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GOING BEYOND VARIATION OF SETS

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Abstract. We study integralgeometric representations of variations of general sets $A \subset \mathbb{R}^n$ without any regularity assumptions. If we assume, for example, that just one partial derivative of its characteristic function χ^A is a signed Borel measure on $\mathbb{R}^n$ with finite total variation, can we provide a nice integralgeometric representation of this variation? This is a delicate question, as the Gauss-Green type theorems of De Giorgi and Federer are not available in this generality. We will show that a 'measure-theoretic boundary' plays its role in such representations similarly as for the sets of finite variation. There is a variety of suitable notions of 'measure-theoretic boundary' and one can address the question to find notions of measure-theoretic boundary that are as fine as possible.

1. INTRODUCTION

A recurring theme in Geometric Measure Theory and in the study of geometric variational problems is the theory of sets of finite perimeter. The best known classical result about such sets due to De Giorgi and Federer [Gio54, Gio55, Fed45, Fed58] says that a (measurable) set $A \subset \mathbb{R}^n$ has finite perimeter if and only if its measure-theoretic boundary has finite area ($(n-1)$-dimensional Hausdorff measure), and more precisely the perimeter agrees with the area of the measure-theoretic boundary of A.

Our focus here is on more general framework of sets. The main result in this paper (Theorem 4.5) states that a set A has finite variation in a given direction τ (that is, the distributional derivative of the characteristic function of A in the direction τ is a finite measure) if and only if a suitably defined $(n-1)$-dimensional measure of a suitably defined measure-theoretic boundary is finite, and more precisely the variation of A in the direction τ agrees with the measure of such boundary. Interestingly, our results give also a relatively elementary proof of the classical result of De Giorgi and Federer mentioned above (Theorem 4.9) . The results show quite clearly that the natural notion of area is not the $(n-1)$-dimensional Hausdorff measure, but the integralgeometric measure (which of course agree in case of rectifiable sets).

A set $A \subset \mathbb{R}^n$ is said to be of finite perimeter if it is Lebesgue measurable and the gradient $D\chi^A$ in the sense of distributions of its characteristic function χ^A is an $\mathbb{R}^n$ valued Borel measure on $\mathbb{R}^n$ with finite total variation. The value of the perimeter of A, denoted by $P(A)$, is then the total variation $||D\chi^A||$ of the vector measure $D\chi^A$. Otherwise, let the perimeter of A be equal to $+\infty$. (Another equivalent definition of perimeter was given in [Gio54], see also [Gio55] and [Fed58].)

Given a direction $\tau \in S^{n-1}$ a set $A \subset \mathbb{R}^n$ is said to have bounded variation at the direction τ if it is Lebesgue measurable and the directional derivative in the sense of distributions $\partial_\tau \chi^A$ of its characteristic function χ^A is a signed Borel measure with finite total variation on $\mathbb{R}^n$. The value of the variation at direction τ of A, denoted by $P_\tau(A)$, is then the total variation $||\partial_\tau \chi^A||$ of the signed measure $\partial_\tau \chi^A$. Otherwise, let $P_\tau(A) = +\infty$.

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It is well known that, for a Lebesgue measurable set A and $\tau = e_i$ being the standard orthonormal basis direction (and writing briefly P_i instead of P_{e_i}),

$$P_i(A) = \int m_i^A(z)\, dz$$

where $m_i^A(z)$ is the infimum of the variations in x_i of all functions defined on the line $L_i(z)$ (paralel to the x_i axis and meeting z) which are equivalent to $\chi^A | L_i(z)$ and the integration is over the $(n-1)$ space orthogonal to the x_i axis.

It is known that the perimeter of A (if it is finite) is equal to the $(n-1)$ measure of the set $fr_r A$ that is called the reduced boundary (see [Fed58]) or equivalently it is equal to $(n-1)$ measure of the essential boundary $fr_e A$ of A (see [Vol67] or [Fed69, 4.5.6]). Specifically, $x \in fr_r A$ iff there is an $(n-1)$ plane π through x such that the symmetric difference of A and one of the halfspaces determined by π has density zero at x. Further, $x \in fr_e A$ iff both A and complement of A have positive outer upper density at x.

Moreover, if the $(n-1)$ measure of $fr_e A$ is finite then A is of finite perimeter [Fed69, 4.5.11]. Hence $(n-1)$ measure of $fr_e A$ is equal to the perimeter of A for a general set $A \subset \mathbb{R}^n$ (Our method also offers a simple self-contained proof of this fact for an integralgeometric $(n-1)$ measure.)

The main purpose of this paper is to show that the directional variation of a general set $A \subset \mathbb{R}^n$ (without any assumptions on regularity of A) is equal to the measure of projection (with multiplicities taken into account) of the 'measure-theoretic boundary'. The essential boundary $fr_e A$ can play here the role of such a 'measure-theoretic boundary', but one can aim to replace it even with finer notions of 'measure-theoretic boundary'. We show that one can replace $fr_e A$ by finer preponderant boundary $fr_{pr} A$ (see 4.5, 4.8, 4.9 and 4.10). Specifically, $x \in fr_{pr} A$ iff both A and complement of A have the outer upper density at x greater than or equal to $\frac{1}{2}$.

2. NOTATION AND TERMINOLOGY

Throughout the whole paper we deal with the sets in the n-dimensional Euclidean space $\mathbb{R}^n$. We tacitly assume that $n \geq 2$ but results trivially hold in the case $n = 1$.

Let $e_1, e_2, \ldots, e_n$ stand for the orthonormal base in $\mathbb{R}^n$, $e_1 = (1, 0, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0)$, ..., $e_n = (0, 0, 0, \ldots, 1)$. The symbol $\mathbf{0}$ has also the meaning of the zero vector $(0, 0, \ldots, 0)$ in $\mathbb{R}^n$. For $x, y \in \mathbb{R}^n$ we denote by $|x|$ the euclidean norm of x and by $x \circ y$ the inner product of x and y. The symbol $[x, y]$ stands for the convex hull of the set $\{x, y\}$ and $]x, y[$ means $[x, y] \setminus \{x, y\}$.

Whenever $x \in \mathbb{R}^n$ and $r > 0$ $B(x, r)$ and $U(x, r)$ stand for the closed and open balls, respectively, with center x and radius r and $Q(x, r)$ stands for the cubic interval

$$\{ \ y \in \mathbb{R}^n : \ |y_i - x_i| \leq r \ , \ 1 \leq i \leq n \ \} .$$

We put

$$S^{n-1} = \{ \ x \in \mathbb{R}^n : \ |x| = 1 \ \} \quad \text{and} \quad L_\tau(x) = \{ \ x + t\tau : \ t \in \mathbb{R} \ \} \ \text{ for } x \in \mathbb{R}^n \text{ and } \tau \in \mathbb{R}^n \setminus \{\mathbf{0}\} .$$

For $\tau \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ we denote by $\mathbb{R}^{n-1}(\tau)$ the orthogonal complement in $\mathbb{R}^n$ to the one dimensional subspace $\{t\tau : t \in \mathbb{R}\}$ and by p_τ the orthogonal projection of $\mathbb{R}^n$ onto $\mathbb{R}^{n-1}(\tau)$. We write briefly $L_i(x)$, $\mathbb{R}^{n-1}(i)$ and p_i in the case $\tau = e_i$.

For $A, B \subset \mathbb{R}^n$ we denote by $A \triangle B$ the symmetric difference of A and B, $A \triangle B = (A \setminus B) \cup (B \setminus A)$, and by A^c and χ^A the complement of A to $\mathbb{R}^n$ and the characteristic function of A (on $\mathbb{R}^n$), respectively.

For an open set $\Omega \subset \mathbb{R}^n$ we will denote by $C_0^\infty(\Omega)$ (and $C_0^\infty(\Omega, \mathbb{R}^n)$) the space of all infinitely differentiable real valued functions with compact support in Ω (and the space of all infinitely differentiable $\mathbb{R}^n$ valued vector functions with compact support in Ω, respectively). These spaces are considered to be equipped with the "sup norm".

For any function f, any set A and any value y, the multiplicity $N(f, A, y)$ is defined as the number of elements (possibly $+\infty$) of the set $\{x \in A : f(x) = y\}$.

For $A \subset \mathbb{R}^n$ we denote by $intA, clA$ and frA the interior, closure and boundary of A, respectively.

For any outer measure μ on $\mathbb{R}^n$ and for any set $X \subset \mathbb{R}^n$ we define the outer measure $\mu|X$ on $\mathbb{R}^n$ by the formula

$$(\mu|X)(A) = \mu\,(X \cap A) \quad \text{for every } A \subset \mathbb{R}^n.$$

For a signed Borel measure μ (or for a vector Borel measure $\mu = (\mu_1, \mu_2, \dots, \mu_n)$ on $\mathbb{R}^n$ the symbol $|\mu|$ stands for Borel measure which is the variation of μ (or the variation of μ in the sense of vector measures, respectively).

2.1 Hausdorff measures. For an integer $k = 0, 1, \dots, n$ let H_k stand for the k-dimensional Hausdorff outer measure on $\mathbb{R}^n$, which is normalized in such a way that

$$H_k\{\ x \in \mathbb{R}^n :\ 0 \le x_i \le 1 \quad \text{for} \quad 1 \le i \le k \quad \text{and } x_i = 0 \quad \text{for} \quad k < i \le n\ \} = 1.$$

In particular, H_0 is the counting measure and H_n coincides with the Lebesgue outer measure on $\mathbb{R}^n$.

The constant $V(n) = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$ means the volume of the unit ball in $\mathbb{R}^n$ (with $V(0) = 1$), and the constant $A(n) = nV(n) = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$ means the area of S^{n-1}.

We define the equivalence relation $\sim$ for subsets of $\mathbb{R}^n$ by the prescription

$$A \sim B \quad \text{iff} \quad H_n\,[A \triangle B] = 0.$$

2.2 Projection measures μ_τ. For $\tau \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ the result of Caratheodory's construction from the set function

$$B \longmapsto H_{n-1}[\ p_\tau(B)\ ]$$

which is defined on the covering family of all Borel sets in $\mathbb{R}^n$ will be called the projection measure at the direction τ and denoted by μ_τ. Then μ_τ is a Borel regular outer measure on $\mathbb{R}^n$ and $\mu_\tau \le H_{n-1}$.

From Fubini theorem it follows that $H_n(C) = 0$ whenever $C \subset \mathbb{R}^n$ is such that $\mu_\tau(C) < \infty$.

2.3 Integralgeometric measure $\Im_1^{n-1}$. The result of Caratheodory's construction from the set function

$$B \longrightarrow \frac{1}{2V(n-1)} \int_{S^{n-1}} H_{n-1}[\ p_\tau(B)\ ]\, dH_{n-1}(\tau)$$

which is defined on the covering family of all Borel sets in $\mathbb{R}^n$ is usually termed $(n-1)$ dimensional integral geometric measure with exponent 1 on $\mathbb{R}^n$ and denoted by $\Im_1^{n-1}$. (For the existence of the above integral see e.g. [Fed69, 2.10.5].)

$\Im_1^{n-1}$is a Borel regular outer measure on $\mathbb{R}^n$ and $2V(n-1)\Im_1^{n-1} \le A(n)H_{n-1}$. Moreover $\Im_1^{n-1} \le H_{n-1}$ by [Fed69, 3.3.16].

2.4 Densities. For every set $A \subset \mathbb{R}^n$ and each $x \in \mathbb{R}^n$ we define the upper outer density $\overline{d}(x,A)$ and the lower outer density $\underline{d}(x,A)$ of A at x by the formulas

$$\overline{d}(x,A) = \overline{\lim}_{r\to 0+} \frac{H_n[A \cap B(x,r)]}{H_n[B(x,r)]} \ ,$$
$$\underline{d}(x,A) = \underline{\lim}_{r\to 0+} \frac{H_n[A \cap B(x,r)]}{H_n[B(x,r)]}.$$

In the case $\overline{d}(x,A) = \underline{d}(x,A)$ this common value is termed the outer density of A at x and it is denoted by $d(x,A)$.

A point x for which $\underline{d}(x,A) = 1$ is termed the outer density point of A. (We may drop the adjective "outer" from this terminology whenever the set A is Lebesgue measurable.)

2.5 Essential and preponderant interior and boundary. We define the essential interior $int_e A$ and the essential boundary $fr_e A$ of the set $A \subset \mathbb{R}^n$ by the formulas

$$int_e A = \{ \ x \in \mathbb{R}^n : \ d(x,A^c) = 0\},$$
$$fr_e A = [\ int_e A \cup int_e A^c \ ]^c = \{ \ x \in \mathbb{R}^n : \ \overline{d}(x,A) > 0 \quad \text{and} \quad \overline{d}(x,A^c) > 0 \ \}.$$

It is easy to see that $int_e A \cap int_e(A^c) = \emptyset$, $int_e A$ is of type $F_{\sigma\delta}$ and $fr_e A$ is of type $G_{\sigma\delta}$. We also define the preponderant interior $int_{pr} A$ and the preponderant boundary $fr_{pr} A$ of $A \subset \mathbb{R}^n$ by the formulas

$$int_{pr} A = \left\{ x \in \mathbb{R}^n : \overline{d}(x,A^c) < \frac{1}{2} \right\},$$
$$fr_{pr} A = [\ int_{pr} A \cup int_{pr} A^c \ ]^c = \left\{ x \in \mathbb{R}^n : \overline{d}(x,A) \ \geq \frac{1}{2} \quad \text{and} \quad \overline{d}(x,A^c) \geq \frac{1}{2} \right\}.$$

It is easy to see that $int_{pr} A \cap int_{pr} A^c = \emptyset$, $int_{pr} A$ is of type F_σ and $fr_{pr} A$ is of type G_δ.

2.6 BV functions. For a nonempty open set $\Omega \subset \mathbb{R}^n$ and for any $\tau \in \mathbb{R}^n$ we define the space $BV(\Omega,\tau)$ of all locally (in Ω) H_n summable functions g for which there exists a finite signed Borel measure $\Phi^g_{\Omega,\tau}$ on Ω with the equality

$$\int_\Omega g(x) \cdot \tau \circ \operatorname{grad} \varphi(x)\, dx = - \int_\Omega \varphi(x)\, d\Phi^g_{\Omega,\tau}(x)$$

whenever $\varphi \in C_0^\infty(\Omega)$. $BV(\Omega)$ is defined as the space of all locally (in Ω) H_n summable functions g such that there exist the finite signed Borel measures $\Phi^g_{\Omega,1}, \Phi^g_{\Omega,2}, \ldots, \Phi^g_{\Omega,n}$ with the equality

$$\int_\Omega g(x) \cdot \operatorname{div} \psi(x)\, dx = - \sum_{i=1}^n \int_\Omega \psi_i(x)\, d\Phi^g_{\Omega,i}(x)$$

whenever $\psi = (\psi_1, \psi_2, \ldots, \psi_n) \in C_0^\infty(\Omega, \mathbb{R}^n)$.

We also define the space $BV_{loc}(\Omega)$ (and the space $BV_{loc}(\Omega,\tau)$ analogously) by the following: $g \in BV_{loc}(\Omega)$ iff $g|\tilde{\Omega} \in BV(\tilde{\Omega})$ whenever $\tilde{\Omega}$ is nonempty open set the closure of which is a compact subset of Ω

2.7 Directional variation and perimeter of sets. For a nonempty open set $\Omega \subset \mathbb{R}^n$ and for any $\tau \in \mathbb{R}^n$ the set functions $P_{\Omega,\tau}$ and P_Ω over the subsets of $\mathbb{R}^n$ are defined for $A \subset \mathbb{R}^n$ by the following:

- If $A \cap \Omega$ is not H_n measurable then we put

$$P_{\Omega,\tau}(A) = P_\Omega(A) = \infty.$$

- If $A \cap \Omega$ is H_n measurable then we put

$$P_{\Omega,\tau}(A) = \sup \left\{ \int_\Omega \chi^A(x) \tau \circ D\varphi(x)\, dx : \ \varphi \in C_0^\infty(\Omega) \quad \text{and} \quad |\varphi| \le 1 \right\},$$

$$P_\Omega(A) = \sup \left\{ \int_\Omega \chi^A(x) \operatorname{div} \psi(x)\, dx : \ \psi \in C_0^\infty(\Omega, \mathbb{R}^n) \quad \text{and} \quad |\psi| \le 1 \right\}.$$

The value $P_{\Omega,\tau}(A)$ is termed the variation at direction τ of the set A in Ω, and $P_\Omega(A)$ is the perimeter of A in Ω.

In the case $\chi^A|\Omega \in BV(\Omega, \tau)$ the symbol $\Phi^A_{\Omega,\tau}$ stand for the (uniquely determined) signed Borel measure on Ω such that

$$\int_\Omega \chi^A(x) \tau \circ D\varphi(x)\, dx = - \int_\Omega \varphi(x)\, d\Phi^A_{\Omega,\tau}(x)$$

holds whenever $\varphi \in C_0^\infty(\Omega)$. We write briefly $\Phi^A_{\Omega,i}$ in the case $\tau = e_i$.

3. AUXILIARY RESULTS

3.1 Lebesgue outer density theorem. For any set $A \subset \mathbb{R}^n$ H_n almost every point of A is an outer density point of A.

3.2 Remark. Let $\Omega \subset \mathbb{R}^n$ nonempty open and let $A \subset \mathbb{R}^n$ arbitrary. Using Lebesgue density theorem and Borel regularity of Lebesgue outer measure one can easily show that the following statements are true:
If $A \cap \Omega$ is H_n measurable then

$$\Omega \cap int_e A \sim \Omega \cap int_{pr} A \sim \Omega \cap A \quad \text{and} \quad H_n(\Omega \cap fr_e A) = H_n(\Omega \cap fr_{pr} A) = 0.$$

If $A \cap \Omega$ is not H_n measurable then

$$H_n\{\ x \in \Omega\, d(x, A) = 1 \quad \text{and} \quad d(x, A^c) = 1\ \} > 0$$

and especially

$$H_n(\Omega \cap fr_e A) \ge H_n(\Omega \cap fr_{pr} A) > 0\,.$$

3.3 Observations.

(1) If $g \in BV(\Omega)$ then $g \in BV(\Omega, \tau)$ for every $\tau \in \mathbb{R}^n \setminus \{\mathbf{0}\}$.
(2) If $\tau_j \in \mathbb{R}^n, \alpha_j \in \mathbb{R}, g \in BV(\Omega, \tau_j)$ $(j = 1, 2, \ldots, r)$ and $\tau = \sum_{i=1}^r \alpha_j \tau_j$ then $g \in BV(\Omega, \tau)$ and $\Phi^g_{\Omega,\tau} = \sum_{i=1}^r \alpha_j \Phi^g_{\Omega,\tau_j}$.
(3) If $\tau_1, \tau_2, \ldots, \tau_n \in \mathbb{R}^n$ are linearly independent and $g \in BV(\Omega, \tau_j)$ $(j = 1, 2, \ldots, n)$ then $g \in BV(\Omega)$.
(4) $P_{\Omega,\tau}(A) < \infty$ holds if and only if $\chi_A|\Omega \in BV(\Omega, \tau)$. In this case $P_{\Omega,\tau}(A) = |\Phi^A_{\Omega,\tau}|(\Omega)$ holds.
(5) $P_\Omega(A) < \infty$ holds if and only if $\chi_A|\Omega \in BV(\Omega)$. In this case $P_\Omega(A) = |\Phi^A_\Omega|(\Omega)$, where $\Phi^A_\Omega = (\Phi^A_{\Omega,1}, \Phi^A_{\Omega,2}, \ldots, \Phi^A_{\Omega,n})$.

(6) If $P_\Omega(A) = \infty$ then the set $\{\tau \in \mathbb{R}^n : P_{\Omega,\tau}(A) < \infty\}$ is contained in an $(n-1)$-dimensional linear subspace of $\mathbb{R}^n$.

3.4 Lemma. Let $B \subset \mathbb{R}^n$ be a Borel set.

(i) For any $\tau \in S^{n-1}$ the function $z \longmapsto N(p_\tau, B, z)$ defined on $\mathbb{R}^{n-1}(\tau)$ is H_{n-1} measurable and

$$\mu_\tau(B) = \int\limits_{\mathbb{R}^{n-1}(\tau)} N(p_\tau, B, z)\, dH_{n-1}(z).$$

(ii) The function $\tau \longmapsto \mu_\tau(B)$ defined on S^{n-1} is H_{n-1} measurable and

$$\mathfrak{I}_1^{n-1}(B) = \frac{1}{2V(n-1)} \int\limits_{S^{n-1}} \mu_\tau(B)\, dH_{n-1}(\tau).$$

Proof. See [Fed69, 2.10.10] and [Fed69, 2.10.15]. □

3.5 Definition. Let L_0 be a line in $\mathbb{R}^n$ and let $L \subset L_0$ be relatively open in L_0. For any set $A \subset \mathbb{R}^n$ the point $x \in L$ is termed a hit of L on A provided both $L \cap A \cap U(x,r)$ and $(L \setminus A) \cap U(x,r)$ have a positive H_1 measure for every $r > 0$.

For $\Omega \subset \mathbb{R}^n$ nonempty open, $A \subset \mathbb{R}^n$, $z \in \mathbb{R}^n$ and $\tau \in S^{n-1}$ the symbol $M^A_{\Omega,\tau}(z)$ stands for the set of all hits of $L_\tau(z) \cap \Omega$ on A , $m^A_{\Omega,\tau}(z)$ stands for the number of elements (possibly $+\infty$) of $M^A_{\Omega,\tau}(z)$ and we put

$$M^A_{\Omega,\tau} = \cup \{\ M^A_{\Omega,\tau}(z)\ z \in \mathbb{R}^n\ \}.$$

We write briefly $M^A_{\Omega,i}(z)$, $m^A_{\Omega,i}(z)$ and $M^A_{\Omega,i}$ in the case $\tau = e_i$.

The starting point to our results is the following known integral representation of directional variation of a set.

3.6 Lemma. Let $\Omega \subset \mathbb{R}^n$ be nonempty open and let $A \subset \mathbb{R}^n$ be such that $A \cap \Omega$ is H_n measurable. Then for every $\tau \in S^{n-1}$ the function $z \longmapsto m^A_{\Omega,\tau}(z)$ defined on $\mathbb{R}^{n-1}(\tau)$ is H_{n-1} measurable and

$$P_{\Omega,\tau}(A) = \int\limits_{\mathbb{R}^{n-1}(\tau)} m^A_{\Omega,\tau}(z)\, dH_{n-1}(z).$$

Proof. See e.g. [Mar58] and Chap. 7 of [Kri57]. □

4. INTEGRAL-GEOMETRIC CHARACTERIZATION OF VARIATIONS

4.1 Notation. Let $\Omega \subset \mathbb{R}^n$ be nonempty open and $A \subset \mathbb{R}^n$ be such that $A \cap \Omega$ is H_n measurable. Let us identify $\mathbb{R}^n$ with $\mathbb{R}^{n-1} \times \mathbb{R}$. For any α, β such that $-\infty \le \alpha < \beta \le +\infty$ put

$$E_\Omega(\alpha,\beta;A) = \{\ z \in \mathbb{R}^{n-1}\ :\ \{z\} \times (\alpha,\beta) \subset \Omega \quad \text{and} \quad H_1(\{z\} \times (\alpha,\beta) \setminus A) = 0\ \}.$$

It easily follows from Fubini's theorem that all the sets $E_\Omega(\alpha,\beta;A)$ in $\mathbb{R}^{n-1}$ are H_{n-1} measurable.

4.2 Lemma. Let $\Omega \subset \mathbb{R}^n$ be nonempty open and $A \subset \mathbb{R}^n$ be such that $A \cap \Omega$ is H_n measurable. Then there is an H_{n-1} null set $N \subset \mathbb{R}^{n-1}$ such that every $z \in \mathbb{R}^{n-1} \setminus N$ has the following properties:

(a) If $\alpha, \beta \in \mathbb{Q} \cup \{-\infty, +\infty\}, -\infty \le \alpha < \beta \le +\infty$ ($\mathbb{Q}$ being the set of rationals) are such that $z \in E_\Omega(\alpha,\beta;A)$ ($z \in E_\Omega(\alpha,\beta;A^c)$, respectively) then z is a density point in $\mathbb{R}^{n-1}$ of $E_\Omega(\alpha,\beta;A)$ (of $E_\Omega(\alpha,\beta;A^c)$, respectively).

(b) If $-\infty \le \alpha < \beta \le +\infty$ are such that $\{z\} \times (\alpha, \beta) \subset \Omega$ and $H_1(\{z\} \times (\alpha, \beta) \setminus A) = 0$ ($H_1(\{z\} \times (\alpha, \beta) \setminus A^c) = 0$, respectively) then $\{z\} \times (\alpha, \beta) \subset int_e A$ ($\{z\} \times (\alpha, \beta) \subset int_e A^c$, respectively).

(c) $\{x \in \Omega \cap fr_e A : \; p_n(x) = z\} \subset M^A_{\Omega,n}$.

Proof. (a) For any H_{n-1} measurable set $B \subset \mathbb{R}^{n-1}$ put $\tilde{B} = \{z \in B : z \text{ is not a density point of } B\}$. Due to the Lebesgue density theorem $\tilde{B}$ is an H_{n-1} null set. Hence the set

$$N = \cup\{\tilde{E}_\Omega(\alpha, \beta; A) \cup \tilde{E}_\Omega(\alpha, \beta; A^c) : \alpha, \beta \in \mathbb{Q} \cup \{-\infty, +\infty\}, \alpha < \beta\}$$

is an H_{n-1} null set and each $z \in \mathbb{R}^{n-1} \setminus N$ has the property (a).

(b) If $z \in \mathbb{R}^{n-1} \setminus N$ and $-\infty \le \alpha < \beta \le +\infty$ are such that $\{z\} \times (\alpha, \beta) \subset \Omega$ and $H_1(\{z\} \times (\alpha, \beta) \setminus A) = 0$ ($H_1(\{z\} \times (\alpha, \beta) \setminus A^c) = 0$, respectively), then z is a density point in $\mathbb{R}^{n-1}$ of $E_\Omega(\alpha_1, \beta_1; A)$ (of $E_\Omega(\alpha_1, \beta_1; A^c)$, respectively) whenever $\alpha_1, \beta_1 \in \mathbb{Q} \cup \{-\infty, +\infty\}$ with $\alpha \le \alpha_1 < \beta_1 \le \beta$. From Fubini's theorem it follows that $\{z\} \times (\alpha, \beta) \subset int_e A$ ($\{z\} \times (\alpha, \beta) \subset int_e A^c$, respectively), hence (b) holds true.

(c) Let us keep $z \in \mathbb{R}^{n-1} \setminus N$ fixed and assume that $x \in \Omega \setminus M^A_{\Omega,n}$ is such that $p_n(x) = z$. Our aim is to prove that then necessarily $x \notin fr_e A$. As $x \in \Omega \setminus M^A_{\Omega,n}$ we can find real numbers $\alpha < \beta$ such that $x \in \{z\} \times (\alpha, \beta) \subset \Omega$ and either $H_1(\{z\} \times (\alpha, \beta) \setminus A) = 0$ or $H_1(\{z\} \times (\alpha, \beta) \setminus A^c) = 0$. From (b) it follows that either $x \in int_e A$ or $x \in int_e A^c$, hence $x \notin fr_e A$. This completes the proof.

□

4.3 Lemma. Let $X, Y \subset \mathbb{R}$ be two disjoint sets of type F_σ such that every $x \in X$ is a bilateral accumulation point of $\mathbb{R} \setminus Y$ and every $y \in Y$ is a bilateral accumulation point of $\mathbb{R} \setminus X$. Then $]\,a, c\,[\setminus (X \cup Y)$ is nonempty whenever $a \in X$ and $c \in Y$.

Proof. We have $X = \cup_{k=1}^\infty X_k$ and $Y = \cup_{k=1}^\infty Y_k$ where $X_1 \subset X_2 \subset X_3 \subset \dots$ and $Y_1 \subset Y_2 \subset Y_3 \subset \dots$ are closed. Let $a \in X$ and $c \in Y$ be arbitrarily chosen. Suppose that $X \cup Y \supset]a, c[$. Our assumptions imply that $X \cap]a, c[$ and $Y \cap]a, c[$ are nonempty and that every $x \in X$ is a bilateral accumulation point of X and every $y \in Y$ is a bilateral accumulation point of Y. We can construct by induction an infinite sequence of nonnegative integers $0 = k_0 < k_1 < k_2 < \dots$ and the sequences $\{a_r\}_{r=0}^\infty$ and $\{c_r\}_{r=0}^\infty$ of real numbers such that $a_0 = a, c_0 = c$ and, for every positive integer r,

$$a_r \in X_{k_r} \cap]\,a_{r-1}, c_{r-1}\,[\;, \;\; c_r \in Y_{k_r} \cap]\,a_{r-1}, c_{r-1}\,[\quad \text{and} \quad (X_{k_r} \cup Y_{k_r}) \cap]\,a_r, c_r\,[\; = \emptyset.$$

as follows. Assume that $a_0 = a, c_0 = c, k_0 = 0$ and that $a_{r-1}, c_{r-1}, k_{r-1}$ (for a positive integer r) have been constructed. Choose $\tilde{a}_r \in X \cap]\,a_{r-1}, c_{r-1}\,[$ and $\tilde{c}_r \in Y \cap]\,a_{r-1}, c_{r-1}\,[$ arbitrarily and an integer k_r so large that $k_r > k_{r-1}$, $\tilde{a}_r \in X_{k_r}$ and $\tilde{c}_r \in Y_{k_r}$. As $[\tilde{a}_r, \tilde{c}_r] \cap X_{k_r}$ and $[\tilde{a}_r, \tilde{c}_r] \cap Y_{k_r}$ are two disjoint compact sets, we can choose their points a_r and c_r, respectively, such that they realize the distance between these sets. Then we have $a_r \in X_{k_r} \cap]\,a_{r-1}, c_{r-1}\,[$, $c_r \in Y_{k_r} \cap]\,a_{r-1}, c_{r-1}\,[$ and $(X_{k_r} \cup Y_{k_r}) \cap]\,a_r, c_r\,[\; = \emptyset$.

Now it is easy to see that for our constructed sequence of intervals $[a_r, c_r]$ we have

$$\emptyset \neq \bigcap_{r=1}^{\infty} [\,a_r, c_r\,] = \bigcap_{r=1}^{\infty}]\,a_r, c_r\,[\; \subset]\,a, c\,[\setminus (X \cup Y) \; .$$

That completes the proof.

□

4.4 Definition. As the density of the ball $B(0,1) \subset \mathbb{R}^n$ is equal to $\frac{1}{2}$ at every point of its boundary, we can fix for any positive integer k a constant $\delta(k)$ (depending only on k and on dimension n) such that $0 < \delta(k) \leq 1$ and

$$H_n[B(e_1, \delta(k)) \cap B(0,1)] \geq \frac{V(n)}{2}\left(1 - \frac{1}{8k}\right)[\delta(k)]^n .$$

As the function

$$y \longmapsto H_n[B(y, \delta(k)) \cap B(0,1)]$$

is continuous on $\mathbb{R}^n$ we can fix for k and $\delta(k)$ as above a constant $\varepsilon(k) > 0$ such that

$$H_n[B(y, \delta(k)) \cap B(0,1)] \geq \frac{V(n)}{2}\left(1 - \frac{1}{4k}\right)[\delta(k)]^n$$

whenever $y \in [e_1, (1 + \varepsilon(k))e_1]$.

According to the homogeneity and the invariance under Euclidean isometries of H_n we see that

$$H_n[B(y, \delta(k)) \cap B(x,r)] \geq \frac{V(n)}{2}\left(1 - \frac{1}{4k}\right)[\delta(k)r]^n$$

whenever k is a positive integer, $0 < r < \infty, x \in \mathbb{R}^n$ and $y \in B(x, (1+\varepsilon(k))r) \setminus U(x,r)$.

4.5 Theorem. Let $\Omega \subset \mathbb{R}^n$be nonempty open, $A \subset \mathbb{R}^n$ be arbitrary and $\tau \in S^{n-1}$. Then

$$P_{\Omega,\tau}(A) = \mu_\tau(\Omega \cap fr_e A) = \mu_\tau(\Omega \cap fr_{pr} A).$$

Proof. Since $fr_{pr} A \subset fr_e A$, it is sufficient to prove the inequalities

$$\mu_\tau(\Omega \cap fr_e A) \leq P_{\Omega,\tau}(A) \leq \mu_\tau(\Omega \cap fr_{pr} A).$$

(i) To prove the first inequality we may assume $P_{\Omega,\tau}(A) < \infty$ and therefore we may assume $A \cap \Omega$ is H_n measurable. According to Lemma 4.2(c) we have

$$N(p_\tau, \Omega \cap fr_e A, z) \leq m^A_{\Omega,\tau} \quad \text{for} \quad H_{n-1} \quad \text{a.e.} \quad z \in \mathbb{R}^{n-1}(\tau).$$

By using of Lemma 3.4 and 3.8 we see after the integration of the above inequality that

$$\mu_\tau(\Omega \cap fr_e A) = \int\limits_{\mathbb{R}^{n-1}(\tau)} N(p_\tau, \Omega \cap fr_e A, z)\, dH_{n-1}(z) \leq \int\limits_{\mathbb{R}^{n-1}(\tau)} m^A_{\Omega,\tau}(z)\, dH_{n-1}(z) = P_{\Omega,\tau}(A).$$

(ii) To prove the inequality

$$P_{\Omega,\tau}(A) \leq \mu_\tau(\Omega \cap fr_{pr} A)$$

it is sufficient to assume that $\tau = e_n$ and

$$\mu_n(\Omega \cap fr_{pr} A) < \infty.$$

We see that then $H_n(\Omega \cap fr_{pr} A) = 0$, according to 3.2 the set $\Omega \cap A$ is H_n measurable and $A \cap \Omega \sim \Omega \cap int_{pr} A$. Hence

$$P_{\Omega,n}(A) = P_{\Omega,n}(int_{pr} A)$$

and according to 3.4–3.8 it is sufficient to prove that

$$m^{int_{pr} A}_{\Omega,n}(z) \leq N(p_n, \Omega \cap fr_{pr} A, z) \quad \text{for} \quad H_{n-1} \quad \text{a.e.} \quad z \in \mathbb{R}^{n-1}(n). \tag{1}$$

For any positive integer k we put

$$A(k)=\left\{\ x\in\mathbb{R}^n\ H_n(B(x,r)\setminus A)\le\frac{V(n)}{2}\left(1-\frac{1}{k}\right)r^n\quad\text{if}\quad r\in\left]0,\frac{1}{k}\right]\ \right\},$$

$$C(k)=\left\{\ x\in\mathbb{R}^n\ H_n(B(x,r)\cap A)\le\frac{V(n)}{2}\left(1-\frac{1}{k}\right)r^n\quad\text{if}\quad r\in\left]0,\frac{1}{k}\right]\ \right\}.$$

Obviously $A(k)$ and $C(k)$ are closed and $A(k)\uparrow int_{pr}A, C(k)\uparrow int_{pr}A^c$ with $k\uparrow+\infty$. For any pair of positive integers (k,m) we put

$$A^+(k,m)=\left\{\ x\in A(k)\colon Q\left(x,\frac{8}{m}\right)\subset\Omega,\left]x,x+\frac{8}{m}e_n\right[\subset int_{pr}A^c\ \right\},$$

$$A^-(k,m)=\left\{\ x\in A(k)\colon Q\left(x,\frac{8}{m}\right)\subset\Omega,\left]x,x-\frac{8}{m}e_n\right[\subset int_{pr}A^c\ \right\},$$

$$C^+(k,m)=\left\{\ x\in C(k)\colon Q\left(x,\frac{8}{m}\right)\subset\Omega,\left]x,x+\frac{8}{m}e_n\right[\subset int_{pr}A\ \right\},$$

$$C^-(k,m)=\left\{\ x\in C(k)\colon Q\left(x,\frac{8}{m}\right)\subset\Omega,\left]x,x-\frac{8}{m}e_n\right[\subset int_{pr}A\ \right\},$$

$$B=\bigcup_{k=1}^{\infty}\bigcup_{m=1}^{\infty}\left[A^+(k,m)\cup A^-(k,m)\cup C^+(k,m)\cup C^-(k,m)\right].$$

To prove (1) it is sufficient to prove that

(2) $$m^{int_{pr}A}_{\Omega,n}(z)\le N(p_n,\Omega\cap fr_{pr}A,z)\quad\text{if } z\in\mathbb{R}^{n-1}(n)\setminus p_n(B),\text{ and}$$

(3) $$H_{n-1}[\,p_n(B)\,]=0\ .$$

Firstly we make the following observation:
If $z\in\mathbb{R}^{n-1}(n)\setminus p_n(B)$ then the assumptions of 4.3 are fulfiled with $L_n(z), L_n(z)\cap\Omega\cap int_{pr}A$ and $L_n(z)\cap\Omega\cap int_{pr}A^c$ instead of $\mathbb{R}$, X and Y, respectively. Therefore for such z there exists $b\in\,]a,c[\,\cap\,fr_{pr}A$ whenever $a\in L_n(z)\cap int_{pr}A$ and $c\in L_n(z)\cap int_{pr}A^c$ are such that $[a,c]\subset\Omega$.

To prove (2) we fix $z\in\mathbb{R}^{n-1}(n)\setminus p_n(B)$. We may assume that

(4) $$N(p_n,\Omega\cap fr_{pr}A,z)<\infty$$

Then even the inclusion

(5) $$M^{int_{pr}A}_{\Omega,n}(z)\subset fr_{pr}A$$

holds. To prove it we fix a point $x\in(L_n(z)\cap\Omega)\setminus fr_{pr}A$. According to (4) we may fix $\varepsilon>0$ such that

$$[x-\varepsilon e_n,x+\varepsilon e_n]\subset\Omega\quad\text{and}\quad[x-\varepsilon e_n,x+\varepsilon e_n]\cap fr_{pr}A=0.$$

According to the observation made above, we get either

$$[x-\varepsilon e_n,x+\varepsilon e_n]\subset int_{pr}A\quad\text{or}\quad[x-\varepsilon e_n,x+\varepsilon e_n]\subset int_{pr}A^c\subset(int_{pr}A)^c.$$

Both cases imply that x does not belong to $M^{int_{pr}A}_{\Omega,n}(z)$. This completes the proof of (5) and (2).

To prove (3) we fix the positive integers k, m and we shall prove that

(6) $$H_{n-1}\{\ p_n[A^+(k,m)]\ \}=0\ .$$

(One can analogously prove that $p_n[A^-(k,m)], p_n[C^+(k,m)]$ and $p_n[C^-(k,m)]$ are H_{n-1} null sets.)

To prove (6) we put for any integer s

$$A^+(k,m,s) = \left\{ \, x \in A^+(k,m) : \; \frac{s-1}{m} < x_n \leq \frac{s}{m} \, \right\}$$

and assume, on the contrary, that for some fixed s we have

$$H_{n-1}\{ \, p_n[A^+(k,m,s)] \, \} > 0. \tag{7}$$

From Lebesgue outer density theorem we can fix $z_0 \in p_n[A^+(k,m,s)]$ which is an outer density point (in the space $\mathbb{R}^{n-1}(n)$) of $p_n[A^+(k,m,s)]$.

For every $z \in p_n[A^+(k,m,s)]$ obviously there exists the point $x \in A^+(k,m,s)$ (depending on z) such that

$$p_n(x) = z, \quad Q\left(x, \frac{8}{m}\right) \subset \Omega \quad \text{and} \quad \left]z + \frac{s}{m}e_n, z + \frac{s+7}{m}e_n\right[\subset \left]x, z + \frac{s+7}{m}e_n\right[\subset int_{pr}A^c.$$

We put $x_1 = z_0 + \frac{s+1}{m}e_n$. According to the choice of z_0 we can fix positive r_0 such that $r_0 \leq \frac{1}{m}, r_0 \leq \frac{1}{k}$ and

$$\frac{1}{V(n-1)r_0^{n-1}} H_{n-1}\{ \, p_n[U(x_1,r_0)] \cap p_n[A^+(k,m,s)] \, \} \geq 1 - \frac{V(n)}{16kV(n-1)}[\delta(k)]^n, \tag{8}$$

where $\delta(k)$ is the constant from 4.4. Putting $S = p_n[U(x_1,r_0)]$, from (8) we get

$$H_{n-1}\{ \, S \cap p_n[A^+(k,m,s)] \, \} \geq H_{n-1}(S) - \frac{V(n)}{16k}[\delta(k)]^n r_0^{n-1}. \tag{9}$$

According to the choice of x_1 and r_0 we see that $U(x_1,r_0) \cap A^+(k,m,s) = 0$. We can define the number $t_0 \in \left]\frac{s-1}{m}, \frac{s+1}{m}\right]$ by the formula

$$t_0 = \sup \left\{ \, t \in \left]\frac{s-1}{m}, \frac{s+1}{m}\right] U(z_0 + te_n, r_0) \cap A^+(k,m,s) \neq \emptyset \, \right\}$$

and we put $x_0 = z_0 + t_0 e_n$. The ball $U(x_0,r_0)$ has the following properties:

$$L_n \cap U(x_0,r_0) \subset int_{pr}A^c \quad \text{whenever} \quad z \in p_n[A^+(k,m,s)] \;, \tag{10}$$

$U(x_0,r_0) \subset \Omega$, especially $A \cap U(x_0,r_0)$ is H_n measurable,

$$A^+(k,m,s) \cap [\, B(x_0,(1+\varepsilon)r_0) \setminus U(x_0,r_0) \,] \neq \emptyset$$

whenever $\varepsilon > 0$.

We fix some $y \in A^+(k,m,s) \cap [\, B(x_0,(1+\varepsilon(k))r_0) \setminus U(x_0,r_0) \,]$, where $\varepsilon(k)$ is as in 4.4.

From 4.4 we see that

$$H_n\{ \, [\, B(y,\delta(k)r_0) \cap B(x_0,r_0) \,] \, \} \geq \frac{V(n)}{2}\left(1 - \frac{1}{4k}\right)[\delta(k)r_0]^n \,. \tag{11}$$

We define the function

$$g : \mathbb{R}^{n-1}(n) \longrightarrow [0, 2r_0]$$

by the formula

$$g(z) = H_1\{ \, [\, L_n(z) \cap U(x_0,r_0) \,] \setminus int_{pr}A^c \, \} \quad , z \in \mathbb{R}^{n-1}(n).$$

According to 3.2 we have

$$[U(x_0,r_0) \setminus int_{pr}A^c] \quad \sim \quad [U(x_0,r_0) \cap A] \;,$$

and by using of Fubini's theorem we get that g is H_{n-1} measurable and

$$H_n\,[U(x_0,r_0)\cap A] = \int\limits_{\mathbb{R}^{n-1}(n)} g(z)\,dH_{n-1}(z)\ . \tag{12}$$

From (10) we see that

$$\begin{aligned} &g(z)=0 \quad \text{whenever} \quad z\in p_n[A^+(k,m,s)], \quad \text{and obviously}\\ &g(z)=0 \quad \text{whenever} \quad z\in\mathbb{R}^{n-1}(n)\setminus S. \end{aligned}$$

Especially the set

$$\{\ z\in\mathbb{R}^{n-1}(n)\ g(z)>0\ \} = S\setminus\{\ z\in S\ g(z)=0\ \}$$

is H_{n-1} measurable and from (9) we get

$$\begin{aligned} H_{n-1}\{\ z\in\mathbb{R}^{n-1}(n)\ \ g(z)>0\ \} &= H_{n-1}(S) - H_{n-1}\{z\in S\ \ g(z)=0\} \le \\ &\le H_{n-1}(S) - H_{n-1}\{S\cap p_n[A^+(k,m,s)]\} \le \frac{V(n)}{16k}[\delta(k)]^n {r_0}^{n-1} \end{aligned} \tag{13}$$

From (12) and (13) we see that

$$H_n[U(x_0,r_0)\cap A] \le 2r_0 H_{n-1}\{z\in\mathbb{R}^{n-1}(n)\ \ g(z)>0\} \le \frac{V(n)}{8k}[\delta(k)r_0]^n. \tag{14}$$

We see that

$$H_n[\,B(y,\delta(k)r_0)\setminus A\,] \ge H_n[\,B(x_0,r_0)\cap B(y,\delta(k)r_0)\,] - H_n[\,A\cap U(x_0,r_0)\,]. \tag{15}$$

According to (11), (14) and (15) we eventually get

$$H_n[B(y,\delta(k)r_0)\setminus A] \ge \frac{V(n)}{2}\left(1-\frac{1}{2k}\right)[\delta(k)r_0]^n\ . \tag{16}$$

As $y\in A^+(k,m,s)\subset A(k)$ and $\delta(k)r_0\le r_0\le\frac{1}{k}$, the equality (16) contradicts with our definition of $A(k)$. Hence the assumption made in (7) leads to the contradiction and consequently (6) and (3) hold. This completes the proof. □

4.6 Corollary. Let $\Omega\subset\mathbb{R}^n$ be nonempty open and $A\subset\mathbb{R}^n$ be arbitrary. Then the following are equivalent :

(i) $P_\Omega(A)<\infty$.
(ii) There exist linearly independent vectors $\tau_1,\tau_2,\dots,\tau_n\in\mathbb{R}^n$ such that $\mu_{\tau_i}(\Omega\cap fr_{pr}A)<\infty$ for $i=1,2,\dots,n$.

4.7. Lemma. Let $\Omega\subseteq\mathbb{R}^n$ a nonempty open set and $\Phi=(\Phi_1,\dots,\Phi_n)$ be an $\mathbb{R}^n$ valued Borel measure on Ω with finite total variation. For any $\tau=(\tau_1,\tau_2,\dots,\tau_n)\in S^{n-1}$ let Φ_τ stand for the signed Borel measure $\sum_{i=1}^n \tau_i\Phi_i$. Then

$$\|\Phi\| = \frac{1}{2V(n-1)}\int_{S^{n-1}}\|\Phi_\tau\|\,dH_{n-1}(\tau).$$

Proof. Let $v\colon \Omega \to \mathbb{R}^n$ be the Radon-Nikodym derivative of Φ with respect to its variation measure $|\Phi|$. Then v is a $|\Phi|$ measurable $\mathbb{R}^n$ valued function and $|v(x)| = 1$ for $|\Phi|$ a.e. $x \in \Omega$ (see [Fed69, 2.5.12]).

As $\Phi_\tau(B) = \int_B \tau \circ v(x)\, d|\Phi|(x)$ for any Borel $B \subseteq \Omega$, clearly

$$\|\Phi_\tau\| = \int_\Omega |\tau \circ v(x)|\, d|\Phi|(x).$$

Integrating over S^{n-1} and using Fubini's theorem we get

$$\int_{S^{n-1}} \|\Phi_\tau\|\, dH_{n-1}(\tau) = \int_\Omega \left(\int_{S^{n-1}} |\tau \circ v(x)|\, dH_{n-1}(\tau) \right) d|\Phi|(x).$$

As S^{n-1} and H_{n-1} are invariant under orthonormal transformations of $\mathbb{R}^n$,

$$\int_{S^{n-1}} |\tau \circ w|\, dH_{n-1}(\tau) = |w| \int_{S^{n-1}} |\tau_1|\, dH_{n-1}(\tau) = 2V(n-1)|w| \quad \text{for any } w \in \mathbb{R}^n.$$

(See [Fed69, 3.2.13] for the exact values of constants $V(n-1)$ and $\int_{S^{n-1}} |\tau_1|\, dH_{n-1}(\tau)$.) Hence

$$\int_{S^{n-1}} \|\Phi_\tau\|\, dH_{n-1}(\tau) = 2V(n-1) \int_\Omega |v(x)|\, d|\Phi|(x) = 2V(n-1)\|\Phi\|,$$

that completes the proof. □

4.8 Theorem. Let $\Omega \subset \mathbb{R}^n$ be nonempty open and $A \subset \mathbb{R}^n$ be arbitrary. Then

$$P_\Omega(A) = \frac{1}{2V(n-1)} \int_{S^{n-1}} P_{\Omega,\tau}(A)\, dH_{n-1}(\tau).$$

Proof. If $P_\Omega(A) = +\infty$ then clearly $P_{\Omega,\tau}(A) = +\infty$ for H_{n-1} a.e. $\tau \in S^{n-1}$ and the equality holds. If $P_\Omega(A) < +\infty$ then the gradient $D\chi^A$ in the sense of distributions over Ω of the characteristic function χ^A is an $\mathbb{R}^n$ valued Borel measure over Ω with finite total variation. As $P_\Omega(A) = ||D\chi^A||(\Omega)$ and $P_{\Omega,\tau}(A) = ||\tau \circ D\chi^A||(\Omega)$, the equality holds due to the previous lemma applied to the restriction of the vector measure $D\chi^A$ to Ω. □

4.9 Theorem. Let $\Omega \subset \mathbb{R}^n$ be nonempty open and $A \subset \mathbb{R}^n$ be arbitrary. Then the following equalities hold:

$$P_\Omega(A) = \Im_1^{n-1}(\Omega \cap fr_e A) = \Im_1^{n-1}(\Omega \cap fr_{pr} A).$$

Proof. Integrating the equalities from Theorem 4.5 over S^{n-1} and using Lemma 3.3(ii) we get

$$\frac{1}{2V(n-1)} \int_{S^{n-1}} P_{\Omega,\tau}(A)\, dH_{n-1}(\tau) = \Im_1^{n-1}(\Omega \cap fr_e A) = \Im_1^{n-1}(\Omega \cap fr_{pr} A).$$

Due to Lemma 4.8 the first term is equal to $P_\Omega(A)$. That completes the proof. □

4.10 Remark. The equality $P_\Omega(A) = \Im_1^{n-1}(\Omega \cap fr_e A)$ for an arbitrary set $A \subset \mathbb{R}^n$ is known (see [Fed69, 4.5.6] and [Fed69, 4.5.11]), but our simple and seff-contained proof does not depend on the deep results of De Giorgi, Federer and Volpert on the sets with finite perimeter, and it can be of independent interest. It is easy to combine our results with other known facts and then replace integralgeometric measure $\Im_1^{n-1}$ in the theorem above by the Hausdorff measure H_{n-1} and to prove in full generality that also $P_\Omega(A) = H_{n-1}(\Omega \cap fr_e A) = H_{n-1}(\Omega \cap fr_{pr} A)$.

4.11 Some open questions. We have seen that there is a variety of notions of 'measure theoretic boundary' that play an important role in integralgeometric representations of various notions of variation of a general set $A \subset \mathbb{R}^n$. We demonstrated this here using the essential boundary, and the slightly finer preponderant boundary. While for the sets of bounded variation there is plenty of such notions of boundary that can be used, much less is known about which notions of 'boundary' can be used for integral representations of variations of more general sets. Even for the usual notion of the perimeter $P(A)$ of a set $A \subset \mathbb{R}^n$ we aim to understand for which notions of 'fine boundary', $fr_{\mathit{fine}}(A)$, we can say that $P(A)$ is equal to $(n-1)$-dimensional measure of $fr_{\mathit{fine}}(A)$ for fully general sets $A \subset \mathbb{R}^n$. One of natural choices for such finer notions of 'boundary' that need to be understand for general sets is the following 'strong boundary',

$$fr_s(A) = \{\ x \in \mathbb{R}^n :\ \underline{d}(x, A) > 0 \quad \text{and} \quad \underline{d}(x, A^c) > 0\ \}.$$

Or one can suggest its finer version, $fr_{s,\delta}(A)$ for $0 < \delta \le 0.5$,

$$fr_{s,\delta}(A) = \{\ x \in \mathbb{R}^n :\ \underline{d}(x, A) \ge \delta \quad \text{and} \quad \underline{d}(x, A^c) \ge \delta\ \}.$$

For finite variation sets these 'boundaries' can be used to represent their variation, but what can be said about them for general sets? Is their $(n-1)$-dimensional measure always equal to $P(A)$, or can it be 'small' for some set A of infinite variation? We know the answers in dimension $n = 1$ only, for higher dimensions these questions about 'strong boundaries' of general sets are open.

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