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Converse theorems on contraction metrics for an equilibrium

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Abstract

The stability and basin of attraction of an equilibrium can be determined by a contraction metric. A contraction metric is a Riemannian metric with respect to which the distance between adjacent trajectories decreases. The advantage of a contraction metric over, e.g., a Lyapunov function is that the contraction condition is robust under perturbations of the system.

While the sufficiency of a contraction metric for the existence, stability and basin of attraction of an equilibrium has been extensively studied, in this paper we will prove converse theorems, showing the existence of several different contraction metrics. This will be useful to develop algorithms for the construction of contraction metrics.

Keywords: Stability; basin of attraction; contraction metric; converse theorem.
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1. Introduction

Stability and the basin of attraction of attractors are important properties for the analysis of nonlinear systems. One way of establishing stability and obtaining information about the basin of attraction is to use Lyapunov functions, which are scalar-valued functions, decreasing along solutions and thus measuring in some way the distance to the attractor. Lyapunov functions thus require some knowledge of the attractor and, moreover, are not robust under perturbations of the system.

A different way to study stability and attraction, which does not require knowledge about the attractor, is to compare adjacent trajectories with respect to a certain metric. This is a local property. We assume that the distance between adjacent trajectories decreases with respect to the metric, thus, the metric is called *contraction metric*. Then one can obtain information about the attractor and its basin of attraction. The advantage compared to Lyapunov functions is that no information about the attractor is required and the contraction metric is robust under perturbations of the system.

Contraction analysis can be used to study the distance between trajectories, without reference to an attractor, establishing (exponential) attraction of adjacent trajectories, see [21, 17]. This can be expanded to study the geodesic distance between solutions. For global attraction one needs to assume that the Riemannian metric is uniformly positive definite, see [1, 3]. We, however, will focus on the case of compact, positively invariant sets in this paper, where this assumption is not necessary.

Contraction analysis can be generalised to the study of a Finsler-Lyapunov function [6], which is a scalar-valued function on the tangential bundle $T\mathcal{M}$ of a smooth manifold $\mathcal{M}$, where $V(\mathbf{x}, \delta\mathbf{x})$ measures the distance between solutions through the point $\mathbf{x}$ and adjacent solutions in the tangential direction $\delta\mathbf{x}$.

If contraction to a trajectory through $\mathbf{x}$ occurs with respect to all adjacent trajectories, then solutions converge to an equilibrium point. If the attractor is, e.g., a periodic orbit, then contraction cannot occur in the direction tangential to the trajectories. Hence, contraction analysis for periodic orbits assumes contraction only to occur in a suitable $(n-1)$-dimensional subspace of the tangent space. This can be generalised to subspaces of other dimensions with applications to systems with symmetries, this is called horizontal contraction in [6].

Contraction metrics for periodic orbits have been studied by Borg [4] with the Euclidean metric and Stenström [27] with a general Riemannian metric. Further results using a contraction metric to establish existence, uniqueness, stability and information about the basin of attraction of a periodic orbit have been obtained in [18, 19, 22, 23, 24]. Generalisations to time-periodic [9], almost periodic [14] and non-smooth systems [10] have been made.

As in the case of Lyapunov functions, one is interested in the existence and construction of contraction metrics for a given system. While in the case of Lyapunov functions, converse theorems have been established, only few converse theorems for contraction metrics have been obtained, establishing the existence of a contraction metric. These include the case of periodic orbits in autonomous [8], time-periodic [9] and non-smooth systems [12]. [25, Theorem 3.5] gave a converse theorem, but the Riemannian metric $M(t, \mathbf{x})$ depends on t and, in general, can become unbounded as $t \to \infty$.

Constructive converse theorems, providing algorithms for the explicit construction of a contraction metric, are given in [3] for the global stability of an equilibrium in polynomial systems, using Linear Matrix Inequalities (LMI) and sums of squares (SOS). Meshless collocation, in particular Radial Basis Functions, has been used to construct a contraction metric for the basin of attraction of a periodic orbit in two-dimensional [13] and higher-dimensional systems [15]. Finally, an algorithm to construct a continuous piecewise affine (CPA) contraction metric for periodic orbits in time-periodic systems using semi-definite optimization has been proposed in [16].

Motivated by theoretical and computational studies of contraction metrics, we prove converse statements in this paper, showing the existence of contraction metrics. We consider a general autonomous ODE in $\mathbb{R}^n$ and restrict ourselves to the case that the attractor is an equilibrium. We consider Riemannian contraction metrics, described by a symmetric and positive definite matrix $M(\mathbf{x})$ for $\mathbf{x} \in G \subset \mathbb{R}^n$, which defines a point-dependent scalar product through $\langle \mathbf{v}, \mathbf{w} \rangle_{\mathbf{x}} = \mathbf{v}^T M(\mathbf{x}) \mathbf{w}$ for all $\mathbf{v}, \mathbf{w}$ of the tangent space at $\mathbf{x}$, which can be identified with $\mathbb{R}^n$.

We will study the case of compact as well as unbounded sets G. Note, however, that we do not prove converse theorems for uniformly positive definite contraction metrics; as a main application of these results will be the construction, we are more interested in compact sets, for which any positive definite contraction metric is uniformly positive definite. The results also focus on certain properties of the contraction metrics which will be useful for computation, such as smoothness of the metric or equations, which are satisfied by the metric, rather than inequalities.

Let us give an overview over this paper. After the definition of contraction metrics in Section 2, we give a short proof of the sufficiency in Section 3, i.e. a contraction metric gives information about the existence, uniqueness and exponential stability of an equilibrium, as well as its basin of attraction. In Section 4 we prove the main results of this paper, namely three converse theorems: In Theorem 4.1 we show the existence of a contraction metric on a compact subset of the basin of attraction, satisfying an inequality. In Theorems 4.2 and 4.4 we prove the existence of a contraction metric on the whole basin of attraction, which satisfies an equation for the minimal contraction (Theorem 4.2) or a matrix equation (Theorem 4.4), similar to a Lyapunov equation, respectively. The latter two results will be particularly useful for constructive methods. While the metric in Theorem 4.2 is only continuous and orbitally continuously differentiable, the metric in Theorem 4.4 is smooth. In Section 5 we calculate these metrics in examples and we conclude with a summary and an outlook in Section 6.

2. Contraction metric

Throughout the paper we will study the autonomous ODE

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{2.1}$$

where $\mathbf{f} \in C^1(\mathbb{R}^n, \mathbb{R}^n)$; further assumptions on the smoothness of $\mathbf{f}$ will be made later. The solution $\mathbf{x}(t)$ with initial condition $\mathbf{x}(0) = \boldsymbol{\xi}$ is denoted by $\mathbf{x}(t) =: S_t\boldsymbol{\xi}$ and is assumed to exist for all $t \geq 0$. Throughout the paper, $\|\mathbf{x}\|$ denotes the Euclidean norm of a vector $\mathbf{x} \in \mathbb{R}^n$ and $\|A\| = \sup_{\mathbf{x}\neq\mathbf{0}} \frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|}$ denotes the induced matrix norm.

Definition 2.1 (Riemannian metric). *Let $G \subset \mathbb{R}^n$. A Riemannian metric is a matrix-valued function $M \in C^0(G, \mathbb{S}^n)$, where $\mathbb{S}^n$ denotes the symmetric $n \times n$ matrices with real entries, such that $M(\mathbf{x})$ is positive definite for all $\mathbf{x} \in G$. Then $\mathbf{v}^T M(\mathbf{x})\mathbf{w}$ defines a scalar product for each $\mathbf{x} \in G$ and $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$.*

Furthermore, we assume that M is orbitally continuously differentiable with respect to (2.1), i.e.

$$M'(\mathbf{x}) = \frac{d}{dt} M(S_t\mathbf{x})\Big|_{t=0}$$

exists and is continuous. Note that a sufficient condition for this is that $M \in C^1(G, \mathbb{S}^n)$; then $M'_{ij}(\mathbf{x}) = \nabla M_{ij}(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x})$ for all $i, j \in \{1, \ldots, n\}$.

Remark 2.2. *In the case of a general smooth manifold $\mathcal{M}$, $\mathbf{v}$ and $\mathbf{w}$ are elements of the tangent space $T_{\mathbf{x}}\mathcal{M}$ to $\mathcal{M}$ at $\mathbf{x}$.*

Remark 2.3. *Note that if G is compact, then M is uniformly positive definite, i.e. there exists $\epsilon > 0$ such that $\mathbf{v}^T M(\mathbf{x})\mathbf{v} \geq \epsilon\, \mathbf{v}^T I \mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^n$ and all $\mathbf{x} \in G$.*

In the next definition we will define a Riemannian contraction metric.

Definition 2.4 (Riemannian contraction metric). *Let M be a Riemannian metric. For $\mathbf{v} \in \mathbb{R}^n$ define*

$$L_M(\mathbf{x};\mathbf{v}) \;\; := \;\; \frac{1}{2}\mathbf{v}^T \left[M(\mathbf{x})D\mathbf{f}(\mathbf{x}) + D\mathbf{f}(\mathbf{x})^T M(\mathbf{x}) + M'(\mathbf{x})\right]\mathbf{v}.$$

The Riemannian metric is called **contracting in $G \subset \mathbb{R}^n$ with exponent** $-\nu < 0$ *if*

$$\mathcal{L}_M(\mathbf{x}) \;\; \leq \;\; -\nu \text{ for all } \mathbf{x} \in G, \text{ where} \qquad (2.2)$$

$$\mathcal{L}_M(\mathbf{x}) \;\; := \;\; \max_{\mathbf{v}^T M(\mathbf{x})\mathbf{v}=1} L_M(\mathbf{x};\mathbf{v}). \qquad (2.3)$$

Remark 2.5. *Fix $\mathbf{x} \in G$. Note that (2.2) is equivalent to*

$$M(\mathbf{x})D\mathbf{f}(\mathbf{x}) + D\mathbf{f}(\mathbf{x})^T M(\mathbf{x}) + M'(\mathbf{x}) \preceq -2\nu\, M(\mathbf{x}) \qquad (2.4)$$

where $A \preceq B$ for $A, B \in \mathbb{S}^n$ means $A - B$ is negative semi-definite, i.e. $\mathbf{w}^T(A-B)\mathbf{w} \leq 0$ for all $\mathbf{w} \in \mathbb{R}^n$.

Indeed, assume that (2.2) holds. We want to show for all $\mathbf{w} \in \mathbb{R}^n$

$$\mathbf{w}^T[M(\mathbf{x})D\mathbf{f}(\mathbf{x}) + D\mathbf{f}(\mathbf{x})^T M(\mathbf{x}) + M'(\mathbf{x}) + 2\nu\, M(\mathbf{x})]\mathbf{w} \;\; \leq \;\; 0. \quad (2.5)$$

If $\mathbf{w} = \mathbf{0}$, this is true. Assume now that $\mathbf{w} \neq \mathbf{0}$ and set $\mathbf{v} = \frac{\mathbf{w}}{\sqrt{\mathbf{w}^T M(\mathbf{x})\mathbf{w}}}$. Then $\mathbf{v}^T M(\mathbf{x})\mathbf{v} = 1$ and thus by (2.2) we have

$$\mathbf{v}^T[M(\mathbf{x})D\mathbf{f}(\mathbf{x}) + D\mathbf{f}(\mathbf{x})^T M(\mathbf{x}) + M'(\mathbf{x})]\mathbf{v} \;\; \leq \;\; -2\nu \;\; = \;\; -2\nu\; \mathbf{v}^T M(\mathbf{x})\,\mathbf{v}.$$

Multiplication by $\mathbf{w}^T M(\mathbf{x})\mathbf{w}$ gives (2.5).

Now assume that (2.4) holds. Then for all $\mathbf{v}$ with $\mathbf{v}^T M(\mathbf{x})\mathbf{v} = 1$ we have

$$\frac{1}{2}\mathbf{v}^T[M(\mathbf{x})D\mathbf{f}(\mathbf{x}) + D\mathbf{f}(\mathbf{x})^T M(\mathbf{x}) + M'(\mathbf{x})]\mathbf{v} \;\; \leq \;\; -\nu\; \mathbf{v}^T M(\mathbf{x})\mathbf{v} = -\nu,$$

which shows (2.2).

Lemma 2.6. *Let $G \subset \mathbb{R}^n$, $\mathbf{f} \in C^1(G, \mathbb{R}^n)$, and M be a Riemannian metric in the sense of Definition 2.1. Then the function $\mathcal{L}_M(\mathbf{x})$ in (2.3) is continuous with respect to $\mathbf{x}$.*

Proof: The function $L_M(\mathbf{x};\mathbf{v})$ is continuous with respect to $\mathbf{x}$ and $\mathbf{v}$ under the above smoothness assumptions on $\mathbf{f}$ and M, so the maximum in (2.3) exists. Now assume that the function $\mathcal{L}_M$ is not continuous with respect to $\mathbf{x}$, so there exists $\epsilon > 0$, $\mathbf{x} \in G$ and a sequence $\boldsymbol{\xi}_n \to 0$ such that $\mathbf{x}+\boldsymbol{\xi}_n \in G$ and $|\mathcal{L}_M(\mathbf{x}+\boldsymbol{\xi}_n)-\mathcal{L}_M(\mathbf{x})| \geq \epsilon$ holds for all $n \in \mathbb{N}$. Hence, for a subsequence we have either $\mathcal{L}_M(\mathbf{x}+\boldsymbol{\xi}_n) \geq \mathcal{L}_M(\mathbf{x})+\epsilon$ or $\mathcal{L}_M(\mathbf{x}+\boldsymbol{\xi}_n) \leq \mathcal{L}_M(\mathbf{x})-\epsilon$; we consider these cases separately later in Case 1 and 2.

Define

$$\begin{aligned}
\widetilde{L_M}(\mathbf{x};\mathbf{v}) &= \frac{\frac{1}{2}\mathbf{v}^T[M(\mathbf{x})D\mathbf{f}(\mathbf{x})+D\mathbf{f}(\mathbf{x})^T M(\mathbf{x})+M'(\mathbf{x})]\mathbf{v}}{\mathbf{v}^T M(\mathbf{x})\mathbf{v}} \\
&= \frac{L_M(\mathbf{x};\mathbf{v})}{\mathbf{v}^T M(\mathbf{x})\mathbf{v}}. \qquad (2.6)
\end{aligned}$$

$\widetilde{L_M}(\mathbf{x};\mathbf{v})$ is a continuous function with respect to $\mathbf{x}$ and $\mathbf{v} \neq \mathbf{0}$.

Fix $\mathbf{x}$. Note that $\widetilde{L_M}(\mathbf{x};\mathbf{v}) = \widetilde{L_M}(\mathbf{x};\lambda\mathbf{v})$ for any $\lambda \neq 0$, i.e. that $\widetilde{L_M}(\mathbf{x};\mathbf{v})$ is actually a function on $(\mathbf{x};\mathbf{v}) \in \mathbb{R}^n \times \{\mathbf{v} \in \mathbb{R}^n \mid \|\mathbf{v}\| = 1\}$. For $\boldsymbol{\xi}_n$ small enough, i.e. $n \geq N$ for suitable $N \in \mathbb{N}$, we have

$$|\widetilde{L_M}(\mathbf{x}+\boldsymbol{\xi}_n;\mathbf{v}) - \widetilde{L_M}(\mathbf{x};\mathbf{v})| < \frac{\epsilon}{2} \qquad (2.7)$$

uniformly for all $\{\mathbf{v} \in \mathbb{R}^n \mid \|\mathbf{v}\| = 1\}$, which is a compact set, and thus by the above remark also for all $\mathbf{v} \neq \mathbf{0}$. Moreover, as $\widetilde{L_M}(\mathbf{x};\mathbf{v}) = \widetilde{L_M}(\mathbf{x};\lambda\mathbf{v})$ for any $\lambda \neq 0$, we have

$$\sup_{\mathbf{v}\neq\mathbf{0}} \widetilde{L_M}(\mathbf{x};\mathbf{v}) = \max_{\mathbf{w}^T M(\mathbf{x})\mathbf{w}=1} L_M(\mathbf{x};\mathbf{w}) = \mathcal{L}_M(\mathbf{x}). \qquad (2.8)$$

Case 1: Assume that $n \geq N$ and $\mathcal{L}_M(\mathbf{x}+\boldsymbol{\xi}_n) \geq \mathcal{L}_M(\mathbf{x})+\epsilon$. Let $\mathbf{v}_n$ be a vector such that $\mathbf{v}_n^T M(\mathbf{x}+\boldsymbol{\xi}_n)\mathbf{v}_n = 1$ and $\mathcal{L}_M(\mathbf{x}+\boldsymbol{\xi}_n) = L_M(\mathbf{x}+\boldsymbol{\xi}_n;\mathbf{v}_n)$. Then

$$\begin{aligned}
\widetilde{L_M}(\mathbf{x};\mathbf{v}_n) &> \widetilde{L_M}(\mathbf{x}+\boldsymbol{\xi}_n;\mathbf{v}_n) - \frac{\epsilon}{2} \text{ by (2.7)} \\
&= L_M(\mathbf{x}+\boldsymbol{\xi}_n;\mathbf{v}_n) - \frac{\epsilon}{2} \\
&= \mathcal{L}_M(\mathbf{x}+\boldsymbol{\xi}_n) - \frac{\epsilon}{2} \\
&\geq \mathcal{L}_M(\mathbf{x}) + \frac{\epsilon}{2} \text{ by assumption} \\
&= \sup_{\mathbf{v}\neq\mathbf{0}} \widetilde{L_M}(\mathbf{x};\mathbf{v}) + \frac{\epsilon}{2} \text{ by (2.8)},
\end{aligned}$$

which is a contraction.

Case 2: Assume that $n \geq N$ and $\mathcal{L}_M(\mathbf{x}+\boldsymbol{\xi}_n) \leq \mathcal{L}_M(\mathbf{x}) - \epsilon$. Let $\mathbf{v}$ be a vector such that $\mathbf{v}^T M(\mathbf{x})\mathbf{v} = 1$ and $\mathcal{L}_M(\mathbf{x}) = L_M(\mathbf{x};\mathbf{v})$. Then

$$\begin{aligned}
\widetilde{L_M}(\mathbf{x}+\boldsymbol{\xi}_n;\mathbf{v}) &> \widetilde{L_M}(\mathbf{x};\mathbf{v}) - \frac{\epsilon}{2} \text{ by (2.7)} \\
&= L_M(\mathbf{x};\mathbf{v}) - \frac{\epsilon}{2} \\
&= \mathcal{L}_M(\mathbf{x}) - \frac{\epsilon}{2} \\
&\geq \mathcal{L}_M(\mathbf{x}+\boldsymbol{\xi}_n) + \frac{\epsilon}{2} \text{ by assumption} \\
&= \sup_{\mathbf{w}\neq\mathbf{0}} \widetilde{L_M}(\mathbf{x}+\boldsymbol{\xi}_n;\mathbf{w}) + \frac{\epsilon}{2} \text{ by (2.8)},
\end{aligned}$$

which is a contraction. □

If N is a Riemannian metric and V is a scalar-valued function, then we can define $M(\mathbf{x}) = e^{2V(\mathbf{x})}N(\mathbf{x})$. In the following lemma we show that M is also a Riemannian metric and we calculate $\mathcal{L}_M$ in terms of $\mathcal{L}_N$ and V'.

Lemma 2.7. *Let $G \subset \mathbb{R}^n$, $N\colon G \to \mathbb{S}^n$ be a Riemannian metric and $V\colon G \to \mathbb{R}$ be a continuous and orbitally continuously differentiable function.*

Then $M(\mathbf{x}) = e^{2V(\mathbf{x})}N(\mathbf{x})$ is a Riemannian metric and

$$\mathcal{L}_M(\mathbf{x}) = \mathcal{L}_N(\mathbf{x}) + V'(\mathbf{x}).$$

PROOF: It is clear that $M(\mathbf{x})$ is a Riemannian metric because $e^{2V(\mathbf{x})} > 0$. Let $\mathbf{v} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$. We have

$$\begin{aligned}
L_M(\mathbf{x};\mathbf{v}) &= \frac{1}{2}\mathbf{v}^T \left(M(\mathbf{x})D\mathbf{f}(\mathbf{x}) + D\mathbf{f}(\mathbf{x})^T M(\mathbf{x}) + M'(\mathbf{x})\right)\mathbf{v} \\
&= \frac{1}{2}\mathbf{v}^T \Big(e^{2V(\mathbf{x})}N(\mathbf{x})D\mathbf{f}(\mathbf{x}) + e^{2V(\mathbf{x})}D\mathbf{f}(\mathbf{x})^T N(\mathbf{x}) \\
&\qquad + e^{2V(\mathbf{x})}(2V'(\mathbf{x})N(\mathbf{x}) + N'(\mathbf{x}))\Big)\mathbf{v} \\
&= L_N(\mathbf{x};\mathbf{w}) + \mathbf{w}^T N(\mathbf{x})\mathbf{w}\ V'(\mathbf{x}),
\end{aligned} \tag{2.9}$$

where $\mathbf{w} = e^{V(\mathbf{x})}\mathbf{v}$.

Now let $\mathbf{v}$ be such that $\mathbf{v}^T M(\mathbf{x})\mathbf{v} = 1$ and $\mathcal{L}_M(\mathbf{x}) = L_M(\mathbf{x};\mathbf{v})$. Define $\mathbf{w} = e^{V(\mathbf{x})}\mathbf{v}$; note that $\mathbf{w}^T N(\mathbf{x})\mathbf{w} = 1$. Then by (2.9) we have

$$\begin{aligned}
\mathcal{L}_M(\mathbf{x}) &= L_M(\mathbf{x};\mathbf{v}) \\
&\le \max_{\mathbf{w}^T N(\mathbf{x})\mathbf{w}=1} [L_N(\mathbf{x};\mathbf{w}) + V'(\mathbf{x})] \\
&= \mathcal{L}_N(\mathbf{x}) + V'(\mathbf{x}).
\end{aligned}$$

Conversely, let $\mathbf{w}$ be such that $\mathbf{w}^T N(\mathbf{x})\mathbf{w} = 1$ and $\mathcal{L}_N(\mathbf{x}) = L_N(\mathbf{x};\mathbf{w})$. Define $\mathbf{v} = e^{-V(\mathbf{x})}\mathbf{w}$; note that $\mathbf{v}^T M(\mathbf{x})\mathbf{v} = 1$. Then by (2.9) we have

$$\begin{aligned}
\mathcal{L}_N(\mathbf{x}) + V'(\mathbf{x}) &= L_N(\mathbf{x};\mathbf{w}) + \mathbf{w}^T N(\mathbf{x})\mathbf{w}\ V'(\mathbf{x}) \\
&\le \max_{\mathbf{v}^T M(\mathbf{x})\mathbf{v}=1} L_M(\mathbf{x};\mathbf{v}) \\
&= \mathcal{L}_M(\mathbf{x}).
\end{aligned}$$

This shows the lemma. □

3. Sufficiency

Theorem 1 of [3] states that if a Riemannian (exponential) contraction metric exists in $\mathbb{R}^n$, then there exists a unique equilibrium and all trajectories converge to this equilibrium, see [1]; because of the unbounded space $\mathbb{R}^n$ their definition of contraction metrics includes *uniform* positive definiteness of M and *uniform* negative definiteness for the contraction property. These publications include results about the evolution of the distance between adjacent solutions, which is exponentially decreasing.

In this paper, however, we focus on a somewhat weaker result. We consider a contraction metric on a compact, connected and positively invariant set, and thus can prove similar results much shorter.

We will later prove a converse statement to the following Theorem 3.1 in Theorem 4.1. The other converse results in Theorems 4.2 and 4.4 even establish the existence of a contraction metric in the whole basin of attraction $A(\mathbf{x}_0)$, however, as we show in Section 5, it is not uniformly positive definite in general. The proof of Theorem 3.1 follows the ideas of [7].

Theorem 3.1. *Let $\emptyset \neq G \subset \mathbb{R}^n$ be a compact, connected and positively invariant set and M be a Riemannian contraction metric in G with exponent $-\nu$ in the sense of Definition 2.4.*

Then there exists one and only one equilibrium in $\mathbf{x}_0$ in G; $\mathbf{x}_0$ is exponentially stable, the real part of all eigenvalues of $D\mathbf{f}(\mathbf{x}_0)$ is at most $-\nu$, and $G \subset A(\mathbf{x}_0)$.

PROOF: **Step 1.** We first assume that $\mathbf{x}_0 \in G$ is an arbitrary equilibrium point and show that the maximal real part μ of all eigenvalues of $D\mathbf{f}(\mathbf{x}_0)$ satisfies

$$\mu \leq \mathcal{L}_M(\mathbf{x}_0).$$

Case 1: Let λ first be a real eigenvalue of $D\mathbf{f}(\mathbf{x}_0)$, i.e. there exists an eigenvector $\mathbf{v} \in \mathbb{R}^n$ with $\mathbf{v}^T M(\mathbf{x}_0)\mathbf{v} = 1$ such that

$$D\mathbf{f}(\mathbf{x}_0)\mathbf{v} = \lambda\,\mathbf{v}.$$

Hence, as $M'(\mathbf{x}_0) = 0$

$$\begin{aligned} L_M(\mathbf{x}_0; \mathbf{v}) &= \frac{1}{2}\mathbf{v}^T[M(\mathbf{x}_0)D\mathbf{f}(\mathbf{x}_0) + D\mathbf{f}(\mathbf{x}_0)^T M(\mathbf{x}_0) + M'(\mathbf{x}_0)]\mathbf{v} \\ &= \lambda\,\mathbf{v}^T M(\mathbf{x}_0)\mathbf{v} \\ &= \lambda, \end{aligned}$$

which means that $\mathcal{L}_M(\mathbf{x}_0) \geq \lambda$.
Case 2: Now let $\lambda = \alpha + i\beta$, $\alpha, \beta \in \mathbb{R}$, be a complex eigenvalue of $D\mathbf{f}(\mathbf{x}_0)$ with eigenvector $\tilde{\mathbf{u}} + i\tilde{\mathbf{v}}$, $\tilde{\mathbf{u}}, \tilde{\mathbf{v}} \in \mathbb{R}^n$, $\tilde{\mathbf{v}} \neq \mathbf{0}$, such that

$$\begin{aligned} D\mathbf{f}(\mathbf{x}_0)\tilde{\mathbf{u}} &= \alpha\,\tilde{\mathbf{u}} - \beta\,\tilde{\mathbf{v}}, \\ D\mathbf{f}(\mathbf{x}_0)\tilde{\mathbf{v}} &= \alpha\,\tilde{\mathbf{v}} + \beta\,\tilde{\mathbf{u}}. \end{aligned}$$

Depending on the sign of $\gamma := \beta\,\tilde{\mathbf{u}}^T M(\mathbf{x}_0)\tilde{\mathbf{v}}$ we consider:

- Case A ($\gamma \geq 0$): the eigenvector $\mathbf{u} + i\mathbf{v}$, where $\mathbf{u} = \frac{\tilde{\mathbf{u}}}{\sqrt{\tilde{\mathbf{v}}^T M(\mathbf{x}_0)\tilde{\mathbf{v}}}}$ and $\mathbf{v} = \frac{\tilde{\mathbf{v}}}{\sqrt{\tilde{\mathbf{v}}^T M(\mathbf{x}_0)\tilde{\mathbf{v}}}}$.
- Case B ($\gamma < 0$): the eigenvector $\mathbf{u} + i\mathbf{v}$, where $\mathbf{u} = \frac{\tilde{\mathbf{u}}}{\sqrt{\tilde{\mathbf{u}}^T M(\mathbf{x}_0)\tilde{\mathbf{u}}}}$ and $\mathbf{v} = \frac{\tilde{\mathbf{v}}}{\sqrt{\tilde{\mathbf{u}}^T M(\mathbf{x}_0)\tilde{\mathbf{u}}}}$; note that $\tilde{\mathbf{u}} \neq \mathbf{0}$ as $\gamma \neq 0$.

In both cases we have

$$\begin{aligned} D\mathbf{f}(\mathbf{x}_0)\mathbf{u} &= \alpha\,\mathbf{u} - \beta\,\mathbf{v}, \\ D\mathbf{f}(\mathbf{x}_0)\mathbf{v} &= \alpha\,\mathbf{v} + \beta\,\mathbf{u} \end{aligned}$$

and

$$\begin{aligned}
L_M(\mathbf{x}_0;\mathbf{u}) &= \frac{1}{2}\mathbf{u}^T[M(\mathbf{x}_0)D\mathbf{f}(\mathbf{x}_0)+D\mathbf{f}(\mathbf{x}_0)^TM(\mathbf{x}_0)+M'(\mathbf{x}_0)]\mathbf{u}\\
&= \alpha\,\mathbf{u}^TM(\mathbf{x}_0)\mathbf{u}-\beta\,\mathbf{u}^TM(\mathbf{x}_0)\mathbf{v},\\
L_M(\mathbf{x}_0;\mathbf{v}) &= \frac{1}{2}\mathbf{v}^T[M(\mathbf{x}_0)D\mathbf{f}(\mathbf{x}_0)+D\mathbf{f}(\mathbf{x}_0)^TM(\mathbf{x}_0)+M'(\mathbf{x}_0)]\mathbf{v}\\
&= \alpha\,\mathbf{v}^TM(\mathbf{x}_0)\mathbf{v}+\beta\,\mathbf{u}^TM(\mathbf{x}_0)\mathbf{v}.
\end{aligned}$$

This shows that in Case A

$$\begin{aligned}
\mathcal{L}_M(\mathbf{x}_0) &= \max_{\mathbf{w}^TM(\mathbf{x}_0)\mathbf{w}=1} L_M(\mathbf{x}_0;\mathbf{w})\\
&\geq L_M(\mathbf{x}_0;\mathbf{v})\\
&\geq \alpha+\frac{\gamma}{\tilde{\mathbf{v}}^TM(\mathbf{x}_0)\tilde{\mathbf{v}}}\\
&\geq \alpha
\end{aligned}$$

and in Case B

$$\begin{aligned}
\mathcal{L}_M(\mathbf{x}_0) &= \max_{\mathbf{w}^TM(\mathbf{x}_0)\mathbf{w}=1} L_M(\mathbf{x}_0;\mathbf{w})\\
&\geq L_M(\mathbf{x}_0;\mathbf{u})\\
&\geq \alpha-\frac{\gamma}{\tilde{\mathbf{u}}^TM(\mathbf{x}_0)\tilde{\mathbf{u}}}\\
&\geq \alpha,
\end{aligned}$$

which means in both cases that $\mathcal{L}_M(\mathbf{x}_0)\geq\alpha$.

Step 2. Denote by $\mathbf{x}_i$, $i\in I$ all equilibria in G and by $E=\bigcup_{i\in I}\{\mathbf{x}_i\}$ the set of all equilibria in G. As all equilibria are exponentially stable by Step 1, they each have a basin of attraction $A(\mathbf{x}_i)$, and we denote $A_i=A(\mathbf{x}_i)\cap G$; note that A_i is open in G.

We will now show that $\bigcup_{i\in I}A_i=G$. We define the function $V(\mathbf{x})=\mathbf{f}(\mathbf{x})^TM(\mathbf{x})\mathbf{f}(\mathbf{x})$. We have that $V(\mathbf{x})\geq 0$ for all $\mathbf{x}\in G$ and $V(\mathbf{x})=0$ if and only if $\mathbf{x}\in E$. Moreover, V is non-increasing along solution trajectories. Indeed,

$$\begin{aligned}
&V'(\mathbf{x})\\
&= (D\mathbf{f}(\mathbf{x})\mathbf{f}(\mathbf{x}))^TM(\mathbf{x})\mathbf{f}(\mathbf{x})+\mathbf{f}(\mathbf{x})^TM'(\mathbf{x})\mathbf{f}(\mathbf{x})+\mathbf{f}(\mathbf{x})^TM(\mathbf{x})D\mathbf{f}(\mathbf{x})\mathbf{f}(\mathbf{x})\\
&= \mathbf{f}(\mathbf{x})^T\left[D\mathbf{f}(\mathbf{x})^TM(\mathbf{x})+M'(\mathbf{x})+M(\mathbf{x})D\mathbf{f}(\mathbf{x})\right]\mathbf{f}(\mathbf{x})\\
&= 2L_M(\mathbf{x};\mathbf{f}(\mathbf{x}))\leq 0
\end{aligned}$$

and we have equality if and only if $\mathbf{f}(\mathbf{x}) = \mathbf{0}$, i.e. $\mathbf{x} \in E$.

Now let $\mathbf{x} \in G$. Since G is compact and positively invariant, $\emptyset \neq \omega(\mathbf{x}) \subset G$, and, by LaSalle's principle, we have for all $\mathbf{y} \in \omega(\mathbf{x})$ that V is constant on $\omega(\mathbf{x})$, so $V'(\mathbf{y}) = 0$ for all $\omega(\mathbf{x})$. Hence, $\omega(\mathbf{x}) \subset E$, i.e., all points in the omega-limit set are equilibria. In particular, there exists at least one equilibrium $\mathbf{x}_i \in \omega(\mathbf{x})$. By Step 1, $\mathbf{x}_i$ is exponentially asymptotically stable, so $\mathbf{x} \in A_i$, which shows the statement.

Step 3. As all sets A_i are open and disjoint, and G is connected, there is only one equilibrium point $\mathbf{x}_0$ and $G = A_0$. This shows the theorem. □

4. Necessity

For this section we assume that $\mathbf{x}_0$ is an exponentially stable equilibrium with basin of attraction $A(\mathbf{x}_0)$. We want to construct a Riemannian metric M such that $\mathcal{L}_M(\mathbf{x}) < 0$ holds; ideally we want $\mathcal{L}_M(\mathbf{x})$ to be negative for all $\mathbf{x} \in A(\mathbf{x}_0)$ and we also seek to bound it away from zero by $\mathcal{L}_M(\mathbf{x}) \leq -\nu < 0$, where $-\nu$ is the largest real part of the eigenvalues of $D\mathbf{f}(\mathbf{x}_0)$. For the explicit construction of the metric it is also desirable to establish equations rather than inequalities.

We will prove three results in this direction; for the first one (Theorem 4.1) we choose a compact set $K \subset A(\mathbf{x}_0)$, which we can also assume to be positively invariant and connected, and we can construct a Riemannian contraction metric satisfying $\mathcal{L}_M(\mathbf{x}) \leq -\nu + \epsilon$ for all $\mathbf{x} \in K$, where $\epsilon > 0$ is arbitrary.

The second result, Theorem 4.2, constructs a Riemannian contraction metric on the whole basin of attraction with an equation for $\mathcal{L}_M$: it satisfies $\mathcal{L}_M(\mathbf{x}) = -\nu + \epsilon$ for all $\mathbf{x} \in A(\mathbf{x}_0)$, where $\epsilon > 0$ is arbitrary.

The third result, Theorem 4.4, establishes the existence of a Riemannian contraction metric, again on the whole basin of attraction, satisfying a Lyapunov-type matrix equation for M, namely

$$D\mathbf{f}(\mathbf{x})^T M(\mathbf{x}) + M(\mathbf{x}) D\mathbf{f}(\mathbf{x}) + M'(\mathbf{x}) = -C$$

where $C \in \mathbb{S}^n$ is an arbitrary positive definite matrix. However, in general we cannot obtain a bound on $\mathcal{L}_M$ on the whole basin of attraction.

4.1. Compact case

We choose a compact set $K \subset A(\mathbf{x}_0)$, which we can also assume to be positively invariant and connected, and we construct a Riemannian contraction metric; this is thus a converse theorem to Theorem 3.1. The proof follows the ideas of [7, 13].

Theorem 4.1. *Let $\mathbf{x}_0$ be an exponentially stable equilibrium with basin of attraction $A(\mathbf{x}_0)$, let $\mathbf{f} \in C^\sigma(\mathbb{R}^n, \mathbb{R}^n)$, $\sigma \geq 1$, let $-\nu < 0$ be the largest real part of all eigenvalues of $D\mathbf{f}(\mathbf{x}_0)$ and let $K \subset A(\mathbf{x}_0)$ be a compact neighbourhood of $\mathbf{x}_0$.*

Then for every $\epsilon > 0$ there exists a Riemannian contraction metric $M \in C^\sigma(A(\mathbf{x}_0), \mathbb{S}^n)$ such that

$$\mathcal{L}_M(\mathbf{x}) \leq -\nu + \epsilon \text{ for all } \mathbf{x} \in K.$$

PROOF: **Step 1: Local construction**

At $\mathbf{x}_0$, we will define a matrix M_0 such that $\mathcal{L}_{M_0}(\mathbf{x}_0) = -\nu + \frac{\epsilon}{2}$ or even $\mathcal{L}_{M_0}(\mathbf{x}_0) = -\nu$ if all eigenvalues are semi-simple. Here, $\mathcal{L}_{M_0}$ uses the constant Riemannian metric M_0. Let $T \in \mathbb{R}^{n\times n}$ be an invertible matrix such that $T^{-1}D\mathbf{f}(\mathbf{x}_0)T = \text{blockdiag}(J_1, J_2, \ldots, J_r)$ is in a special real Jordan normal form with blocks of the form $J_i = \begin{pmatrix} \lambda_i & \epsilon/2 & 0 & \ldots & 0 \\ & \lambda_i & \epsilon/2 & 0 & 0 \\ & & \ddots & \ddots & \\ & & & \lambda_i & \epsilon/2 \\ & & & & \lambda_i \end{pmatrix}$ for a real eigenvalue $\lambda_i \in \mathbb{R}$ and $J_i = \begin{pmatrix} \mu_i & \nu_i & \epsilon/2 & 0 & \ldots & \ldots & \ldots & \ldots & 0 \\ -\nu_i & \mu_i & 0 & \epsilon/2 & 0 & \ldots & \ldots & \ldots & 0 \\ & & \mu_i & \nu_i & \epsilon/2 & 0 & \ldots & \ldots & 0 \\ & & -\nu_i & \mu_i & 0 & \epsilon/2 & 0 & \ldots & 0 \\ & & & & \ddots & & \ddots & \ddots & \vdots \\ & & & & & \mu_i & \nu_i & \epsilon/2 & 0 \\ & & & & & -\nu_i & \mu_i & 0 & \epsilon/2 \\ & & & & & & & \mu_i & \nu_i \\ & & & & & & & -\nu_i & \mu_i \end{pmatrix}$ for a pair of complex eigenvalues $\lambda_i = \mu_i \pm i\nu_i \in \mathbb{C}$.

In particular, we have

$$\max_{\|\mathbf{w}\|=1} \mathbf{w}^T T^{-1} D\mathbf{f}(\mathbf{x}_0) T\mathbf{w} = -\nu + \frac{\epsilon}{2}.$$

(see [8], but without taking out one direction). Note that if all eigenvalues are semi-simple, all Jordan blocks have size 1 for real and size 2 for complex conjugate eigenvalues, so that in this case we have

$$\max_{\|\mathbf{w}\|=1} \mathbf{w}^T T^{-1} D\mathbf{f}(\mathbf{x}_0) T\mathbf{w} = -\nu.$$

Define $M_0 := (T^{-1})^T T^{-1}$. It is clear that M_0 defines a Riemannian metric, since M_0 is symmetric, and all eigenvalues of M_0 are strictly positive, since T^{-1} is non-singular. Then, denoting $\mathbf{w} = T^{-1}\mathbf{v}$ we have

$$\begin{aligned}
&\mathcal{L}_{M_0}(\mathbf{x}_0)\\
&= \frac{1}{2}\max_{\mathbf{v}^T M_0 \mathbf{v}=1} \mathbf{v}^T \left(M_0 D\mathbf{f}(\mathbf{x}_0) + D\mathbf{f}(\mathbf{x}_0)^T M_0\right)\mathbf{v}\\
&= \frac{1}{2}\max_{\mathbf{w}^T T^T M_0 T\mathbf{w}=1} \mathbf{w}^T T^T \left((T^{-1})^T T^{-1} D\mathbf{f}(\mathbf{x}_0) + D\mathbf{f}(\mathbf{x}_0)^T (T^{-1})^T T^{-1}\right) T\mathbf{w}\\
&= \frac{1}{2}\max_{\mathbf{w}^T\mathbf{w}=1} \mathbf{w}^T \left(T^{-1} D\mathbf{f}(\mathbf{x}_0) T + T^T D\mathbf{f}(\mathbf{x}_0)^T (T^{-1})^T\right)\mathbf{w}\\
&= \max_{\|\mathbf{w}\|=1} \mathbf{w}^T T^{-1} D\mathbf{f}(\mathbf{x}_0) T\mathbf{w}\\
&\leq -\nu + \frac{\epsilon}{2}.
\end{aligned}$$

Since $\mathcal{L}_{M_0}(\cdot)$ is a continuous function by Lemma 2.6, there is a $\delta > 0$ such that $B_\delta(\mathbf{x}_0) \subset K$ and

$$\mathcal{L}_{M_0}(\mathbf{x}) \leq -\nu + \epsilon \tag{4.1}$$

holds for all $\mathbf{x} \in B_\delta(\mathbf{x}_0)$. Set $\mu := \max_{\mathbf{x}\in K} \mathcal{L}_{M_0}(\mathbf{x})$, which exists since K is compact and $\mathcal{L}_{M_0}$ is continuous, see Lemma 2.6. If $\mu \leq -\nu + \epsilon$, then we can choose $M(\mathbf{x}) := M_0$ and the statement holds.

Step 2: Lyapunov function

Now let us assume that $\mu > -\nu + \epsilon$. There exists a Lyapunov function $V \in C^\sigma(A(\mathbf{x}_0), \mathbb{R})$ satisfying $V'(\mathbf{x}) = -\|\mathbf{x} - \mathbf{x}_0\|^2$, cf. e.g. [11], hence, we have

$$\max_{\mathbf{x}\in K\setminus B_\delta(\mathbf{x}_0)} V'(\mathbf{x}) = -\delta^2.$$

Now set $c := \frac{\mu+\nu-\epsilon}{\delta^2} > 0$ and define

$$M(\mathbf{x}) := e^{2cV(\mathbf{x})} M_0.$$

By Lemma 2.7, $M(\mathbf{x})$ is a Riemannian metric and we have $\mathcal{L}_M(\mathbf{x}) = \mathcal{L}_{M_0}(\mathbf{x}) + cV'(\mathbf{x})$; moreover, $M \in C^\sigma(A(\mathbf{x}_0), \mathbb{S}^n)$.

We calculate $\mathcal{L}_M(\mathbf{x})$ and distinguish between the cases (i) $\mathbf{x} \in B_\delta(\mathbf{x}_0)$ and (ii) $\mathbf{x} \in K \setminus B_\delta(\mathbf{x}_0)$.

(i) If $\mathbf{x} \in B_\delta(\mathbf{x}_0)$, then $\mathcal{L}_M(\mathbf{x}) = \mathcal{L}_{M_0}(\mathbf{x}) + cV'(\mathbf{x}) \leq -\nu + \epsilon + 0$ by (4.1).

(ii) If $\mathbf{x} \in K \setminus B_\delta(\mathbf{x}_0)$, then $\mathcal{L}_M(\mathbf{x}) = \mathcal{L}_{M_0}(\mathbf{x}) + cV'(\mathbf{x}) \leq \mu - c\delta^2 = -\nu + \epsilon$.

This shows the theorem. □

4.2. Contraction metric on $A(\mathbf{x}_0)$

In this section we prove the existence of a contraction metric on the whole basin of attraction; it will satisfy an equation for $\mathcal{L}_M(\mathbf{x})$. However, the metric is only continuous with continuous orbital derivative.

Theorem 4.2. *Let $\mathbf{x}_0$ be an exponentially stable equilibrium, and let $-\nu$ be the maximal real part of all eigenvalues of $D\mathbf{f}(\mathbf{x}_0)$. Denote its basin of attraction by $A(\mathbf{x}_0)$. Let $\mathbf{f}$ be continuously differentiable, and let $D\mathbf{f}$ be locally Lipschitz-continuous.*

If all eigenvalues of $D\mathbf{f}(\mathbf{x}_0)$ are semi-simple, then there exists a Riemannian contraction metric M on $A(\mathbf{x}_0)$ such that

$$\mathcal{L}_M(\mathbf{x}) = -\nu$$

holds for all $\mathbf{x} \in A(\mathbf{x}_0)$.

Otherwise for every $\epsilon > 0$ there exists a Riemannian contraction metric M on $A(\mathbf{x}_0)$ such that

$$\mathcal{L}_M(\mathbf{x}) = -\nu + \epsilon$$

holds for all $\mathbf{x} \in A(\mathbf{x}_0)$.

Proof: **Step 1: Local construction**

At $\mathbf{x}_0$, we define a matrix M_0 such that $\mathcal{L}_{M_0}(\mathbf{x}_0) = -\nu$ if all eigenvalues of $D\mathbf{f}(\mathbf{x}_0)$ are semi-simple, or $\mathcal{L}_{M_0}(\mathbf{x}_0) = -\nu + \epsilon$ otherwise; this is as Step 1 in the proof of Theorem 4.1.

Step 2: Global construction

Define the function

$$V(\mathbf{x}) \;=\; \int_0^\infty \left(\mathcal{L}_{M_0}(S_t\mathbf{x}) - \mathcal{L}_{M_0}(\mathbf{x}_0)\right)\, dt. \tag{4.2}$$

This is inspired by [13]. In Step 4 we will prove that V is well defined for all $\mathbf{x} \in A(\mathbf{x}_0)$, continuous, the orbital derivative exists and is continuous, and that (see (4.5))

$$V'(\mathbf{x}) = -\mathcal{L}_{M_0}(\mathbf{x}) + \mathcal{L}_{M_0}(\mathbf{x}_0).$$

Now we define

$$M(\mathbf{x}) = e^{2V(\mathbf{x})}M_0.$$

Thus, M has the required smoothness for a Riemannian contraction metric. Moreover, we have by Lemma 2.7

$$\begin{aligned}
\mathcal{L}_M(\mathbf{x}) &= \mathcal{L}_{M_0}(\mathbf{x}) + V'(\mathbf{x}) \\
&= \mathcal{L}_{M_0}(\mathbf{x}) + [-\mathcal{L}_{M_0}(\mathbf{x}) + \mathcal{L}_{M_0}(\mathbf{x}_0)] \\
&= \mathcal{L}_{M_0}(\mathbf{x}_0),
\end{aligned}$$

which is $-\nu$ if all eigenvalues of $D\mathbf{f}(\mathbf{x}_0)$ are semi-simple, and $-\nu+\epsilon$ otherwise.

Step 3: Properties of $\mathcal{L}_{M_0}$

We will show that $\mathcal{L}_{M_0}(\mathbf{x})$ is locally Lipschitz-continuous with respect to $\mathbf{x}$. We can express $\mathcal{L}_{M_0}(\mathbf{x})$, using $M_0 = (T^{-1})^T T^{-1}$ and $\mathbf{w} = T^{-1}\mathbf{v}$ similar to Step 1 in the proof of Theorem 4.1, as

$$\begin{aligned}
\mathcal{L}_{M_0}(\mathbf{x}) &= \frac{1}{2}\max_{\mathbf{v}^T M_0 \mathbf{v}=1} \mathbf{v}^T \left(M_0 D\mathbf{f}(\mathbf{x}) + D\mathbf{f}(\mathbf{x})^T M_0\right)\mathbf{v} \\
&= \frac{1}{2}\max_{\|\mathbf{w}\|=1} \mathbf{w}^T \left(T^{-1}D\mathbf{f}(\mathbf{x})T + T^T D\mathbf{f}(\mathbf{x})^T (T^{-1})^T\right)\mathbf{w} \\
&= \lambda_{max}\left(\frac{1}{2}\left(T^{-1}D\mathbf{f}(\mathbf{x})T + T^T D\mathbf{f}(\mathbf{x})^T (T^{-1})^T\right)\right),
\end{aligned}$$

where $\lambda_{max}(A)$ denotes the maximal eigenvalue of $A \in \mathbb{S}^n$; this follows from the fact that there exists an orthonormal basis of eigenvectors for a symmetric matrix A. Note, that for $A, B \in \mathbb{S}^n$ we have $\lambda_{max}(A+B) \leq \lambda_{max}(A) + \lambda_{max}(B)$ and $|\lambda_{max}(A)| \leq \|A\|$ (see, e.g., [16]). Thus,

$$\begin{aligned}
\lambda_{max}(A+B) - \lambda_{max}(A) &\leq \lambda_{max}(B) \leq \|B\| \\
\lambda_{max}(A) - \lambda_{max}(A+B) &= \lambda_{max}(A+B-B) - \lambda_{max}(A+B) \\
&\leq \lambda_{max}(-B) \leq \|B\|,
\end{aligned}$$

so with $N_1 = A + B$ and $N_2 = A$ we have

$$|\lambda_{max}(N_1) - \lambda_{max}(N_2)| \leq \|N_1 - N_2\|.$$

In particular, $l_1 \colon A \mapsto \lambda_{max}(A)$ is globally Lipschitz-continuous.

Since $l_2 \colon \mathbf{x} \mapsto \frac{1}{2}\left(T^{-1}D\mathbf{f}(\mathbf{x})T + (T^{-1})^T D\mathbf{f}(\mathbf{x})^T (T^{-1})^T\right)$ is a locally Lipschitz-continuous function by assumption on $\mathbf{f}$, the composition $\mathcal{L}_{M_0}(\mathbf{x}) = (l_1 \circ l_2)(\mathbf{x})$ is locally Lipschitz-continuous.

Step 4: Properties of V

We define

$$g_T(\theta, \mathbf{x}) := \int_\theta^{T+\theta} \left(\mathcal{L}_{M_0}(S_t\mathbf{x}) - \mathcal{L}_{M_0}(\mathbf{x}_0)\right) dt$$

and show that $g_T(\theta, \mathbf{x})$ converges pointwise and $\frac{d}{d\theta} g_T(\theta, \mathbf{x})$ converges uniformly in θ as $T \to \infty$. Hence, $\frac{d}{d\theta} \lim_{T\to\infty} g_T(\theta, \mathbf{x}) = \lim_{T\to\infty} \frac{d}{d\theta} g_T(\theta, \mathbf{x})$.

By the exponential stability of $\mathbf{x}_0$ there is a positively invariant neighbourhood U of $\mathbf{x}_0$ such that $\|S_t\mathbf{x} - \mathbf{x}_0\| \leq Ke^{-\mu t}$ holds for all $\mathbf{x} \in U$ and $t \geq 0$ with a constant $K > 0$. By Step 3 we can also assume that U is so small that $\mathcal{L}_{M_0}(\cdot)$ is Lipschitz-continuous in U with Lipschitz constant L.

Fix $\mathbf{x} \in A(\mathbf{x}_0)$. Then there is a time $T^* > 0$ such that $S_{T^*}\mathbf{x} \in U$. Now, we have for all $t \geq T^*$

$$|\mathcal{L}_{M_0}(S_t\mathbf{x}) - \mathcal{L}_{M_0}(\mathbf{x}_0)| \;\leq\; L\; \|S_t\mathbf{x} - \mathbf{x}_0\| \;\leq\; LKe^{-\mu t}, \tag{4.3}$$

which is integrable over $[0, \infty)$. Hence, by Lebesgue's dominated convergence theorem, the function $g_T(\theta, \mathbf{x})$ converges pointwise for $T \to \infty$.

Also,

$$\frac{d}{d\theta} g_T(\theta, \mathbf{x}) \;=\; \mathcal{L}_{M_0}(S_{T+\theta}\mathbf{x}) - \mathcal{L}_{M_0}(S_\theta\mathbf{x}) \tag{4.4}$$

converges to $\mathcal{L}_{M_0}(\mathbf{x}_0) - \mathcal{L}_{M_0}(S_\theta\mathbf{x})$ for $T \to \infty$ by (4.3), uniformly in θ in some neighbourhood of $\theta = 0$.

Altogether, we thus have

$$\begin{aligned}
V'(\mathbf{x}) &= \frac{d}{d\theta} V(S_\theta\mathbf{x})\Big|_{\theta=0} \\
&= \frac{d}{d\theta} \int_0^\infty \left(\mathcal{L}_{M_0}(S_{t+\theta}\mathbf{x}) - \mathcal{L}_{M_0}(\mathbf{x}_0)\right) dt\Big|_{\theta=0} \\
&= \frac{d}{d\theta} \lim_{T\to\infty} \int_0^T \left(\mathcal{L}_{M_0}(S_{t+\theta}\mathbf{x}) - \mathcal{L}_{M_0}(\mathbf{x}_0)\right) dt\Big|_{\theta=0} \\
&= \frac{d}{d\theta} \lim_{T\to\infty} \int_\theta^{T+\theta} \left(\mathcal{L}_{M_0}(S_t\mathbf{x}) - \mathcal{L}_{M_0}(\mathbf{x}_0)\right) dt\Big|_{\theta=0} \\
&= \frac{d}{d\theta} \lim_{T\to\infty} g_T(\theta, \mathbf{x})\big|_{\theta=0}
\end{aligned}$$

$$\begin{aligned}
&= \lim_{T\to\infty} \frac{d}{d\theta} g_T(\theta, \mathbf{x})\big|_{\theta=0} \\
&= \lim_{T\to\infty} [\mathcal{L}_{M_0}(S_{T+\theta}\mathbf{x}) - \mathcal{L}_{M_0}(S_\theta \mathbf{x})]\big|_{\theta=0} \text{ by (4.4)} \\
&= [\mathcal{L}_{M_0}(\mathbf{x}_0) - \mathcal{L}_{M_0}(S_\theta \mathbf{x})]\big|_{\theta=0} \\
&= \mathcal{L}_{M_0}(\mathbf{x}_0) - \mathcal{L}_{M_0}(\mathbf{x}) \qquad (4.5)
\end{aligned}$$

and in particular, that V is continuously orbitally differentiable. □

4.3. Matrix equation

The idea for the proof of the following theorem comes from the classical Lyapunov equation, constructing a quadratic Lyapunov function, and also a constant contraction metric, for a linear ODE.

Consider the linear ODE $\dot{\mathbf{x}} = A\mathbf{x}$ such that the origin is an exponentially stable equilibrium, i.e. all eigenvalues of A have negative real part. Then for any symmetric, positive definite matrix C the Lyapunov equation (matrix equation)

$$A^T B + BA = -C \qquad (4.6)$$

has a unique solution B, which is also symmetric and positive definite, cf. e.g. [20, Theorem 4.6]. The quadratic form $V(\mathbf{x}) = \mathbf{x}^T B\mathbf{x}$ is a strict Lyapunov function for the origin of the linear ODE $\dot{\mathbf{x}} = A\mathbf{x}$. B is given by the formula

$$B = \int_0^\infty \exp(A^T\tau) C \exp(A\tau)\, d\tau;$$

note that $\exp(A\tau)$ is the fundamental matrix solution of the initial value problem $\dot{\mathbf{y}} = A\mathbf{y}$, $\mathbf{y}(0) = I$; compare this with (4.13).

Moreover, $M(\mathbf{x}) = B$ is a (constant) contraction metric, as it satisfies

$$\begin{aligned}
\mathbf{v}^T[D\mathbf{f}(\mathbf{x})^T M(\mathbf{x}) + M(\mathbf{x})D\mathbf{f}(\mathbf{x}) + M'(\mathbf{x})]\mathbf{v} &= \mathbf{v}^T[A^T B + BA]\mathbf{v} \\
&= -\mathbf{v}^T C\mathbf{v}
\end{aligned}$$

where we have used that $M'(\mathbf{x}) = 0$ and $D\mathbf{f}(\mathbf{x}) = A$. Hence,

$$\mathcal{L}_M(\mathbf{x}) \le -\frac{\lambda_{min}(C)}{2\lambda_{max}(B)} < 0.$$

Note, however, that in general this bound is not optimal in the sense that it is not close to the maximal real part $-\nu$ of all eigenvalues of A, see the following example.

Example 4.3. *Consider the linear equation* $\dot{\mathbf{x}} = A\mathbf{x}$ *where* $A = \begin{pmatrix} -1 & 4c \\ 0 & -1 \end{pmatrix}$ *with* $c \in \mathbb{R}$ *and* $C = I$*. Then* $B = \begin{pmatrix} 1/2 & c \\ c & 1/2 + 4c^2 \end{pmatrix}$ *is the solution of the Lyapunov equation (4.6) and*

$$\lambda_{max}(B) = \frac{1}{2} + 2c^2 + \sqrt{c^2 + 4c^4}.$$

Since $\lambda_{min}(C) = 1$*, we have*

$$-\frac{\lambda_{min}(C)}{2\lambda_{max}(B)} = -(1 + 4c^2 + 2\sqrt{c^2 + 4c^4})^{-1},$$

which becomes arbitrarily close to 0 *for* $c \to \infty$*, while the maximal real part of all eigenvalues of* A *is* $-\nu = -1$*.*

In the special case that $\mathbf{f}(\mathbf{x}) = A\mathbf{x}$ in the following Theorem 4.4 is linear and $M(\mathbf{x}) = B$ is constant, equation (4.8) reduces to the Lyapunov equation (4.6). Now, however, we assume that $\mathbf{f}(\mathbf{x})$ is a general nonlinear function, and we generalise the classical result to the nonlinear case.

Note that the definition of $M(\mathbf{x})$ coincides with the solution of the time-dependent Lyapunov matrix equation, see e.g. [2]. If we restrict ourselves to one trajectory $S_t\mathbf{x}$, then the solution M of equation (4.8) satisfies $M(S_t\mathbf{x}) = P(t)$, where P is the solution of the differential Lyapunov equation

$$-\dot{P} = PA(t) + A(t)^T P + L^T L$$

for the time-dependent system

$$\dot{\mathbf{y}} = A(t)\mathbf{y}$$

where $A(t) = D\mathbf{f}(S_t\mathbf{x})$ and $C = L^T L$, cf. [2, equation (47)]. In the proof of the following Theorem 4.4, we prove a similar construction in Step 1, compare [2, equation (45)] with (4.13), but we go further and prove the dependence and smoothness of M on $\mathbf{x}$.

If $\sup_{\mathbf{x} \in A(\mathbf{x}_0)} \lambda_{max}(M(\mathbf{x})) < \infty$, then (4.8) gives us a negative bound on $\mathcal{L}_M(\mathbf{x})$ for all $\mathbf{x} \in A(\mathbf{x}_0)$ by using

$$\begin{aligned} \mathcal{L}_M(\mathbf{x}) &= \max_{\mathbf{v}^T M(\mathbf{x})\mathbf{v}=1} L_M(\mathbf{x};\mathbf{v}) \\ &\leq \frac{1}{2} \max_{\mathbf{v}^T M(\mathbf{x})\mathbf{v}=1} (-\mathbf{v}^T C\mathbf{v}) \\ &\leq -\frac{\lambda_{min}(C)}{2\sup_{\mathbf{x} \in A(\mathbf{x}_0)} \lambda_{max}(M(\mathbf{x}))}. \end{aligned} \tag{4.7}$$

We will later give an example (see Example 5.2) to show that, however, $\lambda_{max}(M(\mathbf{x}))$ does not need to be bounded. To obtain an estimate of the exponential rate of attraction of the equilibrium, however, it is enough obtain an estimate $\mathcal{L}_M(\mathbf{x})$ in a compact neighbourhood of the equilibrium, which is always possible. Note, however, that even in the linear case, we cannot expect to obtain an *optimal* estimate in general, cf. Example 4.3.

Theorem 4.4. *Consider the dynamical system given by* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$*,* $\mathbf{f} \in C^\sigma(\mathbb{R}^n, \mathbb{R}^n)$*,* $\sigma \geq 2$*, and assume that* $\mathbf{x}_0$ *is an exponentially stable equilibrium with basin of attraction* $A(\mathbf{x}_0)$*. Let* $C \in \mathbb{S}^n$ *be a positive definite matrix.*

Then there exists a Riemannian contraction metric $M \in C^{\sigma-1}(A(\mathbf{x}_0), \mathbb{S}^n)$ *such that*

$$D\mathbf{f}(\mathbf{x})^T M(\mathbf{x}) + M(\mathbf{x}) D\mathbf{f}(\mathbf{x}) + M'(\mathbf{x}) = -C \text{ for all } \mathbf{x} \in A(\mathbf{x}_0). \tag{4.8}$$

PROOF: **Step 1: Definition of M**

We consider the linear, non-autonomous ODE

$$\dot{\mathbf{y}} = D\mathbf{f}(S_t\mathbf{x})\mathbf{y}.$$

As $A(t) := D\mathbf{f}(S_t\mathbf{x})$ is defined and continuous for all $\mathbf{x} \in A(\mathbf{x}_0)$ and $t \geq 0$, the principal fundamental matrix solution of the initial value problem with $\mathbf{y}(t_0) = I$ exists and we denote it by

$$\phi(t, t_0; \mathbf{x}).$$

Note that for fixed $\mathbf{x}$ there exists a $\theta_0 > 0$ such that $S_t\mathbf{x}$, $A(t)$ and thus also $\phi(t, t_0; \mathbf{x})$ are defined for all $t, t_0 \geq -\theta_0$. We will discuss the smoothness in more detail: the solution $S_t\mathbf{x}$ of the ODE is C^σ with respect to $\mathbf{x}$ and thus $D\mathbf{f}(S_t\mathbf{x})$ is $C^{\sigma-1}$ with respect to $\mathbf{x}$ and t. Hence, $\phi(t, t_0; \mathbf{x})$ is $C^{\sigma-1}$ with respect to $\mathbf{x}$, t and t_0.

By the Chapman-Kolmogorov identities, cf. e.g. [5], p. 151, we have

$$\begin{aligned}
\frac{d}{dt}\phi(t, t_0; \mathbf{x}) &= D\mathbf{f}(S_t\mathbf{x})\phi(t, t_0; \mathbf{x}), && (4.9)\\
\frac{d}{dt_0}\phi(t, t_0; \mathbf{x}) &= -\phi(t, t_0; \mathbf{x}) D\mathbf{f}(S_{t_0}\mathbf{x}), && (4.10)\\
\phi(t_0, t_0; \mathbf{x}) &= I, && (4.11)\\
\phi(t, 0; S_\theta\mathbf{x}) &= \phi(t + \theta, \theta; \mathbf{x}). && (4.12)
\end{aligned}$$

The last equation holds for all $t, t+\theta \geq -\theta_0$ and follows from the fact that both functions satisfy the same initial value problem $\frac{d}{dt}\mathbf{y}(t) = D\mathbf{f}(S_{t+\theta}\mathbf{x})\mathbf{y}(t)$ with $\mathbf{y}(0) = I$; for this we use the semi-group property of the solution of the original ODE $S_{t+\theta}\mathbf{x} = S_t(S_\theta\mathbf{x})$.

Now we define the function

$$M(\mathbf{x}) = \int_0^\infty \phi(\tau, 0; \mathbf{x})^T C \phi(\tau, 0; \mathbf{x})\, d\tau. \tag{4.13}$$

We need to check that M is well defined for all $\mathbf{x} \in A(\mathbf{x}_0)$, symmetric, positive definite and $C^{\sigma-1}$, and that it satisfies the equation (4.8).

From the definition, it is clear that $M(\mathbf{x})$ is symmetric, since C is. Moreover, since C is positive definite, we have

$$\mathbf{v}^T M(\mathbf{x})\mathbf{v} = \int_0^\infty [\phi(\tau, 0; \mathbf{x})\mathbf{v}]^T C \phi(\tau, 0; \mathbf{x})\mathbf{v}\, d\tau \geq 0$$

and the expression is equal to zero if and only if $\mathbf{v} = 0$, as both $\phi(\tau, 0; \mathbf{x})$ and C are non-singular. We will show that M satisfies the equation (4.8) in Step 2, and we will prove the smoothness of M in Step 3.

Step 2: Matrix equation

Define

$$g_T(\theta, \mathbf{x}) = \int_\theta^{T+\theta} \phi(\tau, \theta; \mathbf{x})^T C \phi(\tau, \theta; \mathbf{x})\, d\tau. \tag{4.14}$$

We have for all $\theta \geq -\theta_0$ by a change of variables and (4.12)

$$g_T(\theta, \mathbf{x}) = \int_0^T \phi(\tau+\theta, \theta; \mathbf{x})^T C \phi(\tau+\theta, \theta; \mathbf{x})\, d\tau \tag{4.15}$$

$$= \int_0^T \phi(\tau, 0; S_\theta\mathbf{x})^T C \phi(\tau, 0; S_\theta\mathbf{x})\, d\tau. \tag{4.16}$$

We will show that $g_T(\theta, \mathbf{x})$ converges pointwise and $\frac{d}{d\theta} g_T(\theta, \mathbf{x})$ converges uniformly in θ as $T \to \infty$. Hence, $\frac{d}{d\theta} \lim_{T\to\infty} g_T(\theta, \mathbf{x}) = \lim_{T\to\infty} \frac{d}{d\theta} g_T(\theta, \mathbf{x})$.

By the exponential stability of $\mathbf{x}_0$ there is a positively invariant, compact neighbourhood U of $\mathbf{x}_0$ such that $\|S_t\mathbf{x} - \mathbf{x}_0\| \leq Ke^{-\mu t}$ holds for all $\mathbf{x} \in U$ and $t \geq 0$.

Fix $\mathbf{x} \in A(\mathbf{x}_0)$. Then there is a time $T^* > 0$ such that $S_{T^*}\mathbf{x} \in U$. Since $D\mathbf{f}$ is locally Lipschitz-continuous at $\mathbf{x}_0$ we have

$$\|D\mathbf{f}(S_t\mathbf{x}) - D\mathbf{f}(\mathbf{x}_0)\| \leq L\|S_t\mathbf{x} - \mathbf{x}_0\| \leq de^{-\mu t}$$

for all $t \geq 0$, when choosing d appropriately. As discussed above, we choose $\theta_0 > 0$ such that $S_t\mathbf{x}$ exists for all $t \geq -\theta_0$. Then we also obtain similarly

$$\|D\mathbf{f}(S_{t+\theta}\mathbf{x}) - D\mathbf{f}(\mathbf{x}_0)\| \leq d_1 e^{-\mu t} \tag{4.17}$$

for all $t \geq 0$ and all $|\theta| \leq \theta_0$. Now, we can apply Lemma Appendix A.2 from the appendix to $A(t) = D\mathbf{f}(S_t S_\theta \mathbf{x})$ and $A = D\mathbf{f}(\mathbf{x}_0)$ to obtain

$$\|\phi(t, 0; S_\theta\mathbf{x})\| \leq ce^{-\rho t} \tag{4.18}$$

for all $|\theta| \leq \theta_0$ and $t \geq 0$. Thus we have, with a new constant c,

$$\|\phi(\tau, 0; S_\theta\mathbf{x})^T C\phi(\tau, 0; S_\theta\mathbf{x})\| \leq ce^{-2\rho\tau}, \tag{4.19}$$

which is integrable over $\tau \in [0, \infty)$. Hence, by Lebesgue's dominated convergence theorem, the function $g_T(\theta, \mathbf{x})$, see (4.16), converges pointwise for $T \to \infty$.

Also, using (4.14), (4.11) and (4.10), we have

$$\begin{aligned}
\frac{d}{d\theta} g_T(\theta, \mathbf{x}) &= \phi(T+\theta, \theta; \mathbf{x})^T C\phi(T+\theta, \theta; \mathbf{x}) - C \\
&\quad -D\mathbf{f}(S_\theta\mathbf{x})^T \int_\theta^{T+\theta} \phi(\tau, \theta; \mathbf{x})^T C\phi(\tau, \theta; \mathbf{x})\, d\tau \\
&\quad - \int_\theta^{T+\theta} \phi(\tau, \theta; \mathbf{x})^T C\phi(\tau, \theta; \mathbf{x})\, d\tau \; D\mathbf{f}(S_\theta\mathbf{x}) \\
&= \phi(T+\theta, \theta; \mathbf{x})^T C\phi(T+\theta, \theta; \mathbf{x}) - C \\
&\quad -D\mathbf{f}(S_\theta\mathbf{x})^T \int_0^{T} \phi(\tau+\theta, \theta; \mathbf{x})^T C\phi(\tau+\theta, \theta; \mathbf{x})\, d\tau \\
&\quad - \int_0^{T} \phi(\tau+\theta, \theta; \mathbf{x})^T C\phi(\tau+\theta, \theta; \mathbf{x})\, d\tau \; D\mathbf{f}(S_\theta\mathbf{x}) \\
&= \phi(T, 0; S_\theta\mathbf{x})^T C\phi(T, 0; S_\theta\mathbf{x}) - C \\
&\quad -D\mathbf{f}(S_\theta\mathbf{x})^T \int_0^{T} \phi(\tau, 0; S_\theta\mathbf{x})^T C\phi(\tau, 0; S_\theta\mathbf{x})\, d\tau \\
&\quad - \int_0^{T} \phi(\tau, 0; S_\theta\mathbf{x})^T C\phi(\tau, 0; S_\theta\mathbf{x})\, d\tau \; D\mathbf{f}(S_\theta\mathbf{x}) \text{ by (4.12).}
\end{aligned}$$

By (4.19), this converges uniformly in $|\theta| \leq \theta_0$ as $T \to \infty$ in some neighbourhood of $\theta = 0$. Hence, we can exchange limit and derivative, obtaining

$$\begin{aligned}
\frac{d}{d\theta}\lim_{T\to\infty} g_T(\theta,\mathbf{x}) &= \lim_{T\to\infty}\frac{d}{d\theta}g_T(\theta,\mathbf{x}) \\
&= -C - D\mathbf{f}(S_\theta\mathbf{x})^T\int_0^\infty \phi(\tau,0;S_\theta\mathbf{x})^T C\phi(\tau,0;S_\theta\mathbf{x})\,d\tau \\
&\quad -\int_0^\infty \phi(\tau,0;S_\theta\mathbf{x})^T C\phi(\tau,0;S_\theta\mathbf{x})\,d\tau\ D\mathbf{f}(S_\theta\mathbf{x}). \qquad (4.20)
\end{aligned}$$

Altogether, we thus have

$$\begin{aligned}
M'(\mathbf{x}) &= \frac{d}{d\theta}M(S_\theta\mathbf{x})\Big|_{\theta=0} \\
&= \frac{d}{d\theta}\lim_{T\to\infty}\left[\int_0^T \phi(\tau,0;S_\theta\mathbf{x})^T C\phi(\tau,0;S_\theta\mathbf{x})\,d\tau\right]\Bigg|_{\theta=0} \\
&= \frac{d}{d\theta}\lim_{T\to\infty}\left[\int_0^T \phi(\tau+\theta,\theta;\mathbf{x})^T C\phi(\tau+\theta,\theta;\mathbf{x})\,d\tau\right]\Bigg|_{\theta=0} \quad \text{by (4.12)} \\
&= \frac{d}{d\theta}\lim_{T\to\infty} g_T(\theta,\mathbf{x})\Big|_{\theta=0} \quad \text{by (4.15)} \\
&= \Bigg[-C - D\mathbf{f}(S_\theta\mathbf{x})^T\int_0^\infty \phi(\tau,0;S_\theta\mathbf{x})^T C\phi(\tau,0;S_\theta\mathbf{x})\,d\tau \\
&\quad -\int_0^\infty \phi(\tau,0;S_\theta\mathbf{x})^T C\phi(\tau,0;S_\theta\mathbf{x})\,d\tau\ D\mathbf{f}(S_\theta\mathbf{x})\Bigg]\Bigg|_{\theta=0} \quad \text{by (4.20)} \\
&= -C - D\mathbf{f}(\mathbf{x})^T M(\mathbf{x}) - M(\mathbf{x})D\mathbf{f}(\mathbf{x}) \text{ by (4.13).}
\end{aligned}$$

This shows the matrix equation (4.8).

Step 3: Smoothness of M

Let $-\nu < 0$ be the largest real part of all eigenvalues of $D\mathbf{f}(\mathbf{x}_0)$ and let $\epsilon = \frac{\nu}{2}$. By Step 1 of Theorem 4.1 there is an invertible matrix $T \in \mathbb{R}^{n\times n}$ such that

$$\max_{\|\mathbf{w}\|=1} \mathbf{w}^T T^{-1} D\mathbf{f}(\mathbf{x}_0) T\mathbf{w} \ \le\ -\nu + \frac{\epsilon}{2}. \qquad (4.21)$$

Since $\mathbf{f} \in C^1(\mathbb{R}^n,\mathbb{R}^n)$ we can choose a positively invariant, compact neighborhood U of $\mathbf{x}_0$ so small that

$$\|T^{-1}[D\mathbf{f}(\mathbf{x}) - D\mathbf{f}(\mathbf{x}_0)]T\| \ \le\ \frac{\epsilon}{2} \qquad (4.22)$$

holds for all $\mathbf{x} \in U$.

Let $\nu' = \min\left(\frac{\nu}{4}, \rho\right)$, where ρ was defined in (4.18). Now we show that

$$\|T^{-1}\partial_{\mathbf{x}}^{\boldsymbol{\alpha}}\phi(t,0;\mathbf{x})\| \quad \leq \quad C_{\boldsymbol{\alpha}}e^{-\nu' t} \tag{4.23}$$

holds for all $\boldsymbol{\alpha} \in \mathbb{N}_0^n$ with $|\boldsymbol{\alpha}| := \sum_{i=1}^n |\alpha_i| \leq \sigma - 1$, $\mathbf{x} \in U$ and $t \geq 0$. We prove this by induction with respect to $k = |\boldsymbol{\alpha}|$.

For $k = 0$ the inequality follows from (4.18); note that d_1 in (4.17) can be chosen uniformly for all $\mathbf{x} \in U$ and $\theta = 0$. Now assume that (4.23) holds for all $|\boldsymbol{\alpha}| = k-1$ with $1 \leq k \leq \sigma - 1$. Let $|\boldsymbol{\alpha}'| = k \leq \sigma - 1$. Write $\boldsymbol{\alpha}' = \boldsymbol{\alpha} + \mathbf{e}_i$, where $|\boldsymbol{\alpha}| = k-1$ and $\mathbf{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with the 1 at i-th position. We have for all $\mathbf{x} \in U$ and $t \geq 0$

$$\begin{aligned}
&\|T^{-1}\partial_{\mathbf{x}}^{\boldsymbol{\alpha}'}\phi(t,0;\mathbf{x})\| \cdot \frac{d}{dt}\|T^{-1}\partial_{\mathbf{x}}^{\boldsymbol{\alpha}'}\phi(t,0;\mathbf{x})\| \\
= \;& \frac{1}{2}\frac{d}{dt}\|T^{-1}\partial_{\mathbf{x}}^{\boldsymbol{\alpha}'}\phi(t,0;\mathbf{x})\|^2 \\
= \;& (T^{-1}\partial_{\mathbf{x}}^{\boldsymbol{\alpha}'}\phi(t,0;\mathbf{x}))^T T^{-1}\Bigg(\partial_{\mathbf{x}}^{\boldsymbol{\alpha}'} \underbrace{\frac{d}{dt}\phi(t,0;\mathbf{x})}_{= D\mathbf{f}(S_t x)\phi(t,0;\mathbf{x})}\Bigg)
\end{aligned}$$

by (4.9). Note that we could exchange $\partial_{\mathbf{x}}^{\boldsymbol{\alpha}'}$ and $\frac{d}{dt}$ since the solution ϕ is smooth enough, cf. e.g. [18], Chapter V, Theorem 4.1. We have, using a generalised Leibniz rule,

$$\begin{aligned}
\partial_{\mathbf{x}}^{\boldsymbol{\alpha}'}(D\mathbf{f}(S_t x)\phi(t,0;\mathbf{x})) \;=\; & D\mathbf{f}(S_t x)\partial_{\mathbf{x}}^{\boldsymbol{\alpha}'}\phi(t,0;\mathbf{x}) \\
& + \sum_{\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2=\boldsymbol{\alpha}', |\boldsymbol{\alpha}_1|\geq 1} c_{\boldsymbol{\alpha}_1}\partial_{\mathbf{x}}^{\boldsymbol{\alpha}_1}D\mathbf{f}(S_t\mathbf{x})\partial_{\mathbf{x}}^{\boldsymbol{\alpha}_2}\phi(t,0;\mathbf{x}).
\end{aligned}$$

As $|\boldsymbol{\alpha}_1| \leq \sigma - 1$ and $\mathbf{f} \in C^\sigma$, we have a uniform bound for all $t \geq 0$ and $\mathbf{x} \in U$ on

$$\sum_{\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2=\boldsymbol{\alpha}', |\boldsymbol{\alpha}_1|\geq 1} |c_{\boldsymbol{\alpha}_1}|\|\partial_{\mathbf{x}}^{\boldsymbol{\alpha}_1}D\mathbf{f}(S_t\mathbf{x})\|.$$

Hence, we have, using the induction hypothesis (4.23) for $\|\partial_{\mathbf{x}}^{\boldsymbol{\alpha}_2}\phi(t,0;\mathbf{x})\|$, noting that $|\boldsymbol{\alpha}_2| \leq |\boldsymbol{\alpha}'| - 1 = k-1$

$$\begin{aligned}
&\|T^{-1}\partial_{\mathbf{x}}^{\boldsymbol{\alpha}'}\phi(t,0;\mathbf{x})\| \cdot \frac{d}{dt}\|T^{-1}\partial_{\mathbf{x}}^{\boldsymbol{\alpha}'}\phi(t,0;\mathbf{x})\| \\
&\quad = (T^{-1}\partial_{\mathbf{x}}^{\boldsymbol{\alpha}'}\phi(t,0;\mathbf{x}))^T T^{-1} D\mathbf{f}(\mathbf{x}_0) T (T^{-1}\partial_{\mathbf{x}}^{\boldsymbol{\alpha}'}\phi(t,0;\mathbf{x})) \\
&\qquad + \|T^{-1}[D\mathbf{f}(S_t\mathbf{x}) - D\mathbf{f}(\mathbf{x}_0)]T\| \|T^{-1}\partial_{\mathbf{x}}^{\boldsymbol{\alpha}'}\phi(t,0;\mathbf{x})\|^2 \\
&\qquad + c\|T^{-1}\partial_{\mathbf{x}}^{\boldsymbol{\alpha}'}\phi(t,0;\mathbf{x})\| e^{-\nu' t} \\
&\quad \leq (-\nu+\epsilon)\|T^{-1}\partial_{\mathbf{x}}^{\boldsymbol{\alpha}'}\phi(t,0;\mathbf{x})\|^2 + c\|T^{-1}\partial_{\mathbf{x}}^{\boldsymbol{\alpha}'}\phi(t,0;\mathbf{x})\| e^{-\nu' t} \\
&\qquad\qquad \text{by (4.21) and (4.22)} \\
&\quad \leq -2\nu'\|T^{-1}\partial_{\mathbf{x}}^{\boldsymbol{\alpha}'}\phi(t,0;\mathbf{x})\|^2 + c\|T^{-1}\partial_{\mathbf{x}}^{\boldsymbol{\alpha}'}\phi(t,0;\mathbf{x})\| e^{-\nu' t}
\end{aligned}$$

by definition of ν'.

Now we use Gronwall's Lemma for $G(t) = \|T^{-1}\partial_{\mathbf{x}}^{\boldsymbol{\alpha}'}\phi(t,0;\mathbf{x})\| \geq 0$: as $\frac{d}{dt}G(t) \leq -2\nu' G(t) + ce^{-\nu' t}$ we have, by integration from 0 to t,

$$\begin{aligned}
\frac{d}{d\tau}\left(e^{2\nu'\tau}G(\tau)\right) &\leq ce^{\nu'\tau} \\
e^{2\nu' t}G(t) - G(0) &\leq \frac{c}{\nu'}\left(e^{\nu' t} - 1\right) \\
G(t) &\leq e^{-2\nu' t}G(0) + \frac{c}{\nu'}\left(e^{-\nu' t} - e^{-2\nu' t}\right) \\
&\leq e^{-\nu' t}\left[G(0) + \frac{c}{\nu'}\right]
\end{aligned}$$

for $t \geq 0$. This shows (4.23).

Next, we show that $\int_0^T \partial_{\mathbf{x}}^{\boldsymbol{\alpha}}(\phi(\tau,0;\mathbf{x})^T C\phi(\tau,0;\mathbf{x}))\,d\tau$ converges uniformly with respect to $\mathbf{x}$ as $T \to \infty$ for $1 \leq |\boldsymbol{\alpha}| \leq \sigma - 1$. Let $\mathbf{x} \in A(\mathbf{x}_0)$ and let O be a bounded, open neighborhood of $\mathbf{x}$, such that $\overline{O} \subset A(\mathbf{x}_0)$. Since $\overline{O}$ is compact, there is a $T \in \mathbb{R}_0^+$ such that $S_{T+t}\overline{O} \subset U$ holds for all $t \geq 0$. Hence, it is sufficient to show the statement for all $\mathbf{x} \in U$. Using (4.23) and a generalised Leibniz rule we have

$$\int_0^T \|\partial_{\mathbf{x}}^{\boldsymbol{\alpha}}(\phi(\tau,0;\mathbf{x})^T C\phi(\tau,0;\mathbf{x}))\|\,d\tau \leq \int_0^T \tilde{c}e^{-2\nu'\tau}\,d\tau$$

for all $\mathbf{x} \in U$, $T \geq 0$. Hence, $\int_0^T \partial_{\mathbf{x}}^{\boldsymbol{\alpha}}(\phi(\tau,0;\mathbf{x})^T C\phi(\tau,0;\mathbf{x}))\,d\tau$ converges uniformly as $T \to \infty$. This proves that $M \in C^{\sigma-1}(A(\mathbf{x}_0), \mathbb{S}^n)$. □

5. Examples

In this section we explicitly calculate the contraction metrics from Theorems 4.2 and 4.4 for particular, one- and two-dimensional examples. Example 5.1 shows that in general the solution $M(\mathbf{x})$ will not be uniformly positive definite. Example 5.2 shows that in general we cannot expect that $\sup_{\mathbf{x}\in A(\mathbf{x}_0)} \lambda_{max}(M(\mathbf{x}))$ is bounded for the contraction metric from Theorem 4.4. Finally, we compare the contraction metrics from Theorems 4.2 and 4.4 in the two-dimensional Example 5.3.

Example 5.1. *Consider the one-dimensional ODE*

$$\dot{x} = -x - x^3.$$

We will compute the contraction metrics from Theorem 4.2 and 4.4 for this example. Note that $x = 0$ is the only equilibrium and $A(0) = \mathbb{R}$. The solution of the ODE with initial condition x is given by

$$S_t x = \frac{x}{\sqrt{e^{2t}(1 + x^2) - x^2}}.$$

We start with the contraction metric from Theorem 4.4. We have $Df(x) = -1 - 3x^2$ and

$$Df(S_t x) = -1 - 3\frac{x^2}{e^{2t}(1 + x^2) - x^2}.$$

The solution of $\dot{y} = Df(S_t x)y$, $y(0) = 1$ is given by

$$\phi(t, 0; x) = \frac{e^{-t}}{(1 + x^2(1 - e^{-2t}))^{\frac{3}{2}}}.$$

The solution of (4.8) with $C = 1$ is given by (4.13), hence

$$\begin{aligned}
M(x) &= \int_0^\infty \phi(t, 0; x)^2 \, dt \\
&= \int_0^\infty \frac{e^{-2t}}{(1 + x^2(1 - e^{-2t}))^3} \, dt \\
&= -\frac{1}{4x^2}(1 + x^2(1 - e^{-2t}))^{-2}\Big|_0^\infty \\
&= \frac{2 + x^2}{4(1 + x^2)^2}.
\end{aligned}$$

Note that this also holds for $x = 0$. One can easily check that

$$2M(x)Df(x) + M'(x) = -1.$$

Now let us compute the contraction metric from Theorem 4.2. We have $M_0 = 1$ and

$$\mathcal{L}_{M_0}(x) = Df(x) = -1 - 3x^2.$$

Hence,

$$\mathcal{L}_{M_0}(S_t x) = -1 - 3\frac{x^2}{e^{2t}(1+x^2) - x^2}.$$

Then

$$\begin{aligned} V(x) &= -3\int_0^\infty \frac{x^2 e^{-2t}}{1 + x^2 - e^{-2t}x^2}\, dt \\ &= -\frac{3}{2}\ln(1 + x^2 - e^{-2t}x^2)\Big|_0^\infty \\ &= -\frac{3}{2}\ln(1+x^2), \\ M(x) &= e^{2V(x)}M_0 \\ &= (1+x^2)^{-3}. \end{aligned}$$

We can check that

$$\begin{aligned} L_M(x;v) &= \left(\frac{1}{2}M'(x) + Df(x)M(x)\right)v^2 \\ &= \left(-\frac{3}{2}(1+x^2)^{-4}2x(-x-x^3) + (-1-3x^2)(1+x^2)^{-3}\right)v^2 \\ &= -(1+x^2)^{-3}v^2, \\ \mathcal{L}_M(x) &= \max_{v^2 M(x)=1}[-(1+x^2)^{-3}v^2] \\ &= -1. \end{aligned}$$

Note that for both contraction metrics we have $M(x) > 0$ for all x, but $\lim_{|x|\to\infty} M(x) = 0$. There is, however, a uniformly positive definite contraction metric for this example, namely $M(x) = 1$, which satisfies

$$\mathcal{L}_M(x) = Df(x) = -1 - 3x^2 \leq -1$$

for all $x \in \mathbb{R}$.

Example 5.2. *Consider the one-dimensional ODE*

$$\dot{x} = f(x) := \begin{cases} -\operatorname{sign} x & \text{for } |x| \geq 1 \\ -\frac{3}{2}x + \frac{x^3}{2} & \text{for } |x| < 1 \end{cases}$$

We will compute the contraction metric M from Theorem 4.4 for this example; we will observe that in this case this is possible and $M \in C^1(\mathbb{R}, \mathbb{S})$, although we only have $f \in C^1(\mathbb{R}, \mathbb{R})$. Note that $x = 0$ is the only equilibrium and $A(0) = \mathbb{R}$.

We first consider the case $|x| < 1$. The solution of the ODE with initial condition $|x| < 1$ is given by

$$S_t x = \frac{x e^{-\frac{3}{2}t}}{\sqrt{1 + \frac{x^2}{3}(e^{-3t} - 1)}}.$$

We have $Df(x) = -\frac{3}{2} + \frac{3}{2}x^2$ and

$$Df(S_t x) = -\frac{3}{2} + \frac{3}{2} \frac{x^2 e^{-3t}}{1 + \frac{x^2}{3}(e^{-3t} - 1)}.$$

The solution of $\dot{y} = Df(S_t x)y$, $y(0) = 1$ is given by (4.13), hence

$$\phi(t, 0; x) = \frac{e^{-\frac{3}{2}t}}{(1 + \frac{x^2}{3}(e^{-3t} - 1))^{3/2}}.$$

The solution of (4.8) with $C = 1$ is given by

$$\begin{aligned} M(x) &= \int_0^\infty \phi(t, 0; x)^2 \, dt \\ &= \int_0^\infty \frac{e^{-3t}}{(1 + \frac{x^2}{3}(e^{-3t} - 1))^3} \, dt \\ &= \frac{1}{2x^2}(1 + \frac{x^2}{3}(e^{-3t} - 1))^{-2} \Big|_0^\infty \\ &= \frac{12 - 2x^2}{(6 - 2x^2)^2}. \end{aligned}$$

Note that this holds also for $x = 0$.

For $|x| > 1$ we can solve the equation $M'(x)+2Df(x)M(x) = \mp M_x = -1$ directly to obtain $M(x) = c_\pm + |x|$. We match the constants $c_\pm = \frac{5}{8}$ to obtain a continuous function M as follows

$$M(x) \;=\; \begin{cases} -\frac{3}{8} + |x| & \text{for } |x| \geq 1, \\ \frac{12-2x^2}{(6-2x^2)^2} & \text{for } |x| < 1. \end{cases}$$

Moreover, M is C^1. One can easily check that

$$2M(x)Df(x) + M'(x) = -1.$$

Note that $M(x) > 0$ for all x, but $\lim_{|x|\to\infty} M(x) = \infty$. Hence, there is no negative bound on the value of $\mathcal{L}_M(\mathbf{x})$ for all $\mathbf{x} \in A(0)$ as given in (4.7) since $\sup_{x\in A(0)} \lambda_{max}(M(x)) = \infty$.

Example 5.3. *Consider the linear, two-dimensional ODE*

$$\dot{\mathbf{x}} = \begin{pmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix} \mathbf{x},$$

where $\lambda_1, \lambda_2 > 0$. We will compute the contraction metrics from Theorem 4.2 and 4.4 for this example and observe that the contraction metric from Theorem 4.2 is independent of λ_1 and λ_2, whereas the contraction metric from Theorem 4.4 takes the different contraction in directions $\mathbf{v} = \mathbf{e}_1$ and $\mathbf{v} = \mathbf{e}_2$ into account. Note that $\mathbf{x}_0 = \mathbf{0}$ is the only equilibrium and $A(\mathbf{0}) = \mathbb{R}^2$.

For the contraction metric from Theorem 4.2 we obtain $M_0 = I$ and $V(\mathbf{x}) = 0$; hence

$$M(\mathbf{x}) = I,$$

independent of λ_1, λ_2. We can check that

$$\begin{aligned} L_M(\mathbf{x};\mathbf{v}) &= \mathbf{v}^T \begin{pmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix} \mathbf{v} \\ &= -\lambda_1 v_1^2 - \lambda_2 v_2^2 \\ \mathcal{L}_M(x) &= \max_{\|\mathbf{v}\|=1} L_M(\mathbf{x};\mathbf{v}) \\ &= -\min(\lambda_1, \lambda_2). \end{aligned}$$

For the contraction metric from Theorem 4.4 we solve the Lyapunov equation for $C = I$ and obtain

$$M(\mathbf{x}) = B = \frac{1}{2} \begin{pmatrix} \frac{1}{\lambda_1} & 0 \\ 0 & \frac{1}{\lambda_2} \end{pmatrix}.$$

This metric takes the different contraction for different $\mathbf{v}$ *into account and results in a uniform contraction, i.e.* $L_M(\mathbf{x};\mathbf{v}) = -\frac{1}{2}$ *for all* $\|\mathbf{v}\| = 1$*. For* $\mathcal{L}_M$*, however, for this example, we have the same result as for the metric from Theorem 4.1, namely*

$$\mathcal{L}_M(\mathbf{x}) = \max_{\mathbf{v}^T M(\mathbf{x})\mathbf{v}=1} L_M(\mathbf{x};\mathbf{v}) = -\min(\lambda_1, \lambda_2).$$

Summarising, considering (2.6), the contraction metric from Theorem 4.2 takes account of the different eigenvalues $-\lambda_1, -\lambda_2$ *in the nominator of* $\widetilde{L}$*, whereas the contraction metric from Theorem 4.4 deals with them in the denominator.*

6. Summary and outlook

In this paper we have established three converse theorems, showing the existence of a contraction metric for an exponentially stable equilibrium. In Theorem 4.1, the metric is only defined in a compact subset of the basin of attraction and $\mathcal{L}_M$ satisfies an inequality, arbitrarily close to the expected bound. Moreover, M is C^σ, i.e. as smooth as $\mathbf{f} \in C^\sigma$.

The metric in Theorem 4.2 is defined in the whole basin of attraction and $\mathcal{L}_M$ satisfies an equation, which is beneficial for its construction, arbitrarily close to the expected bound $-\nu$. However, M is only continuous with continuous orbital derivative.

Finally, the metric in Theorem 4.4 is defined in the whole basin of attraction and M satisfies a (matrix) equation; this is the best starting point for a construction, as the equation does not involve any maximum. Moreover, M is $C^{\sigma-1}$, so nearly as smooth as $\mathbf{f} \in C^\sigma$ is. This contraction metric takes the contraction in different directions into account. However, $\mathcal{L}_M(\mathbf{x})$ might not be bounded away from 0 for all $\mathbf{x} \in A(\mathbf{x}_0)$. But the exponential attraction is determined at the equilibrium point, and here $\mathcal{L}_M$ is bounded, although the bound may not be optimal, see Example 4.3.

These converse theorems can be used to develop construction algorithms for contraction metrics and prove that these algorithms will always succeed. Moreover, similar results could be established for other attractors such as periodic orbits, where the contraction is restricted to a subspace.

Appendix A. Exponential bounds on variation equation

In this appendix we will show that the solution of the linear, time-dependent ODE $\dot{\mathbf{y}} = D\mathbf{f}(S_t\mathbf{x})\mathbf{y}$ is exponentially decreasing, if $\mathbf{x}$ is in the basin of attraction of an exponentially stable equilibrium $\mathbf{x}_0$. To establish this result, we use a special type of Gronwall-Lemma, see e.g. [26], Lemma D.2.

Lemma Appendix A.1. *Let $r, K, a \in L^1_{loc}([0,\infty),\mathbb{R})$ be nonnegative functions and let $b \in L^\infty_{loc}([0,\infty),\mathbb{R})$ be a continuous nonnegative function such that*

$$r(t) \leq a(t) + K(t)\int_0^t b(s)r(s)\,ds$$

holds for almost all $t \geq 0$.

Then

$$r(t) \leq a(t) + K(t)\int_0^t a(s)b(s)\,ds \cdot \exp\left(\int_0^t K(s)b(s)\,ds\right)$$

holds for almost all $t \geq 0$.

Lemma Appendix A.2. *Let $A\colon [0,\infty) \to \mathbb{R}^{n\times n}$ be a continuous matrix-valued function such that*

$$\|A(t) - A\| \leq d_1 e^{-\mu t} \text{ holds for all } t \geq 0,$$

where $A \in \mathbb{R}^{n\times n}$ and $\mu > 0$.

Moreover, assume that all eigenvalues of A have negative real part, i.e. in particular $\|e^{At}\| \leq d_2 e^{-\rho t}$ holds for all $t \geq 0$ with $\rho > 0$.

Then there is a constant $c > 0$, depending only on d_1, d_2 and μ, such that for all solutions ϕ of

$$\dot{\phi} = A(t)\phi$$

the following estimate

$$\|\phi(t)\| \leq ce^{-\rho t}\|\phi(0)\| \text{ holds for all } t \geq 0.$$

Proof: For $\phi(0) = 0$ the statement is clear, so we now assume $\|\phi(0)\| > 0$ and, from ODE theory, know that $\|\phi(t)\| > 0$ for all $t \geq 0$. We have for all $s > 0$, using the ODE for ϕ

$$\frac{d}{ds}\left(e^{-As}\phi(s)\right) \;=\; e^{-As}\left(A(s) - A\right)\phi(s).$$

Integrating both sides from 0 to $t \geq 0$ we obtain

$$\begin{aligned}
e^{-At}\phi(t) - \phi(0) &= \int_0^t e^{-As}\,(A(s) - A)\,\phi(s)\,ds \\
\phi(t) &= e^{At}\phi(0) + \int_0^t e^{A(t-s)}\,(A(s) - A)\,\phi(s)\,ds \\
\|\phi(t)\| &\leq \|e^{At}\|\|\phi(0)\| + \int_0^t \|e^{A(t-s)}\|\|A(s) - A\|\|\phi(s)\|\,ds \\
&\leq d_2 e^{-\rho t}\|\phi(0)\| + \int_0^t d_2 e^{-\rho(t-s)} d_1 e^{-\mu s}\|\phi(s)\|\,ds \\
&= d_2 e^{-\rho t}\|\phi(0)\| + d_1 d_2 e^{-\rho t}\int_0^t e^{s(\rho-\mu)}\|\phi(s)\|\,ds.
\end{aligned}$$

Now we apply Lemma Appendix A.1 with $r(t) = \|\phi(t)\| > 0$, $a(t) = d_2 e^{-\rho t}\|\phi(0)\| > 0$, $K(t) = d_1 d_2 e^{-\rho t} > 0$ and $b(t) = e^{t(\rho-\mu)} > 0$, giving

$$\begin{aligned}
\|\phi(t)\| &\leq d_2 e^{-\rho t}\|\phi(0)\| \\
&\quad + d_1 d_2 e^{-\rho t}\int_0^t d_2 e^{-\rho s}\|\phi(0)\| e^{s(\rho-\mu)}\,ds \cdot \exp\left(d_1 d_2 \int_0^t e^{-\rho s} e^{s(\rho-\mu)}\,ds\right) \\
&= d_2 e^{-\rho t}\|\phi(0)\| + d_1 d_2^2 \|\phi(0)\| e^{-\rho t}\int_0^t e^{-\mu s}\,ds \cdot \exp\left(d_1 d_2 \int_0^t e^{-\mu s}\,ds\right) \\
&\leq d_2 e^{-\rho t}\|\phi(0)\| + \frac{d_1 d_2^2}{\mu}\|\phi(0)\| e^{-\rho t} \cdot \exp\left(\frac{d_1 d_2}{\mu}\right)
\end{aligned}$$

using $\int_0^t e^{-\mu s}\,ds = \frac{1}{\mu}(1 - e^{-\mu t}) \leq \frac{1}{\mu}$. Note that this holds for all $t \geq 0$ since $\phi(t)$ is continuous. This shows that statement with $c = d_2 + \frac{d_1 d_2^2}{\mu} \cdot \exp\left(\frac{d_1 d_2}{\mu}\right)$. $\square$

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