E_{\boldsymbol{f}}\left[T_j(t,0)\right] \leq \pi_j t + \delta_j \tag{24}$$

$$q_{ij}\pi_i t - q_{ij}\delta_i \leq E_{\boldsymbol{f}}\left[N_{ij}(t,0)\right] \leq q_{ij}\pi_i t + q_{ij}\delta_i,\ i \neq j \tag{25}$$

$$-q_{jj}\pi_j t + \delta_j q_{jj} \leq E_{\boldsymbol{f}}\left[\overline{N}_j(t,0)\right] \leq -q_{jj}\pi_j t - \delta_j q_{jj} + 1 \tag{26}$$

For a special case of a two-mode Markov process, i.e., $N = 2$, $[(Q + \mathbf{1}_{2\times 2})^{-1}]_j = [b_{1j}\ \ b_{2j}]^T$, $j = 1,2$. From the definition of $Q$, it follows $(Q + \mathbf{1}_{2\times 2})\alpha = 2\alpha$ and $(Q + \mathbf{1}_{2\times 2})^{-1}\alpha = \alpha/2$, where $\alpha = [1\ \ 1]^T$. Therefore $b_{11} + b_{12} = b_{21} + b_{22}$. This implies that $\delta_1 = \left|b_{11} - b_{21}\right| = \left|b_{22} - b_{12}\right| = \delta_2$. □

## IV. AS STABILITY FOR V-SMJLS

Based on Lemma 2, two sufficient conditions for the EAS-stability of V-SMJLSs are given for the cases when the deterministic switching satisfies the constraints of minimal dwell-time and the average dwell time respectively.

*Theorem 1:* The V-SMJLS (1) is EAS-stable if the minimum dwell time $T_{D\min}^{[j]}$ satisfies:

$$T_{D\min}^{[j]} > \frac{\xi^{[j]}}{\mu^{[j]}},\quad \mu^{[j]} > 0,\quad \forall j \in \mathcal{M} \tag{27}$$

where $\xi^{[j]} = \sum_{i=1}^N \alpha_i^{[j]}(1 - q_{ii}^{[j]}\delta_i^{[j]}) + \sum_{i\in\mathcal{N}_j^+}\beta_i^{[j]}\delta_i^{[j]} -$

$\sum_{i\in\mathcal{N}_j^-}\beta_i^{[j]}\delta_i^{[j]}$, $\mu^{[j]}=\sum_{i=1}^N\pi_i^{[j]}(\alpha_i^{[j]}q_{ii}^{[j]}+\beta_i^{[j]})$, $\mathcal{N}_j^+=\{i\in\mathcal{N}:\beta_i^{[j]}\geq 0\}$, $\mathcal{N}_j^-=\{i\in\mathcal{N}:\beta_i^{[j]}<0\}$. $q_{ii}^{[j]}$ is defined in Section II, $\delta_i^{[j]}$ is specified in Remark 2. Parameter $\alpha_i^{[j]}\geq 0$ and $\beta_i^{[j]}$ satisfy

$$\left\|e^{A_i^{[j]}t}\right\|\leq e^{\alpha_i^{[j]}-\beta_i^{[j]}t},\ \forall t\geq 0,\ i\in\mathcal{N},\ j\in\mathcal{M} \tag{28}$$

*Proof:* Assume that $\gamma(t)=j,\ t\in[t^a,t^b)$ and $\gamma(t^{a^-})\neq j$; hence $t^a$ is a deterministic switching instant; denote $\Phi^{[j]}(t^b,t^a)$ the stochastic transition matrix in $[t^a,t^b)$, and define $\bar{N}_i^{[j]}(t^b,t^a)$ and $T_i^{[j]}(t^b,t^a)$ the activation number and the total sojourn time of subsystem $\dot{x}(t)=A_i^{[j]}x(t)$ in $[t^a,t^b)$, respectively. Noticing there is no deterministic switching in $[t^a,t^b)$, it follows from (28)

$$\begin{aligned}\ln\left\|\Phi^{[j]}(t^b,t^a)\right\|&\leq\sum_{i=1}^N\left(\alpha_i^{[j]}\bar{N}_i^{[j]}(t^b,t^a)-\beta_i^{[j]}T_i^{[j]}(t^b,t^a)\right)\\&=\sum_{i=1}^N\alpha_i^{[j]}\bar{N}_i^{[j]}(t^b,t^a)-\sum_{i\in\mathcal{N}_j^+}\beta_i^{[j]}T_i^{[j]}(t^b,t^a)\\&\quad-\sum_{i\in\mathcal{N}_j^-}\beta_i^{[j]}T_i^{[j]}(t^b,t^a)\end{aligned} \tag{29}$$

Thus

$$\begin{aligned}&E_{\boldsymbol{f}(t^a)}\left[\ln\left\|\Phi^{[j]}(t^b,t^a)\right\|\right]\leq\sum_{i=1}^N\alpha_i^{[j]}E_{\boldsymbol{f}(t^a)}\left[\bar{N}_i^{[j]}(t^b,t^a)\right]\\&-\sum_{i\in\mathcal{N}_j^+}\beta_i^{[j]}E_{\boldsymbol{f}(t^a)}\left[T_i^{[j]}(t^b,t^a)\right]-\sum_{i\in\mathcal{N}_j^-}\beta_i^{[j]}E_{\boldsymbol{f}(t^a)}\left[T_i^{[j]}(t^b,t^a)\right]\end{aligned} \tag{30}$$

where $\boldsymbol{f}(t^a)$ is the absolute probability distribution of each mode of the sub–MJLS being activated at the deterministic switching instant $t^a$.

Notice $\gamma(t^a)=j$ and from Lemma 2

$$E_{\boldsymbol{f}(t^a)}\left[\bar{N}_i^{[j]}(t^b,t^a)\right]=-q_{ii}^{[j]}\pi_i^{[j]}\tau-h_i^{[j]}(\tau)q_{ii}^{[j]}+\boldsymbol{f}(t^a)\left[P^{[j]}(\tau)\right]_i$$

$$E_{\boldsymbol{f}(t^a)}\left[T_i^{[j]}(t^b,t^a)\right]=\pi_i^{[j]}\tau+h_i^{[j]}(\tau)$$

where $h_i^{[j]}(\tau)=\boldsymbol{f}(t^a)\left(P^{[j]}(\tau)-I\right)\left[(Q^{[j]}+\mathbf{1}_{N\times N})^{-1}\right]_i$, $\tau=t^b-t^a$, $\boldsymbol{\pi}^{[j]}=[\pi_1^{[j]}\ \cdots\ \pi_N^{[j]}]$ and $P^{[j]}(t)$ are the invariant distribution and the transition probability matrix of the $j$-th sub-MJLS, respectively. Then, from (24) and (26)

$$E_{\boldsymbol{f}(t^a)}\left[\bar{N}_i^{[j]}(t^b,t^a)\right]\leq-q_{ii}^{[j]}\pi_i^{[j]}\tau-q_{ii}^{[j]}\delta_i^{[j]}+1 \tag{31}$$

$$\pi_i^{[j]}\tau-\delta_i^{[j]}\leq E_{\boldsymbol{f}(t^a)}\left[T_i^{[j]}(t^b,t^a)\right]\leq\pi_i^{[j]}\tau+\delta_i^{[j]} \tag{32}$$

Applying (31) and (32) to(30),

$$\begin{aligned}E_{\boldsymbol{f}(t^a)}\left[\ln\left\|\Phi^{[j]}(t^b,t^a)\right\|\right]&\leq\sum_{i=1}^N\alpha_i^{[j]}(-q_{ii}^{[j]}\pi_i^{[j]}\tau-q_{ii}^{[j]}\delta_i^{[j]}+1)\\&-\sum_{i\in\mathcal{N}_j^+}\beta_i^{[j]}(\pi_i^{[j]}\tau-\delta_i^{[j]})-\sum_{i\in\mathcal{N}_j^-}\beta_i^{[j]}(\pi_i^{[j]}\tau+\delta_i^{[j]})\\&=-\tau\mu^{[j]}+\xi^{[j]}\end{aligned} \tag{33}$$

where $\mu^{[j]}=\sum_{i=1}^N(\alpha_i^{[j]}q_{ii}^{[j]}\pi_i^{[j]}+\beta_i^{[j]}\pi_i^{[j]})$, $\xi^{[j]}=\sum_{i=1}^N\alpha_i^{[j]}(1-q_{ii}^{[j]}\delta_i^{[j]})+\sum_{i\in\mathcal{N}_j^+}\beta_i^{[j]}\delta_i^{[j]}-\sum_{i\in\mathcal{N}_j^-}\beta_i^{[j]}\delta_i^{[j]}$.

Assume that there are $k$ deterministic switches in $[0,t)$ and the corresponding switching instants are $t_0$, $t_1$, ..., $t_{k-1}$, $t_k$, then the transition matrix $\Phi(t,0)$ of the V-SMJLS (1) is

$$\Phi(t,0)=\Phi^{[\gamma(t_k)]}(t,t_k)\Phi^{[\gamma(t_{k-1})]}(t_k,t_{k-1})\cdots\Phi^{[\gamma(t_0)]}(t_1,t_0) \tag{34}$$

Hence

$$\ln\left\|\Phi(t,0)\right\|\leq\ln\left\|\Phi^{[\gamma(t_k)]}(t,t_k)\right\|+\sum_{m=0}^{k-1}\ln\left\|\Phi^{[\gamma(t_m)]}(t_{m+1},t_m)\right\|$$

Then it follows from (33) that

$$\begin{aligned}E_{\boldsymbol{f}^{[\gamma_0]}}\left[\ln\left\|\Phi(t,0)\right\|\right]&\leq E_{\boldsymbol{f}(t_k)}\left[\ln\left\|\Phi^{[\gamma(t_k)]}(t,t_k)\right\|\right]\\&\quad+\sum_{m=0}^{k-1}E_{\boldsymbol{f}(t_m)}\left[\ln\left\|\Phi^{[\gamma(t_m)]}(t_{m+1},t_m)\right\|\right]\\&\leq-(t-t_k)\mu^{[\gamma(t_k)]}+\xi^{[\gamma(t_k)]}\\&\quad+\sum_{m=0}^{k-1}\left[-(t_{m+1}-t_m)\mu^{[\gamma(t_m)]}+\xi^{[\gamma(t_m)]}\right]\\&=-(t-t_k)\mu^{[\gamma(t_k)]}-\sum_{m=0}^{k-1}(t_{m+1}-t_m)\mu^{[\gamma(t_m)]}+\sum_{m=0}^{k}\xi^{[\gamma(t_m)]}\\&\leq\sum_{j=1}^M\left(-\mu^{[j]}T_D^{[j]}(t,0)+\xi^{[j]}D^{[j]}(t,0)\right)\end{aligned} \tag{35}$$

where $T_D^{[j]}(t,0)$ and $D^{[j]}(t,0)$ are the total dwell time and the total number of visits of the $j$-th sub-MJLS in $[0,t)$, respectively.

Denote $r^{[j]}(t)$ the fraction of total sojourn time of the $j$-th sub-MJLS in $[0,t)$, then

$$T_D^{[j]}(t,0)=r^{[j]}(t)t \tag{36}$$

From the definition of $T_{D\min}^{[j]}$:

$$D^{[j]}(t,0)\leq r^{[j]}(t)t\big/T_{D\min}^{[j]} \tag{37}$$

Noticing $\xi^{[j]}>0$ and substituting (36) and (37) into (35),

$$E_{\boldsymbol{f}^{[\gamma_0]}}\left[\ln\left\|\Phi(t,0)\right\|\right]\leq t\left(\sum_{j=1}^M-\mu^{[j]}r^{[j]}(t)+\sum_{j=1}^M\frac{\xi^{[j]}r^{[j]}(t)}{T_{D\min}^{[j]}}\right) \tag{38}$$

From (27), it leads to

$$\begin{aligned}&E_{\boldsymbol{f}^{[\gamma_0]}}\left[\limsup_{t\to\infty}\frac{\ln\left\|\Phi(t,0)\right\|}{t}\right]\\&\leq\limsup_{t\to\infty}\sum_{j=1}^M\left(-\mu^{[j]}r^{[j]}(t)+\frac{\xi^{[j]}r^{[j]}(t)}{T_{D\min}^{[j]}}\right)\\&=\sum_{j=1}^M r^{[j]}\left(-\mu^{[j]}+\xi^{[j]}\big/T_{D\min}^{[j]}\right)<0\end{aligned} \tag{39}$$

where $\forall j\in\mathcal{M}$, $r^{[j]}=\limsup_{t\to\infty}r^{[j]}(t)\geq 0$, and at least one $r^{[j]}$ is strictly greater than zero.

Hence,

$$E\left[\limsup_{t\to\infty}\frac{\ln\left\|\Phi(t,0)\right\|}{t}\right]<0 \tag{40}$$

Based on Lemma 1, the V-SMJLS (1) is EAS-stable. □

*Remark 3:* Theorem 1 is for a V-SMJLS where the transition rate of Markov process varies with the deterministic switching.

For an F-SMJLS, the transition rate of the Markov process is fixed, therefore based on the ergodicity of the Markov process, $E_{\boldsymbol{\pi}}[T_j(t,0)] = \pi_j t$ and $E_{\boldsymbol{\pi}}[\bar{N}_j(t,0)] = -q_{jj}\pi_j t + \pi_j$. This naturally leads to the following corollary for F-SMJLSs.

*Corollary 1:* A F-SMJLS $\dot{x}(t) = A_{\sigma(t)}^{[\gamma(t)]}x(t)$ is EAS-stable if

$$T_{D\min}^{[j]} > \frac{\hat{\xi}^{[j]}}{\hat{\mu}^{[j]}}, \quad \hat{\mu}^{[j]} > 0, \quad \forall j \in \mathcal{M} \tag{41}$$

where $\hat{\xi}^{[j]} = \sum_{i=1}^{N}\alpha_i^{[j]}\pi_i$, $\hat{\mu}^{[j]} = \sum_{i=1}^{N}(\alpha_i^{[j]}q_{ii}\pi_i + \beta_i^{[j]}\pi_i)$, parameter $\alpha_i^{[j]}$, $\beta_i^{[j]}$ and $\pi_i$ are defined as in Theorem 1. □

*Remark 4:* Let $\bar{\alpha}_i = \max_j \alpha_i^{[j]}$ and $\tilde{\eta}^{[j]} = \sum_{i=1}^{N}(\bar{\alpha}_i q_{ii}\pi_i + \beta_i^{[j]}\pi_i)$, then $\hat{\xi}^{[j]} < \max_i \bar{\alpha}_i$, $\hat{\mu}^{[j]} > \tilde{\eta}^{[j]} \geq \min_j \tilde{\eta}^{[j]}$. The inequality $\max_i \bar{\alpha}_i / \min_j \tilde{\eta}^{[j]} > \hat{\xi}^{[j]}/\hat{\mu}^{[j]}$ means that the constraint on minimum dwell time in Corollary 1 is less restrictive than that in [15].

*Remark 5:* Even if one or more sub-MJLSs are not activated from a certain instant onwards, the dwell time constraints $T_{D\min}^{[j]} > \xi^{[j]}/\mu^{[j]}$ can still guarantee the overall system to be EAS stable.

In Theorem 1, the condition $\mu^{[j]} > 0$, $\forall j \in \mathcal{M}$ implies that all sub-MJLS need to be EAS stable [15, Lemma 2]. When some sub-MJLSs are not EAS stable, the following result can make that V-SMJLS (1) to be EAS stable if the deterministic switching satisfies the specified average dwell time constraint. Denote $\mathcal{H}^+ = \{j \in \mathcal{M}: \mu^{[j]} > 0\}$ and $\mathcal{H}^- = \{j \in \mathcal{M}: \mu^{[j]} \leq 0\}$. Let $\tilde{\mu}^+ = \min_{j\in\mathcal{H}^+}\mu^{[j]}$, $\tilde{\mu}^- = \min_{j\in\mathcal{H}^-}\mu^{[j]}$. Clearly $\tilde{\mu}^+ > 0$, $\tilde{\mu}^- \leq 0$, where $\tilde{\mu}^- = 0$ when $\mathcal{H}^-$ is empty.

*Theorem 2:* When the deterministic switching function $\gamma(t)$ satisfies an average dwell time constraint with a parameter $\tau_a$, and if (42) and (43) below are satisfied, then the V-SMJLS (1) is EAS-stable,

$$\frac{\sum_{j\in\mathcal{H}^+} r^{[j]}}{\sum_{j\in\mathcal{H}^-} r^{[j]}} \geq -\frac{\theta + \tilde{\mu}^-}{\theta + \tilde{\mu}^+}, \quad -\tilde{\mu}^+ < \theta < 0 \tag{42}$$

$$\tau_a\theta + \tilde{\xi} < 0 \tag{43}$$

where $\tilde{\xi} = \max_{1\leq j\leq M}\xi^{[j]}$, $\xi^{[j]}$ is as in Theorem 1, $r^{[j]} = \limsup_{t\to\infty} r^{[j]}(t)$, $r^{[j]}(t)$ is the ratio of total sojourn time of the $j$-th sub-MJLS in $[0,t)$, parameter $\theta$ is to be determined.

*Proof:* From (35) and (36) it follows that

$$\begin{aligned} E_{\boldsymbol{f}^{[\gamma_0]}}\left[\ln\left\|\Phi(t,0)\right\|\right] &\leq \sum_{j=1}^{M} -\mu^{[j]}r^{[j]}(t)t + \sum_{j=1}^{M}\xi^{[j]}D^{[j]}(t,0) \\ &\leq \sum_{j=1}^{M} -\mu^{[j]}r^{[j]}(t)t + \tilde{\xi}D(t,0) \end{aligned} \tag{44}$$

where $D^{[j]}(t,0)$ is the total number of visits of $j$-th sub-MJLS in $[0,t)$, $D(t,0)$ is the total number of the deterministic switching in $[0,t)$, $\sum_{j=1}^{M}D^{[j]}(t,0) = D(t,0)$. Recall the definition of the ADT, $D(t,0) \leq t/\tau_a$. Noticing $\tilde{\xi} > 0$, $r^{[j]} > 0$ and $\sum_{j=1}^{M} r^{[j]} \geq 1$, it follows from (42) and (43) that

$$\begin{aligned} E_{\boldsymbol{f}^{[\gamma_0]}}\left[\limsup_{t\to\infty}\frac{\ln\left\|\Phi(t,0)\right\|}{t}\right] &\leq \sum_{j=1}^{M} -\mu^{[j]}r^{[j]} + \tilde{\xi}/\tau_a \\ &= -\sum_{j\in\mathcal{H}^+}\mu^{[j]}r^{[j]} - \sum_{j\in\mathcal{H}^-}\mu^{[j]}r^{[j]} + \tilde{\xi}/\tau_a \\ &\leq -\tilde{\mu}^+\sum_{j\in\mathcal{H}^+}r^{[j]} - \tilde{\mu}^-\sum_{j\in\mathcal{H}^-}r^{[j]} + \tilde{\xi}/\tau_a \\ &\leq \theta\left(\sum_{j\in\mathcal{H}^+}r^{[j]} + \sum_{j\in\mathcal{H}^-}r^{[j]}\right) + \tilde{\xi}/\tau_a \\ &= \theta\sum_{j=1}^{M}r^{[j]} + \tilde{\xi}/\tau_a \leq \theta + \tilde{\xi}/\tau_a < 0 \end{aligned} \tag{45}$$

Hence

$$E\left[\limsup_{t\to\infty}\frac{\ln\left\|\Phi(t,0)\right\|}{t}\right] < 0 \tag{46}$$

Based on Lemma 1, the V-SMJLS (1) is EAS-stable. □

## V. Two Numerical Examples

*Example 1:* Consider a two-mode Markov process $\{\sigma(t),\ t \geq 0\}$ with a transition rate matrix $Q = \begin{bmatrix} -1 & 1 \\ 5 & -5 \end{bmatrix}$, the initial probability distribution $\boldsymbol{f} = [0.6\ \ 0.4]$. By (4), it follows that $\boldsymbol{\pi} = [5/6\ \ 1/6]$. Noticing $P(t) = e^{Qt}$ and decomposing $Q$ into Jordan form, it leads to

$$P(t) = \begin{bmatrix} \frac{5}{6} + \frac{1}{6}e^{-6t} & \frac{1}{6} - \frac{1}{6}e^{-6t} \\ \frac{5}{6} - \frac{5}{6}e^{-6t} & \frac{1}{6} + \frac{5}{6}e^{-6t} \end{bmatrix}$$

Substituting $P(t)$ into Lemma 2, the theoretical values of $E_{\boldsymbol{f}}[T_j(t,0)]$, $E_{\boldsymbol{f}}[N_{ij}(t,0)]$ and $E_{\boldsymbol{f}}[\bar{N}_j(t,0)]$ can be calculated for any given $t$. These theoretical values (TVs) are consistent with the calculated values (CVs) obtained from the average values of 6000 samples of the Markov process–as shown in Table I. Clearly, $E_{\boldsymbol{f}}[T_j(t,0)]$, $E_{\boldsymbol{f}}[N_{ij}(t,0)]$ and $E_{\boldsymbol{f}}[\bar{N}_j(t,0)]$ are dependent on both $t$ and initial distribution $\boldsymbol{f}$. Remark 2 shows that the entries containing $\boldsymbol{f}$ are bounded, therefore the effects of $\boldsymbol{f}$ on the aboved three expectations are limited.

TABLE I
EXPECTATIONS OF SOJOURN TIME, SWITCHING NUMBER AND ACTIVATION NUMBER

| t | | 1 | 10 | 50 | 100 |
|---|---|---|---|---|---|
| $E_f[T_1(t,0)]$ | CV | 0.7947 | 8.2894 | 41.6466 | 83.2978 |
| | TV | 0.7945 | 8.2944 | 41.6278 | 83.2944 |
| $E_f[N_{12}(t,0)]$ | CV | 0.7955 | 8.3190 | 41.6795 | 83.2825 |
| | TV | 0.7945 | 8.2944 | 41.6278 | 83.2944 |
| $E_f[\bar{N}_1(t,0)]$ | CV | 1.6262 | 9.1492 | 42.5190 | 84.1117 |
| | TV | 1.6273 | 9.1278 | 42.4611 | 84.1278 |

*Example 2*: Consider a V-SMJLS (1) $M = N = 2$:

$$A_1^{[1]} = \begin{bmatrix} -3 & 0 \\ 2 & 2 \end{bmatrix}, A_2^{[1]} = \begin{bmatrix} -4 & -2 \\ 0 & -2 \end{bmatrix}, Q^{[1]} = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix};$$

$$A_1^{[2]} = \begin{bmatrix} 3 & -1 \\ 0 & -1 \end{bmatrix}, A_2^{[2]} = \begin{bmatrix} -5 & 1 \\ -1 & -2 \end{bmatrix}, Q^{[2]} = \begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix};$$

Then, $\pi^{[1]} = [1/3 \;\; 2/3]$ and $\pi^{[2]} = [3/4 \;\; 1/4]$.

By Lemma 3.1 in [20], choose

$$\alpha_1^{[1]} = 0.05,\ \alpha_2^{[1]} = 0.1,\ \beta_1^{[1]} = -2.05,\ \beta_2^{[1]} = 1.8$$

$$\alpha_1^{[2]} = 0,\ \alpha_2^{[2]} = 0,\ \beta_1^{[2]} = -3.1,\ \beta_2^{[2]} = 2.1$$

By Remark 2, $\delta_1^{[1]} = \delta_2^{[1]} = 0.333$, $\delta_1^{[2]} = \delta_2^{[2]} = 0.25$, and $\mu^{[1]} = 0.417$, $\xi^{[1]} = 1.499$, $\mu^{[2]} = -1.8$, $\xi^{[2]} = 1.3$, $\tilde{\mu}^{+} = 0.417$, $\tilde{\mu}^{-} = -1.8$, $\tilde{\xi} = 1.499$. Noticing $\mu^{[1]} > 0$ fulfills the sufficient condition of EAS stability [15, Lemma 2], therefore the first sub-MJLS is EAS-stable. However, for the second sub-MJLS, the sufficient condition of the EAS stability in [15, Lemma 2] is not satisfied due to $\mu^{[2]} < 0$. Meanwhile, this sub-MJLS does not violate the necessary conditions of EAS stability [11, Theorem 5]. Thus theoretically we are not able to assert that the second sub-MJLS is EAS stable or not.

From Theorem 2 and choose $\theta = -0.15$, the system is EAS stable if $r^{[1]}/r^{[2]} \geq 7.303$ and $\tau_a > 9.993$. Choose a piece-wise constant periodic switching signal $\gamma(t)$ satisfying the constraints of ADT obtained in Theorem 2

$$\gamma(t) = \begin{cases} 2 & t \in [nT, nT+3) \\ 1 & t \in [nT+3, nT+26) \end{cases},$$

where the period $T = 26$, $n = 0, 1, 2, \ldots$, $r^{[1]}/r^{[2]} = 7.667$ and $\tau_a = 13$. Fig. 4 shows the V-SMJLS is EAS stable as stated by Theorem 2.

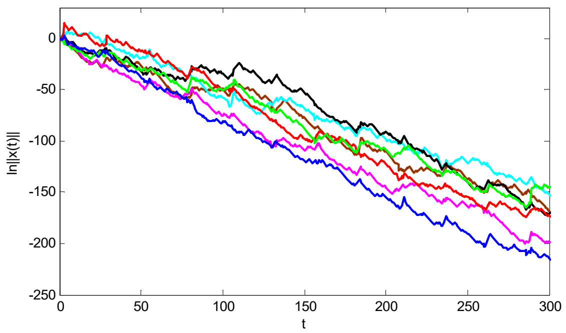


Fig.4. Seven realizations of $\ln\|x(t)\|$ with $x(0) = [1 \;\; 3]^T$.

## VI. Conclusion

This paper studies almost sure stability of Switching Markov Jump Linear Systems (SMJLSs). The main contributions are: (i) Transient analysis of Markov process to obtain the expectations of the sojourn time, the activation number of any mode, and the switching number between any two modes; and (ii) Two sufficient conditions of the exponential almost sure stability for a SMJLS where the transition rate matrix of Markov process changes when the deterministic switching function switches between different sub-MJLSs.

## Reference


[1] P. Brémaud, *Markov chains: Gibbs fields, Monte Carlo simulation, and queues* vol. 31: Springer Science & Business Media, 2013.

[2] H. Ji, S. Yang, and Z. Xue, "Active fault tolerant control systems by the semi-Markov model approach," *Int. J. Adapt. Control Signal Process.*, vol. 28, pp. 833-47, 2014.

[3] J. Zhu, L. Wang, and M. Spiryagin, "Control and decision strategy for a class of Markovian jump systems in failure prone manufacturing process," *IET Control Theory Appl.*, vol. 6, pp. 1803-1811, 2012.

[4] P. Minero, L. Coviello, and M. Franceschetti, "Stabilization over Markov feedback channels: the general case," *IEEE Trans. Autom. Control*, vol. 58, pp. 349-362, 2013.

[5] O. L. V. Costa, M. D. Fragoso, and R. P. Marques, *Discrete-Time Markov Jump Linear Systems*: Springer-Verlag, London, 2006.

[6] Y. Fang, "A new general sufficient condition for almost sure stability of jump linear systems," *IEEE Trans. Autom. Control*, vol. 42, pp. 378-382, 1997.

[7] Y. Fang and K. A. Loparo, "Stabilization of continuous-time jump linear systems," *IEEE Trans. Autom. Control*, vol. 47, pp. 1590-1603, 2002.

[8] C. Li, M. Z. Chen, J. Lam, and X. Mao, "On exponential almost sure stability of random jump systems," *IEEE Trans. Autom. Control*, vol. 57, pp. 3064-3077, 2012.

[9] P. Bolzern, P. Colaneri, and G. De Nicolao, "Stochastic stability of positive Markov jump linear systems," *Automatica*, vol. 50, pp. 1181-1187, 2014.

[10] H. Huang, G. Feng, and X. Chen, "Stability and stabilization of Markovian jump systems with time delay via new Lyapunov functionals," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 59, pp. 2413-2421, 2012.

[11] P. Bolzern, P. Colaneri, and G. De Nicolao, "On almost sure stability of continuous-time Markov jump linear systems," *Automatica*, vol. 42, pp. 983-988, 2006.

[12] Y. Song, H. Dong, T. Yang, and M. Fei, "Almost sure stability of discrete-time Markov jump linear systems," *IET Control Theory Appl.*, vol. 8, pp. 901-906, 2014.

[13] P. Colaneri, "Dwell time analysis of deterministic and stochastic switched systems," *Eur. J. Control*, vol. 15, pp. 228-248, 2009.

[14] P. Bolzern, P. Colaneri, and G. De Nicolao, "Markov jump linear systems with switching transition rates: mean square stability with dwell-time," *Automatica*, vol. 46, pp. 1081-1088, 2010.

[15] P. Bolzern, P. Colaneri, and G. De Nicolao, "Almost sure stability of Markov jump linear systems with deterministic switching," *IEEE Trans. Autom. Control*, vol. 58, pp. 209-214, 2013.

[16] S. R. Arwade, M. A. Lackner, and M. D. Grigoriu, "Probabilistic Models for Wind Turbine and Wind Farm Performance," *J. Solar Energy Eng.*, vol. 133, Nov 2011.

[17] D. Palejiya, J. Hall, C. Mecklenborg, and D. Chen, "Stability of Wind Turbine Switching Control in an Integrated Wind Turbine and Rechargeable Battery System: A Common Quadratic Lyapunov Function Approach," *J. Dyn. Syst., Meas., Control*, vol. 135, p. 021018, 2013.

[18] S. M. Ross, *Introduction to Probability Models*, 11th ed.: New York, NY, USA: Academic Press, 2014.

[19] S. I. Resnick, *Adventures in Stochastic Processes*: Basel, Switzerland: Birkhauser Verlag, 1992.

[20] M. Tanelli, B. Picasso, P. Bolzern, and P. Colaneri, "Almost sure stabilization of uncertain continuous-time Markov jump linear systems," *IEEE Trans. Autom. Control*, vol. 55, pp. 195-201, 2010.