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Multiple solutions of the quasirelativistic Choquard equation

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We prove existence of multiple solutions to the quasirelativistic Choquard equation with a scalar potential. © *2012 American Institute of Physics*. [http://dx.doi.org/10.1063/1.3695991]

I. INTRODUCTION

We study the nonlocal and nonlinear problem

$$L\phi + V\phi - |\phi|^2 * W\phi = -\lambda\phi, \tag{1.1}$$

$$\|\phi\|_{L^2(\mathbb{R}^3)} = 1 \tag{1.2}$$

for a large class of potentials V and W, and $L = \sqrt{-\alpha^{-2}\Delta + \alpha^{-4}} - \alpha^{-2}$ (the quasirelativistic Laplacian) with α being Sommerfeld's fine structure constant. This Hartree-like Choquard equation arises as the Euler-Lagrange equation associated with an energy functional $\mathcal{E}(\cdot)$ introduced in (3.2). We prove the existence of multiple solutions for two separate cases. Theorem 3.2 concerns the unconstrained problem (1.1), and Theorem 3.4 treats the constrained problems (1.1) and (1.2).

By replacing L by the negative Laplacian and by choosing $V = 0$, and $W(x) = 1/|x|$, we obtain the nonrelativistic Choquard equation which models an electron trapped in its own hole and was proposed by Choquard in 1976 as an approximation to Hartree-Fock theory of a one-component plasma.[6] In a meson nucleon theory a system similar to this equation, but with $W(x) = \frac{e^{-\mu|x|}}{|x|}$, arises when one includes the nucleon recoil caused by surrounding mesons;[8] this classical model provides solitary waves. A quantum theory of gravitating particles yields another example.[2] Furthermore, the Choquard equation has become a prototype of nonlocal problems, which arise in many situations.[16]

For the nonrelativistic Choquard equation (in the special case $W(x) = 1/|x|$) Lieb[10] proved existence and uniqueness (modulo translations) of a minimizer (for some λ) by using symmetric decreasing rearrangement inequalities. His existence proof can be extended to more general W provided W is symmetric decreasing which, in some sense, has to be considered a severe restriction; regularity of the solution was subsequently studied by Menzala.[13]

Within the same setting, always for the negative Laplacian, Lions[12] proved existence of infinitely many spherically symmetric solutions by application of abstract critical point theory both without the constraint (here it suffices that W is spherical symmetric) and with the constraint (more severe restrictions on W must be assumed). Zhang[17,18] has studied existence of solutions for the nonhomogeneous Choquard equation; considering $\lambda = 1$, a negative V which tends to zero at infinity,

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and adding a positive function g on the right-hand side of (1.1). Küpper *et al.*[9] have studied positive solutions and the bifurcation problem arising when one adds a term $\mu f(x)$ to (1.1); $\mu > 0$ and f being non-negative. Furthermore, Zhang *et al.* have studied existence of solutions, when the right-hand side is multiplied by a positive function which tends to a constant at infinity.[19]

For $V = 0$ and $W = 1/|x|$, the first rigorous study of (1.1) was performed by Lieb and Yau[11] in a slightly different context, when the constraint is replaced by $\|\phi\|_{L^2} = N$. They established the existence of a symmetric decreasing minimizer provided $N < N_{\rm b}$ for some number $N_{\rm b}$.

We prove existence of multiple solutions, including a minimizer of the corresponding energy functional $\mathcal{E}$. Moreover, we prove some additional properties of the solutions. Our proofs are based upon two classic theorems of critical point theory: in the unconstrained case we apply the mountain pass theorem by Ambrosetti and Rabinowitz,[3] and for the constrained case, we apply a suitable variant due to Berestycki and Lions.[5]

II. PRELIMINARIES

Throughout the paper we denote by C (with or without indices) various constants whose precise value is of no importance. Let $\mathbb{R}^N$ be the N-dimensional Euclidean space. We set

$$B_R = \{\, x \in \mathbb{R}^N \,:\, |x| < R\,\}, \quad B(x, R) = \{\, y \in \mathbb{R}^N \,:\, |x - y| < R\,\}.$$

By $\mathbb{S}^{N-1}$ we will denote the unit sphere in $\mathbb{R}^N$.

Functions. By C_0^∞, C^∞, and L^p we refer to the standard function spaces. For a measure space $\langle M, \mu\rangle$, μ being a σ-finite measure, the weak L^p space (or Marcinkiewicz space) is defined as the space of measurable functions ϕ such that

$$\sup_{t>0} t\,\mu\left(\{\, x \,:\, |\phi(x)| > t\}\right)^{1/p} < \infty.$$

The space of bounded measures is denoted $\mathcal{M}_{\rm b}$.

Sobolev spaces. Denoting the Fourier-Plancherel transform of $u \in L^2(\mathbb{R}^3)$ by $\hat{u}$, we define

$$\mathbf{H}^{1/2}(\mathbb{R}^3) = \{\, \phi \in L^2(\mathbb{R}^3) \,:\, (1 + |\xi|)^{1/2}\hat{\phi} \in L^2(\mathbb{R}^3)\,\}, \tag{2.1}$$

which, equipped with the scalar product

$$\langle \phi, \psi\rangle_{\mathbf{H}^{1/2}(\mathbb{R}^3)} = \int_{\mathbb{R}^3} (1 + |\xi|)\hat{\phi}(\xi)\overline{\hat{\psi}(\xi)}\,{\rm d}\xi,$$

becomes a Hilbert space; evidently, $\mathbf{H}^1(\mathbb{R}^3) \subset \mathbf{H}^{1/2}(\mathbb{R}^3)$. We have that $C_0^\infty(\mathbb{R}^3)$ is dense in $\mathbf{H}^{1/2}(\mathbb{R}^3)$ and the continuous embedding $\mathbf{H}^{1/2}(\mathbb{R}^3) \hookrightarrow L^r(\mathbb{R}^3)$ holds whenever $r \in [2, 3]$.[1] Theorem 7.57] Moreover, we shall use that any weakly convergent sequence in $\mathbf{H}^{1/2}(\mathbb{R}^3)$ has a pointwise convergent subsequence. The space of radial (i.e., spherically symmetric) functions belonging to $\mathbf{H}^{1/2}(\mathbb{R}^3)$ will be denoted $\mathbf{H}_{\rm r}^{1/2}(\mathbb{R}^3)$.

Auxiliary results. We need the following "radial" lemma by Lions.[4]

Lemma 2.1: If $u \in L^p(\mathbb{R}^N)$, $1 \le p < \infty$, is a radial non-increasing function (i.e., $0 \le u(x) \le u(y)$ whenever $|x| \ge |y|$), then

$$|u(x)| \le |x|^{-N/p}\left(\frac{N}{|\mathbb{S}^{N-1}|}\right)^{1/p} \|u\|_{L^p(\mathbb{R}^N)}, \quad x \ne 0.$$

Moreover, we will apply the following compactness lemma due to Strauss.[14]

Lemma 2.2: Let P and $Q : \mathbb{R} \to \mathbb{R}$ be two continuous functions satisfying $P(s)/Q(s) \to 0$ as $s \to +\infty$. Let (u_n) be a sequence of measurable functions from $\mathbb{R}^N$ into $\mathbb{R}$ such that

$$\sup_n \int_{\mathbb{R}^N} |Q(u_n(x))|\,dx < \infty$$

and

$$P(u_n(x)) \to v(x) \text{ a.e. } in\ \mathbb{R}^N,\ as\ n \to +\infty.$$

Then for any bounded Borel set Ω *one has*

$$\int_\Omega |P(u_n(x)) - v(x)|\,dx \longrightarrow 0 \ as\ n \to +\infty.$$

If, moreover, one assumes that $P(s)/Q(s) \to 0$ *as* $s \to 0$ *and* $u_n(x) \to 0$ *as* $|x| \to +\infty$ *uniformly with respect to n, then* $P(u_n)$ *converges to* v *in* $L^1(\mathbb{R}^N)$ *as* $n \to \infty$.

Genus. The genus of any compact symmetric subset A of $\mathbf{H}_{\mathrm{r}}^{1/2}(\mathbb{R}^3) \setminus \{0\}$ will be denoted by $\gamma(A)$. Bear in mind that the boundary ∂A of a symmetric bounded neighborhood of 0 in a d-dimensional space has a genus equal to d. For the definition and properties of the genus, we refer to Struwe.[15]

III. ASSUMPTIONS AND MAIN THEOREMS

Functionals. The kinetic energy is defined by

$$\tilde{\mathfrak{t}}_0[\phi] := \alpha^{-1}\|\hat{\phi}(k)\|^2_{L^2(\mathbb{R}^3,(\sqrt{(2\pi|k|)^2+\alpha^{-2}}-\alpha^{-1})\mathrm{d}k)}$$

on $\mathbf{H}^{1/2}(\mathbb{R}^3)$. It is convenient to introduce

$$\mathfrak{t}_0[\phi] := \alpha^{-1}\|\hat{\phi}(k)\|^2_{L^2(\mathbb{R}^3,\sqrt{(2\pi|k|)^2+\alpha^{-2}}\mathrm{d}k)}.$$

Moreover, we introduce

$$\mathfrak{s}_V\ :\ \mathbf{H}^{1/2}(\mathbb{R}^3) \to \mathbb{R} \text{ by } \phi \mapsto \int_{\mathbb{R}^3} V(x)|\phi(x)|^2\mathrm{d}x \tag{3.1}$$

along with (arising from the direct Coulomb energy)

$$\mathcal{J}_W(\psi,\phi) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \psi(x)\phi(y)W(x-y)\mathrm{d}x\mathrm{d}y,$$

whenever it makes sense. We consider the following functional $\mathcal{E}\ :\ \mathbf{H}^{1/2}(\mathbb{R}^3) \to \mathbb{R}$ defined by

$$\phi \mapsto \frac{1}{2}\mathfrak{t}_0[\phi] + \frac{1}{2}\mathfrak{s}_V[\phi] + \frac{1}{2}(\lambda - \alpha^{-2})\|\phi\|^2_{L^2} - \frac{1}{4}\mathcal{J}_W(|\phi|^2, |\phi|^2), \tag{3.2}$$

At this place we do not focus on whether the functionals are well defined or not, this will be discussed in detail in the sequel.

Assumptions. We impose the following conditions.

Assumption 3.1: Let V be a real-valued measurable function on $\mathbb{R}^3$ such that V is non-negative, the associated form $\mathfrak{s}_V$ is $\mathfrak{t}_0$-bounded with bound less than one, and $\mathfrak{t}_0 + \mathfrak{s}_V$ is weakly lower semicontinuous on $\mathbf{H}^{1/2}(\mathbb{R}^3)$. Let W be a non-negative, nonzero, spherically symmetric measure such that there exist $K \geq 1$, $p_k \in (1, \infty)$, with $k \in [1, K]$, and functions W_k satisfying

$$\begin{cases} W = \nu + \sum_{k=1}^K W_k \\ \nu \in \mathcal{M}_{\mathrm{b}}(\mathbb{R}^3), \quad W_k \in L_{\mathrm{w}}^{p_k}(\mathbb{R}^3) \end{cases}.$$

We have the following.

Theorem 3.2: *Let Assumption 3.1 be satisfied. Then, for* $\lambda > \alpha^{-2}$, *there exists a sequence of (nontrivial) solutions* $(u_j)_{j\geq 1}$ *of (1.1) satisfying:*

1. *The functions* u_j *are radial and non-increasing.*
2. *The function* u_1 *is positive and decreasing provided W is non-increasing and V is non-negative and bounded from above.*

3. *One has*

$$0 < \mathcal{E}(u_{j-1}) \le \mathcal{E}(u_j) \xrightarrow[j\to\infty]{} \infty.$$

The general case. We introduce, for $N > 0$, the set

$$\mathcal{C} = \{ u \in \mathbf{H}_{\mathrm{r}}^{1/2}(\mathbb{R}^3) \,:\, \|u\|_{L^2} = N \}.$$

We seek critical points of $\mathcal{E}$ restricted to $\mathcal{C}$.

Assumption 3.3: Let V satisfy the hypotheses in Assumption 3.1. Let W be a non-negative, nonzero, spherically symmetric measure such that there exist $K \ge 1$, $p_k \in (3/2, \infty)$, with $k \in [1, K]$, and functions W_k satisfying

$$W = \sum_{k=1}^{K} W_k, \qquad W_k \in L_{\mathrm{w}}^{p_k}(\mathbb{R}^3).$$

The main result is as follows.

Theorem 3.4: *Let Assumption 3.3 be satisfied and let $d \ge 1$. Suppose there exists a compact symmetric set Ω such that*

$$\Omega \subset \mathcal{C}\,; \quad \gamma(\Omega) \ge d\,, : \quad \mathcal{E}(u) < 0 \text{ for } u \in \Omega. \tag{3.3}$$

Then there exists a sequence of pairs $(\lambda_j, u_j)_{1 \le j \le d}$ satisfying

$$\begin{cases} \alpha^{-2} < \lambda_j < \infty \\ u_j \text{ is a solution of } (I.1) \text{ with } \lambda = \lambda_j \end{cases},$$

and, furthermore, one has the following:

1. *The function u_1 is positive and*

$$\mathcal{E}(u_1) = \min_{\phi \in \mathcal{C}} \mathcal{E}(\phi) < 0.$$

2. *The functions u_j belong to $\mathcal{C}$.*
3. *One has $\mathcal{E}(u_1) \le \mathcal{E}(u_2) \le \cdots \le \mathcal{E}(u_j) < 0$.*
4. *All u_j are distinct.*

If (3.3) holds for all d, then assertions 1–3 are valid for $j \ge 1$ and $\mathcal{E}(u_j) \nearrow 0$ as $j \to \infty$.

IV. UNCONSTRAINED PROBLEM: PROOF OF THEOREM 3.2

We begin with the following auxiliary result.

Lemma 4.1: For every $u \in \mathbf{H}^{1/2}(\mathbb{R}^3)$ we have

$$\frac{1}{2}\|u\|^2_{\mathbf{H}^{1/2}} \le \langle u, (\sqrt{-\Delta + \alpha^{-2}})u\rangle \le \alpha^{-1}\|u\|^2_{\mathbf{H}^{1/2}}. \tag{4.1}$$

Proof: For every real $a \ge 0$ and $b \ge 1$ we have the following inequality:

$$\frac{a+1}{2} \le \sqrt{a^2 + b^2} \le b(a+1). \tag{4.2}$$

Letting $a = 2\pi|k|$ and $b = \alpha^{-1}$ in (4.2) we get

$$\frac{2\pi|k| + 1}{2} \le \sqrt{(2\pi|k|)^2 + \alpha^{-2}} \le \alpha^{-1}(2\pi|k| + 1),$$

and, consequently,

$$\frac{1}{2}\langle(2\pi|k|+\alpha^{-1})\hat{u},\hat{u}\rangle_{L^2}\le\langle\sqrt{(2\pi|k|)^2+\alpha^{-2}}\hat{u},\hat{u}\rangle_{L^2}\le\alpha^{-1}\langle(2\pi|k|+\alpha^{-1})\hat{u},\hat{u}\rangle_{L^2}.$$

Since $\langle u,(\sqrt{-\Delta+\alpha^{-2}})u\rangle_{L^2}=\langle\sqrt{(2\pi|k|)^2+\alpha^{-2}}\hat{u},\hat{u}\rangle_{L^2}$ we obtain (4.1). □

Proof of Theorem 3.2: We apply Theorems 2.1 and 2.8 of Ambrosetti and Rabinowitz.[3] For this purpose we need to verify several conditions. We divide the proof into three steps but first we fix some notation. Let $\mathcal{K}=\mathbf{H}^{1/2}(\mathbb{R}^3)$ and make the decomposition $\mathcal{K}=\mathcal{X}\oplus\mathcal{V}$, where $\mathcal{V}$ is a finite dimensional subspace of $\mathcal{K}$. Moreover, we let $B_\rho=\{u\in\mathcal{X}\,:\,\|u\|_{\mathbf{H}^{1/2}}=\rho\}$.

1. First we show that there exist $\rho,\sigma>0$ such that $\mathcal{E}_{|\partial B_\rho\cap\mathcal{X}}>\sigma$. For any $u\in\mathcal{X}$, the weak Young inequality implies that

$$\mathcal{J}_W(u^2,u^2)\le\|W\|_{L^p_w}\|u^2\|_{L^1}\|u^2\|_{L^r}=\|W\|_{L^p_w}\|u\|^2_{L^2}\|u\|^2_{L^{2r}}$$

with $1/p\;+\;1/r\;+\;1=2$ and $r\in[1,\,3/2]$; the latter is a consequence of the Sobolev embedding $\mathbf{H}^{1/2}(\mathbb{R}^3)\hookrightarrow L^s(\mathbb{R}^3)$ valid for $s\;+\;[2,3]$. In particular, $\|u\|_{L^2}\le C_1\|u\|_{\mathbf{H}^{1/2}}$ and $\|u\|_{L^r}\le C_2\|u\|_{\mathbf{H}^{1/2}}$ and, therefore,

$$\mathcal{J}_W(u^2,u^2)\le C\|u\|^4_{\mathbf{H}^{1/2}}. \tag{4.3}$$

From the latter inequality, Lemma 4.1, and $\lambda>\alpha^{-2}$, we get that

$$\begin{aligned}\mathcal{E}(u)&\ge\frac{\alpha^{-1}}{4}\|u\|^2_{\mathbf{H}^{1/2}}+\frac{1}{2}(\lambda-\alpha^{-2})\|u\|^2_{L^2}-C\|u\|^4_{\mathbf{H}^{1/2}}\\&\ge\frac{\alpha^{-1}}{4}\|u\|^2_{\mathbf{H}^{1/2}}-C\|u\|^4_{\mathbf{H}^{1/2}}\\&\ge\|u\|^2_{\mathbf{H}^{1/2}}\left(\frac{\alpha^{-1}}{4}-C\|u\|^2_{\mathbf{H}^{1/2}}\right).\end{aligned}$$

Next we choose codim $\mathcal{X}$ such that, for every $u\in\mathcal{X}$, $\|u\|^2_{\mathbf{H}^{1/2}}<\frac{\alpha^{-1}}{4C}$. Then, for every $u\in\partial B_\rho\cap\mathcal{X}$, we conclude that $\mathcal{E}(u)>\sigma>0$ with $\sigma=\rho^2(\frac{\alpha^{-1}}{4}-C\rho^2)$.
2. For each finite dimensional subspace $\mathcal{V}$ of $\mathcal{K}$ there exists $R=R(\mathcal{V})$ such that $\mathcal{E}<0$ on $\mathcal{V}\setminus B_R$; B_R is defined similarly to B_ρ above. With a slight abuse of notation we let $\mathsf{J}(u)=\mathcal{J}_W(u^2,u^2)$. Then we see that $\mathsf{J}'(u)u=4\mathsf{J}(u)$ for all $u\in\mathcal{K}$. Let $\mathcal{V}$ be a finite dimensional subspace of $\mathcal{K}$. For every $u\in\mathcal{K}$ with $\|u\|_{\mathbf{H}^{1/2}}\ge 1$ and, for any $t>0$, let $g(t)=\mathsf{J}(tu/\|u\|_{\mathbf{H}^{1/2}})$. Then $g(t)>0$ and

$$\begin{aligned}g'(t)&=\mathsf{J}'\left(\frac{tu}{\|u\|_{\mathbf{H}^{1/2}}}\right)\frac{u}{\|u\|_{\mathbf{H}^{1/2}}}=\frac{1}{t}\mathsf{J}'\left(\frac{tu}{\|u\|_{\mathbf{H}^{1/2}}}\right)\frac{tu}{\|u\|_{\mathbf{H}^{1/2}}}\\&=\frac{4}{t}\mathsf{J}\left(\frac{tu}{\|u\|_{\mathbf{H}^{1/2}}}\right)=4t^{-1}g(t).\end{aligned}$$

Thus,

$$\frac{g'(t)}{g(t)}=\frac{4}{t}\Rightarrow\int_1^{\|u\|_{\mathbf{H}^{1/2}}}\frac{g'(t)}{g(t)}\,dt=\int_1^{\|u\|_{\mathbf{H}^{1/2}}}\frac{4}{t}\,dt,$$

and, consequently,

$$\begin{aligned}&\ln[\mathsf{J}(u)]-\ln[\mathsf{J}(tu/\|u\|_{\mathbf{H}^{1/2}})]=\ln[\|u\|^4_{\mathbf{H}^{1/2}}]\\&\Rightarrow\qquad \mathsf{J}(u)=\|u\|^4_{\mathbf{H}^{1/2}}\mathsf{J}\left(\frac{tu}{\|u\|_{\mathbf{H}^{1/2}}}\right).\end{aligned} \tag{4.4}$$

Let $\delta=\inf\{\mathsf{J}(u)\,:\,\|u\|_{\mathbf{H}^{1/2}}=1,\;u\in\mathcal{V}\}$, $\mathcal{S}_\mathcal{V}$ be the unit sphere of $\mathcal{V}$, and $(u_j)_{j\ge1}$ be a sequence in $\mathcal{S}_\mathcal{V}$. Then (u_j) is bounded and therefore there exists a subsequence of (u_j) still denoted by (u_j) that converges weakly to u in $\mathcal{K}$. Since $\dim\mathcal{V}<\infty$ we can assume that (u_j) is a minimizing sequence of $\mathsf{J}(\cdot)$ and also (u_j) converges strongly to u in $\mathcal{V}$. The weakly lower semicontinuity of $\mathsf{J}(\cdot)$ implies that

$$\delta = \inf_{v\in\mathcal{S}_{\mathcal{V}}} \mathsf{J}(v) = \liminf_j \mathsf{J}(u_j) \geq \mathsf{J}(u) > 0, \quad \text{because } u \neq 0.$$

From (4.4) and above it follows that

$$\mathsf{J}(u) \geq \|u\|^4_{\mathbf{H}^{1/2}} \inf_{\mathcal{S}_{\mathcal{V}}} \mathsf{J}(u) \text{ i.e. } \mathsf{J}(u) \geq \delta\|u\|^4_{\mathbf{H}^{1/2}}.$$

This, in conjunction with Lemma 4.1, gives us that

$$\mathcal{E}(u) \leq \alpha^{-1}\|u\|^2_{\mathbf{H}^{1/2}} + (\lambda - \alpha^{-2})\|u\|^2_{L^2} - \delta\|u\|^4_{\mathbf{H}^{1/2}}.$$

It is not hard to see that $\mathcal{E}(u) \to -\infty$ as $\|u\|_{\mathbf{H}^{1/2}} \to +\infty$. This ends step 2.

3. Within the framework of Ambrosetti and Rabinowitz we look for critical points of $\mathcal{E}(\cdot)$ in $\mathbf{H}_r^{1/2}(\mathbb{R}^3)$. It is easy to see that $\mathcal{E} \in C^1(\mathbf{H}^{1/2}(\mathbb{R}^3);\mathbb{R})$. It remains to check the Palais-Smale (PS) condition, i.e., if $(u_j)_{j\geq 1}$ is a sequence of non-increasing functions in $\mathbf{H}_r^{1/2}(\mathbb{R}^3)$ such that

$$\begin{cases} \mathcal{E}(u_j) \text{ is bounded} \\ \mathcal{E}'(u_j) = \left(\alpha^{-1}\sqrt{-\Delta+\alpha^{-2}}\right)u_j + (\lambda-\alpha^{-2})u_j + Vu_j - (W*|u_j|^2)u_j \xrightarrow[\mathbf{H}^{-1/2}]{} 0 \end{cases},$$

then there exists a subsequence of (u_j) which converges in $\mathbf{H}^{1/2}(\mathbb{R}^3)$.

Let $(u_j)_{j\geq 1}$ be such a sequence and let $\epsilon_j = \mathcal{E}'(u_j)$. We begin by proving that $(u_j)_{j\geq 1}$ is a bounded sequence in $\mathbf{H}^{1/2}(\mathbb{R}^3)$. Now,

$$\mathfrak{t}_0[u_j] + (\lambda-\alpha^{-2})\|u_j\|^2_{L^2} + \mathfrak{s}[u_j] - \mathcal{J}_W(u_j^2, u_j^2) = \langle \epsilon_j, u_j\rangle_{\mathbf{H}^{-1/2},\mathbf{H}^{1/2}}. \tag{4.5}$$

Since, by hypothesis, $\mathcal{E}(u_j)$ is bounded, we have that

$$\begin{aligned} \mathfrak{t}_0[u_j] + (\lambda-\alpha^{-2})\|u_j\|^2_{L^2} + \mathfrak{s}[u_j] &= 2\mathcal{E}(u_j) + \frac{1}{2}\mathcal{J}_W(u_j^2, u_j^2) \\ &\leq C + \frac{1}{2}\mathcal{J}_W(u_j^2, u_j^2). \end{aligned} \tag{4.6}$$

On the other hand,

$$\langle \mathcal{E}'(u_j), u_j\rangle = \mathfrak{t}_0[u_j] + (\lambda-\alpha^{-2})\|u_j\|^2_{L^2} + \mathfrak{s}[u_j] + \mathcal{J}_W(u_j^2, u_j^2),$$

i.e.,

$$\langle \mathcal{E}'(u_j), u_j\rangle = 2\mathcal{E}(u_j) - \frac{1}{2}\mathcal{J}_W(u_j^2, u_j^2),$$

which implies that

$$\langle \epsilon_j, u_j\rangle + \frac{1}{2}\mathcal{J}_W(u_j^2, u_j^2) = 2\mathcal{E}(u_j) \leq C,$$

and, consequently,

$$|\langle \mathcal{E}'(u_j), u_j\rangle| \leq C \text{ and } \frac{1}{2}\mathcal{J}_W(u_j^2, u_j^2) \leq C.$$

This, in conjunction with (4.6) implies that

$$\mathfrak{t}_0[u_j] + (\lambda-\alpha^{-2})\|u_j\|^2_{L^2} + \mathfrak{s}[u_j] \leq C,$$

whence

$$\mathfrak{t}_0[u_j] + \alpha(\lambda-\alpha^{-2})\|u_n\|^2_{L^2} \leq C$$

because V is non-negative. Then by (4.1) we obtain

$$\frac{1}{2}\|u_j\|^2_{\mathbf{H}^{1/2}} + \alpha(\lambda-\alpha^{-2})\|u_j\|^2_{L^2} \leq C.$$

Since $\lambda - \alpha^{-2} \geq 0$, then we immediately conclude that $\|u_j\|_{\mathbf{H}^{1/2}} \leq C$.

Now, by the Banach-Alaoglu theorem there exists a subsequence of u_j (still denoted u_j) such that $u_j \rightharpoonup u$ in $\mathbf{H}^{1/2}(\mathbb{R}^3)$ and a.e. on $\mathbb{R}^3$. It is worth to mention that u is radial and non-increasing because all u_j are. Since u_j is radial and non-increasing, Lemma 2.1 implies that

$$|u_j(x)| \leq c|x|^{-3/2}, \quad x \neq 0.$$

Therefore, $\lim_{|x|\to\infty} u_j(x) = 0$ and, consequently, $\lim_{|x|\to\infty} u(x) = 0$. Let $v_j = u_j - u$. Then it is not hard to see that $(v_j)_{j\geq 1}$ is bounded in $\mathbf{H}^{1/2}$ and $\lim_{|x|\to\infty} v_j(x) = 0$. An application of Sobolev's embedding theorem shows that each v_j belongs to $L^p(\mathbb{R}^3)$, $p \in [2, 3]$. Hence, we can apply Lemma 2.2, i.e., Strauss' compactness principle,[14] wherein we choose $P(s) = |s|^r$ and $Q(s) = |s|^2 + |s|^3$, and $v = 0$. It follows that

$$\int_{\mathbb{R}^3} |v_n|^r \, dx \xrightarrow[n\to\infty]{} 0, \text{ i.e., } \|u_j - u\|_{L^r} \xrightarrow[n\to\infty]{} 0, \quad r \in [2, 3].$$

Next we show that $\mathcal{E}'(u_j) \to \mathcal{E}'(u)$ in $\mathbf{H}^{-1/2}(\mathbb{R}^3)$. We have $(u_j^2)_{j\geq 1}$ bounded in $L^s(\mathbb{R}^3)$, $s \in [1, \frac{3}{2}]$ since u_j is bounded in $L^r(\mathbb{R}^3)$, $r \in [2, 3]$ and, together with $W \in L_{\mathrm{w}}^{p_k}(\mathbb{R}^3)$ and the generalized Young inequality, we deduce that $W * u_j^2$ is bounded in $L^q(\mathbb{R}^3)$ with $3/2 < q < \infty$. Moreover, by the dominated convergence theorem we infer that $W * u_j^2$ converges strongly to $W * |u|^2$ in $L^q(\mathbb{R}^3)$. Let $\psi_j = W * |u_j|^2$, and $w \in \mathbf{H}^{1/2}$. Then

$$\begin{aligned} |\langle \psi_j u_j - \psi u, w\rangle_{\mathbf{H}^{-1/2},\mathbf{H}^{1/2}}| &= |\langle \psi_j u_j - \psi_j u + \psi_j u - \psi u, w\rangle_{\mathbf{H}^{-1/2},\mathbf{H}^{1/2}}| \\ &\leq C\left[\|\psi_j(u_j - u)\|_{L^2} + \|(\psi_j - \psi)u\|_{L^2}\right]. \end{aligned}$$

By Hölder's inequality we have that

$$\|\psi_j(u_j - u)\|_{L^2} \leq \|\psi_j^2\|_{L^l}\|(u_j - u)^2\|_{L^m}$$

with $(1/l) + (1/m) = 1$; valid because $m \in [1, 3/2]$ and $l \in (3/4, \infty)$. Then, by the uniform boundedness of ψ_j in $L^q(\mathbb{R}^3)$, $q \in (3/2, \infty)$, and the strong convergence of u_j to u in L^r, $r \in [2, 3]$, and the strong convergence of ψ_j to ψ in $L^q(\mathbb{R}^3)$, it follows that $\langle \psi_j u_j - \psi u, w\rangle_{\mathbf{H}^{-1/2},\mathbf{H}^{1/2}} \to 0$ as $j \to \infty$. Hence,

$$\psi_j u_j = (W * u_j^2)u_j \underset{\mathbf{H}^{-1/2}}{\rightharpoonup} \psi u = (W * u^2)u. \tag{4.7}$$

On the other hand, by the boundedness of u_j in $\mathbf{H}^{1/2}(\mathbb{R}^3)$ and the boundedness of $W * u_j^2$ in L^q, we have that $(W * u_j^2)u_j^2$ is bounded in L^1. These facts, together with the pointwise convergence of $(W * u_j^2)u_j^2$ to $(W * u^2)u^2$ in $\mathbb{R}^3$ imply that Lebesgue's dominated convergence theorem yields

$$\mathcal{J}_W(u_j^2, u_j^2) \longrightarrow \mathcal{J}_W(u^2, u^2).$$

By passing to the limit in (4.5) as $j \to \infty$, we get that

$$\lim_j \left\{\mathfrak{t}_0[u_j] + (\lambda - \alpha^{-2})\|u_j\|_{L^2}^2 + \mathfrak{s}[u_j]\right\} = \mathcal{J}_W(u^2, u^2).$$

An application of Fatou's lemma yields

$$\begin{aligned} \mathfrak{t}_0[u] + (\lambda - \alpha^{-2})\|u\|_{L^2}^2 + \mathfrak{s}[u] &\leq \liminf_j \left\{\mathfrak{t}_0[u_j] + (\lambda - \alpha^{-2})\|u_j\|_{L^2}^2 + \mathfrak{s}[u_j]\right\} \\ &= \lim_j \left\{\mathfrak{t}_0[u_j] + (\lambda - \alpha^{-2})\|u_j\|_{L^2}^2 + \mathfrak{s}[u_j]\right\} \\ &= \mathcal{J}_W(u^2, u^2). \end{aligned}$$

Moreover, since u_j converges strongly to u in $L^r(\mathbb{R}^3)$, $r \in [2, 3]$, we have that

$$\alpha^{-1}\left(\sqrt{-\Delta + \alpha^{-2}} - \alpha^{-1}\right)u_j + \lambda u_j + Vu_j \xrightarrow[\mathbf{H}^{-1/2}]{} \alpha^{-1}\left(\sqrt{-\Delta + \alpha^{-2}} - \alpha^{-1}\right)u + \lambda u + Vu$$

in the sense of distributions. The latter, in conjunction with (4.7), implies that

$$\mathcal{E}'(u_j) \xrightarrow[\mathbf{H}^{-1/2}]{} \mathcal{E}'(u) = \left(\sqrt{-\alpha^{-2}\Delta + \alpha^{-4}} - \alpha^{-2}\right) u + \lambda u + Vu + (W * u^2)u.$$

Then, by hypothesis, we deduce that $\mathcal{E}'(u) = 0$. In particular, $\langle \mathcal{E}'(u), u\rangle = 0$ and we infer that

$$\mathfrak{l}_0[u] + (\lambda - \alpha^{-2})\|u\|_{L^2}^2 + \mathfrak{s}[u] = \mathcal{J}_W(u^2, u^2).$$

Furthermore,

$$\begin{aligned}
&\langle u_j - u, \sqrt{-\alpha^{-2}\Delta + \alpha^{-4}}(u_j - u)\rangle \\
&\quad = \langle \sqrt{-\alpha^{-2}\Delta + \alpha^{-4}}u, u - u_j\rangle - \langle \sqrt{-\alpha^{-2}\Delta + \alpha^{-4}}u_j, u - u_j\rangle \\
&\quad = \langle \left(\sqrt{-\alpha^{-2}\Delta + \alpha^{-4}} - \alpha^{-2}\right) u + \lambda u + Vu - (W * u^2)u, u - u_j\rangle + \int (W * |u|^2)u(u - u_j)\, dx \\
&\qquad +(\alpha^{-2} - \lambda)\langle u, u - u_j\rangle - \langle Vu, u - u_j\rangle - \langle \sqrt{-\alpha^{-2}\Delta + \alpha^{-4}}u_j, u - u_j\rangle.
\end{aligned}$$

The first term on the right-hand side is equal to $\langle \mathcal{E}'(u), u - u_j\rangle_{\mathbf{H}^{-1/2},\mathbf{H}^{1/2}} = 0$, the third term from the right-hand side, viz., $\langle u, u - u_j\rangle$ tends to zero (because u_j converges weakly to u in $\mathbf{H}^{1/2}$), the same argument applies to fourth term. As for the second term we apply Hölder's inequality twice. Since both $W * u^2$ and u are bounded in L^q, $3/2 < q < \infty$ and u_j converges strongly to u in L^r, $r \in [2, 3]$, this implies that the second term tends to zero. For the last term we need the uniform boundedness of $\sqrt{-\alpha^{-2}\Delta + \alpha^{-4}}u_j$ in $L^2(\mathbb{R}^3)$, together with the strong convergence of u_j to u in $L^2(\mathbb{R}^3)$ to conclude. In view of the above, we obtain

$$\langle \sqrt{-\alpha^{-2}\Delta + \alpha^{-4}}(u_j - u), u_j - u\rangle_{L^2} \longrightarrow 0.$$

Since $\langle \sqrt{-\alpha^{-2}\Delta + \alpha^{-4}}(u_j - u), u_j - u\rangle \geq \big(|\nabla|(u_j - u), u_j - u\big)\rangle$, we have $\langle |\nabla|(u_j - u), u_j - u\rangle \to 0$. We conclude that $\|u_j - u\|_{\mathbf{H}^{1/2}} \to 0$. □

It is worth to mention that Assumption 3.1 is optimal for a non-negative, radial W because there exists $W \in L^\infty(\mathbb{R}^3)$ such that (1.1) has no $\mathbf{H}^{1/2}(\mathbb{R}^3)$ solutions. For instance, we may choose $W \equiv 1$. Then (1.1), with $V \equiv 0$, takes the form $Lu + (1 - \|u\|_{L^2}^2)u = 0$ and this implies that $u \equiv 0$.

V. CONSTRAINED PROBLEM: PROOF OF THEOREM 3.4

We prove Theorem 3.4 and we establish two corollaries.

Proof of Theorem 3.4: Without loss of generality we consider $W \in L_w^{p_i}(\mathbb{R}^3)$. The idea is to apply the critical point theory by Berestycki and Lions[5] in the following framework: $\mathcal{H} = L^2(\mathbb{R}^3)$ and $\mathcal{K} = \mathbf{H}_{\mathrm{r}}^{1/2}(\mathbb{R}^3)$. In order to apply the abstract theorem, we need to establish the following requirements:

1. $\mathcal{E}_{|\mathcal{C}}$ is bounded below;
2. $\mathcal{E}$ is weakly lower semicontinuous on $\mathcal{T} = \{u \in \mathcal{C} : \mathcal{E}(u) \leq 0\}$; and
3. $\mathcal{E}_{|\mathcal{C}}$ satisfies the $(\mathrm{PS})_-$ condition.

Verification of item 1. From Lemma 4.1 we find that

$$\mathcal{E}(u) \geq \frac{\alpha^{-1}}{4}(\|u\|_{\mathbf{H}^{1/2}}^2 - \|u\|_{L^2}^2) - 1/4\mathcal{J}_W(u^2, u^2). \tag{5.1}$$

An application of the weak Young inequality and Sobolev's inequality yields

$$\mathcal{J}_W(u^2, u^2) \leq \|W\|_{L_w^p}\|u^2\|_{L^1}\|u^2\|_{L^{r/2}} \leq CN^2\|W\|_{L_w^p}\|u\|_{\mathbf{H}^{1/2}}^2, \tag{5.2}$$

where $1/p + 2/r + 1 = 2$, i.e., $1/p + 2/r = 1$ which is possible to satisfy because $r \in [2, 3]$ and $p \geq 3$. Since u belongs to $\mathcal{C}$, it is not hard to see that $\|u^2\|_{L^1} = \|u\|_{L^2}^2 = N^2$. Moreover,

$\|u^2\|_{L^{r/2}} = \|u\|^2_{L^r} \le C\|u\|^2_{\mathbf{H}^{1/2}}$. Without loss of generality, we choose $\|W\|_{L^p_w} = 1/2\alpha C N^2$. Then inequality (5.2) becomes

$$\mathcal{J}_W(u^2, u^2) \le \frac{\alpha^{-1}}{2}\|u\|^2_{H^{1/2}},$$

while (5.1) becomes simply $\mathcal{E}(u) \ge -N^2$.

Verification of item 2. Let $(u_j) \subset \mathcal{T} := \{u \in \mathcal{C} : \mathcal{E}(u) \le 0\}$ such that $u_j \rightharpoonup u$ in $\mathbf{H}^{1/2}(\mathbb{R}^3)$. Obviously, as for item 1, it follows that

$$\sup_j \mathcal{J}_W(u_j^2, u_j^2) < \infty$$

and, by Fatou's lemma, we get that

$$\mathcal{J}_W(u^2, u^2) \le \liminf_j \mathcal{J}_W(u_j^2, u_j^2).$$

Since the remaining terms are obviously weakly lower semicontinuous, it follows that $\mathcal{E}$ is weakly lower semicontinuous on $\mathcal{T}$.

Verification of item 3. Let $(u_j)_{j\ge 1}$ be a sequence in $\mathcal{C}$ satisfying

$$\begin{cases} -\infty < \beta \le \mathcal{E}(u_j) \le \sigma < 0 \\ (\sqrt{-\alpha^{-2}\Delta - \alpha^{-4}} - \alpha^{-2})u_j + Vu_j - (W * u_j^2)u_j + \lambda_j u_j = \epsilon_j \xrightarrow[\mathbf{H}^{-1/2}]{} 0 \end{cases},$$

where

$$-\lambda_j = \mathcal{E}(u_j) = \frac{1}{2}\mathfrak{t}_0[u_j] + \frac{1}{2}\mathfrak{s}[u_j] - \frac{1}{4}\mathcal{J}_W(u_j^2, u_j^2).$$

We have

$$\begin{aligned}&\frac{1}{2}\langle(\sqrt{-\alpha^{-2}\Delta - \alpha^{-4}} - \alpha^{-2})u_j, u_j\rangle + \frac{1}{2}(\lambda - \alpha^{-2})\|u_j\|^2_{L^2}\\ &+\frac{1}{2}\int_{\mathbb{R}^3} V(x)|u_j(x)|^2\,dx - \frac{1}{4}\int\int W(x-y)|u_j(x)|^2|u_j(y)|^2\,dxdy \le \sigma.\end{aligned}$$

Since we have already proved that, for any $v \in \mathcal{C}$, $\mathcal{J}_W(v^2, v^2) \le C$, we obtain

$$\frac{1}{2}\langle\sqrt{-\alpha^{-2}\Delta - \alpha^{-4}}u_j, u_j\rangle + \frac{1}{2}(\lambda - \alpha^{-2})\|u_j\|^2_{L^2} + \frac{1}{2}\int_{\mathbb{R}^3} V(x)|u_j(x)|^2\,dx \le C,$$

whence

$$C \ge \frac{1}{2}\langle(\sqrt{-\alpha^{-2}\Delta - \alpha^{-4}})u_j, u_j\rangle \ge \|u_j\|^2_{\mathbf{H}^{1/2}}.$$

Therefore, $C \ge \|u_j\|^2_{\mathbf{H}^{1/2}}$, i.e., (u_j) is bounded in $\mathbf{H}_r^{1/2}(\mathbb{R}^3)$. Furthermore,

$$-\lambda_j \le 2\mathcal{E}(u_j) \le 2\sigma, \qquad -2\sigma \le \lambda_j \le \lambda.$$

Indeed,

$$\begin{aligned}\frac{-1}{2}\lambda_j &= \frac{1}{2}\langle(\sqrt{-\alpha^{-2}\Delta + \alpha^{-4}} - \alpha^{-2})u_j, u_j\rangle + \frac{1}{2}\int_{\mathbb{R}^3} V(x)|u_j(x)|^2\,dx\\ &-\frac{1}{4}\int\int W(x-y)|u_j(x)|^2|u_j(y)|^2\,dxdy - \frac{1}{4}\int\int W(x-y)|u_n(x)|^2|u_n(y)|^2\,dxdy,\end{aligned}$$

i.e.,

$$\frac{-1}{2}\lambda_j = \mathcal{E}(u_j) - \frac{1}{4}\int\int W(x-y)|u_j(x)|^2|u_j(y)|^2\,dxdy.$$

This shows that $\frac{-1}{2}\lambda_j \le \mathcal{E}(u_j)$ and then $-\lambda_j \le 2\mathcal{E} \le 2\sigma$.

On the other hand, since $\mathcal{J}_W(u_j^2, u_j^2)$ is uniformly bounded with respect to j and from the facts above we conclude that $\lambda_j \le \lambda$. Now we can follow the proof of Theorem 3.2 and conclude that u_j

converges strongly to u in $\mathbf{H}_{\mathrm{r}}^{1/2}(\mathbb{R}^3)$. This verifies item 3. Then the assertions of the theorem follows immediately from Berestycki and Lions[5] (Theorems 7 and 9). $\square$

Corollary 5.1: Let the hypotheses of Theorem 3.4 be satisfied. Then there exists a nondecreasing and positive sequence $(N_d)_{d\geq 1}$ such that, if $N \geq N_d$, then the conclusions of Theorem 3.4 hold.

Proof: Let $(\mathcal{V}_d)_{d\geq 1}$ be a sequence of d-dimensional subspaces of $\mathbf{H}_{\mathrm{r}}^{1/2}$ such that $\mathcal{V}_d \subset \mathcal{V}_{d+1}$ and let $\mathcal{C}_1 = \{u \in \mathbf{H}_{\mathrm{r}}^{1/2} \,:\, \|u\|_{L^2} = 1\}$. By definition of the genus, $\gamma(\mathcal{C}_1 \cap \mathcal{V}_d) = d$. For any positive real number N and any $u \in \mathcal{C}_1 \cap \mathcal{V}_d$, we have that

$$\begin{aligned}\mathcal{E}(Nu) &\leq \frac{N^2}{2}\mathfrak{t}_0[u] + \frac{N^2}{2}\mathfrak{s}[u] - \frac{N^4}{4}\mathcal{J}_W(u^2, u^2)\\ &\leq \frac{N^2}{2}\left\{\sup_{u\in\mathcal{C}_1\cap\mathcal{V}_d}(\mathfrak{t}_0[u] + \mathfrak{s}[u]) - \frac{N^2}{2}\inf_{u\in\mathcal{C}_1\cap\mathcal{V}_d}\mathcal{J}_W(u^2, u^2))\right\}.\end{aligned}$$

Then there exists N_d such that for $N \geq N_d$ the right-hand side is negative and, therefore, $\mathcal{E}$ is negative. Thus, for $N \geq N_d$, $\tilde{A} = \{Nu \,:\, u \in \mathcal{C}_1 \cap \mathcal{V}_d\}$ satisfies (3.3) and, consequently, the assertions of Theorem 3.4 hold true. $\square$

Corollary 5.2: Let the hypotheses of Theorem 3.4 be satisfied. If, moreover,

$$\liminf_{r\to+\infty} r^2 W(r) \geq L, \tag{5.3}$$

then there exists L_d such that (3.3) holds true provided $L \geq L_d$. If $L = +\infty$, then (3.3) holds true for all $d \geq 1$. In particular, the assertions of Theorem 3.4 are valid.

Proof: Without loss of generality we may suppose $N = 1$. Let $A = \mathcal{C}_1 \cap \mathcal{V}_d$ where $(\mathcal{V}_d)_{d\geq 1}$ is a sequence of d-dimensional subspaces of $\mathbf{H}_{\mathrm{r}}^{1/2}$ (to be specified below) such that $\mathcal{V}_d \subset \mathcal{V}_{d+1}$.

Choose $u \in A$ and let $u_\kappa(x) = u(x/\kappa)$. Then $\|\kappa^{-3/2}u_\kappa\|_{L^2} = 1$ and

$$\mathcal{E}(\kappa^{-3/2}u_\kappa) \leq \frac{1}{2}\mathfrak{t}_0[\kappa^{-3/2}u_\kappa] + \frac{1}{2}\int_{R^3} V(\kappa x)|u(x)|^2\,dx - \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} u^2(x)u^2(y)W(\kappa|x-y|)\,dxdy.$$

Using that $\mathbf{H}^1(\mathbb{R}^3) \subset \mathbf{H}^{1/2}$ and, specifically,

$$\mathfrak{t}_0[\phi] \leq C\|\phi\|^2_{\mathbf{H}^1}, \quad \forall\phi \in \mathbf{H}^1(\mathbb{R}^3),$$

in conjunction with

$$\int\int_{\frac{1}{2}\leq|x-y|\leq 1} u^2(x)u^2(y)W(\kappa|x-y|)\,dxdy \leq \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} u^2(x)u^2 W(\kappa|x-y|)\,dxdy,$$

we have that

$$\begin{aligned}\mathcal{E}(\kappa^{-3/2}u_\kappa) &\leq \frac{C}{2}\lambda^{-2}\int_{\mathbb{R}^3}|\nabla u|^2 + 1 + \frac{1}{2}\int_{\mathbb{R}^3} V(\kappa x)|u(x)|^2\,dx\\ &\quad -\frac{1}{4}\int\int_{1/2\leq|x-y|\leq 1} u^2(x)u^2(y)W(\kappa|x-y|)\,dxdy\\ &\leq \frac{C_1}{2}\kappa^{-2}\left\{\int_{\mathbb{R}^3}|\nabla u|^2 - \frac{\kappa^2}{2}\int\int_{1/2\leq|x-y|\leq 1} u^2(x)u^2(y)W(\kappa|x-y|)\,dxdy\right\} + C_2\\ &\leq \frac{C_1}{2}\kappa^{-2}\left\{\int_{\mathbb{R}^3}|\nabla u|^2 - \frac{L}{2}\int\int_{1/2\leq|x-y|\leq 1} u^2(x)u^2(y)\,dxdy\right\} + C_2,\end{aligned}$$

where, in the last inequality, we used the assumption in (5.3). For $u \in A$ we may suppose that $u^2(x) > 0$ for $\Xi = \{|x| \leq 2\}$. Indeed, we may choose $\mathcal{V}_d$ to be the subspace spanned by the first d eigenfunctions

u_n of $-\Delta$ with Dirichlet boundary conditions on $\partial\Xi$. Since each $u_n \in \mathbf{H}^1(\mathbb{R}^3) \subset \mathbf{H}^{1/2}(\mathbb{R}^3)$ is radial, we have that $u_n \in \mathbf{H}^1_{\mathrm{r}}(\mathbb{R}^3) \subset \mathbf{H}^{1/2}_{\mathrm{r}}(\mathbb{R}^3)$ as required. This choice of $\mathcal{V}_d$ will ensure that

$$\inf_{u \in \mathcal{C}_1 \cap \mathcal{V}_d} \int \int_{1/2 \le |x-y| \le 1} u^2(x) u^2(y)\, dx dy > 0$$

and, by taking L large enough, we find that

$$\sup_{u \in \mathcal{C}_1 \cap \mathcal{V}_d} \mathcal{E}(\kappa^{-3/2} u_\kappa) < 0 \quad \text{for } \kappa \ge \kappa_0.$$

Finally, with $\tilde{A} = \{\kappa_0^{-3/2} u_{\kappa_0} : u \in \mathcal{C}_1 \cap \mathcal{V}_d\}$ we conclude that $\gamma(\tilde{A}) = \gamma(A) = d$ and, therefore, (3.3) is satisfied for $\tilde{A}$. $\square$

If one takes $W(x) = 1/|x|^\alpha$, $2 < \alpha < 4$, then $\mathcal{E}$ is not even bounded below; this observation alone shows that Assumption 3.3 is necessary.

A posteriori it can be shown that solutions u_j of (1.1) satisfy the following properties:

(i) $u_j \in C^\infty(\mathbb{R}^3 \setminus \{0\})$;

(ii) For all $R > 0$ and $\beta < \nu := \sqrt{\lambda(2\alpha^{-2} - \lambda)}$, there exists $C = C(\beta, R) > 0$ such that

$$|u_j(x)| \le C e^{-\beta|x|}, \qquad \text{for} \quad |x| \ge R.$$

Indeed, the proof of properties (i) and (ii) for the quasirelativistic Choquard equation (1.1) is carried over, with minor changes, from the proof of similar properties, valid for the quasirelativistic Hartree-Fock equations, found in Ref. 7.

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