The Bayesian Approach to Inverse Problems

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Abstract. These lecture notes highlight the mathematical and computational structure relating to the formulation of, and development of algorithms for, the Bayesian approach to inverse problems in differential equations. This approach is fundamental in the quantification of uncertainty within applications involving the blending of mathematical models with data. The finite dimensional situation is described first, along with some motivational examples. Then the development of probability measures on separable Banach space is undertaken, using a random series over an infinite set of functions to construct draws; these probability measures are used as priors in the Bayesian approach to inverse problems. Regularity of draws from the priors is studied in the natural Sobolev or Besov spaces implied by the choice of functions in the random series construction, and the Kolmogorov continuity theorem is used to extend regularity considerations to the space of Hölder continuous functions. Bayes' theorem is derived in this prior setting, and here interpreted as finding conditions under which the posterior is absolutely continuous with respect to the prior, and determining a formula for the Radon-Nikodym derivative in terms of the likelihood of the data. Having established the form of the posterior, we then describe various properties common to it in the infinite dimensional setting. These properties include well-posedness, approximation theory, and the existence of maximum a posteriori estimators. We then describe measure-preserving dynamics, again on the infinite dimensional space, including Markov chain-Monte Carlo and sequential Monte Carlo methods, and measure-preserving reversible stochastic differential equations. By formulating the theory and algorithms on the underlying infinite dimensional space, we obtain a framework suitable for rigorous analysis of the accuracy of reconstructions, of computational complexity, as well as naturally constructing algorithms which perform well under mesh refinement, since they are inherently well-defined in infinite dimensions.

Keywords. Inverse problems, Bayesian inversion, Tikhonov regularization and MAP estimators, Markov chain Monte Carlo, sequential Monte Carlo, Langevin stochastic partial differential equations.

*The authors are grateful to EPSRC, ERC and ONR for financial support which led to the work described in these lecture notes.

1. Introduction

Many uncertainty quantification problems arising in the sciences and engineering require the incorporation of data into a model; indeed doing so can significantly reduce the uncertainty in model predictions and is hence a very important step in many applications. Bayes' formula provides the natural way to do this. The purpose of these lecture notes is to develop the Bayesian approach to inverse problems in order to provide a rigorous framework for the development of uncertainty quantification in the presence of data. Of course it is possible to simply discretize the inverse problem and apply Bayes' formula on a finite dimensional space. However we adopt a different approach: we formulate Bayes' formula on a separable Banach space and study its properties in this infinite dimensional setting. This approach, of course, requires considerably more mathematical sophistication and it is important to ask whether this is justified. The answer, of course, is "yes". The formulation of the Bayesian approach on a separable Banach space has numerous benefits: (i) it reveals an attractive well-posedness framework for the inverse problem, allowing for the study of robustness to changes in the observed data, or to numerical approximation of the forward model; (ii) it allows for direct links to be established with the classical theory of regularization, which has been developed in a separable Banach space setting; (iii) and it leads to new algorithmic approaches which build on the full power of analysis and numerical analysis to leverage the structure of the infinite dimensional inference problem.

The remainder of this section contains a discussion of Bayesian inversion in finite dimensions, for motivational purposes, and two examples of partial differential equation (PDE) inverse problems. In section 2 we describe the construction of priors on separable Banach spaces, using random series and employing the random series to discuss various Sobolev, Besov and Hölder regularity results. Section 3 is concerned with the statement and derivation of Bayes' theorem in this separable Banach space setting. In section 4 we describe various properties common to the posterior, including well-posedness in the Hellinger metric, a related approximation theory which leverages well-posedness to deliver the required stability estimate, and the existence of maximum a posteriori (MAP) estimators; these address points (i) and (ii) above, respectively. Then, in section 5, we discuss various discrete and continuous time Markov processes which preserve the posterior probability measure, including Markov chain-Monte Carlo methods (MCMC), sequential Monte-Carlo methods (SMC) and reversible stochastic partial differential equations, addressing point (iii) above. The infinite dimensional perspective on algorithms is beneficial as it provides a direct way to construct algorithms which behave well under refinement of finite dimensional approximations of the underlying separable Banach space. We conclude in section 6 and then an appendix collects together a variety of basic definitions and results from the theory of differential equations and probability. Each section is accompanied by bibliographical notes connecting the developments herein to the wider literature. The notes complement and build on other overviews of Bayesian inversion, and its relations to uncertainty quantification, which may be found in [93, 94]. All results (lemmas, theorems etc.) which are quoted without proof are given pointers to the literature, where proofs may be

found, within the bibliography of the section containing the result.

1.1. Bayesian Inversion on $\mathbb{R}^n$. Consider the problem of finding $u \in \mathbb{R}^n$ from $y \in \mathbb{R}^J$ where u and y are related by the equation

$$y = G(u).$$

We refer to y as *observed data* and to u as the *unknown*. This problem may be difficult for a number of reasons. We highlight two of these, both particularly relevant to our future developments.

1. The first difficulty, which may be illustrated in the case where $n = J$, concerns the fact that often the equation is perturbed by noise and so we should really consider the equation
$$y = G(u) + \eta, \tag{1.1}$$
where $\eta \in \mathbb{R}^J$ represents the *observational noise* which enters the observed data. Assume further that G maps $\mathbb{R}^J$ into a proper subset of itself, Im_G, and that G has a unique inverse as a map from Im_G into $\mathbb{R}^J$. It may then be the case that, because of the noise, $y \notin \mathrm{Im}_G$ so that simply inverting G on the data y will not be possible. Furthermore, the specific instance of η which enters the data may not be known to us; typically, at best, only the statistical properties of a typical noise η are known. Thus we cannot subtract η from the observed data y to obtain something in Im_G. Even if $y \in \mathrm{Im}_G$ the uncertainty caused by the presence of noise η causes problems for the inversion.

2. The second difficulty is manifest in the case where $n > J$ so that the system is *underdetermined*: the number of equations is smaller than the number of unknowns. How do we attach a sensible meaning to the concept of solution in this case where, generically, there will be many solutions?

Thinking probabilistically enables us to overcome both of these difficulties. We will treat u, y and η as random variables and determine the joint probability distribution of (u, y). We then define the "solution" of the inverse problem to be the probability distribution of u given y, denoted $u|y$. This allows us to model the noise via its statistical properties, even if we do not know the exact instance of the noise entering the given data. And it also allows us to specify *a priori* the form of solutions that we believe to be more likely, thereby enabling us to attach weights to multiple solutions which explain the data. This is the *Bayesian approach* to inverse problems.

To this end, we define a random variable $(u, y) \in \mathbb{R}^n \times \mathbb{R}^J$ as follows. We let $u \in \mathbb{R}^n$ be a random variable with (Lebesgue) density $\rho_0(u)$. Assume that $y|u$ (y given u) is defined via the formula (1.1) where $G : \mathbb{R}^n \to \mathbb{R}^J$ is measurable, and η is independent of u (we sometimes write this as $\eta \perp u$) and distributed according to measure $\mathbb{Q}_0$ with Lebesgue density $\rho(\eta)$. Then $y|u$ is simply found by shifting

$\mathbb{Q}_0$ by $G(u)$ to measure $\mathbb{Q}_u$ with Lebesgue density $\rho(y - G(u))$. It follows that $(u, y) \in \mathbb{R}^n \times \mathbb{R}^J$ is a random variable with Lebesgue density $\rho(y - G(u))\rho_0(u)$. The following theorem allows us to calculate the distribution of the random variable $u|y$:

Theorem 1.1. Bayes' Theorem. *Assume that*

$$Z := \int_{\mathbb{R}^n} \rho\big(y - G(u)\big)\rho_0(u)du > 0.$$

Then $u|y$ is a random variable with Lebesgue density $\rho^y(u)$ given by

$$\rho^y(u) = \frac{1}{Z}\rho\big(y - G(u)\big)\rho_0(u).$$

Remarks 1.2. The following remarks establish the nomenclature of Bayesian statistics, and also frame the previous theorem in a manner which generalizes to the infinite dimensional setting.

- $\rho_0(u)$ is the **prior density**.
- $\rho\big(y - G(u)\big)$ is the **likelihood**.
- $\rho^y(u)$ is the **posterior density**.
- It will be useful in what follows to define

  $$\Phi(u; y) = -\log \rho\big(y - G(u)\big).$$

  We call Φ the **potential**. This is the **negative log likelihood.**
- Note that Z is the probability of y. Bayes' formula expresses

  $$\mathbb{P}(u|y) = \frac{1}{\mathbb{P}(y)}\mathbb{P}(y|u)\mathbb{P}(u).$$

- Let μ^y be a measure on $\mathbb{R}^n$ with density ρ^y and μ_0 a measure on $\mathbb{R}^n$ with density ρ_0. Then the conclusion of Theorem 1.1 may be written as:

  $$\begin{aligned} \frac{d\mu^y}{d\mu_0}(u) &= \frac{1}{Z}\exp\big(-\Phi(u; y)\big), \\ Z &= \int_{\mathbb{R}^n} \exp\big(-\Phi(u; y)\big)\mu_0(du). \end{aligned} \tag{1.2}$$

  Thus the posterior is absolutely continuous with respect to the prior, and the Radon-Nikodym derivative is proportional to the likelihood. This is rewriting Bayes' formula in the form

  $$\frac{1}{\mathbb{P}(u)}\mathbb{P}(u|y) = \frac{1}{\mathbb{P}(y)}\mathbb{P}(y|u).$$

- The expression for the Radon-Nikodym derivative is to be interpreted as the statement that, for all measurable $f : \mathbb{R}^n \to \mathbb{R}$,

$$\mathbb{E}^{\mu^y} f(u) = \mathbb{E}^{\mu_0}\Big(\frac{d\mu^y}{d\mu_0}(u) f(u)\Big).$$

  Alternatively we may write this in integral form as

$$\begin{aligned}\int_{\mathbb{R}^n} f(u)\mu^y(du) &= \int_{\mathbb{R}^n}\Big(\frac{1}{Z}\exp\big(-\Phi(u;y)\big) f(u)\Big)\mu_0(du) \\ &= \frac{\int_{\mathbb{R}^n}\exp\big(-\Phi(u;y)\big) f(u)\mu_0(du)}{\int_{\mathbb{R}^n}\exp\big(-\Phi(u;y)\big)\mu_0(du)}.\end{aligned}$$

□

1.2. Inverse Heat Equation. This inverse problem illustrates the first difficulty, labelled 1. in the previous subsection, which motivates the Bayesian approach to inverse problems. Let $D \subset \mathbb{R}^d$ be a bounded open set, with Lipschitz boundary ∂D. Then define the Hilbert space H and operator A as follows:

$$\begin{aligned}&H = \Big(L^2(D), \langle\cdot,\cdot\rangle, \|\cdot\|\Big); \\ &A = -\triangle, \quad \mathcal{D}(A) = H^2(D) \cap H_0^1(D).\end{aligned}$$

We make the following assumption about the spectrum of A which is easily verified for simple geometries, but in fact holds quite generally.

Assumption 1.3. The eigenvalue problem

$$A\varphi_j = \alpha_j \varphi_j,$$

has a countably infinite set of solutions, indexed by $j \in \mathbb{Z}^+$. They may be normalized to satisfy the L^2-orthonormality condition

$$\langle \varphi_j, \varphi_k \rangle = \begin{cases} 1, & j = k \\ 0, & j \neq k, \end{cases}$$

and form a basis for H. Furthermore, the eigenvalues are positive and, if ordered to be increasing, satisfy $\alpha_j \asymp j^{\frac{2}{d}}$. □

Here and in the remainder of the notes, the notation $\asymp$ denotes the existence of constants $C^{\pm} > 0$ such that

$$C^- j^{2/d} \leq \alpha_j \leq C^+ j^{2/d} \tag{1.3}$$

for all $j \in \mathbb{N}$.

Any $w \in H$ can be written as

$$w = \sum_{j=1}^{\infty} \langle w, \varphi_j \rangle \varphi_j$$

and we can define the Hilbert scale of spaces $\mathcal{H}^t = \mathcal{D}(A^{t/2})$ as explained in Section 7.1.3 for any $t > 0$ and with the norm

$$\|w\|_{\mathcal{H}^t}^2 = \sum_{j=1}^{\infty} j^{\frac{2t}{d}} |w_j|^2$$

where $w_j = \langle w, \varphi_j \rangle$.

Consider the heat conduction equation on D, with Dirichlet boundary conditions, writing it as an ordinary differential equation in H:

$$\frac{dv}{dt} + Av = 0, \quad v(0) = u. \tag{1.4}$$

We have the following:

Lemma 1.4. *Let Assumption 1.3 hold. Then for every $u \in H$ and every $s > 0$ there is a unique solution v of equation* (1.4) *in the space $C([0,\infty); H) \cap C((0,\infty); \mathcal{H}^s)$. We write $v(t) = \exp(-At)u$.*

To motivate this statement, and in particular the high degree of regularity seen at each fixed t, we argue as follows. Note that, if the initial condition is expanded in the eigenbasis as

$$u = \sum_{j=1}^{\infty} u_j \varphi_j, \quad u_j = \langle u, \varphi_j \rangle,$$

then the solution of (1.4) has the form

$$v(t) = \sum_{j=1}^{\infty} u_j e^{-\alpha_j t} \varphi_j.$$

Thus

$$\begin{aligned} \|v(t)\|_{\mathcal{H}^s}^2 &= \sum_{j=1}^{\infty} j^{2s/d} e^{-2\alpha_j t} |u_j|^2 \asymp \sum_{j=1}^{\infty} \alpha_j^s e^{-2\alpha_j t} |u_j|^2 \\ &= t^{-s} \sum_{j=1}^{\infty} (\alpha_j t)^s e^{-2\alpha_j t} |u_j|^2 \le C t^{-s} \sum_{j=1}^{\infty} |u_j|^2 \\ &= C t^{-s} \|u\|_H^2. \end{aligned}$$

It follows that $v(t) \in \mathcal{H}^s$ for any $s > 0$, provided $u \in H$.

We are interested in the inverse problem of finding u from y where

$$y = v(1) + \eta = G(u) + \eta = e^{-A} u + \eta \, .$$

Here $\eta \in H$ is noise and $G(u) := v(1) = e^{-A}u$. Formally this looks like an infinite dimensional linear version of the inverse problem (1.1), extended from finite dimensions to a Hilbert space setting. However, the infinite dimensional setting throws up significant new issues. To see this, assume that there is $\beta_c > 0$ such that η has regularity $\mathcal{H}^\beta$ if and only if $\beta < \beta_c$. Then y is not in the image space of G which is, of course, contained in $\cap_{s>0}\mathcal{H}^s$. Applying the formal inverse of G to y results in an object which is not in H.

To overcome this problem, we will apply a Bayesian approach and hence will need to put probability measures on the Hilbert space H; in particular we will want to study $\mathbb{P}(u)$, $\mathbb{P}(y|u)$ and $\mathbb{P}(u|y)$, all probability measures on H.

1.3. Elliptic Inverse Problem. One motivation for adopting the Bayesian approach to inverse problems is that prior modelling is a transparent approach to dealing with under-determined inverse problems; it forms a rational approach to dealing with the second difficulty, labelled 2. in subsection 1.1. The elliptic inverse problem we now describe is a concrete example of an under-determined inverse problem.

As in subsection 1.2, $D \subset \mathbb{R}^d$ denotes a bounded open set, with Lipschitz boundary ∂D. We define the Gelfand triple of Hilbert spaces $V \subset H \subset V^*$ by

$$H = \left(L^2(D), \langle\cdot,\cdot\rangle, \|\cdot\|\right), \qquad V = \left(H_0^1(D), \langle\nabla\cdot,\nabla\cdot\rangle, \|\cdot\|_V = \|\nabla\cdot\|\right). \tag{1.5}$$

and V^* the dual of V with respect to the pairing induced by H. Note that $\|\cdot\| \leq C_{\mathsf{p}}\|\cdot\|_V$ for some constant C_{p}: the Poincaré inequality.

Let $\kappa \in X := L^\infty(D)$ satisfy

$$\operatorname{ess}\inf_{x\in D} \kappa(x) = \kappa_{\min} > 0. \tag{1.6}$$

Now consider the equation

$$-\nabla\cdot(\kappa\nabla p) = f, \quad x \in D, \tag{1.7a}$$

$$p = 0, \quad x \in \partial D. \tag{1.7b}$$

Lax-Milgram theory yields the following:

Lemma 1.5. *Assume that $f \in V^*$ and that κ satisfies* (1.6)*. Then (1.7) has a unique weak solution $p \in V$. This solution satisfies*

$$\|p\|_V \leq \|f\|_{V^*}/\kappa_{\min}$$

and, if $f \in H$,

$$\|p\|_V \leq C_p\|f\|/\kappa_{\min}.$$

We will be interested in the inverse problem of finding κ from y where

$$y_j = l_j(p) + \eta_j, \quad j = 1, \cdots, J. \tag{1.8}$$

Here $l_j \in V^*$ is a continuous linear functional on V and η_j is a noise.

Notice that the unknown, $\kappa \in X$, is a function (infinite dimensional) whereas the data from which we wish to determine κ is finite dimensional: $y \in \mathbb{R}^J$. The problem is severely under-determined, illustrating point 2. from subsection 1.1. One way to treat such problems is by adopting the Bayesian framework, using prior modelling to fill-in missing information. We will take the unknown function to be u where either $u = \kappa$ or $u = \log \kappa$. In either case, we will define $G_j(u) = l_j(p)$ and, noting that p is then a nonlinear function of u, (1.8) may be written as

$$y = G(u) + \eta \tag{1.9}$$

where $y, \eta \in \mathbb{R}^J$ and $G : X^+ \subseteq X \to \mathbb{R}^J$. The set X^+ is introduced because G may not be defined on the whole of X. In particular, the positivity constraint (1.6) is only satisfied on

$$X^+ := \left\{ u \in X : \operatorname{ess\,inf}_{x \in D} u(x) > 0 \right\} \subset X \tag{1.10}$$

in the case where $\kappa = u$. On the other hand if $\kappa = \exp(u)$ then the positivity constraint (1.6) is satisfied for any $u \in X$ and we may take $X^+ = X$.

Notice that we again need probability measures on function space, here the Banach space $X = L^\infty(D)$. Furthermore, in the case where $u = \kappa$, these probability measures should charge only positive functions, in view of the desired inequality (1.6). Probability on Banach spaces of functions is most naturally developed in the setting of separable spaces, which $L^\infty(D)$ is not. This difficulty can be circumvented in various different ways as we describe in what follows.

1.4. Bibliographic Notes.

- Subsection 1.1. See [11] for a general overview of the Bayesian approach to statistics in the finite dimensional setting. The Bayesian approach to linear inverse problems with Gaussian noise and prior in finite dimensions is discussed in [93, Chapters 2 and 6] and, with a more algorithmic flavour, in the book [53].

- Subsection 1.2. For details on the heat equation as an ODE in Hilbert space, and the regularity estimates of Lemma 1.4, see [80, 70]. The classical approach to linear inverse problems is described in numerous books; see, for example, [51, 32]. The case where the spectrum of the forward map G decays exponentially, as arises for the heat equation, is sometimes termed *severely ill-posed*. The Bayesian approach to linear inverse problems was developed systematically in [71, 68], following from the seminal paper [36] in which the approach was first described; for further reading on ill-posed linear problems see [93, Chapters 3 and 6]. Recovering the truth underlying the data from the Bayesian approach, known as *Bayesian posterior consistency*, is the topic of [55, 3]; generalizations to severely ill-posed problems, such as the heat equation, may be found in [56, 4].

- Subsection 1.3. See [33] for the Lax-Milgram theory which gives rise to Lemma 1.5. For classical inversion theory for the elliptic inverse problem – determining the permeability from the pressure in a Darcy model of flow in a porous medium – see [87, 8]; for Bayesian formulations see [25, 24]. For posterior consistency results see [100].

2. Prior Modeling

In this section we show how to construct probability measures on a function space, adopting a constructive approach based on random series. As explained in section 6.2, the natural setting for probability in a function space is that of a separable Banach space. A countable infinite sequence in the Banach space X will be used for our random series; in the case where X is not separable the resulting probability measure will be constructed on a separable subspace X' of X (see the discussion in subsection 2.1).

Subsection 2.1 describes this general setting, and subsections 2.2, 2.3 and 2.4 consider, in turn, three classes of priors termed uniform, Besov and Gaussian. In subsection 2.5 we link the random series construction to the widely used random field perspective on spatial stochastic processes and we summarize in subsection 2.6. We denote the prior measures constructed in this section by μ_0.

2.1. General Setting. We let $\{\phi_j\}_{j=1}^{\infty}$ denote an infinite sequence in the Banach space X, with norm $\|\cdot\|$, of $\mathbb{R}$-valued functions defined on a domain D. We will either take $D \subset \mathbb{R}^d$, a bounded, open set with Lipschitz boundary, or $D = \mathbb{T}^d$ the d-dimensional torus. We normalize these functions so that $\|\phi_j\| = 1$ for $j = 1, \cdots, \infty$. We also introduce another element $m_0 \in X$, not necessarily normalized to 1. Define the function u by

$$u = m_0 + \sum_{j=1}^{\infty} u_j \phi_j. \tag{2.1}$$

By randomizing $\mathsf{u} := \{u_j\}_{j=1}^{\infty}$ we create real-valued random functions on D. (The extension to $\mathbb{R}^n$-valued random functions is straightforward, but omitted for brevity.)

We now define the deterministic sequence $\gamma = \{\gamma_j\}_{j=1}^{\infty}$ and the i.i.d. random sequence $\xi = \{\xi_j\}_{j=1}^{\infty}$, and set $u_j = \gamma_j \xi_j$. We assume that ξ is centred, i.e. that it has mean zero. Formally we see that the average value of u is then m_0 so that this element of X should be thought of as the *mean function*. We assume that $\gamma \in \ell^p_w$ for some $p \in [1, \infty)$ and some positive weight sequence $\{w_j\}$ (see subsection 7.1.1). We define $\Omega = \mathbb{R}^{\infty}$ and view ξ as a random element in the probability space $\big(\Omega, \mathsf{B}(\Omega), \mathbb{P}\big)$ of i.i.d. sequences equipped with the product σ-algebra; we let $\mathbb{E}$ denote expectation. This sigma algebra can be generated by cylinder sets if an

appropriate distance d is defined on sequences. However the distance d captures nothing of the properties of the random function u itself. For this reason we will be interested in the pushforward of the measure $\mathbb{P}$ on the measure space $\big(\Omega, \mathsf{B}(\Omega)\big)$ into a measure μ on $\big(X', \mathsf{B}(X')\big)$, where X' is a separable Banach space and $\mathsf{B}(X')$ denotes its Borel σ-algebra. Sometimes X' will be the same as X but not always: the space X may not be separable; and, although we have stated the normalization of the ϕ_j in X, they may of course live in smaller spaces X', and u may do so too. For either of these reasons X' may be a proper subspace of X.

In the next three subsections we demonstrate how this general setting may be adapted to create a variety of useful prior measures on function space; the fourth subsection, which follows these three, relates the random series construction, in the Gaussian case, to the standard construction of Gaussian random fields. We will express many of our results in terms of the probability measure $\mathbb{P}$ on i.i.d sequences, but all such results will, of course, have direct implications for the induced pushforward measures on the function spaces where the random functions u live. We discuss this perspective in the summary section 2.6. In dealing with the random series construction we will also find it useful to consider the truncated random functions

$$u^N = m_0 + \sum_{j=1}^{N} u_j \phi_j, \quad u_j = \gamma_j \xi_j. \tag{2.2}$$

2.2. Uniform Priors. To construct the random functions (2.1) we take $X = L^\infty(D)$, choose the deterministic sequence $\gamma = \{\gamma_j\}_{j=1}^\infty \in \ell^1$ and specify the i.i.d. sequence $\xi = \{\xi_j\}_{j=1}^\infty$ by $\xi_1 \sim U[-1,1]$, uniform random variables on $[-1,1]$. Assume further that there are finite, strictly positive constants $m_{\min}$, $m_{\max}$, and δ such that

$$\begin{aligned} &\operatorname{ess\,inf}_{x\in D} m_0(x) \geq m_{\min}; \\ &\operatorname{ess\,sup}_{x\in D} m_0(x) \leq m_{\max}; \\ &\|\gamma\|_{\ell^1} = \frac{\delta}{1+\delta} m_{\min}. \end{aligned}$$

The space X is not separable and so, instead, we work with the space X' found as the closure of the linear span of the functions $(m_0, \{\phi_j\}_{j=1}^\infty)$ with respect to the norm $\|\cdot\|_\infty$ on X. The Banach space $\big(X', \|\cdot\|_\infty\big)$ is separable.

Theorem 2.1. *The following holds $\mathbb{P}$-almost surely: the sequence of functions $\{u^N\}_{N=1}^\infty$ given by (2.2) is Cauchy in X' and the limiting function u given by (2.1) satisfies*

$$\frac{1}{1+\delta} m_{\min} \leq u(x) \leq m_{\max} + \frac{\delta}{1+\delta} m_{\min} \quad a.e. \quad x \in D.$$

Proof. Let $N > M$. Then, $\mathbb{P}$-a.s.,

$$\begin{aligned}\|u^N - u^M\|_\infty &= \Big\| \sum_{j=M+1}^{N} u_j \phi_j \Big\|_\infty \\ &\le \Big\| \sum_{j=M+1}^{N} \gamma_j \xi_j \phi_j \Big\|_\infty \\ &\le \sum_{j=M+1}^{\infty} |\gamma_j||\xi_j| \|\phi_j\|_\infty \\ &\le \sum_{j=M+1}^{\infty} |\gamma_j|.\end{aligned}$$

The right hand side tends to zero as $M \to \infty$ by the dominated convergence theorem and hence the sequence is Cauchy in X'.

We have $\mathbb{P}$-a.s. and for a.e. $x \in D$,

$$\begin{aligned}u(x) &\ge m_0(x) - \sum_{j=1}^{\infty} |u_j| \|\phi_j\|_\infty \\ &\ge \operatorname*{ess\,inf}_{x \in D} m_0(x) - \sum_{j=1}^{\infty} |\gamma_j| \\ &\ge m_{\min} - \|\gamma\|_{\ell^1} \\ &= \frac{1}{1+\delta} m_{\min}.\end{aligned}$$

Proof of the upper bound is similar. □

Example 2.2. Consider the random function (2.1) as specified in this section. By Theorem 2.1 we have that, $\mathbb{P}$-a.s.,

$$u(x) \ge \frac{1}{1+\delta} m_{\min} > 0, \quad \text{a.e.} \quad x \in D. \tag{2.3}$$

Set $\kappa = u$ in the elliptic equation (1.6), so that the coefficient κ in the equation and the solution p are random variables on $\big(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty), \mathbb{P}\big)$. Since (2.3) holds $\mathbb{P}$-a.s., Lemma 1.5 shows that, again $\mathbb{P}$-a.s.,

$$\|p\|_V \le (1+\delta)\|f\|_{V^*}/m_{\min}.$$

Since the r.h.s. is non-random we have that for all $r \in \mathbb{Z}^+$ the random variable $p \in L^r_{\mathbb{P}}(\Omega; V)$:

$$\mathbb{E}\|p\|_V^r < \infty.$$

In fact $\mathbb{E}\exp(\alpha\|p\|_V^r) < \infty$ for all $r \in \mathbb{Z}^+$ and $\alpha \in (0, \infty)$. □

We now consider the situation where the family $\{\phi_j\}_{j=1}^\infty$ have a uniform Hölder exponent α and study the implications for Hölder continuity of the random function u. Specifically we assume that there are $C, a > 0$ and $\alpha \in (0, 1]$ such that, for all $j \geq 1$,

$$|\phi_j(x) - \phi_j(y)| \leq Cj^a|x - y|^\alpha,\ x, y \in D. \tag{2.4}$$

and

$$|m_0(x) - m_0(y)| \leq C|x - y|^\alpha,\ x, y \in D. \tag{2.5}$$

Theorem 2.3. *Assume that u is given by (2.1) where the collection of functions $(m_0, \{\phi_j\}_{j=1}^\infty)$ satisfy (2.4) and (2.5). Assume further that $\sum_{j=1}^\infty |\gamma_j|^2 j^{a\theta} < \infty$ for some $\theta \in (0, 2)$. Then $\mathbb{P}$-a.s. we have $u \in C^{0,\beta}(D)$ for all $\beta < \frac{\alpha\theta}{2}$.*

Proof. This is an application of Corollary 7.22 of the Kolmogorov continuity theorem and S_1 and S_2 are as defined there. We use θ in place of the parameter δ appearing in Corollary 7.22 in order to avoid confusion with δ appearing in Theorem 2.1 above and in (2.7) below. Note that, since m_0 has assumed Hölder regularity α, which exceeds $\frac{\alpha\theta}{2}$ since $\theta \in (0, 2)$, it suffices to consider the centred case where $m_0 \equiv 0$. We let $f_j = \gamma_j\phi_j$ and complete the proof by noting that

$$S_1 = \sum_{j=1}^\infty |\gamma_j|^2 \leq S_2 \leq \sum_{j=1}^\infty |\gamma_j|^2 j^{a\theta} < \infty.$$

□

Example 2.4. Let $\{\phi_j\}$ denote the Fourier basis for $D = [0, 1]^d$. Then we may take $a = \alpha = 1$. If $\gamma_j = j^{-s}$ then $s > 1$ ensures $\gamma \in \ell^1$. Furthermore

$$\sum_{j=1}^\infty |\gamma_j|^2 j^{a\theta} = \sum_{j=1} j^{\theta-2s} < \infty$$

for $\theta < 2s - 1$. We thus deduce that $u \in C^{0,\beta}([0, 1]^d)$ for all $\beta < \min\{s - \frac{1}{2}, 1\}$.

2.3. Besov Priors.

For this construction of random functions we take X to be the Hilbert space

$$X := \dot{L}^2(\mathbb{T}^d) = \Big\{u : \mathbb{T}^d \to \mathbb{R}\Big| \int_{\mathbb{T}^d} |u(x)|^2 dx < \infty, \int_{\mathbb{T}^d} u(x)dx = 0\Big\}$$

of real valued periodic functions in dimension $d \leq 3$ with inner-product and norm denoted by $\langle\cdot, \cdot\rangle$ and $\|\cdot\|$ respectively. We then set $m_0 = 0$ and let $\{\phi_j\}_{j=1}^\infty$ be an orthonormal basis for X. Consequently, for any $u \in X$, we have for a.e. $x \in \mathbb{T}^d$,

$$u(x) = \sum_{j=1}^\infty u_j\phi_j(x), \quad u_j = \langle u, \phi_j\rangle. \tag{2.6}$$

Given a function $u : \mathbb{T}^d \to \mathbb{R}$ and the $\{u_j\}$ as defined in (2.6) we define the Banach space $X^{t,q}$ by

$$X^{t,q} = \Big\{ u : \mathbb{T}^d \to \mathbb{R} \Big| \|u\|_{X^{t,q}} < \infty, \int_{\mathbb{T}^d} u(x)dx = 0 \Big\}$$

where

$$\|u\|_{X^{t,q}} = \Big(\sum_{j=1}^{\infty} j^{(\frac{tq}{d} + \frac{q}{2} - 1)} |u_j|^q \Big)^{\frac{1}{q}}$$

with $q \in [1, \infty)$ and $s > 0$. If $\{\phi_j\}$ form the Fourier basis and $q = 2$ then $X^{t,2}$ is the Sobolev space $\dot{H}^t(\mathbb{T}^d)$ of mean-zero periodic functions with t (possibly non-integer) square-integrable derivatives; in particular $X^{0,2} = \dot{L}^2(\mathbb{T}^d)$. On the other hand, if the $\{\phi_j\}$ form certain wavelet bases, then $X^{t,q}$ is the Besov space B^t_{qq}.

As described above, we assume that $u_j = \gamma_j \xi_j$ where $\xi = \{\xi_j\}_{j=1}^{\infty}$ is an i.i.d. sequence and $\gamma = \{\gamma_j\}_{j=1}^{\infty}$ is deterministic. Here we assume that ξ_1 is drawn from the centred measure on $\mathbb{R}$ with density proportional to $\exp\big(-\frac{1}{2}|x|^q\big)$ for some $1 \le q < \infty$ – we refer to this as a *q-exponential distribution*, noting that $q = 2$ gives a Gaussian and $q = 1$ a Laplace-distributed random variable. Then for $s > 0$ and $\delta > 0$ we define

$$\gamma_j = j^{-(\frac{s}{d} + \frac{1}{2} - \frac{1}{q})} \big(\frac{1}{\delta}\big)^{\frac{1}{q}}. \tag{2.7}$$

The parameter δ is a key scaling parameter which will appear in the statement of exponential moment bounds below.

We now prove convergence of the series (found from (2.2) with $m_0 = 0$)

$$u^N = \sum_{j=1}^{N} u_j \phi_j, \quad u_j = \gamma_j \xi_j \tag{2.8}$$

to the limit function

$$u(x) = \sum_{j=1}^{\infty} u_j \phi_j(x), \quad u_j = \gamma_j \xi_j, \tag{2.9}$$

in an appropriate space. To understand the sequence of functions $\{u^N\}$, it is useful to introduce the following function space:

$$L^q_{\mathbb{P}}(\Omega; X^{t,q}) := \Big\{ v : D \times \Omega \to \mathbb{R} \Big| \mathbb{E}\big(\|v\|^q_{X^{t,q}}\big) < \infty \Big\}.$$

This is a Banach space, when equipped with the norm $\Big(\mathbb{E}\big(\|v\|_{X^{t,q}}\big)^q\Big)^{\frac{1}{q}}$. Thus every Cauchy sequence is convergent in this space.

Theorem 2.5. *For $t < s - \frac{d}{q}$ the sequence of functions $\{u^N\}_{N=1}^{\infty}$, given by (2.8) and (2.7) with ξ_1 drawn from a centred q-exponential distribution, is Cauchy in the Banach space $L^q_{\mathbb{P}}(\Omega; X^{t,q})$. Thus the infinite series* (2.9) *exists as an $L^q_{\mathbb{P}}$-limit and takes values in $X^{t,q}$ almost surely, for all $t < s - \frac{d}{q}$.*

Proof. For $N > M$,

$$\begin{aligned}\mathbb{E}\|u^N - u^M\|^q_{X^{t,q}} &= \delta^{-1}\mathbb{E}\sum_{j=M+1}^{N} j^{\frac{(t-s)q}{d}}|\xi_j|^q \\ &\asymp \sum_{j=M+1}^{N} j^{\frac{(t-s)q}{d}} \le \sum_{j=M+1}^{\infty} j^{\frac{(t-s)q}{d}}.\end{aligned}$$

The sum on the right hand side tends to 0 as $M \to \infty$, provided $\frac{(t-s)q}{d} < -1$, by the dominated convergence theorem. This completes the proof. □

The previous theorem gives a sufficient condition, on t, for existence of the limiting random function. The following theorem refines this to an if and only if statement, in the context of almost sure convergence.

Theorem 2.6. *Assume that u is given by* (2.9) *and* (2.7) *with ξ_1 drawn from a centred q-exponential distribution. Then the following are equivalent:*

i) $\|u\|_{X^{t,q}} < \infty$ $\mathbb{P}$-a.s.;

ii) $\mathbb{E}\big(\exp(\alpha\|u\|^q_{X^{t,q}})\big) < \infty$ for any $\alpha \in [0, \frac{\delta}{2})$;

iii) $t < s - \frac{d}{q}$.

Proof. We first note that, for the random function in question,

$$\|u\|^q_{X^{t,q}} = \sum_{j=1}^{\infty} j^{(\frac{tq}{d}+\frac{q}{2}-1)}|u_j|^q = \sum_{j=1}^{\infty} \delta^{-1} j^{-\frac{(s-t)q}{d}}|\xi_j|^q.$$

Now, for $\alpha < \frac{1}{2}$,

$$\begin{aligned}\mathbb{E}\exp\big(\alpha|\xi_1|^q\big) &= \int_{\mathbb{R}} \exp\Big(-(\frac{1}{2}-\alpha)|x|^q\Big)dx \Big/ \int_{\mathbb{R}} \exp\Big(-\frac{1}{2}|x|^q\Big)dx \\ &= (1-2\alpha)^{-\frac{1}{q}}.\end{aligned}$$

iii) ⇒ ii).

$$\begin{aligned}\mathbb{E}\Big(\exp(\alpha\|u\|^q_{X^{t,q}})\Big) &= \mathbb{E}\Big(\exp(\alpha\sum_{j=1}^{\infty}\delta^{-1}j^{-\frac{(s-t)q}{d}}|\xi_j|^q)\Big) \\ &= \prod_{j=1}^{\infty}\Big(1-\frac{2\alpha}{\delta}j^{-\frac{(s-t)q}{d}}\Big)^{-\frac{1}{q}}.\end{aligned}$$

For $\alpha < \frac{\delta}{2}$ the product converges if $\frac{(s-t)q}{d} > 1$ i.e. $t < s - \frac{d}{q}$ as required.

ii) ⇒ i).

If (i) does not hold, $Z := \|u\|^q_{X^{t,q}}$ is positive infinite on a set of positive measure S. Then, since for $\alpha > 0$, $\exp(\alpha Z) = +\infty$ if $Z = +\infty$, and $\mathbb{E}\exp(\alpha Z) \geq \mathbb{E}(\mathbb{1}_S \exp(\alpha Z))$ we get a contradiction.

i) $\Rightarrow$ iii).

To show that (i) implies (iii) note that (i) implies that, almost surely,

$$\sum_{j=1}^{\infty} j^{(t-s)q/d}|\xi_j|^q < \infty.$$

This implies that $t < s$. To see this assume for contradiction that $t \geq s$. Then, almost surely,

$$\sum_{j=1}^{\infty} |\xi_j|^q < \infty.$$

Since there is a constant $c > 0$ with $\mathbb{E}|\xi_j|^q = c$ for any $j \in \mathbb{N}$, this contradicts the law of large numbers.

Now define $\zeta_j = j^{(t-s)q/d}|\xi_j|^q$. Using the fact that the ζ_j are non-negative and independent we deduce from Lemma 2.7 (below) that

$$\sum_{j=1}^{\infty} \mathbb{E}\big(\zeta_j \wedge 1\big) = \sum_{j=1}^{\infty} \mathbb{E}\Big(j^{(t-s)q/d}|\xi_j|^q \wedge 1\Big) < \infty.$$

Since $t < s$ we note that then

$$\begin{aligned}
\mathbb{E}\zeta_j &= \mathbb{E}\Big(j^{-(s-t)q/d}|\xi_j|^q\Big) \\
&= \mathbb{E}\Big(j^{-(s-t)q/d}|\xi_j|^q \mathbb{I}_{\{|\xi_j| \leq j^{(s-t)/d}\}}\Big) + \mathbb{E}\Big(j^{-(s-t)q/d}|\xi_j|^q \mathbb{I}_{\{|\xi_j| > j^{(s-t)/d}\}}\Big) \\
&\leq \mathbb{E}\Big(\big(\zeta_j \wedge 1\big)\mathbb{I}_{\{|\xi_j| \leq j^{(s-t)/d}\}}\Big) + I \\
&\leq \mathbb{E}\Big(\zeta_j \wedge 1\Big) + I,
\end{aligned}$$

where

$$I \propto j^{-(s-t)q/d} \int_{j^{(s-t)/d}}^{\infty} x^q e^{-x^q/2} dx.$$

Noting that, since $q \geq 1$, the function $x \mapsto x^q e^{-x^q/2}$ is bounded, up to a constant of proportionality, by the function $x \mapsto e^{-\alpha x}$ for any $\alpha < \frac{1}{2}$, we see that there is a positive constant K such that

$$\begin{aligned}
I &\leq K j^{-(s-t)q/d} \int_{j^{(s-t)/d}}^{\infty} e^{-\alpha x} dx \\
&= \frac{1}{\alpha} K j^{-(s-t)q/d} \exp\big(-\alpha j^{(s-t)/d}\big) \\
&:= \iota_j.
\end{aligned}$$

Thus we have shown that

$$\sum_{j=1}^{\infty}\mathbb{E}\Big(j^{-(s-t)q/d}|\xi_j|^q\Big)\le\sum_{j=1}^{\infty}\mathbb{E}\Big(\zeta_j\wedge 1\Big)+\sum_{j=1}^{\infty}\iota_j<\infty.$$

Since the ξ_j are i.i.d. this implies that

$$\sum_{j=1}^{\infty}j^{(t-s)q/d}<\infty,$$

from which it follows that $(s-t)q/d>1$ and (iii) follows. □

Lemma 2.7. *Let $\{I_j\}_{j=1}^{\infty}$ be an independent sequence of $\mathbb{R}^+$-valued random variables. Then*

$$\sum_{j=1}^{\infty}I_j<\infty\quad a.s.\Leftrightarrow\sum_{j=1}^{\infty}\mathbb{E}(I_j\wedge 1)<\infty.$$

As in the previous subsection, we now study the situation where the family $\{\phi_j\}$ have a uniform Hölder exponent α and study the implications for Hölder continuity of the random function u. In this case, however, the basis functions are normalized in $\dot{L}^2$ and not L^∞; thus we must make additional assumptions on the possible growth of the L^∞ norms of $\{\phi_j\}$ with j. We assume that there are $C,a,b>0$ and $\alpha\in(0,1]$ such that, for all $j\ge 0$,

$$|\phi_j(x)|=\beta_j\le Cj^b,\ x\in D. \tag{2.10a}$$

$$|\phi_j(x)-\phi_j(y)|\le Cj^a|x-y|^\alpha,\ x,y\in D. \tag{2.10b}$$

We also assume that $a>b$ as, since $\|\phi_j\|_{L^2}=1$, it is natural that the pre-multiplication constant in the Hölder estimate on the $\{\phi_j\}$ grows in j at least as fast as the bound on the functions themselves.

Theorem 2.8. *Assume that u is given by* (2.9) *and* (2.7) *with ξ_1 drawn from a centred q-exponential distribution. Suppose also that* (2.10) *hold and that $s>d\big(b+q^{-1}+\frac{1}{2}\theta(a-b)\big)$ for some $\theta\in(0,2)$. Then $\mathbb{P}$-a.s. we have $u\in C^{0,\beta}(\mathbb{T}^d)$ for all $\beta<\frac{\alpha\theta}{2}$.*

Proof. We apply Corollary 7.22 of the Kolmogorov continuity theorem and S_1 and S_2 are as defined there. We use θ in place of the parameter δ appearing in Corollary 7.22 in order to avoid confusion with δ appearing in Theorem 2.1 and (2.7) above. Let $f_j=\gamma_j\phi_j$ and note that

$$S_1=\sum_{j=1}^{\infty}|\gamma_j|^2\beta_j^2\lesssim\sum_{j=1}^{\infty}j^{-c_1}$$

$$S_2=\sum_{j=1}^{\infty}|\gamma_j|^{2-\theta}\beta_j^{2-\theta}\gamma_j^\theta j^{a\theta}\lesssim\sum_{j=1}^{\infty}j^{-c_2}.$$

Short calculation shows that

$$c_1 = \frac{2s}{d} + 1 - \frac{2}{q} - 2b,$$
$$c_2 = \frac{2s}{d} + 1 - \frac{2}{q} - 2b - \theta(a-b).$$

We require $c_1 > 1$ and $c_2 > 1$ and since $a > b$ satisfaction of the second of these will imply the first. Satisfaction of the second gives the desired lower bound on s. □

We note that the result of Theorem 2.8 holds true when the mean function is nonzero if it satisfies

$$|m_0(x)| \leq C,\ x \in D.$$
$$|m_0(x) - m_0(y)| \leq C|x-y|^\alpha,\ x, y \in D.$$

We have the following sharper result if the family $\{\phi_j\}$ is regular enough to be a basis for B^t_{qq} instead of satisfying (2.10):

Theorem 2.9. *Assume that u is given by (2.9) and (2.7) with ξ_1 drawn from a centred q-exponential distribution. Suppose also that $\{\phi_j\}_{j\in\mathbb{N}}$ form a basis for B^t_{qq} for some $t < s - \frac{d}{q}$. Then $u \in C^{0,t}(\mathbb{T}^d)$ $\mathbb{P}$-almost surely.*

Proof. For any $m \geq 1$, using the definition of $X^{t,q}$-norm we can write

$$\|u\|^{mq}_{B^t_{mq,mq}} = (\tfrac{1}{\delta})^m \sum_{j=1}^{\infty} j^{\frac{mqt}{d} + \frac{mq}{2} - 1} j^{-mq(\frac{s}{d} + \frac{1}{2} - \frac{1}{q})} |\xi_j|^{mq}.$$

For every $m \in \mathbb{N}$ there exists a constant C_m with $\mathbb{E}|\xi_j|^{mq} = C_m$. Since each term of the above series is measurable we can swap the sum and the integration and write

$$\mathbb{E}\|u\|^{mq}_{B^t_{mq,mq}} = C_m(\tfrac{1}{\delta})^m \sum_{j=1}^{\infty} j^{\frac{mq}{d}(t-s)+m-1} \leq \tilde{C}_m,$$

noting that the exponent of j is smaller than -1 (since $t < s - d/q$). Now for a given $t < s - d/q$, one can choose m large enough so that $\frac{d}{mq} < s - d/q - t$. Then the embedding $B^{t_1}_{mq,mq} \subset C^t$ for any t_1 satisfying $t + \frac{d}{mq} < t_1 < s - d/q$ implies that $\mathbb{E}\|u\|^{mq}_{C^t(\mathbb{T}^d)} < \infty$. It follows that $u \in C^t$ $\mathbb{P}$-almost surely.

□

If the mean function m_0 is t-Hölder continuous, the result of the above theorem holds for a random series with nonzero mean function as well.

2.4. Gaussian Priors. Let X be a Hilbert space $\mathcal{H}$ of real-valued functions on bounded open $D \subset \mathbb{R}^d$ with Lipschitz boundary, and with inner-product and norm denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ respectively; for example $\mathcal{H} = L^2(D; \mathbb{R})$. Assume that $\{\phi_j\}_{j=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$. We study the Gaussian case where $\xi_1 \sim N(0,1)$ and then equation (2.1) with $u_j = \gamma_j \xi_j$ generates random draws from the Gaussian measure $N(m_0, \mathcal{C})$ on $\mathcal{H}$ where the covariance operator $\mathcal{C}$ depends on the sequence $\gamma = \{\gamma_j\}_{j=1}^{\infty}$. See the Appendix for background on Gaussian measures in a Hilbert space. As in Section 2.3, we consider the setting in which $m_0 = 0$ so that the function u is given by (2.6) and has mean zero. We thus focus on identifying $\mathcal{C}$ from the random series (2.6), and studying the regularity of random draws from $N(0, \mathcal{C})$.

Define the Hilbert scale of spaces $\mathcal{H}^t$ as in Subsection 7.1.3 with, recall, norm

$$\|u\|_{\mathcal{H}^t}^2 = \sum_{j=1}^{\infty} j^{\frac{2t}{d}} |u_j|^2.$$

We choose $\xi_1 \sim N(0,1)$ and study convergence of the series (2.8) for u^N to a limit function u given by (2.9); the spaces in which this convergence occurs will depend upon the sequence γ. To understand the sequence of functions $\{u^N\}$, it is useful to introduce the following function space:

$$L^2_{\mathbb{P}}(\Omega; \mathcal{H}^t) := \Big\{ v : D \times \Omega \to \mathbb{R} \Big| \mathbb{E}\big(\|v\|_{\mathcal{H}^t}\big)^2 < \infty \Big\}.$$

This is in fact a Hilbert space, although we will not use the Hilbert space structure. We will only use the fact that $L^2_{\mathbb{P}}$ is a Banach space when equipped with the norm $\Big(\mathbb{E}\big(\|v\|_{\mathcal{H}^t}^2\big)\Big)^{\frac{1}{2}}$ and that hence every Cauchy sequence is convergent.

Theorem 2.10. *Assume that $\gamma_j \asymp j^{-\frac{s}{d}}$. Then the sequence of functions $\{u^N\}_{N=1}^{\infty}$ given by* (2.8) *is Cauchy in the Hilbert space $L^2_{\mathbb{P}}(\Omega; \mathcal{H}^t)$, $t < s - \frac{d}{2}$. Thus the infinite series* (2.9) *exists as an $L^2_{\mathbb{P}}$-limit and takes values in $\mathcal{H}^t$ almost surely, for $t < s - \frac{d}{2}$.*

Proof. For $N > M$,

$$\begin{aligned} \mathbb{E}\|u^N - u^M\|_{\mathcal{H}^t}^2 &= \mathbb{E} \sum_{j=M+1}^{N} j^{\frac{2t}{d}} |u_j|^2 \\ &\asymp \sum_{j=M+1}^{N} j^{\frac{2(t-s)}{d}} \leq \sum_{j=M+1}^{\infty} j^{\frac{2(t-s)}{d}}. \end{aligned}$$

The sum on the right hand side tends to 0 as $M \to \infty$, provided $\frac{2(t-s)}{d} < -1$, by the dominated convergence theorem. This completes the proof. □

Remarks 2.11. We make the following remarks concerning the Gaussian random functions constructed in the preceding theorem.

- The preceding theorem shows that the sum (2.8) has an $L^2_{\mathbb{P}}$ limit in $\mathcal{H}^t$ when $t < s - d/2$, as one can also see from the following direct calculation

$$\begin{aligned}\mathbb{E}\|u\|^2_{\mathcal{H}^t} &= \sum_{j=1}^{\infty} j^{\frac{2t}{d}}\,\mathbb{E}(\gamma_j^2\xi_j^2) \\ &= \sum_{j=1}^{\infty} j^{\frac{2t}{d}}\gamma_j^2 \\ &\asymp \sum_{j=1}^{\infty} j^{\frac{2(t-s)}{d}} < \infty.\end{aligned}$$

  Thus $u \in \mathcal{H}^t$ a.s., for $t < s - \frac{d}{2}$.

- From the preceding theorem we see that, provided $s > \frac{d}{2}$, the random function in (2.9) generates a mean zero Gaussian measure on $\mathcal{H}$. The expression (2.9) is known as the *Karhunen-Loève expansion*, and the eigenfunctions $\{\phi_j\}_{j=1}^{\infty}$ as the *Karhunen-Loève basis*.

- The covariance operator $\mathcal{C}$ of a measure μ on $\mathcal{H}$ may then be viewed as a bounded linear operator from $\mathcal{H}$ into itself defined to satisfy

$$\mathcal{C}\ell = \int_{\mathcal{H}} \langle \ell, u\rangle u\, \mu(du)\,, \tag{2.14}$$

  for all $\ell \in \mathcal{H}$. Thus

$$\mathcal{C} = \int_{\mathcal{H}} u \otimes u\, \mu(du)\,. \tag{2.15}$$

  The following formal calculation, which can be made rigorous if $\mathcal{C}$ is trace-class on $\mathcal{H}$, gives an expression for the covariance operator:

$$\begin{aligned}\mathcal{C} &= \mathbb{E}u \otimes u \\ &= \mathbb{E}\Big(\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \gamma_j\gamma_k\xi_j\xi_k\phi_j \otimes \phi_k\Big) \\ &= \Big(\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \gamma_j\gamma_k\mathbb{E}(\xi_j\xi_k)\phi_j \otimes \phi_k\Big) \\ &= \Big(\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \gamma_j\gamma_k\delta_{jk}\phi_j \otimes \phi_k\Big) \\ &= \sum_{j=1}^{\infty} \gamma_j^2\phi_j \otimes \phi_j.\end{aligned}$$

From this expression for the covariance, we may find eigenpairs explicitly:

$$\begin{aligned} \mathcal{C}\phi_k &= \Big(\sum_{j=1}^{\infty} \gamma_j^2 \phi_j \otimes \phi_j\Big)\phi_k \\ &= \sum_{j=1}^{\infty} \gamma_j^2 \langle \phi_j, \phi_k\rangle \phi_j = \sum_{j=1}^{\infty} \gamma_j^2 \delta_{jk}\phi_k = \gamma_k^2 \phi_k. \end{aligned}$$

- The Gaussian measure is denoted by $\mu_0 := N(0, \mathcal{C})$, a Gaussian with mean function 0 and covariance operator $\mathcal{C}$. The eigenfunctions of $\mathcal{C}$, $\{\phi_j\}_{j=1}^{\infty}$, are known as the *Karhunen-Loève* basis for measure μ_0. The γ_j^2 are the eigenvalues associated with this eigenbasis, and thus γ_j is the standard deviation of the Gaussian measure in the direction ϕ_j.

In the case where $\mathcal{H} = \dot{L}^2(\mathbb{T}^d)$ we are in the setting of Section 2.3 and we briefly consider this case. We assume that the $\{\phi_j\}_{j=1}^{\infty}$ constitute the Fourier basis. Let $A = -\triangle$ denote the negative Laplacian equipped with periodic boundary conditions on $[0,1)^d$, and restricted to functions which integrate to zero over $[0,1)^d$. This operator is positive self-adjoint and has eigenvalues which grow like $j^{2/d}$, analogously to the Assumption 1.3 made in the case of Dirichlet boundary conditions. It then follows that $\mathcal{H}^t = \mathcal{D}(A^{t/2}) = \dot{H}^t(\mathbb{T}^d)$, the Sobolev space of periodic functions on $[0,1)^d$ with spatial mean equal to zero and t (possibly negative or fractional) square integrable derivatives. Thus, by the preceding Remarks 2.11, u defined by (2.9) is in the space $\dot{H}^t$ a.s., $t < s - \frac{d}{2}$. In fact we can say more about regularity, using the Kolmogorov continuity test and Corollary 7.20; this we now do.

Theorem 2.12. *Consider the Karhunen-Loève expansion* (2.9) *so that u is a sample from the measure $N(0,\mathcal{C})$ in the case where $\mathcal{C} = A^{-s}$ with $A = -\triangle$, $\mathcal{D}(A) = \dot{H}^2(\mathbb{T}^d)$ and $s > \frac{d}{2}$. Then, $\mathbb{P}$-a.s., $u \in \dot{H}^t$, $t < s - \frac{d}{2}$, and $u \in C^{0,t}(\mathbb{T}^d)$ a.s., $t < 1 \wedge (s - \frac{d}{2})$.*

Proof. Because of the stated properties of the eigenvalues of the Laplacian, it follows that the eigenvalues of $\mathcal{C}$ satisfy $\gamma_j^2 \asymp j^{-\frac{2s}{d}}$ and the eigenbasis $\{\phi_j\}$ is the Fourier basis. Thus we may apply the conclusions stated in Remarks 2.11 to deduce that $u \in \dot{H}^t$, $t < \alpha - \frac{d}{2}$. Furthermore we may apply Corollary 7.22 to obtain Hölder regularity of u. To do this we note that the $\{\phi_j\}$ are bounded in $L^\infty(\mathbb{T}^d)$, and are Lipschitz with constants which grow like $j^{1/d}$. We apply that corollary with $\alpha = 1$ and obtain

$$S_1 = \sum_{j=1}^{\infty} \gamma_j^2, \quad S_2 = \sum_{j=1}^{\infty} \gamma_j^2 j^{\delta/d}.$$

The corollary delivers the desired result after noting that any $\delta < 2s - d$ will make S_2, and hence S_1, summable. □

The previous example illustrates the fact that, although we have constructed Gaussian measures in a Hilbert space setting, and that they are naturally defined

on a range of Hilbert (Sobolev-like) spaces defined through fractional powers of the Laplacian, they may also be defined on Banach spaces, such as the space of Hölder continuous functions. We now return to the setting of the general domain D, rather than the d-dimensional torus. In this general context it is important to highlight the Fernique Theorem, here restated from the Appendix because of its importance:

Theorem 2.13 (Fernique Theorem). *Let μ_0 be a Gaussian measure on the separable Banach space X. Then there exists $\beta_c \in (0, \infty)$ such that, for all $\beta \in (0, \beta_c)$,*

$$\mathbb{E}^{\mu_0} \exp\left(\beta \|u\|_X^2\right) < \infty.$$

Remarks 2.14. We make two remarks concerning the Fernique Theorem.

- Theorem 2.13, when combined with Theorem 2.12, shows that, with β sufficiently small, $\mathbb{E}^{\mu_0} \exp\left(\beta \|u\|_X^2\right) < \infty$ for both $X = \dot{H}^t$ and $X = C^{0,t}(\mathbb{T}^d)$, if $t < s - \frac{d}{2}$.
- Let $\mu_0 = N(0, A^{-s})$ where A is as in Theorem 2.12. Then Theorem 2.6 proves the Fernique Theorem 2.13 for $X = X^{t,2} = \dot{H}^t$, if $t < s - \frac{d}{2}$; the proof in the case of the torus is very different from the general proof of the result in the abstract setting of Theorem 2.13.
- Theorem 2.6ii) gives, in the Gaussian case, the Fernique Theorem in the case that X is the Hilbert space $X^{t,2}$. Furthermore, the constant β_c is specified explicitly in that setting. More explicit versions of the general Fernique Theorem 2.13 are possible, but the characterization of β_c is more involved.

Example 2.15. Consider the random function (2.1) in the case where $\mathcal{H} = \dot{L}^2(\mathbb{T}^d)$ and $\mu_0 = N(0, A^{-s})$, $s > \frac{d}{2}$ as in the preceding example. Then we know that, μ_0-a.s., $u \in C^{0,t}$, $t < 1 \wedge (s - \frac{d}{2})$. Set $\kappa = e^u$ in the elliptic PDE (1.7) so that the coefficient κ, and hence the solution p, are random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then $\kappa_{\min}$ given in (1.4) satisfies

$$\kappa_{\min} \geq \exp\big(-\|u\|_\infty\big).$$

By Lemma 1.5 we obtain

$$\|p\|_V \leq \exp\big(\|u\|_\infty\big)\|f\|_{V^*}.$$

Since $C^{0,t} \subset L^\infty(\mathbb{T}^d)$, $t \in (0,1)$, we deduce that,

$$\|u\|_{L^\infty} \leq K_1 \|u\|_{C^{0,t}}.$$

Furthermore, for any $\epsilon > 0$, there is constant $K_2 = K_2(\epsilon)$ such that $\exp(K_1 r x) \leq K_2 \exp(\epsilon x^2)$ for all $x \geq 0$. Thus

$$\begin{aligned}\|p\|_V^r &\leq \exp\left(K_1 r \|u\|_{C^{0,t}}\right)\|f\|_{V^*}^r \\ &\leq K_2 \exp\left(\epsilon \|u\|_{C^{0,t}}^2\right)\|f\|_{V^*}^r.\end{aligned}$$

Hence, by Theorem 2.13, we deduce that

$$\mathbb{E}\|p\|_V^r < \infty, \quad \text{i.e.} \quad p \in L^r_{\mathbb{P}}(\Omega; V) \quad \forall\, r \in \mathbb{Z}^+.$$

This result holds for any $r \geq 0$. Thus, when the coefficient of the elliptic PDE is *log-normal*, that is κ is the exponential of a Gaussian function, moments of all orders exist for the random variable p. However, unlike the case of the uniform prior, we cannot obtain exponential moments on $\mathbb{E}\exp(\alpha\|p\|_V^r)$ for any $(r, \alpha) \in \mathbb{Z}^+ \times (0, \infty)$. This is because the coefficient κ, whilst positive a.s., does not satisfy a uniform positive lower bound across the probability space. □

2.5. Random Field Perspective. In this subsection we link the preceding constructions of random functions, through randomized series, to the notion of *random fields*. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, with expectation denoted by $\mathbb{E}$, and $D \subseteq \mathbb{R}^d$ an open set. For the random series constructions developed in the preceding subsections $\Omega = \mathbb{R}^\infty$ and $\mathcal{F} = \mathsf{B}(\Omega)$; however the development of the general theory of random fields does not require this specific choice. A *random field* on D is a measurable mapping $u : D \times \Omega \to \mathbb{R}^n$. Thus, for any $x \in D$, $u(x; \cdot)$ is an $\mathbb{R}^n$-valued random variable; on the other hand, for any $\omega \in \Omega$, $u(\cdot; \omega) : D \to \mathbb{R}^n$ is a vector field. In the construction of random fields it is commonplace to first construct the *finite dimensional distributions*. These are found by choosing any integer $K \geq 1$, and any set of points $\{x_k\}_{k=1}^K$ in D, and then considering the random vector $(u(x_1; \cdot)^*, \cdots, u(x_K; \cdot)^*)^* \in \mathbb{R}^{nK}$. From the finite dimensional distributions of this collection of random vectors we would like to be able to make sense of the probability measure μ on X, a separable Banach space equipped with the Borel σ-algebra $\mathsf{B}(X)$, via the formula

$$\mu(A) = \mathbb{P}(u(\cdot; \omega) \in A), \quad A \in \mathsf{B}(X), \tag{2.16}$$

where ω is taken from a common probability space on which the random element $u \in X$ is defined. It is thus necessary to study the joint distribution of a set of K $\mathbb{R}^n$-valued random variables, all on a common probability space. Such $\mathbb{R}^{nK}$-valued random variables are, of course, only defined up to a set of zero measure. It is desirable that all such finite dimensional distributions are defined on a common subset $\Omega_0 \subset \Omega$ with full measure, so that u may be viewed as a function $u : D \times \Omega_0 \to \mathbb{R}^n$; such a choice of random field is termed a *modification* When reinterpreting the previous subsections in terms of random fields, statements about almost sure (regularity) properties should be viewed as statements concerning the existence of a modification possessing the stated almost sure regularity property.

We may define the space of functions

$$L^q_{\mathbb{P}}(\Omega; X) := \Big\{ v : D \times \Omega \to \mathbb{R}^n \Big| \mathbb{E}\big(\|v\|_X^q\big) < \infty \Big\}.$$

This is a Banach space, when equipped with the norm $\Big(\mathbb{E}\big(\|v\|_X^q\big)\Big)^{\frac{1}{q}}$. We have used such spaces in the preceding subsections when demonstrating convergence of the

randomized series. Note that we often simply write $u(x)$, suppressing the explicit dependence on the probability space.

A *Gaussian random field* is one where, for any integer $K \geq 1$, and any set of points $\{x_k\}_{k=1}^K$ in D, the random vector $(u(x_1;\cdot)^*, \cdots, u(x_K;\cdot)^*)^* \in \mathbb{R}^{nK}$ is a Gaussian random vector. The *mean function* of a Gaussian random field is $m(x) = \mathbb{E}u(x)$. The *covariance function* is $c(x,y) = \mathbb{E}\big(u(x) - m(x)\big)\big(u(y) - m(y)\big)^*$. For Gaussian random fields the mean function $m : D \to \mathbb{R}^n$ and the covariance function $c : D \times D \to \mathbb{R}^{n\times n}$ together completely specify the joint probability distribution for $(u(x_1;\cdot)^*, \cdots, u(x_K)^*)^* \in \mathbb{R}^{nK}$. Furthermore, if we view the Gaussian random field as a Gaussian measure on $L^2(D;\mathbb{R}^n)$ then the covariance operator can be constructed from the covariance function as follows. Without loss of generality we consider the mean zero case; the more general case follows by shift of origin. Since the field has mean zero we have, from (2.14), that for all $h_1, h_2 \in L^2(D;\mathbb{R}^n)$,

$$\begin{aligned}
\langle h_1, \mathcal{C}h_2\rangle &= \mathbb{E}\langle h_1, u\rangle\langle u, h_2\rangle \\
&= \mathbb{E}\int_D\int_D h_1(x)^*\big(u(x)u(y)^*\big)h_2(y)dydx \\
&= \mathbb{E}\int_D h_1(x)^*\Big(\int_D\big(u(x)u(y)^*\big)h_2(y)dy\Big)dx \\
&= \int_D h_1(x)^*\Big(\int_D c(x,y)h_2(y)dy\Big)dx
\end{aligned}$$

and we deduce that, for all $\psi \in L^2(D;\mathbb{R}^n)$,

$$\big(\mathcal{C}\psi\big)(x) = \int_D c(x,y)\psi(y)dy. \tag{2.17}$$

Thus the covariance operator of a Gaussian random field is an integral operator with kernel given by the covariance function. As such we may also view the covariance function as the Green's function of the inverse covariance, or *precision*.

A mean-zero Gaussian random field is termed *stationary* if $c(x,y) = s(x-y)$ for some matrix-valued function s, so that shifting the field by a fixed random vector does not change the statistics. It is *isotropic* if it is stationary and, in addition, $s(\cdot) = \iota(|\cdot|)$, for some matrix-valued function ι.

In the previous subsection we demonstrated how the regularity of random fields maybe established from the properties of the sequences γ (deterministic, with decay) and ξ (i.i.d. random). Here we show similar results but express them in terms of properties of the covariance function and covariance operator.

Theorem 2.16. *Consider an $\mathbb{R}^n$-valued Gaussian random field u on $D \subset \mathbb{R}^d$ with mean zero and with isotropic correlation function $c : D \times D \to \mathbb{R}^{n\times n}$. Assume that D is bounded and that $\operatorname{Tr} c(x,y) = k\big(|x-y|\big)$ where $k : \mathbb{R}^+ \to \mathbb{R}$ is Hölder with any exponent $\alpha \leq 1$. Then u is almost surely Hölder continuous on D with any exponent smaller than $\frac{1}{2}\alpha$.*

Proof. We have

$$\begin{aligned}\mathbb{E}|u(x)-u(y)|^2 &= \mathbb{E}|u(x)|^2 + \mathbb{E}|u(y)|^2 - 2\mathbb{E}\langle u(x), u(y)\rangle \\ &= \mathrm{Tr}\Big(c(x,x)+c(y,y)-2c(x,y)\Big) \\ &= 2\Big(k\big(0\big)-k\big(|x-y|\big)\Big) \\ &\leq C|x-y|^{\alpha}.\end{aligned}$$

Since u is Gaussian it follows that, for any integer $r>0$,

$$\mathbb{E}|u(x)-u(y)|^{2r} \leq C_r|x-y|^{\alpha r}.$$

Let $p=2r$ and noting that

$$\alpha r = p\Big(\frac{\alpha}{2}-\frac{d}{p}\Big)+d$$

we deduce from Corollary 7.20 that u is Hölder continuous on D with any exponent smaller than

$$\sup_{p\in\mathbb{N}} \min\Big\{1, \frac{\alpha}{2}-\frac{d}{p}\Big\} = \frac{\alpha}{2}\,,$$

which is precisely what we claimed. □

It is often convenient both algorithmically and theoretically to define the covariance operator through fractional inverse powers of a differential operator. Indeed in the previous subsection we showed that our assumptions on the random series construction we used could be interpreted as having a covariance operator which was an inverse fractional power of the Laplacian on zero spatial average functions with periodic boundary conditions. We now generalize this perspective and consider covariance operators which are a fractional power of an operator A satisfying the following.

Assumption 2.17. The operator A, densely defined on the Hilbert space $\mathcal{H} = L^2(D;\mathbb{R}^n)$, satisfies the following properties:

1. A is positive-definite, self-adjoint and invertible;
2. the eigenfunctions $\{\phi_j\}_{j\in\mathbb{N}}$ of A form an orthonormal basis for $\mathcal{H}$;
3. the eigenvalues of A satisfy $\alpha_j \asymp j^{2/d}$;
4. there is $C>0$ such that

$$\sup_{j\in\mathbb{N}}\Big(\|\phi_j\|_{L^\infty} + \frac{1}{j^{1/d}}\mathrm{Lip}(\phi_j)\Big) \leq C.$$

These properties are satisfied by the Laplacian on a torus, when applied to functions with spatial mean zero. But they are in fact satisfied for a much wider range of differential operators which are *Laplacian-like*. For example the Dirichlet Laplacian on a bounded open set D in $\mathbb{R}^d$, together with various Laplacian operators perturbed by lower order terms; for example Schrödinger operators. Inspection of the proof of Theorem 2.12 reveals that it only uses the properties of Assumptions 2.17. Thus we have:

Theorem 2.18. *Let u be a sample from the measure $N(0, \mathcal{C})$ in the case where $\mathcal{C} = A^{-s}$ with A satisfying Assumptions 2.17 and $s > \frac{d}{2}$. Then, $\mathbb{P}$-a.s., $u \in \dot{H}^t$, for $t < s - \frac{d}{2}$, and $u \in C^{0,t}(D)$, for $t < 1 \wedge (s - \frac{d}{2})$.*

Example 2.19. Consider the case $d = 2, n = 1$ and $D = [0,1]^2$. Define the Gaussian random field through the measure $\mu = N(0, \left(-\triangle\right)^{-\alpha})$ where $\triangle$ is the Laplacian with domain $H_0^1(D) \cap H^2(D)$. Then Assumptions 2.17 are satisfied by $-\triangle$. By Theorem 2.18 it follows that choosing $\alpha > 1$ suffices to ensure that draws from μ are almost surely in $L^2(D)$. It also follows that, in fact, draws from μ are almost surely in $C(D)$.

2.6. Summary. In the preceding four subsections we have shown how to create random functions by randomizing the coefficients of a series of functions. Using these random series we have also studied the regularity properties of the resulting functions. Furthermore we have extended our perspective in the Gaussian case to determine regularity properties from the properties of the covariance function or the covariance operator.

For the uniform prior we have shown that the random functions all live in a subset of $X = L^\infty$ characterized by the upper and lower bounds given in Theorem 2.1 and found as the closure of the linear span of the set of functions $(m_0, \{\phi_j\}_{j=1}^\infty)$; denote this subset, which is a separable Banach space, by X'. For the Besov priors we have shown in Theorem 2.6 that the random functions live in the separable Banach spaces $X^{t,q}$ for all $t < s - d/q$; denote any one of these Banach spaces by X'. And finally for the Gaussian priors we have shown in Theorem 2.10 that the random function exists as an L^2-limit in any of the Hilbert spaces $\mathcal{H}^t$ for $t < s - d/2$. Furthermore, we have indicated that, by use of the Kolmogorov continuity theorem, we can also show that the Gaussian random functions lie in certain Hölder spaces; these Hölder spaces are not separable but, by the discussion in subsection 7.1.2, we can embed the spaces $C^{0,\gamma'}$ in the separable uniform Hölder spaces $C_0^{0,\gamma}$ for any $\gamma < \gamma'$; since the upper bound on the range of Hölder exponents established by use of Kolmogorov continuity theorem is open, this means we can work in the same range of Hölder exponents, but restricted to uniform Hölder spaces, thereby regaining separability. In this Gaussian case we denote any of the separable Hilbert or Banach spaces where the Gaussian random function lives almost surely by X'.

Thus, in all of these examples, we have created a probability measure μ_0 which is the pushforward of the measure $\mathbb{P}$ on the i.i.d. sequence ξ under the map which takes the sequence into the random function. The resulting measure lives on the

separable Banach space X', and we will often write $\mu_0(X') = 1$ to denote this fact. This is shorthand for saying that functions drawn from μ_0 are in X' almost surely. Separability of X' naturally leads to the use of the Borel σ-algebra to define a canonical measurable space, and to the development of an integration theory – Bochner integration – which is natural on this space; see subsection 7.2.2.

2.7. Bibliographic Notes.

- Subsection 2.1. For general discussion of the properties of random functions constructed via randomization of coefficients in a series expansion see [49]. The construction of probability measure on infinite sequences of i.i.d. random variables may be found in [27].

- Subsection 2.2. These uniform priors have been extensively studied in the context of the field of Uncertainty Quantification and the reader is directed to [18, 19] for more details. Uncertainty Quantification in this context does not concern inverse problems, but rather studies the effect, on the solution of an equation, of randomizing the input data. Thus the interest is in the pushforward of a measure on input parameter space onto a measure on solution space, for a differential equation. Recently, however, these priors have been used to study the inverse problem; see [91].

- Subsection 2.3. Besov priors were introduced in the paper [69] and Theorem 2.6 is taken from that paper. We notice that the theorem constitutes a special case of the Fernique Theorem in the Gaussian case $q = 2$; it is restricted to a specific class of Hilbert space norms, however, whereas the Fernique Theorem in full generality applies in all norms on Banach spaces which have full Gaussian measure. See [35, 40] for proof of the Fernique Theorem. A more general Fernique-like property of the Besov measures is proved in [24] but it remains open to determine the appropriate complete generalization of the Fernique Theorem to Besov measures. For proof of Lemma 2.7 see [54, Chapter 4]. For properties of families of functions that can form a basis for a Besov space, and examples of such families see [31, 74].

- Subsection 2.4. The general theory of Gaussian measures on Banach spaces is contained in [67, 14]. The text [28], concerning the theory of stochastic PDEs, also has a useful overview of the subject. The Karhunen-Loève expansion (2.9) is contained in [1]. The formal calculation concerning the covariance operator of the Gaussian measure which follows Theorem 2.10 leads to the answer which may be rigorously justified by using characteristic functions; see, for example, Proposition 2.18 in [28]. All three texts include statement and proof of the Fernique Theorem in the generality given here. The Kolmogorov continuity theorem is discussed in [28] and [1]. Proof of Hölder regularity adapted to the case of the periodic setting may be found in [40] and [93, Chapter 6]. For further reading on Gaussian measures see [27].

- Subsection 2.5. A key tool in making the random field perspective rigorous is the Kolmogorov Extension Theorem 7.4.

- Subsection 2.6. For a discussion of measure theory on general spaces see [15]. The notion of Bochner integral is introduced in [13]; we discuss it in subsection 7.2.2.

3. Posterior Distribution

In this section we prove a Bayes' theorem appropriate for combining a likelihood with prior measures on separable Banach spaces as constructed in the previous section. In subsection 3.1 we start with some general remarks about conditioned random variables. Subsection 3.2 contains our statement and proof of a Bayes' theorem, and specifically its application to Bayesian inversion. We note here that, in our setting, the posterior μ^y will always be absolutely continuous with respect to the prior μ_0, and we use the standard notation $\mu^y \ll \mu_0$ to denote this. It is possible to construct examples, for instance in the purely Gaussian setting, where the posterior is not absolutely continuous with respect to the prior. Thus it is certainly not necessary to work in the setting where $\mu^y \ll \mu_0$. However it is quite natural, from a modelling point of view, to work in this setting: absolute continuity ensures that almost sure properties built into the prior will be inherited by the posterior. For these almost sure properties to be changed by the data would require that the data contains an infinite ammount of information, something which is unnatural in most applications.

In subsection 3.3 we study the example of the heat equation, introduced in subsection 1.2, from the perspective of Bayesian inversion and in subsection 3.4 we do the same for the elliptic inverse problem of subsection 1.3.

3.1. Conditioned Random Variables. Key to the development of Bayes' Theorem, and the posterior distribution, is the notion of conditional random variables. In this section we state an important theorem concerning conditioning.

Let (X, A) and (Y, B) denote a pair of measurable spaces and let ν and π be probability measures on $X \times Y$. We assume that $\nu \ll \pi$. Thus there exists π-measurable $\phi : X \times Y \to \mathbb{R}$ with $\phi \in L^1_\pi$ (see section 7.1.4 for definition of L^1_π) and

$$\frac{d\nu}{d\pi}(x, y) = \phi(x, y). \tag{3.1}$$

That is, for $(x, y) \in X \times Y$,

$$\mathbb{E}^\nu f(x, y) = \mathbb{E}^\pi \big(\phi(x, y) f(x, y)\big),$$

or, equivalently,

$$\int_{X\times Y} f(x,y)\nu(dx,dy) = \int_{X\times Y} \phi(x,y)f(x,y)\pi(dx,dy).$$

Theorem 3.1. *Assume that the conditional random variable $x|y$ exists under π with probability distribution denoted $\pi^y(dx)$. Then the conditional random variable $x|y$ under ν exists, with probability distribution denoted by $\nu^y(dx)$. Furthermore, $\nu^y \ll \pi^y$ and if $c(y) := \int_X \phi(x,y)d\pi^y(x) > 0$ then*

$$\frac{d\nu^y}{d\pi^y}(x) = \frac{1}{c(y)}\phi(x,y).$$

Example 3.2. Let $X = C\big([0,1];\mathbb{R}\big)$, $Y = \mathbb{R}$. Let π denote the measure on $X \times Y$ induced by the random variable $\big(w(\cdot), w(1)\big)$, where w is a draw from standard unit Wiener measure on $\mathbb{R}$, starting from $w(0) = z$. Let π^y denote measure on X found by conditioning Brownian motion to satisfy $w(1) = y$, thus π^y is a Brownian bridge measure with $w(0) = z, w(1) = y$.

Assume that $\nu \ll \pi$ with

$$\frac{d\nu}{d\pi}(x,y) = \exp\big(-\Phi(x,y)\big).$$

Assume further that

$$\sup_{x\in X} \Phi(x,y) = \Phi^+(y) < \infty$$

for every $y \in \mathbb{R}$. Then

$$c(y) = \int_{\mathbb{R}} \exp\big(-\Phi(x,y)\big)d\pi^y(x) > \exp\big(-\Phi^+(y)\big) > 0.$$

Thus $\nu^y(dx)$ exists and

$$\frac{d\nu^y}{d\pi^y}(x) = \frac{1}{c(y)}\exp\big(-\Phi(x,y)\big). \quad \square$$

We will use the preceding theorem to go from a construction of the joint probability distribution on unknown and data to the conditional distribution of the unknown, given data. In constructing the joint probability distribution we will need to establish measurability of the likelihood, for which the following will be useful:

Lemma 3.3. *Let (Z, B) be a Borel measurable topological space and assume that $G \in C(Z;\mathbb{R})$ and that $\pi(Z) = 1$ for some probability measure π on (Z, B). Then G is a π-measurable function.*

3.2. Bayes' Theorem for Inverse Problems. Let X, Y be separable Banach spaces, equipped with the Borel σ-algebra, and $G : X \to Y$ a measurable mapping. We wish to solve the inverse problem of finding u from y where

$$y = G(u) + \eta \tag{3.2}$$

and $\eta \in Y$ denotes noise. We employ a Bayesian approach to this problem in which we let $(u, y) \in X \times Y$ be a random variable and compute $u|y$. We specify the random variable (u, y) as follows:

- **Prior**: $u \sim \mu_0$ measure on X.
- **Noise**: $\eta \sim \mathbb{Q}_0$ measure on Y, and (recalling that $\perp$ denotes independence) $\eta \perp u$.

The random variable $y|u$ is then distributed according to the measure $\mathbb{Q}_u$, the translate of $\mathbb{Q}_0$ by $G(u)$. We *assume* throughout the following that $\mathbb{Q}_u \ll \mathbb{Q}_0$ for u μ_0- a.s. Thus, for some **potential** $\Phi : X \times Y \to \mathbb{R}$,

$$\frac{d\mathbb{Q}_u}{d\mathbb{Q}_0}(y) = \exp\big(-\Phi(u; y)\big). \tag{3.3}$$

Thus, for fixed u, $\Phi(u; \cdot) : Y \to \mathbb{R}$ is measurable and $\mathbb{E}^{\mathbb{Q}_0} \exp\big(-\Phi(u; y)\big) = 1$. For given instance of the data y, $-\Phi(\cdot; y)$ is termed the **log likelihood**.

Define ν_0 to be the product measure

$$\nu_0(du, dy) = \mu_0(du)\mathbb{Q}_0(dy). \tag{3.4}$$

We *assume* in what follows that $\Phi(\cdot, \cdot)$ is ν_0 measurable. Then the random variable $(u, y) \in X \times Y$ is distributed according to measure $\nu(du, dy) = \mu_0(du)\mathbb{Q}_u(dy)$. Furthermore, it then follows that $\nu \ll \nu_0$ with

$$\frac{d\nu}{d\nu_0}(u, y) = \exp\big(-\Phi(u; y)\big).$$

We have the following infinite dimensional analogue of Theorem 1.1.

Theorem 3.4 (Bayes' Theorem). *Assume that $\Phi : X \times Y \to \mathbb{R}$ is ν_0 measurable and that, for y $\mathbb{Q}_0$-a.s.,*

$$Z := \int_X \exp\big(-\Phi(u; y)\big)\mu_0(du) > 0. \tag{3.5}$$

Then the conditional distribution of $u|y$ exists under ν, and is denoted by μ^y. Furthermore $\mu^y \ll \mu_0$ and, for y ν-a.s.,

$$\frac{d\mu^y}{d\mu_0}(u) = \frac{1}{Z}\exp\big(-\Phi(u; y)\big). \tag{3.6}$$

Proof. First note that the positivity of Z holds for y ν_0-almost surely, and hence by absolute continuity of ν with respect to ν_0, for y ν-almost surely. The proof is an application of Theorem 3.1 with π replaced by ν_0, $\phi(x,y) = \exp\big(-\Phi(u,y)\big)$ and $(x,y) = (u,y)$. Since $\nu_0(du,dy)$ has product form, the conditional distribution of $u|y$ under ν_0 is simply μ_0. The result follows. □

Remarks 3.5. In order to implement the derivation of Bayes' formula (3.6) four essential steps are required:

- Define a suitable prior measure μ_0 and noise measure $\mathbb{Q}_0$ whose independent product form the reference measure ν_0.
- Determine the potential Φ such that formula (3.3) holds.
- Show that Φ is ν_0 measurable.
- Show that the normalization constant Z given by (3.5) is positive almost surely with respect to $y \sim \mathbb{Q}_0$.

We will show how to carry out this program for two examples in the following subsections. The following remark will also be used in studying one of the examples.

Remarks 3.6. The following comments on the set-up above may be useful.

- In formula (3.6) we can shift $\Phi(u,y)$ by any constant $c(y)$, independent of u, provided the constant is finite $\mathbb{Q}_0$-a.s. and hence ν-a.s. Such a shift can be absorbed into a redefinition of the normalization constant Z.
- Our Bayes' Theorem only asserts that the posterior is absolutely continuous with respect to the prior μ_0. In fact equivalence (mutual absolute continuity) will occur when $\Phi(\cdot;y)$ is finite everywhere in X.

3.3. Heat Equation.

We apply Bayesian inversion to the heat equation from subsection 1.2. Recall that for $G(u) = e^{-A}u$, we have the relationship

$$y = G(u) + \eta,$$

which we wish to invert. Let $X = H$ and define

$$\mathcal{H}^t = \mathcal{D}(A^{t/2}) = \Big\{ w \big| w = A^{-t/2} w_0, w_0 \in H \Big\}.$$

Under Assumptions 1.3 we have $\alpha_j \asymp j^{\frac{2}{d}}$ so that this family of spaces is identical with the Hilbert scale of spaces $\mathcal{H}^t$ as defined in subsections 1.2 and 2.4.

We choose the prior $\mu_0 = N(0, A^{-\alpha})$, $\alpha > \frac{d}{2}$. Thus $\mu_0(X) = \mu_0(H) = 1$. Indeed the analysis in subsection 2.4 shows that $\mu_0(\mathcal{H}^t) = 1$, $t < \alpha - \frac{d}{2}$. For the likelihood we assume that $\eta \perp u$ with $\eta \sim \mathbb{Q}_0 = N(0, A^{-\beta})$, and $\beta \in \mathbb{R}$. This

measure satisfies $\mathbb{Q}_0(\mathcal{H}^t) = 1$ for $t < \beta - \frac{d}{2}$ and we thus choose $Y = \mathcal{H}^{t'}$ for some $t' < \beta - \frac{d}{2}$. Notice that our analysis includes the case of white observational noise, for which $\beta = 0$. The Cameron-Martin Theorem 7.27, together with the fact that $e^{-\lambda A}$ commutes with arbitrary fractional powers of A, can be used to show that $y|u \sim \mathbb{Q}_u := N(G(u), A^{-\beta})$ where $\mathbb{Q}_u \ll \mathbb{Q}_0$ with

$$\frac{d\mathbb{Q}_u}{d\mathbb{Q}_0}(y) = \exp\big(-\Phi(u;y)\big),$$

and

$$\Phi(u;y) = \frac{1}{2}\|A^{\frac{\beta}{2}}e^{-A}u\|^2 - \langle A^{\frac{\beta}{2}}e^{-\frac{A}{2}}y, A^{\frac{\beta}{2}}e^{-\frac{A}{2}}u\rangle.$$

In the following we repeatedly use the fact that $A^\gamma e^{-\lambda A}$, $\lambda > 0$, is a bounded linear operator from $\mathcal{H}^a$ to $\mathcal{H}^b$, any $a, b, \gamma \in \mathbb{R}$. Recall that $\nu_0(du, dy) = \mu_0(du)\mathbb{Q}_0(dy)$. Note that $\nu_0(H \times \mathcal{H}^{t'}) = 1$. Using the boundedness of $A^\gamma e^{-\lambda A}$ it may be shown that

$$\Phi : H \times \mathcal{H}^{t'} \to \mathbb{R}$$

is continuous, and hence ν_0-measurable by Lemma 3.3.

Theorem 3.4 shows that the posterior is given by μ^y where

$$\frac{d\mu^y}{d\mu_0}(u) = \frac{1}{Z}\exp\big(-\Phi(u;y)\big),$$

$$Z = \int_H \exp\big(-\Phi(u;y)\big)\mu_0(du),$$

provided that $Z > 0$ for y $\mathbb{Q}_0$-a.s. We establish this positivity in the remainder of the proof. Since $y \in \mathcal{H}^t$ for any $t < \beta - \frac{d}{2}$, $\mathbb{Q}_0$-a.s., we have that $y = A^{-t'/2}w_0$ for some $w_0 \in H$ and $t' < \beta - \frac{d}{2}$. Thus we may write

$$\Phi(u;y) = \frac{1}{2}\|A^{\frac{\beta}{2}}e^{-A}u\|^2 - \langle A^{\frac{\beta-t'}{2}}e^{-\frac{A}{2}}w_0, A^{\frac{\beta}{2}}e^{-\frac{A}{2}}u\rangle. \tag{3.7}$$

Then, using the boundedness of $A^\gamma e^{-\lambda A}$, $\lambda > 0$, together with (3.7), we have

$$\Phi(u;y) \le C(\|u\|^2 + \|w_0\|^2)$$

where $\|w_0\|$ is finite $\mathbb{Q}_0$-a.s. Thus

$$Z \ge \int_{\|u\|^2 \le 1} \exp\big(-C(1 + \|w_0\|^2)\big)\mu_0(du)$$

and, since $\mu_0(\|u\|^2 \le 1) > 0$ (by Theorem 7.28 all balls have positive measure for Gaussians on a separable Banach space) the required positivity follows.

3.4. Elliptic Inverse Problem. We consider the elliptic inverse problem from subsection 1.3 from the Bayesian perspective. We consider the use of both uniform and Gaussian priors. Before studying the inverse problem, however, it is important to derive some continuity properties of the forward problem. Throughout this section we consider equation (1.7) under the assumption that $f \in V^*$.

3.4.1. Forward Problem. Recall that in subsection 1.3, equation (1.10), we defined

$$X^+ = \Big\{ v \in L^\infty(D) \Big| \operatorname*{ess\,inf}_{x\in D} v(x) > 0 \Big\}. \tag{3.8}$$

Then the map $\mathcal{R} : X^+ \to V$ by $\mathcal{R}(\kappa) = p$. This map is well-defined by Lemma 1.5 and we have the following result.

Lemma 3.7. *For $i = 1, 2$, let*

$$\begin{aligned} -\nabla \cdot (\kappa_i \nabla p_i) &= f, \quad x \in D, \\ p_i &= 0, \quad x \in \partial D. \end{aligned}$$

Then

$$\|p_1 - p_2\|_V \le \frac{1}{\kappa_{\min}^2} \|f\|_{V^*} \|\kappa_1 - \kappa_2\|_{L^\infty}$$

where we assume that

$$\kappa_{\min} := \operatorname*{ess\,inf}_{x\in D} \kappa_1(x) \wedge \operatorname*{ess\,inf}_{x\in D} \kappa_2(x) > 0.$$

Thus the function $\mathcal{R} : X^+ \to V$ is locally Lipschitz.

Proof. Let $e = \kappa_1 - \kappa_2$, $r = p_1 - p_2$. Then

$$\begin{aligned} -\nabla \cdot (\kappa_1 \nabla r) &= \nabla \cdot \big((\kappa_1 - \kappa_2) \nabla p_2 \big), \quad x \in D \\ r &= 0, \quad x \in \partial D. \end{aligned}$$

Multiplying by r and integrating by parts on both sides of the identity gives

$$\kappa_{\min} \int_D |\nabla r|^2 \, dx \le \|(\kappa_2 - \kappa_1)\nabla p_2\| \|\nabla r\|.$$

Using the fact that $\|\varphi\|_V = \|\nabla \varphi\|$, and applying Lemma 1.5 to bound p_2 in V, we find that

$$\begin{aligned} \|r\|_V &\le \|(\kappa_2 - \kappa_1)\nabla p_2\| / \kappa_{\min} \\ &\le \|\kappa_2 - \kappa_1\|_{L^\infty} \|p_2\|_V / \kappa_{\min} \\ &\le \frac{1}{\kappa_{\min}^2} \|f\|_{V^*} \|e\|_{L^\infty}. \end{aligned}$$

□

3.4.2. Uniform Priors. We now study the inverse problem of finding κ from a finite set of continuous linear functionals $\{l_j\}_{j=1}^J$ on V, representing measurements of p; thus $l_j \in V^*$. To match the notation from subsection 3.2 we take $\kappa = u$ and we define the separable Banach space X' as in subsection 2.2. It is straightforward to see that Lemma 3.7 extends to the case where X^+ given by (3.8) is replaced by

$$X^+ = \Big\{ v \in X' \Big| \operatorname*{ess\,inf}_{x\in D} v(x) > 0 \Big\} \tag{3.9}$$

since $X' \subset L^\infty(D)$. When considering uniform priors for the elliptic problem we work with this definition of X^+.

We define $G : X^+ \to \mathbb{R}^J$ by

$$G_j(u) = l_j\big(\mathcal{R}(u)\big), \quad j = 1, \dots, J$$

where, recall, the l_j are elements of V^*: bounded linear functionals on V. Then $G(u) = \big(G_1(u), \cdots, G_J(u)\big)$ and we are interested in the inverse problem of finding $u \in X^+$ from y where

$$y = G(u) + \eta$$

and η is the noise. We assume $\eta \sim N(0, \Gamma)$, for positive symmetric $\Gamma \in \mathbb{R}^{J\times J}$. (Use of other statistical assumptions on η is a straightforward extension of what follows whenever η has a smooth density on $\mathbb{R}^J$.)

Let μ_0 denote the prior measure constructed in subsection 2.2. Then μ_0-almost surely we have, by Theorem 2.1,

$$u \in X_0^+ := \Big\{ v \in X' \Big| \frac{1}{1+\delta} m_{\min} \le v(x) \le m_{\max} + \frac{\delta}{1+\delta} m_{\min} \quad \text{a.e.} \quad x \in D \Big\}. \tag{3.10}$$

Thus $\mu_0(X_0^+) = 1$.

The likelihood is defined as follows. Since $\eta \sim N(0, \Gamma)$ it follows that $\mathbb{Q}_0 = N(0, \Gamma)$, $\mathbb{Q}_u = N\big(G(u), \Gamma\big)$ and

$$\frac{d\mathbb{Q}_u}{d\mathbb{Q}_0}(y) = \exp\big(-\Phi(u; y)\big),$$

$$\Phi(u; y) = \frac{1}{2}\big|\Gamma^{-\frac{1}{2}}(y - G(u))\big|^2 - \frac{1}{2}\big|\Gamma^{-\frac{1}{2}} y\big|^2.$$

Recall that $\nu_0(dy, du) = \mathbb{Q}_0(dy)\mu_0(du)$. Since $G : X^+ \to \mathbb{R}^J$ is locally Lipschitz by Lemma 3.7, Lemma 3.3 implies that $\Phi : X^+ \times Y \to \mathbb{R}$ is ν_0-measurable. Thus Theorem 3.4 shows that $u|y \sim \mu^y$ where

$$\frac{d\mu^y}{d\mu_0}(u) = \frac{1}{Z} \exp\big(-\Phi(u; y)\big) \tag{3.11}$$

$$Z = \int_{X^+} \exp\big(-\Phi(u; y)\big)\mu_0(du),$$

provided $Z > 0$ for y $\mathbb{Q}_0$-almost surely. To see that $Z > 0$ note that

$$Z = \int_{X_0^+} \exp\big(-\Phi(u; y)\big)\mu_0(du),$$

since $\mu_0(X_0^+) = 1$. On X_0^+ we have that $\mathcal{R}(\cdot)$ is bounded in V, and hence G is bounded in $\mathbb{R}^J$. Furthermore y is finite $\mathbb{Q}_0$-almost surely. Thus $\mathbb{Q}_0$-almost surely with respect to y, $\Phi(\cdot; y)$ is bounded on X_0^+; we denote the resulting bound by $M = M(y) < \infty$. Hence

$$Z \ge \int_{X_0^+} \exp(-M)\mu_0(du) = \exp(-M) > 0.$$

and the result is proved.

We may use Remark 3.6 to shift Φ by $\frac{1}{2}|\Gamma^{-\frac{1}{2}}y|^2$, since this is almost surely finite under $\mathbb{Q}_0$ and hence under $\nu(du,dy) = \mathbb{Q}_u(dy)\mu_0(du)$. We then obtain the equivalent form for the posterior distribution μ^y:

$$\frac{d\mu^y}{d\mu_0}(u) = \frac{1}{Z}\exp\Big(-\frac{1}{2}\big|\Gamma^{-\frac{1}{2}}\big(y-G(u)\big)\big|^2\Big), \tag{3.12a}$$

$$Z = \int_X \exp\Big(-\frac{1}{2}\big|\Gamma^{-\frac{1}{2}}\big(y-G(u)\big)\big|^2\Big)\mu_0(du). \tag{3.12b}$$

3.4.3. Gaussian Priors. We conclude this subsection by discussing the same inverse problem, but using Gaussian priors from subsection 2.4. We now set $X = C(\overline{D})$, $Y = \mathbb{R}^J$ and we note that X embeds continuously into $L^\infty(D)$. We assume that we can find an operator A which satisfies Assumptions 2.17. We now take $\kappa = \exp(u)$, and define $G : X \to \mathbb{R}^J$ by

$$G_j(u) = l_j\Big(\mathcal{R}\big(\exp(u)\big)\Big), \quad j = 1,\dots,J.$$

We take as prior on u the measure $N(0, A^{-s})$ with $s > d/2$. Then Theorem 2.18 shows that $\mu(X) = 1$. The likelihood is unchanged by the prior, since it concerns y given u, and is hence identical to that in the case of the uniform prior, although the mean shift from $\mathbb{Q}_0$ to $\mathbb{Q}_u$ by $G(u)$ now has a different interpretation since $\kappa = \exp(u)$ rather than $\kappa = u$. Thus we again obtain (3.11) for the posterior distribution (albeit with a different definition of $G(u)$) provided that we can establish that, $\mathbb{Q}_0$-a.s.,

$$Z = \int_X \exp\Big(\frac{1}{2}\big|\Gamma^{-\frac{1}{2}}y\big|^2 - \frac{1}{2}\big|\Gamma^{-\frac{1}{2}}\big(y-G(u)\big)\big|^2\Big)\mu_0(du) > 0.$$

To this end we use the fact that the unit ball in X, denoted B, has positive measure by Theorem 7.28, and that on this ball $\mathcal{R}\big(\exp(u)\big)$ is bounded in V by $e^{-a}\|f\|_{V^*}$, by Lemma 1.5, for some finite positive constant a. This follows from the continuous embedding of X into L^∞ and since the infimum of $\kappa = \exp(u)$ is bounded below by $e^{-\|u\|_{L^\infty}}$. Thus G is bounded on B and, noting that y is $\mathbb{Q}_0$-a.s. finite, we have for some $M = M(y) < \infty$,

$$\sup_{u\in B}\Big(\frac{1}{2}\big|\Gamma^{-\frac{1}{2}}\big(y-G(u)\big)\big|^2 - \frac{1}{2}\big|\Gamma^{-\frac{1}{2}}y\big|^2\Big) < M.$$

Hence

$$Z \geq \int_B \exp(-R)\mu_0(du) = \exp(-R)\mu_0(B) > 0$$

since all balls have positive measure for Gaussian measure on a separable Banach space. Thus we again obtain (3.12) for the posterior measure, now with the new definition of G, and hence Φ.

3.5. Bibliographic Notes.

- Subsection 3.1. Theorem 3.1 is taken from [43] where it is used to compute expressions for the measure induced by various conditionings applied to SDEs. The existence of regular conditional probability distributions is discussed in [54], Theorem 6.3. The Example 3.2, concerning end-point conditioning of measures defined via a density with respect to Wiener measure, finds application to problems from molecular dynamics in [83, 82]. Further material concerning the equivalence of posterior with respect to the prior may be found in [93, Chapters 3 and 6], [3], [4]. The equivalence of Gaussian measures is studied via the Feldman-Hájeki theorem; see [28] and the Appendix. A proof of Lemma 3.3 can be found in [89, Chapter 1, Theorem 1.12]. See also [54, Lemma 1.5].

- Subsection 3.2. General development of Bayes' Theorem for inverse problems on function space, along the lines described here, may be found in [17, 93]. The reader is also directed to the papers [61, 62] for earlier related material, and to [63, 64, 65] for recent developments.

- Subsection 3.3. The inverse problem for the heat equation was one of the first infinite dimensional inverse problems to receive Bayesian treatment; see [36], leading to further developments in [71, 68]. The problem is worked through in detail in [93]. To fully understand the details the reader will need to study the Cameron-Martin theorem (concerning shifts in the mean of Gaussian measures) and the Feldman-Hájek theorem (concerning equivalence of Gaussian measures); both of these may be found in [28, 67, 14] and are also discussed in [93].

- Subsection 3.4. The elliptic inverse problem with the uniform prior is studied in [91]. A Gaussian prior is adopted in [25], and a Besov prior in [24].

4. Common Structure

In this section we discuss various common features of the posterior distribution arising from the Bayesian approach to inverse problems. We start, in subsection 4.1, by studying the continuity properties of the posterior with respect to changes in data, proving a form of well-posedness; indeed we show that the posterior is Lipschitz in the data with respect to the Hellinger metric. In subsection 4.2 we use similar ideas to study the effect of approximation on the posterior distribution, showing that small changes in the potential Φ lead to small changes in the posterior distribution, again in the Hellinger metric; this work may be used to translate error analysis pertaining to the forward problem into estimates on errors in the posterior distribution. In the final subsection 4.3 we study an important link between the

Bayesian approach to inverse problems and classical regularization techniques for inverse problems; specifically we link the Bayesian MAP estimator to a Tikhonov-Phillips regularized least squares problem. The first two subsections work with general priors, whilst the final one is concerned with Gaussians only.

4.1. Well-Posedness. In many classical inverse problems small changes in the data can induce arbitrarily large changes in the solution, and some form of regularization is needed to counteract this ill-posedness. We illustrate this effect with the inverse heat equation example. We then proceed to show that the Bayesian approach to inversion has the property that small changes in the data lead to small changes in the posterior distribution. Thus working with probability measures on the solution space, and adopting suitable priors, provides a form of regularization.

Example 4.1. Consider the heat equation introduced in subsection 1.2 and both perfect data $y = e^{-A}u$, derived from the forward model with no noise, and noisy data $y' = e^{-A}u + \eta$. Consider the case where $\eta = \epsilon\varphi_j$ with ϵ small and φ_j a normalized eigenfuction of A. Thus $\|\eta\| = \epsilon$. Obviously application of the inverse of e^{-A} to y returns the point u which gave rise to the perfect data. It is natural to apply the inverse of e^{-A} to both y and to y' to understand the effect of the noise. Doing so yields the identity

$$\|e^{A}y - e^{A}y'\| = \|e^{A}(y - y')\| = \|e^{A}\eta\| = \epsilon\|e^{A}\varphi_j\| = \epsilon e^{\alpha_j} \ .$$

Recall Assumption 1.3 which gives $\alpha_j \asymp j^{2/d}$. Now fix any $a > 0$ and choose j large enough to ensure that $\alpha_j = (a+1)\log(\epsilon^{-1})$. It then follows that $\|y - y'\| = \mathcal{O}(\epsilon)$ whilst $\|e^{A}y - e^{A}y'\| = \mathcal{O}(\epsilon^{-a})$. This is a manifestation of ill-posedness. Furthermore, since $a > 0$ is arbitrary, the ill-posedness can be made arbitrarily bad by considering $a \to \infty$. □

Our aim in this section is to show that this ill-posedness effect does not occur in the Bayesian posterior distribution: small changes in the data y lead to small changes in the measure μ^y. Let X, Y be separable Banach spaces, equipped with the Borel σ-algebra, and μ_0 a measure on X. We will work under assumptions which enable us to make sense of the following measure $\mu^y \ll \mu_0$ defined, for some $\Phi : X \times Y \to \mathbb{R}$, by

$$\frac{d\mu^y}{d\mu_0}(u) = \frac{1}{Z(y)}\exp\big(-\Phi(u;y)\big), \tag{4.1a}$$

$$Z(y) = \int_X \exp\big(-\Phi(u;y)\big)\mu_0(du). \tag{4.1b}$$

We make the following assumptions concerning Φ :

Assumptions 4.2. Let $X' \subseteq X$ and assume that $\Phi \in C(X' \times Y; \mathbb{R})$. Assume further that there are functions $M_i : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, $i = 1, 2$, monotonic non-decreasing separately in each argument, and with M_2 strictly positive, such that

for all $u \in X'$, $y, y_1, y_2 \in B_Y(0,r)$,

$$\Phi(u;y) \geq -M_1(r, \|u\|_X),$$

$$|\Phi(u;y_1) - \Phi(u;y_2)| \leq M_2(r, \|u\|_X)\|y_1 - y_2\|_Y. \quad \square$$

In order to measure the effect of changes in y on the measure μ^y we need a metric on measures. We use the Hellinger metric defined in subsection 7.2.4.

Theorem 4.3. *Let Assumptions 4.2 hold. Assume that $\mu_0(X') = 1$ and that $\mu_0(X' \cap B) > 0$ for some bounded set B in X. Assume additionally that, for every fixed $r > 0$,*

$$\exp\big(M_1(r, \|u\|_X)\big) \in L^1_{\mu_0}(X;\mathbb{R}).$$

Then, for every $y \in Y$, $Z(y)$ given by (4.1b) is positive and finite and the probability measure μ^y given by (4.1) is well-defined.

Proof. The boundedness of $Z(y)$ follows directly from the lower bound on Φ in Assumptions 4.2, together with the assumed integrability condition in the theorem. Since $u \sim \mu_0$ satisfies $u \in X'$ a.s., we have

$$Z(y) = \int_{X'} \exp\big(-\Phi(u;y)\big)\mu_0(du).$$

Note that $B' = X' \cap B$ is bounded in X. Define

$$R_1 := \sup_{u \in B'} \|u\|_X < \infty.$$

Since $\Phi : X' \times Y \to \mathbb{R}$ is continuous it is finite at every point in $B' \times \{y\}$. Thus, by the continuity of $\Phi(\cdot;\cdot)$ implied by Assumptions 4.2, we see that

$$\sup_{(u,y) \in B' \times B_Y(0,r)} \Phi(u;y) = R_2 < \infty.$$

Hence

$$Z(y) \geq \int_{B'} \exp(-R_2)\mu_0(du) = \exp(-R_2)\mu_0(B') > 0. \tag{4.2}$$

Since $\mu_0(B')$ is assumed positive and R_2 is finite we deduce that $Z(y) > 0$. $\square$

Remarks 4.4. The following remarks apply to the preceding and following theorem.

- In the preceding theorem we are not explicitly working in a Bayesian setting: we are showing that, under the stated conditions on Φ, the measure is well-defined and normalizable. In Theorem 3.4 we did not need to check normalizability because μ^y was defined as a regular conditional probability, via Theorem 3.1, and therefore automatically normalizable.
- The lower bound (4.2) is used repeatedly in what follows, without comment.

- Establishing the integrability conditions for both the preceding and following theorem is often achieved for Gaussian μ_0 by appealing to the Fernique theorem.

Theorem 4.5. *Let Assumptions 4.2 hold. Assume that $\mu_0(X') = 1$ and that $\mu_0(X' \cap B) > 0$ for some bounded set B in X. Assume additionally that, for every fixed $r > 0$,*

$$\exp\big(M_1(r, \|u\|_X)\big)\Big(1 + M_2(r, \|u\|_X)^2\Big) \in L^1_{\mu_0}(X; \mathbb{R}).$$

Then there is $C = C(r) > 0$ such that, for all $y, y' \in B_Y(0, r)$

$$d_{\text{Hell}}(\mu^y, \mu^{y'}) \le C\|y - y'\|_Y.$$

Proof. Throughout this proof we use C to denote a constant independent of u, but possibly depending on the fixed value of r; it may change from occurence to occurence. We use the fact that, since $M_2(r, \cdot)$ is monotonic non-decreasing and strictly positive on $[0, \infty)$,

$$\exp\big(M_1(r, \|u\|_X)\big)M_2(r, \|u\|_X) \le \exp\big(M_1(r, \|u\|_X)\big)\Big(1 + M_2(r, \|u\|_X)^2\Big), \quad (4.3a)$$

$$\exp\big(M_1(r, \|u\|_X)\big) \le \exp\big(M_1(r, \|u\|_X)\big)\Big(1 + M_2(r, \|u\|_X)^2\Big). \quad (4.3b)$$

Let $Z = Z(y)$ and $Z' = Z(y')$ denote the normalization constants for μ^y and $\mu^{y'}$ so that, by Theorem 4.3,

$$Z = \int_{X'} \exp\Big(-\Phi(u; y)\Big)\mu_0(du) > 0,$$
$$Z' = \int_{X'} \exp\Big(-\Phi(u; y')\Big)\mu_0(du) > 0.$$

Then, using the local Lipschitz property of the exponential and the assumed Lipschitz continuity of $\Phi(u; \cdot)$, together with (4.3a), we have

$$\begin{aligned}
|Z - Z'| &\le \int_{X'} |\exp\big(-\Phi(u; y)\big) - \exp\big(-\Phi(u; y')\big)|\mu_0(du) \\
&\le \int_{X'} \exp\big(M_1(r, \|u\|_X)\big)|\Phi(u; y) - \Phi(u; y')|\mu_0(du) \\
&\le \Big(\int_{X'} \exp\big(M_1(r, \|u\|_X)\big)M_2(r, \|u\|_X)\mu_0(du)\Big)\|y - y'\|_Y \\
&\le \Big(\int_{X'} \exp\big(M_1(r, \|u\|_X)\big)(1 + M_2(r, \|u\|_X)^2)\mu_0(du)\Big)\|y - y'\|_Y \\
&\le C\|y - y'\|_Y.
\end{aligned}$$

The last line follows because the integrand is in $L^1_{\mu_0}$ by assumption. From the definition of Hellinger distance we have

$$\Big(d_{\text{Hell}}(\mu^y, \mu^{y'})\Big)^2 \le I_1 + I_2,$$

where

$$I_1 = \frac{1}{Z}\int_{X'}\Big(\exp\big(-\frac{1}{2}\Phi(u;y)\big) - \exp(-\frac{1}{2}\Phi(u;y'))\Big)^2\mu_0(du),$$
$$I_2 = \big|Z^{-\frac{1}{2}} - (Z')^{-\frac{1}{2}}\big|^2\int_{X'}\exp(-\Phi(u;y'))\mu_0(du).$$

Note that, again using similar Lipschitz calculations to those above, using the fact that $Z > 0$ and Assumptions 4.2,

$$\begin{aligned} I_1 &\leq \frac{1}{4Z}\int_{X'}\exp\big(M_1(r,\|u\|_X)\big)|\Phi(u;y)-\Phi(u;y')|^2\mu_0(du) \\ &\leq \frac{1}{Z}\Big(\int_{X'}\exp\big(M_1(r,\|u\|_X)\big)M_2(r,\|u\|_X)^2\mu_0(du)\Big)\|y-y'\|_Y^2 \\ &\leq C\|y-y'\|_Y^2. \end{aligned}$$

Also, using Assumptions 4.2, together with (4.3b),

$$\begin{aligned} \int_{X'}\exp\big(-\Phi(u;y')\big)\mu_0(du) &\leq \int_{X'}\exp\big(M_1(r,\|u\|_X)\big)\mu_0(du) \\ &< \infty. \end{aligned}$$

Hence

$$I_2 \leq C\big(Z^{-3}\vee(Z')^{-3}\big)|Z-Z'|^2 \leq C\|y-y'\|_Y^2.$$

The result is complete. □

Remark 4.6. The Hellinger metric has the very desirable property that it translates directly into bounds on expectations. For functions f which are in $L^2_{\mu^y}(X;\mathbb{R})$ and $L^2_{\mu^{y'}}(X;\mathbb{R})$ the closeness of the Hellinger metric implies closeness of expectations of f. To be precise, for $y, y' \in B_Y(0,r)$ we have

$$|\mathbb{E}^{\mu^y}f(u) - \mathbb{E}^{\mu^{y'}}f(u)| \leq Cd_{\text{Hell}}(\mu^y,\mu^{y'})$$

where constant C depends on r and on the expectations of $|f|^2$ under μ^y and $\mu^{y'}$. It follows that

$$|\mathbb{E}^{\mu^y}f(u) - \mathbb{E}^{\mu^{y'}}f(u)| \leq C\|y-y'\|,$$

for a possibly different constant C which also depends on r and on the expectations of $|f|^2$ under μ^y and $\mu^{y'}$.

4.2. Approximation.

In this section we concentrate on continuity properties of the posterior measure with respect to approximation of the potential Φ. The methods used are very similar to those in the previous subsection, and we establish

a continuity property of the posterior distribution, in the Hellinger metric, with respect to small changes in the potential Φ.

Because the data y plays no explicit role in this discussion, we drop explicit reference to it. Let X be a Banach space and μ_0 a measure on X. Assume that μ and μ^N are both absolutely continuous with respect to μ_0 and given by

$$\frac{d\mu}{d\mu_0}(u) = \frac{1}{Z}\exp\big(-\Phi(u)\big), \tag{4.4a}$$

$$Z = \int_X \exp\big(-\Phi(u)\big)\mu_0(du) \tag{4.4b}$$

and

$$\frac{d\mu^N}{d\mu_0}(u) = \frac{1}{Z^N}\exp\big(-\Phi^N(u)\big), \tag{4.5a}$$

$$Z^N = \int_X \exp\big(-\Phi^N(u)\big)\mu_0(du) \tag{4.5b}$$

respectively. The measure μ^N might arise, for example, through an approximation of the forward map G underlying an inverse problem of the form (3.2). It is natural to ask whether closeness of the forward map and its approximation imply closeness of the posterior measures. We now address this question.

Assumptions 4.7. Let $X' \subseteq X$ and assume that $\Phi \in C(X';\mathbb{R})$. Assume further that there are functions $M_i : \mathbb{R}^+ \to \mathbb{R}^+$, $i = 1, 2$, independent of N and monotonic non-decreasing separately in each argument, and with M_2 strictly positive, such that for all $u \in X'$,

$$\Phi(u) \geq -M_1(\|u\|_X),$$
$$\Phi^N(u) \geq -M_1(\|u\|_X),$$
$$|\Phi(u) - \Phi^N(u)| \leq M_2(\|u\|_X)\psi(N),$$

where $\psi(N) \to 0$ as $N \to \infty$. □

The following two theorems are very similar to Theorems 4.3, 4.5 and the proofs are adapted to estimate changes in the posterior caused by changes in the potential Φ, rather than the data y.

Theorem 4.8. *Let Assumptions 4.7 hold. Assume that $\mu_0(X') = 1$ and that $\mu_0(X' \cap B) > 0$ for some bounded set B in X. Assume additionally that, for every fixed $r > 0$,*

$$\exp\big(M_1(r, \|u\|_X)\big) \in L^1_{\mu_0}(X;\mathbb{R}).$$

Then Z, Z^N given by (4.1b), (4.4b) are positive and finite and the probability measures μ and μ^N given by (4.1), (4.4) are well-defined. Furthermore, for sufficiently large N, Z^N given by (4.5b) is bounded below by a positive constant independent of N.

Proof. Finiteness of the normalization constants Z and Z^N follows from the lower bounds on Φ and Φ^N given in Assumptions 4.7, together with the integrability condition in the theorem. Since $u \sim \mu_0$ satisfies $u \in X'$ a.s., we have

$$Z = \int_{X'} \exp\big(-\Phi(u)\big)\mu_0(du).$$

Note that $B' = X' \cap B$ is bounded in X. Thus

$$R_1 := \sup_{u\in B'} \|u\|_X < \infty.$$

Since $\Phi : X' \to \mathbb{R}$ is continuous it is finite at every point in B'. Thus, by the properties of $|\Phi(\cdot) - \Phi^N(\cdot)|$ implied by Assumptions 4.7, we see that

$$\sup_{u\in B'} \Phi(u) = R_2 < \infty.$$

Hence

$$Z \geq \int_{B'} \exp(-R_2)\mu_0(du) = \exp(-R_2)\mu_0(B').$$

Since $\mu_0(B')$ is assumed positive and R_2 is finite we deduce that $Z > 0$. By Assumptions 4.7 we may choose N large enough so that

$$\sup_{u\in B'} |\Phi(u) - \Phi^N(u)| \leq R_2$$

so that

$$\sup_{u\in B'} \Phi^N(u) \leq 2R_2 < \infty.$$

Hence

$$Z^N \geq \int_{B'} \exp(-2R_2)\mu_0(du) = \exp(-2R_2)\mu_0(B').$$

Since $\mu_0(B')$ is assumed positive and R_2 is finite we deduce that $Z^N > 0$. Furthermore, the lower bound is independent of N, as required. □

Theorem 4.9. *Let Assumptions 4.7 hold. Assume that $\mu_0(X') = 1$ and that $\mu_0(X' \cap B) > 0$ for some bounded set B in X. Assume additionally that*

$$\exp\big(M_1(\|u\|_X)\big)\Big(1 + M_2(\|u\|_X)^2\Big) \in L^1_{\mu_0}(X;\mathbb{R}).$$

Then there is $C > 0$ such that, for all N sufficiently large,

$$d_{\text{Hell}}(\mu, \mu^N) \leq C\psi(N).$$

Proof. Throughout this proof we use C to denote a constant independent of u, and N; it may change from occurrence to occurrence. We use the fact that, since $M_2(\cdot)$ is monotonic non-decreasing and since it is strictly positive on $[0, \infty)$,

$$\exp\big(M_1(\|u\|_X)\big)M_2(\|u\|_X) \leq \exp\big(M_1(\|u\|_X)\big)\Big(1 + M_2(\|u\|_X)^2\Big), \tag{4.6a}$$

$$\exp\big(M_1(\|u\|_X)\big) \leq \exp\big(M_1(\|u\|_X)\big)\Big(1 + M_2(\|u\|_X)^2\Big). \tag{4.6b}$$

Let Z and Z^N denote the normalization constants for μ and μ^N so that for all N sufficiently large, by Theorem 4.8,

$$Z = \int_{X'} \exp\Big(-\Phi(u)\Big)\mu_0(du) > 0,$$
$$Z^N = \int_{X'} \exp\Big(-\Phi^N(u)\Big)\mu_0(du) > 0,$$

with positive lower bounds independent of N. Then, using the local Lipschitz property of the exponential and the approximation property of $\Phi^N(\cdot)$ from Assumptions 4.7, together with (4.6a), we have

$$\begin{aligned}
|Z - Z^N| &\leq \int_{X'} |\exp\big(-\Phi(u)\big) - \exp\big(-\Phi^N(u)\big)|\mu_0(du) \\
&\leq \int_{X'} \exp\big(M_1(\|u\|_X)\big)|\Phi(u) - \Phi^N(u)|\mu_0(du) \\
&\leq \Big(\int_{X'} \exp\big(M_1(\|u\|_X)\big)M_2(\|u\|_X)\mu_0(du)\Big)\Big)\psi(N) \\
&\leq \Big(\int_{X'} \exp\big(M_1(\|u\|_X)\big)(1 + M_2(\|u\|_X)^2)\mu_0(du)\Big)\Big)\psi(N) \\
&\leq C\psi(N).
\end{aligned}$$

The last line follows because the integrand is in $L^1_{\mu_0}$ by assumption. From the definition of Hellinger distance we have

$$\Big(d_{\text{Hell}}(\mu^y, \mu^{y'})\Big)^2 \leq I_1 + I_2,$$

where

$$I_1 = \frac{1}{Z}\int_{X'} \Big(\exp\big(-\frac{1}{2}\Phi(u)\big) - \exp(-\frac{1}{2}\Phi^N(u))\Big)^2 \mu_0(du),$$
$$I_2 = \big|Z^{-\frac{1}{2}} - (Z^N)^{-\frac{1}{2}}\big|^2 \int_{X'} \exp(-\Phi^N(u))\mu_0(du).$$

Note that, again by means of similar Lipschitz calculations to those above, using the fact that $Z, Z^N > 0$ uniformly for N sufficiently large by Theorem 4.8, and Assumptions 4.7,

$$\begin{aligned}
I_1 &\leq \frac{1}{4Z}\int_{X'} \exp\big(M_1(\|u\|_X)|\Phi(u) - \Phi^N(u)|^2\mu_0(du) \\
&\leq \frac{1}{Z}\Big(\int_{X'} \exp\big(M_1(\|u\|_X)\big)M_2(\|u\|_X)^2\mu_0(du)\Big)\psi(N)^2 \\
&\leq C\psi(N)^2.
\end{aligned}$$

Also, using Assumptions 4.7, together with (4.6b),

$$\begin{aligned}
\int_{X'} \exp\big(-\Phi^N(u)\big)\mu_0(du) &\leq \int_{X'} \exp\big(M_1(\|u\|_X)\big)\mu_0(du) \\
&< \infty,
\end{aligned}$$

and the upper bound is independent of N. Hence

$$I_2 \leq C\big(Z^{-3} \vee (Z^N)^{-3}\big)|Z - Z^N|^2 \leq C\psi(N)^2.$$

The result is complete. □

Remarks 4.10. The following two remarks are relevant to establishing the conditions of the preceding two theorems, and to applying them.

- As mentioned in the previous susbection concerning well-posedness, the Fernique Theorem can frequently be used to establish integrability conditions, such as those in the two preceding theorems when μ_0 is Gaussian.
- Using the ideas underlying Remark 4.6, the preceding theorem enables us to translate errors arising from approximation of the forward problem into errors in the Bayesian solution of the inverse problem. Furthermore, the errors in the forward and inverse problems scale the same way with respect to N. For functions f which are in L^2_μ and $L^2_{\mu^N}$, uniformly with respect to N, the closeness of the Hellinger metric implies closeness of expectations of f:

$$|\mathbb{E}^\mu f(u) - \mathbb{E}^{\mu^N} f(u)| \leq C\psi(N).$$

4.3. MAP Estimators and Tikhonov Regularization. The aim of this section is to connect the probabilistic approach to inverse problems with the classical method of Tikhonov regularization. We consider the setting in which the prior measure μ_0 is a Gaussian measure. We then show that MAP estimators, points of maximal probability, coincide with minimizers of a Tikhonov-Phillips regularized least-squares function, with regularization being with respect to the Cameron-Martin norm of the Gaussian prior. The data y plays no explicit role in our developments here and so we work in the setting of equation (4.4). Recall, however, that in the context of inverse problems, a classical methodology is to simply try and minimize (subject to some regularization) $\Phi(u)$. Indeed for finite data and Gaussian observational noise with Gaussian distribution $N(0, \Gamma)$ we have

$$\Phi(u) = \frac{1}{2}\big|\Gamma^{-\frac{1}{2}}\big(y - G(u)\big)\big|^2.$$

Thus Φ is simply a covariance weighted model-data misfit least squares function.

In this section we show that maximizing probability under μ (in a sense that we will make precise in what follows) is equivalent to minimizing

$$I(u) = \begin{cases} \Phi(u) + \frac{1}{2}\|u\|^2_E & \text{if } u \in E, \text{ and} \\ +\infty & \text{else.} \end{cases} \tag{4.7}$$

Here $(E, \|\cdot\|_E)$ denotes the Cameron-Martin space associated to the Gaussian prior μ_0. We view μ_0 as a Gaussian probability measure on a separable Banach space $(X, \|\cdot\|_X)$ so that $\mu_0(X) = 1$. We make the following assumptions about the function Φ :

Assumption 4.11. The function $\Phi\colon X \to \mathbb{R}$ satisfies the following conditions:

(i) For every $\epsilon > 0$ there is an $M = M(\epsilon) \in \mathbb{R}$, such that for all $u \in X$,

$$\Phi(u) \geq M - \epsilon\|u\|_X^2.$$

(ii) Φ is locally bounded from above, *i.e.* for every $r > 0$ there exists $K = K(r) > 0$ such that, for all $u \in X$ with $\|u\|_X < r$ we have

$$\Phi(u) \leq K.$$

(iii) Φ is locally Lipschitz continuous, *i.e.* for every $r > 0$ there exists $L = L(r) > 0$ such that for all $u_1, u_2 \in X$ with $\|u_1\|_X, \|u_2\|_X < r$ we have

$$|\Phi(u_1) - \Phi(u_2)| \leq L\|u_1 - u_2\|_X.$$

In finite dimensions, for measures which have a continuous density with respect to Lebesgue measure, there is an obvious notion of most likely point(s): simply the point(s) at which the Lebesgue density is maximized. This way of thinking does not translate into the infinite dimensional context, but there is a way of restating it which does. Fix a small radius $\delta > 0$ and identify centres of balls of radius δ which have maximal probability. Letting $\delta \to 0$ then recovers the preceding definition, when there is a continuous Lebesgue density. We adopt this small ball approach in the infinite dimensional setting.

For $z \in E$, let $B^\delta(z) \subset X$ be the open ball centred at $z \in X$ with radius δ in X. Let

$$J^\delta(z) = \mu\big(B^\delta(z)\big)$$

be the mass of the ball $B^\delta(z)$ under the measure μ. Similarly we define

$$J_0^\delta(z) = \mu_0\big(B^\delta(z)\big)$$

the mass of the ball $B^\delta(z)$ under the Gaussian prior. Recall that all balls in a separable Banach space have positive Gaussian measure, by Theorem 7.28; it thus follows that $J_0^\delta(z)$ is finite and positive for any $z \in E$. By Assumptions 4.11(i),(ii) together with the Fernique Theorem 2.13 the same is true for $J^\delta(z)$. Our first theorem encapsulates the idea that probability is maximized where I is minimized. To see this, fix any point z_2 in the Cameron-Martin space E and notice that the probability of the small ball at z_1 is maximized, asymptotically as the radius of the ball tends to zero, at minimizers of I.

Theorem 4.12. *Let Assumptions 4.11 hold and assume that $\mu_0(X) = 1$. Then the function I defined by* (4.7) *satisfies, for any $z_1, z_2 \in E$,*

$$\lim_{\delta\to 0} \frac{J^\delta(z_1)}{J^\delta(z_2)} = \exp\big(I(z_2) - I(z_1)\big).$$

Proof. Since $J^\delta(z)$ is finite and positive for any $z \in E$ the ratio of interest is finite and positive. The key estimate in the proof is given in Theorem 7.30:

$$\lim_{\delta\to 0} \frac{J_0^\delta(z_1)}{J_0^\delta(z_2)} = \exp\left(\frac{1}{2}\|z_2\|_E^2 - \frac{1}{2}\|z_1\|_E^2\right). \tag{4.8}$$

This estimate transfers questions about probability, naturally asked on the space X of full measure under μ_0, into statements concerning the Cameron-Martin norm of μ_0; note that under this norm a random variable distributed as μ_0 is almost surely infinite so the result is non-trivial.

We have

$$\begin{aligned}\frac{J^\delta(z_1)}{J^\delta(z_2)} &= \frac{\int_{B^\delta(z_1)} \exp(-\Phi(u))\,\mu_0(\mathrm{d}u)}{\int_{B^\delta(z_2)} \exp(-\Phi(v))\,\mu_0(\mathrm{d}v)} \\ &= \frac{\int_{B^\delta(z_1)} \exp(-\Phi(u)+\Phi(z_1))\exp(-\Phi(z_1))\,\mu_0(\mathrm{d}u)}{\int_{B^\delta(z_2)} \exp(-\Phi(v)+\Phi(z_2))\exp(-\Phi(z_2))\,\mu_0(\mathrm{d}v)}.\end{aligned}$$

By Assumption 4.11 (iii) there is $L = L(r)$ such that, for all $u, v \in X$ with $\max\{\|u\|_X, \|v\|_X\} < r$,

$$-L\,\|u-v\|_X \;\le\; \Phi(u) - \Phi(v) \;\le\; L\,\|u-v\|_X.$$

If we define $L_1 = L(\|z_1\|_X + \delta)$ and $L_2 = L(\|z_2\|_X + \delta)$ then we have

$$\begin{aligned}\frac{J^\delta(z_1)}{J^\delta(z_2)} &\le \mathrm{e}^{\delta(L_1+L_2)} \frac{\int_{B^\delta(z_1)} \exp(-\Phi(z_1))\,\mu_0(\mathrm{d}u)}{\int_{B^\delta(z_2)} \exp(-\Phi(z_2))\,\mu_0(\mathrm{d}v)} \\ &= \mathrm{e}^{\delta(L_1+L_2)}\mathrm{e}^{-\Phi(z_1)+\Phi(z_2)} \frac{\int_{B^\delta(z_1)}\,\mu_0(\mathrm{d}u)}{\int_{B^\delta(z_2)}\,\mu_0(\mathrm{d}v)}.\end{aligned}$$

Now, by (4.8), we have

$$\frac{J^\delta(z_1)}{J^\delta(z_2)} \le r_1(\delta)\,\mathrm{e}^{\delta(L_2+L_1)}\mathrm{e}^{-I(z_1)+I(z_2)}$$

with $r_1(\delta) \to 1$ as $\delta \to 0$. Thus

$$\limsup_{\delta\to 0} \frac{J^\delta(z_1)}{J^\delta(z_2)} \;\le\; \mathrm{e}^{-I(z_1)+I(z_2)}. \tag{4.9}$$

Similarly we obtain

$$\frac{J^\delta(z_1)}{J^\delta(z_2)} \ge \frac{1}{r_2(\delta)}\,\mathrm{e}^{-\delta(L_2+L_1)}\mathrm{e}^{-I(z_1)+I(z_2)}$$

with $r_2(\delta) \to 1$ as $\delta \to 0$ and deduce that

$$\liminf_{\delta\to 0} \frac{J^\delta(z_1)}{J^\delta(z_2)} \;\ge\; \mathrm{e}^{-I(z_1)+I(z_2)} \tag{4.10}$$

Inequalities (4.9) and (4.10) give the desired result. □

We have thus linked the Bayesian approach to inverse problems with a classical regularization technique. We conclude the subsection by showing that, under the prevailing Assumptions 4.11, the minimization problem for I is well-defined. We first recall a basic definition and lemma from the calculus of variations.

Definition 4.13. The function $I : E \to \mathbb{R}$ is *weakly lower semicontinuous* if

$$\liminf_{n\to\infty} I(u_n) \geq I(u)$$

whenever $u_n \rightharpoonup u$ in E. The function $I : E \to \mathbb{R}$ is *weakly continuous* if

$$\lim_{n\to\infty} I(u_n) = I(u)$$

whenever $u_n \rightharpoonup u$ in E. □

Clearly weak continuity implies weak lower semicontinuity.

Lemma 4.14. *If $(E, \langle\cdot,\cdot\rangle_E)$ is a Hilbert space with induced norm $\|\cdot\|_E$ then the quadratic form $J(u) := \frac{1}{2}\|u\|_E^2$ is weakly lower semicontinuous.*

Proof. The result follows from the fact that

$$\begin{aligned} J(u_n) - J(u) &= \frac{1}{2}\|u_n\|_E^2 - \frac{1}{2}\|u\|_E^2 \\ &= \frac{1}{2}\langle u_n - u, u_n + u\rangle_E \\ &= \frac{1}{2}\langle u_n - u, 2u\rangle_E + \frac{1}{2}\|u_n - u\|_E^2 \\ &\geq \frac{1}{2}\langle u_n - u, 2u\rangle_E. \end{aligned}$$

But the right hand side tends to zero since $u_n \rightharpoonup u$ in E. Hence the result follows. □

Theorem 4.15. *Suppose that Assumptions 4.11 hold and let E be a Hilbert space compactly embedded in X. Then there exists $\overline{u} \in E$ such that*

$$I(\overline{u}) = \overline{I} := \inf\{I(u) : u \in E\}.$$

Furthermore, if $\{u_n\}$ is a minimizing sequence satisfying $I(u_n) \to I(\overline{u})$ then there is a subsequence $\{u_{n'}\}$ that converges strongly to $\overline{u}$ in E.

Proof. Compactness of E in X implies that, for some universal constant C,

$$\|u\|_X^2 \leq C\|u\|_E^2.$$

Hence, by Assumption 4.11(i), it follows that, for any $\epsilon > 0$, there is $M(\epsilon) \in \mathbb{R}$ such that

$$\Big(\frac{1}{2} - C\epsilon\Big)\|u\|_E^2 + M(\epsilon) \leq I(u).$$

By choosing ϵ sufficiently small, we deduce that there is $M \in \mathbb{R}$ such that, for all $u \in E$,

$$\frac{1}{4}\|u\|_E^2 + M \leq I(u). \tag{4.11}$$

Let u_n be an infimizing sequence satisfying $I(u_n) \to \overline{I}$ as $n \to \infty$. For any $\delta > 0$ there is $N = N_1(\delta)$:

$$\overline{I} \leq I(u_n) \leq \overline{I} + \delta, \quad \forall n \geq N_1. \tag{4.12}$$

Using (4.11) we deduce that the sequence $\{u_n\}$ is bounded in E and, since E is a Hilbert space, there exists $\overline{u} \in E$ such that $u_n \rightharpoonup \overline{u}$ in E. By the compact embedding of E in X we deduce that $u_n \to \overline{u}$, strongly in X. By the Lipschitz continuity of Φ in X (Assumption 4.11(iii)) we deduce that $\Phi(u_n) \to \Phi(\overline{u})$. Thus Φ is weakly continuous on E. The functional $J(u) := \frac{1}{2}\|u\|_E^2$ is weakly lower semicontinuous on E by Lemma 4.14. Hence $I(u) = J(u) + \Phi(u)$ is weakly lower semicontinuous on E. Using this fact in (4.12) it follows that, for any $\delta > 0$,

$$\overline{I} \leq I(\overline{u}) \leq \overline{I} + \delta.$$

Since δ is arbitrary the first result follows.

By passing to a further subsequence, and for $n, \ell \geq N_2(\delta)$,

$$\begin{aligned}
\frac{1}{4}\|u_n - u_\ell\|_E^2 &= \frac{1}{2}\|u_n\|_E^2 + \frac{1}{2}\|u_\ell\|_E^2 - \frac{1}{4}\|u_n + u_\ell\|_E^2 \\
&= I(u_n) + I(u_\ell) - 2I\Big(\frac{1}{2}(u_n + u_\ell)\Big) - \Phi(u_n) - \Phi(u_\ell) + 2\Phi\Big(\frac{1}{2}(u_n + u_\ell)\Big) \\
&\leq 2(\overline{I} + \delta) - 2\overline{I} - \Phi(u_n) - \Phi(u_\ell) + 2\Phi\Big(\frac{1}{2}(u_n + u_\ell)\Big) \\
&\leq 2\delta - \Phi(u_n) - \Phi(u_\ell) + 2\Phi\Big(\frac{1}{2}(u_n + u_\ell)\Big).
\end{aligned}$$

But u_n, u_ℓ and $\frac{1}{2}(u_n + u_\ell)$ all converge strongly to $\overline{u}$ in X. Thus, by continuity of Φ, we deduce that for all $n, \ell \geq N_3(\delta)$,

$$\frac{1}{4}\|u_n - u_\ell\|_E^2 \leq 3\delta.$$

Hence the sequence is Cauchy in E and converges strongly and the proof is complete. □

Corollary 4.16. *Suppose that Assumptions 4.11 hold and the Gaussian measure μ_0 with Cameron-Martin space space E satisfies $\mu_0(X) = 1$. Then there exists $\overline{u} \in E$ such that*

$$I(\overline{u}) = \overline{I} := \inf\{I(u) : u \in E\}.$$

Furthermore, if $\{u_n\}$ is a minimizing sequence satisfying $I(u_n) \to I(\overline{u})$ then there is a subsequence $\{u_{n'}\}$ that converges strongly to $\overline{u}$ in E.

Proof. By Theorem 7.29, E is compactly embedded in X. Hence the result follows by Theorem 4.15.

□

4.4. Bibliographic Notes.

- Subsection 4.1. The well-posedness theory described here was introduced in the papers [17] and [93]. Relationships between the Hellinger dostance on probability measures, and the Total Variation distance and Kullback-Leibler divergence may be found in [38], [84], as well as in [93].

- Subsection 4.2. Generalization of the well-posedness theory to study the effect of numerical approximation of the forward model on the inverse problem may be found in [20]. The relationship between expectations and Hellinger distance, as used in Remark 4.10, is demonstrated in [93].

- Subsection 4.3. The connection between Tikhonov-Phillips regularization and MAP estimators is widely appreciated in computational Bayesian inverse problems; see [53]. Making the connection rigorous in the separable Banach space setting is the subject of the paper [30]; further references to the historical development of the subject may be found therein. Related to Lemma 4.14 see also [23, Chapter 3].

5. Measure Preserving Dynamics

The aim of this section is to study Markov processes, in continuous time, and Markov chains, in discrete time, which preserve the measure μ given by (4.4). The overall setting is described in subsection 5.1, and introduces the role of detailed balance and reversibility in constructing measure-preserving Markov chains/processes. Subsection 5.2 concerns Markov chain-Monte Carlo (MCMC) methods; these are Markov chains which are invariant with respect to μ. Metropolis-Hastings methods are introduced and the role of detailed balance in their construction is explained. The benefit of conceiving MCMC methods which are defined on the infinite dimensional space is emphasized. In particular, the idea of using proposals which preserve the prior, more specificallty which are prior reversible, is introduced as an example. In subsection 5.3 we show how sequential Monte Carlo (SMC) methods can be used to construct approximate samples from the measure μ given by (4.4). Again our perspective is to construct algorithms which are provably well-defined on the infinite dimensional space and in fact we find an upper bound for the approximation error of the SMC method which proves its convergence on an infinite dimensional space. The MCMC methods from the previous section play an important role in the construction of these SMC methods. Subsections 5.4–5.6 concern continuous time μ-reversible processes. In particular they concern derivation and study of a Langevin equation which is invariant with respect to the measure μ. (Note that this is called the overdamped Langevin equation for a physicist, the plain Langevin equation for a statistician.) In continuous time we work entirely in

the case of Gaussian prior measure $\mu_0 = N(0, \mathcal{C})$ on Hilbert space $\mathcal{H}$ with inner-product and norm denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ respectively; however in discrete time our analysis is more general, applying on a separable Banach space $(X, \| \cdot \|)$ and for quite general prior measure.

5.1. General Setting. This section is devoted to Banach space valued Markov chains or processes which are invariant with respect to the posterior measure μ^y constructed in subsection 3.2. Within this section, the data y arising in the inverse problems plays no explicit role; indeed the theory applies to a wide range of measures μ on separable Banach space X. Thus the discussion in this chapter includes, but is not limited to, Bayesian inverse problems. All of the Markov chains we construct will exploit structure in a reference measure μ_0 with respect to which the measure μ is absolutely continuous; thus μ has a density with respect to μ_0. In continuous time we will explicitly use the Gaussianity of μ_0, but in discrete time we will be more general.

Let μ_0 be a reference measure on the separable Banach space X equipped with the Borel σ-algebra $\mathsf{B}(X)$. We assume that $\mu \ll \mu_0$ is given by

$$\frac{d\mu}{d\mu_0}(u) = \frac{1}{Z} \exp\big(-\Phi(u)\big), \tag{5.1a}$$

$$Z = \int_X \exp\big(-\Phi(u)\big)\mu_0(du), \tag{5.1b}$$

where $Z \in (0, \infty)$. In the following we let $P(u, dv)$ denote a Markov transition kernel so that $P(u, \cdot)$ is a probability measure on $\big(X, \mathsf{B}(X)\big)$ for each $u \in X$. Our interest is in probability kernels which preserve μ.

Definition 5.1. The Markov chain with transition kernel P is *invariant* with respect to μ if

$$\int_X \mu(du)P(u, \cdot) = \mu(\cdot)$$

as measures on $\big(X, \mathsf{B}(X)\big)$. The Markov kernel is said to satisfy *detailed balance* with respect to μ if

$$\mu(du)P(u, dv) = \mu(dv)P(v, du)$$

as measures on $\big(X \times X, \mathsf{B}(X) \otimes \mathsf{B}(X)\big)$. The resulting Markov chain is then said to be *reversible* with respect to μ. □

It is straightforward to see, by integrating the detailed balance condition with respect to u and using the fact that $P(v, du)$ is a Markov kernel, the following:

Lemma 5.2. *A Markov chain which is reversible with respect to μ is also invariant with respect to μ.*

Reversible Markov chains and processes arise naturally in many physical systems which are in statistical equilibrium. They are also important, however, as a

means of *constructing* Markov chains which are invariant with respect to a given probability measure. We demonstrate this in subsection 5.2 where we consider the Metropolis-Hastings variant of MCMC methods. Then, in subsections 5.4, 5.5 and 5.6, we move to continuous time Markov processes. In particular we show that the equation

$$\frac{du}{dt} = -u - \mathcal{C}D\Phi(u) + \sqrt{2}\frac{dW}{dt}, \quad u(0) = u_0, \tag{5.2}$$

preserves the measure μ, where W is a $\mathcal{C}$-Wiener process, defined below in subsection 7.4. Precisely we show that if $u_0 \sim \mu$, independently of the driving Wiener process, then $\mathbb{E}\varphi\big(u(t)\big) = \mathbb{E}\varphi(u_0)$ for all $t > 0$ for continuous bounded φ defined on an appropriately chosen subspaces, under boundedness conditions on Φ and its derivatives.

Example 5.3. Consider the (measurable) Hilbert space $\big(\mathcal{H}, \mathsf{B}(\mathcal{H})\big)$ equipped, as usual, with the Borel σ-algebra. Let μ denote the Gaussian measure $N(0, \mathcal{C})$ on $\mathcal{H}$ and, for fixed u, let $P(u, dv)$ denote the Gaussian measure $N\big((1-\beta^2)^{\frac{1}{2}}u, \beta^2\mathcal{C}\big)$, also viewed as a probability measure on $\mathcal{H}$. Thus $v \sim P(u, dv)$ can be expressed as $v = (1-\beta^2)^{\frac{1}{2}}u + \beta\xi$ where $\xi \sim N(0, \mathcal{C})$ is independent of u. We show that P is reversible, and hence invariant, with respect to μ. To see this we note that $\mu(du)P(u, dv)$ is a centred Gaussian measure on $\mathcal{H} \times \mathcal{H}$, equipped with the σ-algebra $\mathsf{B}(\mathcal{H}) \otimes \mathsf{B}(\mathcal{H})$. The covariance of the jointly varying random variable is characterized by the identities

$$\mathbb{E}u \otimes u = \mathcal{C}, \ \mathbb{E}v \otimes v = \mathcal{C}, \ \mathbb{E}u \otimes v = (1-\beta^2)^{\frac{1}{2}}\mathcal{C}. \tag{5.3}$$

Indeed, letting $\nu(du, dv) := \mu(du)P(u, dv)$, and with $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ the inner product and norm on $\mathcal{H}$ respectively, we can write, using (7.17),

$$\begin{aligned}
\hat{\nu}(d\xi, d\eta) &= \int_{\mathcal{H}\times\mathcal{H}} \mathrm{e}^{i\langle u,\xi\rangle + i\langle v,\eta\rangle} \mu(du)P(u, dv) \\
&= \int_{\mathcal{H}} \mathrm{e}^{i\langle u,\xi\rangle} \int_{\mathcal{H}} \mathrm{e}^{i\langle v,\eta\rangle} \, P(u, dv) \, \mu(du) \\
&= \int_{\mathcal{H}} \mathrm{e}^{i\langle u,\xi\rangle} \mathrm{e}^{i\sqrt{1-\beta^2}\langle u,\eta\rangle - \frac{1}{2}\|\beta\mathcal{C}^{\frac{1}{2}}\eta\|^2} \mu(du) \\
&= \mathrm{e}^{-\frac{\beta^2}{2}\|\mathcal{C}^{\frac{1}{2}}\eta\|^2} \int_{\mathcal{H}} \mathrm{e}^{i\langle u, \sqrt{1-\beta^2}\,\eta + \xi\rangle} \mu(du) \\
&= \mathrm{e}^{-\frac{\beta^2}{2}\|\mathcal{C}^{\frac{1}{2}}\eta\|^2} \mathrm{e}^{-\frac{1}{2}\|\mathcal{C}^{\frac{1}{2}}(\sqrt{1-\beta^2}\,\eta + \xi)\|^2} \\
&= \exp\left(-\frac{1}{2}\|\mathcal{C}^{\frac{1}{2}}\eta\|^2 - \frac{1}{2}\|\mathcal{C}^{\frac{1}{2}}\xi\|^2 - (1-\beta^2)^{\frac{1}{2}} \langle \mathcal{C}^{\frac{1}{2}}\xi, \mathcal{C}^{\frac{1}{2}}\eta \rangle \right).
\end{aligned}$$

Hence, by Lemma 7.9 and equation (7.17), $\mu(du)P(u, dv)$ is a centred Gaussian measure with the covariance operator given by (5.3). Since the expression in the last line of the above equation is symmetric in ξ and η, $\mu(dv)P(v, du)$ is a centred Gaussian measure with the same covariance as $\mu(du)P(u, dv)$ and so the reversibility is proved. □

Example 5.4. Consider the equation

$$\frac{du}{dt} = -u + \sqrt{2}\frac{dW}{dt},\ u(0) = u_0, \tag{5.4}$$

where W is a $\mathcal{C}$-Wiener process (defined in subsection 7.4 below). Then

$$u(t) = e^{-t}u_0 + \sqrt{2}\int_0^t e^{-(t-s)}dW(s).$$

Use of the Itô isometry demonstrates that $u(t)$ is distributed according to the Gaussian $N\big(e^{-t}u_0, (1-e^{-2t})\mathcal{C}\big)$. Setting $\beta^2 = 1-e^{-2t}$ and employing the previous example shows that the Markov process is reversible since, for every $t > 0$, the transition kernel of the process is reversible. □

5.2. Metropolis-Hastings Methods.

In this section we study Metropolis-Hastings methods designed to sample from the probability measure μ given by (5.1). The perspective that we have described on inverse problems, specifically the formulation of Bayesian inversion on function space, leads to new sampling methods which are specifically tailored to the high dimensional problems which arise from discretization of the infinite dimensional setting. In particular it leads naturally to the philosophy that it is advantageous to design algorithms which, in principle, make sense in infinite dimensions; it is these methods which will perform well under refinement of finite dimensional approximations. Most Metropolis-Hastings methods which are defined in finite dimensions will not make sense in the infinite dimensional limit. This is because the acceptance probability for Metropolis-Hastings methods is defined as the Radon-Nikodym derivative between two measures describing the behaviour of the Markov chain in stationarity. Since measures in infinite dimensions have a tendency to be mutually singular, only carefully designed methods will have interpretations in infinite dimensions. To simplify the presentation we work with the following assumptions throughout:

Assumptions 5.5. The function $\Phi : X \to \mathbb{R}$ is bounded on bounded subsets of X. □

We now consider the following prototype Metropolis-Hastings method which accept-rejects proposals from a Markov kernel Q to produce a Markov chain with kernel P which is reversible with respect to μ.

Algorithm 5.6. Given $a : X \times X \to [0,1]$ generate $\{u^{(k)}\}_{k\geq 0}$ as follows:

1 Set $k = 0$ and pick $u^{(0)} \in X$.

2 Propose $v^{(k)} \sim Q(u^{(k)}, dv)$.

3 Set $u^{(k+1)} = v^{(k)}$ with probability $a(u^{(k)}, v^{(k)})$, independently of $(u^{(k)}, v^{(k)})$.

4 Set $u^{(k+1)} = u^{(k)}$ otherwise.

5 $k \to k+1$ and return to 2.

□

Given a proposal kernel Q, a key question in the design of MCMC methods is the question of how to choose $a(u,v)$ to ensure that $P(u,dv)$ satisfies detailed balance with respect to μ. If the resulting Markov chain is ergodic this then yields an algorithm which, asymptotically, samples from μ, and can be used to estimate expectations against μ.

To determine conditions on a which are necessary and sufficient for detailed balance we first note that the Markov kernel which arises from accepting/rejecting proposals from Q is given by

$$P(u,dv) = Q(u,dv)a(u,v) + \delta_u(dv)\int_X \big(1-a(u,w)\big)Q(u,dw). \tag{5.5}$$

Notice that

$$\int_X P(u,dv) = 1$$

as required. Substituting the expression for P into the detailed balance condition from Definition 5.1 we obtain

$$\begin{aligned} &\mu(du)Q(u,dv)a(u,v) + \mu(du)\delta_u(dv)\textstyle\int_X\big(1-a(u,w)\big)Q(u,dw) \\ = \\ &\mu(dv)Q(v,du)a(v,u) + \mu(dv)\delta_v(du)\textstyle\int_X\big(1-a(v,w)\big)Q(v,dw). \end{aligned}$$

We now note that the measure $\mu(du)\delta_u(dv)$ is in fact symmetric in the pair (u,v) and that $u=v$ almost surely under it. As a consequence the identity reduces to

$$\mu(du)Q(u,dv)a(u,v) = \mu(dv)Q(v,du)a(v,u). \tag{5.6}$$

Our aim now is to identify choices of a which ensure that (5.6) is satisfied. This will then ensure that the prototype algorithm does indeed lead to a Markov chain for which μ is invariant. To this end we define the measures

$$\nu(du,dv) = \mu(du)Q(u,dv)$$

and

$$\nu^{\mathsf{T}}(du,dv) = \mu(dv)Q(v,du)$$

on $\big(X\times X, \mathsf{B}(X)\otimes\mathsf{B}(X)\big)$. The following theorem determines a necessary and sufficient condition for the choice of a to make the algorithm μ reversible, and identifies the canonical Metropolis-Hastings choice.

Theorem 5.7. *Assume that ν and ν^{T} are equivalent as measures on $X\times X$, equipped with the σ-algebra $\mathsf{B}(X)\otimes\mathsf{B}(X)$, and that $\nu(du,dv) = r(u,v)\nu^{\mathsf{T}}(du,dv)$. Then the probability kernel* (5.5) *satisfies detailed balance if and only if*

$$r(u,v)a(u,v) = a(v,u), \quad \nu\text{-a.s.}. \tag{5.7}$$

In particular the choice $\alpha_{\mathrm{mh}}(u,v) = \min\{1, r(v,u)\}$ will imply detailed balance.

Proof. Since ν and ν^{T} are equivalent (5.6) holds if and only if

$$\frac{d\nu}{d\nu^{\mathsf{T}}}(u,v)a(u,v) = a(v,u).$$

This is precisely (5.7). Now note that $\nu(du,dv) = r(u,v)\nu^{\mathsf{T}}(du,dv)$ and $\nu^{\mathsf{T}}(du,dv) = r(v,u)\nu(du,dv)$ since ν and ν^{T} are equivalent. Thus $r(u,v)r(v,u) = 1$. It follows that

$$\begin{aligned} r(u,v)\alpha_{\mathrm{mh}}(u,v) &= \min\{r(u,v), r(u,v)r(v,u)\} \\ &= \min\{r(u,v), 1\} \\ &= \alpha_{\mathrm{mh}}(v,u) \end{aligned}$$

as required. □

A good example of the resulting methodology arises in the case where $Q(u,dv)$ is reversible with respect to μ_0 :

Theorem 5.8. *Let Assumption 5.5 hold. Consider Algorithm 5.6 applied to* (5.1) *in the case where the proposal kernel Q is reversible with respect to the prior μ_0. Then the resulting Markov kernel P given by* (5.5) *is reversible with respect to μ if $a(u,v) = \min\{1, \exp\big(\Phi(u) - \Phi(v)\big)\}$.*

Proof. Prior reversibility implies that

$$\mu_0(du)Q(u,dv) = \mu_0(dv)Q(v,du).$$

Multiplying both sides by $\exp\big(-\Phi(u)\big)$ gives

$$\mu(du)Q(u,dv) = \exp\big(-\Phi(u)\big)\mu_0(dv)Q(v,du)$$

and then multiplication by $\exp\big(-\Phi(v)\big)$ gives

$$\exp\big(-\Phi(v)\big)\mu(du)Q(u,dv) = \exp\big(-\Phi(u)\big)\mu(dv)Q(v,du).$$

This is the statement that

$$\exp\big(-\Phi(v)\big)\nu(du,dv) = \exp\big(-\Phi(u)\big)\nu^{\mathsf{T}}(du,dv).$$

Since Φ is bounded on bounded sets by Assumption 5.5 we deduce that

$$\frac{d\nu}{d\nu^{\mathsf{T}}}(u,v) = r(u,v) = \exp\big(\Phi(v) - \Phi(u)\big).$$

Theorem 5.7 gives the desired result. □

We provide two examples of prior reversible proposals, the first applying in the general Banach space setting, and the second when the prior is a Gaussian measure.

Algorithm 5.9. Independence Sampler The independence sampler arises when $Q(u, dv) = \mu_0(dv)$ so that proposals are independent draws from the prior. Clearly prior reversibility is satisfied. The following algorithm results. Define

$$a(u, v) = \min\{1, \exp\big(\Phi(u) - \Phi(v)\big)\}$$

and generate $\{u^{(k)}\}_{k\geq 0}$ as follows:

1. Set $k = 0$ and pick $u^{(0)} \in X$.
2. Propose $v^{(k)} \sim \mu_0$ independently of $u^{(k)}$.
3. Set $u^{(k+1)} = v^{(k)}$ with probability $a(u^{(k)}, v^{(k)})$, independently of $(u^{(k)}, v^{(k)})$.
4. Set $u^{(k+1)} = u^{(k)}$ otherwise.
5. $k \to k + 1$ and return to 2.

□

The preceding algorithm works well when the likelihood is not *too* informative; however when the information in the likelihood is substantial, and $\Phi(\cdot)$ varies significantly depending on where it is evaluated, the independence sampler will not work well. In such a situation it is typically the case that *local* proposals are needed, with a parameter controlling the degree of locality; this parameter can then be optimized by choosing it as large as possible, consistent with achieving a reasonable acceptance probability. The following algorithm is an example of this concept, with parameter β playing the role of the locality parameter. The algorithm may be viewed as the natural generalization of the random Walk Metropolis method, for targets defined by density with respect to Lebesgue measure, to the situation where the targets are defined by density with respect to Gaussian measure. The name pCN is used because of the original derivation of the algorithm via a Crank-Nicolson discretization of the Hilbert space valued SDE (5.4).

Algorithm 5.10. pCN Method Assume that X is a Hilbert space $\big(\mathcal{H}, \mathsf{B}(\mathcal{H})\big)$ and that $\mu_0 = N(0, \mathcal{C})$ is a Gaussian prior on $\mathcal{H}$. Now define $Q(u, dv)$ to be the Gaussian measure $N\big((1-\beta^2)^{\frac{1}{2}}u, \beta^2\mathcal{C}\big)$, also on $\mathcal{H}$. Example 5.3 shows that Q is μ_0 reversible. The following algorithm results.

Define

$$a(u, v) = \min\{1, \exp\big(\Phi(u) - \Phi(v)\big)\}$$

and generate $\{u^{(k)}\}_{k\geq 0}$ as follows:

1. Set $k = 0$ and pick $u^{(0)} \in X$.
2. Propose $v^{(k)} = \sqrt{(1-\beta^2)}u^{(k)} + \beta\xi^{(k)}, \quad \xi^{(k)} \sim N(0, \mathcal{C})$.
3. Set $u^{(k+1)} = v^{(k)}$ with probability $a(u^{(k)}, v^{(k)})$, independently of $(u^{(k)}, \xi^{(k)})$.
4. Set $u^{(k+1)} = u^{(k)}$ otherwise.

5. $k \to k+1$ and return to 2.

□

Example 5.11. Example 5.4 shows that using the proposal from Example 5.3 within a Metropolis-Hastings context may be viewed as using a proposal based on the μ-measure-preserving equation (5.2), but with the $D\Phi$ term dropped. The accept-reject mechanism of Algorithm 5.10, which is based on differences of Φ, then compensates for the missing $D\Phi$ term.

5.3. Sequential Monte Carlo Methods. In this section we introduce sequential Monte Carlo methods and show how these may be viewed as a generic tool for sampling the posterior distribution arising in Bayesian inverse problems. These methods have their origin in filtering of dynamical systems but, as we will demonstrate, have the potential as algorithms for probing a very wide class of probability measures. The key idea is to introduce a sequence of measures which evolve the prior distribution into the posterior distribution. Particle filtering methods are then applied to this sequence of measures in order to evolve a set of particles that are prior distributed into a set of particles that are approximately posterior distributed. From a practical perspective, a key step in the construction of these methods is the use of MCMC methods which preserve the measure of interest, and other measures closely related to it; furthermore, our interest is in designing SMC methods which, in principle, are well-defined on the infinite dimensional space; for these two reasons the MCMC methods from the previous subsection play a central role in what follows.

Given integer J, let $h = J^{-1}$ and for non-negative integer $j \leq J$ define the sequence of measures $\mu_j \ll \mu_0$ by

$$\frac{d\mu_j}{d\mu_0}(u) = \frac{1}{Z_j}\exp\big(-jh\Phi(u)\big), \tag{5.8a}$$

$$Z_j = \int_X \exp\big(-jh\Phi(u)\big)\mu_0(du). \tag{5.8b}$$

Then $\mu_J = \mu$ given by (5.1); thus our interest is in approximating μ_J and we will achieve this by approximating the sequence of measures $\{\mu_j\}_{j=0}^J$, using information about μ_j to inform approximation of μ_{j+1}. To simplify the analysis we assume that Φ is bounded above and below on X so that there is $\phi^{\pm} \in \mathbb{R}$ such that

$$\phi^- \leq \Phi(u) \leq \phi^+ \quad \forall u \in X. \tag{5.9}$$

Without loss of generality we assume that $\phi^- \leq 0$ and that $\phi^+ \geq 0$, which may be achieved by normalization. Note that then the family of measures $\{\mu_j\}_{j=0}^J$ are mutually absolutely continuous and, furthermore,

$$\frac{d\mu_{j+1}}{d\mu_j}(u) = \frac{Z_j}{Z_{j+1}}\exp\big(-h\Phi(u)\big). \tag{5.10}$$

An important idea here is that, whilst μ_0 and μ may be quite far apart as measures, the pair of measures μ_j, μ_{j+1} can be quite close, for sufficiently small h. This fact can be used to incrementally evolve samples from μ_0 into approximate samples of μ_J.

Let L denote the operator on probability measures which corresponds to application of Bayes' theorem with likelihood proportional to $\exp\big(-h\Phi(u)\big)$ and let P_j denote any Markov kernel which preserves the measure μ_j; such kernels arise, for example, from the MCMC methods of the previous subsection. These considerations imply that

$$\mu_{j+1} = \mathsf{L}P_j\mu_j. \tag{5.11}$$

Sequential Monte Carlo methods proceed by approximating the sequence $\{\mu_j\}$ by a set of Dirac measures, as we now describe. It is useful to break up the iteration (5.11) and write it as

$$\widehat{\mu}_{j+1} = P_j\mu_j, \tag{5.12a}$$

$$\mu_{j+1} = \mathsf{L}\widehat{\mu}_{j+1}. \tag{5.12b}$$

We approximate each of the two steps in (5.12) separately. To this end it helps to note that, since P_j preserves μ_j,

$$\frac{d\mu_{j+1}}{d\widehat{\mu}_{j+1}}(u) = \frac{Z_j}{Z_{j+1}}\exp\big(-h\Phi(u)\big). \tag{5.13}$$

To define the method, we write an N-particle Dirac measure approximation of the form

$$\mu_j \approx \mu_j^N := \sum_{n=1}^{N} w_j^{(n)}\delta(v_j - v_j^{(n)}). \tag{5.14}$$

The approximate distribution is completely defined by particle positions $v_j^{(n)}$ and weights $w_j^{(n)}$ respectively. Thus the objective of the method is to find an update rule for $\{v_j^{(n)}, w_j^{(n)}\}_{n=1}^N \mapsto \{v_{j+1}^{(n)}, w_{j+1}^{(n)}\}_{n=1}^N$. The weights must sum to one. To do this we proceed as follows. First each particle $v_j^{(n)}$ is updated by proposing a new candidate particle $\widehat{v}_{j+1}^{(n)}$ according to the Markov kernel P_j; this corresponds to (5.12a) and creates an approximation to $\widehat{\mu}_{j+1}$. (See the last two parts of Remark 5.14 for a discussion on the role of P_j in the algorithm.) We can think of this approximation as a prior distribution for application of Bayes' rule in the form (5.12b), or equivalently (5.13). Secondly, each new particle is re-weighted according to the desired distribution μ_{j+1} given by (5.13). The required calculations are very straightforward because of the assumed form of the measures as sums of Dirac's, as we now explain.

The first step of the algorithm has made the approximation

$$\widehat{\mu}_{j+1} \approx \widehat{\mu}_{j+1}^N = \sum_{n=1}^{N} w_j^{(n)}\delta(v_{j+1} - \widehat{v}_{j+1}^{(n)}). \tag{5.15}$$

We now apply Bayes' formula in the form (5.13). Using an approximation proportional to (5.15) for $\widehat{\mu}_{j+1}$ we obtain

$$\mu_{j+1} \approx \mu_{j+1}^N := \sum_{n=1}^N w_{j+1}^{(n)} \delta(v_{j+1} - \widehat{v}_{j+1}^{(n)}). \tag{5.16}$$

where

$$\widehat{w}_{j+1}^{(n)} = \exp\bigl(-h\Phi(\widehat{v}_{j+1}^{(n)})\bigr) w_j^{(n)} \tag{5.17}$$

and normalization requires

$$w_{j+1}^{(n)} = \widehat{w}_{j+1}^{(n)} / \bigl(\sum_{n=1}^N \widehat{w}_{j+1}^{(n)}\bigr). \tag{5.18}$$

Practical experience shows that some weights become very small and for this reason it is desirable to add a *resampling step* to determine the $\{v_{j+1}^{(n)}\}$ by drawing from (5.16); this has the effect of removing particles with very low weights and replacing them with multiple copies of the particles with higher weights. Because the initial measure $\mathbb{P}(v_0)$ is not in Dirac form it is convenient to place this resampling step at the *start* of each iteration, rather than at the end as we have presented here, as this naturally introduces a particle approximation of the initial measure. This reordering makes no difference to the iteration we have described and results in the following algorithm.

Algorithm 5.12.

1. Let $\mu_0^N = \mu_0$ and set $j = 0$.
2. Draw $v_j^{(n)} \sim \mu_j^N$, $n = 1, \ldots, N$.
3. Set $w_j^{(n)} = 1/N$, $n = 1, \ldots, N$ and define μ_j^N by (5.14).
4. Draw $\widehat{v}_{j+1}^{(n)} \sim P_j(v_j^{(n)}, \cdot)$.
5. Define $w_{j+1}^{(n)}$ by (5.17), (5.18) and μ_{j+1}^N by (5.16).
6. $j \to j+1$ and return to 2.

□

We define S^N to be the mapping between probability measures defined by sampling N i.i.d. points from a measure and approximating that measure by an equally weighted sum of Dirac's at the sample points. Then the preceding algorithm may be written as

$$\mu_{j+1}^N = \mathsf{L} S^N P_j \mu_j^N. \tag{5.19}$$

Although we have written the sampling step S^N *after* application of P_j, some reflection shows that this is well-justified: applying P_j followed by S^N can be shown, by first conditioning on the initial point and sampling with respect to P_j, and then sampling over the distribution of the initial point, to be the algorithm as defined. The sequence of distributions that we wish to approximate simply satisfies

the iteration (5.11). Thus analyzing the particle filter requires estimation of the error induced by application of S^N (the *resampling error*) together with estimation of the rate of accumulation of this error in time.

The operators L, P_j and S^N map the space $\mathsf{P}(X)$ of probability measures on X into itself according to the following:

$$\begin{aligned}(\mathsf{L}\mu)(dv) &= \frac{\exp\bigl(-h\Phi(v)\bigr)\mu(dv)}{\int_X \exp\bigl(-h\Phi(v)\bigr)\mu(dv)},\\ (P_j\mu)(dv) &= \int_X P_j(v',dv)\mu(dv'),\\ (S^N\mu)(dv) &= \frac{1}{N}\sum_{n=1}^N \delta(v-v^{(n)})dv, \quad v^{(n)}\sim\mu \quad \text{i.i.d..}\end{aligned}$$

where P_j is the kernel associated with the μ_j-invariant Markov chain.

Let $\mu = \mu^{(\omega)}$ denote, for each ω, an element of $\mathsf{P}(X)$. If we assume that ω is a random variable, and let $\mathbb{E}^\omega$ denote expectation over ω, then we may define a distance $d(\cdot,\cdot)$ between two random probability measures $\mu^{(\omega)}, \nu^{(\omega)}$, as follows:

$$d(\mu,\nu) = \sup_{|f|_\infty\le 1}\sqrt{\mathbb{E}^\omega|\mu(f)-\nu(f)|^2},$$

with $|f|_\infty := \sup_{v\in X}|f(v)|$, and where we have used the convention that $\mu(f) = \int_X f(v)\mu(dv)$ for measurable $f : X \to \mathbb{R}$, and similar for ν. This distance does indeed generate a metric and, in particular, satisfies the triangle inequality. In fact it is simply the total variation distance in the case of measures which are not random.

With respect to this distance between random probability measures we may prove that the SMC method generates a good approximation of the true measure μ, in the limit $N\to\infty$. We use the fact that, under (5.9), we have

$$\exp\bigl(-h\phi^+\bigr) < \exp\bigl(-h\Phi(v)\bigr) < \exp\bigl(-h\phi^-\bigr).$$

Since $\phi^- \le 0$ and $\phi^+ \ge 0$ we deduce that there exists $\kappa\in(0,1)$ such that for all $v\in X$

$$\kappa < \exp\bigl(-h\Phi(v)\bigr) < \kappa^{-1}.$$

This constant κ appears in the following.

Theorem 5.13. *We assume in the following that* (5.9) *holds. Then*

$$d(\mu_J^N,\mu_J) \le \sum_{j=1}^J (2\kappa^{-2})^j \frac{1}{\sqrt{N}}.$$

Proof. The desired result is a consequence of the following three facts, whose proof we postpone to three lemmas at the end of the subsection:

$$\begin{aligned}\sup_{\mu\in\mathsf{P}(X)} d(S^N\mu,\mu) &\le \frac{1}{\sqrt{N}},\\ d(P_j\nu,P_j\mu) &\le d(\nu,\mu),\\ d(\mathsf{L}\nu,\mathsf{L}\mu) &\le 2\kappa^{-2}d(\nu,\mu).\end{aligned}$$

By the triangle inequality we have, for $\nu_j^N = P\mu_j^N$,

$$\begin{aligned} d(\mu_{j+1}^N, \mu_{j+1}) &= d(\mathsf{L}S^N P_j \mu_j^N, \mathsf{L}P_j\mu_j) \\ &\leq d(\mathsf{L}P_j\mu_j^N, \mathsf{L}P_j\mu_j) + d(\mathsf{L}S^N P_j\mu_j^N, \mathsf{L}P_j\mu_j^N) \\ &\leq 2\kappa^{-2}\Big(d(\mu_j^N, \mu_j) + d(S^N\nu_j^N, \nu_j^N)\Big) \\ &\leq 2\kappa^{-2}\Big(d(\mu_j^N, \mu_j) + \frac{1}{\sqrt{N}}\Big). \end{aligned}$$

Iterating, after noting that $\mu_0^N = \mu_0$, gives the desired result. □

Remarks 5.14. This theorem shows that the sequential particle filter actually reproduces the true posterior distribution $\mu = \mu_J$, in the limit $N \to \infty$. We make some comments about this.

- The measure $\mu = \mu_J$ is well-approximated by μ_j^N in the sense that, as the number of particles $N \to \infty$, the approximating measure converges to the true measure. The result holds in the infinite dimensional setting. As a consequence the algorithm as stated is robust to finite dimensional approximation.

- Note that $\kappa = \kappa(J)$ and that $\kappa \to 1$ as $J \to \infty$. Using this fact shows that the error constant in Theorem 5.13 behaves as $\sum_{j=1}^{J}(2\kappa^{-2})^j \asymp J\,2^J$. Optimizing this upper bound does not give a useful rule-of-thumb for choosing J, and in fact suggests choosing $J = 1$. In any case in applications Φ is not bounded from above, or even below in general, and a more refined analysis is then required.

- In principle the theory applies even if the Markov kernel P_j is simply the identity mapping on probability measures. However, moving the particles according to a non-trivial μ_j-invariant measure is absolutely essential for the methodology to work in practice. This can be seen by noting that if P_j is indeed taken to be the identity map on measures then the particle positions will be unchanged as j changes, meaning that the measure $\mu = \mu_J$ is approximated by weighted samples from the prior, clearly undesirable in general.

- In fact, if the Markov kernel P_j is ergodic then it is sometimes possible to obtain bounds which are *uniform* in J.

We now prove the three lemmas which underly the convergence proof.

Lemma 5.15. *The sampling operator satisfies*

$$\sup_{\mu\in\mathsf{P}(X)} d(S^N\mu, \mu) \leq \frac{1}{\sqrt{N}}.$$

Proof. Let ν be an element of $\mathsf{P}(X)$ and $\{v^{(k)}\}_{k=1}^N$ a set of i.i.d. samples with $v^{(1)} \sim \nu$; the randomness entering the probability measures is through these samples, expectation with respect to which we denote by $\mathbb{E}^\omega$ in what follows. Then

$$S^N\nu(f) = \frac{1}{N}\sum_{k=1}^N f(v^{(k)})$$

and, defining $\overline{f} = f - \nu(f)$, we deduce that

$$S^N\nu(f) - \nu(f) = \frac{1}{N}\sum_{k=1}^N \overline{f}(v^{(k)}).$$

It is straightforward to see that

$$\mathbb{E}^\omega \overline{f}(v^{(k)})\overline{f}(v^{(l)}) = \delta_{kl}\mathbb{E}^\omega|\overline{f}(v^{(k)})|^2.$$

Furthermore, for $|f|_\infty \le 1$,

$$\mathbb{E}^\omega|\overline{f}(v^{(1)})|^2 = \mathbb{E}^\omega|f(v^{(1)})|^2 - |\mathbb{E}^\omega f(v^{(1)})|^2 \le 1.$$

It follows that, for $|f|_\infty \le 1$,

$$\mathbb{E}^\omega|\nu(f) - S^N\nu(f)|^2 = \frac{1}{N^2}\sum_{k=1}^N \mathbb{E}^\omega|\overline{f}(v^{(k)})|^2 \le \frac{1}{N}.$$

Since the result is independent of ν we may take the supremum over all probability measures and obtain the desired result. □

Lemma 5.16. *Since P_j is a Markov kernel we have*

$$d(P_j\nu, P_j\nu') \le d(\nu, \nu').$$

Proof. The result is generic for any Markov kernel P, so we drop the index j on P_j for the duration of the proof. Define

$$q(v') = \int_X P(v', dv)f(v),$$

that is the expected value of f under one-step of the Markov chain started from v'. Clearly, since

$$|q(v')| \le \Big(\int_X P(v', dv)\Big)\sup_v |f(v)| = \sup_v |f(v)|$$

it follows that

$$\sup_v |q(v)| \le \sup_v |f(v)|.$$

Also, since

$$P\nu(f) = \int_X f(v)\Big(\int_X P(v', dv)\nu(dv')\Big),$$

exchanging the order of integration shows that

$$|P\nu(f) - P\nu'(f)| = |\nu(q) - \nu'(q)|.$$

Thus

$$\begin{aligned} d(P\nu, P\nu') &= \sup_{|f|_\infty \le 1} \Big(\mathbb{E}^\omega |P\nu(f) - P\nu'(f)|^2\Big)^{\frac{1}{2}} \\ &\le \sup_{|q|_\infty \le 1} \Big(\mathbb{E}^\omega |\nu(q) - \nu'(q)|^2\Big)^{\frac{1}{2}} \\ &= d(\nu, \nu') \end{aligned}$$

as required. □

Lemma 5.17. *Under the Assumptions of Theorem 5.13 we have*

$$d(\mathsf{L}\nu, \mathsf{L}\mu) \le 2\kappa^{-2} d(\nu, \mu).$$

Proof. Define $g(v) = \exp\big(-h\Phi(v)\big)$. Notice that for $|f|_\infty < \infty$ we can rewrite

$$\begin{aligned} (\mathsf{L}\nu)(f) - (\mathsf{L}\mu)(f) =& \frac{\nu(fg)}{\nu(g)} - \frac{\mu(fg)}{\mu(g)} \\ =& \frac{\nu(fg)}{\nu(g)} - \frac{\mu(fg)}{\nu(g)} + \frac{\mu(fg)}{\nu(g)} - \frac{\mu(fg)}{\mu(g)} \\ =& \frac{\kappa^{-1}}{\nu(g)}[\nu(\kappa fg) - \mu(\kappa fg)] + \frac{\mu(fg)}{\mu(g)}\frac{\kappa^{-1}}{\nu(g)}[\mu(\kappa g) - \nu(\kappa g)]. \end{aligned}$$

Now notice that $\nu(g)^{-1} \le \kappa^{-1}$ and that, for $|f|_\infty \le 1$, $\mu(fg)/\mu(g) \le 1$ since the expression corresponds to an expectation with respect to measure found from μ by reweighting with likelihood proportional to g. Thus

$$|(\mathsf{L}\nu)(f) - (\mathsf{L}\mu)(f)| \le \kappa^{-2}|\nu(\kappa fg) - \mu(\kappa fg)| + \kappa^{-2}|\nu(\kappa g) - \mu(\kappa g)|.$$

Since $|\kappa g| \le 1$ it follows that

$$\mathbb{E}^\omega |(\mathsf{L}\nu)(f) - (\mathsf{L}\mu)(f)|^2 \le 4\kappa^{-4} \sup_{|f|_\infty \le 1} \mathbb{E}^\omega |\nu(f) - \mu(f)|^2$$

and the desired result follows. □

5.4. Continuous Time Markov Processes. In the remainder of this section we shift our attention to continuous time processes which preserve μ; these are important in the construction of proposals for MCMC methods, and also as diffusion limits for MCMC. Our main goal is to show that the equation (5.2) preserves μ. Our setting is to work in the separable Hilbert space $\mathcal{H}$ with inner-product and norm denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ respectively. We assume that the prior μ_0 is a Gaussian on $\mathcal{H}$ and, furthermore, we specify the space $X \subset \mathcal{H}$ that will play a central role in this continuous time setting. This choice of space X will link the properties of the reference measure μ_0 and the potential Φ. We assume that $\mathcal{C}$ has eigendecomposition

$$\mathcal{C}\phi_j = \gamma_j^2 \phi_j \tag{5.23}$$

where $\{\phi_j\}_{j=1}^{\infty}$ forms an orthonormal basis for $\mathcal{H}$, and where $\gamma_j \asymp j^{-s}$. Necessarily $s > \frac{1}{2}$ since $\mathcal{C}$ must be trace-class to be a covariance on $\mathcal{H}$. We define the following scale of Hilbert subspaces, defined for $r > 0$, by

$$\mathcal{X}^r = \Big\{ u \in \mathcal{H} \big| \sum_{j=1}^{\infty} j^{2r} |\langle u, \phi_j\rangle|^2 < \infty \Big\}$$

and then extend to superspaces $r < 0$ by duality. We use $\|\cdot\|_r$ to denote the norm induced by the inner-product

$$\langle u, v\rangle_r = \sum_{j=1}^{\infty} j^{2r} u_j v_j$$

for $u_j = \langle u, \phi_j\rangle$ and $v_j = \langle v, \phi_j\rangle$. Application of Theorem 2.6 with $d = 1$ and $q = 2$ shows that $\mu_0(\mathcal{X}^r) = 1$ for all $r \in [0, s - \frac{1}{2})$. In what follows we will take $X = \mathcal{X}^t$ for some fixed $t \in [0, s - \frac{1}{2})$.

Notice that we have not assumed that the underlying Hilbert space is comprised of L^2 functions mapping $D \subset \mathbb{R}^d$ into $\mathbb{R}$, and hence we have not introduced the dimension d of an underlying physical space $\mathbb{R}^d$ into either the decay assumptions on the γ_j or the spaces $\mathcal{X}^r$. However, note that the spaces $\mathcal{H}^t$ introduced in subsection 2.4 are, in the case where $\mathcal{H} = L^2(D;\mathbb{R})$, the same as the spaces $\mathcal{X}^{t/d}$.

We now break our developments into introductory discussion of the finite dimensional setting, in subsection 5.5, and into the Hilbert space setting in subsection 5.6. In subsection 5.5.1 we introduce a family of Langevin equations which are invariant with respect to a given measure with smooth Lebesgue density. Using this, in subsection 5.5.2, we motivate equation (5.2) showing that, in finite dimensions, it corresponds to a particular choice of Langevin equation. In subsection 5.6.1, for the infinite-dimensional setting, we describe the precise assumptions under which we will prove invariance of measure μ under the dynamics (5.2). Subsection 5.6.2 describes the elements of the finite dimensional approximation of (5.2) which will underly our proof of invariance. Finally, subsection 5.6.3 contains statement of the measure invariance result as Theorem 5.28, together with its proof; this is preceded by Theorem 5.26 which establishes existence and uniqueness of a solution to (5.2), as well as continuous dependence of the solution on the initial condition and

Brownian forcing. Theorems 5.20 and 5.18 are the finite dimensional analogues of Theorems 5.28 and 5.26 respectively and play a useful role in motivating the infinite dimensional theory.

5.5. Finite Dimensional Langevin Equation.

5.5.1. Background Theory. Before setting up the (rather involved) technical assumptions required for our proof of measure invariance, we give some finite-dimensional intuition. Recall that $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^n$ and we also use this notation for the induced matrix norm on $\mathbb{R}^n$. We assume that

$$I \in C^2(\mathbb{R}^n, \mathbb{R}^+), \quad \int_{\mathbb{R}^n} e^{-I(u)} du = 1.$$

Thus $\rho(u) = e^{-I(u)}$ is the Lebesgue density corresponding to a random variable on $\mathbb{R}^n$. Let μ be the corresponding measure.

Let $\mathbb{W}$ denote standard Wiener measure on $\mathbb{R}^n$. Thus $B \sim \mathbb{W}$ is a standard Brownian motion in $C([0,\infty);\mathbb{R}^n)$. Let $u \in C([0,\infty);\mathbb{R}^n)$ satisfy the SDE

$$\frac{du}{dt} = -A\, DI(u) + \sqrt{2A}\frac{dB}{dt}, \; u(0) = u_0 \tag{5.24}$$

where $A \in \mathbb{R}^{n\times n}$ is symmetric and strictly positive definite and $DI \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ is the gradient of I. Assume that $\exists M > 0 : \forall u \in \mathbb{R}^n$, the Hessian of I satisfies

$$|D^2 I(u)| \le M.$$

We refer to equations of the form (5.24) as *Langevin equations* (as mentioned earlier they correspond to overdamped Langevin equations in the physics literature, and to Langevin equations in the statistics literature), and the matrix A as a *preconditioner*.

Theorem 5.18. *For every $u_0 \in \mathbb{R}^n$ and $\mathbb{W}$-a.s., equation (5.24) has a unique global in time solution $u \in C([0,\infty);\mathbb{R}^n)$.*

Proof. A solution of the SDE is a solution of the integral equation

$$u(t) = u_0 - \int_0^t A\, DI\big(u(s)\big) ds + \sqrt{2A}B(t). \tag{5.25}$$

Define $X = C([0,T];\mathbb{R}^n)$ and $\mathcal{F} : X \to X$ by

$$(\mathcal{F}v)(t) = u_0 - \int_0^t A\, DI\big(v(s)\big) ds + \sqrt{2A}B(t). \tag{5.26}$$

Thus $u \in X$ solving (5.25) is a fixed point of $\mathcal{F}$. We show that $\mathcal{F}$ has a unique fixed point, for T sufficiently small. To this end we study a contraction property

of $\mathcal{F}$:

$$\begin{aligned}\|(\mathcal{F}v_1)-(\mathcal{F}v_2)\|_X &= \sup_{0\le t\le T}\Big|\int_0^t \Big(A\,DI\big(v_1(s)\big)-A\,DI\big(v_2(s)\big)\Big)ds\Big| \\ &\le \int_0^T \Big|A\,DI\big(v_1(s)\big)-A\,DI\big(v_2(s)\big)\Big|ds \\ &\le \int_0^T |A|M|v_1(s)-v_2(s)|ds \\ &\le T|A|M\|v_1-v_2\|_X.\end{aligned}$$

Choosing $T : T|A|M < 1$ shows that $\mathcal{F}$ is a contraction on X. This argument may be repeated on successive intervals $[T, 2T], [2T, 3T], \ldots$ to obtain a unique global solution in $C([0,\infty);\mathbb{R}^n)$. □

Remark 5.19. Note that, since A is positive-definite symmetric, its eigenvectors e_j form an orthonormal basis for $\mathbb{R}^n$. We write $Ae_j = \alpha_j^2 e_j$. Thus

$$B(t) = \sum_{j=1}^n \beta_j(t)e_j$$

where the $\{\beta_j\}_{j=1}^n$ are an i.i.d. collection of standard unit Brownian motions on $\mathbb{R}$. Thus we obtain

$$\sqrt{A}B(t) = \sum_{j=1}^n \alpha_j\beta_j e_j =: W(t).$$

We refer to W as an A-Wiener process. Such a process is Gaussian with mean zero and covariance structure

$$\mathbb{E}W(t)\otimes W(s) = A(t\wedge s).$$

The equation (5.24) may be written as

$$\frac{du}{dt} = -ADI(u) + \sqrt{2}\frac{dW}{dt},\ u(0)=u_0. \tag{5.27}$$

Theorem 5.20. *Let $u(t)$ solve (5.24). If $u_0 \sim \mu$ then $u(t) \sim \mu$ for all $t > 0$. More precisely, for all $\varphi : \mathbb{R}^n \to \mathbb{R}^+$ bounded and continuous, $u_0 \sim \mu$ implies*

$$\mathbb{E}\varphi\big(u(t)\big) = \mathbb{E}\varphi(u_0),\ \forall t > 0.$$

Proof. Consider the additive noise SDE, for additive noise with strictly positive-definite diffusion matrix Σ,

$$\frac{du}{dt} = f(u) + \sqrt{2\Sigma}\frac{dB}{dt},\ u(0) = u_0 \sim \nu_0.$$

If ν_0 has pdf ρ_0, then the Fokker-Planck equation for this SDE is

$$\begin{aligned} \frac{\partial \overline{\rho}}{\partial t} &= \nabla \cdot (-f\overline{\rho} + \Sigma \nabla \overline{\rho}),\ (u,t) \in \mathbb{R}^n \times \mathbb{R}^+, \\ \overline{\rho}|_{t=0} &= \rho_0. \end{aligned}$$

At time $t > 0$ the solution of the SDE is distributed according to measure $\nu(t)$ with density $\overline{\rho}(u,t)$ solving the Fokker-Planck equation. Thus the initial measure ν_0 is preserved if

$$\nabla \cdot (-f\rho_0 + \Sigma \nabla \rho_0) = 0$$

and then $\overline{\rho}(\cdot, t) = \rho_0,\ \forall t \geq 0$.

We apply this Fokker-Planck equation to show that μ is invariant for equation (5.25). We need to show that

$$\nabla \cdot \big(ADI(u)\rho + A\,\nabla \rho \big) = 0$$

if $\rho = e^{-I(u)}$. With this choice of ρ we have

$$\nabla \rho = -DI(u)e^{-I(u)} = -DI(u)\rho.$$

Thus

$$A\,DI(u)\rho + A\,\nabla \rho = A\,DI(u)\rho - A\,DI(u)\rho = 0,$$

so that

$$\nabla \cdot \big(A\,DI(u)\rho + A\,\nabla \rho \big) = \nabla \cdot (0) = 0.$$

Hence the proof is complete. □

5.5.2. Motivation for Equation (5.2). Using the preceding finite dimensional development, we now motivate the form of equation (5.2). For (5.1) we have, if $\mathcal{H}$ is $\mathbb{R}^n$,

$$\mu(du) = \exp\big(-I(u)\big)\,du\ , \qquad I(u) = \frac{1}{2}|\mathcal{C}^{-\frac{1}{2}}u|^2 + \Phi(u) + \ln Z\ .$$

Thus

$$DI(u) = \mathcal{C}^{-1}u + D\Phi(u)$$

and equation (5.24), which preserves μ, is

$$\frac{du}{dt} = -A\big(\mathcal{C}^{-1}u + D\Phi(u)\big) + \sqrt{2A}\frac{dB}{dt}.$$

Choosing the preconditioner $A = \mathcal{C}$ gives

$$\frac{du}{dt} = -u - \mathcal{C}D\Phi(u) + \sqrt{2\mathcal{C}}\frac{dB}{dt}.$$

This is exactly (5.2) provided $W = \sqrt{\mathcal{C}}B$, where B is a Brownian motion with covariance $\mathcal{I}$. Then W is a Brownian motion with covariance $\mathcal{C}$. This is the finite dimensional analogue of the construction of a $\mathcal{C}$-Wiener process in the Appendix. We are now in a position to prove Theorems 5.26 and 5.28 which are the infinite dimensional analogues of Theorems 5.18 and 5.20.

5.6. Infinite Dimensional Langevin Equation.

5.6.1. Assumptions on Change of Measure. Recall that $\mu_0(\mathcal{X}^r) = 1$ for all $r \in [0, s - \frac{1}{2})$. The functional $\Phi(\cdot)$ is assumed to be defined on $\mathcal{X}^t$ for some $t \in [0, s - \frac{1}{2})$, and indeed we will assume appropriate bounds on the first and second derivatives, building on this assumption. (Thus, in this subsection 5.6.1, t does not denote time; instead we use τ to denote the generic time-argument.) These regularity assumptions on $\Phi(\cdot)$ ensure that the probability distribution μ is not too different from μ_0, when projected into directions associated with ϕ_j for j large.

For each $u \in \mathcal{X}^t$ the derivative $D\Phi(u)$ is an element of the dual $(\mathcal{X}^t)^*$ of $\mathcal{X}^t$ comprising continuous linear functionals on $\mathcal{X}^t$. However, we may identify $(\mathcal{X}^t)^*$ with $\mathcal{X}^{-t}$ and view $D\Phi(u)$ as an element of $\mathcal{X}^{-t}$ for each $u \in \mathcal{X}^t$. With this identification, the following identity holds

$$\|D\Phi(u)\|_{\mathcal{L}(\mathcal{X}^t,\mathbb{R})} = \|D\Phi(u)\|_{-t}$$

and the second derivative $D^2\Phi(u)$ can be identified as an element of $\mathcal{L}(\mathcal{X}^t, \mathcal{X}^{-t})$. To avoid technicalities we assume that $\Phi(\cdot)$ is quadratically bounded, with first derivative linearly bounded and second derivative globally bounded. Weaker assumptions could be dealt with by use of stopping time arguments.

Assumptions 5.21. There exist constants $M_i \in \mathbb{R}^+, i \leq 4$ and $t \in [0, s - 1/2)$ such that, for all $u \in \mathcal{X}^t$, the functional $\Phi : \mathcal{X}^t \to \mathbb{R}$ satisfies

$$\begin{aligned} -M_1 \leq \Phi(u) &\leq M_2 \Big(1 + \|u\|_t^2\Big); \\ \|D\Phi(u)\|_{-t} &\leq M_3 \Big(1 + \|u\|_t\Big); \\ \|D^2\Phi(u)\|_{\mathcal{L}(\mathcal{X}^t,\mathcal{X}^{-t})} &\leq M_4. \end{aligned}$$

□

Example 5.22. The functional $\Phi(u) = \frac{1}{2}\|u\|_t^2$ satisfies Assumptions 5.21. To see this note that we may write $\Phi(u) = \frac{1}{2}\langle u, \mathcal{K}u\rangle$ where

$$\mathcal{K} = \frac{1}{2}\sum_{j=1}^{\infty} j^{2t}\phi_j\phi_j^*.$$

The functional $\Phi : \mathcal{X}^t \to \mathbb{R}^+$ is clearly well-defined by definition. Its derivative at $u \in \mathcal{X}^t$ is given by $\mathcal{K}u = D\Phi(u) = \sum_{j\geq 1} j^{2t}u_j\phi_j$, where $u_j = \langle \phi_j, u\rangle$. Furthermore $D\Phi(u) \in \mathcal{X}^{-t}$ with $\|D\Phi(u)\|_{-t} = \|u\|_t$. The second derivative $D^2\Phi(u) \in \mathcal{L}(\mathcal{X}^t, \mathcal{X}^{-t})$ is the linear operator $\mathcal{K}$, that is the operator that maps $u \in \mathcal{X}^t$ to $\sum_{j\geq 1} j^{2t}\langle u, \phi_j\rangle\phi_j \in \mathcal{X}^t$: its norm satisfies $\|D^2\Phi(u)\|_{\mathcal{L}(\mathcal{X}^t,\mathcal{X}^{-t})} = 1$ for any $u \in \mathcal{X}^t$. □

Since the eigenvalues γ_j^2 of $\mathcal{C}$ decrease as $\gamma_j \asymp j^{-s}$, the operator $\mathcal{C}$ has a smoothing effect: $\mathcal{C}^\alpha h$ gains $2\alpha s$ orders of regularity in the sense that the $\mathcal{X}^\beta$-norm

of $\mathcal{C}^\alpha h$ is controlled by the $\mathcal{X}^{\beta-2\alpha s}$-norm of $h \in \mathcal{H}$. Indeed it is straightforward to show the following:

Lemma 5.23. *Under Assumptions 5.21, the following estimates hold:*

1. *The operator $\mathcal{C}$ satisfies*

$$\|\mathcal{C}^\alpha h\|_\beta \asymp \|h\|_{\beta-2\alpha s}.$$

2. *The function $\mathcal{C}D\Phi : \mathcal{X}^t \to \mathcal{X}^t$ is globally Lipschitz on $\mathcal{X}^t$: there exists a constant $M_5 > 0$ such that*

$$\|\mathcal{C}D\Phi(u) - \mathcal{C}D\Phi(v)\|_t \le M_5 \|u - v\|_t \qquad \forall u, v \in \mathcal{X}^t. \tag{5.28}$$

3. *The function $F : \mathcal{X}^t \to \mathcal{X}^t$ defined by*

$$F(u) = -u - \mathcal{C}D\Phi(u) \tag{5.29}$$

is globally Lipschitz on $\mathcal{X}^t$.

4. *The functional $\Phi(\cdot) : \mathcal{X}^t \to \mathbb{R}$ satisfies a second order Taylor formula (for which we extend $\langle \cdot, \cdot \rangle$ from an inner-product on $\mathcal{X}$ to the dual pairing between $\mathcal{X}^{-t}$ and $\mathcal{X}^t$.) There exists a constant $M_6 > 0$ such that*

$$\Phi(v) - \Big(\Phi(u) + \langle D\Phi(u), v - u\rangle\Big) \le M_6 \|u - v\|_t^2 \qquad \forall u, v \in \mathcal{X}^t. \tag{5.30}$$

5.6.2. Finite Dimensional Approximation. Our analysis now proceeds as follows. First we introduce an approximation of the measure μ, denoted by μ^N. To this end we let P^N denote orthogonal projection in $\mathcal{H}$ onto $X^N := \text{span}\{\phi_1, \cdots, \phi_N\}$ and denote by Q^N orthogonal projection in $\mathcal{H}$ onto $X^\perp := \text{span}\{\phi_{N+1}, \phi_{N+2}, \cdots\}$. Thus $Q^N = I - P^N$. Then define the measure μ^N by

$$\frac{d\mu^N}{d\mu_0}(u) = \frac{1}{Z^N} \exp\big(-\Phi(P^N u)\big), \tag{5.31a}$$

$$Z^N = \int_{X'} \exp\big(-\Phi(P^N u)\big)\mu_0(du). \tag{5.31b}$$

This is a specific example of the approximating family in (4.5) if we define

$$\Phi^N := \Phi \circ P^N. \tag{5.32}$$

Indeed if we take $X = \mathcal{X}^\tau$ for any $\tau \in (t, s - \frac{1}{2})$ we see that $\|P^N\|_{\mathcal{L}(X,X)} = 1$ and that, for any $u \in X$,

$$\begin{aligned}\|\Phi(u) - \Phi^N(u)\| &= \|\Phi(u) - \Phi(P^N u)\| \\ &\le M_3(1 + \|u\|_t)\|(I - P^N)u\|_t \\ &\le CM_3(1 + \|u\|_\tau)\|u\|_\tau N^{-(\tau - t)}.\end{aligned}$$

Since Φ, and hence Φ^N, are bounded below by $-M_1$, and since the function $1 + \|u\|_\tau^2$ is integrable by the Fernique Theorem 2.13, the approximation Theorem 4.9 applies. We deduce that the Hellinger distance between μ and μ^N is bounded above by $\mathcal{O}(N^{-r})$ for any $r < s - \frac{1}{2} - t$ since $\tau - t \in (0, s - \frac{1}{2} - t)$.

We will not use this explicit convergence rate in what follows, but we will use the idea that μ^N converges to μ in order to prove invariance of the measure μ under the SDE (5.2). The measure μ^N has a product structure that we will exploit in the following. We note that any element $u \in \mathcal{H}$ is uniquely decomposed as $u = p + q$ where $p \in X^N$ and $q \in X^\perp$. Thus we will write $\mu^N(du) = \mu^N(dp, dq)$, and similar expressions for μ_0 and so forth, in what follows.

Lemma 5.24. *Define $\mathcal{C}^N = P^N \mathcal{C} P^N$ and $\mathcal{C}^\perp = Q^N \mathcal{C} Q^N$. Then μ_0 factors as the product of measures $\mu_{0,P} = N(0, \mathcal{C}^N)$ and $\mu_{0,Q} = N(0, \mathcal{C}^\perp)$ on X^N and $X^\perp$ respectively. Furthermore μ^N itself also factors as a product measure on $X^N \oplus X^\perp$: $\mu^N(dp, dq) = \mu_P(dp)\mu_Q(dq)$ with $\mu_Q = \mu_{0,Q}$ and*

$$\frac{d\mu_P}{d\mu_{0,P}}(u) \propto \exp\big(-\Phi(p)\big).$$

Proof. Because P^N and Q^N commute with $\mathcal{C}$, and because $P^N Q^N = Q^N P^N = 0$, the factorization of the reference measure μ_0 follows automatically. The factorization of the measure μ follows from the fact that $\Phi^N(u) = \Phi(p)$ and hence does not depend on q. □

To facilitate the proof of the desired measure preservation property, we introduce the equation

$$\frac{du^N}{dt} = -u^N - \mathcal{C} P^N D\Phi^N(u^N) + \sqrt{2}\frac{dW}{dt}. \tag{5.33}$$

By using well-known properties of finite dimensional SDEs, we will show that, if $u^N(0) \sim \mu^N$, then $u^N(t) \sim \mu^N$ for any $t > 0$. By passing to the limit $N = \infty$ we will deduce that for (5.2), if $u(0) \sim \mu$, then $u(t) \sim \mu$ for any $t > 0$.

The next lemma gathers various regularity estimates on the functional $\Phi^N(\cdot)$ that are repeatedly used in the sequel; they follow from the analogous properties of Φ by using the structure $\Phi^N = \Phi \circ P^N$.

Lemma 5.25. *Under Assumptions 5.21, the following estimates hold with all constants uniform in N*

1. *The estimates of Assumptions 5.21 hold with Φ replaced by Φ^N.*
2. *The function $\mathcal{C} D\Phi^N : \mathcal{X}^t \to \mathcal{X}^t$ is globally Lipschitz on $\mathcal{X}^t$: there exists a constant $M_5 > 0$ such that*

$$\|\mathcal{C} D\Phi^N(u) - \mathcal{C} D\Phi^N(v)\|_t \le M_5 \|u - v\|_t \qquad \forall u, v \in \mathcal{X}^t.$$

3. *The function $F^N : \mathcal{X}^t \to \mathcal{X}^t$ defined by*

$$F^N(u) = -u - \mathcal{C} P^N D\Phi^N(u) \tag{5.34}$$

is globally Lipschitz on $\mathcal{X}^t$.

4. *The functional $\Phi^N(\cdot) : \mathcal{X}^t \to \mathbb{R}$ satisfies a second order Taylor formula (for which we extend $\langle \cdot, \cdot \rangle$ from an inner-product on $\mathcal{X}$ to the dual pairing between $\mathcal{X}^{-t}$ and $\mathcal{X}^t$.) There exists a constant $M_6 > 0$ such that*

$$\Phi^N(v) - \Big(\Phi^N(u) + \langle D\Phi^N(u), v - u\rangle\Big) \le M_6 \|u - v\|_t^2 \qquad \forall u, v \in \mathcal{X}^t. \tag{5.35}$$

5.6.3. Main Theorem and Proof. Fix a function $W \in C([0,T]; \mathcal{X}^t)$. Recalling F defined by (5.29), we define a solution of (5.2) to be a function $u \in C([0,T]; \mathcal{X}^t)$ satisfying the integral equation

$$u(\tau) = u_0 + \int_0^\tau F\big(u(s)\big)\, ds + \sqrt{2}\, W(\tau) \qquad \forall \tau \in [0,T]. \tag{5.36}$$

The solution is said to be *global* if $T > 0$ is arbitrary. For us, W will be a $\mathcal{C}$-Wiener process and hence random; we look for existence of a global solution, almost surely with respect to the Wiener measure, Similarly a solution of (5.33) is a function $u^N \in C([0,T]; \mathcal{X}^t)$ satisfying the integral equation

$$u^N(\tau) = u_0 + \int_0^\tau F^N\big(u^N(s)\big)\, ds + \sqrt{2}\, W(\tau) \qquad \forall t \in [0,T]. \tag{5.37}$$

Again, the solution is random because W is a $\mathcal{C}$-Wiener process. Note that the solution to this equation is not confined to X^N, because both u_0 and W have non-trivial components in $X^\perp$. However within $X^\perp$ the behaviour is purely Gaussian and within X^N it is finite dimensional. We will exploit these two facts.

The following establishes basic existence, uniqueness, continuity and approximation properties of the solutions of (5.36) and (5.37).

Theorem 5.26. *For every $u_0 \in \mathcal{X}^t$ and for almost every $\mathcal{C}$-Wiener process W, equation* (5.36) *(respectively* (5.37)*) has a unique global solution. For any pair $(u_0, W) \in \mathcal{X}^t \times C([0,T]; \mathcal{X}^t)$ we define the Itô map*

$$\Theta\colon \mathcal{X}^t \times C([0,T]; \mathcal{X}^t) \to C([0,T]; \mathcal{X}^t)$$

which maps (u_0, W) to the unique solution u (resp. u^N for (5.37)*) of the integral equation* (5.36) *(resp. Θ^N for* (5.37)*). The map Θ (resp. Θ^N) is globally Lipschitz continuous. Finally we have that $\Theta^N(u_0, W) \to \Theta(u_0, W)$ strongly in $C([0,T]; \mathcal{X}^t)$ for every pair $(u_0, W) \in \mathcal{X}^t \times C([0,T]; \mathcal{X}^t)$.*

Proof. The existence and uniqueness of local solutions to the integral equation (5.36) is a simple application of the contraction mapping principle, following arguments similar to those employed in the proof of Theorem 5.18. Extension to a global solution may be achieved by repeating the local argument on successive intervals.

Now let $u^{(i)}$ solve

$$u^{(i)} = u_0^{(i)} + \int_0^\tau F(u^{(i)})(s) ds + \sqrt{2} W^{(i)}(\tau), \quad \tau \in [0,T],$$

for $i = 1, 2$. Subtracting and using the Lipschitz property of F shows that $e = u^{(1)} - u^{(2)}$ satisfies

$$\|e(\tau)\|_t \le \|u_0^{(1)} - u_0^{(2)}\|_t + L\int_0^\tau \|e(s)\|_t ds + \sqrt{2}\|W^{(1)}(\tau) - W^{(2)}(\tau)\|_t$$
$$\le \|u_0^{(1)} - u_0^{(2)}\|_t + L\int_0^\tau \|e(s)\|_t ds + \sqrt{2}\sup_{0\le s\le T}\|W^{(1)}(s) - W^{(2)}(s)\|_t.$$

By application of the Gronwall inequality we find that

$$\sup_{0\le\tau\le T}\|e(\tau)\|_t \le C(T)\big(\|u_0^{(1)} - u_0^{(2)}\|_t + \sup_{0\le s\le T}\|W^{(1)}(s) - W^{(2)}(s)\|_t\big)$$

and the desired continuity is established.

Now we prove pointwise convergence of Θ^N to Θ. Let $e = u - u^N$ where u and u^N solve (5.36), (5.37) respectively. The pointwise convergence of Θ^N to Θ is established by proving that $e \to 0$ in $C([0,T];\mathcal{X}^t)$. Note that

$$F(u) - F^N(u^N) = \big(F^N(u) - F^N(u^N)\big) + \big(F(u) - F^N(u)\big).$$

Also, by Lemma 5.25, $\|F^N(u) - F^N(u^N)\|_t \le L\|e\|_t$. Thus we have

$$\|e\|_t \le L\int_0^\tau \|e(s)\|_t ds + \int_0^\tau \|F\big(u(s)\big) - F^N\big(u(s)\big)\|_t ds.$$

Thus, by Gronwall, it suffices to show that

$$\delta^N := \sup_{0\le s\le T}\|F\big(u(s)\big) - F^N\big(u(s)\big)\|_t$$

tends to zero as $N \to \infty$. Note that

$$\begin{aligned} F(u) - F^N(u) &= \mathcal{C}D\Phi(u) - \mathcal{C}P^N D\Phi(P^N u) \\ &= (I - P^N)\mathcal{C}D\Phi(u) + P^N\big(\mathcal{C}D\Phi(u) - \mathcal{C}D\Phi(P^N u)\big). \end{aligned}$$

Thus, since $\mathcal{C}D\Phi$ is globally Lipschitz on $\mathcal{X}^t$, by Lemma 5.23, and P^N has norm one as a mapping from $\mathcal{X}^t$ into itself,

$$\|F(u) - F^N(u)\|_t \le \|(I - P^N)\mathcal{C}D\Phi(u)\|_t + C\|(I - P^N)u\|_t.$$

By dominated convergence $\|(I - P_N)a\|_t \to 0$ for any fixed element $a \in \mathcal{X}^t$. Thus, because $\mathcal{C}D\Phi$ is globally Lipschitz, by Lemma 5.23, and as $u \in C([0,T];\mathcal{X}^t)$, we deduce that it suffices to bound $\sup_{0\le s\le T}\|u(s)\|_t$. But such a bound is a consequence of the existence theory outlined at the start of the proof, based on the proof of Theorem 5.18. □

The following is a straightforward corollary of the preceding theorem:

Corollary 5.27. *For any pair $(u_0, W) \in \mathcal{X}^t \times C([0,T];\mathcal{X}^t)$ we define the point Itô map*

$$\Theta_\tau \colon \mathcal{X}^t \times C([0,T];\mathcal{X}^t) \to \mathcal{X}^t$$

(resp. Θ_τ^N for (5.37)) which maps (u_0, W) to the unique solution $u(\tau)$ of the integral equation (5.36) (resp. $u^N(\tau)$ for (5.37)) at time τ . The map Θ_τ (resp. Θ_τ^N) is globally Lipschitz continuous. Finally we have that $\Theta_\tau^N(u_0, W) \to \Theta_\tau(u_0, W)$ for every pair $(u_0, W) \in \mathcal{X}^t \times C([0,T]; \mathcal{X}^t)$.

Theorem 5.28. *Let Assumptions 5.21 hold. Then the measure μ given by (4.4) is invariant for (5.2): for all continuous bounded functions $\varphi : \mathcal{X}^t \to \mathbb{R}$ it follows that, if $\mathbb{E}$ denotes expectation with respect to the product measure found from initial condition $u_0 \sim \mu$ and $W \sim \mathbb{W}$, the $\mathcal{C}$-Wiener measure on $\mathcal{X}^t$, then $\mathbb{E}\varphi\big(u(\tau)\big) = \mathbb{E}\varphi(u_0)$.*

Proof. We have that

$$\mathbb{E}\varphi\big(u(\tau)\big) = \int \varphi\big(\Theta_\tau(u_0, W)\big)\mu(du_0)\mathbb{W}(dW), \tag{5.38}$$

$$\mathbb{E}\varphi(u_0) = \int \varphi(u_0)\mu(du_0). \tag{5.39}$$

If we solve equation (5.33) with $u_0 \sim \mu^N$ then, using $\mathbb{E}^N$ with the obvious notation,

$$\mathbb{E}^N\varphi\big(u^N(\tau)\big) = \int \varphi\big(\Theta_\tau^N(u_0, W)\big)\mu^N(du_0)\mathbb{W}(dW), \tag{5.40}$$

$$\mathbb{E}^N\varphi(u_0) = \int \varphi(u_0)\mu^N(du_0). \tag{5.41}$$

Lemma 5.29 below shows that, in fact,

$$\mathbb{E}^N\varphi\big(u^N(\tau)\big) = \mathbb{E}^N\varphi(u_0).$$

Thus it suffices to show that

$$\mathbb{E}^N\varphi\big(u^N(\tau)\big) \to \mathbb{E}\varphi\big(u(\tau)\big) \tag{5.42}$$

and

$$\mathbb{E}^N\varphi(u_0) \to \mathbb{E}\varphi(u_0). \tag{5.43}$$

Both of these facts follow from the dominated convergence theorem as we now show. First note that

$$\mathbb{E}^N\varphi(u_0) = \int \varphi(u_0)e^{-\Phi(P^N u_0)}\mu_0(du_0).$$

Since $\varphi(\cdot)e^{-\Phi\circ P^N}$ is bounded independently of N, by $(\sup\varphi)e^{M_1}$, and since $(\Phi \circ P^N)(u)$ converges pointwise to $\Phi(u)$ on $\mathcal{X}^t$, we deduce that

$$\mathbb{E}^N\varphi(u_0) \to \int \varphi(u_0)e^{-\Phi(u_0)}\mu_0(du_0) = \mathbb{E}\varphi(u_0)$$

so that (5.43) holds. The convergence in (5.42) holds by a similar argument. From (5.40) we have

$$\mathbb{E}^N \varphi\big(u^N(\tau)\big) = \int \varphi\big(\Theta_\tau^N(u_0, W)\big) e^{-\Phi(P^N u_0)} \mu_0(du_0) \mathbb{W}(dW). \tag{5.44}$$

The integrand is again dominated by $(\sup \varphi) e^{M_1}$. Using the pointwise convergence of Θ_τ^N to Θ_τ on $\mathcal{X}^t \times C([0,T]; \mathcal{X}^t)$, as proved in Corollary 5.27, as well as the pointwise convergence of $(\Phi \circ P^N)(u)$ to $\Phi(u)$, the desired result follows from dominated convergence: we find that

$$\mathbb{E}^N \varphi\big(u^N(\tau)\big) \to \int \varphi\big(\Theta_\tau(u_0, W)\big) e^{-\Phi(u_0)} \mu_0(du_0) \mathbb{W}(dW) = \mathbb{E} \varphi\big(u(\tau)\big).$$

The desired result follows. □

Lemma 5.29. *Let Assumptions 5.21 hold. Then the measure μ^N given by* (5.31) *is invariant for* (5.33)*: for all continuous bounded functions $\varphi : \mathcal{X}^t \to \mathbb{R}$ it follows that, if $\mathbb{E}^N$ denotes expectation with respect to the product measure found from initial condition $u_0 \sim \mu^N$ and $W \sim \mathbb{W}$, the $\mathcal{C}$-Wiener measure on $\mathcal{X}^t$, then $\mathbb{E}^N \varphi\big(u^N(\tau)\big) = \mathbb{E}^N \varphi(u_0)$.*

Proof. Recall from Lemma 5.24 that measure μ^N given by (5.31) factors as the independent product of two measures on μ_P on X^N and μ_Q on $X^\perp$. On $X^\perp$ the measure is simply the Gaussian $\mu_Q = \mathcal{N}(0, \mathcal{C}^\perp)$, whilst X^N the measure μ_P is finite dimensional with density proportional to

$$\exp\Big(- \Phi(p) - \frac{1}{2} \|(\mathcal{C}^N)^{-\frac{1}{2}} p\|^2 \Big). \tag{5.45}$$

The equation (5.33) also decouples on the spaces X^N and $X^\perp$. On $X^\perp$ it gives the integral equation

$$q(\tau) = -\int_0^\tau q(s) + \sqrt{2} Q^N W(\tau) \tag{5.46}$$

whilst on X^N it gives the integral equation

$$p(\tau) = -\int_0^\tau \Big(p(s) + \mathcal{C}^N D\Phi\big(p(s)\big) \Big) ds + \sqrt{2} P^N W(\tau). \tag{5.47}$$

Measure μ_Q is preserved by (5.46), because (5.46) simply gives an (integral equation formulation of) the Ornstein-Uhlenbeck process with desired Gaussian invariant measure. On the other hand, equation (5.47) is simply (an integral equation formulation of)the Langevin equation for measure on $\mathbb{R}^N$ with density (5.45) and a calculation with the Fokker-Planck equation, as in Theorem 5.20, demonstrates the required invariance of μ_P. □

5.7. Bibliographic Notes.

- Subsection 5.1 describes general background on Markov processes and invariant measures. The book [78] is a good starting point in this area. The book [75] provides a good overview of this subject area, from an applied and computational statistics perspective. For continuous time Markov chains see [102].

- Subsection 5.2 concerns MCMC methods. The standard RWM was introduced in [73] and led, via the paper [46], to the development of the more general class of Metropolis-Hastings methods. The paper [95] is a key reference which provides a framework for the study of Metropolis-Hastings methods on general state spaces. The subject of MCMC methods which are invariant with respect to the target measure μ on infinite dimensional spaces is overviewed in the paper [21]. The specific idea behind the Algorithm 5.10 is contained in [76, equation (15)], in the finite dimensional setting. It is possible to show that, in the limit $\beta \to 0$, suitably interpolated output of Algorithm 5.10 converges to solution of the equation (5.2): see [82]. Furthermore it is also possible to compute a spectral gap for the Algorithm 5.10 in the infinite dimensional setting [100]. This implies the existence of a dimension independent spectral gap when finite dimensional approximation is used; in contrast standard Metropolis-Hastings methods, such as Random Walk Metropolis, have a dimension-dependent spectral gap which shrinks with increasing dimension [100].

- Subsection 5.3 concerns SMC methods and the foundational work in this area is overviewed in the book [26]. The application of those ideas to the solution of PDE inverse problems was first demonstrated in [50], where the inverse problem is to determine the initial condition of the Navier-Stokes equations from observations. The method is applied to the elliptic inverse problem, with uniform priors, in [10]. The proof of Theorem 5.13 follows the very clear exposition given in [85] in the context of filtering for hidden Markov models.

- Subsections 5.4–5.6 concern measure preserving continuous time dynamics. The finite dimensional aspects of this subsection, which we introduce for motivation, are covered in the texts [79] and [37]; the first of these books is an excellent introduction to the basic existence and uniqueness theory, outlined in a simple case in Theorem 5.18, whilst the second provides an in depth treatment of the subject from the viewpoint of the Fokker-Planck equation, as used in Theorem 5.20. This subject has a long history which is overviewed in the paper [41] where the idea is applied to finding SPDEs which are invariant with respect to the measure generated by a conditioned diffusion process. This idea is generalized to certain conditioned hypoelliptic diffusions in [42]. It is also possible to study deterministic Hamiltonian dynamics which preserves the same measure. This idea is described in [9] in the same set-up as employed here; that paper also contains references to the wider

literature. Lemma 5.23 is proved in [72] and Lemma 5.25 in [82] Lemma 5.29 requires knowledge of the invariance of Ornstein-Uhlenbeck processes together with invariance of finite dimensional first order Langevin equations with the form of gradient dynamics subject to additive noise. The invariance of the Ornstein-Uhlenbeck process is covered in [29] and invariance of finite dimensional SDEs using the Fokker-Planck equation is discussed in [37]. The $\mathcal{C}$-Wiener process, and its properties, are described in [28].

- The primary focus of this section has been on the theory of measure-preserving dynamics, and its relations to algorithms. The SPDEs are of interest in their own right as a theoretical object, but have particular importance in the construction of MCMC methods, and in understanding the limiting behaviour of MCMC methods. It is also important to appreciate that MCMC and SMC methods are by no means the only tools available to study the Bayesian inverse problem. In this context we note that computing the expectation with respect to the posterior can be reformulated as computing the ratio of two expectations with respect to the prior, the denominator being the normalization constant. effectively in some such high dimensional integration problems; [59] and [77] are general references on the QMC methodology. The paper [57] is a survey on the theory of QMC for bounded integration domains and is relevant for uniform priors. The paper [60] contains theoretical results for unbounded integration domains and is relevant to, for example, Gaussian priors. The use of QMC in plain uncertainty quantification (calculating the pushforward of a measure through a map) is studied for elliptic PDEs with random coefficients in [58] (uniform) and [39] (Gaussian). More sophisticated integration tools can be employed, using polynomial chaos representations of the prior measure, and computing posterior expectations in a manner which exploits sparsity in the map from unknown random coefficients to measured data; see [91, 90]. Much of this work, viewing uncertainty quantification from the point of high dimensional integration, has its roots in early papers concerning plain uncertainty quantification in elliptic PDEs with random coefficients; the paper [7] was foundational in this area.

6. Conclusions

We have highlighted a theoretical treatment for Bayesian inversion over infinite dimensional spaces. The resulting framework is appropriate for the mathematical analysis of inverse problems, as well as the development of algorithms. For example, on the analysis side, the idea of MAP estimators, which links the Bayesian approach with classical regularization, developed for Gaussian priors in [30], has recently been extended to other prior models in [47]; the study of contraction of the posterior distribution to a Dirac measure on the truth underlying the data is undertaken in [3, 4, 100]. On the algorithmic side algorithms for Bayesian inversion in geophysical applications are formulated in [16, 81], and on the computational statistics side methods for optimal experimental design are formulated in [5, 6]. All of these

cited papers build on the framework developed in detail here, and first outlined in [93]. It is thus anticipated that the framework herein will form the bedrock of other, related, developments of both the theory and computational practice of Bayesian inverse problems.

7. Appendix

7.1. Function Spaces. In this subsection we briefly define the Hilbert and Banach spaces that will be important in our developments of probability and integration in infinite dimensional spaces. As a consequence we pay particular attention to the issue of separability (the existence of a countable dense subset) which we require in that context. We primarily restrict our discussion to $\mathbb{R}$ or $\mathbb{C}$-valued functions, but the reader will easily be able to extend to $\mathbb{R}^n$-valued or $\mathbb{R}^{n\times n}$-valued situations, and we discuss Banach-space valued functions at the end of the subsection.

7.1.1. ℓ^p and L^p Spaces. Consider real-valued sequences $u = \{u_j\}_{j=1}^{\infty} \in \mathbb{R}^{\infty}$. Let $w \in \mathbb{R}^{\infty}$ denote a positive sequence so that $w_j > 0$ for each $j \in \mathbb{N}$. For every $p \in [1, \infty)$ we define

$$\ell^p_w = \ell^p_w(\mathbb{N}; \mathbb{R}) = \Big\{ u \in \mathbb{R} \Big| \sum_{j=1}^{\infty} w_j |u_j|^p < \infty \Big\}.$$

Then ℓ^p_w is a Banach space when equipped with the norm

$$\|u\|_{\ell^p_w} = \Big(\sum_{j=1}^{\infty} w_j |u_j|^p \Big)^{\frac{1}{p}}.$$

In the case $p = 2$ the resulting spaces are Hilbert spaces when equipped with the inner-product

$$\langle u, v \rangle = \sum_{j=1}^{\infty} w_j u_j v_j.$$

These ℓ^p spaces, with $p \in [1, \infty)$, are separable. Throughout we simply write ℓ^p for the spaces ℓ^p_w with $w_j \equiv 1$. In the case $w_j \equiv 1$ we extend the definition of Banach spaces to the case $p = \infty$ by defining

$$\ell^{\infty} = \ell^{\infty}(\mathbb{N}; \mathbb{R}) = \Big\{ u \in \mathbb{R} \Big| \sup_{j\in\mathbb{N}}(|u_j|) < \infty \Big\}$$

and

$$\|u\|_{\ell^{\infty}} = \sup_{j\in\mathbb{N}}(|u_j|).$$

The space ℓ^{∞} of bounded sequences is *not* separable. Each element of the sequence u_j is real-valued, but the definitions may be readily extended to complex-valued,

$\mathbb{R}^n$-valued and $\mathbb{R}^{n\times n}$-valued sequences, replacing $|\cdot|$ by the complex modulus, the vector ℓ^p norm and the operator ℓ^p norm on matrices respectively.

We now extend the idea of p-summability to functions, and to p-integrability. Let D be a bounded open set in $\mathbb{R}^d$ with Lipschitz boundary and define the space $L^p = L^p(D;\mathbb{R})$ of Lebesgue measurable functions $f : D \to \mathbb{R}$ with norm $\|\cdot\|_{L^p(D)}$ defined by

$$\|f\|_{L^p(D)} := \begin{cases} \left(\int_D |f|^p\, dx\right)^{\frac{1}{p}} & \text{for } 1 \le p < \infty \\ \operatorname{ess\,sup}_D |f| & \text{for } p = \infty. \end{cases}$$

In the above definition we have used the notation

$$\operatorname*{ess\,sup}_D |f| = \inf\{C : |f| \le C \text{ a.e. on } D\}.$$

Here *a.e.* is with respect to Lebesgue measure and the integral is, of course, the Lebesgue integral. Sometimes we drop explicit reference to the set D in the norm and simply write $\|\cdot\|_{L^p}$. For Lebesgue measurable functions $f : D \to \mathbb{R}^n$ the norm is readily extended replacing $|f|$ under the integral by the vector p-norm on $\mathbb{R}^n$. Likewise we may consider Lebegue measurable $f : D \to \mathbb{R}^{n\times n}$, using the operator p-norm on $\mathbb{R}^{n\times n}$. In all these cases we write $L^p(D)$ as shorthand for $L^p(D;X)$ where $X = \mathbb{R}$, $\mathbb{R}^n$ or $\mathbb{R}^{n\times n}$. Then $L^p(D)$ is the vector space of all (equivalence classes of) measurable functions $f : D \to \mathbb{R}$ for which $\|f\|_{L^p(D)} < \infty$. The space $L^p(D)$ is separable for $p \in [1,\infty)$ whilst $L^\infty(D)$ is not separable. We define periodic versions of $L^p(D)$, denoted by $L^p_{\text{per}}(D)$, in the case where D is a unit cube; these spaces are defined as the completion of C^∞ periodic functions on the unit cube, with respect to the L^p-norm. If we define $\mathbb{T}^d$ to be the d-dimensional unit torus then we write $L^p_{\text{per}}([0,1]^d) = L^p(\mathbb{T}^d)$. Again these spaces are separable for $1 \le p < \infty$, but not for $p = \infty$.

7.1.2. Continuous and Hölder Continuous Functions. Let D be an open and bounded set in $\mathbb{R}^d$ with Lipschitz boundary. We will denote by $C(\overline{D},\mathbb{R})$, or simply $C(\overline{D})$, the space of continuous functions $f : \overline{D} \to \mathbb{R}$. When equipped with the supremum norm,

$$\|f\|_{C(\overline{D})} = \sup_{x\in\overline{D}} |f(x)|,$$

$C(\overline{D})$ is a Banach spaces. Building on this we define the space $C^{0,\gamma}(\overline{D})$ to be the space of functions in $C(\overline{D})$ which are Hölder with any exponent $\gamma \in (0,1]$ with norm

$$\|f\|_{C^{0,\gamma}(\overline{D})} = \sup_{x\in\overline{D}} |f(x)| + \sup_{x,y\in\overline{D}} \Big(\frac{|f(x)-f(y)|}{|x-y|^\gamma}\Big). \tag{7.1}$$

The case $\gamma = 1$ corresponds to Lipschitz functions.

We remark that $C(\overline{D})$ is separable since $\overline{D} \subset \mathbb{R}^d$ is compact here. The space of Hölder functions $C^{0,\gamma}(\overline{D};\mathbb{R})$ is, however, *not* separable. Separability can be recovered by working in the subset of $C^{0,\gamma}(\overline{D};\mathbb{R})$ where, in addition to (7.1) being

finite,

$$\lim_{y \to x} \frac{|f(x) - f(y)|}{|x - y|^\gamma} = 0,$$

uniformly in x; we denote the resulting separable space by $C_0^{0,\gamma}(\overline{D}, \mathbb{R})$. This is analogous to the fact that the space of bounded measurable functions is not separable, while the space of continuous functions on a compact domain is. Furthermore it may be shown that $C^{0,\gamma'} \subset C_0^{0,\gamma}$ for every $\gamma' > \gamma$. All of the preceding spaces can be generalized to functions $C^{0,\gamma}(\overline{D}, \mathbb{R}^n)$ and $C_0^{0,\gamma}(\overline{D}, \mathbb{R}^n)$; they may also be extended to periodic functions on the unit torus $\mathbb{T}^d$ found by identifying opposite faces of the unit cube $[0,1]^d$. The same separability issues arise for these generalizations.

7.1.3. Sobolev Spaces. We define Sobolev spaces of functions with integer number of derivatives, extend to fractional and negative derivatives, and make the connection with Hilbert scales. Here D is a bounded open set in $\mathbb{R}^d$ with Lipschitz boundary. In the context of a function $u \in L^2(D)$ we will use the notation $\frac{\partial u}{\partial x_i}$ to denote the weak derivative with respect to x_i and the notation ∇u for the weak gradient.

The *Sobolev space* $W^{r,p}(D)$ consists of all L^p-integrable functions $u : D \to \mathbb{R}$ whose α^{th} order weak derivatives exist and are L^p-integrable for all $|\alpha| \leq r$:

$$W^{r,p}(D) = \left\{ u \middle| D^\alpha u \in L^p(D) \text{ for } |\alpha| \leq r \right\} \tag{7.2}$$

with norm

$$\|u\|_{W^{r,p}(D)} = \begin{cases} \left(\sum_{|\alpha| \leq r} \|D^\alpha u\|_{L^p(D)}^p \right)^{\frac{1}{p}} & \text{for } 1 \leq p < \infty, \\ \sum_{|\alpha| \leq r} \|D^\alpha u\|_{L^\infty(D)} & \text{for } p = \infty. \end{cases} \tag{7.3}$$

We denote $W^{r,2}(D)$ by $H^r(D)$. We define periodic versions of $H^s(D)$, denoted by $H^s_{\text{per}}(D)$, in the case where D is a unit cube $[0,1]^d$; these spaces are defined as the completion of C^∞ periodic functions on the unit cube, with respect to the H^s-norm. If we define $\mathbb{T}^d$ to be d-dimensional unit torus, we then write $H^s(\mathbb{T}^d) = H^s_{\text{per}}([0,1]^d)$.

The spaces $H^s(D)$ with D a bounded open set in $\mathbb{R}^d$, and $H^s_{\text{per}}([0,1]^d)$, are separable Hilbert spaces. In particular if we define the inner-product $(\cdot, \cdot)_{L^2(D)}$ on $L^2(D)$ by

$$(u, v)_{L^2(D)} := \int_D u(x) v(x) dx$$

and define the resulting norm $\| \cdot \|_{L^2(D)}$ by the identity

$$\|u\|^2_{L^2(D)} = (u, u)_{L^2(D)}$$

then the space $H^1(D)$ is a separable Hilbert space with inner product

$$\langle u, v \rangle_{H^1(D)} = (u, v)_{L^2(D)} + (\nabla u, \nabla v)_{L^2(D)}$$

and norm (7.3) with $p = 2$. Likewise the space $H_0^1(D)$ is a separable Hilbert space with inner product

$$\langle u, v\rangle_{H_0^1(D)} = (\nabla u, \nabla v)_{L^2(D)}$$

and norm

$$\|u\|_{H_0^1(D)} = \|\nabla u\|_{L^2(D)}. \tag{7.4}$$

As defined above, Sobolev spaces concern integer numbers of derivatives. However the concept can be extended to fractional derivatives and there is then a natural connection to Hilbert scales of functions. To explain this we start our development in the periodic setting. Recall that, given an element u in $L^2(\mathbb{T}^d)$, we can decompose it as a Fourier series:

$$u(x) = \sum_{k\in\mathbb{Z}^d} u_k e^{2\pi i\langle k,x\rangle} ,$$

where the identity holds for (Lebesgue) almost every $x \in \mathbb{T}^d$. Furthermore, the L^2 norm of u is given by Parseval's identity $\|u\|_{L^2}^2 = \sum |u_k|^2$. The fractional Sobolev space $H^s(\mathbb{T}^d)$ for $s \geq 0$ is given by the subspace of functions $u \in L^2(\mathbb{T}^d)$ such that

$$\|u\|_{H^s}^2 := \sum_{k\in\mathbb{Z}^d} (1 + 4\pi^2|k|^2)^s |u_k|^2 < \infty . \tag{7.5}$$

Note that this is a separable Hilbert space by virtue of ℓ_w^2 being separable. Note also that $H^0(\mathbb{T}^d) = L^2(\mathbb{T}^d)$ and that, for positive integer s, the definition agrees with the definition $H^s(\mathbb{T}^d) = W^{s,2}(\mathbb{T}^d)$ obtained from (7.2) with the obvious generalization from D to $\mathbb{T}^d$. For $s < 0$, we define $H^s(\mathbb{T}^d)$ as the closure of L^2 under the norm (7.5). The spaces $H^s(\mathbb{T}^d)$ for $s < 0$ may also be defined via duality. The resulting spaces H^s are separable for all $s \in \mathbb{R}$.

We now link the spaces $H^s(\mathbb{T}^d)$ to a specific Hilbert scale of spaces. Hilbert scales are families of spaces defined by $\mathcal{D}(A^{s/2})$ for A a positive, unbounded, self-adjoint operator on a Hilbert space. To view the fractional Sobolev spaces from this perspective let $A = I - \triangle$ with domain $H^2(\mathbb{T}^d)$, noting that the eigenvalues of A are simply $1 + 4\pi^2|k|^2$ for $k \in \mathbb{Z}^d$. We thus see that, by the spectral decomposition theorem, $H^s = \mathcal{D}(A^{s/2})$, and we have $\|u\|_{H^s} = \|A^{s/2}u\|_{L^2}$. Note that we may work in the space of real-valued functions where the eigenfunctions of A, $\{\varphi_j\}_{j=1}^{\infty}$, comprise sine and cosine functions; the eigenvalues of A, when ordered on a one-dimensional lattice, then satisfy $\alpha_j \asymp j^{2/d}$. This is relevant to the more general perpsective of Hilbert scales that we now introduce.

We can now generalize the previous construction of fractional Sobolev spaces to more general domains than the torus. The resulting spaces do not, in general, coincide with Sobolev spaces, because of the effect of the boundary conditions of the operator A used in the construction. On an arbitrary bounded open set $D \subset \mathbb{R}^d$ with Lipschitz boundary we consider a positive self-adjoint operator A satisfying Assumption 1.3 so that its eigenvalues satisfy $\alpha_j \asymp j^{2/d}$; then we define the spaces $\mathcal{H}^s = \mathcal{D}(A^{s/2})$ for $s > 0$. Given a Hilbert space $(H, \langle\cdot,\cdot\rangle, \|\cdot\|)$ of real-valued functions on a bounded open set D in $\mathbb{R}^d$, we recall from Assumption 1.3

the orthonormal basis for H denoted by $\{\varphi_j\}_{j=1}^{\infty}$. Any $u \in H$ can be written as

$$u = \sum_{j=1}^{\infty} \langle u, \varphi_j \rangle \varphi_j.$$

Thus

$$\mathcal{H}^s = \left\{ u : D \to \mathbb{R} \middle| \|w\|_{\mathcal{H}^s}^2 < \infty \right\} \tag{7.6}$$

where, for $u_j = \langle u, \varphi_j \rangle$,

$$\|u\|_{\mathcal{H}^s}^2 = \sum_{j=1}^{\infty} j^{\frac{2s}{d}} |u_j|^2.$$

In fact $\mathcal{H}^s$ is a Hilbert space: for $v_j = \langle v, \varphi_j \rangle$ we may define the inner-product

$$\langle u, v \rangle_{\mathcal{H}^s} = \sum_{j=1}^{\infty} j^{\frac{2s}{d}} u_j v_j.$$

For any $s > 0$, the Hilbert space $(\mathcal{H}^s, \langle \cdot, \cdot \rangle_{\mathcal{H}^t}, \| \cdot \|_{\mathcal{H}^t})$ is a subset of the original Hilbert space H; for $s < 0$ the spaces are defined by duality and are supersets of H. Note also that we have Parseval-like identities showing that the $\mathcal{H}^s$ norm on a function u is equivalent to the ℓ_w^2 norm on the sequence $\{u_j\}_{j=1}^{\infty}$ with the choice $w_j = j^{2s/d}$. The spaces $\mathcal{H}^s$ are separable Hilbert spaces for any $s \in \mathbb{R}$.

7.1.4. Other Useful Function Spaces. As mentioned in passing, all of the preceding function spaces can be extended to functions taking values in $\mathbb{R}^n, \mathbb{R}^{n \times n}$; thus we may then write $C(D; \mathbb{R}^n), L^p(D; \mathbb{R}^n), H^s(D; \mathbb{R}^n)$, for example. More generally we may wish to consider functions taking values in a separable Banach space E. For example when we are interested in solutions of time-dependent PDEs then these may be formulated as ordinary differential equations taking values in a separable Banach space E, with norm $\| \cdot \|_E$. It is then natural to consider Banach spaces such as $L^2((0,T); E)$ and $C([0,T]; E)$ with norms

$$\|u\|_{L^2((0,T);E)} = \sqrt{\left(\int_0^T \|u(\cdot,t)\|_E^2 dt \right)}, \quad \|u\|_{C([0,T];E)} = \sup_{t \in [0,T]} \|u(\cdot,t)\|_E.$$

These norms can be generalized in a variety of ways, by generalizing the norm on the time variable.

The preceding idea of defining Banach space-valued L^p spaces defined on an interval $(0,T)$ can be taken further to define Banach space-valued L^p spaces defined on a measure space. Let $(\mathcal{M}, \nu)$ any countably generated measure space, like for example any Polish space (a separable completely metrizable topological space) equipped with a positive Radon measure ν. Again let E denote a separable Banach space. Then $L_\nu^p(\mathcal{M}; E)$ is the space of functions $u : \mathcal{M} \to E$ with norm (in this defintion of norm we use Bochner integration, defined in the next subsection)

$$\|u\|_{L_\nu^p(\mathcal{M};E)} = \left(\int_{\mathcal{M}} \|u(x)\|_E^p \nu(dx) \right)^{\frac{1}{p}}.$$

For $p \in (1, \infty)$ these spaces are separable. However, separability fails to hold for $p = \infty$. We will use these Banach spaces in the case where ν is a probability measure $\mathbb{P}$, with corresponding expectation $\mathbb{E}$, and we then have

$$\|u\|_{L^p_{\mathbb{P}}(\mathcal{M};E)} = \Big(\mathbb{E}\big(\|u\|_E^p\big)\Big)^{\frac{1}{p}}.$$

7.1.5. Interpolation Inequalities and Sobolev Embeddings. Here we state some useful interpolation inequalities, and use them to prove a Sobolev embedding result, all in the context of fractional Sobolev spaces, in the generalized sense defined through a Hilbert scale of functions.

Let $p, q \in [1, \infty]$ be a pair of conjugate exponents so that $p^{-1} + q^{-1} = 1$. Then for any positive real a, b we have the Young inequality

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}\,.$$

As a corollary of this elementary bound, we obtain the following Hölder inequality Let $(\mathcal{M}, \mu)$ be a measure space and denote the norm $\|\cdot\|_{L^p_\nu(\mathcal{M};\mathbb{R})}$ by $\|\cdot\|_p$. For $p, q \in [1, \infty]$ as above and $u, v \colon \mathcal{M} \to \mathbb{R}$ a pair of measurable functions we have

$$\int_{\mathcal{M}} |u(x)v(x)|\,\mu(dx) \le \|u\|_p\,\|v\|_q. \tag{7.7}$$

From this Hölder-like inequality the following interpolation bound results: let $\alpha \in [0, 1]$ and let L denote a (possibly unbounded) self-adjoint operator on the Hilbert space $(H, \langle\cdot,\cdot\rangle, \|\cdot\|)$. Then, the bound

$$\|L^\alpha u\| \le \|Lu\|^\alpha \|u\|^{1-\alpha} \tag{7.8}$$

holds for every $u \in \mathcal{D}(L) \subset H$.

Now assume that A is a self-adjoint unbounded operator on $L^2(D)$ with $D \subset \mathbb{R}^d$ a bounded open set with Lipschitz boundary. Assume further that A has eigenvalues $\alpha_j \asymp j^{\frac{2}{d}}$ and define the Hilbert scale of spaces $\mathcal{H}^t = \mathcal{D}(A^{\frac{t}{2}})$. An immediate corollary of the bound (7.8), obtained by choosing $H = \mathcal{H}^s$, $L = A^{\frac{t-s}{2}}$, and $\alpha = (r-s)/(t-s)$, is:

Lemma 7.1. *Let Assumption 1.3 hold. Then for any $t > s$, any $r \in [s, t]$ and any $u \in \mathcal{H}^t$ it follows that*

$$\|u\|_{\mathcal{H}^r}^{t-s} \le \|u\|_{\mathcal{H}^t}^{r-s} \|u\|_{\mathcal{H}^s}^{t-r}.$$

It is of interest to bound the L^p norm of a function in terms of one of the fractional Sobolev norms, or more generally in terms of norms from a Hilbert scale. To do this we need to not only make assumptions on the eigenvalues of the operator A which defines the Hilbert scale, but also on the behaviour of the corresponding orthonormal basis of eigenfunctions in L^∞. To this end we let Assumption 2.17 hold. It then turns out that bounding the L^∞ norm is rather straightforward and we start with this case.

Lemma 7.2. *Let Assumption 2.17 hold and define the resulting Hilbert scale of spaces $\mathcal{H}^s$ by (7.6). Then for every $s > \frac{d}{2}$, the space $\mathcal{H}^s$ is contained in the space $L^\infty(D)$ and there exists a constant K_1 such that $\|u\|_{L^\infty} \leq K_1 \|u\|_{\mathcal{H}^s}$.*

Proof. It follows from Cauchy-Schwarz that

$$\frac{1}{C}\|u\|_{L^\infty} \leq \sum_{k\in\mathbb{Z}^d} |u_k| \leq \Big(\sum_{k\in\mathbb{Z}^d} (1+|k|^2)^s |u_k|^2\Big)^{1/2} \Big(\sum_{k\in\mathbb{Z}^d} (1+|k|^2)^{-s}\Big)^{1/2} .$$

Since the sum in the second factor converges if and only if $s > \frac{d}{2}$, the claim follows. □

As a consequence of Lemma 7.2, we are able to obtain a more general Sobolev embedding for all L^p spaces:

Theorem 7.3 (**Sobolev Embeddings**). *Let Assumption 2.17 hold, define the resulting Hilbert scale of spaces $\mathcal{H}^s$ by (7.6) and assume that $p \in [2,\infty]$. Then, for every $s > \frac{d}{2} - \frac{d}{p}$, the space $\mathcal{H}^s$ is contained in the space $L^p(D)$ and there exists a constant K_2 such that $\|u\|_{L^p} \leq K_2 \|u\|_{\mathcal{H}^s}$.*

Proof. The case $p = 2$ is obvious and the case $p = \infty$ has already been shown, so it remains to show the claim for $p \in (2,\infty)$. The idea is to divide the space of eigenfunctions into "blocks" and to estimate separately the L^p norm of every block. More precisely, we define a sequence of functions $u^{(n)}$ by

$$u^{(-1)} = u_0\,\varphi_0 \;, \quad u^{(n)} = \sum_{2^n \leq j < 2^{n+1}} u_j\,\varphi_j \;,$$

where the φ_j are an orthonormal basis of eigenfunctions for A, so that $u = \sum_{n\geq -1} u^{(n)}$. For $n \geq 0$ the Hölder inequality gives

$$\|u^{(n)}\|_{L^p}^p \leq \|u^{(n)}\|_{L^2}^2 \|u^{(n)}\|_{L^\infty}^{p-2} . \tag{7.9}$$

Now set $s' = \frac{d}{2} + \epsilon$ for some $\epsilon > 0$ and note that the construction of $u^{(n)}$, together with Lemma 7.2, gives the bounds

$$\|u^{(n)}\|_{L^2} \leq K 2^{-ns/d} \|u^{(n)}\|_{\mathcal{H}^s} \;, \quad \|u^{(n)}\|_{L^\infty} \leq K_1 \|u^{(n)}\|_{\mathcal{H}^{s'}} \leq K 2^{n(s'-s)/d} \|u^{(n)}\|_{\mathcal{H}^s} . \tag{7.10}$$

Inserting this into (7.9), we obtain (possibly for an enlarged K)

$$\begin{aligned}\|u^{(n)}\|_{L^p} &\leq K \|u^{(n)}\|_{\mathcal{H}^s} 2^{n\left((s'-s)\frac{p-2}{p} - \frac{2s}{p}\right)/d} = K \|u^{(n)}\|_{\mathcal{H}^s} 2^{n\left(\epsilon\frac{p-2}{p} + \frac{d}{2} - \frac{d}{p} - s\right)/d} \\ &\leq K \|u\|_{\mathcal{H}^s} 2^{n\left(\epsilon + \frac{d}{2} - \frac{d}{p} - s\right)/d} .\end{aligned}$$

It follows that $\|u\|_{L^p} \leq |u_0| + \sum_{n\geq 0} \|u^{(n)}\|_{L^p} \leq K_2 \|u\|_{\mathcal{H}^s}$, provided that the exponent appearing in this expression is negative which, since ϵ can be chosen arbitrarily small, is precisely the case whenever $s > \frac{d}{2} - \frac{d}{p}$. □

7.2. Probability and Integration In Infinite Dimensions.

7.2.1. Product Measure for i.i.d. Sequences. Perhaps the most straightforward setting in which probability measures in infinite dimensions are encountered is when studying i.i.d. sequences of real-valued random variables. Furthermore, this is our basic building block for the construction of random functions – see subsection 2.1 – so we briefly overview the subject. Let $\mathbb{P}_0$ be a probability measure on $\mathbb{R}$ so that $(\mathbb{R}, \mathsf{B}(\mathbb{R}), \mathbb{P}_0)$ is a probability space and consider the i.i.d. sequence $\xi := \{\xi_j\}_{j=1}^{\infty}$ with $\xi_1 \sim \mathbb{P}_0$.

The construction of such a sequence can be formalised as follows. We consider ξ as a random variable taking values in the space $\mathbb{R}^\infty$ endowed with the product topology, i.e. the smallest topology for which the projection maps $\ell_n \colon \xi \mapsto \xi_n$ are continuous for every n. This is a complete metric space; an example of a distance generating the product topology is given by

$$d(x,y) = \sum_{n=1}^{\infty} 2^{-n} \frac{|x_n - y_n|}{1 + |x_n - y_n|} \ .$$

Since we are considering a *countable* product, the resulting σ-algebra $\mathsf{B}(\mathbb{R}^\infty)$ coincides with the product σ-algebra, which is the smallest σ-algebra for which all ℓ_n's are measurable.

In what follows we need the notion of the *pushforward* of a probability measure under a measurable map. If $f : B_1 \to B_2$ is a measurable map between two measurable spaces $\big(B_i, \mathsf{B}(B_i)\big)$ $i = 1, 2$ and μ_1 is a probability measure on B_1 then $\mu_2 = f^\sharp \mu_1$ denotes the pushforward probability measure on B_2 defined by $\mu_2(A) = \mu_1\big(f^{-1}(A)\big)$ for all $A \in \mathsf{B}(B_2)$. (The notation $f^*\mu$ is sometimes used in place of $f^\sharp\mu$, but we reserve this notation for adjoints.) Recall that in section 2 we construct random functions via the random series (2.1) whose coefficients are constructed from an i.i.d sequence. Our interest is in studying the pushforward measure $\mathcal{F}^\sharp \mathbb{P}_0$ where $\mathcal{F} : \mathbb{R}^\infty \to X'$ is defined by

$$\mathcal{F}\xi = m_0 + \sum_{j=1}^{\infty} \gamma_j \xi_j \phi_j. \tag{7.11}$$

In particular section 2 is devoted to determing suitable separable Banach spaces X' on which to define the pushforward measure.

With the pushforward notation at hand, we may also describe Kolmogorov's extension theorem which can be stated as follows.

Theorem 7.4. (Kolmogorov Extension) *Let X be a Polish space and let I be an arbitrary set. Assume that, for any finite subset $A \subset I$, we are given a probability measure $\mathbb{P}_A$ on the finite product space X^A. Assume furthermore that the family of measures $\{\mathbb{P}_A\}$ is consistent in the sense that if $B \subset A$ and $\Pi_{A,B} \colon X^A \to X^B$ denotes the natural projection map, then $\Pi^\sharp_{A,B}\mathbb{P}_A = \mathbb{P}_B$. Then, there exists a unique probability measure $\mathbb{P}$ on X^I endowed with the product σ-algebra with the property that $\Pi^\sharp_{I,A}\mathbb{P} = \mathbb{P}_A$ for every finite subset $A \subset I$.*

Loosely speaking, one can interpret this theorem as stating that if one knows the law of any *finite* number of components of a random vector or function then this determines the law of the *whole* random vector or function; in particular in the case of the random function this comprises *uncountably* many components. This statement is thus highly non-trivial as soon as the set I is infinite since we have *a priori* defined $\mathbb{P}_A$ only for finite subsets $A \subset I$ and the theorem allows us to extend this uniquely also to infinite subsets.

As a simple application, we can use this theorem to define the infinite product measure $\mathbb{P} = \bigotimes_{k=1}^{\infty} \mathbb{P}_0$ as the measure given by Kolmogorov's Extension Theorem 7.4 if we take as our family of specifications $\mathbb{P}_A = \bigotimes_{k \in A} \mathbb{P}_0$. Our i.i.d. sequence ξ is then naturally defined as a random sample taken from the probability space $\big(\mathbb{R}^\infty, \mathsf{B}(\mathbb{R}^\infty), \mathbb{P}\big)$. A more complicated example follows from making sense of the random field perspective on random functions as explained in subsection 2.5.

7.2.2. Probability and Integration on Separable Banach Spaces. We now study probability and integration on separable Banach spaces B; we let B^* denote the dual space of bounded linear functionals on B. The assumption of separability rules out some important function spaces like $L^\infty(D;\mathbb{R})$, but is required in order for the basic results of integration theory to hold. This is because, when considering a non-separable Banach space B, it is not clear what the "natural" σ-algebra on B is. One natural candidate is the Borel σ-algebra, denoted $\mathsf{B}(B)$, namely the smallest σ-algebra containing all open sets; another is the cylindrical σ-algebra, namely the smallest σ-algebra for which all bounded linear functionals on B are measurable. For i.i.d. sequences, the analogues of these two σ-algebras can be identified whereas, in the general setting, the cylindrical σ-algebra can be strictly smaller than the Borel σ-algebra. In the case of separable Banach spaces however, both σ-algebras agree:

Lemma 7.5. *Let B be a separable Banach space and let μ and ν be two Borel probability measures on B. If $\ell^\sharp \mu = \ell^\sharp \nu$ for every $\ell \in B^*$, then $\mu = \nu$.*

Thus, as for i.i.d. sequences, there is therefore a canonical notion of measurability. Whenever we refer to (probability) measures on a separable Banach space B in the sequel, we really mean (probability) measures on $\big(B, \mathsf{B}(B)\big)$.

We now turn to the definition of integration with respect to probability measures on B. Given a (Borel) measurable function $f\colon \Omega \to B$ where $(\Omega, \mathcal{F}, \mathbb{P})$ is a standard probability space, we say that f is integrable with respect to $\mathbb{P}$ if the map $\omega \mapsto \|f(\omega)\|$ belongs to $L^1_{\mathbb{P}}(\Omega;\mathbb{R})$. (Note that this map is certainly Borel measurable since the norm $\|\cdot\|\colon B \to \mathbb{R}$ is a continuous, and therefore also Borel measurable, function.) Given such an integrable function f, we *define* its Bochner integral by

$$\int f(\omega)\, \mathbb{P}(d\omega) = \lim_{n \to \infty} \int f_n(\omega)\, \mathbb{P}(d\omega)\ ,$$

where f_n is a sequence of simple functions, for which the integral on the right-hand

side may be defined in the usual way, chosen such that

$$\lim_{n\to\infty} \int \|f_n(\omega) - f(\omega)\| \, \mathbb{P}(d\omega) = 0.$$

With this definition the value of the integral does not depend on the approximating sequence, it is linear in f, and

$$\int \ell(f(\omega)) \, \mathbb{P}(d\omega) = \ell\Big(\int f(\omega) \, \mathbb{P}(d\omega) \Big) \,, \tag{7.12}$$

for every element ℓ in the dual space B^*.

Given a probability measure μ on a separable Banach space B, we now say that μ has *finite expectation* if the identity function $x \mapsto x$ is integrable with respect to μ. If this is the case, we define the expectation of μ as

$$\int_B x \, \mu(dx) \,,$$

where the integral is interpreted as a Bochner integral.

Similarly, it is natural to say that μ has *finite variance* if the map $x \mapsto \|x\|^2$ is integrable with respect to μ. Regarding the covariance C_μ of μ itself, it is natural to define it as a bounded linear operator $C_\mu \colon B^* \to B$ with the property that

$$C_\mu \ell = \int_B x \ell(x) \, \mu(dx) \,, \tag{7.13}$$

for every $\ell \in B^*$. At this stage however, it is not clear whether such an operator C_μ always exists solely under the assumption that μ has finite variance. For any $x \in B$, we define the projection operator $P_x \colon B^* \to B$ by

$$P_x \ell = x \, \ell(x) \,, \tag{7.14}$$

suggesting that we define

$$C_\mu := \int_B P_x \, \mu(dx) \,. \tag{7.15}$$

The problem with this definition is that if we view the map $x \mapsto P_x$ as a map taking values in the space $\mathcal{L}(B^*, B)$ of bounded linear operators from $B^* \to B$ then, since this space is not separable in general, it is not clear a priori whether (7.15) makes sense as a Bochner integral. This suggests to define the subspace $B_\star(B) \subset \mathcal{L}(B^*, B)$ given by the closure (in the usual operator norm) of the linear span of operators of the type P_x given in (7.14) for $x \in B$. We then have:

Lemma 7.6. *If B is separable, then $B_\star(B)$ is also separable. Furthermore, $B_\star(B)$ consists of compact operators.*

This leads to the following corollary:

Corollary 7.7. *Assume that μ has finite variance so that the map $x \mapsto \|x\|^2$ is integrable with respect to μ. Then the covariance operator C_μ defiend by* (7.15) *exists as a Bochner integral in $B_\star(B)$.*

Remark 7.8. Once the covariance is defined, the fact that (7.13) holds is then an immediate consequence of (7.12). In general, not every element $C \in B_\star(B)$ can be realised as the covariance of some probability measure. This is the case even if we impose the positivity condition $\ell(C\ell) \geq 0$, which by (7.13) is a condition satisfied by every covariance operator. For further insight into this issue, see Lemma 7.32 which characteritzes precisely the covariance operators of a Gaussian measure in separable Hilbert space. □

Given any probability measure μ on B, we can define its *Fourier transform* $\hat{\mu}\colon B^* \to \mathbb{C}$ by

$$\hat{\mu}(\ell) := \int_B e^{i\ell(x)}\, \mu(dx)\ . \tag{7.16}$$

For a Gaussian measure μ_0 on B with mean a and covariance operator C, it may be shown show that, for any $\ell \in B^*$, the characteristic function is given by

$$\hat{\mu}_0(\ell) = \mathrm{e}^{i\ell(a) - \frac{1}{2}\ell(C\ell)}. \tag{7.17}$$

As a consequence of Lemma 7.5, it is almost immediate that a measure is uniquely determined by its Fourier transform, and this is the content of the following result.

Lemma 7.9. *Let μ and ν be any two probability measures on a separable Banach space B. If $\hat{\mu}(\ell) = \hat{\nu}(\ell)$ for every $\ell \in B^*$, then $\mu = \nu$.*

7.2.3. Probability and Integration on Separable Hilbert Spaces. We will frequently be interested in the case where $B = \mathcal{H}$ for $\big(\mathcal{H}, \langle\cdot,\cdot\rangle, \|\cdot\|\big)$ some separable Hilbert space. Bochner integration can then, of course, be defined as a special case of the preceding development on separable Banach spaces. We make use of the Riesz representation theorem to identify $\mathcal{H}$ with its dual and $\mathcal{H} \otimes \mathcal{H}$ with a subspace of the space of linear operators on $\mathcal{H}$. The covariance operator of a measure μ on $\mathcal{H}$ may then be viewed as a bounded linear operator from $\mathcal{H}$ into itself. The definition (7.13) of C_μ becomes

$$C_\mu \ell = \int_{\mathcal{H}} \langle \ell, x\rangle x\, \mu(dx)\ , \tag{7.18}$$

for all $\ell \in \mathcal{H}$ and (7.15) becomes

$$C_\mu = \int_{\mathcal{H}} x \otimes x\, \mu(dx)\ . \tag{7.19}$$

Corollary 7.7 shows that we can indeed make sense of the second formulation as a Bochner integral, provided that μ has finite variance in $\mathcal{H}$.

7.2.4. Metrics on Probability Measures. When discussing well-posedness and approximation theory for the posterior distribution, it is of interest to estimate the distance between two probability measures and thus we will be interested in metrics between probability measures. In this subsection we introduce two useful metrics on measures: the *total variation distance* and the *Hellinger distance*. We

discuss the relationships between the metrics and indicate how they may be used to estimate differences between expectations of random variables under two different measures. We also discuss the *Kullback-Leibler divergence*, a useful distance measure which does not satisfy the axioms of a metric, but which may be used to bound both the Hellinger and total variation distances, and which is also useful in defining algorithms for finding the best approximation to a given measure from within some restricted class of measures, such as Gaussians.

Assume that we have two probability measures μ and μ' on a separable Banach space denoted by B (actually the considerations here apply on a Polish space but we do not need this level of generality). Assume that μ and μ' are both absolutely continuous with respect to a common reference measure ν, also defined on the same measure space. Such a measure always exists – take $\nu = \frac{1}{2}(\mu + \mu')$ for example. In the following, all integrals of real-valued functions over B are simply denoted by $\int$. The following define two concepts of distance between μ and μ'. The resulting metrics that we define are independent of the choice of this common reference measure.

Definition 7.10. The *total variation distance* between μ and μ' is

$$d_{\mathrm{TV}}(\mu, \mu') = \frac{1}{2} \int \Big| \frac{d\mu}{d\nu} - \frac{d\mu'}{d\nu} \Big| d\nu. \quad \square$$

In particular, if μ' is absolutely continuous with respect to μ then

$$d_{\mathrm{TV}}(\mu, \mu') = \frac{1}{2} \int \Big| 1 - \frac{d\mu'}{d\mu} \Big| d\mu. \tag{7.20}$$

Definition 7.11. The *Hellinger distance* between μ and μ' is

$$d_{\mathrm{Hell}}(\mu, \mu') = \sqrt{\frac{1}{2} \int \Big(\sqrt{\frac{d\mu}{d\nu}} - \sqrt{\frac{d\mu'}{d\nu}} \Big)^2 d\nu}. \quad \square$$

In particular, if μ' is absolutely continuous with respect to μ then

$$d_{\mathrm{Hell}}(\mu, \mu') = \sqrt{\frac{1}{2} \int \Big(1 - \sqrt{\frac{d\mu'}{d\mu}} \Big)^2 d\mu}. \tag{7.21}$$

Note that the numerical constant $\frac{1}{2}$ appearing in both definitions is chosen in such a way as to ensure the bounds

$$0 \le d_{\mathrm{TV}}(\mu, \mu') \le 1 \,, \qquad 0 \le d_{\mathrm{Hell}}(\mu, \mu') \le 1 \,.$$

In the case of the total variation inequality this is an immediate consequence of the triangle inequality, combined with the fact that both μ and μ' are probability measures, so that $\int \frac{d\mu}{d\nu}\, d\nu = 1$ and similarly for μ'. In the case of the Hellinger distance, it follows by expanding the square and applying similar considerations.

The Hellinger and total variation distances are related as follows, which shows in particular that they both generate the same topology:

Lemma 7.12. *The total variation and Hellinger metrics are related by the inequalities*

$$\frac{1}{\sqrt{2}} d_{\mathrm{TV}}(\mu, \mu') \leq d_{\mathrm{Hell}}(\mu, \mu') \leq d_{\mathrm{TV}}(\mu, \mu')^{\frac{1}{2}}.$$

Proof. We have

$$\begin{aligned} d_{\mathrm{TV}}(\mu, \mu') &= \frac{1}{2} \int \Big| \sqrt{\frac{d\mu}{d\nu}} - \sqrt{\frac{d\mu'}{d\nu}} \Big| \Big| \sqrt{\frac{d\mu}{d\nu}} + \sqrt{\frac{d\mu'}{d\nu}} \Big| d\nu \\ &\leq \sqrt{\Big(\frac{1}{2} \int \Big(\sqrt{\frac{d\mu}{d\nu}} - \sqrt{\frac{d\mu'}{d\nu}} \Big)^2 d\nu \Big)} \sqrt{\Big(\frac{1}{2} \int \Big(\sqrt{\frac{d\mu}{d\nu}} + \sqrt{\frac{d\mu'}{d\nu}} \Big)^2 d\nu \Big)} \\ &\leq \sqrt{\Big(\frac{1}{2} \int \Big(\sqrt{\frac{d\mu}{d\nu}} - \sqrt{\frac{d\mu'}{d\nu}} \Big)^2 d\nu \Big)} \sqrt{\Big(\int \Big(\frac{d\mu}{d\nu} + \frac{d\mu'}{d\nu} \Big) d\nu \Big)} \\ &= \sqrt{2} d_{\mathrm{Hell}}(\mu, \mu') \end{aligned}$$

as required for the first bound.

For the second bound note that, for any positive a and b, one has the bound $|\sqrt{a} - \sqrt{b}| \leq \sqrt{a} + \sqrt{b}$. As a consequence, we have the bound

$$\begin{aligned} d_{\mathrm{Hell}}(\mu, \mu')^2 &\leq \frac{1}{2} \int \Big| \sqrt{\frac{d\mu}{d\nu}} - \sqrt{\frac{d\mu'}{d\nu}} \Big| \Big| \sqrt{\frac{d\mu}{d\nu}} + \sqrt{\frac{d\mu'}{d\nu}} \Big| d\nu \\ &= \frac{1}{2} \int \Big| \frac{d\mu}{d\nu} - \frac{d\mu'}{d\nu} \Big| d\nu \\ &= d_{\mathrm{TV}}(\mu, \mu') \,, \end{aligned}$$

as required. □

Example 7.13. Consider two Gaussian densities on $\mathbb{R}$: $N(m_1, \sigma_1^2)$ and $N(m_2, \sigma_2^2)$. The Hellinger distance between them is given by

$$d_{\mathrm{Hell}}(\mu, \mu')^2 = 1 - \sqrt{\exp\Big(-\frac{(m_1 - m_2)^2}{2(\sigma_1^2 + \sigma_2^2)} \Big) \frac{2\sigma_1 \sigma_2}{(\sigma_1^2 + \sigma_2^2)}}.$$

To see this note that

$$d_{\mathrm{Hell}}(\mu, \mu')^2 = 1 - \frac{1}{(2\pi\sigma_1\sigma_2)^{\frac{1}{2}}} \int_{\mathbb{R}} \exp(-Q) dx$$

where

$$Q = \frac{1}{4\sigma_1^2}(x - m_1)^2 + \frac{1}{4\sigma_2^2}(x - m_2)^2.$$

Define σ^2 by

$$\frac{1}{2\sigma^2} = \frac{1}{4\sigma_1^2} + \frac{1}{4\sigma_2^2}.$$

We change variable under the integral to y given by

$$y = x - \frac{m_1 + m_2}{2}$$

and note that then, by completing the square,

$$Q = \frac{1}{2\sigma^2}(y-m)^2 + \frac{1}{4(\sigma_1^2+\sigma_2^2)}(m_2-m_1)^2$$

where m does not appear in what follows and so we do not detail it. Noting that the integral is then a multiple of a standard Gaussian $N(m,\sigma^2)$ gives the desired result. In particular this calculation shows that the Hellinger distance between two Gaussians on $\mathbb{R}$ tends to zero if and only if the means and variances of the two Gaussians approach one another. Furthermore, by the previous lemma, the same is true for the total variation distance. □

The preceding example generalizes to higher dimension and shows that, for example, the total variation and Hellinger metrics cannot metrize weak convergence of probability measures (as one can also show that convergence in total variation metric implies strong convergence). They are nonetheless useful distance measures, for example between families of measures which are mutually absolutely continuous. Furthermore, the Hellinger distance is particularly useful for estimating the difference between expectation values of functions of random variables under different measures. This is encapsulated in the following lemma:

Lemma 7.14. *Let μ and μ' be two probability measures on a separable Banach space X. Assume also that $f : X \to E$, where $(E, \|\cdot\|)$ is a separable Banach space, is measurable and has second moments with respect to both μ and μ'. Then*

$$\|\mathbb{E}^{\mu} f - \mathbb{E}^{\mu'} f\| \le 2\Big(\mathbb{E}^{\mu}\|f\|^2 + \mathbb{E}^{\mu'}\|f\|^2\Big)^{\frac{1}{2}} d_{\mathrm{Hell}}(\mu,\mu').$$

Furthermore, if E is a separable Hilbert space and $f : X \to E$ as before has fourth moments, then

$$\|\mathbb{E}^{\mu}(f\otimes f) - \mathbb{E}^{\mu'}(f\otimes f)\| \le 2\Big(\mathbb{E}^{\mu}\|f\|^4 + \mathbb{E}^{\mu'}\|f\|^4\Big)^{\frac{1}{2}} d_{\mathrm{Hell}}(\mu,\mu').$$

Proof. Let ν be a reference probability measure as above. We then have the bound

$$\begin{aligned}
\|\mathbb{E}^{\mu} f - \mathbb{E}^{\mu'} f\| &\le \int \|f\| \Big| \frac{d\mu}{d\nu} - \frac{d\mu'}{d\nu} \Big| d\nu \\
&= \int \Big(\frac{1}{\sqrt{2}} \Big| \sqrt{\frac{d\mu}{d\nu}} - \sqrt{\frac{d\mu'}{d\nu}} \Big| \Big) \Big(\sqrt{2}\|f\| \Big| \sqrt{\frac{d\mu}{d\nu}} + \sqrt{\frac{d\mu'}{d\nu}} \Big| \Big) d\nu \\
&\le \sqrt{\Big(\frac{1}{2} \int \Big(\sqrt{\frac{d\mu}{d\nu}} - \sqrt{\frac{d\mu'}{d\nu}} \Big)^2 d\nu \Big)} \sqrt{\Big(2 \int \|f\|^2 \Big(\sqrt{\frac{d\mu}{d\nu}} + \sqrt{\frac{d\mu'}{d\nu}} \Big)^2 d\nu \Big)} \\
&\le \sqrt{\Big(\frac{1}{2} \int \Big(\sqrt{\frac{d\mu}{d\nu}} - \sqrt{\frac{d\mu'}{d\nu}} \Big)^2 d\nu \Big)} \sqrt{\Big(4 \int \|f\|^2 \Big(\frac{d\mu}{d\nu} + \frac{d\mu'}{d\nu} \Big) d\nu \Big)} \\
&= 2\Big(\mathbb{E}^{\mu}\|f\|^2 + \mathbb{E}^{\mu'}\|f\|^2\Big)^{\frac{1}{2}} d_{\mathrm{Hell}}(\mu,\mu')
\end{aligned}$$

as required.

The proof for $f \otimes f$ follows from the bound

$$\begin{aligned}\|\mathbb{E}^{\mu}(f\otimes f) - \mathbb{E}^{\mu'}(f\otimes f)\| &= \sup_{\|h\|=1} \|\mathbb{E}^{\mu}\langle f,h\rangle f - \mathbb{E}^{\mu'}\langle f,h\rangle f\| \\ &\leq \int \|f\|^2 \Big|\frac{d\mu}{d\nu} - \frac{d\mu'}{d\nu}\Big| d\nu\,,\end{aligned}$$

and then arguing similarly to the first case but with $\|f\|$ replaced by $\|f\|^2$. □

Remark 7.15. Note, in particular, that choosing $X = E$, and with f chosen to be the identity mapping, we deduce that the differences between the mean (resp. covariance operator) of two measures are bounded above by their Hellinger distance, provided that one has some *a priori* control on the second (resp. fourth) moments. □

We now define a third widely used distance concept for comparing two probability measures. Note, however, that it does not give rise to a metric in the strict sense, because it violates both symmetry and the triangle inequality.

Definition 7.16. The Kullback-Leibler divergence between two measures μ' and μ, with μ' absolutely continuous with respect to μ, is

$$D_{\mathrm{KL}}(\mu'||\mu) = \int \frac{d\mu'}{d\mu} \log\Big(\frac{d\mu'}{d\mu}\Big) d\mu. \quad \square$$

If μ is also absolutely continuous with respect to μ', so that the two measures are equivalent, then

$$D_{\mathrm{KL}}(\mu'||\mu) = -\int \log\Big(\frac{d\mu}{d\mu'}\Big) d\mu'$$

and the two definitions coincide.

Example 7.17. Consider two Gaussian densities on $\mathbb{R}$: $N(m_1, \sigma_1^2)$ and $N(m_2, \sigma_2^2)$. The Kullback-Leibler divergence between them is given by

$$D_{\mathrm{KL}}(\mu_1||\mu_2) = \ln\Big(\frac{\sigma_2}{\sigma_1}\Big) + \frac{1}{2}\Big(\frac{\sigma_1^2}{\sigma_2^2} - 1\Big) + \frac{(m_2-m_1)^2}{2\sigma_2^2}.$$

To see this note that

$$\begin{aligned}D_{\mathrm{KL}}(\mu_1||\mu_2) &= \mathbb{E}^{\mu_1}\Big(\ln\sqrt{\frac{\sigma_2^2}{\sigma_1^2}} + \frac{1}{2\sigma_2^2}|x-m_2|^2 - \frac{1}{2\sigma_1^2}|x-m_1|^2\Big) \\ &= \ln\frac{\sigma_2}{\sigma_1} + \mathbb{E}^{\mu_1}\Big((\frac{1}{2\sigma_2^2} - \frac{1}{2\sigma_1^2})|x-m_1|^2\Big) + \mathbb{E}^{\mu_1}\frac{1}{2\sigma_2^2}\Big(|x-m_2|^2 - |x-m_1|^2\Big) \\ &= \ln\frac{\sigma_2}{\sigma_1} + \frac{1}{2}\Big(\frac{\sigma_1^2}{\sigma_2^2} - 1\Big) + \frac{1}{2\sigma_2^2}\mathbb{E}^{\mu_1}\Big(m_2^2 - m_1^2 + 2x(m_1-m_2)\Big) \\ &= \ln\frac{\sigma_2}{\sigma_1} + \frac{1}{2}\Big(\frac{\sigma_1^2}{\sigma_2^2} - 1\Big) + \frac{1}{2\sigma_2^2}(m_2-m_1)^2\end{aligned}$$

as required. □

As for Hellinger distance, this example shows that two Gaussians on $\mathbb{R}$ approach one another in the Kullback-Leibler divergence if and only if their means and variances approach one another. This generalizes to higher dimensions. The Kullback-Leibler divergence provides an upper bound for the square of the Hellinger distance and for the square of the total variation distance.

Lemma 7.18. *Assume that two measures μ and μ' are equivalent. Then the bounds*

$$d_{\mathrm{Hell}}(\mu,\mu')^2 \le \frac{1}{2} D_{\mathrm{KL}}(\mu||\mu') \,, \qquad d_{\mathrm{TV}}(\mu,\mu')^2 \le D_{\mathrm{KL}}(\mu||\mu') \,,$$

hold.

Proof. The second bound follows from the first by using Lemma 7.12, thus it suffices to proof the first. In the following we use the fact that

$$x - 1 \ge \log(x) \qquad \forall x \ge 0 \,,$$

so that

$$\sqrt{x} - 1 \ge \frac{1}{2}\log(x) \qquad \forall x \ge 0 \,.$$

This yields the bound

$$\begin{aligned} d_{\mathrm{Hell}}(\mu,\mu')^2 &= \frac{1}{2}\int \Big(\sqrt{\frac{d\mu'}{d\mu}} - 1\Big)^2 d\mu = \frac{1}{2}\int \Big(\frac{d\mu'}{d\mu} + 1 - 2\sqrt{\frac{d\mu'}{d\mu}}\Big) d\mu \\ &= \int \Big(1 - \sqrt{\frac{d\mu'}{d\mu}}\Big) d\mu \le \frac{1}{2}\int \Big(-\log\frac{d\mu'}{d\mu}\Big) d\mu \\ &= \frac{1}{2} D_{\mathrm{KL}}(\mu||\mu') \,, \end{aligned}$$

as required. □

7.2.5. Kolmogorov Continuity Test. The setting of Kolmogorov's continuity test is the following. We assume that we are given a compact domain $D \subset \mathbb{R}^d$, a complete separable metric space X, as well as a collection of X-valued random variables $u : x \in D \mapsto X$. At this stage we assume no regularity whatsoever on the parameter x: the distribution of this collection of random variables is a measure μ_0 on the space X^D of all functions from D to X endowed with the product σ-algebra. Any consistent family of marginal distributions does yield such a measure by Kolmogorov's extension Theorem 7.4 . With these notations at hand, Kolmogorov's continuity test can be formulated as follows, and enables the extraction of regularity with respect to variation of $u(x)$ with respect to x.

Theorem 7.19 (**Kolmogorov Continuity Test).** *Let D and u be as above and assume that there exist $p > 1$, $\alpha > 0$ and $K > 0$ such that*

$$\mathbb{E}\mathsf{d}\big(u(x),u(y)\big)^p \le K|x-y|^{p\alpha+d} \,, \qquad \forall x,y \in D \,, \tag{7.22}$$

where d *denotes the distance function on* X*, and* d *the dimension of the compact domain* D*. Then, for every* $\beta < \alpha$*, there exists a unique measure* μ *on* $C^{0,\beta}(D, X)$ *such that the canonical process under* μ *has the same law as* u*.*

We have here generalized the notion of Hölder spaces from subsection 7.1.2 to functions taking values in a Polish space; such generalizations are discussed in subsection 7.1.4. The notion of *canonical process* is defined in subsection 7.4.

We will frequently use Kolmogorov's continuity test in the following setting: we again assume that we are given a compact domain $D \subset \mathbb{R}^d$, and now a collection $u(x)$ of $\mathbb{R}^n$-valued random variables indexed by $x \in D$. We have the following:

Corollary 7.20. *Assume that there exist* $p > 1$*,* $\alpha > 0$ *and* $K > 0$ *such that*

$$\mathbb{E}|u(x) - u(y)|^p \leq K|x - y|^{p\alpha + d}\,, \qquad \forall x, y \in D\,.$$

Then, for every $\beta < \alpha$*, there exists a unique measure* μ *on* $C^{0,\beta}(D)$ *such that the canonical process under* μ *has the same law as* u*.*

Remark 7.21. Recall that $C^{0,\gamma'}(D) \subset C_0^{0,\gamma}(D)$ for all $\gamma' > \gamma$ so that, since the interval $\beta < \alpha$ for this theorem is open, we may interpret the result as giving an equivalent measure defined on a separable Banach space.

A very useful consequence of Kolmogorov's continuity criterion is the following result. The setting is to consider a random function f given by the random series

$$u = \sum_{k \geq 0} \xi_k \psi_k \tag{7.23}$$

where $\{\xi_k\}_{k\geq 0}$ is an i.i.d. sequence and the ψ_k are real- or complex-valued Hölder functions on bounded open $D \subset \mathbb{R}^d$ satisfying, for some $\alpha \in (0, 1]$,

$$|\psi_k(x) - \psi_k(y)| \leq h(\alpha, \psi_k)|x - y|^\alpha \quad x, y \in D; \tag{7.24}$$

of course if $\alpha = 1$ the functions are Lipschitz.

Corollary 7.22. *Let* $\{\xi_k\}_{k\geq 0}$ *be countably many centred i.i.d. random variables (real or complex) with bounded moments of all orders. Moreover let* $\{\psi_k\}_{k\geq 0}$ *satisfy* (7.24)*. Suppose there is some* $\delta \in (0, 2)$ *such that*

$$S_1 := \sum_{k\geq 0} \|\psi_k\|_{L^\infty}^2 < \infty \quad \text{and} \quad S_2 := \sum_{k\geq 0} \|\psi_k\|_{L^\infty}^{2-\delta} h(\alpha, \psi_k)^\delta < \infty\,. \tag{7.25}$$

Then u *defined by* (7.23) *is almost surely finite for every* $x \in D$*, and* u *is Hölder continuous for every Hölder exponent smaller than* $\alpha\delta/2$*.*

Proof. Let us denote by $\kappa_n(X)$ the nth cumulant of a random variable X. The odd cumulants of centred random variables are zero. Furthermore, using the fact

that the cumulants of independent random variables simply add up and that the cumulants of ξ_k are all finite by assumption, we obtain for $p \geq 1$ the bound

$$\begin{aligned}
\left|\kappa_{2p}\big(u(x)-u(y)\big)\right| &= \Big|\sum_{k\geq 0}\kappa_{2p}(\xi_k)\left(\psi_k(x)-\psi_k(y)\right)^{2p}\Big| \\
&\lesssim C_p \sum_{k\geq 0} \min\{2^{2p}\|\psi_k\|_{L^\infty}^{2p}, h(\alpha,\psi_k)^{2p}|x-y|^{2p\alpha}\} \\
&\lesssim C_p \sum_{k\geq 0} \|\psi_k\|_{L^\infty}^{(1-\frac{\delta}{2})2p} h(\alpha,\psi_k)^{2p\cdot\frac{\delta}{2}}|x-y|^{2p\alpha\cdot\frac{\delta}{2}} \\
&\lesssim C_p|x-y|^{p\alpha\delta}\,,
\end{aligned}$$

with C_p denoting positive constants depending on p which can change from occurrence to occurrence, and where we used that $\min\{a,bx^2\} \leq a^{1-\delta/2}b^{\delta/2}|x|^\delta$ for any $a,b\geq 0$ and the finiteness of S_2. In a similar way, we obtain $\left|\kappa_{2p}u(x)\right| < \infty$ for every $p \geq 1$. Since the random variables $u(x)$ are centred, all moments of even order $2p$, $p\geq 1$, can be expressed in terms of homogeneous polynomials of the even cumulants of order upto $2p$, so that

$$\mathbb{E}|u(x)-u(y)|^{2p} \lesssim C_p|x-y|^{p\alpha\delta}\,, \qquad \mathbb{E}|u(x)|^{2p} < \infty\,,$$

uniformly over $x,y\in D$. The almost sure boundedness on L^∞ follows from the second bound. The Hölder continuity claim follows from Kolmogorov's continuity test in the form of Corollary 7.20, after noting that $p\alpha\delta = 2p\big(\frac{1}{2}\alpha\delta - \frac{d}{2p}\big) + d$ and choosing p arbitrarily large. □

Remark 7.23. Note that (7.23) is simply a rewrite of (2.1), with $\psi_0 = m_0$, $\xi_0 = 1$ and $\psi_k = \gamma_k\phi_k$. In the case where the ξ_k are standard normal then the ψ_k's in Corollary 7.22 form an orthonormal basis of the Cameron-Martin space (see Definition 7.26) of a Gaussian measure. The criterion (7.25) then provides an effective way of showing that the measure in question can be realised on a space of Hölder continuous functions. □

7.3. Gaussian Measures.

7.3.1. Separable Banach Space Setting. We start with the definition of a Gaussian measure on a separable Banach space B. There is no equivalent to Lebesgue measure in infinite dimensions (as it could not be σ-additive), and so we cannot define a Gaussian measure by prescribing the form of its density. However, note that Gaussian measures on $\mathbb{R}^n$ can be characterised by prescribing that the projections of the measure onto any one-dimensional subspace of $\mathbb{R}^n$ are all Gaussian. This is a property that can readily be generalised to infinite-dimensional spaces:

Definition 7.24. A *Gaussian probability measure* μ on a separable Banach space B is a Borel measure such that $\ell^\sharp\mu$ is a Gaussian probability measure on $\mathbb{R}$ for every continuous linear functional $\ell\colon B\to\mathbb{R}$. (Here, Dirac measures are considered

to be Gaussian measures with zero variance.) The measure is said to be *centred* if $\ell^\sharp\mu$ has mean zero for every ℓ. $\square$

This is a reasonable definition since, provided that B is separable, the one-dimensional projections of any probability measure carry sufficient information to characterise it – see Lemma 7.5. We now state an important result which controls the tails of Gaussian distributions:

Theorem 7.25 (Fernique). *Let μ be a Gaussian probability measure on a separable Banach space B. Then, there exists $\alpha > 0$ such that $\int_B \exp(\alpha\|x\|^2)\,\mu(dx) < \infty$.*

As a consequence of Fernique's theorem and the Corollary 7.7, every Gaussian measure μ admits a compact covariance operator C_μ given by (7.15), because the second moment is bounded. In fact the techniques used to prove the Fernique theorem show that, if $M = \int_B \|x\|\,\mu(dx)$, then there is a global constant $K > 0$ such that

$$\int_B \|x\|^{2n}\,\mu(dx) \leq n!K\alpha^{-n}M^{2n}. \tag{7.26}$$

Since the covariance operator, and hence the mean, exist for a Gaussian measure, and since they may be shown to characterize the measure completely, we write $N(m, C_\mu)$ for a Gaussian with mean m and covariance operator C_μ.

Measures in infinite dimensional spaces are typically mutually singular. Furthermore, two Gaussian measures are either mutually singular or equivalent (mutually absolutely continuous). The Cameron-Martin space plays a key role in characterizing whether or not two Gaussians are equivalent.

Definition 7.26. The *Cameron-Martin space* $\mathcal{H}_\mu$ of measure μ on a separable Banach space B is the completion of the linear subspace $\mathring{\mathcal{H}}_\mu \subset B$ defined by

$$\mathring{\mathcal{H}}_\mu = \{h \in B \,:\, \exists\, h^* \in B^* \text{ with } h = C_\mu h^*\}\,, \tag{7.27}$$

under the norm $\|h\|_\mu^2 = \langle h, h\rangle_\mu = h^*(C_\mu h^*)$. It is a Hilbert space when endowed with the scalar product $\langle h, k\rangle_\mu = h^*(C_\mu k^*) = h^*(k) = k^*(h)$.

The Cameron-Martin space is actually independent of the space B in the sense that, although we may view the measure as living on a range of separable Hilbert or Banach spaces, the Cameron-Martin space will be the same in all cases. The space characterizes exactly the directions in which a centred Gaussian measure may be shifted to obtain an equivalent Gaussian measure:

Theorem 7.27 (Cameron-Martin). *For $h \in B$, define the map $T_h\colon B \to B$ by $T_h(x) = x + h$. Then, the measure $T_h^\sharp\mu$ is absolutely continuous with respect to μ if and only if $h \in \mathcal{H}_\mu$. Furthermore, in the latter case, its Radon-Nikodym derivative is given by*

$$\frac{dT_h^\sharp\mu}{d\mu}(u) = \exp\bigl(h^*(u) - \tfrac{1}{2}\|h\|_\mu^2\bigr)$$

where $h = C_\mu h^*$.

Thus this theorem characterizes the Radon-Nikodym derivative of the measure $N(h, C_\mu)$ with respect to the measure $N(0, C_\mu)$. Below, in the Hilbert space setting, we also consider changes in the covariance operator which lead to equivalent Gaussian measures. However, before moving to the Hilbert space setting, we conclude this subsection with several useful observations concerning Gaussians on separable Banach spaces. The *topological support* of measure μ on the separable Banach space B is the set of all $u \in B$ such that any neighborhood of u has a positive measure.

Theorem 7.28. *The topological support of a centred Gaussian measure μ on B is the closure of the Cameron-Martin space in B. Furthemore the Cameron-Martin space is dense in X. Therefore all balls in B have positive μ-measure.*

Since the Cameron-Martin space of Gaussian measure μ is independent of the space on which we view the measure as living, this following useful theorem shows that the unit ball in the Cameron-Martin space is compact in any separable Banach space X for which $\mu(X) = 1$:

Theorem 7.29. *The closed unit ball in the Cameron-Martin space $\mathcal{H}_\mu$ is compactly embedded into the separable Banach space B.*

In the setting of Gaussian measures on a separable Banach space, all balls have positive probability. The Cameron-Martin norm is useful in the characterization of small-ball properties of Gaussians. Let $B^\delta(z)$ denote a ball of radius δ in B centred at a point $z \in \mathcal{H}_\mu$.

Theorem 7.30. *The ratio of small ball probabilities under Gaussian measure μ satisfy*

$$\lim_{\delta\to 0} \frac{\mu\big(B^\delta(z_1)\big)}{\mu\big(B^\delta(z_2)\big)} = \exp\left(\frac{1}{2}\|z_2\|_\mu^2 - \frac{1}{2}\|z_1\|_\mu^2\right).$$

Example 7.31. Let μ denote the Gaussian measure $N(0, K)$ on $\mathbb{R}^n$ with K positive definite. Then Theorem 7.30 is the statement that

$$\lim_{\delta\to 0} \frac{\mu\big(B^\delta(z_1)\big)}{\mu\big(B^\delta(z_2)\big)} = \exp\left(\frac{1}{2}|K^{-\frac{1}{2}}z_2|^2 - \frac{1}{2}|K^{-\frac{1}{2}}z_1|^2\right)$$

which follows directly from the fact that the Gaussian measure at point $z \in \mathbb{R}^n$ has Lebesgue density proportional to $\exp\left(-\frac{1}{2}|K^{-\frac{1}{2}}z|^2\right)$ and the fact that the Lebesgue density is a continuous function. □

7.3.2. Separable Hilbert Space Setting. In these notes our approach is primarily based on defining Gaussian measures on Hilbert space; the Banach spaces which are of full measure under the Gaussian are then determined via the Kolmogorov continuity theorem. In this subsection we develop the theory of Gaussian measures in greater detail within the Hilbert space setting. Throughout $\big(\mathcal{H}, \langle\cdot,\cdot\rangle, \|\cdot\|\big)$ denotes the separable Hilbert space on which the Gaussian is constructed. Actually, in this Hilbert space setting the covariance operator C_μ has

considerably more structure than just the boundedness implied by (7.26): it is trace-class and hence necessarily compact on $\mathcal{H}$:

Lemma 7.32. *A Gaussian measure μ on a a separable Hilbert space $\mathcal{H}$, has covariance operator $C_\mu \colon \mathcal{H} \to \mathcal{H}$ which is trace class and satisfies*

$$\int_{\mathcal{H}} \|x\|^2 \, \mu(dx) = \operatorname{Tr} C_\mu \, . \tag{7.28}$$

Conversely, for every positive trace class symmetric operator $K \colon \mathcal{H} \to \mathcal{H}$, there exists a Gaussian measure μ on $\mathcal{H}$ such that $C_\mu = K$.

Since the covariance operator $C_\mu : \mathcal{H} \to \mathcal{H}$ of a Gaussian on $\mathcal{H}$ is a compact operator it follows that if operator $C_\mu : \mathcal{H} \to \mathcal{H}$ has an inverse then it will be a densely-defined unbounded operator on $\mathcal{H}$; we call this the *precision operator*. Both the covariance and the precision operators are self-adjoint on appropriate domains, and fractional powers of them may be defined via the spectral theorem.

Theorem 7.33 (**Cameron-Martin Space on Hilbert Space).** *Let μ be a Gaussian measure on a Hilbert space $\mathcal{H}$ with strictly positive covariance operator K. Then the the Cameron-Martin space $\mathcal{H}_\mu$ consists of the image of $\mathcal{H}$ under $K^{1/2}$ and the Cameron-Martin norm is given by $\|h\|_\mu^2 = \|K^{-\frac{1}{2}} h\|^2$.*

Example 7.34. Consider two Gaussian measures μ_i on $\mathcal{H} = L^2(J), J = (0,1)$ both with precision operator $L = -\frac{d^2}{dx^2}$ where $\mathcal{D}(L) = H_0^1(J) \cap H^2(J)$. (Informally $-L$ is the Laplacian on J with homogeneous Dirichlet boundary conditions.) Let $\mathcal{C}$ denote the inverse of L on $\mathcal{H}$. Assume that $\mu_1 \sim N(m, \mathcal{C})$ and $\mu_2 \sim N(0, \mathcal{C})$. Then $\mathcal{H}_{\mu_i}$ is the image of $\mathcal{H}$ under $\mathcal{C}^{\frac{1}{2}}$ which is the space $= H_0^1(J)$. It follows that the measures are equivalent if and only if $m \in H_0^1(J)$. If this condition is satisfied then, from Theorem 7.33, the Radon-Nikodym derivative between the two measures is given by

$$\frac{d\mu_1}{d\mu_2}(x) = \exp\Big(\langle m, x \rangle_{H_0^1} - \frac{1}{2} \|m\|_{H_0^1}^2 \Big). \quad \square$$

We now turn to the Feldman-Hájek theorem in the Hilbert Space setting. Let $\{\varphi_j\}_{j=1}^\infty$ denote an orthonormal basis for $\mathcal{H}$. Then the *Hilbert-Schmidt norm* of a linear operator $L : \mathcal{H} \to \mathcal{H}$ is defined by

$$\|L\|_{\mathrm{HS}}^2 := \sum_{j=1}^\infty \|L\varphi_j\|^2.$$

The value of the norm is, in fact, independent of the choice of orthonormal basis. In the finite dimensional setting the norm is known as the *Frobenius norm*.

Theorem 7.35 (**Feldman-Hájek on Hilbert Space**). *Let μ_i with $i = 1, 2$ be two centred Gaussian measures on some fixed Hilbert space $\mathcal{H}$ with means m_i and strictly positive covariance operators $\mathcal{C}_i$. Then the following hold:*

1. *μ_1 and μ_2 are either singular or equivalent.*

2. *The measures μ_1 and μ_2 are equivalent Gaussian measures if and only if:*

   (a) *The images of $\mathcal{H}$ under $\mathcal{C}_i^{\frac{1}{2}}$ coincide for $i = 1, 2$, and we denote this common image space by E;*

   (b) $m_1 - m_2 \in E$;

   (c) $\|(\mathcal{C}_1^{-1/2}\mathcal{C}_2^{1/2})(\mathcal{C}_1^{-1/2}\mathcal{C}_2^{1/2})^* - I\|_{\mathrm{HS}} < \infty$.

Remark 7.36. The final condition may be replaced by the condition that

$$\|(\mathcal{C}_1^{1/2}\mathcal{C}_2^{-1/2})(\mathcal{C}_1^{1/2}\mathcal{C}_2^{-1/2})^* - I\|_{\mathrm{HS}} < \infty$$

and the theorem remains true; this formulation is sometimes useful. □

Example 7.37. Consider two mean-zero Gaussian measures μ_i on $\mathcal{H} = L^2(J), J = (0, 1)$ with precision operators $L_1 = -\frac{d^2}{dx^2} + I$ and $L_2 = -\frac{d^2}{dx^2}$ respectively, both with domain $H_0^1(J) \cap H^2(J)$. The operators L_1, L_2 share the same eigenfunctions

$$\phi_k(x) = \sqrt{2}\sin(k\pi x)$$

and have eigenvalues

$$\lambda_k(1) = \lambda_k(2) + 1, \quad \lambda_k(2) = k^2\pi^2,$$

respectively. Thus $\mu_1 \sim N(0, \mathcal{C}_1)$ and $\mu_2 \sim N(0, \mathcal{C}_2)$ where, in the basis of eigenfunctions, $\mathcal{C}_1$ and $\mathcal{C}_2$ are diagonal with eigenvalues

$$\frac{1}{k^2\pi^2 + 1}, \quad \frac{1}{k^2\pi^2}$$

respectively. We have, for $h_k = \langle h, \phi_k \rangle$,

$$\frac{\pi^2}{\pi^2 + 1} \leq \frac{\langle h, \mathcal{C}_1 h\rangle}{\langle h, \mathcal{C}_2 h\rangle} = \frac{\sum_{k\in\mathbb{Z}^+}(1 + k^2\pi^2)^{-1}h_k^2}{\sum_{k\in\mathbb{Z}^+}(k\pi)^{-2}h_k^2} \leq 1.$$

From this it follows that the Cameron-Martin spaces of the two measures coincide, and are equal to $H_0^1(J)$. Notice that

$$T = \mathcal{C}_1^{-\frac{1}{2}}\mathcal{C}_2\mathcal{C}_1^{-\frac{1}{2}} - I$$

is diagonalized in the same basis as the $\mathcal{C}_i$ and has eigenvalues

$$\frac{1}{k^2\pi^2}.$$

These are square summable and so by Theorem 7.35 the two measures are absolutely continuous with respect to one another. □

7.4. Wiener Processes in Infinite Dimensional Spaces. Central to the theory of stochastic PDEs is the notion of a *cylindrical Wiener process*, which can be thought of as an infinite-dimensional generalisation of a standard n-dimensional Wiener process. This leads to the notion of the A-Wiener process for certain classes of operators A. Before we proceed to the definition and construction of such Wiener processes in separable Hilbert spaces, let us recall a few basic facts about stochastic processes in general.

In general, a stochastic process u over a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and taking values in a separable Hilbert space $\mathcal{H}$ is nothing but a collection $\{u(t)\}$ of $\mathcal{H}$-valued random variables indexed by time $t \in \mathbb{R}$ (or taking values in some subset of $\mathbb{R}$). By Kolmogorov's Extension Theorem 7.4, we can also view this as a map $u\colon \Omega \to \mathcal{H}^{\mathbb{R}}$, where $\mathcal{H}^{\mathbb{R}}$ is endowed with the product sigma-algebra. A notable special case which will be of interest here is the case where the probability space is taken to be $\Omega = C([0,T], \mathcal{H})$ (or some other space of $\mathcal{H}$-valued continuous functions) endowed with some Gaussian measure $\mathbb{P}$ and where the process X is given by

$$u(t)(\omega) = \omega(t)\ , \qquad \omega \in \Omega\ .$$

In this case, u is called the canonical process on Ω.

The usual (one-dimensional) Wiener process is a real-valued centred Gaussian process $B(t)$ such that $B(0) = 0$ and $\mathbb{E}|B(t) - B(s)|^2 = |t - s|$ for any pair of times s, t. From our point of view, the Wiener process on any finite time interval I can always be realised as the canonical process for the Gaussian measure on $C(I, \mathbb{R})$ with covariance function $c(s,t) = s \wedge t = \min\{s,t\}$. Note that such a measure exists by the Kolmogorov continuity test, and Corollary 7.20 in particular.

The standard n-dimensional Wiener process $B(t)$ is simply given by n independent copies of a standard one-dimensional Wiener process $\{\beta_j\}_{j=1}^n$, so that its covariance is given by

$$\mathbb{E}\beta_i(s)\beta_j(t) = (s \wedge t)\delta_{i,j}\ .$$

In other words, if u and v are any two elements in $\mathbb{R}^n$, we have

$$\mathbb{E}\langle u, B(s)\rangle\langle B(t), v\rangle = (s \wedge t)\langle u, v\rangle\ .$$

This is the characterisation that we will now extend to an arbitrary separable Hilbert space $\mathcal{H}$. One natural way of constructing such an extension is to fix an orthonormal basis $\{e_n\}_{n\geq 1}$ of $\mathcal{H}$ and a countable collection $\{\beta_j\}_{j=1}^\infty$ of independent one-dimensional Wiener processes, and to set

$$B(t) := \sum_{n=1}^{\infty} \beta_n(t)\, e_n\ . \tag{7.29}$$

If we define

$$B^N(t) := \sum_{n=1}^{N} \beta_n(t)\, e_n$$

then clearly $\mathbb{E}\|B^N(t)\|_{\mathcal{H}}^2 = tN$ and so the series will not converge in $\mathcal{H}$ for fixed $t > 0$. However the expression (7.29) is nonetheless the right way to think of a

cylindrical Wiener process on $\mathcal{H}$; indeed for fixed $t > 0$ the truncated series for B^N will converge in a larger space containing $\mathcal{H}$. We define the following scale of Hilbert subspaces, for $r > 0$, by

$$\mathcal{X}^r = \{u \in \mathcal{H} \big| \sum_{j=1}^{\infty} j^{2r} |\langle u, \phi_j \rangle|^2 < \infty\}$$

and then extend to superspaces $r < 0$ by duality. We use $\|\cdot\|_r$ to denote the norm induced by the inner-product

$$\langle u, v \rangle_r = \sum_{j=1}^{\infty} j^{2r} u_j v_j$$

for $u_j = \langle u, \phi_j \rangle$ and $v_j = \langle v, \phi_j \rangle$. A simple argument, similar to that used to prove Theorem 2.10, shows that $\{B^N(t)\}$ is, for fixed $t > 0$, Cauchy in $\mathcal{X}^r$ for any $r < -\frac{1}{2}$. In fact it is possible to construct a stochastic process as the limit of the truncated series, living on the space $C([0,\infty), \mathcal{X}^r)$ for any $r < -\frac{1}{2}$, by the Kolmogorov Continuity Theorem 7.19 in the setting where $D = [0, T]$ and $X = \mathcal{X}^r$. We give details in the more general setting that follows.

Building on the preceding we now discuss construction of a $\mathcal{C}$-Wiener process W, using the finite dimensional case described in Remark 5.19 to guide us. Here $\mathcal{C} : \mathcal{H} \to \mathcal{H}$ is assumed to be trace-class with eigenvalues γ_j^2. Consider the cylindrical Wiener process given by

$$B(t) = \sum_{j=1}^{\infty} \beta_j e_j,$$

where $\{\beta_j\}_{j=1}^{\infty}$ is an i.i.d. family of unit Brownian motions on $\mathbb{R}$ with $\beta_j \in C([0,\infty); \mathbb{R})$. We note that

$$\mathbb{E}|\beta_j(t) - \beta_j(s)|^2 = |t - s|. \tag{7.30}$$

Since $\sqrt{\mathcal{C}} e_j = \gamma_j e_j$, the $\mathcal{C}$-Wiener process $W = \sqrt{\mathcal{C}} B$ is then

$$W(t) = \sum_{j=1}^{\infty} \gamma_j \beta_j(t) e_j. \tag{7.31}$$

The following formal calculation gives insight into the properties of W:

$$\begin{aligned}
\mathbb{E}W(t) \otimes W(s) &= \mathbb{E}\Big(\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \gamma_j\gamma_k\beta_j(t)\beta_k(s)e_j \otimes e_k\Big) \\
&= \Big(\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \gamma_j\gamma_k\mathbb{E}\big(\beta_j(t)\beta_k(t)\big)e_j \otimes e_k\Big) \\
&= \Big(\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \gamma_j\gamma_k\delta_{jk}(t\wedge s)e_j \otimes e_k\Big) \\
&= \sum_{j=1}^{\infty}\Big(\gamma_j^2\phi_j \otimes \phi_j\Big)t\wedge s \\
&= \mathcal{C}\,(t\wedge s).
\end{aligned}$$

Thus the process has the covariance structure of Brownian motion in time, and covariance operator $\mathcal{C}$ in space. Hence the name $\mathcal{C}$-Wiener process.

Assume now that the sequence $\gamma = \{\gamma_j\}_{j=1}^{\infty}$ is such that $\sum_{j=1}^{\infty} j^{2r}\gamma_j^2 = M < \infty$ for some $r \in \mathbb{R}$. For fixed t it is then possible to construct a stochastic process as the limit of the truncated series

$$W^N(t) = \sum_{j=1}^{N} \gamma_j\beta_j(t)e_j,$$

by means of a Cauchy sequence argument in $L^2_{\mathbb{P}}(\Omega;\mathcal{X}^r)$. Similarly $W(t) - W(s)$ may be defined for any t, s. We may then also discuss the regularity of this process in time. Together equations (7.30),(7.31) give $\mathbb{E}\|W(t) - W(s)\|_r^2 = M^2|t-s|$. It follows that $\mathbb{E}\|W(t) - W(s)\|_r \le M|t-s|^{\frac{1}{2}}$. Furthermore, since $W(t) - W(s)$ is Gaussian, we have by (7.26) that $\mathbb{E}\|W(t) - W(s)\|_r^{2q} \le K_q|t-s|^q$. Applying the Kolmogorov continuity test of Theorem 7.19 then demonstrates that the process given by (7.31) may be viewed as an element of the space $C^{0,\alpha}([0,T];\mathcal{X}^r)$ for any $\alpha < \frac{1}{2}$. Similar arguments may be used to study the cylindrical Wiener process, showing that it lives in $C^{0,\alpha}([0,T];\mathcal{X}^r)$ for $\alpha < \frac{1}{2}$ and $r < -\frac{1}{2}$.

7.5. Bibliographical Notes.

- Subsection 7.1 introduces various Banach and Hilbert spaces, as well as the notion of separablity; see [101]. In the context of PDEs, see [33] and [88], for all of the function spaces defined in subsections 7.1.1–7.1.3; Sobolev spaces are developed in detail in [2]. The nonseparability of the Hölder spaces $C^{0,\beta}$ and the separability of $C_0^{0,\beta}$ is discussed in [40]. For asymptotics of the eigenvalues of the Laplacian operator see [92, Chapter 11]. For discussion of the more general spaces of E-valued functions over a measure space $(\mathcal{M},\nu)$ we refer the reader to [101]. Subsection 7.1.5 concerns Sobolev embedding theorems, building rather explicitly on the case of periodic functions. The corresponding embedding results in domains with more general boundary

conditions or even on more general manifolds or unbounded domains, we refer to the comprehensive series of monographs [96, 97, 98]. The interpolation inequality of (7.8) and Lemma 7.1 may be found in [88]; see also Proposition 6.10 and Corollary 6.11 of [40]. The proof of Theorem 7.3 closely follows that given in [40, Theorem 6.16], and is a slight generalization to the Hilbert scale setting used here.

- Subsection 7.2 briefly introduces the theory of probability measures on infinite dimensional spaces. We refer to the extensive treatise by Bogachev [15], and to the much shorter but more readily accessible book by Billingsley [12], for more details. The subject of independent sequences of random variables, as overviewed in subsection 7.2.1 in the i.i.d. case, is discussed in detail in [27, section 1.5.1]. The Kolmogorov Extension Theorem 7.4 is proved in numerous texts in the setting where $X = \mathbb{R}$ [79]; since any Polish space is isomorphic to $\mathbb{R}$ it may be stated as it is here. Proofs of Lemmas 7.5 and 7.9 may be found in [40], where they appear as Proposition 3.6 and Propostion 3.9 respectively. For (7.17) see [28, Chapter 2]. In subsection 7.2.2 we introduce the Bochner integral; see [13, 48] for further details. Lemma 7.6 and the resulting Corollary 7.7 are stated and proved in [14]. The topic of metrics on probability measures, introduced in subsection 7.2.4 is overviewed in [38], where detailed references to the literature on the subject may also be found; the second inequality in Lemma 7.18 is often termed the *Pinsker inequality* and can be found in [22]. Note that the choice of normalization constants in the definitions of the total variation and Hellinger metrics differs in the literature. For a more detailed account of material on weak convergence of probability measures we refer, for example, to [12, 15, 99]. A proof of the Kolmogorov continuity test as stated in Theorem 7.19 can be found in [86, p. 26] for simple case of D an interval and X a separable Banach space; the generalization given here may be found in a forthcoming uptodate version of [40].

- The subject of Gaussian measures, as introduced in subsection 7.3, is comprehensively studied in [14] in the setting of locally convex topological spaces, including separable Banach spaces as a special case. See also [67] which is concerned with Gaussian random functions. The Fernique Theorem 7.25 is proved in [35] and the reader is directed to [40] for a very clear exposition. In Theorem 7.25 it is possible to take for α any value smaller than $1/(2\|C_\mu\|)$ and this value is sharp: see [66, Thm 4.1]. See [14, 67] for more details on the Cameron-Martin space, and proof of Theorem 7.27. Theorem 7.28 follows from Theorem 3.6.1 and Corollary 3.5.8 of [14]: Theorem 3.6.1 shows that the topological support is the closure of the Cameron-Martin space in B and Corollary 3.5.8 shows that the Cameron-Martin space is dense in B. The *reproducing kernel Hilbert space* for μ (or just *reproducing kernel* for short) appears widely in the literature and is isomorphic to the Cameron-Martin space in a natural way. There is considerable confusion between the two as a result. We retain in these notes the terminology from [14], but the reader

should keep in mind that there are authors who use a slightly different terminology. Theorem 7.30 as stated is a consequence of Proposition 3 in section 18 in [67]. Turning now to the Hilbert space setting we note that Lemma 7.32 is proved as Proposition 3.15, and Theorem 7.33 appears as Exercise 3.34, in [40]. See [14, 28, 52] for alternative developments of the Cameron-Martin and Feldman-Hájek theorems. The original statement of the Feldman-Hájek Theorem 7.35 can be found in [34, 45]. Our statement of Theorem 7.35 mirrors Theorem 2.23 of [28] and Remark 7.36 is Lemma 6.3.1(ii) of [14]. Note that we have not stated a result analogous to Theorem 7.27 in the case where of two equivalent Gaussian measures with differing covariances. Such a result can be stated, but is technically complicated in general because the ratio of normalizations constants of approximating finite dimensional measures can blow-up as the limiting infinite dimensional Radon-Nikodym derivative is attained; see Corollary 6.4.11 in [14].

- Subsection 7.4 contains a discussion of cylindrical and $\mathcal{C}$-Wiener processes. The development is given in more detail in section 3.4 of [40], and in section 4.3 of [28].

Acknowledgements The authors are indebted to Martin Hairer for help in the development of these notes, and in particular for considerable help in structuring the Appendix, for the proof of Theorem 7.3 (which is a slight generalization to Hilbert scales of Theorem 6.16 in [40]) and for the proof of Corollary 7.22 (which is a generalization of Corollary 3.22 in [40] to the non-Gaussian setting and to Hölder, rather than Lipschitz, functions $\{\psi_k\}$). They are also grateful to Joris Bierkens, Patrick Conrad, Matthew Dunlop, Shiwei Lan, Yulong Lu, Daniel Sanz-Alonso, Claudia Schillings and Aretha Teckentrup for careful proof-reading of the notes and related comments. AMS is grateful for various hosts who gave him the opportunity to teach this material in short course form at TIFR-Bangalore (Amit Apte), Göttingen (Axel Munk), PKU-Beijing (Teijun Li), ETH-Zurich (Christoph Schwab) and Cambridge CCA (Arieh Iserles), a process which led to refinements of the material; the authors are also grateful to the students on those courses, who provided useful feedback. The authors would also like to thank Sergios Agapiou and Yuan-Xiang Zhang for help in the preparation of these lecture notes, including type-setting, proof-reading, providing the proof of Lemma 2.7 and delivering problems classes related to the short courses. AMS is also pleased to acknowledge the financial support of EPSRC, ERC and ONR over the last decade whilst the research that underpins this work has been developed.

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