Grassmann phase space theory and the Jaynes–Cummings model

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GRASSMANN VARIABLES AND THE JAYNES-CUMMINGS MODEL

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0.1 Abstract

The Jaynes-Cummings model of a two-level atom in a single mode cavity is of fundamental importance both in quantum optics and in quantum physics generally, involving the interaction of two simple quantum systems - one fermionic system (the TLA), the other bosonic (the cavity mode). Depending on the initial conditions a variety of interesting effects occur, ranging from ongoing oscillations of the atomic population difference at the Rabi frequency when the atom is excited and the cavity is in an n-photon Fock state, to collapses and revivals of these oscillations starting with the atom unexcited and the cavity mode in a coherent state. The observation of revivals for Rydberg atoms in a high-Q microwave cavity is key experimental evidence for quantization of the EM field. Theoretical treatments of the Jaynes-Cummings model based on expanding the state vector in terms on products of atomic and n-photon states and deriving coupled equations for the amplitudes are a well-known and simple method for determining the effects.

In quantum optics however, the behaviour of the bosonic quantum EM field is often treated using phase space methods, where the bosonic mode annihilation and creation operators are represented by c-number phase space variables, with the density operator represented by a distribution function of these variables. Fokker-Planck equations for the distribution function are obtained, and either used directly to determine quantities of experimental interest or used to develop c-number Langevin equations for stochastic versions of the phase space variables from which experimental quantities are obtained as stochastic averages. Phase space methods have also been developed to include atomic systems, with the atomic spin operators being represented by c–number phase space variables, and distribution functions involving these variables and those for any bosonic modes being shown to satisfy Fokker-Planck equations from which c-number Langevin equations are often developed. However, atomic spin operators satisfy the standard angular momentum commutation rules rather than the commutation rules for bosonic annihilation and creation operators, and are in fact second order combinations of fermion annihilation and creation operators. Though phase space methods in which the fermion operators are represented directly by c-number phase space variables have not been successful, the anti-commutation rules for these operators suggests the possibility of using Grassmann variables - which have similar anti-commutation properties. However, in spite of the seminal work by Cahill and Glauber and a few applications, the use of phase space methods in quantum optics to treat fermionic systems by representing fermionic annihilation and creation operators directly by Grassmann phase space variables is rather rare. This paper shows that phase space methods using a positive P type distribution function involving both c-number variables (for the cavity mode) and Grassmann variables (for the TLA) can be used to treat the Jaynes-Cummings model. Although it is a Grassmann function, the distribution function is equivalent to six c-number functions of the two bosonic variables. Experimental quantities are given as bosonic phase space integrals involving the six functions. A Fokker-Planck equation involving

both left and right Grassmann differentiation can be obtained for the distribution function, and is equivalent to six coupled equations for the six c-number functions.

The approach used involves choosing the canonical form of the (non-unique) positive P distribution function, in which the correspondence rules for the bosonic operators are non-standard and hence the Fokker-Planck equation is also unusual. Initial conditions, such as those above for initially uncorrelated states, are discussed and used to determine the initial distribution function. Transformations to new bosonic variables rotating at the cavity frequency enables the six coupled equations for the new c-number functions - that are also equivalent to the canonical Grassmann distribution function - to be solved analytically, based on an ansatz from an earlier paper by Stenholm. It is then shown that the distribution function is exactly the same as that determined from the well-known solution based on coupled amplitude equations.

In quantum-atom optics theories for many atom bosonic and fermionic systems are needed. With large atom numbers, treatments must often take into account many quantum modes - especially for fermions. Generalisations of phase space distribution functions of phase space variables for a few modes to phase space distribution functionals of field functions (which represent the field operators, c-number fields for bosons, Grassmann fields for fermions) are now being developed for large systems. For the fermionic case, the treatment of the simple two mode problem represented by the Jaynes-Cummings model is a useful test case for the future development of phase space Grassmann distribution functional methods for fermionic applications in quantum-atom optics.

1 Introduction

The simplest theoretical model in quantum optics involves a two state atomic system coupled to a single mode quantum electromagmetic (EM) field, in which the system transition frequency is in near resonance with the mode frequency. This system was first treated using the rotating wave approximation and is referred to as the Jaynes-Cummings model [1]. The model can be used not only in cavity quantum electrodynamics (QED) to describe a two level atom (TLA) in a high-Q cavity - which could be two Rydberg atom states in a microwave cavity, or two lower lying atomic states in an optical cavity, but also in laser spectroscopy for treating a two level atom in free space coupled to a single mode laser field in a coherent state. The Jaynes-Cummings model is of fundamental importance both in quantum optics and in quantum physics generally, involving the interaction of two simple quantum systems - one being fermionic (the TLA), the other bosonic (the cavity mode). In laser spectroscopy the prediction [2], interpretation [3] and observation [4], [5], [6] of the three peak resonance fluorescence spectrum for a TLA strongly driven by a narrow bandwidth laser field stimulated much interest in the new research field of quantum optics. In the cavity QED case, a variety of interesting effects occur depending on the initial conditions, ranging from ongoing oscillations of the atomic population difference at the Rabi frequency when the atom is excited and the cavity is in an n-photon Fock state, to collapses and revivals of these oscillations starting with the atom unexcited and the cavity mode in a coherent state [7], [8], [9]. The collapse time corresponds to the time scale for phase factors with photon numbers at the extremes of the photon number distribution to get out of phase by π. The revival time corresponds to when the phase factors for neighboring photon numbers get out of phase by 2π, and hence rephase. For the coherent state case the revival time is $\sqrt{\overline{n}}$ times longer than the collapse time, where $\overline{n}$ is the mean photon number of the coherent field. The observation of revivals [10], [11] for Rydberg atoms in a high-Q microwave cavity is key experimental evidence for quantization of the EM field. Over 1200 papers have been written about the Jaynes-Cummings model indicating its importance in quantum optics, and a comprehensive review is provided by Shore and Knight [12].

Theoretical treatments of the Jaynes-Cummings model based on expanding the state vector in terms of products of atomic and n-photon states and deriving coupled equations for the coefficients are a well-known and simple method for determining the effects [13], [9]. Another simple approach is based on using the actual atom-cavity mode energy eigenstates - the dressed states [3] - to describe the behaviour. In Stenholm's approach [14] the state vector was expanded in products of Bargmann states for the field and spinor states for the TLA. However, in quantum optics the behaviour of the bosonic quantum EM field is often treated using phase space methods [15], [16], [13], [17] with the density operator represented by a *phase space distribution function* and the bosonic mode

annihilation and creation operators represented by c-number *phase space variables*. *Fokker-Planck equations* for the distribution function are obtained, and either used directly to determine quantities of experimental interest or used to develop c-number *Langevin equations* for stochastic versions of the phase space variables, such as when distribution functions of the positive P type [18] are used. Phase space methods have also been developed to include atomic systems [15], [16], with the atomic spin operators being represented by c–number phase space variables, and distribution functions involving these variables and those for any bosonic modes being shown to satisfy Fokker-Planck equations from which c-number Langevin equations are often developed. However, atomic spin operators satisfy the standard angular momentum commutation rules rather than the commutation rules for bosonic annihilation and creation operators, and are in fact second order combinations of fermion annihilation and creation operators. It would perhaps be more natural to use a phase space method in which the phase space variables represent the fermion operators directly. The problem however is that it is not possible for fermion annihilation and creation operators to be represented by c-number variables. The anti-commutation rules for these operators suggests the use of variables which have similar anti-commutation properties, and such variables are *Grassmann numbers*. These numbers were invented in the 19th century by the mathematician Hermann Grassmann, but their properties would be unfamiliar to many physicists. A general account of Grassmann variables is to be found in the book by Berezin [19].

Apart from the seminal work by Cahill and Glauber [20] and a few applications [21], [22], [23], [24] the use of phase space methods in quantum optics to treat fermionic systems by representing fermionic annihilation and creation operators with Grassmann variables [19], [20] is rather rare. In contrast, this approach is widely used in particle physics [25], [26], [27]. This paper shows how phase space methods using a distribution function involving both c-number variables (for the cavity mode) and Grassmann variables (for the TLA) can be used to treat the Jaynes-Cummings model. Quantities of experimental interest such as atomic state populations and coherences or mean numbers of photons are related to *quantum correlation functions*, which are expectation values of normally ordered products of boson, fermion annihhilation and creation operators. Accordingly, the distribution function chosen is of the *positive* P type for the bosonic and is similar to the *complex* P type for the fermionic variables (for short, a positive P distribution). Unlike the more usual Glauber-Sudarshan P, Husimi Q or Wigner W distributions, the positive and complex P distributions involve a *double phase space*, with two phase space variables for each bosonic or fermonic mode. The distribution function is *not unique* but the existence of the distribution function in a canonical form can be established. The distribution function is a *Grassmann function*, and is specified by six c-number functions of the two bosonic variables. Experimental quantities are given as bosonic phase space integrals involving the six functions. The non-uniqueness of the positive P distribution is reflected in different Fokker-Planck equations based on differing correspondence rules. Fokker-Planck equations involving both left and right Grassmann differentiation can be obtained for both the canonical and stan-

dard distribution functions, and these are equivalent to six coupled equations for the six c-number bosonic functions. Initial conditions, such as for initially uncorrelated states, are discussed and used to determine the initial canonical distribution function. The positive P type distribution functions would enable the phase space variables to be replaced by stochastic variables and the Fokker-Planck equation replaced by Langevin stochastic equations of the Ito type. However, this development is not needed for the present paper as the Fokker-Planck equation for the canonical distribution function is solved directly via an adaption of Stenholm's method [14]. The solution is entirely in agreement with the standard quantum optics results. However, the analogous solution of the Fokker-Planck equation for the standard distribution function leads to a distribution function that diverges on the phase space boundary and for large times, indicating that the original Fokker-Planck equation was not valid. Applications of the positive P distribution using the canonical distribution function may be preferable to those where the derivation of the Fokker-Planck equation is based on the standard correspondence rules. For the canonical distribution function more general Fokker-Planck equations involving derivatives higher than second order may occur [28], so that no replacement by Langevin stochastic equations is possible. The Fokker-Planck equation may also not have a positive definite diffusion matrix.

The use of phase space approaches to the Jaynes-Cummings model is itself rather unusual, however the paper by Eiselt and Risken [29] does use such a method. A generalised Wigner distribution for the field mode is constructed from four field mode operators given by the matrix elements of the full density operator with the two atomic states, then combined with the field mode annihilation and creation operators. A generalised density operator for the field with six components is obtained and used to define a six component characteristic function and hence a six component Wigner function. Damping effects are also included. For zero damping, collapse and revival effects are found for the population difference of the atomic states. We note that their approach also requires consideration of six field distribution functions, as is required in the present work. However, their approach cannot be related in any simple way to ours.

Although it is of interest to show that the Jaynes-Cummings model can be treated via Grassmann phase space methods the work presented here has more general relevance. In quantum-atom optics Bose-Einstein condensates and degenerate Fermi gases are being studied and theories for many atom bosonic and fermionic systems are needed. With large atom numbers, treatments must often take into account many quantum modes - especially for fermions in view of the Pauli exclusion principle. Phase space distribution function methods based on separate variables for each mode then start to become unwieldy. Generalisations of phase space distribution function methods involving phase space variables for a few modes to *phase space distribution functional* methods based on field functions which represent the entire field operators are now being developed for large systems. The field functions are c-number fields for the bosonic case, Grassmann fields for the case of fermions. This development is more ad-

vanced for bosonic systems, with applications already made to problems such as optical light in dispersive non-linear media [30], [31], spatial squeezing in quantum EM fields [32] and quantum noise in Bose-Einstein condensates [33]. However, distribution functional methods involving atomic media have also been formulated [34], based on a different approach in which the atomic system is treated in terms of c-number fields representing the atomic spin field operators rather than the particle field creation and annihilation operators. More recently, an alternative phase space approach for fermions, the Gaussian operator basis method [35], originally developed for bosonic systems [36] has been published. Here the fermionic density operators are represented as a positive distribution over a generalised phase space. The phase space variables again are c-numbers, now associated with pairs of fermion annihilation and/or creation operators, and Grassmann numbers are only used to establish properties of the Gaussian operators. However, less progress has been made for the different approach - analogous to that applied to bosonic systems - in which the fermion particle field operators are represented by Grassmann fields, and the treatment of the simple two mode problem represented by the Jaynes-Cummings model is a useful first step in the development of this type of phase space Grassmann distribution functional methods for applications in quantum-atom optics. For such fermion multimode systems, functional Fokker-Planck equations would be obtained for the distribution functional, and these would be converted into equivalent Langevin stochastic field equations of the Ito type. It is expected that the latter step would be important for practical numerical applications, since solving for the full distribution functional would likely be impractical, and unnecessary if the required physical predictions can be made via less computer intensive stochastic methods.

In section 2 the basic physics of the Jaynes-Cummings model is reviewed, with the Hamiltonian, the physical and spin states, atomic population and transition operators, photon number operators and spin operators being treated in terms of fermion and boson creation, annihilation operators. The phase space approach is developed in section 3, starting with defining the quantum correlation functions then introducing the characteristic function and its phase space integral relation to the distribution function. The characteristic and distribution functions are shown to be certain Grassmann functions with c-number coefficients that are functions of the bosonic phase space variables, and the expression for the canonical form of the distribution function is presented. Results for the quantum correlation functions, atomic populations and coherences are obtained in terms of phase space integrals. In section 4 correspondence rules for the effects of boson and fermion operators on the density operator are obtained, and used to derive Fokker-Planck equations for the Jaynes-Cummings model. The standard and canonical correspondence rules are presented and the Fokker-Planck equation is obtained for the canonical distribution function. Sets of coupled equations for the c-number distribution function coefficients are obtained from the Fokker-Planck equations. The original c-number distribution function coefficients are replaced by coefficients in a frame rotating at the cavity freqency and expressed in terms of rotating phase space variables. The initial form of

the distribution function is found for the case of uncorrelated initial states, such as those leading to effects such as Rabi oscillations and collapse/revival phenomena. In section 5 the coupled c-number equations are solved analytically for the canonical distribution case and shown to predict the characteristic the Jaynes-Cummings mode effects. The conclusions for the paper are set out in section 6. Basic results for Grassmann algebra and calculus and for boson and fermion Bargmann states are presented in the Appendices.

2 Basic Physics of Jaynes-Cummings Model

2.1 Atomic System and Cavity Mode

The basic atomic system treated in the Jaynes-Cummings model is a *single atom* system with just two internal states, denoted $|1\rangle$, $|2\rangle$, with energies E_1, E_2. Centre of mass degrees of freedom are ignored. There are therefore just four atomic operators to consider, denoted $\widehat{\sigma}_{ij} = |i\rangle\langle j|$. Two of these are *population* operators, $\widehat{P}_1 = |1\rangle\langle 1|$ and $\widehat{P}_2 = |2\rangle\langle 2|$, and two are atomic *transition* operators $\widehat{\sigma}_+ = |2\rangle\langle 1|$ and $\widehat{\sigma}_- = |1\rangle\langle 2|$. In the simple Jaynes-Cummings model the two-level atom interacts with a cavity mode. Let $\widehat{a}, \widehat{a}^\dagger$ be the boson annihilation, creation operators for the field mode. The states for the cavity mode are the usual n photon states, denoted $|n\rangle$, with $n = 0, 1, 2, ...$ If the atomic transition frequency is $\omega_0 = (E_2 - E_1)/\hbar$, the frequency for the field mode is ω and the atom-field mode coupling constant (one-photon Rabi frequency) is Ω, then the *Hamiltonian* for the *one atom Jaynes-Cummings model* is given by

$$\widehat{H}_{JC} = E_A(\widehat{P}_2 + \widehat{P}_1) + \frac{1}{2}\hbar\omega_0(\widehat{P}_2 - \widehat{P}_1) + \hbar\omega(\widehat{a}^\dagger\widehat{a}) + \frac{1}{2}\hbar\Omega(\widehat{\sigma}_+\widehat{a} + \widehat{a}^\dagger\widehat{\sigma}_-) \quad (1)$$

where the average atomic energy is $E_A = (E_2 + E_1)/2$. The first term is usually ignored as it only introduces a phase factor $\exp(-E_A t/\hbar)$ into the evolution of all states in the one atom Jaynes-Cummimgs model.

The dynamics of the one atom Jaynes-Cummings model is treated in many quantum optics textbooks and papers (see [37], [13], [9]). The state vector is expanded in terms of n photon states for the field and the two internal atomic states as

$$\begin{aligned} \lfloor\Psi\rangle &= \exp(-E_A t/\hbar) \quad (2) \\ &\times \sum_n \left(A_{\overline{n-1}2}(t)\exp(-i(\overline{n-1}\omega + \frac{1}{2}\omega_0)t)\ |2\rangle\,|n-1\rangle + A_{n1}(t)\exp(-i(n\omega - \frac{1}{2}\omega_0)t)\ |1\rangle\,|n\rangle \right) \end{aligned}$$

involving interaction picture amplitudes $A_{n1}, A_{\overline{n-1}2}$. The coupled equations for the amplitudes

$$\begin{aligned} i\frac{\partial}{\partial t}A_{\overline{n-1}2} &= \frac{1}{2}\Omega\sqrt{n}\exp(+i\Delta t)\ A_{n1} \\ i\frac{\partial}{\partial t}A_{n1} &= \frac{1}{2}\Omega\sqrt{n}\exp(-i\Delta t)\ A_{\overline{n-1}2} \quad (3) \end{aligned}$$

involve the detuning

$$\Delta = \omega_0 - \omega \tag{4}$$

and may be solved using Laplace transforms. The solution is

$$\begin{aligned} A_{n1}(t) &= \exp(-\frac{1}{2}i\Delta t)\left(\cos(\frac{1}{2}\omega_n t)\left\{A_{n1}(0)\right\} + i\sin(\frac{1}{2}\omega_n t)\left\{\frac{\Delta\ A_{n1}(0) - \Omega\sqrt{n}\ A_{\overline{n-1}2}(0)}{\omega_n}\right\}\right) \\ A_{\overline{n-1}2}(t) &= \exp(+\frac{1}{2}i\Delta t)\left(\cos(\frac{1}{2}\omega_n t)\left\{A_{\overline{n-1}2}(0)\right\} - i\sin(\frac{1}{2}\omega_n t)\left\{\frac{\Omega\sqrt{n}\ A_{n1}(0) + \Delta\ A_{\overline{n-1}2}(0)}{\omega_n}\right\}\right) \end{aligned} \tag{5}$$

where

$$\omega_n = \sqrt{\Delta^2 + n\Omega^2} \tag{6}$$

is the Rabi frequency. Population and coherence oscillations at various frequencies ω_n are therefore predicted, and which frequencies are observed depends on the inital conditions. For example, if the atom is not excited and the field is in an m photon state, then $A_{n1}(0) = \delta_{nm}$, $A_{\overline{n-1}2}(0) = 0$ and hence oscillations at a single frequency ω_m will occur. On the other hand, if the field was in a coherent state with amplitude η, then oscillation frequencies clustered around the mean frequency $\omega_{\overline{n}}$ associated with mean photon number $\overline{n} = |\eta|^2$ occur, the range of frequencies being associated with the standard deviation $\Delta n = \sqrt{\overline{n}}$ in photon numbers. This results in the collapse, revival effects discussed previously. In an alternative approach, Stenholm [14] expanded the state vector in products of Bargman states for the field and spinor states for the TLA and also demonstrated collapse, revival effects. As we see, his solution can be adapted to determining the Grassmann distribution function in the present treatment.

2.2 Fermion and Bosonic Modes

Instead of treating the Jaynes-Cummings model via the usual elementary quantum optics approach - for example via matrix mechanics with basis states $|1\rangle\,|n\rangle$, $|2\rangle\,|n-1\rangle$, we can replace the original one two-level atom plus cavity mode system with a somewhat *enlarged system* consisting of two fermionic modes 1, 2 interacting with one bosonic mode. This enlarged system includes states that are in one-one correspondence with the states for the original one two-level atom plus single cavity mode system, and the general dynamics for the larger Fermi-Bose system incorporates all possible behaviour for the original Jaynes-Cummings model. Note that relating the two state atomic system to two fermion modes has nothing to do with whether the atom itself is a fermion or a boson.

This process involves the introduction of *fermion annihilation, creation* operators $\widehat{c}_i, \widehat{c}_i^{\dagger}$ $(i = 1, 2)$ for the two fermion modes, which satisfy standard fermion anti-commutation rules. The commutation and anti-commutation rules for the

fermion and boson operators are

$$\begin{aligned} \{\widehat{c}_i, \widehat{c}_j^\dagger\} &= \delta_{ij} \\ \{\widehat{c}_i, \widehat{c}_j\} &= \{\widehat{c}_i^\dagger, \widehat{c}_j^\dagger\} = 0 \\ [\widehat{a}, \widehat{a}^\dagger] &= 1 \end{aligned} \tag{7}$$

with the boson and fermion operators commuting.

We then introduce the four possible fermion *Fock states*

$$|m_1; m_2\rangle = (\widehat{c}_1^\dagger)^{m_1} (\widehat{c}_2^\dagger)^{m_2} |0\rangle \tag{8}$$

where m_1, m_2 are equal to $0, 1$ only and $|0\rangle$ is the vacuum state. For the vacuum state $\widehat{c}_i |0\rangle = 0$, $\langle 0| \widehat{c}_i^\dagger = 0$. Thus $|0;0\rangle$ will be the vacuum state, with no fermions in either mode, $|1;0\rangle$ and $|0;1\rangle$ will be the one fermion states with one fermion in modes $1, 2$ respectively, and $|1;1\rangle$ will be the two fermion state with one fermion in each mode. Naturally, the Pauli exclusion principle excludes states with more than one fermion in any mode.

The full *basis states* for the Jaynes-Cummings model are Fock states of the form

$$|m_1; m_2; n\rangle = (\widehat{c}_1^\dagger)^{m_1} (\widehat{c}_2^\dagger)^{m_2} \frac{(\widehat{a}^\dagger)^n}{\sqrt{n!}} |0\rangle \tag{9}$$

where $m_i = 0, 1$ give the number of atoms in state $i = 1, 2$ and $n = 0, 1, 2, ...$ gives the number of photons in the field mode. Obviously since $(\widehat{c}_i^\dagger)^2 = 0$ there can not be more than one atom in state i. The state with no atoms or photons present is the vacuum state $|0\rangle$.

2.3 Quantum States

The most *general mixed state* for which there may be either zero, one or two atoms has a density operator

$$\begin{aligned} \widehat{\rho} &= \sum_{n,m} (\rho_{00n;00m} |0;0;n\rangle \langle 0;0,m| \\ &\quad + \sum_{n,m} (\rho_{10n;10m} |1;0;n\rangle \langle 1;0,m| + \rho_{10n;01m} |1;0;n\rangle \langle 0;1,m| \\ &\quad + \rho_{01n;10m} |0;1;n\rangle \langle 1;0,m| + \rho_{01n;01m} |0;1;n\rangle \langle 0;1,m|) \\ &\quad + \sum_{n,m} (\rho_{11n;11m} |1;1;n\rangle \langle 1;1,m| \end{aligned} \tag{10}$$

where the $\rho_{ijn;klm}$ $(i, j, k, l = 0, 1)$ are density matrix elements. This corresponds to a statistical mixture of states in which there are no atoms, one atom or two atoms. Pure states which are quantum superpositions of states with differering numbers of atoms are forbidden by the super-selection rule based on conservation of atom number. Statistical mixtures of such forbidden states

would in general lead to the presence of coherences between states with differing numbers of atoms. Thus there are no coherences between states with different numbers of atoms in (10).

The condition that the trace of the density operator is one requires

$$Tr\widehat{\rho} = \sum_n (\rho_{00n;00n} + \rho_{10n;10n} + \rho_{01n;01n} + \rho_{11n;11n}) = 1 \tag{11}$$

Note that in addition to atomic states corresponding to the one atom Jaynes-Cummings model, we now have two extra states in our enlarged system. and we need to consider whether these have any physical significance. The vacuum state $|0;0\rangle$ could correspond to an experiment with only the cavity field present but with no atoms inside the cavity - not a situation of great interest. The two fermion state $|1;1\rangle$ could correspond to a two atom system inside the cavity, with one atom in state $|1\rangle$ and the other in state $|2\rangle$. All four fermion states $|m_1;m_2\rangle$ also have a mathematical interpretation in terms of *spin states* in our enlarged system.

2.4 Normal Ordering and Vacuum State Projector

The vacuum state $|0\rangle$ which contains no bosons or fermions in any mode is associated with the *vacuum state projector* $|0\rangle\langle 0|$. It turns out that the vacuum projector can be written in terms of *normally ordered* forms of products of exponential operators based on mode number operators. The normally ordered form for a *product* of annihilation, creation operators involves placing all creation operators to the left of all annihilation operators, whilst retaining all creation and all annihilation operators in their original order, and multiplying by either $+1$ or -1 depending on whether the required permutation of *fermion* operators from the original product is even or odd [38].

General *functions* of the annihilation, creation operators (such as the Hamiltonian) can always be written as linear combinations of products with c-number coefficients, and the normally ordered form of such a function is just the linear combination of the normally ordered form of the various products. A useful theorem that applies to the normally ordered product of two functions involving the number operators for two different fermion modes is

$$\mathcal{N}(F(\widehat{c}_i^\dagger \widehat{c}_i)\, G(\widehat{c}_j^\dagger \widehat{c}_j)) = \mathcal{N}(F(\widehat{c}_i^\dagger \widehat{c}_i))\, \mathcal{N}(G(\widehat{c}_j^\dagger \widehat{c}_j)) \qquad (i \neq j) \tag{12}$$

where F, G are arbitrary functions. This result only depends on the annihilation and creation operators anticommuting for $i \neq j$.

The vacuum state projector is given in terms of normally ordered operators as

$$|0\rangle\langle 0| = \mathcal{N}(\prod_i \exp(-\widehat{a}_i^\dagger \widehat{a}_i)) = \mathcal{N}(\exp(-\sum_i \widehat{a}_i^\dagger \widehat{a}_i)) \tag{13}$$

$$= \mathcal{N}(\prod_i \exp(-\widehat{c}_i^\dagger \widehat{c}_i)) = \mathcal{N}(\exp(-\sum_i \widehat{c}_i^\dagger \widehat{c}_i)) \tag{14}$$

for bosons and fermions respectively. The (13) result for bosons is given in [13] (see p.45). The result for fermions follows from the vacuum state requirements that $\widehat{c}_i\,|0\rangle = 0$, $\langle 0|\,\widehat{c}_i^{\dagger} = 0$ are also satisfied by the expression in (14), together with the requirement that the vacuum state matrix element for the projector is also unity.

2.5 Population and Transition Operators

We now identify the two one fermion Fock states with the two internal atomic states in the one atom Jaynes-Cummings model as follows:

$$\begin{aligned} \widehat{c}_1^{\dagger}\,|0\rangle &= |1;0\rangle \Longleftrightarrow |1\rangle \\ \widehat{c}_2^{\dagger}\,|0\rangle &= |0;1\rangle \Longleftrightarrow |2\rangle \end{aligned} \tag{15}$$

and identify the four different products involving an annihilation, a creation operator and the vacuum state projector with the four *one atom atomic population* or *projector operator* $\widehat{P}_{1,2}$ and *transtion operators* $\widehat{\sigma}_{\pm}$

$$\begin{aligned} \widehat{c}_1^{\dagger}\,|0\rangle\langle 0|\,\widehat{c}_1 &\Longleftrightarrow \widehat{P}_1 \qquad \widehat{c}_2^{\dagger}\,|0\rangle\langle 0|\,\widehat{c}_2 \Longleftrightarrow \widehat{P}_2 \\ \widehat{c}_1^{\dagger}\,|0\rangle\langle 0|\,\widehat{c}_2 &\Longleftrightarrow \widehat{\sigma}_- \qquad \widehat{c}_2^{\dagger}\,|0\rangle\langle 0|\,\widehat{c}_1 \Longleftrightarrow \widehat{\sigma}_+ \end{aligned} \tag{16}$$

For the expanded Jaynes-Cummings model the *two atom state population* or *projector operator* $\widehat{P}_{12}$ with one atom in each of the atomic states we have

$$\widehat{c}_1^{\dagger}\widehat{c}_2^{\dagger}\,|0\rangle = |1;1\rangle \tag{17}$$

$$\widehat{c}_1^{\dagger}\widehat{c}_2^{\dagger}\,|0\rangle\langle 0|\,\widehat{c}_2\widehat{c}_1 \Longleftrightarrow \widehat{P}_{12} \tag{18}$$

and for the *zero atom population* or *projector operator* $\widehat{P}_0$ with no atoms in either of the internal atomic states we have

$$|0\rangle = |0;0\rangle \tag{19}$$

$$|0\rangle\langle 0| \Longleftrightarrow \widehat{P}_0 \tag{20}$$

This of course is the same as the vacuum state projector.

Using Eq.(14) and the result (12) we can obtain expressions for all the atom state operators just in terms of fermion annihilation, creation operators. We find for the one atom population and transition operators

$$\widehat{P}_1 = \widehat{c}_1^{\dagger}\widehat{c}_1 - \widehat{c}_1^{\dagger}\widehat{c}_2^{\dagger}\widehat{c}_2\widehat{c}_1 \qquad \widehat{P}_2 = \widehat{c}_2^{\dagger}\widehat{c}_2 - \widehat{c}_1^{\dagger}\widehat{c}_2^{\dagger}\widehat{c}_2\widehat{c}_1 \tag{21}$$

$$\widehat{\sigma}_- = \widehat{c}_1^{\dagger}\widehat{c}_2 \qquad \widehat{\sigma}_+ = \widehat{c}_2^{\dagger}\widehat{c}_1 \tag{22}$$

and for the two atom and zero atom population operators

$$\widehat{P}_{12} = \widehat{c}_1^{\dagger}\widehat{c}_2^{\dagger}\widehat{c}_2\widehat{c}_1 \tag{23}$$

$$\widehat{P}_0 = 1 - \widehat{c}_1^{\dagger}\widehat{c}_1 - \widehat{c}_2^{\dagger}\widehat{c}_2 + \widehat{c}_1^{\dagger}\widehat{c}_2^{\dagger}\widehat{c}_2\widehat{c}_1 \tag{24}$$

The population operators thus satisfy the condition

$$\widehat{P}_0 + \widehat{P}_1 + \widehat{P}_2 + \widehat{P}_{12} = \widehat{1} \tag{25}$$

which (as will be seen below), just represents the total probability for any general mixed state (10) that the system is found in either a zero, one or two atom state being equal to unity

2.6 Hamiltonian and Number Operators

The Hamiltonian needs to be modified for the *enlarged system* to allow for states with two or zero atoms. The one atom Jaynes-Cummings model Hamiltonian $\widehat{H}_{JC}$ is replaced by an *enlarged Hamiltonian* $\widehat{H}$ by adding a two atom energy term $(E_2 + E_1)\widehat{P}_{12}$ to the Hamiltonian in (1). This enlarged Hamiltonian has zero matrix elements between two atom states $|1;1\rangle$, one atom states $|0;1\rangle$, $|1;0\rangle$ and zero atom states $|0;0\rangle$. The two and zero atom states are eigenstates of the Hamiltonian with atomic energies $E_2 + E_1$ and 0, as expected. The atom-field coupling term also cannot cause a transtion between states with differing total atom number, consistent with not violating the super-selection rule.

In terms of the bosonic and fermionic annihilation, creation operators the *enlarged Hamiltonian* for the Jaynes-Cummings model may be written as

$$\widehat{H} = E_A(\widehat{c}_2^\dagger\widehat{c}_2 + \widehat{c}_1^\dagger\widehat{c}_1) + \frac{1}{2}\hbar\omega_0(\widehat{c}_2^\dagger\widehat{c}_2 - \widehat{c}_1^\dagger\widehat{c}_1) + \hbar\omega(\widehat{a}^\dagger\widehat{a}) + \frac{1}{2}\hbar\Omega(\widehat{c}_2^\dagger\widehat{c}_1\widehat{a} + \widehat{a}^\dagger\widehat{c}_1^\dagger\widehat{c}_2) \tag{26}$$

where the results (21), (22) and (23) for the one and two atom operators have been substituted into the enlarged Hamiltonian. Terms involving $\widehat{c}_1^\dagger\widehat{c}_2^\dagger\widehat{c}_2\widehat{c}_1$ cancel out.

The *number of photons* present is determined from the operator

$$\widehat{n} = \widehat{a}^\dagger\widehat{a} \tag{27}$$

whose eigenvalues are $n = 0, 1, 2,$ .We can also introduce *number operators* for the two atomic states via

$$\widehat{n}_i = \widehat{c}_i^\dagger\widehat{c}_i \qquad (i = 1, 2) \tag{28}$$

where from the femion anticommutation rules the eigenvalues for the atomic number operators are $0, 1$ only.

The *total number of atoms* present is given by the number operator

$$\widehat{N} = (\widehat{c}_2^\dagger\widehat{c}_2 + \widehat{c}_1^\dagger\widehat{c}_1) \tag{29}$$

$$= 0 \times \widehat{P}_0 + 1 \times (\widehat{P}_1 + \widehat{P}_2) + 2 \times \widehat{P}_{12} \tag{30}$$

the second expression for the number operator being related to the projectors $\widehat{P}_0$, $\widehat{P}_1$, $\widehat{P}_2$ and $\widehat{P}_{12}$ being for zero, one, one and two atom states respectively. The number operator can have eigenvalues $0, 1, 2$ and clearly

$$\widehat{N}\ |m_1; m_2; n\rangle = (m_1 + m_2)\ |m_1; m_2; n\rangle \tag{31}$$

The number operator commutes with the Hamiltonian, so if we initially have a physical state with a specified number of atoms, then state evolution does not change the atom number. Thus one atom states do not change into two atom states and if the initial density operator corresponds to the form in (10) with $\rho_{00n;00m} = \rho_{11n;11m} = 0$ for one atom states it will remain in this form. There is of course no conservation law for the photon number. .

2.7 Probabilities and Coherences

We can obtain expressions for the physical quantities in terms of the annihilation, creation operators. For the *one atom probability* of finding one atom in state $|1\rangle$ and none in state $|2\rangle$ we have

$$P_1 = Tr(\widehat{P}_1\,\widehat{\rho}) = \left\langle \widehat{c}_1^\dagger \widehat{c}_1 \right\rangle - \left\langle \widehat{c}_1^\dagger \widehat{c}_2^\dagger \widehat{c}_2 \widehat{c}_1 \right\rangle = Tr(\widehat{c}_1 \widehat{\rho} \widehat{c}_1^\dagger) - Tr(\widehat{c}_2 \widehat{c}_1 \widehat{\rho} \widehat{c}_1^\dagger \widehat{c}_2^\dagger) \tag{32}$$

where $\left\langle \widehat{\Xi} \right\rangle = Tr(\widehat{\rho}\,\widehat{\Xi})$. Similarly the *one atom probability* of finding one atom in state $|2\rangle$ and none in state $|1\rangle$ is

$$P_2 = Tr(\widehat{P}_2\,\widehat{\rho}) = \left\langle \widehat{c}_2^\dagger \widehat{c}_2 \right\rangle - \left\langle \widehat{c}_1^\dagger \widehat{c}_2^\dagger \widehat{c}_2 \widehat{c}_1 \right\rangle = Tr(\widehat{c}_2 \widehat{\rho} \widehat{c}_2^\dagger) - Tr(\widehat{c}_2 \widehat{c}_1 \widehat{\rho} \widehat{c}_1^\dagger \widehat{c}_2^\dagger) \tag{33}$$

Note that these probabilities are not the same as $\left\langle \widehat{c}_i^\dagger \widehat{c}_i \right\rangle$ as might be expected.

The *one atom coherences* between state $|1\rangle$ and state $|2\rangle$ are given by

$$\begin{aligned} \rho_{12} &= Tr(\widehat{\sigma}_-\,\widehat{\rho}) = \left\langle \widehat{c}_1^\dagger \widehat{c}_2 \right\rangle = Tr(\widehat{c}_2\,\widehat{\rho}\,\widehat{c}_1^\dagger) \\ \rho_{21} &= Tr(\widehat{\sigma}_+\,\widehat{\rho}) = \left\langle \widehat{c}_2^\dagger \widehat{c}_1 \right\rangle = Tr(\widehat{c}_1\,\widehat{\rho}\,\widehat{c}_2^\dagger) \end{aligned} \tag{34}$$

The *mean number of photons* is

$$\overline{n} = \left\langle \widehat{a}^\dagger \widehat{a} \right\rangle \tag{35}$$

Also, the *two atom probability* P_{12} for finding one atom in state $|1\rangle$ and one in state $|2\rangle$, and the *zero atom probability* P_0 for finding no atom either in state $|1\rangle$ or in state $|2\rangle$ are given by

$$P_{12} = Tr(\widehat{P}_{12}\,\widehat{\rho}) = \left\langle \widehat{c}_1^\dagger \widehat{c}_2^\dagger \widehat{c}_2 \widehat{c}_1 \right\rangle = Tr(\widehat{c}_2 \widehat{c}_1 \widehat{\rho} \widehat{c}_1^\dagger \widehat{c}_2^\dagger) \tag{36}$$

$$P_0 = Tr(\widehat{P}_0\,\widehat{\rho}) = 1 - P_1 - P_2 - P_{12} \tag{37}$$

the last result just expressing the fact that all the probabilities must add up to one. Expressions involving the *density matrix elements* for *general mixed states* can be easily obtained for all these results.

Clearly, for *one atom states* where $\rho_{00n;00m} = \rho_{11n;11m} = 0$ we have using (11)

$$P_0 = P_{12} = 0 \qquad P_1 + P_2 = 1 \tag{38}$$

$$\overline{n} = \sum_n n(\rho_{10n;10n} + \rho_{01n;01n}) \tag{39}$$

For general mixed states as in (10) we see that expectation values of odd numbers of fermionic creation and destruction operators are zero. Thus

$$\langle \widehat{c}_i \rangle = \left\langle \widehat{c}_i^{\dagger} \right\rangle = \left\langle \widehat{c}_1^{\dagger} \widehat{c}_2^{\dagger} \widehat{c}_i \right\rangle = \left\langle \widehat{c}_i^{\dagger} \widehat{c}_2 \widehat{c}_1 \right\rangle = 0 \tag{40}$$

For one atom states the only non-zero expectation values are those that involve one fermion annihilation operator and one creation operator of the form $\left\langle \widehat{c}_i^{\dagger} \widehat{c}_j \right\rangle$. Thus in addition to the last results the expectation value involving four fermion operators is zero

$$\left\langle \widehat{c}_1^{\dagger} \widehat{c}_2^{\dagger} \widehat{c}_2 \widehat{c}_1 \right\rangle = P_{12} = 0 \tag{41}$$

since $\rho_{11n;11n} = 0$ for one atom states. This corresponds to the probability of finding two atoms present being zero in the one atom Jaynes-Cummings model. For general mixed states the quantity $\left\langle \widehat{c}_1^{\dagger} \widehat{c}_2^{\dagger} \widehat{c}_2 \widehat{c}_1 \right\rangle$ is conserved, since $\widehat{c}_1^{\dagger} \widehat{c}_2^{\dagger} \widehat{c}_2 \widehat{c}_1$ commutes with the Hamiltonian

Although we will be mainly focused on states corresponding to one atom states in the Jaynes-Cummings model, it will be convenient to introduce *non-physical states* involving coherent superpositions of different total numbers of atoms and even states where the expansion coefficients are Grassmann numbers. For non-physical states involving superpositions of different atom numbers there may be off diagonal elements of the density matrix between states of different atom numbers, and the results in Eq.(40) may not apply.

3 Phase Space Theory

There are several ways to set out *phase space theory*. Here we start with the characteristic function - which is equivalent to the set of all normally ordered quantum correlation functions - and then relate this to the distribution function - whose phase space integral with products of phase space variables determines the correlation functions. The *normally ordered quantum correlation functions* are defined as

$$\begin{aligned} & G(m_1, m_2, n;\, p, l_2, l_1) \\ = & \left\langle (\widehat{c}_1^{\dagger})^{m_1} (\widehat{c}_2^{\dagger})^{m_2} (\widehat{a}^{\dagger})^n (\widehat{a})^p (\widehat{c}_2)^{l_2} (\widehat{c}_1)^{l_1} \right\rangle \\ = & \mathrm{Tr}((\widehat{c}_2)^{l_2} (\widehat{c}_1)^{l_1}\, (\widehat{a})^p\, \hat{\rho}\, (\widehat{a}^{\dagger})^n\, (\widehat{c}_1^{\dagger})^{m_1} (\widehat{c}_2^{\dagger})^{m_2}) \end{aligned} \tag{42}$$

where $m_i, l_i = 0, 1$ only. Normally ordered correlation functions in which the density operator appears in the middle of the trace with annihilation operators on the left and creation operators on the right appear in fundamental treatments of boson and fermion detection processes (see for example [39]).

3.1 Characteristic Function

We define the *characteristic function* $\chi(\xi,\xi^+,h_i,h_i^+)$ via

$$\begin{aligned}\chi(\xi,\xi^+,h,h^+) &= \mathrm{Tr}(\,\hat\Omega_b^+(\xi^+)\,\hat\Omega_f^+(h^+)\,\hat\rho\,\hat\Omega_f^-(h)\,\hat\Omega_b^-(\xi)) &(43)\\ \hat\Omega_b^+(\xi^+) &= \exp i\hat a\xi^+ \qquad \hat\Omega_b^-(\xi)=\exp i\xi\hat a^\dagger &(44)\\ \hat\Omega_f^+(h^+) &= \exp i\sum_{i=1,2}\hat c_i h_i^+ \qquad \hat\Omega_f^-(h)=\exp i\sum_{i=1,2}h_i\hat c_i^\dagger &(45)\end{aligned}$$

For the bosonic mode we associate a pair of *c-numbers* ξ,ξ^+ . For each fermionic mode $i=1,2$ we associate a pair of *Grassmann-numbers* h_i,h_i^+ and $h=\{h_1,h_2\}$, $h^+=\{h_1^+,h_2^+\}$. The characteristic function will be a c-number *analytic* function of ξ,ξ^+ and a *Grassmann function* of h_1,h_1^+,h_2,h_2^+. The factors for the different modes in the last two expressions for $\hat\Omega_f^+(h^+)$ and $\hat\Omega_f^-(h)$ may be put in any order since they commute, but by convention the order will be $2,1$ for $\hat\Omega_f^+(h^+)$ and $,1,2$ for $\hat\Omega_f^-(h)$. This leads to a quasi distribution function of the *positive P* type, which is generally better suited for obtaining Ito stochastic equations.

For the *physical states* where the state vector involves a single number of atoms ($N=0,1,2$), expectation values for cases where the number of fermion annihilation operators differs from the number of creation operators are zero. Using $\hat\Omega_f^+(h^+)=(1+\hat c_2h_2^+)(1+\hat c_1h_1^+)$ and $\hat\Omega_f^-(h)=(1+h_1\hat c_1^\dagger)(1+h_2\hat c_2^\dagger)$ we see that the characteristic function is of the form

$$\chi(\xi,\xi^+,h,h^+)=\chi_0(\xi,\xi^+)+\sum_{i,j=1,2}\chi_2^{i;j}(\xi,\xi^+)\,h_j^+h_i+\chi_4^{12;21}(\xi,\xi^+)\,h_2^+h_1^+h_1h_2 \tag{46}$$

and the relationship with the coefficients for $\xi,\xi^+=0,0$ is

$$\begin{aligned}\chi_0(0,0) &= 1\\ \chi_2^{i,j}(0,0) &= i^2\left\langle\widehat c_i^\dagger\widehat c_j\right\rangle\\ \chi_4^{12,21}(0,0) &= i^4\left\langle\widehat c_1^\dagger\widehat c_2^\dagger\widehat c_2\widehat c_1\right\rangle &(47)\end{aligned}$$

Thus for one atom physical states at most six $(2.2!/(2!2!)=(C_0^2)^2+(C_1^2)^2+(C_2^2)^2$ [40]) c-number coefficients are required to define the characteristic function $\chi(\xi,\xi^+,h,h^+)$ as a Grassmann function.

The various normally ordered quantum correlation functions can be expressed as c-number and *Grassmann derivatives* of the characteristic function

$$\begin{aligned}&G(m_1,m_2,n;\,p,l_2,l_1)\\ =\;&\left(\frac{(\overrightarrow\partial)^{l_2}}{\partial(ih_2^+)^{l_2}}\frac{(\overrightarrow\partial)^{l_1}}{\partial(ih_1^+)^{l_1}}\left(\frac{\partial^n}{\partial(i\xi)^n}\frac{\partial^p}{\partial(i\xi^+)^p}\chi(\xi,\xi^+,h,h^+)\right)\frac{(\overleftarrow\partial)^{m_1}}{\partial(ih_1)^{m_1}}\frac{(\overleftarrow\partial)^{m_2}}{\partial(ih_2)^{m_2}}\right)_{\xi,\xi^+,h,h^+=0}\end{aligned} \tag{48}$$

where $m_i,l_i=0,1$ only, and in the case where l_i or m_i is zero, then no differentiation takes place.

3.2 Distribution Function

The characteristic function $\chi(\xi,\xi^+,h,h^+)$ is related to the *distribution function* $P(\alpha,\alpha^+,\alpha^*,\alpha^{+*},g,g^+)$ via *phase space integrals*, which are c-number integrals for the field mode and *Grassmann integrals* for the two atomic modes. The formula is

$$\begin{aligned}
&\chi(\xi,\xi^+,h,h^+)\\
=\;&\int\prod_{i=1,2}dg_i^+dg_i\int d^2\alpha^+d^2\alpha\\
&\times\exp i\sum_{i=1}^{2}\{g_ih_i^+\}\,\exp i\{\alpha\,\xi^+\}\cdot P(\alpha,\alpha^+,\alpha^*,\alpha^{+*},g,g^+)\cdot\exp i\{\xi\alpha^+\}\,\exp i\sum_{i=1}^{2}\{h_ig_i^+\}
\end{aligned}\tag{49}$$

where for the field mode we associate another pair of c-numbers α,α^+, and for the two atomic modes i we associate another pair of g-numbers g_i,g_i^+ and $g=\{g_1,g_2\}$, $g^+=\{g_1^+,g_2^+\}$. As previously the characteristic function $\chi(\xi,\xi^+,h,h^+)$ is a function of the variables ξ,ξ^+,h_i,h_i^+ and g_i,g_i^+ but *not* of their complex conjugates, whereas the distribution function $P(\alpha,\alpha^+,\alpha^*,\alpha^{+*},g,g^+)$ also depends on the complex conjugates α^*,α^{+*}. Unlike the characteristic function, the distribution function is *non analytic*. The distribution function will be a c-number function of $\alpha,\alpha^+,\alpha^*,\alpha^{+*}$ and a Grassmann function of g_1,g_1^+,g_2,g_2^+, the highest order monomial being $g_1g_2g_2^+g_1^+$. The c-number integrations $d^2\alpha^+$ and $d^2\alpha$ are over the two *complex planes* α,α^+ and thus $d^2\alpha\equiv d\alpha_xd\alpha_y$ and $d^2\alpha^+\equiv d\alpha_x^+d\alpha_y^+$. The Grassmann integrals dg_i^+ and dg_i are over the *single variables* g_i^+ and g_i only, and not over g_i^{+*} and g_i^* as well. The distribution function is of the *positive P* type for the bosonic variables α,α^+ and similar to the *complex P* type for the Grassmann variables g_i,g_i^+. Although pairs of differentials $dg_i^+dg_i$ commute with other pairs, the convention used here is to write $\prod_i dg_i^+dg_i=dg_2^+dg_2\,dg_1^+dg_1$.

The distribution function is of the form

$$\begin{aligned}
&P(\alpha,\alpha^+,\alpha^*,\alpha^{+*},g,g^+)\\
=\;&P_0(\alpha,\alpha^+,\alpha^*,\alpha^{+*})+\sum_{i;j}P_2^{i,j}(\alpha,\alpha^+,\alpha^*,\alpha^{+*})\,g_ig_j^+\\
&+P_{2n}^{12,21}(\alpha,\alpha^+,\alpha^*,\alpha^{+*})\,g_1g_2g_2^+g_1^+
\end{aligned}\tag{50}$$

Other possible terms do not lead to the characteristic function is given by (46) when the Grassmann integration in (49) is carried out using (160) after expanding the Grassmann exponential factors. An ordering convention in which the g_i are arranged in ascending order and the g_j^+ in descending order has been used. The coefficients are functions of the bosonic variables α,α^+. There are six $(2.2)!/(2!2!)=(C_0^2)^2+(C_1^2)^2+(C_2^2)^2$ [40]) c-number coefficients to define the

distribution function $P(\alpha, \alpha^+, g, g^+)$ as a Grassmann function, the same number of course as the characteristic.that it determines. The distribution function is thus an *even Grassmann function* of the order $2^2 = 4$ in the variables g_1, g_1^+, g_2, g_2^+, a feature that is needed later.

The hermiticity of the density operator leads to relationships between the coefficients

$$\begin{aligned} P_0(\widetilde{\boldsymbol{\alpha}})^* &= P_0(\widetilde{\boldsymbol{\alpha}}) \\ P_2^{1,1}(\widetilde{\boldsymbol{\alpha}})^* &= P_2^{1,1}(\widetilde{\boldsymbol{\alpha}}) \qquad P_2^{1,2}(\widetilde{\boldsymbol{\alpha}})^* = P_2^{2,1}(\widetilde{\boldsymbol{\alpha}}) \\ P_2^{2,1}(\widetilde{\boldsymbol{\alpha}})^* &= P_2^{1;2}(\widetilde{\boldsymbol{\alpha}}) \qquad P_2^{2,2}(\widetilde{\boldsymbol{\alpha}})^* = P_2^{2,2}(\widetilde{\boldsymbol{\alpha}}) \\ P_4^{12,21}(\widetilde{\boldsymbol{\alpha}})^* &= P_4^{12,21}(\widetilde{\boldsymbol{\alpha}}) \end{aligned} \tag{51}$$

where for short we write $\widetilde{\boldsymbol{\alpha}} \equiv \{\alpha, \alpha^+, \alpha^*, \alpha^{+*}\}$. This shows that four of the bosonic coefficients are real and the other two $P_2^{1,2}(\widetilde{\boldsymbol{\alpha}}), P_2^{2,1}(\widetilde{\boldsymbol{\alpha}})$ are complex conjugates.

For the Jaynes-Cummings model case the quantum correlation functions are obtained from the characteristic function from Eq. (48). Applying this result by carrying out the differentiations on the formula (49) relating the characteristic and distribution functions gives the *quantum correlation functions* in terms of *phase space integrals*

$$\begin{aligned} & G(m_1, m_2, n;\, p, l_2, l_1) \\ = & \int d^2\alpha^+ d^2\alpha \int \prod_{i=1,2} dg_i^+ dg_i \\ & \times (g_2)^{l_2} (g_1)^{l_1}\, (\alpha)^p\, P(\alpha, \alpha^+, \alpha^*, \alpha^{+*}, g, g^+)\, (\alpha^+)^n\, (g_1^+)^{m_1} (g_2^+)^{m_2} \end{aligned} \tag{52}$$

which involves both c-number and Grassmann number phase space integrals with the Bose-Fermi distribution function. Note that the numbers of fermion annihilation and creation operators are the same. Alternative forms for the quantum correlation function with the distribution function as the left factor in the integrand and all the α_i and g_j placed to the right of the α_i^+ and g_j^+ can be easily found.

3.3 Existence of Distribution Function - Canonical Form

The distribution function $P(\alpha, \alpha^+, \alpha^*, \alpha^{+*}, g, g^+)$ is not required to be *unique*, only that it generates the characteristic function $\chi(\xi, \xi^+, h, h^+)$ - which is unique - via the phase space integral (49). However, it is important to be able to show that a distribution function always *exists*.

The existence of the positive P type distribution function can be shown by introducing the so-called *canonical representation* of the density operator, which

involves the fermion and boson *Bargmann coherent states* (see Appendix 9)

$$\begin{aligned}\hat{\rho} &= \int\int dg^{+}dg \int\int d^{2}\alpha^{+}d^{2}\alpha \; P_{canon}(\alpha,\alpha^{+},\alpha^{*},\alpha^{+*},g,g^{+})\,\widehat{\Lambda}(g,g^{+},\alpha,\alpha^{+}) \\ &= \int\int dg^{+}dg \int\int d^{2}\alpha^{+}d^{2}\alpha\,\widehat{\Lambda}(g,g^{+},\alpha,\alpha^{+})\,P_{canon}(\alpha,\alpha^{+},\alpha^{*},\alpha^{+*},g,g^{+})\end{aligned} \tag{53}$$

where

$$\widehat{\Lambda}(g,g^{+},\alpha,\alpha^{+}) = \frac{|g\rangle_{B}\langle g^{+*}|_{B}}{Tr(|g\rangle_{B}\langle g^{+*}|_{B})}\frac{|\alpha\rangle_{B}\langle\alpha^{+*}|_{B}}{Tr(|\alpha\rangle_{B}\langle\alpha^{+*}|_{B})} \tag{54}$$

is a normalised *projector* and

$$\begin{aligned}&P_{canon}(\alpha,\alpha^{+},\alpha^{*},\alpha^{+*},g,g^{+}) \\ &= \left(\frac{1}{4\pi^{2}}\right)^{n}\int\int dg^{+*}dg^{*}\exp(\sum_{i}(g_{i}g_{i}^{*}+g_{i}^{+*}g_{i}^{+}+g_{i}g_{i}^{+}))\exp(-\frac{1}{2}\sum_{i}(\alpha_{i}\alpha_{i}^{*}+\alpha_{i}^{+*}\alpha_{i}^{+})) \\ &\times\langle g|_{B}\left\langle\frac{\alpha+\alpha^{+*}}{2}\right|_{B}\widehat{\rho}\left|\frac{\alpha+\alpha^{+*}}{2}\right\rangle_{B}|g^{+*}\rangle_{B}\end{aligned} \tag{55}$$

is a canonical representation function for the density operator $\widehat{\rho}$. The two forms for the density operator are the same because the normalised projector $\widehat{\Lambda}(g,g^{+},\alpha,\alpha^{+})$ is an even Grassmann operator and thus commutes with $P_{canon}(\alpha,\alpha^{+},\alpha^{*},\alpha^{+*},g,g^{+})$. By substituting for the canonical form of the density operator (53) into the expression (43) for the characteristic function we can easily show that the characteristic function and distribution function are related as in (49) with the distribution function given by the canonical form (55). For bosons the proof of the existence of the canonical form is given by Drummond and Gardiner ([15], [18]). For fermions the proof is outlined in papers by Cahill and Glauber [20] and Plimak et al [21].

Note also that *irrespective* of whether or not the detailed form for $P(\alpha,\alpha^{+},\alpha^{*},\alpha^{+*},g,g^{+})$ is given by the canonical form (55), the expression (53) is often used to *define* the positive P type distribution function [15], rather than the characteristic function expression (49). If the density operator can be written as

$$\widehat{\rho} = \int\int dg^{+}dg \int\int d^{2}\alpha^{+}d^{2}\alpha\,\widehat{\Lambda}(g,g^{+},\alpha,\alpha^{+})\,P(\alpha,\alpha^{+},\alpha^{*},\alpha^{+*},g,g^{+}) \tag{56}$$

then this defines *a* (non-unique) positive P distribution function. This form of the density operator leads to the previous relationship (49) between the distribution and characteristic functions.

3.4 Probabilities and Coherences as Phase Space Integrals

The Grassmann phase space integrals can be evaluated using the non-zero result $\int\prod_{i}dg_{i}^{+}dg_{i}\,g_{1}g_{2}g_{2}^{+}g_{1}^{+}=1$, giving the probabilities and coherences as bosonic

phase space integrals involving the six coefficients that determine the distribution function (50).

For the *two atom probability* which is given by the fourth order quantum correlation function, we obtain the result

$$P_{12} = \int d^2\alpha^+ d^2\alpha \, P_0(\alpha, \alpha^+, \alpha^*, \alpha^{+*}) \tag{57}$$

This correlation function may be zero without $P_0(\alpha, \alpha^+, \alpha^*, \alpha^{+*})$ being zero.

The *one atom probabilities* for states $1, 2$ are given by

$$P_1 = \int d^2\alpha^+ d^2\alpha \, \left(P_2^{2;2}(\alpha, \alpha^+, \alpha^*, \alpha^{+*}) - P_0(\alpha, \alpha^+, \alpha^*, \alpha^{+*}) \right) \tag{58}$$

$$P_2 = \int d^2\alpha^+ d^2\alpha \, \left(P_2^{1;1}(\alpha, \alpha^+, \alpha^*, \alpha^{+*}) - P_0(\alpha, \alpha^+, \alpha^*, \alpha^{+*}) \right) \tag{59}$$

These quantities are of course real, consistent with (51).

For the *atomic coherences* we have the results

$$\rho_{12} = -\int d^2\alpha^+ d^2\alpha \, P_2^{1;2}(\alpha, \alpha^+, \alpha^*, \alpha^{+*}) \tag{60}$$

$$\rho_{21} = -\int d^2\alpha^+ d^2\alpha \, P_2^{2;1}(\alpha, \alpha^+, \alpha^*, \alpha^{+*}) \tag{61}$$

These quantities are complex conjugates, consistent with (51).

The *mean number of photons* is

$$\overline{n} = \int d^2\alpha^+ d^2\alpha \, (\alpha) \, P_4^{12;21}(\alpha, \alpha^+) \, (\alpha^+) \tag{62}$$

This quantity is of course real, consistent with (51).

Note that the mean photon number result involves the fourth order expansion coefficient $P_4^{12;21}(\alpha, \alpha^+, \alpha^*, \alpha^{+*})$, and the one atom probabilities for states $1, 2$ involve the opposite second order expansion coefficients - $P_2^{2;2}(\alpha, \alpha^+, \alpha^*, \alpha^{+*})$ and $P_2^{1;1}(\alpha, \alpha^+, \alpha^*, \alpha^{+*})$ respectively, minus the quantity $P_0(\alpha, \alpha^+, \alpha^*, \alpha^{+*})$ that determines the two atom probability. The coherences involve the second expansion coefficients $P_2^{1;2}(\alpha, \alpha^+, \alpha^*, \alpha^{+*})$ and $P_2^{2;1}(\alpha, \alpha^+, \alpha^*, \alpha^{+*})$ respectively, multiplied by -1. Thus in their final form, only c-number integrations are needed to determine the quantum correlation functions.

The *normalisation integral* for the combined bose-fermi distribution function is given by

$$Tr\widehat{\rho} = \int\int d^2\alpha^+ d^2\alpha \prod_{i=1}^{2} dg_i^+ dg_i \, P(\alpha, \alpha^+, \alpha^*, \alpha^{+*}, g, g^+) = 1 \tag{63}$$

If we substitute the specific ordered form (50) for the distribution function, we see that the normalisation integral gives

$$\int d^2\alpha^+ d^2\alpha \, P_4^{12;21}(\alpha, \alpha^+, \alpha^*, \alpha^{+*}) = 1 \tag{64}$$

4 Fokker-Planck Equation

The derivation of the Fokker-Planck equation is based on using the *correspondence rules*. As has been noted previously the distribution function is not unique and is a non-analytic function of the bosonic phase space variables. The correspondence rules are also non-unique, but the application of the standard correspondence rules (65) - (72) is required to lead to a distribution function which correctly determines the quantum correlation functions. It turns out however that the Fokker-Planck equation based on the standard correspondence rules leads to a distribution function that is not satisfactory for the Jaynes-Cummings model. However, the canonical form of the distribution function always exists and can be applied to the initial conditions. Furthermore, it turns out to lead to a Fokker-Planck equation that can be solved analytically so the Fokker-Planck equation for the *canonical distribution function* will now be obtained. The relevant correspondence rules will be those given in Eqs. (81) and the distribution function will be written in termsof the new variables (78).

4.1 Correspondence Rules

We replace the Liouville-von Neumann equation for the density operator by a *Fokker-Planck equation* for the distribution function. To do this we make use of so-called *correspondence rules*, which are presented here for the general case of a combined system of bosons and fermions. The correspondence rules state what happens to the distribution function $P(\alpha, \alpha^+, \alpha^*, \alpha^{+*}, g, g^+)$ when the density operator $\hat{\rho}$ is replaced by the product of the density operator with an annihilation or creation operator.

The *standard* correspondence rules are:

$$\hat{\rho} \;\Rightarrow\; \hat{a}_i\,\hat{\rho} \qquad P(\alpha,\alpha^+,\alpha^*,\alpha^{+*},g,g^+) \Rightarrow \alpha_i\,P \tag{65}$$

$$\hat{\rho} \;\Rightarrow\; \hat{\rho}\,\hat{a}_i \qquad P(\alpha,\alpha^+,\alpha^*,\alpha^{+*},g,g^+) \Rightarrow \left(\alpha_i - \frac{\partial}{\partial \alpha_i^+}\right) P \tag{66}$$

$$\hat{\rho} \;\Rightarrow\; \hat{a}_i^\dagger\,\hat{\rho} \qquad P(\alpha,\alpha^+,\alpha^*,\alpha^{+*},g,g^+) \Rightarrow \left(\alpha_i^+ - \frac{\partial}{\partial \alpha_i}\right) P \tag{67}$$

$$\hat{\rho} \;\Rightarrow\; \hat{\rho}\,\hat{a}_i^\dagger \qquad P(\alpha,\alpha^+,\alpha^*,\alpha^{+*},g,g^+) \Rightarrow \alpha_i^+ P \tag{68}$$

$$\hat{\rho} \;\Rightarrow\; \widehat{c}_i\,\hat{\rho} \qquad P(\alpha,\alpha^+,\alpha^*,\alpha^{+*},g,g^+) \Rightarrow g_i\,P = P\,g_i \tag{69}$$

$$\hat{\rho} \;\Rightarrow\; \hat{\rho}\,\widehat{c}_i \qquad P(\alpha,\alpha^+,\alpha^*,\alpha^{+*},g,g^+) \Rightarrow P\left(+\frac{\overleftarrow{\partial}}{\partial g_i^+} - g_i\right) \tag{70}$$

$$= \quad \left(-\frac{\overrightarrow{\partial}}{\partial g_i^+} - g_i\right) P$$

$$\hat{\rho} \;\Rightarrow\; \widehat{c}_i^\dagger\,\hat{\rho} \qquad P(\alpha,\alpha^+,\alpha^*,\alpha^{+*},g,g^+) \Rightarrow \left(+\frac{\overrightarrow{\partial}}{\partial g_i} - g_i^+\right) P \tag{71}$$

$$= \quad P\left(-\frac{\overleftarrow{\partial}}{\partial g_i} - g_i^+\right)$$

$$\hat{\rho} \;\Rightarrow\; \hat{\rho}\widehat{c}_i^\dagger \qquad P(\alpha,\alpha^+,\alpha^*,\alpha^{+*},g,g^+) \Rightarrow P\,g_i^+ = g_i^+\,P \tag{72}$$

$$\frac{\partial \hat{\rho}}{\partial t} \;\rightarrow\; \frac{\partial P(\alpha,\alpha^+,\alpha^*,\alpha^{+*},g,g^+)}{\partial t} \tag{73}$$

The proof of these results can be obtained either starting from the expression (49) for the *characteristic function* or from the expression (56) for the *canonical form* of the density operator. The latter approach is simpler and is based on the effect of the annihilation or creation operators on the *Bargmann state projectors* given in Eqs. (188) and (189). Both proofs involve an *integration by parts* step, which requires the distribution function to go to zero rapidly enough on the bosonic phase space boundary.

The proof of the first expressions for the *fermion results* do not depend on the distribution function $P(\alpha,\alpha^+,\alpha^*,\alpha^{+*},g,g^+)$ being an even Grassmann function. However, the second form of the fermion results uses the feature that the distribution function $P(\alpha,\alpha^+,\alpha^*,\alpha^{+*},g,g^+)$ is an even Grassmann function. If it had been odd, then the second forms would have had all their signs reversed. This point becomes important when *applying the correspondence rules* to derive Fokker-Planck equation terms associated with the effects of *several* fermionic annihilation or creation operators on the density operator. The correspondence rules are carried out *in succession* to give the required term in the Fokker-Planck equation. However, multiplying or differentiating a Grassmann distribution function changes it between being even and being odd, and this has to be taken into account when applying the correspondence rules in succession when dealing with fermionic operators. The safest procedure is to use the correspondence rule form that does not depend on the evenness or oddness of the

distribution function, which is why the *first forms* for the fermion rules in Eqs. (69) - (72) should be used.

The non-uniqueness of the distribution function is associated with the bosonic phase space variables. The standard *bosonic results* for the correspondence rules in Eqs. (65) - (68) can be generalised. Since the bosonic projectors $\widehat{\Lambda}_b(\alpha,\alpha^+)$ are analytic it follows that

$$\begin{aligned} 0 &= \int\int dg^+dg \int\int d^2\alpha^+d^2\alpha \left(\frac{\partial}{\partial\alpha_i^*}\widehat{\Lambda}(g,g^+,\alpha,\alpha^+)\right) P(\alpha,\alpha^+,\alpha^*,\alpha^{+*},g,g^+) \\ &= \int\int dg^+dg \int\int d^2\alpha^+d^2\alpha \left(\frac{\partial}{\partial\alpha_i^{+*}}\widehat{\Lambda}(g,g^+,\alpha,\alpha^+)\right) P(\alpha,\alpha^+,\alpha^*,\alpha^{+*},g,g^+) \end{aligned} \tag{74}$$

and hence in the correspondence rule proof based on the canonical form of the density operator we see that arbitrary linear combinations

$$\lambda\frac{\partial}{\partial\alpha_i^*} + \lambda^+\frac{\partial}{\partial\alpha_i^{+*}} \tag{75}$$

of the derivatives with respect to $\alpha_i^*, \alpha_i^{+*}$ may be added to *each* of the standard bosonic correspondence rules in Eqs. (65) -(68) at the applying integration by parts step. A similar situation occurs for the proof based on the characteristic function. Hence for example, the correspondence rule (66) can be replaced by

$$\hat{\rho} \Rightarrow \hat{\rho}\,\hat{a}_i \qquad P(\alpha,\alpha^+,\alpha^*,\alpha^{+*},g,g^+) \Rightarrow \left(\alpha_i - \frac{\partial}{\partial\alpha_i^+} - \frac{\partial}{\partial\alpha_i^{+*}}\right)P = \left(\alpha_i - \frac{\partial}{\partial\alpha_{ix}^+}\right)P \tag{76}$$

$$\hat{\rho} \Rightarrow \hat{\rho}\,\hat{a}_i \qquad P(\alpha,\alpha^+,\alpha^*,\alpha^{+*},g,g^+) \Rightarrow \left(\alpha_i - \frac{\partial}{\partial i\alpha_i^+} + \frac{\partial}{\partial\alpha_i^{+*}}\right)P = \left(\alpha_i - \frac{\partial}{\partial(i\alpha_{iy}^+)}\right)P \tag{77}$$

choosing $\lambda = 0, \lambda^+ = -1$ or $\lambda = 0, \lambda^+ = +1$ respectively.

There are however, further possibilities - a feature not widely commented upon in other work but which ultimately reflects the non-uniqueness of the positive P distribution for bosons. In particular, the *flexibility* in the correspondence rules is even greater that merely replacing α_i, α_i^+, $\frac{\partial}{\partial\alpha_i}$ or $\frac{\partial}{\partial\alpha_i^+}$ by these quantities plus a particular linear combination of $\frac{\partial}{\partial\alpha_i^*}$ and $\frac{\partial}{\partial\alpha_i^{+*}}$ as in (75) for every term when $\hat{\rho} \Rightarrow \hat{a}_i\,\hat{\rho}$, $\hat{\rho}\,\hat{a}_i$, $\hat{a}_i^\dagger\,\hat{\rho}$ or $\hat{\rho}\,\hat{a}_i^\dagger$ for the creation, anihilation operators associated with a specific mode i. In fact, the linear combination used can be *different* for each α_i, α_i^+, $\frac{\partial}{\partial\alpha_i}$ or $\frac{\partial}{\partial\alpha_i^+}$ *wherever* it occurs! So for example, if in one term where $\hat{\rho} \Rightarrow \hat{\rho}\,\hat{a}_i$ we replace $\left(\alpha_i - \frac{\partial}{\partial\alpha_i^+}\right)$ by $\left(\alpha_i - \frac{\partial}{\partial\alpha_i^+} - \frac{\partial}{\partial\alpha_i^{+*}}\right)$ to give $\left(\alpha_i - \frac{\partial}{\partial\alpha_{ix}^+}\right)$, in another term where $\hat{\rho} \Rightarrow \hat{\rho}\,\hat{a}_i$ we may replace $\left(\alpha_i - \frac{\partial}{\partial\alpha_i^+}\right)$ by $\left(\alpha_i - \frac{\partial}{\partial\alpha_i^+} + \frac{\partial}{\partial\alpha_i^{+*}}\right)$ to give $\left(\alpha_i - \frac{\partial}{\partial(i\alpha_{iy}^+)}\right)$. The reason why this is possible

is that the only requirement is that the equation for the distribution function gives the correct equation for the characteristic function (or the density operator). Additional terms of the form in (75) acting on the distribution function produce zero when in the integration by parts step they act back on either the exponential factor $\exp i\sum_j\{\alpha_j\,\xi_j^+\}$ and $\exp i\sum_j\{\xi_j\alpha_j^+\}$ or the Bargmann state projector $\widehat{\Lambda}_b(\alpha,\alpha^+)$, both of which are analytic functions of the α_i, α_i^+ and hence yield zero when $\frac{\partial}{\partial\alpha_i^*}$ or $\frac{\partial}{\partial\alpha_i^{+*}}$ are applied. This flexibility is important in being able to convert the Fokker-Planck equation based on the standard correspondence rules into a form with a *positive definite* diffusion matrix (see [15], [18]).

For the *canonical distribution function* the flexibility described in the previous paragraph is *not* available for the correspondence rules. They do however represent particular choices of the forms (75). In terms of *new variables* γ_i, γ_i^*, δ_i, δ_i^* which replace $\alpha_i,\alpha_i^+,\alpha_i^*,\alpha_i^{+*}$ via

$$\begin{aligned}
\gamma_i &= \frac{1}{2}(\alpha_i+\alpha_i^{+*}) \qquad \gamma_i^*=\frac{1}{2}(\alpha_i^*+\alpha_i^+)\\
\delta_i &= \frac{1}{2}(\alpha_i-\alpha_i^{+*}) \qquad \delta_i^*=\frac{1}{2}(\alpha_i^*-\alpha_i^+)\\
\alpha_i &= \gamma_i+\delta_i \qquad \alpha_i^*=\gamma_i^*+\delta_i^*\\
\alpha_i^+ &= \gamma_i^*-\delta_i^* \qquad \alpha_i^{+*}=\gamma_i-\delta_i
\end{aligned} \tag{78}$$

the canonical distribution function can be written in the form

$$\begin{aligned}
&P_{canon}(\gamma,\gamma^*,\delta,\delta^*,g,g^+)\\
=\;&\left(\frac{1}{4\pi^2}\right)^n\exp(-\sum_i\delta_i\delta_i^*)\exp(-\sum_i\gamma_i\gamma_i^*)\\
&\times\int\int dg^{+*}dg^*\exp(\sum_i(g_ig_i^*+g_i^{+*}g_i^++g_ig_i^+))\,\langle g|_B\,\langle\gamma|_B\,\widehat{\rho}\,|\gamma\rangle_B\,\left|g^{+*}\right\rangle_B
\end{aligned} \tag{79}$$

The phase space integration is changed:

$$\iint d^2\alpha^+\,d^2\alpha\Rightarrow 4\iint d^2\delta\,d^2\gamma \tag{80}$$

The *canonical correspondence rules* are

$$\begin{aligned}
\hat{\rho} &\Rightarrow \hat{a}_i\,\hat{\rho} \qquad P_{canon}(\gamma,\gamma^*,\delta,\delta^*,g,g^+)\Rightarrow\left(\frac{\partial}{\partial\gamma_i^*}+\gamma_i\right)P_{canon}\\
\hat{\rho} &\Rightarrow \hat{\rho}\,\hat{a}_i \qquad P_{canon}(\gamma,\gamma^*,\delta,\delta^*,g,g^+)\Rightarrow(\gamma_i)\;P_{canon}\\
\hat{\rho} &\Rightarrow \hat{a}_i^\dagger\,\hat{\rho} \qquad P_{canon}(\gamma,\gamma^*,\delta,\delta^*,g,g^+)\Rightarrow(\gamma_i^*)\;P_{canon}\\
\hat{\rho} &\Rightarrow \hat{\rho}\,\hat{a}_i^\dagger \qquad P_{canon}(\gamma,\gamma^*,\delta,\delta^*,g,g^+)\Rightarrow\left(\frac{\partial}{\partial\gamma_i}+\gamma_i^*\right)P_{canon}
\end{aligned} \tag{81}$$

using the results in Eqs. (166), (167), (169) and (167) directly in the expression (79) for the distribution function. These correspondence rules can also be obtained from the standard bosonic correspondence rules by adding the additional terms in Eq. (75). We see that

$$\begin{aligned}
\alpha_i + 2\frac{\partial}{\partial \alpha_i^*} &= \left(\frac{\partial}{\partial \gamma_i^*} + \gamma_i\right) + \left(\frac{\partial}{\partial \delta_i^*} + \delta_i\right) \\
\left(\alpha_i - \frac{\partial}{\partial \alpha_i^+}\right) + 2\frac{\partial}{\partial \alpha_i^*} &= \gamma_i + \left(\frac{\partial}{\partial \delta_i^*} + \delta_i\right) \\
\left(\alpha_i^+ - \frac{\partial}{\partial \alpha_i}\right) + 2\frac{\partial}{\partial \alpha_i^{+*}} &= \gamma_i^* - \left(\frac{\partial}{\partial \delta_i} + \delta_i^*\right) \\
\alpha_i^+ + 2\frac{\partial}{\partial \alpha_i^{+*}} &= \left(\frac{\partial}{\partial \gamma_i} + \gamma_i^*\right) - \left(\frac{\partial}{\partial \delta_i} + \delta_i^*\right)
\end{aligned} \tag{82}$$

using

$$\begin{aligned}
\frac{\partial}{\partial \gamma_i^*} &= \frac{\partial}{\partial \alpha_i^*} + \frac{\partial}{\partial \alpha_i^+} \qquad \frac{\partial}{\partial \gamma_i} = \frac{\partial}{\partial \alpha_i} + \frac{\partial}{\partial \alpha_i^{+*}} \\
\frac{\partial}{\partial \delta_i^*} &= \frac{\partial}{\partial \alpha_i^*} - \frac{\partial}{\partial \alpha_i^+} \qquad \frac{\partial}{\partial \delta_i} = \frac{\partial}{\partial \alpha_i} - \frac{\partial}{\partial \alpha_i^{+*}}
\end{aligned} \tag{83}$$

It follows that applying the modified correspondence rules in Eq. (82) to the canonical form (79) for the distribution function gives the same result as in (81), since the effect of the operators involving δ_i, δ_i^* is zero. Written in terms of the original variables $\alpha_i, \alpha_i^+, \alpha_i^*, \alpha_i^{+*}$ the canonical correspondence rules (81) were originally obtained by Schack and Schenzle [28] (see the Appendix).

4.2 Canonical Distribution Function - Fokker-Planck Equation

The *Fokker-Planck equation* for the canonical distribution function $P_{canon}(\gamma, \gamma^*, \delta, \delta^*, g, g^+)$ is

$$
\begin{aligned}
&\frac{\partial P_{canon}(\gamma, \gamma^*, \delta, \delta^*, g, g^+)}{\partial t} \\
= \; & -i\frac{E_A}{\hbar}\left(\frac{\overrightarrow{\partial}}{\partial g_2}(g_2)P_{canon}(\gamma, \gamma^*, \delta, \delta^*, g, g^+) + \frac{\overrightarrow{\partial}}{\partial g_1}(g_1)P_{canon}(\gamma, \gamma^*, \delta, \delta^*, g, g^+)\right) \\
& +i\frac{E_A}{\hbar}\left(P_{canon}(\gamma, \gamma^*, \delta, \delta^*, g, g^+)(g_2^+)\frac{\overleftarrow{\partial}}{\partial g_2^+} + P_{canon}(\gamma, \gamma^*, \delta, \delta^*, g, g^+)(g_1^+)\frac{\overleftarrow{\partial}}{\partial g_1^+}\right) \\
& -\frac{1}{2}i\omega_0\left(\frac{\overrightarrow{\partial}}{\partial g_2}(g_2)P_{canon}(\gamma, \gamma^*, \delta, \delta^*, g, g^+)) - \frac{\overrightarrow{\partial}}{\partial g_1}(g_1)P_{canon}(\gamma, \gamma^*, \delta, \delta^*, g, g^+)\right) \\
& +\frac{1}{2}i\omega_0\left(P_{canon}(\gamma, \gamma^*, \delta, \delta^*, g, g^+)(g_2^+)\frac{\overleftarrow{\partial}}{\partial g_2^+} - P_{canon}(\gamma, \gamma^*, \delta, \delta^*, g, g^+)(g_1^+)\frac{\overleftarrow{\partial}}{\partial g_1^+}\right) \\
& -i\omega\left(\frac{\partial}{\partial \gamma^*}(\gamma^* P_{canon}(\gamma, \gamma^*, \delta, \delta^*, g, g^+)) - \frac{\partial}{\partial \gamma}(\gamma P_{canon}(\gamma, \gamma^*, \delta, \delta^*, g, g^+))\right) \\
& -\frac{1}{2}i\Omega\left(\frac{\overrightarrow{\partial}}{\partial g_1}(g_2)(\gamma^* P_{canon}(\gamma, \gamma^*, \delta, \delta^*, g, g^+))\right) \\
& +\frac{1}{2}i\Omega\left((\gamma P_{canon}(\gamma, \gamma^*, \delta, \delta^*, g, g^+))(g_2^+)\frac{\overleftarrow{\partial}}{\partial g_1^+}\right) \\
& -\frac{1}{2}i\Omega\frac{\overrightarrow{\partial}}{\partial g_2}(g_1)(\left(\gamma + \frac{\partial}{\partial \gamma^*}\right)P_{canon}(\gamma, \gamma^*, \delta, \delta^*, g, g^+)) \\
& +\frac{1}{2}i\Omega\left((\left(\gamma^* + \frac{\partial}{\partial \gamma}\right)P_{canon}(\gamma, \gamma^*, \delta, \delta^*, g, g^+))(g_1^+)\frac{\overleftarrow{\partial}}{\partial g_2^+}\right) \\
& +\frac{1}{2}i\Omega(g_2^+ g_1)(\left(\frac{\partial}{\partial \gamma^*}\right)P_{canon}(\gamma, \gamma^*, \delta, \delta^*, g, g^+)) \\
& -\frac{1}{2}i\Omega\left((\left(\frac{\partial}{\partial \gamma}\right)P_{canon}(\gamma, \gamma^*, \delta, \delta^*, g, g^+))(g_1^+ g_2)\right)
\end{aligned}
\tag{84}
$$

where the bosonic correspondence rules (81) and the first versions of fermionic correspondence rules (69) - (72) have been used. The Fokker-Planck equation based on the standard correspondence rules is set out in Appendix 10.

4.3 Coupled Distribution Function Coefficients

Writing γ, δ for $\gamma, \gamma^*, \delta, \delta^*$ the expression (50) for the distribution function can now be substituted into the Fokker-Planck equation (84) for the *canonical*

$P+$ distribution to obtain coupled equations for the six c-number coefficients $P_0(\gamma,\delta)$, $P_2^{i;j}(\gamma,\delta)$ and $P_4^{12;21}(\gamma,\delta)$ that specify the distribution function. For convenience we use the same terminology for these coefficients as in the general case and leave the canonical label understood. In the derivation the Grassmann differentiations of various products of Grassmann variables are first carried out using the results in Appendix 8 and we then equate terms involving the the zeroth, second and fourth order monomials in the Grassmann variables g_1, g_2, g_2^+ and g_1^+ to arrive at six separate coupled equations for the c-number coefficients $P_0(\gamma,\delta)$, $P_2^{i;;j}(\gamma,\delta)$ and $P_4^{12;21}(\gamma,\delta)$ that specify the distribution function.

For the *zeroth* order terms we have

$$\frac{\partial}{\partial t}P_0(\gamma,\delta) = -i\omega\,\left(\frac{\partial}{\partial\gamma^*}\gamma^* - \frac{\partial}{\partial\gamma}\gamma\right)P_0(\gamma,\delta) \tag{85}$$

For the *second* order terms we have four equations.

$$\begin{aligned}
&\frac{\partial}{\partial t}P_2^{1;1}(\gamma,\delta)\\
=\;& -i\omega\,\left(\frac{\partial}{\partial\gamma^*}\gamma^* - \frac{\partial}{\partial\gamma}\gamma\right)P_2^{1;1}(\gamma,\delta)\\
&+\frac{1}{2}i\Omega\,\left(\gamma+\frac{\partial}{\partial\gamma^*}\right)P_2^{2;1}(\gamma,\delta) - \frac{1}{2}i\Omega\,\left(\gamma^*+\frac{\partial}{\partial\gamma}\right)P_2^{1;2}(\gamma,\delta)
\end{aligned} \tag{86}$$

$$\begin{aligned}
&\frac{\partial}{\partial t}P_2^{1;2}(\gamma,\delta)\\
=\;& -i\omega\,\left(\frac{\partial}{\partial\gamma^*}\gamma^* - \frac{\partial}{\partial\gamma}\gamma\right)P_2^{1;2}(\gamma,\delta)\\
&-i\omega_0 P_2^{1;2}(\gamma,\delta)\\
&-\frac{1}{2}i\Omega\,\gamma\,\Big(P_2^{1;1}(\gamma,\delta) - P_2^{2;2}(\gamma,\delta)\Big) + \frac{1}{2}i\Omega\,\left(\frac{\partial}{\partial\gamma^*}\right)\Big(P_2^{2;2}(\gamma,\delta) - P_0(\gamma,\delta)\Big)
\end{aligned} \tag{87}$$

$$\begin{aligned}
&\frac{\partial}{\partial t}P_2^{2;1}(\gamma,\delta)\\
=\;& -i\omega\,\left(\frac{\partial}{\partial\gamma^*}\gamma^* - \frac{\partial}{\partial\gamma}\gamma\right)P_2^{2;1}(\gamma,\delta)\\
&+i\omega_0 P_2^{2;1}(\gamma,\delta)\\
&+\frac{1}{2}i\Omega\,\gamma^*\,\Big(P_2^{1;1}(\gamma,\delta) - P_2^{2;2}(\gamma,\delta)\Big) - \frac{1}{2}i\Omega\,\left(\frac{\partial}{\partial\gamma}\right)\Big(P_2^{2;2}(\gamma,\delta) - P_0(\gamma,\delta)\Big)
\end{aligned} \tag{88}$$

$$
\begin{aligned}
&\frac{\partial}{\partial t}P_2^{2;2}(\gamma,\delta)\\
= \;& -i\omega\left(\frac{\partial}{\partial\gamma^*}\gamma^* - \frac{\partial}{\partial\gamma}\gamma\right)P_2^{2;2}(\gamma,\delta)\\
&+\frac{1}{2}i\Omega\left(\gamma^* P_2^{1;2}(\gamma,\delta)\right) - \frac{1}{2}i\Omega\left(\gamma\, P_2^{2;1}(\gamma,\delta)\right)
\end{aligned}
\tag{89}
$$

For the *fourth* order term

$$
\begin{aligned}
&\frac{\partial}{\partial t}P_4^{12;21}(\gamma,\delta)\\
= \;& -i\omega\left(\frac{\partial}{\partial\gamma^*}\gamma^* - \frac{\partial}{\partial\gamma}\gamma\right)P_4^{12;21}(\gamma,\delta)\\
&+\frac{1}{2}i\Omega\left(\frac{\partial}{\partial\gamma^*}\right)P_2^{2;1}(\gamma,\delta) - \frac{1}{2}i\Omega\left(\frac{\partial}{\partial\gamma}\right)P_2^{1;2}(\gamma,\delta)
\end{aligned}
\tag{90}
$$

The corresponding coupled equations for the general distribution function are set out in Appendix 10.

4.4 Initial Conditions for Uncorrelated Case

For the case of *uncorrelated* initial states the density operator is a product $\widehat{\rho} = \widehat{\rho}_f\,\widehat{\rho}_b$ and the overall *initial distribution function* is just the product of the atomic term $P_f(g,g^+)$ with the cavity term $P_b(\alpha,\alpha^+,\alpha^*,\alpha^{+*})$ so that at the initial time the distribution function is

$$
P(\alpha,\alpha^+,\alpha^*,\alpha^{+*},g,g^+)_0 = P_f(g,g^+)P_b(\alpha,\alpha^+,\alpha^*,\alpha^{+*}) \tag{91}
$$

and therefore the coefficients are given by

$$
\begin{aligned}
P_0 &= \left\langle \widehat{c}_1^\dagger\widehat{c}_2^\dagger\widehat{c}_2\widehat{c}_1\right\rangle P_b(\alpha,\alpha^+,\alpha^*,\alpha^{+*}) = 0\\
P_2^{1;1} &= \left\langle \widehat{c}_2^\dagger\widehat{c}_2\right\rangle P_b(\alpha,\alpha^+,\alpha^*,\alpha^{+*}) \qquad P_2^{1;2} = -\left\langle \widehat{c}_1^\dagger\widehat{c}_2\right\rangle P_b(\alpha,\alpha^+,\alpha^*,\alpha^{+*})\\
P_2^{2;1} &= -\left\langle \widehat{c}_2^\dagger\widehat{c}_1\right\rangle P_b(\alpha,\alpha^+,\alpha^*,\alpha^{+*}) \qquad P_2^{2;2} = \left\langle \widehat{c}_1^\dagger\widehat{c}_1\right\rangle P_b(\alpha,\alpha^+,\alpha^*,\alpha^{+*})\\
P_4^{12;21} &= P_b(\alpha,\alpha^+,\alpha^*,\alpha^{+*})
\end{aligned}
\tag{92}
$$

where the initial distribution function for the cavity mode is

$$
P_b(\alpha,\alpha^+,\alpha^*,\alpha^{+*}) = \left(\frac{1}{4\pi^2}\right)\exp(-\frac{1}{4}|\alpha-\alpha^{+*}|^2)\left\langle \frac{\alpha+\alpha^{+*}}{2},\frac{\alpha^*+\alpha^+}{2}\right|\widehat{\rho}_b\left|\frac{\alpha+\alpha^{+*}}{2},\frac{\alpha^*+\alpha^+}{2}\right\rangle \tag{93}
$$

4.5 Rotating Phase Variables and Coefficients

To proceed further it is useful to introduce *rotating phase variables* defined via the transformation

$$\begin{aligned} \alpha &= \beta \exp(-i\omega t) \qquad \alpha^* = \beta^* \exp(+i\omega t) \\ \alpha^+ &= \beta^+ \exp(+i\omega t) \qquad \alpha^{+*} = \beta^{+*} \exp(-i\omega t) \end{aligned} \tag{94}$$

This then gives

$$\begin{aligned} \gamma &= \frac{1}{2}(\beta + \beta^{+*}) \exp(-i\omega t) = \widetilde{\gamma} \exp(-i\omega t) \\ \gamma^* &= \frac{1}{2}(\beta^* + \beta^+) \exp(+i\omega t) = \widetilde{\gamma}^* \exp(+i\omega t) \\ \delta &= \frac{1}{2}(\beta - \beta^{+*}) \exp(-i\omega t) = \widetilde{\delta} \exp(-i\omega t) \\ \delta^* &= \frac{1}{2}(\beta^* - \beta^+) \exp(+i\omega t) = \widetilde{\delta}^* \exp(+i\omega t) \end{aligned} \tag{95}$$

so that we will replace $\gamma, \gamma^*, \delta, \delta^*$ by $\widetilde{\gamma}, \widetilde{\gamma}^*, \widetilde{\delta}, \widetilde{\delta}^*$, or $\widetilde{\gamma}, \widetilde{\delta}$ for short. The canonical distribution function coefficients will be of the form $\widetilde{P}(\widetilde{\gamma}, \widetilde{\delta})$, where we now change the labelling of the coefficients to $\widetilde{P}$ so as to reflect this variable change.

This transformation enables the elimination of the cavity field term from the equations for the coefficients. Explicit time dependences will be left understood. For any $\widetilde{P}(\widetilde{\gamma}, \widetilde{\delta})$

$$\frac{\partial}{\partial t} P(\gamma, \delta) = i\omega \left(\frac{\partial}{\partial \widetilde{\gamma}} \widetilde{\gamma}\, \widetilde{P}(\widetilde{\gamma}, \widetilde{\delta}) - \frac{\partial}{\partial \widetilde{\gamma}^*} \widetilde{\gamma}^*\, \widetilde{P}(\widetilde{\gamma}, \widetilde{\delta}) \right) + \frac{\partial}{\partial t} \widetilde{P}(\widetilde{\gamma}, \widetilde{\delta}) \tag{96}$$

where since each canonical distribution coefficient is of the form $\exp(-\widetilde{\delta}\,\widetilde{\delta}^*)\, F(\widetilde{\gamma}, \widetilde{\gamma}^*)$ it follows that the derivative terms involving $\widetilde{\delta}, \widetilde{\delta}^*$ can be ignored because

$$\frac{\partial}{\partial \widetilde{\delta}} \widetilde{P}(\widetilde{\gamma}, \widetilde{\delta}) \frac{\partial \widetilde{\delta}}{\partial t} + \frac{\partial}{\partial \widetilde{\delta}^*} \widetilde{P}(\widetilde{\gamma}, \widetilde{\delta}) \frac{\partial \widetilde{\delta}^*}{\partial t} = 0 \tag{97}$$

Note also that the original $\gamma, \gamma^*, \delta, \delta^*$ were time independent. On the right side of each coefficient equation we have a term

$$-i\omega \left(\frac{\partial}{\partial \gamma^*} \gamma^* - \frac{\partial}{\partial \gamma} \gamma \right) P(\gamma, \delta) = -i\omega \left(\frac{\partial}{\partial \widetilde{\gamma}^*} \widetilde{\gamma}^*\, \widetilde{P}(\widetilde{\gamma}, \widetilde{\delta}) - \frac{\partial}{\partial \widetilde{\gamma}} \widetilde{\gamma}\, \widetilde{P}(\widetilde{\gamma}, \widetilde{\delta}) \right) \tag{98}$$

This cancels out the equivalent term on the left side so we now have the following results.

For the *zeroth* order terms we have

$$\frac{\partial}{\partial t} \widetilde{P}_0(\widetilde{\gamma}, \widetilde{\delta}) = 0 \tag{99}$$

For the *second* order terms we have four equations.

$$\begin{aligned}
&\frac{\partial}{\partial t}\widetilde{P}_2^{1;1}(\widetilde{\gamma},\widetilde{\delta}) \\
= \;& +\frac{1}{2}i\Omega\left(\widetilde{\gamma}+\frac{\partial}{\partial\widetilde{\gamma}^*}\right)\widetilde{P}_2^{2;1}(\widetilde{\gamma},\widetilde{\delta})\exp(-i\omega t) \\
& -\frac{1}{2}i\Omega\left(\widetilde{\gamma}^*+\frac{\partial}{\partial\widetilde{\gamma}}\right)\widetilde{P}_2^{1;2}(\widetilde{\gamma},\widetilde{\delta})\exp(+i\omega t)
\end{aligned} \tag{100}$$

$$\begin{aligned}
&\frac{\partial}{\partial t}\widetilde{P}_2^{1;2}(\widetilde{\gamma},\widetilde{\delta}) \\
= \;& -i\omega_0\widetilde{P}_2^{1;2}(\widetilde{\gamma},\widetilde{\delta}) \\
& -\frac{1}{2}i\Omega\,\widetilde{\gamma}\,\exp(-i\omega t)\left(\widetilde{P}_2^{1;1}(\widetilde{\gamma},\widetilde{\delta})-\widetilde{P}_2^{2;2}(\widetilde{\gamma},\widetilde{\delta})\right) \\
& +\frac{1}{2}i\Omega\,\exp(-i\omega t)\left(\frac{\partial}{\partial\widetilde{\gamma}^*}\right)\left(\widetilde{P}_2^{2;2}(\widetilde{\gamma},\widetilde{\delta})-\widetilde{P}_0(\widetilde{\gamma},\widetilde{\delta})\right)
\end{aligned} \tag{101}$$

$$\begin{aligned}
&\frac{\partial}{\partial t}\widetilde{P}_2^{2;1}(\widetilde{\gamma},\widetilde{\delta}) \\
= \;& +i\omega_0\widetilde{P}_2^{2;1}(\widetilde{\gamma},\widetilde{\delta}) \\
& +\frac{1}{2}i\Omega\,\widetilde{\gamma}^*\exp(+i\omega t)\left(\widetilde{P}_2^{1;1}(\widetilde{\gamma},\widetilde{\delta})-\widetilde{P}_2^{2;2}(\widetilde{\gamma},\widetilde{\delta})\right) \\
& -\frac{1}{2}i\Omega\;\exp(+i\omega t)\left(\frac{\partial}{\partial\widetilde{\gamma}}\right)\left(\widetilde{P}_2^{2;2}(\widetilde{\gamma},\widetilde{\delta})-\widetilde{P}_0(\widetilde{\gamma},\widetilde{\delta})\right)
\end{aligned} \tag{102}$$

$$\begin{aligned}
&\frac{\partial}{\partial t}\widetilde{P}_2^{2;2}(\widetilde{\gamma},\widetilde{\delta}) \\
= \;& +\frac{1}{2}i\Omega\left(\widetilde{\gamma}^*\widetilde{P}_2^{1;2}(\widetilde{\gamma},\widetilde{\delta})\exp(+i\omega t)\right) \\
& -\frac{1}{2}i\Omega\left(\widetilde{\gamma}\,\widetilde{P}_2^{2;1}(\widetilde{\gamma},\widetilde{\delta})\exp(-i\omega t)\right)
\end{aligned} \tag{103}$$

For the *fourth* order term

$$\begin{aligned}
&\frac{\partial}{\partial t}\widetilde{P}_4^{12;21}(\widetilde{\gamma},\widetilde{\delta}) \\
= \;& +\frac{1}{2}i\Omega\left(\frac{\partial}{\partial\widetilde{\gamma}^*}\right)\widetilde{P}_2^{2;1}(\widetilde{\gamma},\widetilde{\delta})\exp(-i\omega t)-\frac{1}{2}i\Omega\left(\frac{\partial}{\partial\widetilde{\gamma}}\right)\widetilde{P}_2^{1;2}(\widetilde{\gamma},\widetilde{\delta})\exp(+i\omega t)
\end{aligned} \tag{104}$$

Also, equations with *no explicit time dependence* can be obtained via the change of coefficients from $\widetilde{P}$ to $\widetilde{S}$ by incorporating $\exp(\pm i\omega t)$ in the $\widetilde{P}_2^{1;2}(\widetilde{\gamma},\widetilde{\delta})$, $\widetilde{P}_2^{2;1}(\widetilde{\gamma},\widetilde{\delta})$ coefficients, and we can also *factor out* the explicit dependence on $\widetilde{\delta}$ via the overall factor $\exp(-\widetilde{\delta}\,\widetilde{\delta}^*) = \exp(-\delta\,\delta^*)$. We also note that the canonical form (79) of the distribution function (and hence all the coefficients) contains the factor $\exp(-\gamma\gamma^*) = \exp(-\widetilde{\gamma}\widetilde{\gamma}^*)$, so it is useful to *eliminate* this factor via the change from $\widetilde{P}(\widetilde{\gamma},\widetilde{\delta})$ to $\widetilde{S}(\widetilde{\gamma})$. The new coefficients are:

$$\begin{aligned}
\widetilde{P}_2^{1;1}(\widetilde{\gamma},\widetilde{\delta}) &= \widetilde{S}_2^{1;1}(\widetilde{\gamma})\,\exp(-\widetilde{\gamma}\widetilde{\gamma}^*)\,\exp(-\widetilde{\delta}\,\widetilde{\delta}^*) \\
\widetilde{P}_2^{2;2}(\widetilde{\gamma},\widetilde{\delta}) &= \widetilde{S}_2^{2;2}(\widetilde{\gamma})\,\exp(-\widetilde{\gamma}\widetilde{\gamma}^*)\,\exp(-\widetilde{\delta}\,\widetilde{\delta}^*) \\
\widetilde{P}_2^{1;2}(\widetilde{\gamma},\widetilde{\delta}) &= \widetilde{S}_2^{1;2}(\widetilde{\gamma})\,\exp(-\widetilde{\gamma}\widetilde{\gamma}^*)\,\exp(-\widetilde{\delta}\,\widetilde{\delta}^*)\exp(-i\omega t) \\
\widetilde{P}_2^{2;1}(\widetilde{\gamma},\widetilde{\delta}) &= \widetilde{S}_2^{2;1}(\widetilde{\gamma})\,\exp(-\widetilde{\gamma}\widetilde{\gamma}^*)\,\exp(-\widetilde{\delta}\,\widetilde{\delta}^*)\exp(+i\omega t) \\
\widetilde{P}_0(\widetilde{\gamma},\widetilde{\delta}) &= \widetilde{S}_0(\widetilde{\gamma})\,\exp(-\widetilde{\gamma}\widetilde{\gamma}^*)\,\exp(-\widetilde{\delta}\,\widetilde{\delta}^*) \\
\widetilde{P}_4^{12;21}(\widetilde{\gamma},\widetilde{\delta}) &= \widetilde{S}_4^{12;21}(\widetilde{\gamma})\,\exp(-\widetilde{\gamma}\widetilde{\gamma}^*)\,\exp(-\widetilde{\delta}\,\widetilde{\delta}^*)
\end{aligned} \tag{105}$$

The separate cavity and atomic transition frequencies are then incorporated into the *detuning*

$$\Delta = \omega_0 - \omega \tag{106}$$

Also, $\widetilde{\delta}$ now plays no further role in the dynamics and the $\widetilde{S}$ only depend on $\widetilde{\gamma},\widetilde{\gamma}^*$ (or $\widetilde{\gamma}$ for short).

With this substitution the equations for the coefficients are as follows:

For the *zeroth* order term

$$\frac{\partial}{\partial t}\widetilde{S}_0(\widetilde{\gamma}) = 0 \tag{107}$$

The four second *second* order equations are.

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\widetilde{S}_2^{1;1}(\widetilde{\gamma}) - \widetilde{S}_0(\widetilde{\gamma})\right) \\
&= +\frac{1}{2}i\Omega\left(\frac{\partial}{\partial\widetilde{\gamma}^*}\right)\widetilde{S}_2^{2;1}(\widetilde{\gamma}) - \frac{1}{2}i\Omega\left(\frac{\partial}{\partial\widetilde{\gamma}}\right)\widetilde{S}_2^{1;2}(\widetilde{\gamma})
\end{aligned} \tag{108}$$

$$\begin{aligned}
&\frac{\partial}{\partial t}\widetilde{S}_2^{1;2}(\widetilde{\gamma}) \\
=\;& -i\Delta\,\widetilde{S}_2^{1;2}(\widetilde{\gamma}) \\
&-\frac{1}{2}i\Omega\,\widetilde{\gamma}\left(\widetilde{S}_2^{1;1}(\widetilde{\gamma}) - \widetilde{S}_0(\widetilde{\gamma})\right) + \frac{1}{2}i\Omega\left(\frac{\partial}{\partial\widetilde{\gamma}^*}\right)\left(\widetilde{S}_2^{2;2}(\widetilde{\gamma}) - \widetilde{S}_0(\widetilde{\gamma})\right)
\end{aligned} \tag{109}$$

$$\begin{aligned}
&\frac{\partial}{\partial t}\widetilde{S}_2^{2;1}(\widetilde{\gamma}) \\
=\;& +i\Delta\,\widetilde{S}_2^{2;1}(\widetilde{\gamma}) \\
&+\frac{1}{2}i\Omega\,\widetilde{\gamma}^*\left(\widetilde{S}_2^{1;1}(\widetilde{\gamma}) - \widetilde{S}_0(\widetilde{\gamma})\right) - \frac{1}{2}i\Omega\left(\frac{\partial}{\partial\widetilde{\gamma}}\right)\left(\widetilde{S}_2^{2;2}(\widetilde{\gamma}) - \widetilde{S}_0(\widetilde{\gamma})\right)
\end{aligned} \tag{110}$$

$$\begin{aligned} & \frac{\partial}{\partial t}\left(\widetilde{S}_2^{2;2}(\widetilde{\gamma}) - \widetilde{S}_0(\widetilde{\gamma})\right) \\ = \; & +\frac{1}{2}i\Omega\left(\widetilde{\gamma}^*\,\widetilde{S}_2^{1;2}(\widetilde{\gamma})\right) - \frac{1}{2}i\Omega\left(\widetilde{\gamma}\,\widetilde{S}_2^{2;1}(\widetilde{\gamma})\right) \end{aligned} \tag{111}$$

where we have substracted the zero quantity $\frac{\partial}{\partial t}\widetilde{S}_0(\widetilde{\gamma})$ from each side of the first and fourth equation. From subsection 3.4 we see that the quantities $\widetilde{S}_2^{i;i}(\widetilde{\gamma}) - \widetilde{S}_0(\widetilde{\gamma})$ determine the one atom probabilities. However, from (107) and the initial conditions (92) we see that $\widetilde{S}_0(\widetilde{\gamma})$ will be zero for the one atom Jaynes-Cummings model, so we can *ignore* $\widetilde{S}_0(\widetilde{\gamma})$ henceforth.

The *fourth* order equation is

$$\begin{aligned} & \frac{\partial}{\partial t}\widetilde{S}_4^{12;21}(\widetilde{\gamma}) \\ = \; & +\frac{1}{2}i\Omega\left(-\widetilde{\gamma} + \frac{\partial}{\partial\widetilde{\gamma}^*}\right)\widetilde{S}_2^{2;1}(\widetilde{\gamma}) - \frac{1}{2}i\Omega\left(-\widetilde{\gamma}^* + \frac{\partial}{\partial\widetilde{\gamma}}\right)\widetilde{S}_2^{1;2}(\widetilde{\gamma}) \end{aligned} \tag{112}$$

Analogous equations for the standard distribution function are set out in Appendix 10.

5 Solution to Fokker-Planck Equation

The approach used by Stenholm [14] can be adapted to provide an analytical solution for the canonical distribution function for the one atom Jaynes-Cummings model for any initial conditions. The equations (108) - (111) for the $\widetilde{S}_2^{i;j}(\widetilde{\gamma})$ can be solved via the substitution

$$\widetilde{S}_2^{i;j}(\widetilde{\gamma}) = \Psi_i^*(\widetilde{\gamma}^*)\,\Psi_j(\widetilde{\gamma}) \tag{113}$$

where the $\Psi_i^*(\widetilde{\gamma}^*)$ are functions of the $\widetilde{\gamma}^*$ and the $\Psi_i(\widetilde{\gamma})$ are functions of the $\widetilde{\gamma}$, and where $\Psi_i(\widetilde{\gamma})$ satisfy the coupled equations

$$\begin{aligned} \frac{\partial}{\partial t}\Psi_1(\widetilde{\gamma}) &= \frac{1}{2}i\Delta\,\Psi_1(\widetilde{\gamma}) - \frac{1}{2}i\Omega\left(\frac{\partial}{\partial\widetilde{\gamma}}\right)\Psi_2(\widetilde{\gamma}) \\ \frac{\partial}{\partial t}\Psi_2(\widetilde{\gamma}) &= -\frac{1}{2}i\Delta\,\Psi_2(\widetilde{\gamma}) - \frac{1}{2}i\Omega\,\widetilde{\gamma}\,\Psi_1(\widetilde{\gamma}) \end{aligned} \tag{114}$$

This ansatz is consistent with the original equations (108) - (111) for the $\widetilde{S}_2^{i;j}(\widetilde{\gamma})$. As indicated previously we have set $\widetilde{S}_0(\widetilde{\gamma}) = 0$ for the one atom case.

Differentiating the second equation and substituting from the first gives

$$\begin{aligned} \frac{\partial^2}{\partial t^2}\Psi_2(\widetilde{\gamma}) + \frac{1}{4}\Delta^2\,\Psi_2(\widetilde{\gamma}) &= -\frac{1}{4}\Omega^2\,\widetilde{\gamma}\left(\frac{\partial}{\partial\widetilde{\gamma}}\right)\Psi_2(\widetilde{\gamma}) \\ &= -\frac{1}{4}\Omega^2\left(\frac{\partial}{\partial s}\right)\Psi_2(\widetilde{\gamma}) \end{aligned} \tag{115}$$

where the substitution

$$s = \lg \widetilde{\gamma} \qquad \widetilde{\gamma} = \exp\, s \tag{116}$$

has been made.

A solution of the equation (115) can be obtained using *separation* of the variables

$$\Psi_2(\widetilde{\gamma}) = T(t)\, K(s) \tag{117}$$

whence we find that

$$\frac{1}{T(t)}\frac{d^2}{dt^2}T(t) + \frac{1}{4}\Delta^2 = -\frac{1}{4}\Omega^2\, \frac{1}{K(s)}\left(\frac{\partial}{\partial s}\right) K(s) = -\lambda \tag{118}$$

where since the left side is a function of t and the right side is a function of s the quantity λ must be a constant.

The solution of these two equations is straightforward. We have

$$\begin{aligned} K(s) &= C \exp(\frac{4\lambda}{\Omega^2} s) \\ &= C\, (\widetilde{\gamma})^{(4\lambda/\,\Omega^2)} \end{aligned} \tag{119}$$

where C is a constant, and

$$T(t) = A \cos\left(\sqrt{\lambda + \Delta^2/4}\, t\right) + B \sin\left(\sqrt{\lambda + \Delta^2/4}\, t\right) \tag{120}$$

with A and B also constant.

Since we require the overall distribution function to be a non-singular single-valued function of the phase space variables we see from Eq.(119) that there is a restriction on λ such that

$$\frac{4\lambda}{\Omega^2} = n \qquad (n = 0, 1, 2, ..) \tag{121}$$

where n is an *integer*.

Combining the variables to eliminate λ and absorbing C into the other constants we see that a solution for $\Psi_2(\widetilde{\gamma})$ for a particular integer n is

$$\Psi_2(\widetilde{\gamma}) = \widetilde{\gamma}^n \left(A_n \cos\frac{1}{2}\omega_n t + B_n \sin\frac{1}{2}\omega_n t\right) \tag{122}$$

where

$$\omega_n = \sqrt{n\,\Omega^2 + \Delta^2} \tag{123}$$

is the frequency associated with population and coherence oscillations in the one atom Jaynes-Cummings model. The corresponding solution for $\Psi_1(\widetilde{\gamma})$ is then obtained from Eq.(114) and thus

$$\Psi_1(\widetilde{\gamma}) \;=\; i\, \widetilde{\gamma}^{(n-1)} \left(\cos\frac{1}{2}\omega_n t \left\{\frac{\omega_n B_n + i\Delta A_n}{\Omega}\right\} + \sin\frac{1}{2}\omega_n t \left\{\frac{-\omega_n A_n + i\Delta B_n}{\Omega}\right\}\right) \tag{124}$$

However, a solution with $n = 0$ leads to a singular $\widetilde{\gamma}^{-1}$ behaviour, so it follows that n is restricted to the positive integers. Also, as the ansatz equations are linear the general solution is a sum of terms with differing n so that we finally have the solution in the form

$$\begin{aligned}\Psi_1(\widetilde{\gamma}) &= i\sum_{n=1}^{\infty}\widetilde{\gamma}^{(n-1)}\left(\left\{\frac{\omega_n B_n + i\Delta A_n}{\Omega}\right\}\cos\frac{1}{2}\omega_n t + \left\{\frac{-\omega_n A_n + i\Delta B_n}{\Omega}\right\}\sin\frac{1}{2}\omega_n t\right)\\ \Psi_2(\widetilde{\gamma}) &= \sum_{n=1}^{\infty}\widetilde{\gamma}^{n}\left(A_n\cos\frac{1}{2}\omega_n t + B_n\sin\frac{1}{2}\omega_n t\right)\end{aligned} \tag{125}$$

The constants A_n, B_n are chosen to fit the initial conditions.

We can now express the *original distribution function* coefficients in terms of these function using Eqs. (113) and (105). Expressions from which $\widetilde{S}_4^{12;21}(\widetilde{\gamma})$ could be obtained, but these are of little interest. The results can be written in terms of the original phase variables by substituting

$$\widetilde{\gamma} = \frac{1}{2}(\alpha + \alpha^{+*})\exp(+i\omega t) \qquad \widetilde{\delta} = \frac{1}{2}(\alpha - \alpha^{+*})\exp(+i\omega t) \tag{126}$$

from (94) and (95) into the above results.

The analogous solution for the general distribution function is set out in Appendix 10. It turns out that the Fokker-Planck equation based on the standard correspondence rules leads to equations for the coefficients that can also be solved by a similar ansatz. However, the solutions lead to a distribution function that diverges on the phase space boundary and in general diverge at large t, with dependences on hyperbolic functions of $\frac{1}{2}\sqrt{n}\Omega t$ in the case of zero detuning. This then throws the original derivation of the standard Fokker-Planck equation into doubt because the integration by parts step fails. Other cases where this occurs have been studied by Gilchrist et al [41].

Finally, as we will see in subsection 5.1, the solutions based on the canonical distribution function (105) agree with the standard quantum optics result, and in particular the quantities A_n, B_n can be chosen so that the initial conditions are the same as for the canonical distribution function determined from the standard quantum optics solution.

5.1 Comparison with Standard Quantum Optics Result

As a comparison, we now calculate the canonical distribution function (55) as determined from the state vector given in Eq. (2) and (5) obtained from standard quantum optics methods.

The density operator is

$$\begin{aligned}
\widehat{\rho} &= \sum_{n,m} B_{n1}(t)B^*_{m1}(t)\ \exp(-i\overline{n-m}\omega t)\ \widehat{c}_1^\dagger\,|0\rangle\,\langle 0|\,\widehat{c}_1\ \ |n\rangle\,\langle m| \\
&+ \sum_{n,m} B_{\overline{n-1}2}(t)B^*_{\overline{m-1}2}(t)\ \exp(-i\overline{n-m}\omega t)\ \widehat{c}_2^\dagger\,|0\rangle\,\langle 0|\,\widehat{c}_2\ \ \left|\overline{n-1}\right\rangle\left\langle\overline{m-1}\right| \\
&+ \sum_{n,m} B_{n1}(t)B^*_{\overline{m-1}2}(t)\ \exp(-i\overline{n-m}\omega t)\ \widehat{c}_1^\dagger\,|0\rangle\,\langle 0|\,\widehat{c}_2\ \ |n\rangle\left\langle\overline{m-1}\right| \\
&+ \sum_{n,m} B_{\overline{n-1}2}(t)B^*_{m1}(t)\ \exp(-i\overline{n-m}\omega t)\ \widehat{c}_2^\dagger\,|0\rangle\,\langle 0|\,\widehat{c}_1\ \ \left|\overline{n-1}\right\rangle\langle m| \qquad (127)
\end{aligned}$$

where the new amplitudes are given by

$$\begin{aligned}
B_{n1}(t) &= \left(\cos(\frac{1}{2}\omega_n t)\,\{A_{n1}(0)\} + i\sin(\frac{1}{2}\omega_n t)\left\{\frac{\Delta\ A_{n1}(0) - \Omega\sqrt{n}\ A_{\overline{n-1}2}(0)}{\omega_n}\right\}\right) \\
B_{\overline{n-1}2}(t) &= \left(\cos(\frac{1}{2}\omega_n t)\,\left\{A_{\overline{n-1}2}(0)\right\} - i\sin(\frac{1}{2}\omega_n t)\left\{\frac{\Omega\sqrt{n}\ A_{n1}(0) + \Delta\ A_{\overline{n-1}2}(0)}{\omega_n}\right\}\right) \qquad (128)
\end{aligned}$$

noting that

$$\begin{aligned}
\exp(-\frac{1}{2}i\Delta t)\exp(-i(n\omega - \frac{1}{2}\omega_0)t) &= \exp(-i(n-\frac{1}{2})\omega t) \\
\exp(+\frac{1}{2}i\Delta t)\exp(-i(\overline{n-1}\omega + \frac{1}{2}\omega_0)t) &= \exp(-i(n-\frac{1}{2})\omega t) \qquad (129)
\end{aligned}$$

Then with $\lambda = \frac{1}{2}(\alpha + \alpha^{+*})$, $\delta = \frac{1}{2}(\alpha - \alpha^{+*})$ and $\widetilde{\gamma} = \gamma\exp(+i\omega t)$, $\widetilde{\delta} = \delta\exp(+i\omega t)$, as in Eqs. (78) and (95) we have on substituting into Eq. (55) and using expressions for the bosonic Bargmann states from Appendix 9.

$$\begin{aligned}
&{}_B\left\langle\frac{1}{2}(\alpha+\alpha^{+*})\right|\widehat{\rho}\left|\frac{1}{2}(\alpha+\alpha^{+*})\right\rangle_B \\
&= \sum_{n,m} B_{n1}(t)B^*_{m1}(t)\ \widehat{c}_1^\dagger\,|0\rangle\,\langle 0|\,\widehat{c}_1\ \frac{(\widetilde{\gamma}^*)^n}{\sqrt{n!}}\frac{(\widetilde{\gamma})^m}{\sqrt{m!}} \\
&+ \sum_{n,m} B_{\overline{n-1}2}(t)B^*_{\overline{m-1}2}(t)\ \widehat{c}_2^\dagger\,|0\rangle\,\langle 0|\,\widehat{c}_2\ \frac{(\widetilde{\gamma}^*)^{n-1}}{\sqrt{(n-1)!}}\frac{(\widetilde{\gamma})^{m-1}}{\sqrt{(m-1)!}} \\
&+ \sum_{n,m} B_{n1}(t)B^*_{\overline{m-1}2}(t)\ \widehat{c}_1^\dagger\,|0\rangle\,\langle 0|\,\widehat{c}_2\ \frac{(\widetilde{\gamma}^*)^{n}}{\sqrt{n!}}\frac{(\widetilde{\gamma})^{m=1}}{\sqrt{(m-1)!}}\exp(+i\omega t) \\
&+ \sum_{n,m} B_{\overline{n-1}2}(t)B^*_{m1}(t)\ \widehat{c}_2^\dagger\,|0\rangle\,\langle 0|\,\widehat{c}_1\ \frac{(\widetilde{\gamma}^*)^{n-1}}{\sqrt{(n-1)!}}\frac{(\widetilde{\gamma})^{m}}{\sqrt{m!}}\exp(-i\omega t) \qquad (130)
\end{aligned}$$

and

$$\exp(-\frac{1}{2}(\alpha\alpha^* + \alpha^{+*}\alpha^+) = \exp(-\widetilde{\gamma}\,\widetilde{\gamma}^*)\ \exp(-\widetilde{\delta}\,\widetilde{\delta}^*) \qquad (131)$$

Also using expressions for the fermionic Bargmann states from Appendix 9

$$
\begin{aligned}
& {}_B\langle g \mid \left(\widehat{c}_l^{\dagger} \mid 0 \rangle \langle 0 \mid \widehat{c}_k \right) \mid g \rangle_B \\
= & \sum_{i,j} g_i^* g_j^{+*} \, \delta_{il} \delta_{jk}
\end{aligned} \tag{132}
$$

so that

$$
\begin{aligned}
& {}_B\langle g \mid {}_B\left\langle \frac{1}{2}(\alpha + \alpha^{+*}) \right| \widehat{\rho} \left| \frac{1}{2}(\alpha + \alpha^{+*}) \right\rangle_B \mid g \rangle_B \\
= & \sum_{n,m} B_{n1}(t) B_{m1}^*(t) \; g_1^* g_1^{+*} \, \frac{(\widetilde{\gamma}^*)^n}{\sqrt{n!}} \frac{(\widetilde{\gamma})^m}{\sqrt{m!}} \\
& + \sum_{n,m} B_{\overline{n-1}2}(t) B_{\overline{m-1}2}^*(t) \; g_2^* g_2^{+*} \, \frac{(\widetilde{\gamma}^*)^{(n-1)}}{\sqrt{(n-1)!}} \frac{(\widetilde{\gamma})^{(m-1)}}{\sqrt{(m-1)!}} \\
& + \sum_{n,m} B_{n1}(t) B_{\overline{m-1}2}^*(t) \; g_1^* g_2^{+*} \, \frac{(\widetilde{\gamma}^*)^n}{\sqrt{n!}} \frac{(\widetilde{\gamma})^{(m=1)}}{\sqrt{(m-1)!}} \exp(+i\omega t) \\
& + \sum_{n,m} B_{\overline{n-1}2}(t) B_{m1}^*(t) \; g_2^* g_1^{+*} \, \frac{(\widetilde{\gamma}^*)^{(n-1)}}{\sqrt{(n-1)!}} \frac{(\widetilde{\gamma})^m}{\sqrt{m!}} \exp(-i\omega t)
\end{aligned} \tag{133}
$$

The Grassmann phase space integrations can be carried out using results from Appendix 8 and give

$$
\begin{aligned}
& \int \int dg^{+*} dg^* \exp(\sum_i (g_i g_i^* + g_i^{+*} g_i^+ + g_i g_i^+)) \; g_k^* g_l^{+*} \\
= & \; g_1 g_1^+ \, (\delta_{k2}\delta_{l2}) + g_1 g_2^+ \, (-\delta_{k2}\delta_{l1}) + g_2 g_1^+ \, (-\delta_{k1}\delta_{l2}) + g_2 g_2^+ \, (\delta_{k1}\delta_{l1}) \\
& + g_1 g_2 g_2^+ g_1^+ \, (\delta_{k2}\delta_{l2} + \delta_{k1}\delta_{l1})
\end{aligned} \tag{134}
$$

so combining the results we find that the canonical distribution function is

$$
\begin{aligned}
& P_{canon}(\alpha, \alpha^+, \alpha^*, \alpha^{+*}, g, g^+) \\
= & \left(\frac{1}{4\pi^2} \right) \exp(-\widetilde{\gamma}\,\widetilde{\gamma}^*) \, \exp(-\widetilde{\delta}\,\widetilde{\delta}^*) \\
& \times [\sum_{n,m} B_{n1}(t) B_{m1}^*(t) \; \left\{ g_2 g_2^+ + g_1 g_2 g_2^+ g_1^+ \right\} \, \frac{(\widetilde{\gamma}^*)^n}{\sqrt{n!}} \frac{(\widetilde{\gamma})^m}{\sqrt{m!}} \\
& + \sum_{n,m} B_{\overline{n-1}2}(t) B_{\overline{m-1}2}^*(t) \; \left\{ g_1 g_1^+ + g_1 g_2 g_2^+ g_1^+ \right\} \, \frac{(\widetilde{\gamma}^*)^{(n-1)}}{\sqrt{(n-1)!}} \frac{(\widetilde{\gamma})^{(m-1)}}{\sqrt{(m-1)!}} \\
& + \sum_{n,m} B_{n1}(t) B_{\overline{m-1}2}^*(t) \; \left\{ -g_2 g_1^+ \right\} \, \frac{(\widetilde{\gamma}^*)^n}{\sqrt{n!}} \frac{(\widetilde{\gamma})^{(m=1)}}{\sqrt{(m-1)!}} \exp(+i\omega t) \\
& + \sum_{n,m} B_{\overline{n-1}2}(t) B_{m1}^*(t) \; \left\{ -g_1 g_2^+ \right\} \, \frac{(\widetilde{\gamma}^*)^{(n-1)}}{\sqrt{(n-1)!}} \frac{(\widetilde{\gamma})^m}{\sqrt{m!}} \exp(-i\omega t)]
\end{aligned} \tag{135}
$$

From this result we can identify the coefficients

$$\begin{aligned}
\widetilde{P}_2^{1;1}(\widetilde{\gamma},\widetilde{\delta}) &= \left(\frac{1}{4\pi^2}\right)\exp(-\widetilde{\gamma}\,\widetilde{\gamma}^*)\,\exp(-\widetilde{\delta}\,\widetilde{\delta}^*)\sum_{n,m}B_{\overline{n-1}2}(t)B^*_{\overline{m-1}2}(t)\,\frac{(\widetilde{\gamma}^*)^{(n-1)}}{\sqrt{(n-1)!}}\frac{(\widetilde{\gamma})^{(m-1)}}{\sqrt{(m-1)!}}\\
\widetilde{P}_2^{1;2}(\widetilde{\gamma},\widetilde{\delta}) &= -\exp(-i\omega t)\left(\frac{1}{4\pi^2}\right)\exp(-\widetilde{\gamma}\,\widetilde{\gamma}^*)\,\exp(-\widetilde{\delta}\,\widetilde{\delta}^*)\sum_{n,m}B_{\overline{n-1}2}(t)B^*_{m1}(t)\,\frac{(\widetilde{\gamma}^*)^{(n-1)}}{\sqrt{(n-1)!}}\frac{(\widetilde{\gamma})^{m}}{\sqrt{m!}}\\
\widetilde{P}_2^{2;1}(\widetilde{\gamma},\widetilde{\delta}) &= -\exp(+i\omega t)\left(\frac{1}{4\pi^2}\right)\exp(-\widetilde{\gamma}\,\widetilde{\gamma}^*)\,\exp(-\widetilde{\delta}\,\widetilde{\delta}^*)\sum_{n,m}B_{n1}(t)B^*_{\overline{m-1}2}(t)\,\frac{(\widetilde{\gamma}^*)^{n}}{\sqrt{n!}}\frac{(\widetilde{\gamma})^{(m=1)}}{\sqrt{(m-1)!}}\\
\widetilde{P}_2^{2;2}(\widetilde{\gamma},\widetilde{\delta}) &= \left(\frac{1}{4\pi^2}\right)\exp(-\widetilde{\gamma}\,\widetilde{\gamma}^*)\,\exp(-\widetilde{\delta}\,\widetilde{\delta}^*)\sum_{n,m}B_{n1}(t)B^*_{m1}(t)\,\frac{(\widetilde{\gamma}^*)^{n}}{\sqrt{n!}}\frac{(\widetilde{\gamma})^{m}}{\sqrt{m!}} \qquad (136)
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{P}_0(\widetilde{\gamma},\widetilde{\delta}) &= 0\\
\widetilde{P}_4^{12;21}(\widetilde{\gamma},\widetilde{\delta}) &= \left(\frac{1}{4\pi^2}\right)\exp(-\widetilde{\gamma}\,\widetilde{\gamma}^*)\,\exp(-\widetilde{\delta}\,\widetilde{\delta}^*) \qquad (137)\\
&\quad\times[\sum_{n,m}B_{n1}(t)B^*_{m1}(t)\,\frac{(\widetilde{\gamma}^*)^{n}}{\sqrt{n!}}\frac{(\widetilde{\gamma})^{m}}{\sqrt{m!}}+\sum_{n,m}B_{\overline{n-1}2}(t)B^*_{\overline{m-1}2}(t)\,\frac{(\widetilde{\gamma}^*)^{(n-1)}}{\sqrt{(n-1)!}}\frac{(\widetilde{\gamma})^{(m-1)}}{\sqrt{(m-1)!}}]
\end{aligned}$$

If we write

$$\begin{aligned}
\Phi_1(\widetilde{\gamma}) &= \left(\frac{1}{2\pi}\right)\sum_m B^*_{\overline{m-1}2}(t)\,\frac{(\widetilde{\gamma})^{(m-1)}}{\sqrt{(m-1)!}}\\
\Phi_2(\widetilde{\gamma}) &= -\left(\frac{1}{2\pi}\right)\sum_m B^*_{m1}(t)\,\frac{(\widetilde{\gamma})^{m}}{\sqrt{m!}} \qquad (138)
\end{aligned}$$

then

$$\begin{aligned}
\widetilde{P}_2^{1;1}(\widetilde{\gamma},\widetilde{\delta}) &= \exp(-\widetilde{\gamma}\,\widetilde{\gamma}^*)\,\exp(-\widetilde{\delta}\,\widetilde{\delta}^*)\,\Phi_1^*(\widetilde{\gamma}^*)\,\Phi_1(\widetilde{\gamma})\\
\widetilde{P}_2^{1;2}(\widetilde{\gamma},\widetilde{\delta}) &= \exp(-\widetilde{\gamma}\,\widetilde{\gamma}^*)\,\exp(-\widetilde{\delta}\,\widetilde{\delta}^*)\,\Phi_1^*(\widetilde{\gamma}^*)\,\Phi_2(\widetilde{\gamma})\,\exp(-i\omega t)\\
\widetilde{P}_2^{2;1}(\widetilde{\gamma},\widetilde{\delta}) &= \exp(-\widetilde{\gamma}\,\widetilde{\gamma}^*)\,\exp(-\widetilde{\delta}\,\widetilde{\delta}^*)\,\Phi_2^*(\widetilde{\gamma}^*)\,\Phi_1(\widetilde{\gamma})\,\exp(+i\omega t)\\
\widetilde{P}_2^{2;2}(\widetilde{\gamma},\widetilde{\delta}) &= \exp(-\widetilde{\gamma}\,\widetilde{\gamma}^*)\,\exp(-\widetilde{\delta}\,\widetilde{\delta}^*)\,\Phi_2^*(\widetilde{\gamma}^*)\,\Phi_2(\widetilde{\gamma}) \qquad (139)
\end{aligned}$$

To have agreement with the previous results in Eq. (105) so that

$$\begin{aligned}
\Phi_1(\widetilde{\gamma}) &= \Psi_1(\widetilde{\gamma})\\
\Phi_2(\widetilde{\gamma}) &= \Psi_2(\widetilde{\gamma}) \qquad (140)
\end{aligned}$$

where $\Psi_1(\widetilde{\gamma})$ and $\Psi_2(\widetilde{\gamma})$ are as in Eqs. (125), we require

$$\begin{aligned}
i\left\{\frac{\omega_n B_n + i\Delta A_n}{\Omega}\right\} &= \left(\frac{1}{2\pi}\right)\left\{A_{\overline{n-1}2}(0)\right\}^* \frac{1}{\sqrt{(n-1)!}} \\
i\left\{\frac{-\omega_n A_n + i\Delta B_n}{\Omega}\right\} &= \left(\frac{1}{2\pi}\right)(-i)^*\left\{\frac{\Omega\sqrt{n}\,A_{n1}(0) + \Delta\, A_{\overline{n-1}2}(0)}{\omega_n}\right\}^* \frac{1}{\sqrt{(n-1)!}} \\
A_n &= \left(\frac{1}{2\pi}\right)(-)\left\{A_{n1}(0)\right\}^* \frac{1}{\sqrt{(n)!}} \\
B_n &= \left(\frac{1}{2\pi}\right)(-)(+i)^*\left\{\frac{\Delta\, A_{n1}(0) - \Omega\sqrt{n}\,A_{\overline{n-1}2}(0)}{\omega_n}\right\}^* \frac{1}{\sqrt{(n)!}}
\end{aligned} \tag{141}$$

The last two equations give explicit expressions for A_n and B_n. Substituting these expressions into the left side of the first two equations gives the right hand sides, showing that the four equations are consistent. Hence we see that for *any* initial conditions for the one atom Jaynes-Cummings model, the solution given by the Grassmann phase space approach is the same as that from the standard quantum optics treatment.

5.2 Application of Results

As an illustration of how to apply the above results for the canonical distribution function we consider the case where the atom is initially in the *lower* state and the field is in a *coherent* state of amplitude η. In ths case we have from subsection **??**

$$\begin{aligned}
P_2^{1;1} &= 0 \qquad P_2^{1;2} = 0 \\
P_2^{2;1} &= 0 \qquad P_2^{2;2} = P_b(\alpha, \alpha^+)
\end{aligned} \tag{142}$$

with

$$P_b(\alpha,\alpha^+) = \frac{1}{4\pi^2}\exp(-\frac{|\alpha-\alpha^{+*}|^2}{4})\exp\left(-\left|\frac{1}{2}(\alpha+\alpha^{+*})-\eta\right|^2\right)$$

Hence

$$\begin{aligned}
\widetilde{P}_2^{2;2}(\widetilde{\gamma},\widetilde{\delta}) &= \frac{1}{4\pi^2}\exp(-\widetilde{\delta}\,\widetilde{\delta}^*)\exp\left(-\left|\widetilde{\gamma}-\eta\right|^2\right) = \exp(-\widetilde{\delta}\,\widetilde{\delta}^*)\exp(-\widetilde{\gamma}\widetilde{\gamma}^*)\,\widetilde{S}_2^{2;2}(\widetilde{\gamma}) \\
\widetilde{S}_2^{2;2}(\widetilde{\gamma}) &= \frac{1}{4\pi^2}\exp\left(-(\widetilde{\gamma}-\eta)(\widetilde{\gamma}^*-\eta^*)\right)\exp(+\widetilde{\gamma}\widetilde{\gamma}^*) \\
&= \left(\frac{1}{2\pi}\exp(-\frac{1}{2}\eta\eta^*)\exp(-\widetilde{\gamma}^*\eta)\right)\left(\frac{1}{2\pi}\exp(-\frac{1}{2}\eta\eta^*)\exp(-\widetilde{\gamma}\eta^*)\right)
\end{aligned} \tag{143}$$

At $t = 0$ we have

$$\begin{aligned}\Psi_1(\widetilde{\gamma}) &= i\sum_{n=1}^{\infty}\widetilde{\gamma}^{(n-1)}\left(\left\{\frac{\omega_n B_n + i\Delta A_n}{\Omega}\right\}\right) = 0 \\ \Psi_2(\widetilde{\gamma}) &= \sum_{n=1}^{\infty}\widetilde{\gamma}^n\left(A_n\right) = \frac{1}{2\pi}\exp(-\frac{1}{2}\eta\eta^*)\exp(-\widetilde{\gamma}\eta^*) \qquad (144)\end{aligned}$$

so that if we choose

$$\begin{aligned}A_n &= \frac{1}{2\pi}\exp(-\frac{1}{2}\eta\eta^*)\frac{(-\eta^*)^n}{n!} \\ B_n &= \frac{-i\Delta}{\omega_n}A_n \qquad (145)\end{aligned}$$

the solutions for $\Psi_1(\widetilde{\gamma})$ and $\Psi_2(\widetilde{\gamma})$ determines the time dependent distribution function.

6 Conclusion

We have shown that a phase space approach using Grassmann variables to describe the atomic system and c-number variables to describe the cavity mode can be used to treat the Jaynes-Cummings model and to obtain the same results for treating phenomena such as pure Rabi oscillations and collapse, revival effects as those from standard quantum optics methods. The Liouville-von Neumann equation for the density operator was converted into a Fokker-Planck equation for the canonical positive P distribution function using the correspondence rules associated with this choice of distribution function. The distribution function is a Grassmann function involving Grassmann phase space variables g_1, g_1^+ and g_2, g_2^+ for the two fermionic modes associated with the two atomic states, with six c-number functions of the bosonic phase space variables α, α^+ associated with the cavity mode being involved as coefficients in specifying the distribution function. In the context of a general mixed state where there may be zero, one or two atoms present, expressions for the probabilities of finding one atom in one of the two atomic states, one atom in both atomic states and no atom in either atomic state were obtained as bosonic phase space integrals involving the six bosonic coefficients. Coupled equations for the six bosonic coefficients for the canonical distribution function were obtained from the Fokker-Planck equation These equations were solved for the one atom Jaynes-Cummings model using an ansatz similar to that applied by Stenholm [14] in an earlier Bargmann state treatment of the Jaynes-Cummings model, and the results shown to be equivalent to the standard quantum optics treatment based on state vectors and coupled amplitude equations.

Positive P distribution functions have the feature of being non-unique, with different correspondence rules applying in the derivation of the specific Fokker-Planck equation. In this application we also found that applying the standard

correspondence rules (rather than those for the canonical positive P case) leads to a Fokker-Planck equation where the solution for the coupled equations for the bosonic coefficients via a similar ansatz was quite unsuitable. Not only did the solutions diverge for large time t, but the distribution function diverged for large phase space variables α, α^+, thereby throwing into question the derivation of the Fokker-Planck equation. The standard treatment requires the distribution function to vanish on the phase space boundary. As the correspondence rules for the canonical positive P distribution do not require this feature, it is suggested that Fokker-Planck equations based on the canonical positive P distribution may be more reliable. Furthermore, in terms of matching a general solution of the Fokker-Planck equation to the initial conditions, the use of the canonical form of the distribution function is easiest since initial conditions are usually specified via the initial density operator from which the canonical distribution function is directly determined. However, more general Fokker-Planck equations involving derivatives higher than second order may occur using the canonical distribution function [28], so that no replacement by Langevin stochastic equations is then possible. In fact even if the Fokker-Planck equation is only second order, the diffusion matrix may not be positive definite.

The successful treatment of this classic quantum optics system based on phase space methods using Grassmann variables represents an important step in applying such methods to treat more complex problems involving fermionic systems.

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8 Appendix 1 - Grassmann Numbers and Calculus

Since Grassmann variables and their calculus may be unfamiliar to many physicists a short summary may be desirable. More extensive accounts of the properties of Grassmann variables and their calculus are given in Refs. [20, 19].

8.1 1.1 Grassmann Algebra

Grassmann variables satisfy the following *anti-commutation rules* with each other and with fermion annihilation, creation operators

$$\begin{aligned} g_i g_j &= -g_j g_i \qquad g_i \widehat{c}_j = -\widehat{c}_j g_i \qquad g_i \widehat{c}_j^\dagger = -\widehat{c}_j^\dagger g_i \\ \{g_i, g_j\} &= \{g_i, \widehat{c}_j\} = \{g_i, \widehat{c}_j^\dagger\} = 0 \end{aligned} \tag{146}$$

and Grassmann variables and fermion operators commute with c-numbers and boson operators. A key feature of Grassmann variables that immediately follows is that their square and hence all higher powers are zero.

$$g_i^2 = g_i^3 = ... = 0 \tag{147}$$

Grassmann variables also have no inverses, and hence division is undefined.

Grassmann functions involving linear combinations of products of several Grassmann variables with c-numbers coefficients can be defined in an obvious way and are of the form

$$\begin{aligned} f(h_1, h_2, .., h_n) &= f_0 + \sum_i f_i\, h_i + \sum_{i<j} f_{ij}\, h_i h_j + \sum_{i<j<k} f_{ijk}\, h_i h_j h_k + .. \\ &\quad + f_{123..n}\, h_1 h_2 .. h_n \end{aligned} \tag{148}$$

where $f_0, f_i, f_{ij}, .., f_{12..n}$ are c-numbers. Functions may be *even* or *odd* depending on whether all their terms contain an even or odd number of Grassmann variables. A general function is the sum of an even and an odd function. Two even or two odd Grassmann functions commute with each other, whilst an even and an odd function anti-commute.

Grassmann functions have a *linearity* feature in that each Grassmann variable can at most appear in a linear form due to (147). Thus

$$\begin{aligned} & f(h_1, h_2, .., h_n) \\ = \;& a_i(h_1, .., h_{i-1}, h_{i+1}, ..h_n) + b_i(h_1, .., h_{i-1}, h_{i+1}, ..h_n)\, h_i \\ = \;& a_i(h_1, .., h_{i-1}, h_{i+1}, ..h_n) + h_i\, c_i(h_1, .., h_{i-1}, h_{i+1}, ..h_n) \end{aligned} \tag{149}$$

where a_i, b_i and c_i are Grassmann functions that are independent of h_i. In general b_i and c_i are not the same. Linearity has the effect of truncating functions, thus the Grassmann function $\exp g$ is just equal to $1 + g$.

Complex conjugation can also be defined with the property

$$(g_i g_j)^* = g_j^* g_i^* \tag{150}$$

Apart from the conjugation, the anti-commuting multiplication rules and the lack of an inverse, Grassmann variables satisfy the normal algebraic rules (associative laws of addition, multiplication etc).

8.2 1.2 Grassmann Differentiation

The basic rules for *Grassmann differentiation* are

$$\begin{aligned} \frac{\overrightarrow{\partial}}{\partial g_i}1 &= 1\frac{\overleftarrow{\partial}}{\partial g_i} = 0 \\ \frac{\overrightarrow{\partial}}{\partial g_i}g_j &= g_j\frac{\overleftarrow{\partial}}{\partial g_i} = \delta_{i,j} \end{aligned} \tag{151}$$

where both left and right differentiation are defined.

Differentiation proceeds via moving the variable to be differentiated to the left or right of all the other Grassmann variables and then applying the above rules. Thus for the Grassmann function in (149) we have

$$\begin{aligned} \frac{\overrightarrow{\partial}}{\partial h_i}f(h_1,h_2,..,h_n) &= c_i(h_1,..,h_{i-1},h_{i+1},..h_n) \\ f(h_1,h_2,..,h_n)\frac{\overleftarrow{\partial}}{\partial h_i} &= b_i(h_1,..,h_{i-1},h_{i+1},..h_n) \end{aligned} \tag{152}$$

Note that in general, left and right differentiation do not give the same result.

Multiple differentiation is carried out in the order of the derivatives. Thus for left differentiation

$$\frac{\overrightarrow{\partial}}{\partial h_i}\frac{\overrightarrow{\partial}}{\partial h_j}f(h_1,h_2,..,h_n) = \frac{\overrightarrow{\partial}}{\partial h_i}\left(\frac{\overrightarrow{\partial}}{\partial h_j}f(h_1,h_2,..,h_n)\right) \tag{153}$$

with equivalent results for right differentiation. Mixed left and right differentiation also occurs and it turns out the different orders for carrying this out give the same result. Also left and right differentiation are related for even and odd functions.

$$\frac{\overrightarrow{\partial}}{\partial g_i}f_E(g) = (-1)f_E(g)\frac{\overleftarrow{\partial}}{\partial g_i} \tag{154}$$

$$\frac{\overrightarrow{\partial}}{\partial g_i}f_O(g) = (+1)f_O(g)\frac{\overleftarrow{\partial}}{\partial g_i} \tag{155}$$

Product rules for differentiation can be derived. These depend on whether

the factors are even or odd Grassmann functions

$$\begin{aligned}
\frac{\overrightarrow{\partial}}{\partial g_i}(f_1^E f_2) &= (\frac{\overrightarrow{\partial}}{\partial g_i} f_1^E) f_2 + f_1^E (\frac{\overrightarrow{\partial}}{\partial g_i} f_2) \\
\frac{\overrightarrow{\partial}}{\partial g_i}(f_1^O f_2) &= (\frac{\overrightarrow{\partial}}{\partial g_i} f_1^O) f_2 - f_1^O (\frac{\overrightarrow{\partial}}{\partial g_i} f_2) \\
(f_2 f_1^E)\frac{\overleftarrow{\partial}}{\partial g_i} &= f_2 (f_1^E \frac{\overleftarrow{\partial}}{\partial g_i}) + (f_2 \frac{\overleftarrow{\partial}}{\partial g_i}) f_1^E \\
(f_2 f_1^O)\frac{\overleftarrow{\partial}}{\partial g_i} &= f_2 (f_1^O \frac{\overleftarrow{\partial}}{\partial g_i}) - (f_2 \frac{\overleftarrow{\partial}}{\partial g_i}) f_1^O
\end{aligned} \tag{156}$$

Thus the product rule is different in general from that in ordinary calculus. For Grassmann functions that are neither even nor odd the derivative of a product can be obtained from (156) after writing the function as the sum of its even and odd components.

8.3 1.3 Grassmann Integration

The basic rules for *Grassmann integration* are

$$\begin{aligned}
\int dg_i 1 &= \int 1\, dg_i = 0 \\
\int dg_i\, g_j &= \delta_{ij}. \qquad \int g_j\, dg_i = -\delta_{ij}
\end{aligned} \tag{157}$$

where both left and right integration are defined. The different results for left and right integration are due to the differentials thems being anti-commuting Grassmann variables. In the present paper only left integration will be used.

The same outcome from $\frac{\overrightarrow{\partial}}{\partial g_i} g_j = \delta_{i,j}$, $\frac{\overrightarrow{\partial}}{\partial g_i} 1 = 0$ and $\int dg_i\, g_j = \delta_{ij}$, $\int dg_i 1 = 0$ provokes the comment that differentiation and integration are the same, but this is not really the case in view of the different results for right integration and differentiation.

Integration proceeds via moving the variable to be integrated to the left or right of all the other Grassmann variables and then applying the above rules. Thus for the Grassmann function in (149) we have

$$\begin{aligned}
\int dh_i\, f(h_1, h_2, .., h_n) &= c_i(h_1, .., h_{i-1}, h_{i+1}, ..h_n) \\
\int f(h_1, h_2, .., h_n)\, dh_i\, f(h_1, h_2, .., h_n) &= -b_i(h_1, .., h_{i-1}, h_{i+1}, ..h_n)
\end{aligned} \tag{158}$$

Multiple integration is carried out in the order of the differentials, for example in left integration

$$\int\int dh_i\, dh_j\, f(h_1, h_2, .., h_n) = \int dh_i \left(\int dh_j\, f(h_1, h_2, .., h_n) \right) \tag{159}$$

with equivalent results for right integration.

An important results is for the complete Grassmann integral

$$\int\int ..\int dh_1\, dh_2..dh_n\, h_n...h_2\, h_1 = 1 \tag{160}$$

8.4 1.4 Grassmann States and Grassmann Operators

For fermion systems, vectors can be defined in a generalised form of Hilbert space, which involve linear combinations of basis vectors such as Eq.(**??**) but now with Grassmann numbers as the coefficients. Such *Grassmann vectors* are not taken to represent physical states (where the coefficients must be c-numbers) even if a fixed number of fermions are involved, but they have uses in the mathematical manipulations. The fermion coherent states are Grassmann vectors.

Similarly, we may introduce generalised operators in this new Hilbert space by taking linear combinations of the products of the fermion operators with Grassmann numbers as the coefficients, such as in Eq.(45). Such *Grassmann operators* do not represent physical quantities (which must involve c-numbers as coefficients) or symmetry operations, but again are useful mathematically.

Many of the results for Grassmann functions also apply for Grassmann operators and states, in which some of the Grassmann variables are replaced by fermion annihilation or creation operators. The anti-commuting feature of the fermion operators with Grassmann variables enables the same proofs to be made.

In addition to the rules (146) for Grassmann numbers, it is necessary to state the basic rules for multiplying the vacuum state with Grassmann number, since the Fock states involve products of creation operators acting on the vacuum state $|0\rangle$, . The rule is that the Grassmann numbers commute with $|0\rangle$ or $\langle 0|$.

$$g\,|0\rangle = |0\rangle\, g \qquad g\,\langle 0| = \langle 0|\, g \tag{161}$$

Note that as a consequence

$$\begin{aligned} g\,|m_1;m_2;n\rangle &= (-1)^{m_1+m_2}\,|m_1;m_2;n\rangle\, g \\ \langle m_1;m_2;n|\, g &= (-1)^{m_1+m_2} g\,\langle m_1;m_2;n| \end{aligned} \tag{162}$$

so clearly for N fermion states, a Grassmann number anti-commutes with the Fock states for fermion systems with an odd number of fermions, and commutes if the number of fermions is even. The bra and ket vectors for fermion states may be classified as *even* or *odd vectors* depending on whether they only contain terms with even or odd numbers of fermions. Thus $(|0;0;n\rangle + |1;1;n\rangle)$ would be an even vector, whilst $(|0;1;n\rangle + |1;0;n\rangle)$ would be odd. These concepts may be extended to include *Grassmann vectors* as well as ordinary state vectors. In this case as well as the vacuum state $|0\rangle$ or $\langle 0|$, all terms in even (odd) vectors contain an even (odd) number of fermion creation operators, annihilation

operators *and* Grassmann numbers, and consequently commute (anti-commute) with any Grassmann number. Thus the Grassmann vectors $(\widehat{1}+\widehat{c}_1^{\dagger}h_1)\,|0;0;n\rangle = |0;0;n\rangle - h_1\,|1;0;n\rangle$ or $\langle 0;0;n|\,(\widehat{1}+h_1^{+}\widehat{c}_1) = \langle 0;0;n| - \langle 1;0;n|\,h_1^{+}$ are even vectors.

Some operators such as those that represent physical quantities contain an even number of fermion creation and annihilation operators, usually the same number of each (see the Hamiltonian $\widehat{H}$ in (26) for example, which is the sum of terms each containing the same numbers of annihilation and creation operators). These operators are called *even operators*. Consequently even operators will commute with a Grassmann number. Other operators of interest such as the fermion cretaion and annihilation operators themselves involve odd numbers of creation and annihilation operators. These operators are called *odd operators*. Operators either commute or anti-commute with a Grassmann number depending on whether they are even or odd. Arbitary operators can always be expressed as the sum of an even operator and an odd operator and a Grassmann number will commute with the even component and anti-commute with the odd component. These concepts may be extended to include *Grassmann operators* as well as ordinary quantum operators. In this case all terms in even (odd) operators contain an even (odd) number of fermion creation operators, annihilation operators *and* Grassmann numbers, and consequently commute (anti-commute) with a Grassmann number. Thus the operators $\hat{\Omega}_f^{+}(h^{+})$, $\hat{\Omega}_f^{-}(h)$ in (45) are even operators. Operators for which there are no Grassmann variables involved are just a special case where evenness or oddness only depends on the number of fermion creation, annihilation operators.

9 Appendix 2 - Bargmann Coherent States

For both bosonic and fermionic systems we can define the *coherent states*, which are defined by the effect of unitary displacement operators on the vacuum state. These states are parameterised via c-number variables α_i, α_i^* for the bosonic modes and via Grassmann variables g_i, g_i^* for the fermionic modes. The coherent states are normalised to unity and are eigenstates of the boson or fermion annihilation operators with eigenvalues α_i or g_i respectively. A full description of the coherent states is given in [42], [20] for the bosonic and fermionic cases. As in these papers, we will treat the general case of multi-mode systems. For our purposes it will be convenient to use a related set of un-normalised states called the *Bargmann states*, which have the property of only depending on α_i or g_i, and not on the complex conjugates.

9.1 2.1 Bosons

For bosons the Bargmann states are defined as

$$|\alpha\rangle_B = \exp\left(\sum_{i=1}^{n} \hat{a}_i^\dagger \alpha_i\right)|0\rangle = \prod_i \exp\left(\hat{a}_i^\dagger \alpha_i\right)|0\rangle \tag{163}$$

which only depend on the c-number variables $\alpha \equiv \{\alpha_1, \alpha_2, .., \alpha_i, .., \alpha_n\}$, and not on the complex conjugates $\alpha^* \equiv \{\alpha_1^*, \alpha_2^*, .., \alpha_i^*, .., \alpha_n^*\}$. These are related to the coherent states $|\alpha, \alpha^*\rangle$ via

$$|\alpha, \alpha^*\rangle = \exp(-\frac{1}{2}\boldsymbol{\alpha}^* \cdot \boldsymbol{\alpha})\,|\alpha\rangle_B \tag{164}$$

$$= \exp(-\frac{1}{2}\boldsymbol{\alpha}^* \cdot \boldsymbol{\alpha}) \prod_i \sum_{\nu_i=0}^{\infty} \frac{(\alpha_i)^{\nu_i}}{\sqrt{\nu_i!}}\,|\nu_i\rangle\,, \tag{165}$$

where $\boldsymbol{\alpha}^* \cdot \boldsymbol{\beta} = \sum_{i=1}^{n} \alpha_i^* \beta_i$. The second expression is the well-known expansion of the coherent state in terms of Fock states. This c-number expansion represents a Poisson distribution of number states with a mean boson number $\langle \widehat{n}_i \rangle = |\alpha_i|^2$ and a variance $\langle \Delta \widehat{n}_i^2 \rangle = \langle \widehat{n}_i \rangle$. The coherent state is not a physical state except in the case of photons.

As in the case of coherent states, the Bargmann states are eigenstates of the annihilation operator

$$\hat{a}_i\,|\alpha\rangle_B = \alpha_i\,|\alpha\rangle_B \tag{166}$$

$${}_B\langle\alpha|\,\hat{a}_i^\dagger = {}_B\langle\alpha|\,\alpha_i^*. \tag{167}$$

They satisfy normalisation and orthogonality conditions

$${}_B\langle\alpha|\beta\rangle_B = \exp\{\boldsymbol{\alpha}^* \cdot \boldsymbol{\beta}\} \tag{168}$$

Hence Bargmann states are unnormalised versions of the coherent states.

The operation of a creation operator on a Bargmann ket vector or an annihilation operator on a Bargmann bra vector can be written in terms of derivatives of these vectors.

$$\widehat{a}_i^\dagger \, |\alpha\rangle_B = \left(\frac{\partial}{\partial \alpha_i}\right) |\alpha\rangle_B \tag{169}$$

$$\langle\alpha|_B \, \widehat{a}_i = \left(\frac{\partial}{\partial \alpha_i^*}\right) \langle\alpha|_B \tag{170}$$

there being no distinction between left and right differentiation.

9.2 2.2 Fermions

For fermions we define the Bargmann states via

$$|g\rangle_B = \exp\left(\sum_{i=1}^{n} \hat{c}_i^\dagger g_i\right) |0\rangle = \prod_i (1 + \hat{c}_i^\dagger g_i) \, |0\rangle = \prod_i (|0_i\rangle - g_i \, |1_i\rangle) \tag{171}$$

which only depend on n Grassmann numbers $g \equiv \{g_1, g_2, .., g_i, .., g_n\}$, and not on the complex conjugates $g^* \equiv \{g_1^*, g_2^*, .., g_i^*, .., g_n^*\}$. These are related to the fermion coherent states $|g, g^*\rangle$ in a similar way as in the boson case.

$$|g, g^*\rangle = \exp(-\frac{1}{2}\mathbf{g}^* \cdot \mathbf{g}) \, |g\rangle_B \tag{172}$$

$$= \exp(-\frac{1}{2}\mathbf{g}^* \cdot \mathbf{g}) \prod_i (|0_i\rangle - g_i \, |1_i\rangle). \tag{173}$$

where $\mathbf{g}^* \cdot \mathbf{h} = \sum_{i=1}^{n} g_i^* h_i$. This differs from the corresponding bosonic expansion because it only involves a superposition of a zero fermion state with a one fermion state. This is to be expected from the Pauli exclusion principle since any mode can only be occupied by at most one fermion. The fermion coherent state is of course unphysical as it involves Grassmann numbers as expansion coefficients. The Bargmann states are employed by Plimak et al. [21] rather than the fermion coherent states as in Cahill and Glauber [20]. Bargmann states are even Grassmann vectors.

As for the coherent states, the fermion Bargmann states are eigenstates of the annihilation operator

$$\hat{c}_i \, |g\rangle_B = g_i \, |g\rangle_B \tag{174}$$

$${}_B\langle g| \, \hat{c}_i^\dagger = {}_B\langle g| \, g_i^*. \tag{175}$$

In the fermion case we can also find eigenstates of the creation operator, see [20]. The Bargmann states satisfy normalisation and orthogonality conditions

$${}_B\langle g|h\rangle_B = \exp\{\mathbf{g}^* \cdot \mathbf{h}\}. \tag{176}$$

Note the similarity of these results to those for the boson states.

The operation of a creation operator on a Bargmann ket vector or an annihilation operator on a Bargmann bra vector can be written in terms of derivatives of these vectors. In the fermion case

$$\widehat{c}_i^{\dagger}\ |g\rangle_B = \left(-\frac{\overrightarrow{\partial}}{\partial g_i}\right)|g\rangle_B = |g\rangle_B \left(+\frac{\overleftarrow{\partial}}{\partial g_i}\right) \tag{177}$$

$$\langle g|_B\ \widehat{c}_i = \langle g|_B \left(-\frac{\overleftarrow{\partial}}{\partial g_i^*}\right) = \left(+\frac{\overrightarrow{\partial}}{\partial g_i^*}\right)\langle g|_B \tag{178}$$

where we note that both left and right derivatives forms apply for each of $\widehat{c}_i^{\dagger}\ |g\rangle_B$ and $\langle g|_B\ \widehat{c}_i$. This is analogous to the feature that in (174) and (175) the eigenvalues can be placed on either side of the bra or ket Bargmann vector.

The Bargmann states can be used for a representation of quantum operators. For the fermion operator $\hat{\Omega}_f$ we introduce completeness relationships for two sets of fermion Fock states $|\nu_1, \nu_2, .., \nu_n\rangle$, $|\xi_1, \xi_2, .., \xi_n\rangle$, where the occupation numbers for the various single particle states ν_i, ξ_i are $0, 1$ only. We have

$$\langle g|_B\, \hat{\Omega}_f\, |h\rangle_B = \sum_{\nu_1,\nu_2,..,\nu_n} \sum_{\xi_1,\xi_2,..,\xi_n} \langle g|\nu_1, \nu_2, .., \nu_n\rangle_B \langle \nu_1, \nu_2, .., \nu_n|\, \hat{\Omega}_f\, |\xi_1, \xi_2, .., \xi_n\rangle \langle \xi_1, \xi_2, .., \xi_n|h\rangle_B$$

Then from the expression for the Bargmann states

$$\langle \xi_1, \xi_2, .., \xi_n|h\rangle_B = \prod_j \langle \xi_j| \prod_i (|0_i\rangle + |1_i\rangle\, h_i) = \prod_i (\delta_{\xi_j 0} + \delta_{\xi_j 1} h_i) = \prod_i (h_i)^{\xi_i}$$

giving the result

$$\langle g|_B\, \hat{\Omega}_f\, |h\rangle_B = \sum_{\nu_1,\nu_2,..,\nu_n} \sum_{\xi_1,\xi_2,..,\xi_n} \Omega_f(\{\nu\};\{\xi\})(g_{m_1}^*)^{\nu_1}(g_{m_2}^*)^{\nu_2}..(g_{m_n}^*)^{\nu_n}(h_{m_n})^{\xi_n}..(h_{m_2})^{\xi_2}(h_{m_1})^{\xi_1} \tag{179}$$

where $\Omega_f(\{\nu\};\{\xi\}) = \langle \nu_1, \nu_2, .., \nu_n|\, \hat{\Omega}_f\, |\xi_1, \xi_2, .., \xi_n\rangle$.

9.3 2.3 Projectors

The Bargmann states can be used to define normalised projectors as

$$\widehat{\Lambda}_b(\alpha, \beta^*) = \frac{|\alpha\rangle_B \langle\beta|_B}{Tr_b(|\alpha\rangle_B \langle\beta|_B)} \tag{180}$$

$$\widehat{\Lambda}_f(g, h^*) = \frac{|g\rangle_B \langle h|_B}{Tr_f(|g\rangle_B \langle h|_B)}. \tag{181}$$

where the trace of the projectors $|\alpha\rangle_B \langle\beta|_B$ and $|g\rangle_B \langle h|_B$ have a simple form

$$Tr_f(|g\rangle_B \langle h|_B) = \exp\{\mathbf{g}\cdot\mathbf{h}^*\} =_B \langle -h|g\rangle_B =_B \langle h| - g\rangle_B \tag{182}$$

$$Tr_b(|\alpha\rangle_B \langle\beta|_B) = \exp\{\boldsymbol{\alpha}\cdot\boldsymbol{\beta}^*\} =_B \langle\beta|\alpha\rangle_B \tag{183}$$

The normalised projectors have the property that their trace is unity.

$$Tr_b\widehat{\Lambda}_b(\alpha,\beta^*)=1 \qquad Tr_f\widehat{\Lambda}_f(g,h^*)=1. \tag{184}$$

9.4 2.4 Completeness

One of the important features of both the boson and fermion coherent states is that they satisfy completeness relationships. In terms of Bargmann states the completeness relationships are

$$\begin{aligned}\int d^2\mathbf{g}\exp(-\mathbf{g}^*\cdot\mathbf{g})\ |g\rangle_B\langle g|_B &= \hat{1}\\ \int d^2\boldsymbol{\alpha}\exp(-\boldsymbol{\alpha}^*\cdot\boldsymbol{\alpha})\ |\alpha\rangle_B\langle\alpha|_B &= \pi\hat{1}\end{aligned} \tag{185}$$

for fermions and bosons respectively. Here $d^2\mathbf{g}\equiv\prod_i dg_i^*dg_i$ and $d^2\boldsymbol{\alpha}\equiv\prod_i d\alpha_{ix}d\alpha_{iy}$

9.5 2.5 Trace Properties

Cyclic properties of Grassmann operators can be established. For a pair of even Grassmann operators

$$Tr(\hat{\Omega}_f^E(h)\hat{\Delta}_f^E(k))=Tr(\hat{\Delta}_f^E(k)\hat{\Omega}_f^E(-h))=Tr(\hat{\Delta}_f^E(-k)\hat{\Omega}_f^E(h)) \tag{186}$$

For the product of two odd Grassmann operators.

$$Tr(\hat{\Omega}_f^O(h)\hat{\Delta}_f^O(k))=Tr(\hat{\Delta}_f^O(k)\hat{\Omega}_f^O(-h))=Tr(\hat{\Delta}_f^O(-k)\hat{\Omega}_f^O(h)) \tag{187}$$

9.6 2.6 Operator Identities

The effect of bosonic annihilation and creation operators on the Bargmann state projectors involved in the canonical form for the density operator are

$$\begin{aligned}\widehat{a}_i\,\widehat{\Lambda}_b(\alpha,\alpha^+) &= \alpha_i\widehat{\Lambda}_b(\alpha,\alpha^+)\\ \widehat{\Lambda}_b(\alpha,\alpha^+)\,\widehat{a}_i^\dagger &= \alpha_i^+\,\widehat{\Lambda}_b(\alpha,\alpha^+)\\ \widehat{a}_i^\dagger\,\widehat{\Lambda}_b(\alpha,\alpha^+) &= \left(\frac{\partial}{\partial\alpha_i}+\alpha_i^+\right)\widehat{\Lambda}_b(\alpha,\alpha^+)\\ \widehat{\Lambda}_b(\alpha,\alpha^+)\,\widehat{a}_i &= \left(\frac{\partial}{\partial\alpha_i^+}+\alpha_i\right)\widehat{\Lambda}_b(\alpha,\alpha^+)\end{aligned} \tag{188}$$

The effect of fermionic annihilation and creation operators on the Bargmann state projectors involved in the canonical form for the density operator are

$$\begin{aligned}
\widehat{c}_i\,\widehat{\Lambda}_f(g,g^+) &= g_i\,\widehat{\Lambda}_f(g,g^+) = \widehat{\Lambda}_f(g,g^+)\,g_i \\
\widehat{\Lambda}_f(g,g^+)\,\widehat{c}_i^\dagger &= \widehat{\Lambda}_f(g,g^+)\,g_i^+ = g_i^+\,\widehat{\Lambda}_f(g,g^+) \\
\widehat{c}_i^\dagger\,\widehat{\Lambda}_f(g,g^+) &= \left(-\frac{\overrightarrow{\partial}}{\partial g_i} - g_i^+\right)\widehat{\Lambda}_f(g,g^+) = \widehat{\Lambda}_f(g,g^+)\left(+\frac{\overleftarrow{\partial}}{\partial g_i} - g_i^+\right) \\
\widehat{\Lambda}_f(g,g^+)\,\widehat{c}_i &= \widehat{\Lambda}_f(g,g^+)\left(-\frac{\overleftarrow{\partial}}{\partial g_i^+} - g_i\right) = \left(+\frac{\overrightarrow{\partial}}{\partial g_i^+} - g_i\right)\widehat{\Lambda}_f(g,g^+)
\end{aligned} \tag{189}$$

the second form of the result following from $\widehat{\Lambda}_f(g,g^+)$ being an even Grassmann operator.

10 Appendix 5 - Results for Standard Distribution Function

10.1 5.1 Fokker-Planck Equation

The *Fokker-Planck equation* for the general distribution function $P(\alpha, \alpha^+, \alpha^*, \alpha^{+*}, g, g^+)$ is

$$
\begin{aligned}
&\frac{\partial P(\alpha,\alpha^+,g,g^+)}{\partial t} \\
= \;& -i\frac{E_A}{\hbar}\left(\frac{\overrightarrow{\partial}}{\partial g_2}\{g_2\,P(\alpha,\alpha^+,g,g^+)\}+\frac{\overrightarrow{\partial}}{\partial g_1}\{g_1 P(\alpha,\alpha^+,g,g^+)\}\right) \\
& +i\frac{E_A}{\hbar}\left(\{P(\alpha,\alpha^+,g,g^+)\,g_2^+\}\frac{\overleftarrow{\partial}}{\partial g_2^+}+\{P(\alpha,\alpha^+,g,g^+)\,g_1^+\}\frac{\overleftarrow{\partial}}{\partial g_1^+}\right) \\
& -\frac{1}{2}i\,\omega_0\left(\frac{\overrightarrow{\partial}}{\partial g_2}\{g_2\,P(\alpha,\alpha^+,g,g^+)\}-\frac{\overrightarrow{\partial}}{\partial g_1}\{g_1 P(\alpha,\alpha^+,g,g^+)\}\right) \\
& +\frac{1}{2}i\,\omega_0\left(\{P(\alpha,\alpha^+,g,g^+)\,g_2^+\}\frac{\overleftarrow{\partial}}{\partial g_2^+}-\{P(\alpha,\alpha^+,g,g^+)\,g_1^+\}\frac{\overleftarrow{\partial}}{\partial g_1^+}\right) \\
& -i\,\omega\left(-\frac{\partial}{\partial\alpha}\{\alpha\,P(\alpha,\alpha^+,g,g^+)\}+\frac{\partial}{\partial\alpha^+}\{\alpha^+\,P(\alpha,\alpha^+,g,g^+)\}\right) \\
& -\frac{1}{2}i\,\Omega\left(\frac{\overrightarrow{\partial}}{\partial g_1}\{g_2\alpha^+\,P(\alpha,\alpha^+,g,g^+)\}+\frac{\overrightarrow{\partial}}{\partial g_2}\{g_1\alpha\,P(\alpha,\alpha^+,g,g^+)\}\right) \\
& +\frac{1}{2}i\,\Omega\left(\{P(\alpha,\alpha^+,g,g^+)\,g_2^+\,\alpha\}\frac{\overleftarrow{\partial}}{\partial g_1^+}+\{P(\alpha,\alpha^+,g,g^+)\,g_1^+\,\alpha^+\}\frac{\overleftarrow{\partial}}{\partial g_2^+}\right) \\
& +\frac{1}{2}i\,\Omega\left(\frac{\overrightarrow{\partial}}{\partial g_1}\{g_2\left(\frac{\partial}{\partial\alpha}P(\alpha,\alpha^+,g,g^+)\right)\}-\{\left(\frac{\partial}{\partial\alpha^+}P(\alpha,\alpha^+,g,g^+)\right)g_2^+\}\frac{\overleftarrow{\partial}}{\partial g_1^+}\right) \\
& -\frac{1}{2}i\,\Omega\left(\left(g_1^+g_2\right)\left(\frac{\partial}{\partial\alpha}P(\alpha,\alpha^+,g,g^+)\right)-\left(\frac{\partial}{\partial\alpha^+}P(\alpha,\alpha^+,g,g^+)\right)\left(g_2^+g_1\right)\right)
\end{aligned}
\tag{190}
$$

where for simplicity we write $P(\alpha, \alpha^+, g, g^+)$ instead of the full expression $P(\alpha, \alpha^+, \alpha^*, \alpha^{+*}, g, g^+)$ using the notation α, α^+ for $\alpha, \alpha^+, \alpha^*, \alpha^{+*}$.

10.2 5.2 Coupled Expansion Coefficients

The coupled equations for the coefficients specifying the general distribution function are obtained from the Fokker-Planck equation (190) and as follows.

For the *zeroth order* terms

$$
\frac{\partial}{\partial t}P_0(\alpha,\alpha^+)=-i\,\omega\left(-\frac{\partial}{\partial\alpha}\alpha+\frac{\partial}{\partial\alpha^+}\alpha^+\right)P_0(\alpha,\alpha^+) \tag{191}
$$

For the *second order* terms

$$\begin{aligned}\frac{\partial}{\partial t}P_2^{1;1}(\alpha,\alpha^+) &= -i\,\omega\left(-\frac{\partial}{\partial\alpha}\alpha+\frac{\partial}{\partial\alpha^+}\alpha^+\right)P_2^{1;1}(\alpha,\alpha^+)\\ &\quad+\frac{1}{2}i\,\Omega\left(\alpha\,P_2^{2;1}(\alpha,\alpha^+)-\alpha^+P_2^{1;2}(\alpha,\alpha^+)\right) \qquad (192)\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial t}P_2^{1;2}(\alpha,\alpha^+) &= -i\,\omega\left(-\frac{\partial}{\partial\alpha}\alpha+\frac{\partial}{\partial\alpha^+}\alpha^+\right)P_2^{1;2}(\alpha,\alpha^+)\\ &\quad-i\,\omega_0\,P_2^{1;2}(\alpha,\alpha^+)\\ &\quad+\frac{1}{2}i\,\Omega\left(\alpha\,P_2^{2;2}(\alpha,\alpha^+)-\alpha P_2^{1;1}(\alpha,\alpha^+)\right)\\ &\quad+\frac{1}{2}i\,\Omega\left(\frac{\partial}{\partial\alpha^+}\left\{P_2^{1;1}(\alpha,\alpha^+)-P_0(\alpha,\alpha^+)\right\}\right) \qquad (193)\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial t}P_2^{2;1}(\alpha,\alpha^+) &= -i\,\omega\left(-\frac{\partial}{\partial\alpha}\alpha+\frac{\partial}{\partial\alpha^+}\alpha^+\right)P_2^{2;1}(\alpha,\alpha^+)\\ &\quad+i\,\omega_0\,P_2^{2;1}(\alpha,\alpha^+)\\ &\quad+\frac{1}{2}i\,\Omega\left(\alpha^+P_2^{1;1}(\alpha,\alpha^+)-\alpha^+P_2^{2;2}(\alpha,\alpha^+)\right)\\ &\quad-\frac{1}{2}i\,\Omega\left(\frac{\partial}{\partial\alpha}\left\{P_2^{1;1}(\alpha,\alpha^+)-P_0(\alpha,\alpha^+)\right\}\right) \qquad (194)\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial t}P_2^{2;2}(\alpha,\alpha^+) &= -i\,\omega\left(-\frac{\partial}{\partial\alpha}\alpha+\frac{\partial}{\partial\alpha^+}\alpha^+\right)P_2^{2;2}(\alpha,\alpha^+)\\ &\quad+\frac{1}{2}i\,\Omega\left(\alpha^+P_2^{1;2}(\alpha,\alpha^+)-\alpha P_2^{2;1}(\alpha,\alpha^+)\right)\\ &\quad+\frac{1}{2}i\,\Omega\left(-\frac{\partial}{\partial\alpha}P_2^{1;2}(\alpha,\alpha^+)+\frac{\partial}{\partial\alpha^+}P_2^{2;1}(\alpha,\alpha^+)\right) \qquad (195)\end{aligned}$$

For the *fourth order* term

$$\begin{aligned}\frac{\partial}{\partial t}P_4^{12;21}(\alpha,\alpha^+) &= -i\,\omega\left(-\frac{\partial}{\partial\alpha}\alpha+\frac{\partial}{\partial\alpha^+}\alpha^+\right)P_4^{12;21}(\alpha,\alpha^+)\\ &\quad-\frac{1}{2}i\,\Omega\left(\frac{\partial}{\partial\alpha}P_2^{1;2}(\alpha,\alpha^+)-\frac{\partial}{\partial\alpha^+}P_2^{2;1}(\alpha,\alpha^+)\right) \qquad (196)\end{aligned}$$

Thus we see that for the general distribution $P(\alpha,\alpha^+,\alpha^*,\alpha^{+*},g,g^+)$ the coefficients satisfy similar sets of coupled equations as in the canonical distribution case.

10.3 3.3 Rotating Phase Variables and Coefficients

The transformation to rotating phase variables is as in Eq. (94). The new expansion coefficients will be designated $R(\underset{\rightarrow}{\beta})$ where $\underset{\rightarrow}{\beta} = \{\beta, \beta^+, \beta^*, \beta^{+*}\}$ and are related to the original $P(\underset{\rightarrow}{\alpha})$ via

$$\begin{aligned}
P_2^{1;1}(\underset{\rightarrow}{\boldsymbol{\alpha}}) &= R_2^{1;1}(\underset{\rightarrow}{\boldsymbol{\beta}}) \qquad P_2^{2;2}(\underset{\rightarrow}{\boldsymbol{\alpha}}) = R_2^{2;2}(\underset{\rightarrow}{\boldsymbol{\beta}}) \\
P_2^{1;2}(\underset{\rightarrow}{\boldsymbol{\alpha}}) &= R_2^{1;2}(\underset{\rightarrow}{\boldsymbol{\beta}})\exp(-i\omega t) \qquad P_2^{2;1}(\underset{\rightarrow}{\boldsymbol{\alpha}}) = R_2^{2;1}(\underset{\rightarrow}{\boldsymbol{\beta}})\exp(+i\omega t) \\
P_0(\underset{\rightarrow}{\boldsymbol{\alpha}}) &= R_0(\underset{\rightarrow}{\boldsymbol{\beta}}) \qquad P_4^{12;21}(\underset{\rightarrow}{\boldsymbol{\alpha}}) = R_4^{12;21}(\underset{\rightarrow}{\boldsymbol{\beta}})
\end{aligned} \tag{197}$$

Coupled equations for $R_0(\underset{\rightarrow}{\beta}), R_2^{i;j}(\underset{\rightarrow}{\beta})$ and $R_4^{12;21}(\underset{\rightarrow}{\beta})$ are obtained which only involve Ω and Δ. These are as follows.

The *second order* equations are

$$\frac{\partial}{\partial t}\left(R_2^{1;1}(\underset{\rightarrow}{\boldsymbol{\beta}}) - R_0(\underset{\rightarrow}{\boldsymbol{\beta}})\right) = +\frac{1}{2}i\,\Omega\,\left(\beta\,R_2^{2;1}(\underset{\rightarrow}{\boldsymbol{\beta}}) - \beta^+\,R_2^{1;2}(\underset{\rightarrow}{\boldsymbol{\beta}})\right) \tag{198}$$

$$\begin{aligned}
\frac{\partial}{\partial t}R_2^{1;2}(\underset{\rightarrow}{\boldsymbol{\beta}}) &= -i\,\Delta\,R_2^{1;2}(\underset{\rightarrow}{\boldsymbol{\beta}}) \\
&\quad -\frac{1}{2}i\,\Omega\,\beta\,\left(\left\{R_2^{1;1}(\underset{\rightarrow}{\boldsymbol{\beta}}) - R_0(\underset{\rightarrow}{\boldsymbol{\beta}})\right\} - \left\{R_2^{2;2}(\underset{\rightarrow}{\boldsymbol{\beta}}) - R_0(\underset{\rightarrow}{\boldsymbol{\beta}})\right\}\right) \\
&\quad +\frac{1}{2}i\,\Omega\left(\frac{\partial}{\partial\beta^+}\left\{R_2^{1;1}(\underset{\rightarrow}{\boldsymbol{\beta}}) - R_0(\underset{\rightarrow}{\boldsymbol{\beta}})\right\}\right)
\end{aligned} \tag{199}$$

$$\begin{aligned}
\frac{\partial}{\partial t}R_2^{2;1}(\underset{\rightarrow}{\boldsymbol{\beta}}) &= +i\,\Delta\,R_2^{2;1}(\underset{\rightarrow}{\boldsymbol{\beta}}) \\
&\quad +\frac{1}{2}i\,\Omega\,\beta^+\,\left(\left\{R_2^{1;1}(\underset{\rightarrow}{\boldsymbol{\beta}}) - R_0(\underset{\rightarrow}{\boldsymbol{\beta}})\right\} - \left\{R_2^{2;2}(\underset{\rightarrow}{\boldsymbol{\beta}}) - R_0(\underset{\rightarrow}{\boldsymbol{\beta}})\right\}\right) \\
&\quad -\frac{1}{2}i\,\Omega\,\left(\frac{\partial}{\partial\beta}\left\{R_2^{1;1}(\underset{\rightarrow}{\boldsymbol{\beta}}) - R_0(\underset{\rightarrow}{\boldsymbol{\beta}})\right\}\right)
\end{aligned} \tag{200}$$

$$\begin{aligned}
\frac{\partial}{\partial t}\left(R_2^{2;2}(\underset{\rightarrow}{\boldsymbol{\beta}}) - R_0(\underset{\rightarrow}{\boldsymbol{\beta}})\right) &= +\frac{1}{2}i\,\Omega\,\left(\beta^+\,R_2^{1;2}(\underset{\rightarrow}{\boldsymbol{\beta}}) - \beta\,R_2^{2;1}(\underset{\rightarrow}{\boldsymbol{\beta}})\right) \\
&\quad +\frac{1}{2}i\,\Omega\left(-\frac{\partial}{\partial\beta}R_2^{1;2}(\underset{\rightarrow}{\boldsymbol{\beta}}) + \frac{\partial}{\partial\beta^+}R_2^{2;1}(\underset{\rightarrow}{\boldsymbol{\beta}})\right)
\end{aligned} \tag{201}$$

The *zeroth* and *fourth order* equations are

$$\frac{\partial}{\partial t}R_0(\underset{\rightarrow}{\boldsymbol{\beta}}) = 0 \tag{202}$$

$$\frac{\partial}{\partial t}R_4^{12;21}(\underset{\rightarrow}{\boldsymbol{\beta}}) = -\frac{1}{2}i\,\Omega\left(\frac{\partial}{\partial\beta}R_2^{1;2}(\underset{\rightarrow}{\boldsymbol{\beta}}) - \frac{\partial}{\partial\beta^+}R_2^{2;1}(\underset{\rightarrow}{\boldsymbol{\beta}})\right) \tag{203}$$

10.4 3.4 Solution for Distribution Function

The solution to the coupled equation for the rotating distribution function coefficients is found using the ansatz

$$R_2^{i;j}(\underrightarrow{\boldsymbol{\beta}}) = \exp(\beta\beta^+)\ \Phi_i^*(\beta^+)\,\Phi_j(\beta) \tag{204}$$

where the $\Phi_i^*(\beta^+)$ are functions of the β^+ and the $\Phi_i(\beta)$ are functions of the β, and where $\Phi_i(\beta)$ satisfy the coupled equations

$$\begin{aligned}\frac{\partial}{\partial t}\Phi_1(\beta) &= \frac{1}{2}i\Delta\,\Phi_1(\beta) - \frac{1}{2}i\Omega\left(\frac{\partial}{\partial\beta}\right)\Phi_2(\beta)\\ \frac{\partial}{\partial t}\Phi_2(\beta) &= -\frac{1}{2}i\Delta\,\Phi_2(\beta) - \frac{1}{2}i\Omega\,\beta\,\Phi_1(\beta)\end{aligned} \tag{205}$$

This ansatz is consistent with the original equations (198) -(201) for the $R_2^{i;j}(\underrightarrow{\boldsymbol{\beta}})$. As indicated previously we set $R_0(\underrightarrow{\boldsymbol{\beta}})) = 0$ for the one atom case.

Following a similar proceedure as for the canonical distribution function we find a solution for the $\Phi_i(\beta)$ that is single valued and not divergent at the origin which is of the form

$$\Phi_1(\beta) = \beta^m\left(A_m\cos\frac{1}{2}\nu_m t + B_m\sin\frac{1}{2}\nu_m t\right) \tag{206}$$

where

$$\nu_m = \sqrt{-m\,\Omega^2 + \Delta^2} \qquad m = 0, 1, 2, .. \tag{207}$$

However these are *not* the frequenies associated with population and coherence oscillations in the one atom Jaynes-Cummings model. In fact for any detuning and Rabi frequency the quantity ν_m becomes purely imaginary when m is large, leading to solutions for $\Phi_1(\beta)$ and $\Phi_2(\beta)$ involving *hyperbolic functions*. For example with zero detuning we have

$$\Phi_1(\beta) = \beta^m\left(A_m\cosh\frac{1}{2}\sqrt{m}\Omega t - iB_m\sinh\frac{1}{2}\sqrt{m}\Omega t\right) \tag{208}$$

These solutions *diverge* as t becomes large. This indicates that the standard distribution function obtained from the Fokker-Planck equation via the standard correspondence rules becomes infinite for large time. The corresponding solution for $\Phi_2(\beta)$ is obtained from Eq.(205) and thus

$$\Phi_2(\beta) = i\ \beta^{(m-1)}\left(\cos\frac{1}{2}\nu_m t\left\{\frac{\nu_m B_m + i\Delta A_m}{\Omega}\right\} + \sin\frac{1}{2}\nu_m t\left\{\frac{-\nu_m A_m + i\Delta B_m}{\Omega}\right\}\right) \tag{209}$$

However, a solution with $m = 0$ leads to a singular β^{-1} behaviour, so it follows that m is restricted to the positive integers. Also, as the ansatz equations are

linear the general solution is a sum of terms with differing m so that we finally have the solution in the form

$$\begin{aligned}\Phi_1(\beta) &= \sum_{m=1} \beta^m \left(A_m \cos \frac{1}{2}\nu_m t + B_m \sin \frac{1}{2}\nu_m t \right) \\ \Phi_2(\beta) &= i \sum_{m=1} \beta^{(m-1)} \left(\cos \frac{1}{2}\nu_m t \left\{ \frac{\nu_m B_m + i\Delta A_m}{\Omega} \right\} + \sin \frac{1}{2}\nu_m t \left\{ \frac{-\nu_m A_m + i\Delta B_m}{\Omega} \right\} \right)\end{aligned} \tag{210}$$

Overall, this solution in (204) and (210) for the distribution function is unsatisfactory. Not only does it diverge for large t but also the distribution function found diverges at all times as β, β^+ become large due to the $\exp(\beta\beta^+)$ factor. This contradicts the requirement in deriving the Fokker-Planck equation that the distribution function goes to zero for large β, β^+. The conclusion is that the Fokker-Planck equation (190) obtained via the standard correspondence rules must be invalid, since if it was valid then the general solution to it that we have found would have produced a distribution function that was not divergent either in phase space or in time. There is also the question of how to choose the quantities A_n and B_n so that a valid overall distribution function at time $t = 0$ is obtained. The form of the initial distribution function is not known, since the canonical distribution function is not being used in this case where the Fokker-Planck equation is based on the standard correspondence rules. Of course the only requirement is that the choice of distribution function gives the correct characteristic function - the latter being uniquely determined from the density operator. Since the characteristic function is obtained via the phase space integral of Eq. (49) it is clear that there is a problem - the bosonic integral of the distribution function will not converge due to the $\exp(\beta\beta^+)$ factor. No matter how the quantities A_n and B_n are chosen no valid positive P distribution function can therefore be found via the solution given by (210) and (204).