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On Gegenbauer Polynomials and Coefficients $c_j^\ell(\nu)$ $(1 \le j \le \ell,\, \nu > -1/2)$

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Abstract

The Gegenbauer coefficients $c_j^\ell(\nu)$ $(1 \le j \le \ell,\, \nu > -1/2)$ appear in the Maclaurin expansion of the heat kernels on the n-sphere and the real projective n-space. In this note we show that these coefficients can be computed by transforming the higher order derivative formula for the Gegenbauer polynomials C_k^ν $(k \ge 0, \nu > -1/2)$ into a spectral sum involving the powers of the eigenvalues of the associated Gegenbauer operator. We present explicit computations and various implications.

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1 Introduction

The Gegenbauer or ultraspherical polynomials play a fundamental role in the analysis of the Laplace operator on spheres and real projective spaces. They represent the zonal or spherical functions on these symmetric spaces and are the key ingredients in describing the spectral measure and spectral projections associated with the Laplace operator. In this note we strengthen this connection by way of showing that a related set of spectral quantities associated with Gegenbauer operator – hereafter the Gegenbauer coefficients – fully describe the Maclaurin coefficients relating to the Schwartz kernel of invariant operators in the functional calculus of the Laplacian.

Recall that the Gegenbauer polynomial $C_k^\nu(t)$ (with $k \geq 0$, $\nu > -1/2$) is defined by the generating function relation [1]

$$\frac{1}{(1-2xt+x^2)^\nu} = \sum_{k=0}^{\infty} C_k^\nu(t)x^k. \tag{1.1}$$

It is a straightforward matter to show that $y = C_k^\nu(t)$ is a solution to the second-order homogenous differential equation

$$\left(1-t^2\right)\frac{d^2y}{dt^2} - (2\nu+1)t\frac{dy}{dt} + k(k+2\nu)y = 0, \tag{1.2}$$

that in turn constitute a *regular* Sturm-Liouville system with the corresponding Gegenbauer operator a positive selfadjoint second order linear differential operator in the weighted space $L^2[-1,1;(1-t^2)^{\nu-1/2}dt]$. The spectrum here is discrete and given by the set of eigenvalues $\lambda_k^\nu = k(k+2\nu)$ with associated eigenfunctions $y = C_k^\nu(t)$ $(k \geq 0)$. In particular we have the orthogonality relations

$$\begin{aligned}(C_k^\nu, C_m^\nu)_{L^2[-1,1;(1-t^2)^{\nu-1/2}dt]} &= \int_{-1}^{1} C_k^\nu(t)\, C_m^\nu(t)\left(1-t^2\right)^{\nu-\frac{1}{2}}\, dt \\ &= \frac{\pi 2^{1-2\nu}\Gamma(2\nu+m)}{m!(m+\nu)\Gamma(\nu)^2}\delta_{km}, \qquad k,m \geq 0. \end{aligned} \tag{1.3}$$

The Gegenbauer polynomial $y = C_k^\nu(t)$ admits a truncated series representation [resulting from the series solution to (1.2)] in the form

$$C_k^\nu(t) = \sum_{0\leq l\leq[k/2]} \frac{(-1)^l\Gamma(k-l+\nu)}{l!(k-2l)!\Gamma(\nu)}(2t)^{k-2l}, \tag{1.4}$$

and the derivatives of $C_k^\nu(t)$ satisfy the recursive relation

$$\frac{d^m}{dt^m}C_k^\nu(t) = 2^m\frac{\Gamma(\nu+m)}{\Gamma(\nu)}C_{k-m}^{\nu+m}(t). \tag{1.5}$$

Furthermore we have the pointwise (at $t = \pm 1$) and the odd versus even identities

$$C_k^\nu(1) = \frac{(2\nu)_k}{k!}, \qquad C_k^\nu(-t) = (-1)^k C_k^\nu(t), \tag{1.6}$$

where $(x)_k = \Gamma(x+k)/\Gamma(x)$ denotes the rising factorial. For future reference it is often useful to normalise C_k^ν and set

$$\mathscr{C}_k^\nu(t) = \frac{C_k^\nu(t)}{C_k^\nu(1)}, \qquad C_k^\nu(1) = \frac{\Gamma(k+2\nu)}{\Gamma(2\nu)k!}. \tag{1.7}$$

[1] Here we present the main identities and properties needed later. For a more thorough discussion on Gegenbauer polynomials and related topics see [6, 11, 12] and [13].

2 Gegenbauer Coefficients and a Spectral Identity

We start by proving an identity relating the derivatives of the Gegenbauer polynomial C_k^ν to a suitable weighted sum of integer powers of the eigenvalues $\lambda_k^\nu = k(k+2\nu)$ of the Gegenbauer operator [*cf.* (1.2)]. Applications and implications will be discussed afterwards.

Theorem 2.1. (*Gegenbauer coefficients*) *The Gegenbauer polynomial* $C_k^\nu(t)$ *with* $k \geq 0$ *satisfies the differential-spectral identity*

$$\frac{d^{2\ell}}{d\theta^{2\ell}} C_k^\nu(\cos\theta)\Big|_{\theta=0} = C_k^\nu(1) \sum_{j=1}^{\ell} c_j^\ell(\nu)[\lambda_k^\nu]^j = C_k^\nu(1)\mathscr{P}_\ell(\lambda_k^\nu), \qquad \ell \geq 1. \tag{2.1}$$

The scalars $(c_j^\ell(\nu) : 1 \leq j \leq \ell)$ *are referred to as the Gegenbauer coefficients,* $\mathscr{P}_\ell$ *is as in* (2.9) *and* $\lambda_k^\nu = k(k+2\nu)$ *are the eigenvalues associated with the Sturm-Liouville system* (1.2).

Proof. First as a function of $\cos\theta$ the derivative of the even order 2ℓ of $y = C_k^\nu(\cos\theta)$ at $\theta = 0$ is a weighted sum of the derivatives of $C_k^\nu(t)$ of all orders $1 \leq m \leq \ell$ at $t = 1$. Hence for suitable scalars a_m^ℓ we have

$$\frac{d^{2\ell}}{d\theta^{2\ell}} C_k^\nu(\cos\theta)\Big|_{\theta=0} = \left\{ \sum_{m=1}^{\ell} a_m^\ell \frac{d^m}{dt^m} C_k^\nu(t) \right\}\Big|_{t=1}. \tag{2.2}$$

Next using the recursion formula (1.5) on the derivatives of $C_k^\nu(t)$ and the pointwise value identity (1.7) we can write (2.2) as

$$\begin{aligned} \frac{d^{2\ell}}{d\theta^{2\ell}} C_k^\nu(\cos\theta)\Big|_{\theta=0} &= \sum_{m=1}^{\ell} 2^m a_m^\ell \frac{\Gamma(\nu+m)}{\Gamma(\nu)} C_{k-m}^{\nu+m}(1) \\ &= \sum_{m=1}^{\ell} 2^m a_m^\ell \frac{\Gamma(2\nu+k+m)}{\Gamma(2\nu+k)\Gamma(\nu)} \frac{\Gamma(\nu+m)\Gamma(2\nu)k!}{\Gamma(2\nu+2m)(k-m)!} C_k^\nu(1) \\ &= \sum_{m=1}^{\ell} \alpha_m^\ell \frac{\Gamma(2\nu+k+m)k!}{\Gamma(2\nu+k)(k-m)!} C_k^\nu(1) \end{aligned} \tag{2.3}$$

where we have set

$$\alpha_m^\ell(\nu) = 2^m a_m^\ell \frac{\Gamma(\nu+m)\Gamma(2\nu)}{\Gamma(\nu)\Gamma(2\nu+2m)}. \tag{2.4}$$

Finally to complete the proof it suffices to note that for $1 \le m \le \ell$ we have

$$\frac{\Gamma(k+2\nu+m)k!}{\Gamma(k+2\nu)(k-m)!} = \prod_{p=0}^{m-1} \Big(k(k+2\nu) - p(p+2\nu) \Big) = \sum_{j=1}^{m} \beta_j^\ell [k(k+2\nu)]^j$$

for suitable scalars $\beta_j^\ell = \beta_j^\ell(\nu)$. Substitution back in (2.3) gives the required conclusion. □

We now discuss some applications of Theorem 2.1. Consider the Laplace operator $-\Delta$ on the n-sphere and let $\Phi = \Phi(X)$ lie in the functional calculus of $-\Delta$. Then by well known results the Schwartz kernel of $\Phi(-\Delta)$ can be expressed in zonal harmonics as

$$K_\Phi(x,y) = \frac{1}{\omega_n} \sum_{k=0}^{\infty} M_k^n \Phi(\lambda_k^{(n-1)/2}) \mathscr{C}_k^{(n-1)/2}(\cos\theta). \tag{2.5}$$

Here $\lambda_k^{(n-1)/2} = k(n+k-1)$ are the numerically distinct eigenvalues of $-\Delta$ on $\mathbf{S}^n$ each having multiplicity

$$M_k^n = (2k+n-1)\frac{(k+n-2)!}{k!(n-1)!}, \qquad k \ge 0, \tag{2.6}$$

while θ is the geodesic distance between x, y and $\omega_n = \mathrm{Vol}(\mathbf{S}^n)$. Now since by (2.5) K_Φ is an even function of the geodesic distance its Maclaurin expansion about $\theta = 0$ takes the form

$$K_\Phi = \sum_{\ell=0}^{\infty} \frac{\theta^{2\ell}}{(2\ell)!} \frac{\partial^{2\ell}}{\partial\theta^{2\ell}} K_\Phi \Big|_{\theta=0} = \sum_{\ell=0}^{\infty} \frac{b_{2\ell}^n}{(2\ell)!} \theta^{2\ell} \tag{2.7}$$

with $b_{2\ell}^n = b_{2\ell}^n[\Phi]$ $(\ell \ge 0)$ being the usual Maclaurin coefficients. A direct calculation upon invoking (2.1) and (2.5) now shows that these coefficients can be expressed as a suitable operator trace

$$\begin{aligned} b_{2\ell}^n = \frac{\partial^{2\ell}}{\partial\theta^{2\ell}} K_\Phi \Big|_{\theta=0} &= \frac{1}{\omega_n} \sum_{k=0}^{\infty} M_k^n \Phi(\lambda_k^{(n-1)/2}) \frac{d^{2\ell}}{d\theta^{2\ell}} \mathscr{C}_k^{(n-1)/2}(\cos\theta) \Big|_{\theta=0} \\ &= \frac{1}{\omega_n} \sum_{k=0}^{\infty} M_k^n \Phi(\lambda_k^{(n-1)/2}) \sum_{j=1}^{\ell} c_j^\ell \left[\lambda_k^{(n-1)/2} \right]^j \\ &= \frac{1}{\omega_n} \mathrm{tr}\, [\mathscr{P}_\ell \Phi](-\Delta) \end{aligned} \tag{2.8}$$

where $\mathscr{P}_\ell = \mathscr{P}_\ell(X)$ is the polynomial of degree ℓ built out of the Gegenbauer coefficients $(c_j^\ell(\nu) : 1 \le j \le \ell)$, given specifically by,

$$\mathscr{P}_\ell(X) = \sum_{j=1}^{\ell} c_j^\ell(\nu) X^j. \tag{2.9}$$

Note in particular that in case of the heat semigroup with $\Phi(X) = e^{-tX}$ the Maclaurin coefficients of the heat kernel $K_t(x, y)$ can be expressed as $(t > 0)$

$$b_{2\ell}^n(t) = \frac{1}{\omega_n} \sum_{k=0}^{\infty} M_k^n e^{-t\lambda_k^{(n-1)/2}} \mathscr{P}_\ell(\lambda_k^{(n-1)/2}) = \frac{1}{\omega_n} \mathrm{tr}\, \mathscr{P}_\ell(-\Delta) e^{t\Delta}, \tag{2.10}$$

and so

$$b_{2\ell}^n(t) = \sum_{k=0}^{\infty} \frac{M_k^n}{\omega_n} \left\{ \sum_{j=1}^{\ell} (-1)^j c_j^\ell \frac{d^j}{dt^j} \right\} e^{-t\lambda_k^{\frac{n-1}{2}}} = \frac{1}{\omega_n} \left\{ \mathscr{P}_\ell \left(-\frac{d}{dt} \right) \right\} \mathrm{tr}\, e^{t\Delta}. \tag{2.11}$$

This analysis crucially underlines the role of the polynomials $\mathscr{P}_\ell$ and the Gegenbauer coefficients c_j^ℓ in expressing the Maclaurin coefficients $b_{2\ell}^n[\Phi]$ associated with the Schwartz kernel K_Φ of $\Phi(-\Delta)$. (For related but different discussion see also [1–4] as well as [5, 7–10].)

3 Explicit Calculations and the first Gegenbauer Coefficients and polynomials $\mathscr{P}_\ell$

The proof of Theorem 2.1 does not reveal in explicit form the Gegenbauer coefficients. The aim here is to compute the first few in the sequence explicitly. For the sake of brevity hereafter we set $y(t) = C_k^\nu(t)$.

- $(\ell = 1)$ Indeed

$$\begin{aligned} \frac{d^2}{d\theta^2} y(\cos\theta)\bigg|_{\theta=0} = a_1^1 y'(1) &= a_1^1 \frac{\Gamma(\nu+1)}{\Gamma(\nu)} \frac{2\Gamma(k+2\nu+1)}{\Gamma(2\nu+2)(k-1)!} \\ &= a_1^1 \frac{k(k+2\nu)}{2\nu+1} y(1). \end{aligned} \tag{3.1}$$

Then it follows that

$$\frac{d^2}{d\theta^2}y(\cos\theta)\Big|_{\theta=0} = -\frac{k(k+2\nu)}{2\nu+1}y(1) \qquad \Longrightarrow c_1^1(\nu) = -\frac{1}{2\nu+1}. \quad (3.2)$$

- ($\ell = 2$) In this case we have

$$\frac{d^4}{d\theta^4}y(\cos\theta)\Big|_{\theta=0} = a_1^2 y'(1) + a_2^2 y''(1) =$$
$$= a_1^2 \frac{\Gamma(\nu+1)}{\Gamma(\nu)} \frac{2\Gamma(k+2\nu+1)}{\Gamma(2\nu+2)(k-1)!} + a_2^2 \frac{\Gamma(\nu+2)}{\Gamma(\nu)} \frac{4\Gamma(k+2\nu+2)}{\Gamma(2\nu+4)(k-2)!}. \quad (3.3)$$

Now direct calculation gives

$$y''(1) = \frac{(k+2\nu)(k+2\nu+1)\Gamma(k+2\nu)}{(2\nu+1)(2\nu+3)\Gamma(2\nu)(k-2)!} = \frac{(k-1)(k+2\nu+1)}{(2\nu+1)(2\nu+3)} \times$$
$$\times\, k(k+2\nu)y(1) = \left\{ \frac{[k(k+2\nu)]^2}{(2\nu+1)(2\nu+3)} - \frac{(2\nu+1)k(k+2\nu)}{(2\nu+1)(2\nu+3)} \right\} y(1).$$

Hence simplifying further we obtain

$$(3.3) = \left\{ \frac{a_1^2(2\nu+3) - a_2^2(2\nu+1)}{(2\nu+1)(2\nu+3)} k(k+2\nu) + \frac{a_2^2[k(k+2\nu)]^2}{(2\nu+1)(2\nu+3)} \right\} y(1).$$

It is now easily seen that $a_1^2 = 1$, $a_2^2 = 3$ which in turn implies

$$c_1^2(\nu) = -\frac{4\nu}{(2\nu+1)(2\nu+3)}, \qquad c_2^2(\nu) = \frac{3}{(2\nu+1)(2\nu+3)}. \quad (3.4)$$

- ($\ell = 3$) Here we have

$$\frac{d^6}{d\theta^6}y(\cos\theta)\Big|_{\theta=0} = a_1^3 y'(1) + a_2^3 y''(1) + a_3^3 y'''(1) =$$
$$= a_1^3 \frac{\Gamma(\nu+1)}{\Gamma(\nu)} \frac{2\Gamma(k+2\nu+1)}{\Gamma(2\nu+2)(k-1)!} +$$
$$+ a_2^3 \frac{\Gamma(\nu+2)}{\Gamma(\nu)} \frac{4\Gamma(k+2\nu+2)}{\Gamma(2\nu+4)(k-2)!} + a_3^3 \frac{\Gamma(\nu+3)}{\Gamma(\nu)} \frac{8\Gamma(k+2\nu+3)}{\Gamma(2\nu+6)(k-3)!}. \quad (3.5)$$

Further simplification shows that

$$
\begin{aligned}
y'''(1) =& \frac{(k+2\nu)(k+2\nu+1)(k+2\nu+2)\Gamma(k+2\nu)}{(2\nu+1)(2\nu+3)(2\nu+5)\Gamma(2\nu)(k-3)!} \\
=& \frac{(k-1)(k-2)(k+2\nu+1)(k+2\nu+2)}{(2\nu+1)(2\nu+3)(2\nu+5)} k(k+2\nu)y(1) \\
=& \left\{ \frac{[k(k+2\nu)]^3}{(2\nu+1)(2\nu+3)(2\nu+5)} - \frac{(6\nu+5)[k(k+2\nu)]^2}{(2\nu+1)(2\nu+3)(2\nu+5)} + \right. \\
& \left. + \frac{4(\nu+1)(2\nu+1)k(k+2\nu)}{(2\nu+1)(2\nu+3)(2\nu+5)} \right\} y(1). \qquad (3.6)
\end{aligned}
$$

As a result it follows that

$$
\begin{aligned}
(3.5) = & \left\{ \frac{a_1^3(2\nu+3)(2\nu+5) - a_2^3(2\nu+1)(2\nu+5) + a_3^3 4(\nu+1)(2\nu+1)}{(2\nu+1)(2\nu+3)(2\nu+5)} \times \right. \\
& \times k(k+2\nu) + \frac{a_2^3(2\nu+5) - a_3^3(6\nu+5)}{(2\nu+1)(2\nu+3)(2\nu+5)} [k(k+2\nu)]^2 + \\
& \left. + \frac{a_3^3[k(k+2\nu)]^3}{(2\nu+1)(2\nu+3)(2\nu+5)} \right\} y(1). \qquad (3.7)
\end{aligned}
$$

Clearly here we see that $a_1^3 = -1$, $a_2^3 = a_3^3 = -15$ and subsequently we obtain

$$
c_1^3(\nu) = -\frac{16\nu(4\nu+1)}{(2\nu+1)(2\nu+3)(2\nu+5)}, \qquad (3.8)
$$

$$
c_2^3(\nu) = \frac{60\nu}{(2\nu+1)(2\nu+3)(2\nu+5)} \qquad (3.9)
$$

$$
c_3^3(\nu) = -\frac{15}{(2\nu+1)(2\nu+3)(2\nu+5)}. \qquad (3.10)
$$

- ($\ell = 4$) Indeed from (2.3) we have

$$
\begin{aligned}
& \frac{d^8}{d\theta^8} y(\cos\theta)\bigg|_{\theta=0} = a_1^4 y'(1) + a_2^4 y''(1) + a_3^4 y'''(1) + a_4^4 y^{(4)}(1) = \\
& = a_1^4 \frac{\Gamma(\nu+1)}{\Gamma(\nu)} \frac{2\Gamma(k+2\nu+1)}{\Gamma(2\nu+2)(k-1)!} + a_2^4 \frac{\Gamma(\nu+2)}{\Gamma(\nu)} \frac{4\Gamma(k+2\nu+2)}{\Gamma(2\nu+4)(k-2)!} + \\
& + a_3^4 \frac{\Gamma(\nu+3)}{\Gamma(\nu)} \frac{8\Gamma(k+2\nu+3)}{\Gamma(2\nu+6)(k-3)!} + a_4^4 \frac{\Gamma(\nu+4)}{\Gamma(\nu)} \frac{16\Gamma(k+2\nu+4)}{\Gamma(2\nu+8)(k-4)!}. \qquad (3.11)
\end{aligned}
$$

Simplifying further and as before we have

$$y^{(4)}(1) = \frac{(k+2\nu)(k+2\nu+1)(k+2\nu+2)(k+2\nu+3)\Gamma(k+2\nu)}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)\Gamma(2\nu)(k-4)!}$$
$$= \frac{(k-1)(k-2)(k-3)(k+2\nu+1)(k+2\nu+2)(k+2\nu+3)}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)} k(k+2\nu)y(1)$$
$$= \left\{ \frac{[k(k+2\nu)]^4}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)} - \frac{2(6\nu+7)[k(k+2\nu)]^3}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)} \right.$$
$$+ \frac{\left(44\nu^2+96\nu+49\right)[k(k+2\nu)]^2}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)} -$$
$$\left. - \frac{12(\nu+1)(2\nu+1)(2\nu+3)k(k+2\nu)}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)} \right\} y(1). \tag{3.12}$$

It now follows that

$$\left.\frac{d^8}{d\theta^8} y(\cos\theta)\right|_{\theta=0} = \left\{ c_1^4(\nu)k(k+2\nu) + c_2^4(\nu)[k(k+2\nu)]^2 + \right.$$
$$\left. + c_3^4(\nu)[k(k+2\nu)]^3 + c_4^4(\nu)[k(k+2\nu)]^4 \right\} y(1), \tag{3.13}$$

where

$$c_1^4(\nu) = \frac{a_1^4(2\nu+3)(2\nu+5)(2\nu+7) - a_2^4(2\nu+1)(2\nu+5)(2\nu+7)}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)} +$$
$$+ \frac{a_3^4 4(\nu+1)(2\nu+1)((2\nu+7)) - a_4^4 12(\nu+1)(2\nu+1)(2\nu+3)}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)},$$
$$c_2^4(\nu) = \frac{a_2^4(2\nu+5)(2\nu+7) - a_3^4(6\nu+5)(2\nu+7) + a_4^4\left(44\nu^2+96\nu+49\right)}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)},$$
$$c_3^4(\nu) = \frac{a_3^4(2\nu+7) - a_4^4(12\nu+14)}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)},$$
$$c_4^4(\nu) = \frac{a_4^4}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)}.$$

Finally a further differentiation results in $a_1^4 = 1$, $a_2^4 = 63$, $a_3^4 = 210$

and $a_4^4 = 105$ and subsequently we obtain

$$c_1^4(\nu) = -\frac{2176\nu^3 + 1536\nu^2 + 320\nu}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)}, \tag{3.14}$$

$$c_2^4(\nu) = \frac{2352\nu^2 + 672\nu}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)}, \tag{3.15}$$

$$c_3^4(\nu) = -\frac{840\nu}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)}, \tag{3.16}$$

$$c_4^4 = \frac{105}{(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)}. \tag{3.17}$$

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