

Effect of multimode entanglement on lossy optical quantum metrology

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In optical interferometry multimode entanglement is often assumed to be the driving force behind quantum enhanced measurements. Recent work has shown this assumption to be false: single-mode quantum states perform just as well as their multimode entangled counterparts. We go beyond this to show that when photon losses occur, an inevitability in any realistic system, multimode entanglement is actually detrimental to obtaining quantum enhanced measurements. We specifically apply this idea to a superposition of coherent states, demonstrating that these states show a robustness to loss that allows them to significantly outperform their competitors in realistic systems. A practically viable measurement scheme is then presented that allows measurements close to the theoretical bound, even with loss. These results promote an alternate way of approaching optical quantum metrology using single-mode states that we expect to have great implications for the future.

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I. INTRODUCTION

In recent years quantum metrology has begun to fulfill its potential as an important practical method of enhancing precision measurements [1–5]. It has a wide range of applications, from time and frequency measurements [6,7] to lithography [8,9], and is already being used to surpass the classical limit in gravitational wave detectors [10–12]. The precision gains offered by quantum metrology are often attributed to entanglement [13–17] and, specifically in the optical case, entanglement between two modes in an interferometer [18,19]. However, more recently it has been shown that entanglement is not required between the probe and reference systems for Heisenberg limited measurements of a linear phase shift [20–22], and furthermore it has been argued that the important resource for enhancing precision is actually the coherence in the eigenbasis of the phase shift Hamiltonian [23]. We go beyond this to demonstrate that, in some well-known scenarios, single-mode superposition states have a significantly better robustness to loss than their multimode entangled counterparts, which allows them to achieve greatly enhanced precision measurements. Here we discuss an unbalanced-cat state, which can outperform the alternatives and can be created and measured with present-day or near future technology, to a precision close to its theoretical bound.

When measuring fragile systems it is imperative that high precision is achieved from a limited number of probe particles passing *through the sample*. Relevant examples of such systems include spin ensembles [24], biological systems [3,25], atoms [26,27], and single molecules [28]. In this regime we show that single-mode states show significant improvements over the multimode alternatives and furthermore beat the “optimal state” [29–31] for most loss rates, illustrating the importance of our results. We begin by examining the ideal case where no photon losses are present.

II. NO LOSS: SUPERPOSITION STATES ARE SUFFICIENT

The quantum Fisher information (QFI), which quantifies a state’s ability to measure a phase ϕ , for a general density matrix ρ , is given by [32–34]

$$F_Q = \sum_{i,j} \frac{2}{\lambda_i + \lambda_j} |\langle \lambda_i | \partial \rho(\phi) / \partial \phi | \lambda_j \rangle|^2, \quad (1)$$

where λ_i are the eigenvalues and $|\lambda_i\rangle$ a corresponding set of orthonormal eigenvectors of ρ . For a pure state $|\Psi\rangle$ the QFI is [35] $F_Q = 4[\langle \Psi' | \Psi' \rangle - |\langle \Psi' | \Psi \rangle|^2]$, where $|\Psi'\rangle = \frac{\partial}{\partial \phi} |\Psi\rangle$. The fundamental limit to the precision with which the state ρ can measure a phase ϕ is then given by the quantum Cramér-Rao bound (CRB) [32,33]:

$$\delta\phi \geq \frac{1}{\sqrt{m F_Q}}, \quad (2)$$

where m is the number of times that the measurement is independently repeated. From this it is straightforward to show that a linear phase measurement involving N independent particles gives a precision at the shot noise limit (SNL), given by $\delta\phi = 1/\sqrt{N}$ [36–39].

A well-studied state for quantum-enhanced metrology is the NOON state [13,14,40] given by $|\Psi_{\text{NOON}}\rangle = \frac{1}{\sqrt{2}}(|N,0\rangle_{1,2} + |0,N\rangle_{1,2})$, where the subscripts refer to two different modes in a Mach-Zehnder interferometer. This state is maximally entangled and, using Eq. (2), it can be shown that the NOON state can measure a phase with a quantum-enhanced precision of $\delta\phi_{\text{NOON}} = 1/N$, the Heisenberg limit. However, the Heisenberg limit is attainable without the multimode entanglement exhibited by the NOON state, simply by utilizing an analogous single-mode superposition state $|\Psi_{\text{NO}}\rangle = \frac{1}{\sqrt{2}}(|N\rangle + |0\rangle)$ [22], which we refer to as the NO state.

An alternative state that has been shown to be useful for quantum metrology is the entangled coherent state (ECS) $|\Psi_{\text{ECS}}\rangle = \mathcal{N}_e(|\alpha,0\rangle_{1,2} + |0,\alpha\rangle_{1,2})$ [34,41–48], where $\mathcal{N}_e = 1/\sqrt{2 + 2e^{-\alpha^2}}$ and α characterizes the coherent state (we take α to be real throughout without loss of generality). The QFI

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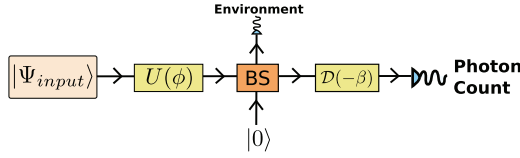


FIG. 1. (Color online) The input state $|\Psi_{\text{input}}\rangle$ first undergoes a phase shift $U(\phi) = e^{i\hat{a}^\dagger \hat{a} \phi}$. We model loss by the addition of a beam splitter with a vacuum port and then trace over the environmental mode. To read out the phase we apply the displacement operator $\mathcal{D}(-\beta)$, with coherent state amplitude $-\beta$ and then count the number of photons in the state.

for this state is given by

$$F_Q = 4\alpha^2 \mathcal{N}_e^2 (1 + \alpha^2 - \alpha^2 \mathcal{N}_e^2), \quad (3)$$

which approximately scales as $F_Q \propto \alpha^4$. However, a very similar QFI can be obtained without the entanglement by utilizing the single-mode analog of the ECS, a balanced-cat state (equally weighted superposition of two coherent states), given by $|\Psi_{\text{cat}}\rangle = \mathcal{N}_c(|\alpha\rangle + |0\rangle)$ where $\mathcal{N}_c = 1/\sqrt{2 + 2e^{-\alpha^2/2}}$. Its QFI is also given by Eq. (3) but with $\mathcal{N}_e \rightarrow \mathcal{N}_c$.

III. ENTANGLEMENT IS DETRIMENTAL WITH LOSS

We now show that in the presence of photon losses the single-mode states have a significant improvement in phase sensitivity over their multimode analogues. We model loss by the addition of a beam splitter after the phase shift [34,35,49], as shown in Fig. 1, which has a probability of transmission η (and therefore a fraction $\mu = 1 - \eta$ of photons are lost). After tracing over the environment we have a mixed state ρ , and from this density matrix the QFI can be determined. The CRB as a function of loss has been calculated for the NOON state [35] and the ECS [50], and we have analytically calculated this for the NO and cat states. The results, in Fig. 2, show that with loss the NO state ($\delta\phi_{\text{NO}}$, black dashed-dotted line) can measure a phase to a higher precision than a NOON state ($\delta\phi_{\text{NOON}}$, blue dots). However, it is not clear how to create a NO state in a physically viable fashion [51].

We can see from Fig. 2 that in the range of reasonable experimental transmission rates, $0.5 \leq \eta \leq 1$ (for example, $\eta = 0.62$ in Ref. [52]), the precision obtained by the cat state ($\delta\phi_{\text{cat}}$, purple dashed line) is significantly better than the ECS ($\delta\phi_{\text{ECS}}$, green solid line). In this region the multimode entanglement in the ECS leads to a more fragile state and a worse precision. Despite this, we can see that for higher loss rates the ECS performs better than the cat state. We now show that the single-mode states can be modified to overcome this issue.

IV. THE UNBALANCED-CAT STATE

We now discuss a single-mode state that generalizes the cat state and displays an improvement in phase sensitivity over the ECS (and the other alternatives) for all values of loss. This unbalanced-cat state (UCS) is given by

$$|\Psi_{\text{UCS}}\rangle = \mathcal{N}_u(|\alpha(a)\rangle + a|0\rangle), \quad (4)$$

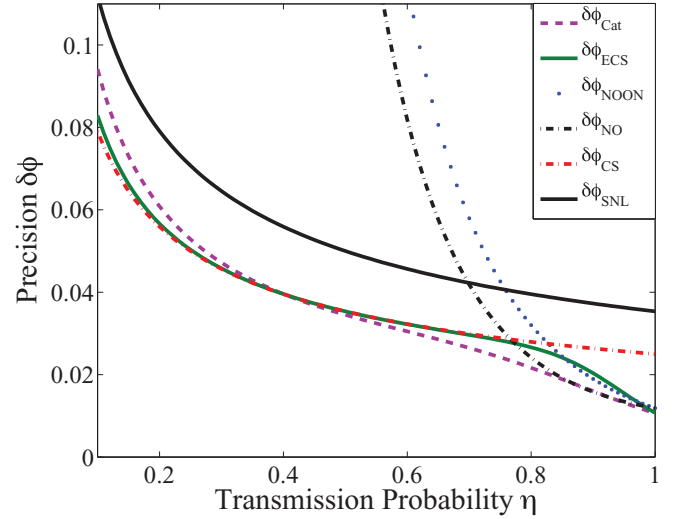


FIG. 2. (Color online) Multimode entanglement reduces phase precision. We show the CRB, calculated from the (analytical) QFI, for the cat state $\delta\phi_{\text{cat}}$, ECS $\delta\phi_{\text{ECS}}$, NOON state $\delta\phi_{\text{NOON}}$, NO state $\delta\phi_{\text{NO}}$, coherent state $\delta\phi_{\text{CS}}$, and SNL $\delta\phi_{\text{SNL}}$. Here $\alpha = 3$, and for fair comparison the NOON and NO states have N such that the number of photons per state through the phase shift \bar{n}_ϕ is equal for each state. Therefore $\bar{n}_\phi(\text{NOON}) = \bar{n}_\phi(\text{NO}) = N/2$ is equal to $\bar{n}_\phi(\text{ECS}) = \bar{n}_\phi(\text{cat}) = \mathcal{N}^2 \alpha^2$. We repeat each state m times so that the total number of photons sent through the phase shift is $R_\phi = m\bar{n}_\phi = 400$ (this is the same throughout our results).

where $0 \leq a \leq 1$, $\mathcal{N}_u = 1/\sqrt{1 + a^2 + 2ae^{-\alpha(a)^2/2}}$, and $\alpha(a)$ is the solution to $\alpha^2(a) = \bar{n}_\phi/\mathcal{N}_u^2[\alpha(a)]$, where \bar{n}_ϕ is the number of photons passing through the phase shift per state. $\alpha(a)$ is defined in such a way as to keep the average number of photons through the phase shift independent of a , and it can be expressed in terms of the Lambert W function. We note that taking $a = 1$ in Eq. (4) gives a balanced-cat state of magnitude $\alpha_{\text{bal}} = \alpha(a = 1)$, and $a = 0$ gives a coherent state. One of the advantages of this state is that the ‘‘quantumness’’ of the state can be altered by varying the parameter a . Loss collapses the quantum superposition, and so when there is high loss we can reduce a so that the state behaves more like a coherent state $|\alpha\rangle$, and with low loss we can set $a \sim 1$ so that we have an equal superposition state.

The reduced density matrix ρ_{UCS} for the UCS after the phase shift and loss is given by

$$\begin{aligned} \rho_{\text{UCS}} = & \mathcal{N}_u^2 [|\alpha_\eta(a)e^{i\phi}\rangle\langle\alpha_\eta(a)e^{i\phi}| + a^2|0\rangle\langle 0| \\ & + ae^{-\alpha_\mu^2(a)/2}(|\alpha_\eta(a)e^{i\phi}\rangle\langle 0| + |0\rangle\langle\alpha_\eta(a)e^{i\phi}|)], \end{aligned} \quad (5)$$

where $\alpha_\eta(a) = \alpha(a)\sqrt{\eta}$ and $\alpha_\mu(a) = \alpha(a)\sqrt{\mu}$. Using a similar method to that in Ref. [50] we can represent and then diagonalize ρ_{UCS} in the orthogonal cat state basis $|\Psi_\pm\rangle = \mathcal{N}_\pm(|\alpha_\eta(a)e^{i\phi}\rangle \pm |0\rangle)$ to find the two nonzero eigenvalues and the corresponding eigenvectors. Using Eq. (1) and Eq. (2) we then analytically calculate the QFI and the CRB, and optimize this over the range of possible choices of a for each value of loss.

We see in Fig. 3 that the CRB for the UCS ($\delta\phi_{\text{UCS}}$, yellow solid line) improves upon the cat state ($\delta\phi_{\text{cat}}$, purple dashed

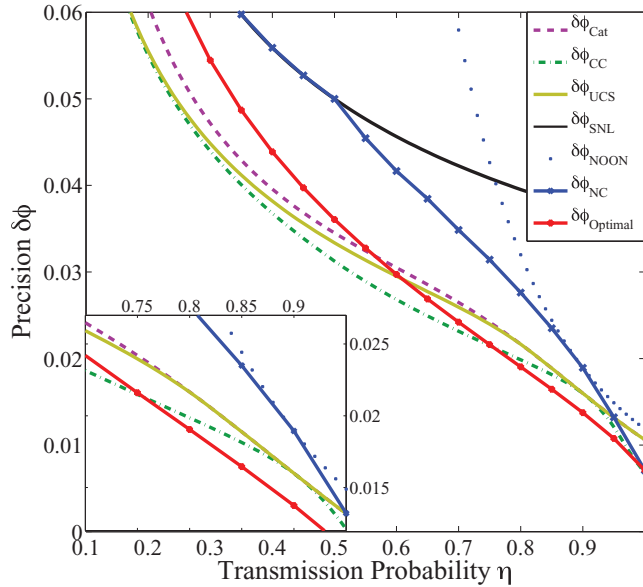


FIG. 3. (Color online) We see the large improvements gained by our single-mode states. We show the (analytical) CRB for the cat state $\delta\phi_{\text{cat}}$, UCS chopping $\delta\phi_{\text{CC}}$, unbalanced-cat state $\delta\phi_{\text{UCS}}$, SNL $\delta\phi_{\text{SNL}}$, NOON state $\delta\phi_{\text{NOON}}$, NOON chopping strategy $\delta\phi_{\text{NC}}$, and the optimal state in Ref. [30] $\delta\phi_{\text{Optimal}}$. Here $\alpha_{\text{bal}} = 3$, and for the chopped states we limit the cat state to $\alpha_{\text{bal}} \leq 5$ (and equivalently limit the NOON state and optimal state).

line). Although this improvement in the CRB is marginal, we will show that with a simple and practical measurement scheme the UCS, unlike the balanced-cat state, can be utilized for phase measurements close to the CRB. We note that both the cat and the UCS show large precision improvements over the SNL.

We can obtain a better precision still by using a “chopping strategy,” introduced in the case of NOON states in Ref. [53], in which different sized UCSs (i.e., different \bar{n}_ϕ) are used for different loss rates. We fix the total number of photons allowed through the phase shift, R_ϕ , and therefore the number of times m that a state is sent through the phase shift is inversely proportional to its average photon number \bar{n}_ϕ . The green dashed-dotted line ($\delta\phi_{\text{CC}}$) in Fig. 3 shows a UCS optimized over \bar{n}_ϕ and the unbalancing parameter a for each loss rate. The chopping strategy utilizes the fact that larger states obtain higher precision for low loss, whereas for high-loss smaller states are more robust and are therefore preferable. For this reason, when there is no loss it is advantageous to take the largest possible state, which we limit here to having magnitude $\alpha_{\text{bal}} = 5$, as larger states than this are physically unrealistic. We see that this chopping strategy applied to the unbalanced cat displays further improvements over all the alternatives, including a vast improvement over the NOON chopping strategy ($\delta\phi_{\text{NC}}$) and the SNL ($\delta\phi_{\text{SNL}}$).

A lot of work has been done in recent years in establishing the ultimate precision bounds of quantum metrology [30,31,38,39]. These bounds have been derived by counting the total number of particles used as the relevant resource. However, as we have discussed above, in probing delicate systems [3,24–28] it is the total number of particles

through the sample that is important. To further emphasize the precision gains achieved by single-mode states in this regime, we have included the “optimal state” in Fig. 3 ($\delta\phi_{\text{Optimal}}$, red line with dots) as given in [29–31]. Our improvement over the “optimal state” for most loss rates illustrates the potential of single-mode states.

By looking at the theoretical limits on the precision (given by the CRB) for various single-mode states, it is clear that these states have huge potential for making quantum-enhanced measurements. Despite this, it is not always clear how to make measurements that saturate this limit, and it is this issue that we turn to next.

V. A MEASUREMENT SCHEME FOR THE UCS WITH LOSS

We will now describe a simple and practical scheme, shown in Fig. 1, for measuring a phase using a UCS, in the presence of loss, that comes close to the theoretical precision limit given by the CRB. The initial resource required is a UCS. There are many examples of cat state generation techniques, such as that given in Ref. [54]. In this scheme a Rydberg atom in a cavity in the state $|g\rangle + |e\rangle$ is coupled to a coherent state via the Jaynes-Cummings Hamiltonian [55]. The atom-cavity system evolves as $|\alpha'\rangle(|g\rangle + |e\rangle) \rightarrow |\alpha'\rangle|g\rangle + |\alpha'e^{i\phi}\rangle|e\rangle$ and, after a transformation and measurement of the Rydberg atom and taking $\phi = \pi$, the resultant state of the field is an even cat state. Alternative schemes are numerous [56–59], and cat states have been created with $\alpha' = 1.76$ and fidelity 0.59 in the laboratory [58]. Some schemes create states of the form $|\Psi_{\text{cat}}\rangle = \mathcal{N}_c(|\alpha\rangle + |0\rangle)$ directly [59], but if the output state is $\mathcal{N}_c(|\alpha'\rangle + |-\alpha'\rangle)$ the application of a displacement operator [60] will create the state $|\Psi_{\text{cat}}\rangle$. The UCS can be created by simple adaptations of these methods for cat state preparation, for example, preparing the Rydberg atom in the unbalanced state $\mathcal{N}_R(|g\rangle + a|e\rangle)$ will give the output state $|\Psi_{\text{UCS}}\rangle$.

The first step in the phase detection scheme is the application of the linear phase shift to the UCS giving $|\Psi_{\text{UCS}}(\phi)\rangle = \mathcal{N}_c(|\alpha(a)e^{i\phi}\rangle + a|0\rangle)$. As discussed earlier, the loss is then modeled by a beam splitter, as shown in Fig. 1, with the resulting mixed state given by Eq. (6). We then apply the displacement operator $\mathcal{D}(-\beta) = e^{\beta^*\hat{a} - \beta\hat{a}^\dagger}$, which can be implemented in a simple manner by mixing the state with a large local oscillator at a highly transmissive beam splitter [60,61]. This gives

$$\begin{aligned} \rho &= \mathcal{D}(-\beta)\rho_{\text{UCS}}\mathcal{D}^\dagger(-\beta) \\ &= \mathcal{N}_c^2[|\sigma\rangle\langle\sigma| + a^2|-\beta\rangle\langle-\beta| \\ &\quad + ae^{-\alpha_a^2(a)/2}(e^{i\theta}|\sigma\rangle\langle-\beta| + e^{-i\theta}|-\beta\rangle\langle\sigma|)], \end{aligned} \quad (6)$$

where $\theta = \alpha_\eta(a)\beta \sin\phi$ and $\sigma = \alpha_\eta(a)e^{i\phi} - \beta$. A photon-number resolving detector (PNRD) is then used to count the number of particles in the state ρ and a Bayesian scheme is used to infer the phase ϕ , and the precision with which it can be measured $\delta\phi$ (we explain this scheme in detail in Ref. [62]). PNRDs are an area of intense research [63], and devices that are highly sensitive in the low-photon regime, the area most relevant for this work, have been demonstrated [64–66].

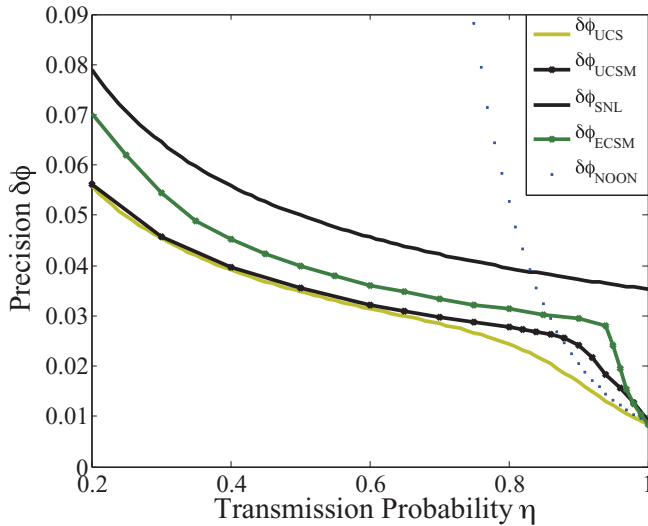


FIG. 4. (Color online) Our measurement scheme, $\delta\phi_{\text{UCSM}}$, comes close to the CRB for the UCS, $\delta\phi_{\text{UCS}}$, and shows large improvements over the ECS measurement scheme in Ref. [62], $\delta\phi_{\text{ECSM}}$. We see that our state surpasses the precision of the NOON state $\delta\phi_{\text{NOON}}$ and the SNL $\delta\phi_{\text{SNL}}$ for most loss rates. Here $\alpha_{\text{bal}} = 4$.

We obtain more precise measurements by taking $\beta > \alpha$, and we optimize over the phase ϕ . The phase precision for this measurement scheme ($\delta\phi_{\text{UCSM}}$), found by simulation, is shown in Fig. 4 (crossed black line) where $\beta = 4\alpha_{\text{bal}}$. Our scheme shows significant improvements over the SNL, the ECS with the measurement scheme in Ref. [62], and the NOON state (for most loss rates). We see that our scheme is much more robust than the NOON state, which is quickly destroyed when the transmission rate drops below $\eta = 0.9$. While the CRB for the UCS shows only a small improvement over the (balanced) cat state, when we consider the measurement scheme the UCS is significantly better.

To understand why the UCS performs so well with this measurement scheme it is instructive to consider the case of a coherent state input, i.e., $|\Psi_{\text{input}}\rangle = |\alpha\rangle$ in Fig 1. To find the phase precision for this input state and measurement we use the propagation of error formula:

$$\delta\phi = \frac{\Delta\hat{X}}{\left|\frac{\partial\langle\hat{X}\rangle}{\partial\phi}\right|}, \quad (7)$$

where $\Delta\hat{X} = \sqrt{\langle\hat{X}^2\rangle - \langle\hat{X}\rangle^2}$, and we take the number counting measurement operator $\hat{X} = \hat{a}^\dagger\hat{a}$. We find that the CRB,

given by $\delta\phi_{\text{CS}} = 1/\sqrt{2\alpha_\eta^2}$ with transmissivity η , is saturated in the limit $\beta \rightarrow \infty$, where β is the displacement parameter. This is a $\sqrt{2}$ improvement over the generic scheme of a coherent state and a vacuum input fed into the arms of a standard Mach-Zehnder interferometer, which can measure at the SNL, $\delta\phi_{\text{SNL}} = 1/\alpha_\eta$. We have found no explicit reference to this optimal measurement for a coherent state in the literature, but note that it bears a similarity to a heterodyne measurement, in which a large reference beam is used to amplify a signal to enhance precision. Since when $a = 0$ a UCS reduces to a coherent state, $\delta\phi_{\text{CS}}$ is the upper bound on the phase precision that will be achieved with a UCS optimized over a . It is not clear how to get this close to saturating the bound for the balanced-cat state, and so the UCS is significantly better when the measurement scheme is considered.

VI. CONCLUSION

High-precision measurements are fundamental to physics, and quantum metrology offers a unique method for improving measurements beyond what is possible with classical physics. We show here that, for optical systems, multimode entanglement is not only unnecessary for phase estimation at the Heisenberg limit, it is actually detrimental to precision measurements when loss is included. Following this principle we introduce a single-mode quantum superposition state: the unbalanced-cat state. This state shows significant improvements over the alternatives and can be created and precisely measured with present-day, or near future, technology. We show that by tuning the degree of superposition in our state, and additionally by ‘‘chopping’’ our states into different sized chunks depending on loss rates, we can produce further improvements to our phase estimation scheme that allow us to surpass the precision obtained by multimode states. This work opens up an alternate approach to optical quantum metrology based on single-mode states which promises huge potential for future precision measurement protocols.

Note added in proof. We would like to acknowledge recent work by Sahota *et al.* [67], which also promotes single-mode optical quantum metrology.

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