Local versus global strategies in multiparameter estimation

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We consider the problem of estimating multiple phases using a multimode interferometer. In this setting we show that while global strategies that estimate all the phases simultaneously can lead to high precision gains, the same enhancements can be obtained with local strategies in which each phase is estimated individually. A key resource for the enhancement is shown to be a large particle-number variance in the probe state, and for states where the total particle number is not fixed, this can be obtained for mode-separable states, and the phases can be read out with local measurements. This has important practical implications because local strategies are generally preferred to global ones for their robustness to local estimation failure, flexibility in the distribution of resources, and comparatively easier state preparation. We obtain our results by analyzing two different schemes: the first uses a set of interferometers, which can be used as a model for a network of quantum sensors, and the second looks at measuring a number of phases relative to a reference, which is concerned primarily with quantum imaging.

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I. INTRODUCTION

Quantum metrology has the potential to revolutionize a diverse range of fields from biological imaging [1] to navigation [2,3] and already plays a crucial role in enhancing the precision of gravitational-wave detectors [4]. In many practical applications it is necessary to estimate multiple parameters [5–8], and hence it is important to understand the potential enhancements that quantum metrology can provide in this setting [9–11]. It has already been shown that in a multimode (multipath) interferometer, measuring all phases simultaneously with a mode-entangled state can enhance the precision [9]. However, in stark contrast to this, in other applications of quantum metrology multimode entanglement can be detrimental, such as when measuring coupled phases [12] or when loss is considered [13]. Furthermore, from a practical point of view, large multimode-entangled states are notoriously difficult to produce and are fragile to experimental imperfections and photon losses.

In this paper we compare local and global strategies for estimating multiple parameters in optical interferometry. We call an estimation procedure a *local estimation strategy* if (i) the input probe state is *separable* with respect to different optical modes and (ii) the measurement of the state can be implemented with only local operations. In a local strategy, each parameter can be estimated individually. In contrast to this, we define a *global estimation strategy* to simply be any estimation procedure which is *not* local; in a global strategy the parameters are estimated simultaneously.

Previous work has shown that a global estimation strategy that estimates all phases simultaneously can give high precision gains over standard quantum metrology protocols [9]. However, in [9], only states with a fixed number of photons were considered. In this paper we relax this constraint and allow for both fixed-number states and indefinite-number states. In this more general setting we demonstrate that the same precision enhancements exhibited by the global strategies can be obtained with mode-separable states and local measurements alone. Local strategies offer a number of advantages over their global counterparts, including robustness to local estimation failure, more flexibility in the distribution of resources, and more realistic methods of state preparation [14–18], measurement, and control.

Our results are obtained by analyzing two different multiparameter estimation schemes which cover a variety of practical applications. First, we consider a collection of (possibly entangled) interferometers which can be used as a model for a network of quantum sensors. This scheme is also relevant to applications such as gravitational-wave astronomy in which multiple parameters of a gravitational wave will be measured simultaneously [5]. Second, we analyze a model for quantum-enhanced imaging [6–8], introduced by Humphreys *et al.* [9], in which many phases are measured relative to a single global reference. In both of these schemes we provide mode-separable states that allow individual phases to be measured with a precision beyond the simultaneous estimation strategy. While previous work has shown that large enhancements are possible in multiparameter estimation [9,19,20], the exact *origin* of this enhancement was not known. Here we shed some light on this by presenting phase precision bounds that explicitly show that, in multimode optical systems with commuting phase generators, the crucial resource for enhanced metrology is a large number variance *within* each mode, which can be obtained without multimode entanglement.

II. MULTIPARAMETER ESTIMATION

Consider the problem of estimating the general vector $\boldsymbol{\phi}$ consisting of d parameters ϕ_i, $i = 1, \ldots, d$. The precision

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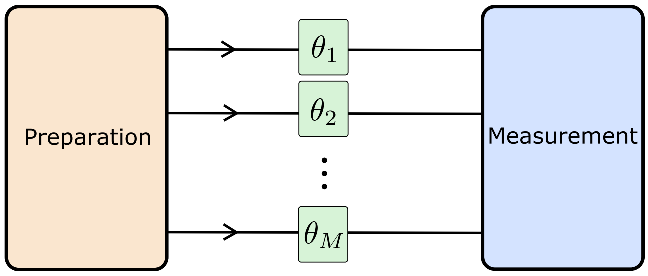

FIG. 1. The general problem under consideration consists of M optical modes with independent linear phase shifts θ_i, $i = 1,\ldots,M$. In optical interferometry the parameters to be estimated, ϕ_i, are given by some function of the M-dimensional vector $\boldsymbol{\theta}$, as described in the main text. For example, ϕ_i could be phase differences between arms.

bound on estimating each parameter ϕ_i is given by the Cramér-Rao bound (CRB) as $\delta\phi_i^2 \geqslant \mu^{-1}(\mathcal{F}^{-1})_{ii}$, where μ is the number of repetitions of the experiment and $\mathcal{F}$ is the quantum Fisher information matrix (QFIM) [21,22]. For a pure state $|\psi_{\boldsymbol{\phi}}\rangle$ which depends on $\boldsymbol{\phi}$ the QFIM is defined by

$$\mathcal{F}_{lm} = \tfrac{1}{2}\langle\psi_{\boldsymbol{\phi}}|(L_l L_m + L_m L_l)|\psi_{\boldsymbol{\phi}}\rangle, \tag{1}$$

where L_l is the symmetric logarithmic derivative, given by

$$L_l = 2(|\partial_l\psi_{\boldsymbol{\phi}}\rangle\langle\psi_{\boldsymbol{\phi}}| + |\psi_{\boldsymbol{\phi}}\rangle\langle\partial_l\psi_{\boldsymbol{\phi}}|), \tag{2}$$

where $|\partial_l\psi_{\boldsymbol{\phi}}\rangle \equiv \frac{\partial}{\partial\phi_l}|\psi_{\boldsymbol{\phi}}\rangle$ [9,23]. Consider the case when $|\psi_{\boldsymbol{\phi}}\rangle = U(\boldsymbol{\phi})|\psi\rangle$ for some $\boldsymbol{\phi}$-independent initial probe state $|\psi\rangle$ and $U(\boldsymbol{\phi}) = \exp(i\sum_{i=1}^{d}\phi_i\hat{O}_i)$, where $\hat{O}_i$ are Hermitian and mutually commuting operators, i.e., $[\hat{O}_i,\hat{O}_j] = 0\ \forall\, i,j$. Then it can be shown that $\mathcal{F}_{lm} = 4\mathrm{Cov}(\hat{O}_l,\hat{O}_m)$, where

$$\mathrm{Cov}(\hat{O}_l,\hat{O}_m) = \langle\hat{O}_l\hat{O}_m\rangle - \langle\hat{O}_l\rangle\langle\hat{O}_m\rangle \tag{3}$$

is the covariance between the two operators $\hat{O}_l$ and $\hat{O}_m$ and the expectation values are taken with respect to the input state $|\psi\rangle$ (this simple QFIM formula will be applicable throughout). The variance, given by $l = m$, will be denoted $\mathrm{Var}(\hat{O}_l) = \mathrm{Cov}(\hat{O}_l,\hat{O}_l)$.

The general scheme for optical multiparameter estimation considered herein is shown in Fig. 1. There are M optical modes with independent linear phase shifts. The unknown phase shifts are imprinted with the unitary operator $U(\boldsymbol{\theta}) = \exp(i\sum_{j=1}^{M}\theta_j\hat{n}_j)$, and the problem is to estimate some number $d \leqslant M$ of independent parameters ϕ_i, which are functions of θ_j, as will become clear when we introduce specific examples below.

III. PARALLEL INTERFEROMETERS

The first scheme we consider is a set of parallel interferometers in which the aim is to measure the phase difference between the two arms in each interferometer, as shown in Fig. 2. One interesting future application is in gravitational-wave astronomy, which will aim to simultaneously measure a number of parameters associated with gravitational waves, such as polarization and direction of origin, and doing so will require multiple interferometers [5].

This parallel interferometer model is a special case of the scheme in Fig. 1 for an even number of modes $M = 2d$, where specifically we take the ith interferometer to consist of modes $2i-1$ and $2i$ ($i = 1,\ldots,d$). The aim is to estimate the d parameters $\phi_i \equiv \phi_{i^-}$, where $\phi_{i^\pm} = \theta_{2i-1} \pm \theta_{2i}$. The phase-shift operator $U(\boldsymbol{\theta})$ can be reparameterized in terms of $\boldsymbol{\phi} = (\phi_{1^-},\ldots,\phi_{d^-},\phi_{1^+},\ldots,\phi_{d^+})$, giving $U(\boldsymbol{\phi}) = \exp[i\sum_{i=1}^{d}(\phi_{i^-}\hat{O}_{i^-} + \phi_{i^+}\hat{O}_{i^+})]$, where the generating operators are $\hat{O}_{i^\pm} = (\hat{n}_{2i-1} \pm \hat{n}_{2i})/2$. Hence, although the estimation is only of d parameters, the relevant QFIM is for the $2d$-dimensional $\boldsymbol{\phi}$ and has the form $\mathcal{F}_{i^\pm j^\pm} = 4\mathrm{Cov}(\hat{O}_{i^\pm},\hat{O}_{j^\pm})$, where the two $\pm$ signs may be chosen independently.

This estimation problem has a symmetry between the interferometers, and furthermore, there is a symmetry between the arms in each interferometer as neither plays a special role. We therefore consider states that are symmetric with respect to swapping interferometer labeling and symmetric with respect to swapping the modes in each interferometer. Using the shorthand $C_{i,j} \equiv \mathrm{Cov}(\hat{n}_i,\hat{n}_j)$ and $V_i \equiv \mathrm{Var}(\hat{n}_i)$, these symmetry assumptions imply that the variances of all the modes are equal, i.e., $V_i = V_j$ for all i and j, and this value may be denoted V. Furthermore, they imply that the covariances between any two modes from the same interferometer are equal, i.e., $C_{2i-1,2i} = C_{2j-1,2j}$ for all i and j, and this value may be denoted C_{Intra}. Given these natural symmetries, it can be shown (see Appendix A) that the precision bound for estimating each parameter ϕ_i is given by

$$\delta\phi_i^2 \geqslant \frac{1}{2(V - C_{\mathrm{Intra}})}. \tag{4}$$

In the literature a single phase-precision parameter $\delta\Phi = \sum_{i=1}^{d}\delta\phi_i$ is sometimes considered (e.g., see [9,19]), which here may be trivially calculated to be $\delta\Phi = d\delta\phi_i$, but throughout this paper we will consider the precision bounds of individual phases $\delta\phi_i$. From Eq. (4) it is clear that the only parameters which directly affect the phase precision are the state's photon-number variance and the correlations between the two modes in an individual interferometer. Hence, entanglement between interferometers provides no direct improvement in the phase precision. It is therefore not necessary to entangle quantum optical sensors in networks, nor entangle multiple gravitational-wave interferometers, which in both cases would be challenging.

It is instructive to rewrite Eq. (4) in terms of the Mandel $\mathcal{Q}$ parameter and the two-mode correlation parameter $\mathcal{J}_{ij}$, which are defined by $\mathcal{Q}_i = (V_i - \bar{n}_i)/\bar{n}_i$ and $\mathcal{J}_{ij} = C_{i,j}/\sqrt{V_iV_j}$, respectively. We denote the Mandel $\mathcal{Q}$ parameter for any mode by $\mathcal{Q}$ (all modes have the same $\mathcal{Q}$) and the two-mode correlation

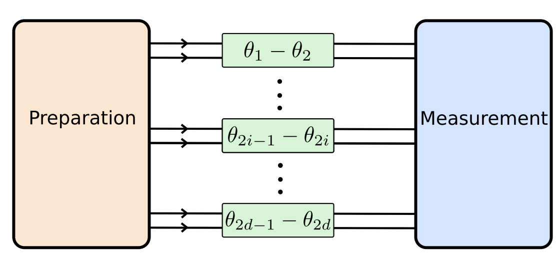

FIG. 2. A network of quantum sensors may be modeled as d parallel interferometers. The parameters to be measured are the phase differences in each interferometer.

between the two modes in any of the interferometers by $\mathcal{J}$, where $\mathcal{J} = C_{\text{Intra}}/V$. Then for all i the phase precision is given by

$$\delta\phi_i^2 \geqslant \frac{1}{2\bar{n}(1+\mathcal{Q})(1-\mathcal{J})}, \tag{5}$$

where $\bar{n}$ is the average number of particles in any *single mode* (i.e., $\bar{n} = \bar{n}_i = \langle \hat{n}_i \rangle$ for any i). For single-parameter estimation this was shown in Ref. [24]. This may also be rewritten in terms of the average *total* photon number $\bar{N}$ using $\bar{N} = 2d\bar{n}$. The two-mode correlation term is bounded by $-1 \leqslant \mathcal{J} \leqslant 1$ and hence provides at most a factor of $1/\sqrt{2}$ improvement in the phase precision.

We now compare local and global phase estimation strategies with examples of both multimode-entangled and mode-separable states. If we consider each interferometer individually, the standard quantum-enhanced precision is the well-known Heisenberg scaling of $\delta\phi_i^2 \geqslant 1/(2\bar{n})^2 = d^2/\bar{N}^2$, where this precision for each individual phase has now been written in terms of the total photon number $\bar{N}$ used to measure all of the phases. Consider a generalized entangled coherent state (GECS) given by

$$|\Psi_{\text{GECS}}\rangle = \mathcal{N}_g \sum_{a\epsilon\mathcal{M}} \hat{D}_a(\alpha_g)|\mathbf{0}\rangle, \tag{6}$$

where $\hat{D}_a(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$ is the displacement operator acting on mode a, $\mathcal{M}$ is the set of $M = 2d$ modes, $|\mathbf{0}\rangle$ is the multimode vacuum state, and $\mathcal{N}_g$ is a normalization factor required due to the nonzero overlap of a coherent state with the vacuum. We find

$$\delta\phi_{\text{GECS}}^2 \geqslant \frac{d}{\bar{N}_g(|\alpha_g|^2+1)} \approx \frac{d}{\bar{N}_g(\bar{N}_g+1)}, \tag{7}$$

where $\bar{N}_g = |\alpha_g|^2/[1+(2d-1)e^{-|\alpha_g|^2}]$ is the total average number in the GECS and the approximation uses $\bar{N}_g \approx |\alpha_g|^2$, which holds for $|\alpha_g| \gg 1$. This is a scaling of $O(d/\bar{N}_g^2)$, which is an $O(d)$ improvement over the expected quantum enhancement. This suggests that, contrary to the evidence of Eq. (5), a global strategy does provide an improvement over the local estimation strategy. However, a local strategy can do just as well or even better, as we will now see.

Consider a multimode but *mode-separable* unbalanced cat state (UCS), given by $|\Psi_{\text{UCS}}\rangle = \mathcal{N}_c(|\alpha_c\rangle + \nu|0\rangle)^{\otimes 2d}$, where ν is a real parameter and again $\mathcal{N}_c$ is the normalization. We find that

$$\delta\phi_{\text{UCS}}^2 \geqslant \frac{d}{\bar{N}_c\left(|\alpha_c|^2+1-\frac{\bar{N}_c}{2d}\right)} \approx \frac{d}{\bar{N}_c\left(\frac{\nu^2}{2d}\bar{N}_c+1\right)}, \tag{8}$$

where $\bar{N}_c = 2d|\alpha_c|^2/(\nu^2+1+2\nu e^{-\frac{1}{2}|\alpha_c|^2})$ is the total average photon number and the approximation is for $|\alpha_c| \gg 1$. For $\nu = 1$ (an ordinary cat state) we find that the precision bound scales as $O(d^2/\bar{N}_c^2)$, as perhaps expected of the local strategy. However, if we instead take ν^2 to scale with d, then it has the form $O(d/\bar{N}_c^2)$. More explicitly, setting photon numbers equal, $\bar{N}_c = \bar{N}_g$, then (for $|\alpha_c|,|\alpha_g| \gg 1$) we have $\delta\phi_{\text{UCS}}^2 < \delta\phi_{\text{GECS}}^2$ when $\nu^2 > 2d$ (this analysis also holds without taking the large-photon-number limit). This shows that for large enough values of ν the UCS can attain a better precision than the GECS. Moreover, with local estimation the CRB can be saturated by mixing the two modes of each interferometer on a beam splitter and photon-number counting on the output modes, as this is the optimal measurement for standard two-mode interferometry with *any* path-symmetric input state [25]. With a global estimation strategy, in general a much more experimentally challenging many-mode measurement is necessary, further reinforcing the advantages of the local strategy. Before further discussion on the conclusions of the parallel interferometer scheme, we will now show that similar conclusions can be drawn for a "quantum imaging" problem.

IV. MULTIMODE QUANTUM-ENHANCED IMAGING

Consider measuring d phase shifts relative to a single reference mode, as described by Humphreys *et al.* [9], which is relevant for a range of applications, including quantum-enhanced imaging [6–8]. This is again a special case of Fig. 1 for $M = d+1$ modes, where the aim is to estimate the d-dimensional vector parameter $\boldsymbol{\phi}$, where $\phi_i = \theta_i - \theta_{d+1}$. For simplicity (and following Humphreys *et al.* [9]) we set $\theta_{d+1} = 0$, in which case the generator of ϕ_i is simply $\hat{n}_i$, and therefore $\mathcal{F}_{ij} = 4\text{Cov}(\hat{n}_i,\hat{n}_j)$ (see Appendix C for a discussion of the role of reference beams in this scheme). As in the case of the parallel interferometers, there is a clear symmetry to this problem, and in this case it is natural to assume symmetry between the d probe modes (but not necessarily between the reference mode and the others). This implies that $V_i = V_j$ for all i and j, which is denoted V, and that $C_{i,j} = C_{m,n}$ for all $i \neq j$ and $m \neq n$, which we denote by C. Using this assumption, it is shown in Appendix B that the precision bound for estimating each parameter ϕ_i is given by

$$\delta\phi_i^2 \geqslant \frac{V+(d-2)C}{4(V-C)(V+(d-1)C)}. \tag{9}$$

Again, the QFIM can be expressed in terms of the Mandel $\mathcal{Q}$ parameter of any mode and the two-mode correlation $\mathcal{J} = C/V$, which gives a phase precision of

$$\delta\phi_i^2 \geqslant \frac{f(d,\mathcal{J})}{4\bar{n}(1+\mathcal{Q})(1-\mathcal{J})}, \tag{10}$$

where $\bar{n}$ is the average photon number in a single mode and the function $f(d,\mathcal{J})$ is given by

$$f(d,\mathcal{J}) = \frac{1+(d-2)\mathcal{J}}{1+(d-1)\mathcal{J}}. \tag{11}$$

When there are many interferometers ($d \gg 1$) then $f(d,\mathcal{J}) \approx 1$, and hence the phase precision has a form very similar to that for the parallel-interferometers case given in Eq. (5). As always, $|\mathcal{J}| \leqslant 1$, and hence as before multimode correlations can provide at most a small constant factor improvement.

In order to explore this further and to understand the relationship to previous work [9,19], examples are now considered. Humphreys *et al.* [9] introduced the generalized NOON state (GNS), given by

$$|\Psi_{\text{GNS}}\rangle = \frac{1}{\sqrt{d+\gamma^2}}(|N,0,\ldots,0,0\rangle + |0,N,\ldots,0,0\rangle + \cdots + |0,0,\ldots,N,0\rangle + \gamma|0,0,\ldots,0,N\rangle),$$

where the real parameter γ is a weighting on the reference mode to be optimized. For each phase, the precision bound is $\delta\phi^2_{\rm GNS} \geqslant (d+\gamma^2)(1+\gamma^2)/4\gamma^2 N^2$, which is optimized for $\gamma = d^{1/4}$ but for which the simpler choice of $\gamma = 1$ provides the same scaling enhancement. The optimal case gives $\delta\phi^2_{\rm GNS} \geqslant (1+\sqrt{d})^2/4N^2$. This is an $O(d)$ enhancement over the expected quantum enhancement [9] (separate NOON states give a precision $\delta\phi^2_{\rm NOON} = d^2/4N^2$), which again suggests that a global strategy does provide an improvement over the local estimation strategy. However, this conclusion holds only if we restrict ourselves to states with a fixed *total* number of photons. We now relax this constraint and consider states with a fixed *average* number of photons (fixed total-number states are a subset of this). In this larger class of states we will demonstrate that the same precision enhancements exhibited by the GNS can be obtained with mode-separable states and local measurements alone.

A mode-separable state that can improve over the GNS is a collection of single-mode unbalanced "NO" (UNO) states, given by

$$|\psi_{\rm UNO}\rangle = \mathcal{N}_{\rm UNO}(|N\rangle + \nu|0\rangle)^{\otimes M}. \tag{12}$$

Choosing $\nu = 1$ returns the same scaling as using separate NOON states. However, if we take $\nu = \sqrt{d+\gamma^2-1}$, or simply $\nu \propto \sqrt{d}$, then we obtain exactly the same precision scaling enhancement as the global estimation strategy with the GNS. Furthermore, the multimode correlations in the GNS die off with increasing d, as $\mathcal{J} = -1/(d+\gamma^2-1)$.

In this quantum imaging setting the optimal measurement required to saturate the CRB for the global estimation strategy is again, in general, some many-mode measurement. The specific optimal measurement depends on the particular input state [26]; for example, see Ref. [9] for the details in the case of a GNS input. However, with the local strategy, the optimal measurement is simply a collection of one-mode measurements on each probe mode, performed after each probe mode has been mixed in some way with the phase reference; the precise procedure again depends on the particular probe state employed. This further highlights the advantages of local estimation strategies.

It is now clear that, for quantum-enhanced optical multiparameter estimation, the essential property required of a pure probe state is large correlations *within* each mode, and this can be obtained without multimode entanglement. The cause of the apparent scaling improvement for the global strategy is that the GNS exhibits the scaling $\mathcal{Q} = O(d\bar{n}) = O(\bar{N})$ rather than $\mathcal{Q} = O(\bar{n})$; that is, the uncertainty in the photon number of each mode grows with the number of modes d for fixed $\bar{n}$. However, the $\mathcal{Q}$ function is simply a local property of each mode, and the desired scaling can also be obtained by a judicious choice of a single-mode state. Generally, for any path-symmetric pure state of M modes $|\Psi\rangle$, consider a pure single-mode state $|\psi(\Psi)\rangle = \sum_{n=0}^{\infty} |\langle n|\Psi\rangle||n\rangle$, with $\langle n|\Psi\rangle$ taken with respect to any mode. Then, by construction, $|\Psi\rangle$ and the M-mode-separable state $|\psi\rangle^{\otimes M}$ contain the same average number of photons and for any mode $\mathcal{Q}(|\Psi\rangle) = \mathcal{Q}(|\psi\rangle^{\otimes M})$. Hence the phase precision as a function of $\bar{n}$ (in either scenario considered herein) for a general multimode state exhibits at most a small constant factor (at best $\sqrt{2}$) improvement over the separable analog (although note that the separable analog can be modified to beat the multimode state, for example, by tuning ν in the UNO or UCS). This argument applies to any global estimation strategy and hence to the extension of Ref. [9] by Liu *et al.* [19] to quantum imaging with a GECS.

V. DISCUSSION

We have shown that in optical multiparameter estimation there is no fundamental improvement in using a global strategy to estimate all of the parameters simultaneously. Local strategies are just as effective, and this has important practical implications because local estimation strategies, which use separable states and local measurements, have a number of advantages. For example, local strategies have greater flexibility in the distribution of resources and are more robust to local estimation failure. Furthermore, single-mode states with a large number variance can be made in experiments [14–17], and realistic schemes have been proposed to produce separable states that improve over the shot-noise limit by more than a factor of 4 [18]. By comparison, multimode-entangled states with large photon numbers are notoriously difficult to make: the largest two-mode optical NOON state that has been made experimentally contains only five photons [27].

The quantum Fisher information (QFI) alone is not always a reliable method for deriving precision scaling bounds that are truly attainable in practice, and a proper consideration of the prior information and the required number of experimental repetitions is needed. Indeed, states with arbitrarily large QFI for a fixed number of photons have been reported in the literature [28], and that effect is relevant here. A further discussion of this is given in Appendix C. However, the precision *scaling* with photon number is often not of direct relevance in an experiment, and a more relevant measure is the absolute precision that can be obtained given an allowed total photon number through the interferometer [4,29,30]. As already noted, there are a range of practical states which improve on the absolute precision of NOON states [14,18], and these are candidates for the multiparameter paradigm using the local estimation strategy considered herein.

To conclude, we have considered the problem of multiparameter estimation in optical interferometry, and we have shown that local estimation strategies where each phase is estimated individually can surpass the precision attained in a global scheme where all the phases are estimated simultaneously. These results hold for quantum sensing, in which a number of phases are measured relative to a reference, and also for a set of parallel interferometers, which can serve as a model for a network of sensors. Local strategies offer many practical advantages over their global counterparts, including flexibility, practicality, and control. Therefore, in the optical systems considered here, local strategies should be considered strong candidates for the practical implementation of multiparameter estimation.

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APPENDIX A: PHASE PRECISION DERIVATIONS, PARALLEL INTERFEROMETERS

We begin with the quantum Fisher information matrix (QFIM) equation from the main text:

$$\mathcal{F}_{i^\pm j^\pm} = 4\text{Cov}(\hat{O}_{i^\pm},\hat{O}_{j^\pm}),$$

where the two $\pm$ signs may be chosen independently and where $\hat{O}_{i^\pm} = (\hat{n}_{2i-1} \pm \hat{n}_{2i})/2$. Now, assume that the input state is symmetric both between and inside interferometers. As interferometer i consists of modes $2i-1$ and $2i$, this implies that

$$\forall\, i \neq j \quad C_{\text{Intra}} \equiv C_{2i-1,2i} = C_{2j-1,2j},$$
$$\forall\, i,j \quad V \equiv V_i = V_j,$$

using the short-hand introduced in the main text that $C_{i,j} \equiv \text{Cov}(\hat{n}_i,\hat{n}_j)$ and $V_i \equiv \text{Var}(\hat{n}_i)$. These equalities state that the covariances between any two modes from the same interferometer are equal and the variances of all the modes are equal. A further implication of the symmetry assumptions is that

$$C_{2i-1,j} = C_{2m-1,n}$$

whenever $j \neq 2i-1,2i$ and $n \neq 2m-1,2m$ (as a covariance is symmetric, this covers all remaining cases), and this value may be denoted C_{Inter} as it represents any correlations between interferometers. Note that total path symmetry can be enforced by letting $C_{\text{Intra}} = C_{\text{Inter}}$, but there is no need to make this assumption (and it is not automatically sensible given the symmetry of the problem).

The QFI is now simplified under these assumptions. The elements of the QFIM when the two independent $\pm$ symbols take the same sign (i.e., both are positive or negative) can be expanded to

$$\mathcal{F}_{i^\pm j^\pm} = C_{2i-1,2j-1} + C_{2i,2j} \pm C_{2i,2j-1} \pm C_{2i-1,2j},$$

where the $\pm$ symbols in this equation are now *not* independent (i.e., they all take the upper or lower symbol). Similarly, the elements of the QFI matrix when the two independent $\pm$ symbols take opposite signs (i.e., one is positive and the other is negative) can be expanded to

$$\mathcal{F}_{i^\pm j^\mp} = C_{2i-1,2j-1} - C_{2i,2j} \pm C_{2i,2j-1} \mp C_{2i-1,2j},$$

where again the $\pm$ symbols in this equation are now *not* independent. Using these equations and the assumptions given above, it is easily confirmed that

$$\mathcal{F}_{i^\pm j^\mp} = 0, \quad \mathcal{F}_{i^- j^-} = 0, \quad i \neq j.$$

The final terms for $i \neq j$ are all equal and given by

$$\mathcal{F}_{i^+ j^+} = 4C_{\text{Inter}}.$$

Consider then $i = j$. It is easily confirmed that

$$\mathcal{F}_{i^\pm i^\mp} = 0,$$

which are the final nondiagonal terms of the QFIM. Finally, consider the remaining $i = j$ terms, which are the diagonal elements of the QFIM and are given by

$$\mathcal{F}_{i^\pm i^\pm} = V + V \pm C_{\text{Intra}} \pm C_{\text{Intra}}.$$

Hence, all of the diagonal terms are one of the two values

$$\mathcal{F}_{i^\pm i^\pm} = 2(V \pm C_{\text{Intra}}),$$

which holds for all i. Hence, combining all of these terms into the QFIM gives

$$\mathcal{F} = \begin{pmatrix} 2(V - C_{\text{Intra}})\mathbb{I} & 0 \\ 0 & M \end{pmatrix},$$

where $\mathbb{I}$ is the $d \times d$ identity matrix and $M = \lambda(\mathbb{I} + \omega\mathcal{I})$, where $\lambda = 2(V + C_{\text{Intra}} - 2C_{\text{Inter}})$, $\omega = 2C_{\text{Inter}}/(V + C_{\text{Intra}} - 2C_{\text{Inter}})$, and $\mathcal{I}$ is the $d \times d$ matrix of all ones. The inverse of any matrix with the form of M is given by

$$M^{-1} = \frac{1}{\lambda}\left(\mathbb{I} - \frac{\omega}{1+\omega d}\mathcal{I}\right), \quad \text{(A1)}$$

which may easily be confirmed directly by noting that $\mathcal{I}^2 = d\mathcal{I}$. However, we are not actually interested in these terms (the parameters of interest are $\phi_i \equiv \phi_i^-$; we are not attempting to also estimate the ϕ_i^+). The inverse of $\mathcal{F}$ may then simply be written as

$$\mathcal{F}^{-1} = \begin{pmatrix} \frac{1}{2(V - C_{\text{Intra}})}\mathbb{I} & 0 \\ 0 & M^{-1} \end{pmatrix}. \quad \text{(A2)}$$

This gives the phase precision bound for the terms of interest (ϕ_i) as

$$\delta\phi_i^2 \geqslant \frac{1}{2(V - C_{\text{Intra}})},$$

as stated in Eq. (4). Note that this is independent of d, and as required, it agrees with the single parameter (i.e., single-interferometer) estimation case ($d = 1$), e.g., see Ref. [14].

APPENDIX B: PHASE PRECISION DERIVATIONS, QUANTUM IMAGING

We begin with the QFIM from the main text $\mathcal{F}_{ij} = 4\text{Cov}(\hat{n}_i,\hat{n}_j) = 4C_{i,j}$. The assumption of path symmetry between the d (probe) modes, as stated in the main text, implies that $V_i = V_j$ for all i and j, which is denoted V, and that $C_{i,j} = C_{m,n}$ for all $i \neq j$ and $m \neq n$, which we denote by C. Then it immediately follows that $\mathcal{F}_{ii} = 4V$ for all i and $\mathcal{F}_{ij} = 4C$ for all $i \neq j$. Hence the QFIM may be written in the form

$$\mathcal{F} = 4(V - C)\left(\mathbb{I} + \frac{C}{V - C}\mathcal{I}\right),$$

where again $\mathcal{I}$ and $\mathbb{I}$ are the $d \times d$ matrix of all ones and the identity, respectively. The inverse of such a matrix is given in Eq. (A1), and using this formula, we have

$$\mathcal{F}^{-1} = \frac{1}{4(V - C)}\left(\mathbb{I} - \frac{C}{V + (d-1)C}\mathcal{I}\right).$$

This then implies that for all i, the phase precision bound for ϕ_i is

$$\delta\phi_i^2 \geqslant \frac{V+(d-2)C}{4(V-C)[V+(d-1)C]},$$

as stated in Eq. (9). Note that for the single-parameter estimation case ($d=1$) this reduces to $1/4V$ as expected.

APPENDIX C: THE QFI AS A FIGURE OF MERIT

The QFI alone is not always a reliable method for deriving precision scaling bounds that are truly attainable in practice. In general, the precision as obtained by the Cramér-Rao bound (CRB), $\delta\phi_i^2 \geqslant \mu^{-1}(\mathcal{F}^{-1})_{ii}$, is achievable given a certain level of prior knowledge of the phase and an asymptotically large number of repetitions μ. Indeed, the unbalanced cat state (UCS), given in the main text by $|\Psi_{\text{UCS}}\rangle = \mathcal{N}_c(|\alpha_c\rangle + \nu|0\rangle)^{\otimes 2d}$, has already been considered in optical quantum metrology, and as shown in [28], it has an unbounded precision for fixed $\bar{n}$. This can be seen by considering Eq. (8) and allowing ν to grow without bound. The root of this strange effect is that the QFI is a measure of how a probe state transforms with an infinitesimal change in the parameter to be estimated and does not take into account any further important details such as the level of prior knowledge required of each phase or the number of experimental repetitions required to obtain this precision. For single-parameter estimation, it is known that states such as the UCS cannot, in practice, provide a "sub-Heisenberg" scaling [31,32]. These results have been extended to the multiparameter case, and it has been shown that a sub-Heisenberg scaling cannot be achieved here either [33,34].

Despite this, the scaling with photon number is often not of direct relevance in an experiment, and a more relevant measure is the absolute precision that can be obtained given an allowed total photon number through the interferometer [4,29,30]. In single-parameter estimation squeezed cat states, which have a large $\mathcal{Q}$, have recently been shown to obtain an improved absolute precision over NOON states [14], and we expect these results to be applicable in the multiparameter case. The squeezed cat states can saturate the CRB from a flat prior knowledge in the region $0 \leqslant \phi \leqslant \pi/2$ using a conceptually simple measurement scheme, which is optimal for most values of the phase shift ϕ [14].

Another limitation of the QFI is that, for indefinite number states, the QFI sometimes assumes the presence of an external reference beam [21]. In particular, in our quantum imaging section, the states with a variable photon number may require additional reference beams to perform the final measurement (note that this is not an issue for the parallel interferometers, in which each phase already has a reference). One possible measurement scheme for the single-mode states (UNO and UCS) is to include one additional reference mode for each probe mode. These additional reference modes would contain states identical to those of the probes (UNO or UCS), and the measurement scheme would be to mix the probe state with its reference at a beam splitter, followed by photon-number counting [25]. However, this will double the numbers of photons used. To overcome this, we can adjust the UCS and UNO so that they have half the photon number while retaining the same precision (for example, this can be done by adjusting N and ν in the UNO). In this way we can always equal the precision attained by the GNS with the same number of photons. Alternatively, in many experiments the main concern is to reduce the number of photons through the sample because the sample itself is fragile [1,29], and in this case we need not count the reference modes in our total resource count.

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