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STABILITY ANALYSIS OF MULTIHARMONIC NONLINEAR VIBRATIONS FOR LARGE MODELS OF GAS-TURBINE ENGINE STRUCTURES WITH FRICTION AND GAPS

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ABSTRACT

An efficient method is proposed for the multiharmonic frequency domain analysis of the stability for nonlinear periodic forced vibrations in gas-turbine engine structures and turbomachines with friction, gaps and other types of nonlinear contact interfaces. The method allows using large-scale finite element models for structural components together with detailed description of nonlinear interactions at contact interfaces between these components. The highly accurate reduced models are applied in the assessment of stability of periodic regimes for large-scale model of gas-turbine structures. An approach is proposed for the highly-accurate calculation of motion of a structure after it is perturbed from the periodic nonlinear forced response.

Efficiency of the developed approach is demonstrated on a set of test cases including simple models and large-scale realistic bladed disc models with different types of nonlinearities: friction, gaps and cubic nonlinear springs.

INTRODUCTION

Gas turbine engines and their major components are usually assembled structures and in many cases they are subjected to essentially nonlinear interactions. The nonlinear interactions are usually localised at contact interfaces and among the most common causes of nonlinear interactions are friction, closing and opening gaps, variable contact areas, cubic nonlinearities due to Hertzian contacts.

Development of methods for analysis of nonlinear vibrations in gas turbine engines represents a major interest for the industry and attracts efforts of many researchers (e.g. see Refs. [1]-[7]). The efficient analysis of periodic vibration can be performed in frequency domain for realistic models of structures which customarily contain hundred thousand degrees of freedom (DOFs). The regimes of nonlinear vibration calculated by frequency-domain methods can be differentiated as being stable or unstable and an important part of any nonlinear analysis is the determination of the stability characteristics for a found nonlinear vibration solution.

The stability analysis of dynamic systems was long time based on the theory developed by a mathematician Gaston Floquet in 1883 (Ref.[8]). A review of some modern interpretations of the use of Floquet theory to the analysis of the dynamic stability is given in Ref.[9] and some important ways of its utilization can be found in [10] and [11].

The Floquet stability analysis requires time-integration of equations for the perturbed motion and, therefore, exhibits insurmountable difficulties in the application to the large scale finite element structures. Because of this, the significant efforts of scientists have been directed recently to the frequency-domain stability analysis methods. One of first papers on such analysis is Ref. [12] where the periodic vibrations of geometrically nonlinear slender mechanical structures are considered. A frequency-domain stability formulation for a Jeffcott rotor was proposed in Ref.[13].

The problems of using the multiharmonic balance method (MHB) in the stability analysis and issues which arise due to the use of many harmonics in the forced response representation were first discussed in the papers devoted to the electrical circuit calculations, e.g. see Refs.[14]-[17]. Some of the ideas reported in these papers were adopted and further developed for the analysis of mechanical systems and applied to nonlinear structures in Refs.[18] - [20]. The size of the models considered in these papers is relatively small: e.g. one-DOF system is analysed in Ref.[19] and 37 DOFs – in Ref.[20].

In the proposed paper, a methodology is developed for the multiharmonic analysis of stability of nonlinear steady-state forced response vibrations in large-scale structures which can have millions DOFs in their finite element models.

First, general multiharmonic stability equations are derived. The equations allow for the explicit dependency of the nonlinear forces on the velocities and accelerations of structural DOFs – the necessity of such formulation has been overlooked so far and all previously known formulations assume implicitly the dependency of the nonlinear forces only on the displacements.

Then, in addition to the commonly known quadratic eigenproblem, a simplified linear eigenvalue problem is formulated for the stability analysis.

An effective method for reduced modelling in the stability analysis is proposed. The method allows using directly the results obtained from the nonlinear forced response analysis. The latter can be based on any type of the model reduction: e.g. using the method proposed in Ref.[21] or any approach from the family of component mode synthesis methods.

The frequency-domain stability equations have been formulated for two major types of damping in linear components of the structure analysed: (i) viscous damping and (ii) structural damping. The introduction of the structural, frequency-independent damping is also a new development: till now only structures with viscous damping have been considered in the stability analysis.

For a first time an approach has been developed and validated for highly-accurate frequency-domain calculation of a time-domain motion of a structure after it is perturbed from the found periodic nonlinear forced response and cases of stable and unstable periodic regimes are considered here.

Major features and numerical properties of the methodology developed are demonstrated on a set of numerical examples, which include a simple 1 DOF nonlinear system and finite element models comprising up to 160,000 DOFs.

FREQUENCY-DOMAIN NONLINEAR FORCED RESPONSE ANALYSIS

The equation of motion for the forced vibrations of a structure with nonlinear interactions at joints can be written in the form:

$$\mathbf{K}\mathbf{x} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{M}\ddot{\mathbf{x}} + \mathbf{f}(\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}}) = \mathbf{p}(t) \quad (1)$$

where $\mathbf{x}(t)$ is a vector of displacements for all degrees of freedom in the structure considered; $\mathbf{K}$, $\mathbf{C}$ and $\mathbf{M}$ are structural stiffness, damping and mass matrices of finite element (FE) model of a structure and $\mathbf{p}(t)$ is a vector of excitation forces. For bladed discs and other structures mounted on a rotating rotor, the stiffness matrix can include terms accounting for the rotation effects (e.g. Coriolis forces, gyroscopic moments, rotation stiffening effects) and, therefore, is generally dependent on the rotation speed, Ω . $\mathbf{f}(\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}})$ is a vector of nonlinear contact interface forces which, for a general case considered here, can be explicitly dependent on displacements, $\mathbf{x}$, velocities, $\dot{\mathbf{x}}$, and accelerations, $\ddot{\mathbf{x}}$, of the structural components. The contact forces occur in gas-turbine structures at the blade root joints of bladed discs, at contact surfaces of underplatform or tip dampers, at contact surfaces of adjacent interlock shrouds and at rubbing contacts between rotor and casing. The causes of nonlinear behaviour are usually friction forces, unilateral interaction at the pairing contact surfaces, gaps, varying contact stiffness properties, as in the case of Hertzian contacts, etc.

A case of periodic excitation forces is considered: $\mathbf{p}(t) = \mathbf{p}(t + 2\pi/\omega)$, where ω is the principal excitation frequency and the steady-state periodic oscillations are studied. A general case is studied here, when the principal excitation frequency may not be related to the rotation speed, Ω , although for the majority of dynamic problems in gas-turbine engines the principal excitation frequency is equal or multiple of the rotation speed: $\omega = k\Omega$ ($k=1, 2, \dots$).

The time variation of displacements, the nonlinear contact forces and the periodic excitation forces for the steady-state periodic regimes with known period $T = 2\pi/\omega$ are represented by a restricted Fourier series:

$$\mathbf{x}(t) = \mathbf{X}_0 + \sum_{j=1}^n (\mathbf{X}_j^c \cos m_j \omega t + \mathbf{X}_j^s \sin m_j \omega t) = \mathbf{H}^T \mathbf{X} \quad (2)$$

$$\mathbf{f}(\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}}) = \mathbf{H}^T \mathbf{F}; \quad \mathbf{p}(t) = \mathbf{H}^T \mathbf{P} \quad (3)$$

where $\mathbf{H} = [\mathbf{I}, \mathbf{I} \cos \omega_1 t, \mathbf{I} \sin \omega_1 t, \dots, \mathbf{I} \sin \omega_n t]^T$; $\omega_j = m_j \omega$; $\mathbf{X} = \{\mathbf{X}_0, \mathbf{X}_1^c, \dots, \mathbf{X}_n^s\}^T$, $\mathbf{F} = \{\mathbf{F}_0, \mathbf{F}_1^c, \dots, \mathbf{F}_n^s\}^T$; $\mathbf{P} = \{\mathbf{P}_0, \mathbf{P}_1^c, \dots, \mathbf{P}_n^s\}^T$. The total number, N_x , of the harmonic coefficients combined in each of vectors, $\mathbf{X}$, $\mathbf{F}$ and $\mathbf{P}$ is $N_x = N(2n+1)$, where N is the total number of DOFs in the structure and $\mathbf{I}(N \times N)$ is the identity matrix. The harmonic numbers, m_j , and their total number, n , are chosen to provide sufficiently accurate description of the periodic motion calculated. For a case when major and superharmonic vibrations are calculated, the harmonic numbers, m_j , are integer numbers. For a case when subharmonic vibrations have to be analysed, some of the harmonic numbers can be represented by fractional numbers which provide the possibility to calculate periodic nonlinear responses with the vibration period larger than the period of the

excitation forces. Applying the standard harmonic balance procedure gives the following nonlinear equation with respect to the vector of harmonic coefficients, $\mathbf{X}$:

$$\mathbf{R}(\mathbf{X}) = [\tilde{\mathbf{K}} + \tilde{\mathbf{C}}\mathbf{E}_1 - \tilde{\mathbf{M}}\mathbf{E}_2] \mathbf{X} + \mathbf{F}(\mathbf{X}) = \mathbf{P} \quad (4)$$

where $\tilde{\mathbf{M}} = \text{diag}(\mathbf{M}, \dots, \mathbf{M})$; $\tilde{\mathbf{C}} = \text{diag}(\mathbf{C}, \dots, \mathbf{C})$; $\tilde{\mathbf{K}} = \text{diag}(\mathbf{K}, \dots, \mathbf{K})$;

$$\mathbf{E}_1 = \text{diag} \left(\mathbf{0}, \omega_1 \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix}, \dots, \omega_n \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix} \right) \quad (5)$$

$$\mathbf{E}_2 = \text{diag}(\mathbf{0}, \omega_1^2 \mathbf{I}, \omega_1^2 \mathbf{I}, \omega_2^2 \mathbf{I}, \omega_2^2 \mathbf{I}, \dots, \omega_n^2 \mathbf{I}) \quad (6)$$

The nonlinear equation is solved by the Newton-Raphson iterations:

$$\left[\tilde{\mathbf{K}} + \tilde{\mathbf{C}}\mathbf{E}_1 - \tilde{\mathbf{M}}\mathbf{E}_2 + \frac{\partial \mathbf{F}(\mathbf{X})}{\partial \mathbf{X}} \right] (\mathbf{X}^{(k)} - \mathbf{X}^{(k-1)}) = \mathbf{R}(\mathbf{X}^{(k-1)}) \quad (7)$$

where k is the iteration number.

STABILITY EQUATIONS IN TIME DOMAIN

The equation describing behaviour of the nonlinear dynamic system under small variations, $\delta \mathbf{x}$, of the displacements in vicinity of the found solution, $\mathbf{x}^*$ can be obtained by the substitution of the perturbed motion, $\mathbf{x} = \mathbf{x}^* + \delta \mathbf{x}$ in Eq.(1) and, then, by its linearization with respect to displacement perturbations, $\delta \mathbf{x}$, and its time derivatives $\delta \dot{\mathbf{x}}$ and $\delta \ddot{\mathbf{x}}$, we obtain:

$$\mathbf{M} \delta \ddot{\mathbf{x}} + \mathbf{C} \delta \dot{\mathbf{x}} + \mathbf{K} \delta \mathbf{x} + \frac{\partial \mathbf{f}^*}{\partial \mathbf{x}} \delta \mathbf{x} + \frac{\partial \mathbf{f}^*}{\partial \dot{\mathbf{x}}} \delta \dot{\mathbf{x}} + \frac{\partial \mathbf{f}^*}{\partial \ddot{\mathbf{x}}} \delta \ddot{\mathbf{x}} = \mathbf{0} \quad (8)$$

where the superscript ‘*’ indicates that the derivatives of the nonlinear forces are calculated for a found solution, i.e. $\partial \mathbf{f}^* / \partial \mathbf{x} = \partial \mathbf{f}(\mathbf{x}^*) / \partial \mathbf{x}$, etc. This equation is an autonomous linear differential equation where the matrices of the nonlinear forces’ derivatives are varying periodically over time. It is known that the solution of this equation can be represented as a product of some periodic function, $\mathbf{s}(t) = \mathbf{s}(t+T)$, which has the same period, T , as the found solution of Eq.(1), and an exponential function of time, i.e.

$$\delta \mathbf{x} = e^{\lambda t} \mathbf{s} \quad (9)$$

and therefore:

$$\delta \dot{\mathbf{x}} = e^{\lambda t} (\lambda \mathbf{s} + \dot{\mathbf{s}}); \quad \delta \ddot{\mathbf{x}} = e^{\lambda t} [\lambda^2 \mathbf{s} + 2\lambda \dot{\mathbf{s}} + \ddot{\mathbf{s}}] \quad (10)$$

Substituting Eqs.(9)-(10) in Eq.(8) we obtain the equation allowing the determination of the stability of the found solution:

$$\left[\mathbf{M} + \frac{\partial \mathbf{f}^*}{\partial \ddot{\mathbf{x}}} \right] (\lambda^2 \mathbf{s} + 2\lambda \dot{\mathbf{s}} + \ddot{\mathbf{s}}) + \left[\mathbf{C} + \frac{\partial \mathbf{f}^*}{\partial \dot{\mathbf{x}}} \right] (\lambda \mathbf{s} + \dot{\mathbf{s}}) + \left[\mathbf{K} + \frac{\partial \mathbf{f}^*}{\partial \mathbf{x}} \right] \mathbf{s} = \mathbf{0} \quad (11)$$

In many practical problems it is possible to assume that the rate of growth (for an unstable solution) or decay (for a stable solution) of amplitudes is small relatively to the variation of displacements over a period: $|\lambda \mathbf{s}| \ll |\dot{\mathbf{s}}|$. For this case the expressions for the time-derivatives of the perturbed motion can be simplified:

$$\delta \dot{\mathbf{x}} \approx e^{\lambda t} \dot{\mathbf{s}}; \quad \delta \ddot{\mathbf{x}} \approx e^{\lambda t} (2\lambda \dot{\mathbf{s}} + \ddot{\mathbf{s}}) \quad (12)$$

And the stability equation is simplified to the form:

$$\left[\mathbf{M} + \frac{\partial \mathbf{f}^*}{\partial \ddot{\mathbf{x}}} \right] (2\lambda \dot{\mathbf{s}} + \ddot{\mathbf{s}}) + \left[\mathbf{C} + \frac{\partial \mathbf{f}^*}{\partial \dot{\mathbf{x}}} \right] \dot{\mathbf{s}} + \left[\mathbf{K} + \frac{\partial \mathbf{f}^*}{\partial \mathbf{x}} \right] \mathbf{s} = \mathbf{0} \quad (13)$$

Eqs. (11) and (13) are eigenvalue equations which have to be solved to obtain the eigenvalues λ and the eigenvectors $\mathbf{s}(t)$. One can notice also that Eq. (11) constitutes a quadratic eigenvalue problem and Eq. (13) constitutes a linear eigenproblem. Naturally, the latter can be solved with significantly smaller computational expense.

STABILITY EQUATIONS IN FREQUENCY-DOMAIN

In order to solve the stability equations formulated above we will represent the eigenvector function, $\mathbf{s}$, using the following multiharmonic expansion:

$$s(t) = S_0 + \sum_{j=1}^{n_s} (S_j^c \cos k_j \omega t + S_j^s \sin k_j \omega t) = H_s^T S \quad (14)$$

where $S (N_s \times 1)$ is the vector of multiharmonic expansion of the periodic eigenvector function and $N_s = N(2n_s + 1)$. It should be noticed that the total number of harmonics, n_s , and the harmonic numbers, k_j , involved in this expansion can be chosen differently from those used for the calculation of the periodic solution of Eq.(1), because of this the vector of harmonic functions, H_s , can generally be different from H . Often the number of harmonics kept in the eigenvector expansion of Eq.(14) can be smaller than such a number used for the search of the periodic solution given by Eq.(2). Let us consider formulation of the stability frequency-domain equations for both eigenvalue problems formulated in the previous section.

1) For the quadratic eigenproblem Eq.(14) is substituted in Eq.(11) and then applying the harmonic balance procedure we obtain the stability equation in the frequency domain:

$$\lambda^2 B_2 S + \lambda B_1 S + B_0 S = 0 \quad (15)$$

This is a quadratic eigenproblem equation with respect to the vector of harmonic coefficients of the perturbed motion, S and the matrices involved here have the form:

$$B_0 = \frac{2}{T} \int_0^T H_s^* K H_s^T dt + \frac{2}{T} \int_0^T H_s^* C \dot{H}_s^T dt + \frac{2}{T} \int_0^T H_s^* M \ddot{H}_s^T dt + \frac{2}{T} \int_0^T H_s^* \left[\frac{\partial f^*}{\partial x} H_s^T + \frac{\partial f^*}{\partial \dot{x}} \dot{H}_s^T + \frac{\partial f^*}{\partial \ddot{x}} \ddot{H}_s^T \right] dt = \tilde{K} + \tilde{C} E_1 - \tilde{M} E_2 + \hat{K} \quad (16)$$

$$B_1 = \frac{2}{T} \int_0^T \left(2 H_s^* \left[M + \frac{\partial f^*}{\partial \dot{x}} \right] \dot{H}_s^T + H_s^* \left[C + \frac{\partial f^*}{\partial x} \right] H_s^T \right) dt = 2 \tilde{M} E_1 + 2 \hat{M}_1 + \tilde{C} + \hat{C} \quad (17)$$

$$B_2 = \frac{2}{T} \int_0^T \left[H_s^* M \ddot{H}_s^T + H_s^* \frac{\partial f^*}{\partial \ddot{x}} \ddot{H}_s^T \right] dt = \tilde{M} + \hat{M}_2 \quad (18)$$

where matrix H_s^* used in the harmonic balance procedure differs from H_s only by a multiplier 0.5 for the components corresponding to zero harmonic (see Ref.[3]). The terms corresponding to the nonlinear forces are indicated by a cap ' $\wedge$ ' above the symbol and they are defined as:

$$\hat{K} = \frac{\partial F}{\partial X} = \frac{2}{T} \int_0^T \left(H_s^* \frac{\partial f^*}{\partial x} H_s^T + H_s^* \frac{\partial f^*}{\partial \dot{x}} \dot{H}_s^T + H_s^* \frac{\partial f^*}{\partial \ddot{x}} \ddot{H}_s^T \right) dt \quad (19)$$

$$\hat{C} = \frac{2}{T} \int_0^T H_s^* \frac{\partial f^*}{\partial \dot{x}} H_s^T dt \quad (20)$$

$$\hat{M}_1 = \frac{2}{T} \int_0^T H_s^* \frac{\partial f^*}{\partial \ddot{x}} \dot{H}_s^T dt; \quad \hat{M}_2 = \frac{2}{T} \int_0^T H_s^* \frac{\partial f^*}{\partial \ddot{x}} H_s^T dt \quad (21)$$

2) For the linear eigenvalue problem Eq.(14) is substituted in Eq.(13) and then we obtain the following equation:

$$\lambda \left[2 \tilde{M} E_1 + 2 \hat{M}_1 \right] S + B_0 S = 0 \quad (22)$$

It can be seen that for the linear eigenvalue problem formulation reduces by factor of two the size of the eigenproblem. Moreover, there is another advantage: there is no necessity to calculate any of the additional matrices $\hat{C}$ and $\hat{M}_2$ and all necessary information about the nonlinear behaviour is provided by the nonlinear stiffness matrix, $\hat{K}$, and, for rare cases when the nonlinear forces are dependent on the accelerations, $\hat{M}_1$.

The stiffness matrix, $\hat{K}$, is calculated during the Newton-Raphson solution of Eq.(4) and, if the set of harmonic numbers used for the stability assessment, $\{k_j\}_1^{n_s}$, is a subset or equal to the harmonic numbers $\{m_j\}_1^n$ used in the solution of the equation of periodic motion, then matrix, $\hat{K}$, is readily available. It can be simply extracted from the matrix calculated at the last iteration of the Newton-Raphson iterative solution in Eq.(7). In the relatively rare cases when additional harmonics have to be added for the stability analysis this matrix is calculated specially for the harmonics included in the stability analysis. In both cases this matrix can be calculated with high accuracy and computational speed using analytical expressions for friction, gaps, Hertzian contacts and other nonlinear contact interactions as shown e.g. in Refs.[3] and [21]. Its calculation does not require any special development, since it can be calculated using the same nonlinear contact interface elements developed for

the periodic response analysis which allow the calculation of the interaction forces and their derivatives with respect to the harmonic coefficients of displacements.

It is important to notice terms $\hat{\mathbf{C}}$, $\hat{\mathbf{M}}_1$ and $\hat{\mathbf{M}}_2$, which appear as a result of the dependency of the nonlinear forces on velocities and accelerations. The necessity of these terms has not been noticed and mentioned in the literature so far although, for example, the nonlinear damping dependent explicitly on the velocity is rather common in the machinery structures. If such nonlinear forces are present in a system, then to assess the stability of nonlinear vibrations, these matrices have to be efficiently calculated. This requires a special development for the nonlinear contact interface elements used for the nonlinear force modelling.

REDUCED MODELLING IN STABILITY ANALYSIS

The large finite element models are customarily used for modelling of structures in industrial applications. These models can contain $10^5 - 10^6$ DOFs and such models can be used for the analysis of nonlinear forced response only together with model reduction techniques. A highly effective model reduction method developed in Ref.[21] allows the exclusion of all linear degrees of freedom (i.e. DOFs where there is no nonlinear interactions) from the resolving equations without loss of accuracy of the nonlinear dynamics analysis. The method allows efficient condensation of the model by reduction of the size for the resolving equation and allows to expand the solution back to the full-sized model.

The tangent stiffness matrix of the nonlinear interactions, $\hat{\mathbf{K}} = \partial \mathbf{F} / \partial \mathbf{X}$, required for stability assessment, is obtained as a by-product of the solution obtained for the reduced model. The matrices $\hat{\mathbf{C}}$, $\hat{\mathbf{M}}_1$ and $\hat{\mathbf{M}}_2$, can be calculated from multiharmonic displacements, $\mathbf{X}$, obtained as a result of the solution of Eq.(4). These matrices are evaluated when forces are explicitly depended on the velocities and accelerations.

The eigenvalue problems formulated in Eqs. (15) and (22) require, for the estimate of solution stability, determination of all eigenvalues. So, this problem is much more time consuming than the solution of the nonlinear equations of motion and, hence, it is not practical to solve it directly for full finite element models due to very large computational time. Therefore, the model reduction for such models becomes a must in the stability analysis.

We will derive the reduced stability equations in the form allowing using the nonlinear solutions obtained from reduced models without a need of returning to a full-sized model. In order to perform such a reduction of the stability equation we will represent the amplitudes of the perturbed motion over the mode shapes of a linear structure:

$$\underset{(N_s \times 1)}{\mathbf{S}} = \underset{(N_s \times N_\phi)}{\tilde{\Phi}} \underset{(N_\phi \times 1)}{\mathbf{c}} \quad (23)$$

where $N_\phi = m(2n_s + 1)$ and m is the number of mode shapes used for the reduction of model. $\tilde{\Phi} = \text{diag}(\underset{N \times m}{\Phi}, \underset{N \times m}{\Phi}, \dots, \underset{N \times m}{\Phi})$; Φ is a matrix of mass-normalised mode shapes of a linear structure calculated for a structure with the absence of the contact interactions by the solution of the following auxiliary eigenproblem:

$$\mathbf{K}\Phi = \mathbf{M}\Phi\Omega^2 \quad (24)$$

where $\Omega = \text{diag}(\omega_1^*, \omega_2^*, \dots, \omega_m^*)$ is the matrix consisting of natural frequencies, ω_j^* , corresponding to the mode shape vectors in Φ . Substituting Eq.(23) in Eqs.(15) and (22) and then projecting these equations on the selected mode shapes, $\tilde{\Phi}$, gives us the following reduced stability equations:

- for the quadratic stability equation:

$$\lambda^2 \mathbf{B}_2^r \mathbf{c} + \lambda \mathbf{B}_1^r \mathbf{c} + \mathbf{B}_0^r \mathbf{c} = \mathbf{0} \quad (25)$$

- for the linear stability equation:

$$\lambda \left[2\mathbf{E}_1^r + 2\tilde{\Phi}^T \hat{\mathbf{M}}_1 \tilde{\Phi} \right] \mathbf{c} + \mathbf{B}_0^r \mathbf{c} = \mathbf{0} \quad (26)$$

where the matrices are expressed in the form:

$$\mathbf{B}_0^r = \tilde{\Phi}^T \left[\tilde{\mathbf{K}} + \tilde{\mathbf{C}}\mathbf{E}_1 - \tilde{\mathbf{M}}\mathbf{E}_2 + \hat{\mathbf{K}} \right] \tilde{\Phi} = \tilde{\Omega}^2 + \tilde{\mathbf{C}}^r \mathbf{E}_1^r - \mathbf{E}_2^r + \tilde{\Phi}^T \hat{\mathbf{K}} \tilde{\Phi} \quad (27)$$

$$\mathbf{B}_1^r = \tilde{\Phi}^T \left[2\tilde{\mathbf{M}}\mathbf{E}_1 + 2\hat{\mathbf{M}}_1 + \tilde{\mathbf{C}} + \hat{\mathbf{C}} \right] \tilde{\Phi} = 2\mathbf{E}_1^r + \tilde{\mathbf{C}}^r + 2\tilde{\Phi}^T \hat{\mathbf{M}}_1 \tilde{\Phi} + \tilde{\Phi}^T \hat{\mathbf{C}} \tilde{\Phi} \quad (28)$$

$$\mathbf{B}_2^r = \tilde{\Phi}^T \left[\tilde{\mathbf{M}} + \hat{\mathbf{M}}_2 \right] \tilde{\Phi} = \tilde{\mathbf{I}}^r + \tilde{\Phi}^T \hat{\mathbf{M}}_2 \tilde{\Phi} \quad (29)$$

$\tilde{\Omega} = \text{diag}(\Omega, \Omega, \dots, \Omega)$; $\tilde{\mathbf{I}}^r$ is the identity matrix of size $(N_\phi \times N_\phi)$ and expressions for matrices $\mathbf{E}_1^r$ and $\mathbf{E}_2^r$ are similar to those given in Eqs.(5) and (6), but the size of the identity matrix, $\mathbf{I}$, used in these expressions is reduced to $(m \times m)$.

Moreover, owing to the fact that the stability analysis is performed in the frequency domain we can consider two cases of damping modelling: (i) viscous damping and (ii) structural frequency-independent damping.

(i) For the viscous damping we express the damping matrix, $\mathbf{C}$, in a such way that at the natural frequency of each j -th mode, ω_j^* , it provides the prescribed value of modal damping, η_j , i.e.

$$\Phi^T \mathbf{C} \Phi = \mathbf{E} \Omega \quad \text{and} \quad \mathbf{C} = \mathbf{M} \Phi \mathbf{E} \Omega \Phi^T \mathbf{M} \quad (30)$$

where $\mathbf{E} = \text{diag}(\eta_1, \eta_2, \dots, \eta_m)$ is the matrix of modal damping factors. For this type of viscous damping, the expression for the reduced linear damping matrix takes the form:

$$\tilde{\mathbf{C}}^r = \text{diag}(\mathbf{E} \Omega, \mathbf{E} \Omega, \dots, \mathbf{E} \Omega) \quad (31)$$

(ii) For the structural damping we can assume that the damping matrix is proportional to the stiffness matrix and ensure that the damping is frequency-independent by dividing the matrix by the excitation frequency:

$$\mathbf{C} = (\eta/\omega) \mathbf{K} \quad (32)$$

and the expression for the reduced linear damping matrix takes the form:

$$\tilde{\mathbf{C}}^r = (\eta/\omega) \text{diag}(\Omega^2, \Omega^2, \dots, \Omega^2) \quad (33)$$

It is possible to consider also a more general case of the frequency independent damping: when the modal damping factors are provided for all modes individually. Assuming that the damping matrix allows the diagonalization by the mode shapes, i.e.

$$\Phi^T \mathbf{C} \Phi = \omega^{-1} \mathbf{E} \Omega^2 \quad (34)$$

we can obtain the reduced frequency-independent damping matrix in the form:

$$\tilde{\mathbf{C}}^r = \omega^{-1} \text{diag}(\mathbf{E} \Omega^2, \mathbf{E} \Omega^2, \dots, \mathbf{E} \Omega^2) \quad (35)$$

CALCULATION AND CHOICE OF STABILITY FACTORS

The solution of eigenvalue problems does not represent a difficulty since standard eigenvalue solvers are available in multitude of public domain and commercial codes. The linear eigenproblem given by Eq.(26) is a generalised eigenproblem which can be solved directly by those codes. The quadratic eigenproblem of Eq.(25) can be transformed to the linear eigenproblem by introduction of the new vector $\mathbf{c}_\lambda = \lambda \mathbf{c}$ and the resolving equation takes the form:

$$\begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{B}_0 & \mathbf{B}_1 \end{bmatrix} \begin{Bmatrix} \mathbf{c} \\ \mathbf{c}_\lambda \end{Bmatrix} = \lambda \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{B}_2 \end{bmatrix} \begin{Bmatrix} \mathbf{c} \\ \mathbf{c}_\lambda \end{Bmatrix} \quad (36)$$

To assess the stability, all eigenvalues, λ_j , of these equations have to be calculated. Since the matrices of the considered eigenproblems comprise only real numbers, the eigenvalues and corresponding them eigenvectors can be: (i) real numbers and (ii) complex numbers. For the latter case, the eigenvalues form pairs since each complex eigenvalue and vector has its complex-conjugate counterpart. The eigenvalues λ_j are exponent values in Eq.(9) and, hence, their real parts define the growth or decay of amplitudes. If any one of the eigenvalues has a positive real part then the found periodic motion is unstable.

The total number of such eigenvalues is $m(2n_s + 1)$ - for the linear eigenproblem and $2m(2n_s + 1)$ - for the quadratic eigenproblem, although only $2m$ of them need to be involved in the assessment of the stability since the Floquet theory requires determination of only $2m$ multipliers. For the choice of the eigenvalues that correspond to 'genuine' Floquet multipliers in Ref.[20] suggested to choose $2m$ eigenvalues with the smallest absolute values of their imaginary parts and in Ref.[19] it is suggested to analyse the found eigenvectors and to choose those with the most symmetric shapes. Our experience indicates that in many cases for systems with one or several DOFs, these approaches can be applied successfully, although should be used with some caution for complex structures where the eigenvalues corresponding to lower harmonics can sometimes have imaginary parts larger than the eigenvalues corresponding to higher harmonics. For the choice of the stability factors a special algorithm has been developed and used in this paper.

CALCULATION OF A PERTURBED MOTION

The stability analysis method developed here allows not only to answer to the question whether the found vibration regime is stable, but also to obtain, with very high accuracy, a law of the motion in time domain of a structure after it is perturbed from the found periodic nonlinear forced response. For a case of a stable vibration regime the perturbed motion is decayed back to this regime and for an unstable regime the perturbed motion is exponentially grows till it falls within the region of attraction of another stable nonlinear vibration regime and then starts to vibrate in accordance to this stable regime.

In order to calculate such perturbed process, first, let us consider the full DOF models containing N DOFs. Since we can obtain from solution of Eq.(15) or Eq.(22) eigenvalues λ_j and corresponding them eigenvectors $\mathbf{S}^{(j)}$ then using Eq.(9) we can express the time variation for all DOFs of the structure in the form:

$$\delta \mathbf{x}(t) = \sum_{j=1}^{2N} d_j e^{\lambda_j t} \mathbf{H}_s^T(t) \mathbf{S}^{(j)} \quad (37)$$

where the coefficients of expansion, d_j , over the eigenvectors, $\mathbf{S}^{(j)}$, can be obtained from the initial conditions which describe the initial perturbation. Without loss of generality we can assume that the vectors of initial displacements, $\delta \mathbf{x}_0$, and velocities, $\delta \dot{\mathbf{x}}_0$ are provided at time instant $t=0$ then from Eq.(37) we obtain the equation for determination of all coefficients, d_j :

$$\delta \mathbf{x}_0 = \sum_{j=1}^{2N} d_j \mathbf{H}_s^T(0) \mathbf{S}^{(j)} ; \delta \dot{\mathbf{x}}_0 = \sum_{j=1}^{2N} d_j [\lambda_j \mathbf{H}_s^T(0) + \dot{\mathbf{H}}_s^T(0)] \mathbf{S}^{(j)} \quad (38)$$

The solution of this equation does not present a problem and then Eq.(37) provides an explicit expression for the perturbed motion of a structure. It should be noted that λ_j and $\mathbf{S}^{(j)}$ in Eq.(38) have generally complex values. However, owing to the fact that all eigenvectors and eigenvalues form complex conjugate pairs or (if they are not complex conjugate) have real values, the coefficients d_j provide real values for displacements and velocities not only at the initial time instant but also for the whole process of the perturbed motion.

For the reduced stability equation the perturbed motion is expressed in the form:

$$\delta \mathbf{x}(t) = \sum_{j=1}^{2m} d_j e^{\lambda_j t} [\mathbf{H}_s^T(t) \tilde{\Phi}] \mathbf{c}^{(j)} \quad (39)$$

Since the number of modes used in the reduced stability equation is usually much smaller than the number of DOFs in the full model (i.e. $m \ll N$) the initial conditions cannot be satisfied for full vectors $\delta \mathbf{x}_0$ and $\delta \dot{\mathbf{x}}_0$, hence, two approaches can be applied:

- (i) to use Eq.(39) together with Eq.(38) but to select only $2m$ DOFs where the initial conditions have to be satisfied and
- (ii) to represent the initial conditions using the modal basis, i.e. $\delta \mathbf{x}_0 = \Phi \mathbf{c}_0$ and $\delta \dot{\mathbf{x}}_0 = \Phi \dot{\mathbf{c}}_0$ and, then, to apply the initial conditions directly to the modal coordinates vectors by providing, $\mathbf{c}_0$ and $\dot{\mathbf{c}}_0$ and to determine d_j from the following equations:

$$\mathbf{c}_0 = \sum_{j=1}^{2m} d_j \mathbf{H}_{s\Phi}^T(0) \mathbf{c}^{(j)} ; \dot{\mathbf{c}}_0 = \sum_{j=1}^{2m} d_j [\lambda_j \mathbf{H}_{s\Phi}^T(0) + \dot{\mathbf{H}}_{s\Phi}^T(0)] \mathbf{c}^{(j)} \quad (40)$$

where $\mathbf{H}_{s\Phi} (N_\Phi \times N_\Phi)$ is the matrix of harmonic expansion of the modal coordinates, $\mathbf{c}(t)$, which is similar to matrix $\mathbf{H}_s$. The sizes of matrices and vectors in the second approach is much smaller because all calculations are performed in the reduced basis which comprises $2m$ vectors.

NUMERICAL EXAMPLES

The methodology developed here has been introduced in a computer code which demonstrated its efficiency for all cases analysed by the author. Some of examples of stability analysis of structures are given below.

A simple model with different nonlinearity types

First example is a one-degree of freedom model described by the Duffing equation:

$$m\ddot{x} + c\dot{x} + kx + f(x) = p \sin \omega t \quad (41)$$

where $k = 40$; $m = 1$; $c = 0.1/\sqrt{40}$; $p = 100$ and $f(x)$ is the nonlinear force. This simple system allows exploring some basis properties of the method and validating the method by comparison of results with the time integration solutions. In the multiharmonic balance method the maximum number of the harmonic used for the displacement representation is 10, with all harmonic numbers included in the analysis: from 0 to 10. The number of non-zero harmonics kept in the stability equations is varied for some cases from 1 to 10. It should also note that since we have here a system with one DOF there are two stability factors defining the periodic vibration regime stability. These two stability factors are selected from the total number of the eigenvalues obtained from the solution of the stability eigenproblem. The total number of eigenvalues, for a case of using 10 harmonics in the stability equation, is 42 – for quadratic eigenproblem and 21 – for linear eigenproblem. The accuracy of the

stability assessment is explored for: (i) different numbers of harmonics in the stability equation; (ii) using the linear and quadratic stability equations; (iii) the determination of a perturbed motion. In the following examples we explored three major types of the nonlinear forces: (i) cubic spring, (ii) gap and (iii) friction force.

Cubic nonlinear spring. For the cubic spring nonlinearity the nonlinear force function is assumed: $f(x) = 10x^3$. The dependency of the maximum forced response on the excitation frequency is shown in **Fig. 1** where blue curve indicates stable vibration regimes and red curve corresponds to unstable vibrations.

Fig. 1 Dependency of maximum displacement on the excitation frequency

The dependencies of the real and imaginary parts of the stability factors obtained for the case when all harmonics from 0 to 10 are included are shown in **Fig. 2** and **Fig. 3** respectively. In these two figures the values of the stability factors obtained from the quadratic and linear eigenproblems are compared. The real parts of the stability factors characterise the rate of exponential growth (for positive values) or decay (for negative values) of the perturbed motion and the imaginary parts characterise the frequency of the perturbed motion. One can see that although there is some discrepancy in the values of stability factors but, as can be seen from **Fig. 2**, the detection of the loss of stability can be efficiently performed using the linear eigenproblem.

Fig. 2 Frequency dependency of the real parts of the stability factors

Fig. 3 Frequency dependency of the imaginary parts of the stability factors

The inclusion of large number of harmonics in the stability equation can increase the calculation time for the solution of the full eigenvalue problem. The effect of reducing the number of harmonics in the stability equation on the accuracy of the stability prediction is illustrated in **Fig. 4** and **Fig. 5** where the highest harmonic number included in the stability analysis is indicated in the legend. It is evident that the stability prediction can be made using the number of harmonics much smaller than the number of harmonics used in the nonlinear forced response calculation: for the case considered here even one harmonics provides sufficient accuracy.

The accuracy of the prediction of the time-domain motion of a structure after it is perturbed from the found periodic nonlinear forced response was validated by comparison of the perturbed motion obtained from Eq.(39) and the results obtained by time integration of the equation of the perturbed motion using Runge-Kutta method. The comparison of perturbed motions obtained by these two approaches are shown in **Fig. 6 - Fig. 8** for three solutions existing for excitation frequency 17rad/s.

Fig. 4 Real parts of the stability factors for different number of harmonics

Fig. 5 Imaginary parts of the stability factors for different number of harmonics

Two of these solutions are stable solutions and one solution is unstable. The quadratic eigenproblem solutions of the frequency-domain stability equation and 10 harmonics are used here to find the perturbed motion. One can see an excellent agreement between the time and frequency domain perturbed motion. It should be noticed that the frequency of the perturbed motion differs from the excitation frequency. The time intervals where the proposed method give a highly accurate prediction of the perturbed motion is sufficiently large to be used in practical analysis of the perturbed motion of structures.

The effect of keeping different numbers of harmonics in the frequency-domain stability equation has been explored and it was found that the perturbed motion can be modelled with rather small number of harmonics included: starting from the inclusion of two harmonics the results become very close and undistinguishable when 3 harmonics and more are used. The linear stability eigenproblem solutions do not provide sufficient accuracy as can be expected from discrepancies in stability factors shown in **Fig. 2** and **Fig. 3**.

Fig. 6 Perturbed motion for 1st solution (stable) at 17 rad/s

Fig. 7 Perturbed motion for 2nd solution (unstable) at 17 rad/s

Fig. 8 Perturbed motion for 3rd solution (stable) at 17 rad/s

1) Gap nonlinearity. For the gap nonlinearity the nonlinear force function is assumed to be $f(x) = 400x$ for $x \geq 10$ and $f(x) = 0$ for $x < 10$. The forced response amplitudes are shown in **Fig. 9** and the real and imaginary parts of the stability factors are given in **Fig. 10**.

Fig. 9 Amplitudes of the simple system with a gap

Fig. 10 Stability factors for the simple system with a gap

One can see that the stability is lost through the appearance of the stability factors having real values. Moreover, the real part of the complex stability factors defining the decay of the perturbed motion stays constant for all frequencies where the gap is not closed and the system behaves as linear. The imaginary part, defining the frequency contents of the perturbed motion, increases linearly after the point where the forced response regime regains its stability after the unstable regime.

2) Friction nonlinearity. For the friction nonlinearity the model developed in Ref.[3] is used with the friction coefficient equal 0.3; the tangential contact stiffness is 100 and the normal force is 800. The calculated forced response is displayed in **Fig. 11** and the stability factors are shown in **Fig. 12**. It should be noted that for this type of the nonlinearity the appearance of the real values in the set of the stability factors for the frequency range [10,12rad/s] (where the friction force causes the energy dissipation) does not lead to the loss of stability, but to the significant increase of the perturbation decay rate. At the peak of the response curve the stability factors become again complex numbers and around 13 rad/s there is another very small frequency range with real-valued stability factors.

Fig. 11 Amplitudes of the simple system with friction

Fig. 12 Stability factors for the simple system with friction

Large finite element models

Two models with a large number of DOFs are used for the demonstration of stability analysis method developed here. Five first harmonics are included in the multiharmonic representation of displacements and in the stability analysis: from 1 to 5.

1st model is a cantilever block with sides 1000×200×100mm and with the following material properties: elasticity modulus $E=10^5 \text{ N/mm}^2$; density $\rho=4.43 \cdot 10^{-9} \text{ Mg/mm}^3$. The three-dimensional solid finite element model is shown in **Fig. 13a**. and the total number of DOFs in the model is 12500. The case of frequency-independent structural damping is considered and the modal damping factors of the block are assumed to be 0.02 for all mode shapes. The nonlinear elements are applied in the middle of the free black end and the excitation force is applied in the middle of the upper edge. The amplitude of the excitation force is 100N.

2nd model is a turbine blade with canals and orifices for air cooling. The finite element model of this blade is shown in **Fig.13b** and the total number of DOFs is 160,000. The blade is fixed at the blade root contact patches (marked in red in the figure); the nonlinear elements are distributed over a contact patch of the blade shroud (marked in green in the figure) and the excitation forces are distributed over blade airfoil surfaces. The damping is frequency-independent with all modal damping factors equal to 0.02.

Fig. 13 Examples of large-scale model analysed: a) a cantilever block; b) a cooled turbine blade

A cantilever block

1) Cubic nonlinearity. For this case one cubic nonlinear spring is applied with the stiffness coefficient 10^6 N/mm^3 . The nonlinear forced response is shown in **Fig. 14** and the real parts of the stability indicators are shown in **Fig. 15**.

Fig. 14 Amplitudes of the block with cubic nonlinearity

Fig. 15 Selected stability factors for the block with cubic nonlinearity: effect of the number of mode shapes

For this structure the reduced stability modelling has been applied with different number of mode shapes used in the reduced model, namely 6, 12 and 24. The total number of stability factors which have to be examined for the stability assessment were for each of these cases respectively: 12, 24 and 48. The two stability factors are selected from these sets to have the largest real parts and the real parts of these two factors are plotted in **Fig. 15** and used as the indicator of the vibration regime stability. One can see that when only 6 modes are used there is some discrepancy in the stability indicator but results obtained with 12 modes and 24 modes included in the reduced model are practically identical, so we can use only 12 modes to assess the stability of the considered model with 12,500 DOFs.

2) Friction nonlinearity. For this case one friction damper is attached to the free end of the block. The forced response and the stability indicators are plotted in **Fig. 16** and **Fig. 17**.

Fig. 16 Amplitudes of the block with friction nonlinearity

Fig. 17 Selected stability factors for the block with friction

For this case we can observe that even 6 modes provides the sufficient accuracy in the stability assessment. Moreover, we can observe the abrupt decrease of the stability indicator when the damper starts slipping and produce the additional energy dissipation. An example of a full set of eigenvalues obtained from the solution of the quadratic eigenproblem for different number of mode shapes kept in the reduced model is plotted in **Fig. 18** for the excitation frequency corresponding to the resonance peak: 85Hz. The number of all eigenvalues for three cases considered here are respectively: 120, 240 and 480. From this plot we can observe two important phenomena: (i) the best correspondence between the stability factors obtained with different numbers of modes is achieved for larger values of the stability factor real parts – i.e. for those that serve as the stability indicator; and (ii) the stability factors distribution over the complex plane is rather intricate and does not obey the simple relationships between the eigenvalues, the stability factors and the excitation frequency, as suggested for simple structures in Refs. [17], [19]

Fig. 18. All stability factors for a block with friction at 85Hz

A turbine blade

For the turbine blade, 22 cubic nonlinear spring are distributed over the contact nodes of the finite element model (see **Fig. 13b**). The calculated forced response is plotted in **Fig. 19** and the stability indicators are shown in **Fig. 20**.

Fig. 19 Amplitudes of the turbine blade with cubic nonlinearity

For this model the number of modes kept in the reduced stability equation are chosen: 12, 24 and 48 and the result show almost identical results for these reduced models which are used instead of the original 160,000 DOFs. An example of a full set of eigenvalues obtained from the solution of quadratic eigenproblem is plotted in **Fig. 21** for the normalised excitation frequency value 1.0. Again, although this case differs from the case plotted in **Fig. 18** by the presence of multitude of nonlinear contacts and another type of the nonlinearity, we can observe the same phenomenon a good correspondence between results obtained with different numbers of modes in the most important part: for larger values of the real part of the stability factors.

Fig. 20 Selected stability factors for the turbine blade: effect of the number of mode shapes

Fig. 21. All stability factors for the turbine blade at the normalised excitation frequency value 1.0

CONCLUSIONS

An efficient frequency-domain method has been developed for analysis of stability of periodic regimes of nonlinear forced vibrations for structures containing large numbers of degrees of freedoms.

The method includes the novel formulation of the frequency-domain stability equations which allow for the dependency of the nonlinear forces on the velocities and accelerations.

The formulations of the stability equations in the form of linear and quadratic eigenvalue problems are presented.

An effective approach for the substantial reduction of the size of the stability equation has been developed. The developed approach allows the stability analysis for realistic finite-element models of gas-turbine structures: containing possibly hundred thousand and million DOFs. It allows using directly the results obtained from the reduced equations used for the forced response analysis and the modal properties of the underlying linear structure.

An approach has been developed for highly-accurate frequency-domain calculation of a time-domain motion of a structure after it is perturbed from the found periodic nonlinear forced response: for cases of stable and unstable periodic regimes.

The efficiency of the method and new approaches are demonstrated on a simple system and on examples of large-scale structures with cubic nonlinear springs, gaps and friction nonlinearities.

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