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Finite Element Convergence for the Joule Heating Problem with Mixed Boundary Conditions

Max Jensen · Axel Målqvist

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Abstract We prove strong convergence of conforming finite element approximations to the stationary Joule heating problem with mixed boundary conditions on Lipschitz domains in three spatial dimensions. We show optimal global regularity estimates on creased domains and prove *a priori* and *a posteriori* bounds for shape regular meshes.

Keywords: Joule heating problem, thermistors, *a posteriori* error analysis, *a priori* error analysis, finite element method.

AMS Classifications: 65N30, 35J60.

1 Introduction

The stationary Joule heating problem is a two way coupled system of non-linear elliptic partial differential equations modelling the heat and electrical potential in a body. The electrical current acts as a heat source in a resistive material while the temperature feeds back to the electrical potential through the electrical conductivity. Joule heating is important in many micro-electromechanical systems, where the effect is used to achieve very exact positioning at the micro scale, e.g. [15]. The Joule heating problem is also studied for the design of semiconductors, in particular in the setting of thermistors. In applications boundary conditions of mixed type are typically used.

The main difficulty in proving the existence of finite energy solutions to the Joule heating problem is that, given a finite energy potential, the source

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term of the heat equation is in general only in L^1 , which means that the usual variational framework is not directly available. This issue has been studied in [7,13,4], for Dirichlet boundary conditions, and later in [8,17], for mixed boundary conditions. Multiplicity of solutions and stability was studied in [9]. Similar questions have also been raised for the time dependent case, see for instance [2,27,22].

There have been several works on the numerical solution of the Joule heating problems in recent years. For the steady state formulation both conforming and non-conforming finite element methods have been studied using homogeneous Dirichlet boundary conditions [30,31] and homogeneous mixed boundary conditions [29]. Under assumption of sufficient regularity of the solution and small data, *a priori* error bounds have been derived with convergence rates. There have also been parallel investigations into numerical methods for the time dependent Joule heating problem, see e.g. [11,1]. The assumption on small data can here be avoided since the Grönwall lemma is available. An *a posteriori* error bound for a time dependent obstacle thermistor problem is presented in [3].

In this paper we prove the strong convergence (of subsequences in case of non-unique exact solutions) of Galerkin approximations to finite energy solutions of the Joule heating problem in three dimensions with mixed boundary conditions, using only very mild assumptions on the computational domain and the data. The analysis covers, in particular, conforming finite element approximations. To this end we introduce a truncation operator in the approximate potential without affecting the consistency of the method. Thereby we avoid the assumption of L^∞ bounds on the discrete potential solution, independent of the mesh size, which are used in [16]. These L^∞ bounds are very difficult to realize in practice in three spatial dimensions. They also impose restrictions on the computational meshes as well as the order of convergence of the method. Under the assumption of a so-called *creased* domain together with a sufficiently weak temperature dependency in the electrical conductivity we also prove optimal global regularity estimates together with local estimates guaranteeing smooth solutions away from the boundary given smooth data. We further prove *a priori* and *a posteriori* error bounds for conforming finite element approximations on shape regular meshes. In our analysis the small data assumption relaxes as the coupling of the equations weakens.

The paper is organized as follows. In Section 2 we introduce the stationary Joule heating problem with mixed boundary conditions. In Section 3 we study the continuity properties of the differential operator to show the convergence of Galerkin approximations to finite energy solutions. In Section 4 study the global and interior regularity of solutions on creased domains. Finally, in Section 5 we derive optimal *a priori* and *a posteriori* error bounds for h-adaptive conforming finite element approximations to the Joule heating problem.

2 The Statement of the Stationary Problem

Let Ω be a bounded Lipschitz domain in \mathbb{R}^3 . Let D_ϕ and D_u be subsets of $\partial\Omega$, whose boundaries ∂D_ϕ and ∂D_u are Lipschitz regular manifolds and set $N_\phi := \partial\Omega \setminus \overline{D_\phi}$ and $R_u := \partial\Omega \setminus \overline{D_u}$. We shall impose Dirichlet boundary conditions for ϕ and u on D_ϕ and D_u , Neumann conditions for ϕ on N_ϕ and Robin conditions for u on R_u .

The scale of Sobolev spaces is denoted by W_s^p . Let, for $s > 1/p$,

$$W_s^p(\Omega; D_\phi) := \{v \in W_s^p(\Omega) : v|_{D_\phi} = 0\}.$$

Define $W_s^p(\Omega; D_u)$ analogously and denote W_1^2 spaces also with H^1 .

Assume for the Dirichlet data that $g_\phi \in W_1^3(\overline{\Omega}) \cap L^\infty(\overline{\Omega})$ and $g_u \in W_{1/2}^2(\partial\Omega)$ and for the Robin data that $h \in W_{-1/2}^2(\partial\Omega)$. Let $\sigma \in C^1(\mathbb{R})$ be bounded from below by a positive $\sigma_\circ \in \mathbb{R}$ and from above by $\sigma^\circ \in \mathbb{R}$ and let $\kappa \in L^\infty(R_u)$ be non-negative. Assume that there are the Poincaré-Friedrichs inequalities

$$\begin{aligned} \|\psi\|_{L^2(\Omega)} &\lesssim \|\nabla\psi\|_{L^2(\Omega, \mathbb{R}^3)} & \forall \psi \in W_1^2(\Omega; D_\phi), \\ \|w\|_{L^2(\Omega)} &\lesssim \|\nabla w\|_{L^2(\Omega, \mathbb{R}^3)} + \|\sqrt{\kappa}w\|_{L^2(R_u)} & \forall w \in W_1^2(\Omega; D_u). \end{aligned} \quad (2.1)$$

Allow $D_u = \emptyset$ provided the Poincaré-Friedrichs inequality remains valid.

The strong formulation of the Joule heating problem is to find $\phi \in W_2^\infty(\Omega)$ and $u \in W_2^\infty(\Omega)$ such that

$$\begin{aligned} -\nabla \cdot (\sigma(u)\nabla\phi) &= 0 & \Leftrightarrow & & -\sigma(u)\Delta u &= \nabla\sigma(u) \cdot \nabla\phi, \\ -\Delta u - \sigma(u)|\nabla\phi|^2 &= 0 \end{aligned} \quad (2.2)$$

with the Dirichlet conditions $\phi|_{D_\phi} = g_\phi$, $u|_{D_u} = g_u$ and the natural boundary conditions $\partial_\nu\phi = 0$ on N_ϕ and $\kappa u + \partial_\nu u = h$ on R_u with the outward unit normal ν .

Remark 2.1 For some applications the Lipschitz assumption on Ω is too restrictive, a good examples being geometries which locally resemble the two-brick domain. We point out that Theorem 3.1 remains valid for domains for which the usual Sobolev embedding holds, a trace operator is available and integration-by-parts can be carried out. For example, see [20] for more information in this direction. For Theorem 4.1, however, the Lipschitz assumption is an essential part of the definition of creased domains.

2.1 The Weak Formulation of the Stationary Problem

A weak solution of the stationary Joule heating problem is a

$$(\phi, u) = (g_\phi + \tilde{\phi}, g_u + \tilde{u}) \in H^1(\Omega) \times H^1(\Omega)$$

such that $\tilde{\phi} \in H^1(\Omega, D_\phi)$, $\tilde{u} \in H^1(\Omega, D_u)$ and

$$\left. \begin{aligned} \langle \sigma(u) \nabla \phi, \nabla \psi \rangle &= 0, \\ \langle \nabla u, \nabla w \rangle + \langle \kappa u, w \rangle_{R_u} &= \langle \sigma(u) \nabla \phi \cdot \nabla \phi, w \rangle + \langle h, w \rangle_{R_u} \end{aligned} \right\} \quad (2.3)$$

for all $\psi \in H^1(\Omega; D_\phi)$ and $\forall w \in W_1^\infty(\Omega; D_u)$. Indeed, the choice of spaces ensures that $\sigma(u) \nabla \phi \cdot \nabla \phi \in L^1(\Omega)$ which guarantees that the second equation is meaningful for all $w \in W_1^\infty(\Omega; D_u)$.

Lemma 2.1 *If (ϕ, u) is a solution of (2.3) then*

$$g_\circ \leq \phi \leq g^\circ, \quad \text{with} \quad g^\circ = \max_{x \in D_\phi} g_\phi, \quad g_\circ = \min_{x \in D_\phi} g_\phi.$$

Proof Define $\chi = \max(0, \phi - g^\circ) \in H^1(\Omega; D_\phi)$. One can use χ as a test function in equation (2.3):

$$\begin{aligned} 0 &= \langle \sigma(u) \nabla \phi, \nabla \chi \rangle = \langle \sigma(u) \nabla (\phi - g^\circ), \nabla \chi \rangle \\ &= \int_{\text{supp}(\chi) \cap \Omega} \sigma(u) \nabla \chi \cdot \nabla \chi \, dx = \langle \sigma(u) \nabla \chi, \nabla \chi \rangle. \end{aligned}$$

Now use the Poincaré-Friedrichs inequality to get $\|\chi\|_{L^2(\Omega)} = 0$, so $\phi \leq g^\circ$. An analogous argument with g_\circ gives $\phi \geq g_\circ$. \square

Because of the maximum principle we may introduce an equivalent weak formulation which employs the cut-off functional

$$[f] := \min(\max(f + g_\phi, g_\circ), g^\circ) - g_\phi.$$

Then $g_\circ - g_\phi \leq [f] \leq g^\circ - g_\phi$ and $[\tilde{\phi}] = \tilde{\phi}$. This functional is essential in the proof of the convergence of Galerkin solutions without the need for a discrete maximum principle; a property desirable from the numerical point of view.

Lemma 2.2 *The set of functions which satisfy*

$$\left. \begin{aligned} \langle \sigma(u) \nabla \phi, \nabla \psi \rangle &= 0, \\ \langle \nabla u, \nabla w \rangle + \langle \kappa u, w \rangle_{R_u} &= \\ -\langle \sigma(u) [\tilde{\phi}] \nabla \phi, \nabla w \rangle + \langle \sigma(u) \nabla g_\phi \cdot \nabla \phi, w \rangle + \langle h, w \rangle_{R_u} \end{aligned} \right\} \quad (2.4)$$

for all $(\psi, w) \in H^1(\Omega; D_\phi) \times H^1(\Omega; D_u)$ is equal to the set of solutions of (2.3).

Proof The identity

$$\langle \sigma(u) \nabla \phi \cdot \nabla \phi, w \rangle = -\langle \sigma(u) \tilde{\phi} \nabla \phi, \nabla w \rangle + \langle \sigma(u) \nabla g_\phi \cdot \nabla \phi, w \rangle$$

follows from Lemma 1 in [17]. The cut-off functional may be used because of Lemma 2.1 above. The larger space of test functions does not change the set of weak solutions due to density and does not lead to infinite terms in (2.4). \square

We define the space $X := H^1(\Omega; D_\phi) \times H^1(\Omega; D_u)$ and the *affine* mapping

$$L : X \rightarrow X^*, (\tilde{\varphi}, \tilde{v}) \mapsto \left((\psi, w) \mapsto \begin{pmatrix} \langle \sigma_\circ \nabla \varphi, \nabla \psi \rangle \\ \langle \nabla v, \nabla w \rangle + \langle \kappa v, w \rangle_{R_u} \end{pmatrix} \right)$$

and the nonlinear mapping

$$N : X \rightarrow X^*, (\tilde{\varphi}, \tilde{v}) \mapsto \left((\psi, w) \mapsto \begin{pmatrix} \langle (\sigma(v) - \sigma_\circ) \nabla \varphi, \nabla \psi \rangle \\ \langle \sigma(v) [\tilde{\varphi}] \nabla \varphi, \nabla w \rangle - \langle \sigma(v) \nabla g_\phi \cdot \nabla \varphi, w \rangle \end{pmatrix} \right)$$

and the functional

$$b : X \rightarrow \mathbb{R}, (\psi, w) \mapsto \begin{pmatrix} 0 \\ \langle h, w \rangle_{R_u} \end{pmatrix}$$

where $\varphi = g_\phi + \tilde{\varphi}$ and $v = g_u + \tilde{v}$. Then equation (2.4) is in operator form

$$Lx + Nx = b \tag{2.5}$$

with $x = (\tilde{\varphi}, \tilde{u}) \in X$.

For $(\tilde{\varphi}, \tilde{v}), (\psi, w) \in X$ one has

$$\left. \begin{aligned} & N((\tilde{\varphi}, \tilde{v}), (\psi, w)) \\ & \leq \sigma^\circ \cdot \left(\|\nabla \tilde{\varphi}\|_{L^2(\Omega)} + \|\nabla g_\phi\|_{L^2(\Omega)} \right) \\ & \quad \cdot \left(\|\nabla \psi\|_{L^2(\Omega)} + (g^\circ - g_\circ) \|\nabla w\|_{L^2(\Omega)} + \|\nabla g_\phi\|_{L^3(\Omega)} \|w\|_{L^6(\Omega)} \right) \\ & \lesssim (1 + \|\nabla \tilde{\varphi}\|_{L^2(\Omega, \mathbb{R}^3)}) \|(\psi, w)\|_X, \end{aligned} \right\} \tag{2.6}$$

where we use $\|(\cdot, \cdot)\|_X$ to denote the natural norm in the product space X , in this case

$$\|(\psi, w)\|_X^2 = \|\psi\|_{H^1(\Omega)}^2 + \|w\|_{H^1(\Omega)}^2.$$

Throughout the text we adopt the notational convention that for a function \mathfrak{b} one understands $\tilde{\mathfrak{b}} = \mathfrak{b} - g_\phi$ if \mathfrak{b} is a Greek letter and $\tilde{\mathfrak{b}} = \mathfrak{b} - g_u$ if \mathfrak{b} is a Latin letter. We call $H^1(\Omega; D_\phi)$ the first and $H^1(\Omega; D_u)$ the second component of X . In this spirit we also refer, for example, to $\langle \sigma_\circ \nabla \varphi, \nabla \psi \rangle$ as the first component of L . Furthermore, we distinguish between ϕ , which is a solution, and φ , which is a generic trial function.

3 Existence and Convergence of Galerkin Approximations

Consider a hierarchical family of subspaces $\{X_n\}_{n \in \mathbb{N}} = \{P_n \times U_n\}_{n \in \mathbb{N}}$ whose union is dense in X . A Galerkin solution $x_n \in X_n$ of (2.5) is a solution of

$$\langle Lx_n + Nx_n, y \rangle = \langle b, y \rangle, \quad \forall y \in X_n. \tag{3.1}$$

Lemma 3.1 examines continuity properties of L and N .

Lemma 3.1 *Let $\{y_n\}_n = \{(\tilde{\varphi}_n, \tilde{v}_n)\}_n$ be a sequence in X and $y = (\tilde{\varphi}, \tilde{v}) \in X$ such that $\tilde{\varphi}_n \rightarrow \tilde{\varphi}, \tilde{v}_n \rightarrow \tilde{v}$ as $n \rightarrow \infty$. Then $Ly_n \rightharpoonup Ly$ weakly and $Ny_n \rightarrow Ny$ strongly in X^* .*

Proof Suppose there is a subsequence $\{v_{n(k)}\}_k$ and an $\varepsilon > 0$ such that

$$\|\sigma(v_{n(k)})\nabla\varphi_{n(k)} - \sigma(v)\nabla\varphi\|_{L^2(\Omega, \mathbb{R}^3)} > \varepsilon \quad \forall k \in \mathbb{N}. \quad (3.2)$$

The compactness of the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ and a corollary of the Riesz-Fischer theorem [21, p. 161] imply that there is a subsequence, also denoted $\{v_{n(k)}\}_k$, which converges pointwise almost everywhere. By possibly passing to another subsequence of indices we may also assume that $\{\nabla\varphi_{n(k)}\}_k$ converges pointwise almost everywhere. The sequence

$$\{\sigma(v_{n(k)})^2 \nabla\varphi_{n(k)} \cdot \nabla\varphi_{n(k)}\}_k$$

is bounded in each component by $(\sigma^\circ)^2 |\nabla\varphi_{n(k)}|^2$. From the dominated convergence theorem, in the form of (Royden, p. 270), it follows that the sequence $\{\sigma(v_{n(k)})\nabla\varphi_{n(k)}\}_k$ converges strongly in $L^2(\Omega, \mathbb{R}^3)$. Observe that almost everywhere the pointwise limit of $\{\sigma(v_{n(k)})\nabla\varphi_{n(k)}\}_k$ is $\sigma(v)\nabla\varphi$, contradicting (3.2). Therefore $Ly_n + Ny_n$ converges, indeed strongly, in the first component. It also follows that the terms

$$\begin{aligned} \sigma(v_n) [\tilde{\varphi}_n] \nabla\varphi_n &\in L^2(\Omega, \mathbb{R}^3), \\ \sigma(v_n) \nabla g_\phi \cdot \nabla\varphi_n &\in L^2(\Omega), \\ \kappa v_n &\in L^2(\partial\Omega) \end{aligned}$$

converge strongly as $n \rightarrow \infty$. Hence $\{Ly_n\}_n$ converges weakly and $\{Ny_n\}_n$ strongly to Ly and Ny in X^* , respectively:

$$\limsup_n \sup_{z \neq 0} \frac{\langle Ny_n - Ny, z \rangle}{\|z\|_X} = 0, \quad \forall z \in X^* : \lim_n \langle Ly_n - Ly, z \rangle = 0,$$

completing the proof. \square

The following lemma establishes a property of $L + N$ which is a variation of condition (S)₀; a concept introduced by Browder, see [5] or [28, IIB, p.583].

Lemma 3.2 *Let $y_n = \{(\tilde{\varphi}_n, \tilde{v}_n)\}_n$ be a sequence in X and $y = (\tilde{\varphi}, \tilde{v}) \in X$ such that*

$$y_n \rightharpoonup y, \quad (3.3)$$

$$Ly_n + Ny_n \rightharpoonup b, \quad (3.4)$$

$$\lim_n \langle Ly_n + Ny_n, (\tilde{\varphi}_n, 0) \rangle = \langle b, (\tilde{\varphi}, 0) \rangle, \quad (3.5)$$

$$\lim_n \langle Ly_n + Ny_n, (0, \tilde{v}_n) \rangle = \langle b, (0, \tilde{v}) \rangle. \quad (3.6)$$

Then $y_n \rightarrow y$ strongly.

Proof Adapting the argument of the proof of the previous lemma it follows analogously that $\{\sigma(v_n)\nabla\psi\}_n$ converges strongly in $L^2(\Omega, \mathbb{R}^3)$. Using the strong convergence in (*), one obtains

$$\begin{aligned}
0 &\leq \limsup_n \langle \sigma(v_n)\nabla(\tilde{\varphi} - \tilde{\varphi}_n), \nabla(\tilde{\varphi} - \tilde{\varphi}_n) \rangle \\
&= \limsup_n (\langle \sigma(v_n)\nabla\tilde{\varphi}, \nabla\tilde{\varphi} \rangle - 2\langle \sigma(v_n)\nabla\tilde{\varphi}_n, \nabla\tilde{\varphi} \rangle + \langle \sigma(v_n)\nabla\tilde{\varphi}_n, \nabla\tilde{\varphi}_n \rangle) \\
&\stackrel{(*)}{=} \limsup_n (\langle \sigma(v_n)\nabla\tilde{\varphi}_n, \nabla\tilde{\varphi} \rangle - 2\langle \sigma(v_n)\nabla\tilde{\varphi}_n, \nabla\tilde{\varphi} \rangle + \langle \sigma(v_n)\nabla\tilde{\varphi}_n, \nabla\tilde{\varphi}_n \rangle) \\
&\stackrel{(3.5)}{=} \limsup_n (-\langle \sigma(v_n)\nabla\tilde{\varphi}_n, \nabla\tilde{\varphi} \rangle + \langle b, (\tilde{\varphi}, 0) \rangle + \langle \sigma(v_n)\nabla g_\phi, \nabla(\tilde{\varphi}_n - \tilde{\varphi}) \rangle) \\
&\stackrel{(3.4)}{=} -\langle b, (\tilde{\varphi}, 0) \rangle + \langle b, (\tilde{\varphi}, 0) \rangle = 0.
\end{aligned}$$

Therefore φ_n converges strongly and Lemma 3.1 becomes available. Hence $\langle Ny_n, y_n \rangle \rightarrow \langle Ny, y \rangle$ and

$$\begin{aligned}
\langle b, y \rangle &\stackrel{(3.4)}{=} \lim_n \langle Ly_n + Ny_n, y \rangle = \lim_n \langle Ly_n, y \rangle + \langle Ny, y \rangle, \\
\langle b, y \rangle &\stackrel{(3.5), (3.6)}{=} \lim_n \langle Ly_n + Ny_n, y_n \rangle = \lim_n \langle Ly_n, y_n \rangle + \langle Ny, y \rangle.
\end{aligned}$$

The weak continuity of L implies that $\lim_n \langle Ly_n, y \rangle = \langle Ly, y \rangle$, cf. [10, p. 422]. Therefore

$$\begin{aligned}
0 &= \lim_n \langle Ly_n, y_n - y \rangle = \lim_n \langle Ly_n, y \rangle - 2\langle Ly_n, y \rangle + \langle Ly_n, y_n \rangle \\
&= \lim_n \langle Ly, y \rangle - 2\langle Ly_n, y \rangle + \langle Ly_n, y_n \rangle = \lim_n \langle L(y - y_n), y - y_n \rangle.
\end{aligned}$$

It follows from the coercivity of the linear part of L that $y_n \rightarrow y$ in X . \square

Let $T_n : X_n \rightarrow X_n$ be defined by $y_n = T\hat{y}_n$, where $y_n = (\tilde{\varphi}_n, \tilde{v}_n) \in X_n$ is given as the solution to

$$\langle Ly_n + N(\tilde{\varphi}_n, \hat{v}_n), (\psi, w) \rangle = \langle b, (\psi, w) \rangle, \quad (\psi, w) \in X_n \quad (3.7)$$

with $\hat{y}_n = (\hat{\varphi}_n, \hat{v}_n)$. Algorithmically an iteration with T corresponds to a method with the primary variable \hat{v}_n and the dummy variable $\hat{\varphi}_n$ as $\hat{\varphi}_n$ does not explicitly appear in the next step of the iteration.

Lemma 3.3 *There exists a radius r , independent of n , such that the range of T_n belongs to*

$$B_r := \{y \in X : \|y\|_X \leq r\}$$

for all $n \in \mathbb{N}$.

Proof Let $y_n = (\tilde{\varphi}_n, \tilde{v}_n) = T_n\hat{y} = T_n(\hat{\varphi}_n, \hat{v}_n)$. The first component of (3.7) gives, with $\psi = \tilde{\varphi}_n$, the identity $0 = \langle \sigma(\hat{v}_n)\nabla(\tilde{\varphi}_n + g_\phi), \nabla\tilde{\varphi}_n \rangle$. Thus, with the above Poincaré-Friedrichs inequality for $H^1(\Omega; D_\phi)$:

$$\|\tilde{\varphi}_n\|_{H^1(\Omega)}^2 \lesssim \langle \sigma(\hat{v}_n)\nabla\tilde{\varphi}_n, \nabla\tilde{\varphi}_n \rangle = -\langle \sigma(\hat{v}_n)\nabla g_\phi, \nabla\tilde{\varphi}_n \rangle.$$

The Cauchy-Schwarz inequality now gives $\|\tilde{\varphi}_n\|_{H^1(\Omega)} \lesssim \|\nabla g_\phi\|_{L^2(\Omega, \mathbb{R}^3)}$. Recall (2.6) with $(\psi, w) = (\tilde{\varphi}_n, \tilde{v}_n)$ and $(\tilde{\varphi}, \tilde{v}) = (\tilde{\varphi}_n, \hat{v}_n)$. The Cauchy-Schwarz inequality, (2.1) and the coercivity of the linear part of L give the boundedness of \tilde{v}_n . \square

Observe that the fixed points of T_n are exactly the Galerkin solutions in the sense of (3.1).

Lemma 3.4 *The mapping T_n has at least one fixed point x_n .*

Proof We have that $T_n : B_r \cap X_n \rightarrow B_r \cap X_n$. The First Lemma of Strang (Braess, 2007, p.106) implies that the Galerkin solution of a linear elliptic equation changes continuously in the H^1 -norm as the diffusion coefficient is varied in the L^∞ -norm. Therefore, looking at the first component in (3.7), $\tilde{\varphi}_n$ depends continuously on \hat{v}_n , taking the equivalence of norms in the finite-dimensional X_n into account. With $\tilde{\varphi}_n$ determined, \tilde{v}_n can be computed from $(\tilde{\varphi}_n, \tilde{v}_n) = L^{-1}(b - N(\tilde{\varphi}_n, \hat{v}_n))$. Lemma 3.1 showed a sequential continuity of property of L and N which guarantees that the finite-dimensional Galerkin restrictions $X_n \rightarrow X_n$ are continuous. Equally the Galerkin restriction $X_n \rightarrow X_n$ of the affine mapping L^{-1} is continuous. This means that T_n is a continuous map $T_n : B_r \cap X_n \rightarrow B_r \cap X_n$, so Brouwer's fixed point theorem gives the existence of a fixed point x_n . \square

It is a direct consequence that Galerkin solutions exist for all $n \in \mathbb{N}$ and (that at least one of them) are contained in B_r . The next theorem conceptually builds upon Proposition 27.4 in [28, vol. II B] where Lemma 3.2 is replaced by (S)₀.

Theorem 3.1 *There exists a subsequence of Galerkin solutions $\{x_{n(k)}\}_k = \{(\tilde{\varphi}_{n(k)}, \tilde{v}_{n(k)})\}_k$ and an $x = (\tilde{\varphi}, \tilde{v})$ in X such that $x_{n(k)} \rightarrow x$ strongly in X and x solves (2.5). If the solution x of (2.5) is unique then the whole sequence converges.*

Proof It follows from (3.1) that $\langle (L + N)x_n, y \rangle \rightarrow \langle b, y \rangle$ for all fixed $y \in X_m$, $m \in \mathbb{N}$. Observe that $L + N$ is a bounded operator, see (2.6) for N . Thus with $\{x_n\}_n$ also the sequence $\{(L + N)x_n\}_n$ is bounded. Consequently, $(L + N)x_n \rightharpoonup b$ in X^* as $n \rightarrow \infty$, see Proposition 21.26 (c),(f) in [28, vol. II A]. The sequence $\{x_n\}_n$ is bounded in the reflexive Banach space X . Thus there exists an $x \in X$ and a subsequence $\{x_{n(k)}\}_k$ with $x_{n(k)} \rightharpoonup x$ as $k \rightarrow \infty$. It follows from (3.1) that

$$\begin{aligned} \lim_k \langle Lx_{n(k)} + Nx_{n(k)}, (\tilde{\varphi}_{n(k)}, 0) \rangle &= \lim_k \langle b, (\tilde{\varphi}_{n(k)}, 0) \rangle = \langle b, (\tilde{\varphi}, 0) \rangle, \\ \lim_k \langle Lx_{n(k)} + Nx_{n(k)}, (0, \tilde{v}_{n(k)}) \rangle &= \lim_k \langle b, (0, \tilde{v}_{n(k)}) \rangle = \langle b, (0, \tilde{v}) \rangle. \end{aligned}$$

Therefore, it follows from Lemma 3.2 that $x_{n(k)} \rightarrow x$ as $k \rightarrow \infty$. The continuity established in Lemma 3.1 yields $(L + N)x_{n(k)} \rightarrow (L + N)x$ as $k \rightarrow \infty$ and therefore $(L + N)x = b$. If the exact solution is unique, then the Galerkin approximations can only have one accumulation point. \square

We remark that this convergence result applies, in particular, to conforming finite element methods.

4 Regularity

In this section we investigate how the regularity estimates for the Poisson problem with mixed boundary conditions, derived in [24], carry over to the Joule heating problem. These bounds are sharp in the Poisson setting. In general they are also sharp for the Joule problem—noting that equation (2.2) takes the form of Poisson’s equation when choosing a constant σ .

The underlying question is whether additional regularity can be gained if the type of the boundary conditions only changes at re-entrant corners:

(C) Ω is a creased domain.

For the full definition of creased domains we refer to [24]; here we only highlight that in the setting of the Joule heating problem the key conditions are that Ω is a bounded Lipschitz domain, that D_ϕ and D_u are open and non-empty and that ∂D_ϕ and ∂D_u are not re-entrant, meaning that the angles between D_ϕ and N_ϕ as well as between D_u and R_u are strictly less than π .

Let $\mathcal{H}_\varepsilon \subset \mathbb{R}^2$ be the interior of the polygon with the vertices

$$(0, 0), \quad (\varepsilon, 0), \quad \left(1, \frac{1}{2} - \varepsilon\right), \quad (1, 1), \quad (1 - \varepsilon, 1), \quad \left(0, \frac{1}{2} + \varepsilon\right).$$

In the statement of the following theorem we let $D \in \{D_\phi, D_u\}$ and $R \in \{N_\phi, R_u\}$. Also $1/p + 1/p' = 1$ and $B_s^{p,q}$ denotes the scale of Besov spaces.

Lemma 4.1 ([24]) *There exists an $\varepsilon = \varepsilon(\partial\Omega, D, R)$ in $(0, 1/2)$ such that Poisson’s equation is well-posed in the spaces*

$$\left. \begin{aligned} v &\in W_{s+\frac{1}{p}}^p(\Omega), & \Delta v &\in \left(W_{2-s-\frac{1}{p}}^{p'}(\Omega; D)\right)^*, \\ v|_D &= g \in B_s^{p,p}(D), & \partial_\nu v|_R &= h \in B_{s-1}^{p,p}(R), \end{aligned} \right\} \quad (4.1)$$

whenever $(s, 1/p) \in \mathcal{H}_\varepsilon$.

Consequently we assume for the remainder of this section that $g_\phi \in B_s^{p,p}(D_\phi)$, $g_u \in B_s^{p,p}(D_u)$ and $h \in B_{s-1}^{p,p}(R_u)$. We also choose

$$\varepsilon := \min\{\varepsilon(\partial\Omega, D_\phi, N_\phi), \varepsilon(\partial\Omega, D_u, R_u)\} \in (0, 1/2).$$

We aim to prove existence of solutions with the smoothness and Lebesgue indices

$$s = \frac{2 - \varepsilon}{2}, \quad t = \frac{8 + \varepsilon}{12}, \quad p = \frac{2}{1 - \varepsilon}, \quad q = \frac{12}{4 - \varepsilon}.$$

Then $(s, 1/p), (t, 1/q) \in \mathcal{H}_\varepsilon$ and

$$W_{t+\frac{1}{q}}^q(\Omega) = W_1^{\frac{12}{4-\varepsilon}}(\Omega) \subset C^{0, \frac{\varepsilon}{4}}(\Omega) \cap W_1^3(\Omega) \subset C(\overline{\Omega}) \cap W_1^3(\Omega) \quad (4.2)$$

and

$$W_{2-s-\frac{1}{p}}^{p'}(\Omega; D) = W_{\varepsilon+\frac{1}{2}}^{\frac{2}{1+\varepsilon}}(\Omega; D) \subset L^{\frac{6}{2+\varepsilon}}(\Omega) \subset L^3(\Omega) \quad (4.3)$$

due to the Sobolev embedding theorem. Note that $W_{2-s-\frac{1}{p}}^{p'}(\Omega)|_{\partial\Omega} = B_{\varepsilon/2}^{p',p'}(\partial\Omega)$ has the dual space $B_{-\varepsilon/2}^{p,p}(\partial\Omega) = B_{s-1}^{p,p}(\partial\Omega)$, cf. [24]. Also

$$W_{s+\frac{1}{p}}^p(\Omega) \subset W_1^{\frac{6}{2-\varepsilon}}(\Omega) \subset W_1^{\frac{12}{4-\varepsilon}}(\Omega) = W_{t+\frac{1}{q}}^q(\Omega).$$

Indeed the embedding of $W_{s+\frac{1}{p}}^p(\Omega)$ into $W_{t+\frac{1}{q}}^q(\Omega)$ is compact for $\varepsilon \in (0, 1/2)$ since

$$\left(s + \frac{1}{p}\right) - \left(t + \frac{1}{q}\right) = \frac{1}{2} - \varepsilon. \quad (4.4)$$

We set

$$Y = W_{t+\frac{1}{q}}^q(\Omega; D_\phi) \times W_{t+\frac{1}{q}}^q(\Omega; D_u), \quad Z = W_{s+\frac{1}{p}}^p(\Omega; D_\phi) \times W_{s+\frac{1}{p}}^p(\Omega; D_u).$$

Corollary 4.1 *Let s, p and $\kappa \in L^\infty(R)$ be as above. There exists an $\varepsilon \in (0, 1/2)$ such that Poisson's equation is well-posed in the spaces*

$$\left. \begin{aligned} v &\in W_{s+\frac{1}{p}}^p(\Omega), & \Delta v &\in \left(W_{2-s-\frac{1}{p}}^{p'}(\Omega; D)\right)^*, \\ v|_D &\in B_s^{p,p}(D), & \kappa v + \partial_\nu v|_R &\in B_{s-1}^{p,p}(R) \end{aligned} \right\} \quad (4.5)$$

whenever $(s, 1/p) \in \mathcal{H}_\varepsilon$.

Proof A standard energy argument ensures that, given Dirichlet boundary conditions on D with data in $B_s^{p,p}(D)$ and Robin boundary conditions with data in $B_{s-1}^{p,p}(R)$, the mixed Poisson problem has a unique solution v in $H^1(\Omega)$. This function v is also the unique solution of the system with Neumann boundary conditions

$$h - \kappa v \in L^2(R) \subset \left(B_{\frac{\varepsilon}{2}, 0}^{\frac{2}{1+\varepsilon}, \frac{2}{1+\varepsilon}}(R)\right)^* = \left(B_{1-s, 0}^{p', p'}(R)\right)^* = B_{s-1}^{p,p}(R) \quad \text{on } R,$$

giving the required regularity by Lemma 4.1. The result now follows since $\|\kappa v\|_{B_{s-1}^{p,p}(R)} \lesssim \|v\|_{H^1(\Omega; D)}$. \square

Now, motivated by (2.2), we consider the following modified weak formulation for $(\tilde{\phi}, \tilde{u}) \in Y$

$$\left. \begin{aligned} \langle \nabla \phi, \nabla \psi \rangle &= \left\langle \frac{\sigma'(u)}{\sigma(u)} \nabla u \cdot \nabla \phi, \psi \right\rangle & \forall \psi, \\ \langle \nabla u, \nabla w \rangle + \langle \kappa u, w \rangle_{R_u} &= \langle \sigma(u) \nabla \phi \cdot \nabla \phi, w \rangle + \langle h, w \rangle_{R_u} & \forall w, \end{aligned} \right\} \quad (4.6)$$

with $(\psi, w) \in Z$. Notice that again $\tilde{\phi} = \phi - g_\phi$ and $\tilde{u} = u - g_u$.

Lemma 4.2 *The set of solutions of (2.3) which belong to Y is equal to the set of solutions of (4.6). Moreover, if $(\tilde{\phi}, \tilde{u}) \in X$ solves (2.3) then it solves (4.6) for all $(\psi, w) \in W_1^\infty(\Omega; D_\phi) \times W_1^\infty(\Omega; D_u)$.*

Proof One only needs to consider the first equation of (2.3) and (4.6). Let $(\tilde{\phi}, \tilde{u}) \in Y$ solve (4.6). Let $\{\vartheta_\varepsilon\}_\varepsilon$ be an approximate identity and $\sigma_\varepsilon := \sigma(u) * \vartheta_\varepsilon$ and $\psi_\varepsilon := \psi \sigma_\varepsilon$. Then

$$\int_{\Omega} \nabla \phi \cdot \nabla \psi \, dx \stackrel{(4.6)}{=} \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \left(\frac{\nabla \sigma(u)}{\sigma_\varepsilon} \cdot \nabla \phi \right) \psi \, dx,$$

as $\sigma_\varepsilon \rightarrow \sigma(u)$ in $L^\infty(\Omega)$. Now choosing ψ_ε as test function

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \nabla \phi \cdot \nabla \psi_\varepsilon \, dx = \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \left(\frac{\nabla \sigma(u)}{\sigma_\varepsilon} \cdot \nabla \phi \right) \psi_\varepsilon \, dx = \int_{\Omega} (\nabla \sigma(u) \cdot \nabla \phi) \psi \, dx,$$

but also

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \nabla \phi \cdot \nabla \psi_\varepsilon \, dx &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \sigma_\varepsilon \nabla \phi \cdot \nabla \psi \, dx + \int_{\Omega} (\nabla \phi \cdot \nabla \sigma_\varepsilon) \psi \, dx \\ &= \int_{\Omega} \sigma(u) \nabla \phi \cdot \nabla \psi \, dx + \int_{\Omega} (\nabla \sigma(u) \cdot \nabla \phi) \psi \, dx. \end{aligned}$$

Subtraction shows that $(\tilde{\phi}, \tilde{u}) \in Y$ solves (2.3). The other direction follows from re-arranging the above identities; the test spaces Z and $W_1^\infty(\Omega; D_\phi) \times W_1^\infty(\Omega; D_u)$ have to be adapted to the size of trial function spaces. \square

It is convenient to define the operators

$$\begin{aligned} S_1 : Y &\rightarrow W_\phi^*, & (\tilde{\varphi}, \tilde{v}) &\mapsto (\psi \mapsto \langle \frac{\sigma'(v)}{\sigma(v)} \nabla v \cdot \nabla \varphi, \psi \rangle), \\ S_2 : Z_\phi &\rightarrow W_\phi^*, & \tilde{\varphi} &\mapsto (\psi \mapsto \langle \nabla \varphi, \nabla \psi \rangle), \\ S_3 : Y &\rightarrow W_u^*, & (\tilde{\varphi}, \tilde{v}) &\mapsto (w \mapsto \langle \sigma(v) \nabla \varphi \cdot \nabla \varphi, w \rangle + \langle h, w \rangle_{R_u}), \\ S_4 : Z_u &\rightarrow W_u^*, & \tilde{v} &\mapsto (w \mapsto \langle \nabla v, \nabla w \rangle + \langle \kappa v, w \rangle_{R_u}), \end{aligned}$$

where

$$\begin{aligned} Z_\phi &= W_{s+\frac{1}{p}}^p(\Omega; D_\phi), & Z_u &= W_{s+\frac{1}{p}}^p(\Omega; D_u), \\ W_\phi &= W_{2-s-\frac{1}{p}}^{p'}(\Omega; D_\phi), & W_u &= W_{2-s-\frac{1}{p}}^{p'}(\Omega; D_u). \end{aligned}$$

The notation indicates that operators map into dual spaces with the associated test functions ψ and v . Let I be the identity map and

$$S := (I, S_4^{-1} \circ S_3) \circ (S_2^{-1} \circ S_1, I). \quad (4.7)$$

Given an initial pair (φ, v) , S_1 returns in the first component the right-hand side of (4.6) and thus $S_2^{-1} \circ S_1$ gives an update of the first component. This, together with an unchanged v , is passed into S_3 and then S_4^{-1} to return first an updated right-hand side and then an updated second component.

Lemma 4.3 *The operators S maps continuously into Z .*

Proof Lemma 4.1 ensures that S_2^{-1} and S_4^{-1} map continuously into $W_{s+\frac{1}{p}}^p(\Omega)$. Notice that, due to (4.3),

$$\frac{\nabla\sigma(v)}{\sigma(v)} \cdot \nabla\varphi, \quad \sigma(v)|\nabla\varphi|^2 \in L^{\frac{6}{4-\varepsilon}}(\Omega) = \left(L^{\frac{6}{2+\varepsilon}}(\Omega)\right)^* \subset \left(W_{2-s-\frac{1}{p}}^{p'}(\Omega)\right)^*.$$

As $h \in B_{s-1}^{p,p}(R_u)$, $\kappa \in L^\infty(R_u)$, $v \in C(\overline{\Omega})$ also $\kappa v, h \in \left(W_{2-s-\frac{1}{p}}^{p'}(\Omega)\right)^*$. Therefore S_1 and S_3 map into W_ϕ^* and W_u^* . \square

Lemma 4.3 gives access to Schauder's fixed point argument provided $\frac{\nabla\sigma}{\sigma}$ is not too large in relation to other parameters of the problem. Let C_1 to C_4 be the embedding constants of

$$\begin{aligned} W_{2-s-\frac{1}{p}}^{p'}(\Omega; D_u) &\hookrightarrow L^{\frac{12}{8+\varepsilon}}(\Omega), & B_s^{p,p}(D_\phi) &\hookrightarrow W_{2-s-\frac{1}{p}}^{p'}(\Omega)^*, \\ W_{s+\frac{1}{p}}^p(\Omega) &\hookrightarrow W_{t+\frac{1}{q}}^q(\Omega), & B_s^{p,p}(D_u) &\hookrightarrow W_{2-s-\frac{1}{p}}^{p'}(\Omega)^*, \end{aligned}$$

respectively. By abuse of notation we denote by $\|S_2^{-1}\|$ and $\|S_4^{-1}\|$ the operator norms of the linear parts of S_2^{-1} and S_4^{-1} ; that is of S_2^{-1} and S_4^{-1} if g_ϕ and g_u were 0.

Lemma 4.4 *There exists a positive constant*

$$C_* = C_*\left(\{C_i\}_{i=1}^4, \|S_2^{-1}\|, \|S_4^{-1}\|, \sigma^\circ, \|g_\phi\|_{B_s^{p,p}}, \|g_u\|_{B_s^{p,p}}, \|h\|_{B_{s-1}^{p,p}}\right) \quad (4.8)$$

such that whenever $\|\frac{\sigma'}{\sigma}\|_{L^\infty(\mathbb{R})} \leq C_*$ then there is a ball $B \subset Y$ such that S maps B into B .

Proof By Hölder's inequality

$$\begin{aligned} \left\| \frac{\nabla\sigma(v)}{\sigma(v)} \cdot \nabla\varphi \right\|_{W_{2-s-\frac{1}{p}}^{p'}(\Omega; D_\phi)^*} &= \sup_{\psi \neq 0} \frac{\langle \frac{\sigma'(v)}{\sigma(v)} \nabla v \cdot \nabla\varphi, \psi \rangle}{\|\psi\|_{W_{2-s-\frac{1}{p}}^{p'}(\Omega; D_\phi)}} \\ &\leq C_1 \left\| \frac{\sigma'}{\sigma} \right\|_{L^\infty(\mathbb{R})} \|\nabla v\|_{L^{\frac{12}{4-\varepsilon}}(\Omega)} \|\nabla\varphi\|_{L^{\frac{12}{4-\varepsilon}}(\Omega)} \\ &\lesssim C_1 \left\| \frac{\sigma'}{\sigma} \right\|_{L^\infty(\mathbb{R})} \left(\|\tilde{\varphi}, \tilde{v}\|_Y^2 + C_2 \|g_\phi\|_{B_s^{p,p}(D_\phi)}^2 + C_4 \|g_u\|_{B_s^{p,p}(D_u)}^2 \right). \end{aligned}$$

Similarly,

$$\begin{aligned} \left\| \sigma(v) \nabla\varphi \cdot \nabla\varphi \right\|_{W_{2-s-\frac{1}{p}}^{p'}(\Omega; D_u)^*} &= \sup_{w \neq 0} \frac{\langle \sigma(v) \nabla\varphi \cdot \nabla\varphi, w \rangle}{\|w\|_{W_{2-s-\frac{1}{p}}^{p'}(\Omega; D_u)}} \\ &\lesssim C_1 \sigma^\circ \left(\|\nabla\tilde{\varphi}\|_{L^{\frac{12}{4-\varepsilon}}(\Omega)}^2 + C_2 \|g_\phi\|_{B_s^{p,p}(D_\phi)}^2 \right). \end{aligned} \quad (4.9)$$

We need to bound the first component $I \circ S_2^{-1} \circ S_1 = S_2^{-1} \circ S_1$ and the second component $S_4^{-1} \circ S_3 \circ (S_2^{-1} \circ S_1, I)$ of S , cf. (4.7). Suppose that $(\tilde{\varphi}, \tilde{v})$

are contained in the ball $B = \{y \in Y : \|y\|_Y \leq r\}$. As S_2 is an invertible affine function, there are generic constants C with a parameter dependence as indicated in (4.8) such that

$$\|(S_2^{-1} \circ S_1)(\tilde{\varphi}, \tilde{v})\|_{W_{t+\frac{1}{q}}^q(\Omega; D_\phi)} \leq C \left\| \frac{\sigma'}{\sigma} \right\|_{L^\infty(\mathbb{R})} r^2 + C. \quad (4.10)$$

For the second component notice that the right-hand side of (4.9) only depends on φ and not v

$$\|(S_4^{-1} \circ S_3 \circ (S_2^{-1} \circ S_1, I))(\tilde{\varphi}, \tilde{v})\|_{W_{t+\frac{1}{q}}^q(\Omega; D_u)} \leq C \underbrace{\left(C \left\| \frac{\sigma'}{\sigma} \right\|_{L^\infty(\mathbb{R})} r^2 + C \right)^2}_{\text{owing to (4.10)}} + C.$$

Therefore $\|(S_2^{-1} \circ S_1)(\tilde{\varphi}, \tilde{v})\|$ is bounded by a quartic polynomial of the form

$$C \left\| \frac{\sigma'}{\sigma} \right\|_{L^\infty(\mathbb{R})}^2 r^4 + C \left\| \frac{\sigma'}{\sigma} \right\|_{L^\infty(\mathbb{R})} r^2 + C.$$

At radii where it intersects the first diagonal $r \mapsto r$ the operator S maps B into B . The existence of such an intersection point is guaranteed if $\left\| \frac{\sigma'}{\sigma} \right\|_{L^\infty(\mathbb{R})}$ is sufficiently small. \square

The above lemma is consistent with the analysis of the linear problem with a constant σ in the sense that a suitable C_* can always be found as $\frac{\sigma'}{\sigma}$ becomes small. We now turn to Schauder's fixed point theorem to obtain the existence of solutions in Z .

Theorem 4.1 *There exists a positive constant*

$$C_* = C_* \left(\{C_i\}_{i=1}^4, \|S_2^{-1}\|, \|S_4^{-1}\|, \sigma^\circ, \|g_\phi\|_{B_s^{p,p}(D_\phi)}, \|g_u\|_{B_s^{p,p}(D_u)}, \|h\|_{B_{s-1}^{p,p}(R_u)} \right)$$

such that whenever $\left\| \frac{\sigma'}{\sigma} \right\|_{L^\infty(\mathbb{R})} \leq C_*$ then (4.6) has a solution in Z .

Proof The compactness of S follows from Lemma 4.3 and (4.4). Now the result is a consequence of Lemma 4.4 and Schauder's fixed point theorem. \square

While the global regularity estimate in Y is sharp in the setting of creased domains (up to the distance of $(t, 1/q)$ to the boundary of \mathcal{H}_ε), as a comparison with the Poisson problem shows, more regularity is seen away from the boundary. For the next theorem we assume $(\tilde{\varphi}, \tilde{u}) \in Y$; however, it is not relevant whether this is established with the above fixed-point argument or otherwise.

Theorem 4.2 *Let Ω_0 be a relatively compact Lipschitz domain in Ω : $\Omega_0 \Subset \Omega$. Let $(\tilde{\varphi}, \tilde{u}) \in Y$ be a solution of (4.6). Then $\tilde{\varphi}, \tilde{u} \in W_2^s(\Omega_0)$ for all $s \in (1, \infty)$. If $\sigma \in C_0^\infty(\mathbb{R})$ then $\tilde{\varphi}, \tilde{u} \in C^\infty(\overline{\Omega_0})$.*

Proof Let $\{\Omega_i\}_{i \in \mathbb{N}}$ and Ω_∞ be smooth domains with $\Omega_i \Subset \Omega_{i+1}$ and $\Omega_i \subset \Omega_\infty \Subset \Omega$ for all $i \in \mathbb{N}$. Without loss of generality we may assume that the boundary data g_ϕ and g_u have extensions from the boundary onto Ω such that $g_\phi, g_u \in C^\infty(\Omega_\infty)$. Fix $i \in \mathbb{N} \setminus \{0\}$. Let ζ_i be a smooth function $\Omega_i \rightarrow [0, 1]$ such that $\zeta_i|_{\partial\Omega_i} = 0$ and $\zeta_i|_{\Omega_{i-1}} = 1$. Then $\zeta_i \tilde{\phi}$ and $\zeta_i \tilde{u}$ solve Poisson's problem on Ω_i with homogeneous Dirichlet boundary conditions and the right-hand sides

$$\begin{aligned} & \zeta_i \left(\frac{\sigma'(\tilde{u})}{\sigma(\tilde{u})} \nabla \tilde{u} \cdot \nabla \tilde{\phi} \right) - \zeta_i \Delta g_\phi - 2\nabla \zeta_i \cdot \nabla \tilde{\phi} - \tilde{\phi} \Delta \zeta_i, \\ & \zeta_i (\sigma(\tilde{u}) \nabla \tilde{\phi} \cdot \nabla \tilde{\phi}) - \zeta_i \Delta g_u - 2\nabla \zeta_i \cdot \nabla v - v \Delta \zeta_i \end{aligned}$$

in $L^{\frac{6}{4-\varepsilon}}(\Omega_i)$, respectively. According to Theorem 9.15 in [14] a solution to Poisson's problem with homogeneous Dirichlet conditions belongs to $W_2^{\frac{6}{4-\varepsilon}}(\Omega_i)$ if the right-hand side belongs to $L^{\frac{6}{4-\varepsilon}}(\Omega_i)$ and $\frac{6}{4-\varepsilon} \in (1, \infty)$. Therefore $\tilde{\phi}, \tilde{u} \in W_2^{\frac{6}{4-\varepsilon}}(\Omega_i) \subset W_1^{\frac{12}{4-2\varepsilon}}(\Omega_i)$. Substituting ε by 2ε one may pass from i to $i-1$ and repeat the argument. We conclude via induction that $\tilde{\phi}, \tilde{u} \in W_2^s(\Omega_1)$ for any $s \in (1, \infty)$, recalling that a negative $\frac{12}{4-2\varepsilon}$ corresponds to a right-hand side in $L^\infty(\Omega_i)$. Indeed $\tilde{\phi}, \tilde{u} \in W_2^s(\Omega_i)$ for all $i \in \mathbb{N}$ and $s \in (1, \infty)$.

Now let $\sigma \in C^\infty(\mathbb{R})$. Theorem 9.19 in [14] states that if the right-hand side in Poisson's problem is in $W_k^s(\Omega_i)$ then the solution belongs to $W_{k+2}^s(\Omega_i)$ with $k \in \{1, 2, 3, \dots\}$ and $s \in (1, \infty)$. Leibniz' rule $(f \cdot g)^{(k)} = \sum_{j=0}^k \binom{k}{j} f^{(j)} \cdot g^{(k-j)}$ shows that for a given $s \in (1, \infty)$ a $s' \in (1, \infty)$ can be chosen such that $\frac{\nabla \sigma(\tilde{u})}{\sigma(\tilde{u})} \nabla \tilde{\phi}, \sigma(\tilde{u}) \nabla \tilde{\phi} \cdot \nabla \tilde{\phi} \in W_k^s(\Omega_i)$ if $\tilde{\phi}, \tilde{u} \in W_{k+1}^{s'}(\Omega_i)$. Hence induction over k , coupled with a shift from Ω_i to Ω_{i-1} as above to impose smooth boundary conditions, shows that $\tilde{\phi}, \tilde{u} \in W_k^s(\Omega_0)$ for all $k \in \mathbb{N}$ and $s \in (1, \infty)$. Use of the Sobolev embedding theorem concludes the proof. \square

Remark 4.1 Assumption (C) is made to establish sufficient regularity of elliptic equations with Lemma 4.1 in the context of non-smooth domains and mixed boundary conditions. Also in other settings corresponding elliptic regularity results are available and the above analysis can be transferred with minor modifications. We point for the pure Dirichlet problem to [19,12], for the Neumann problem to [18,12]. A related approach for the mixed problem on smooth, non-creased domains is proposed in [25].

5 A *Priori* and *a posteriori* Error Analysis

In this section we present *a priori* and *a posteriori* error bounds for conforming finite element approximations. We first present in an abstract form that the Galerkin method is quasi-optimal and that the error $x - x_n$ can be bounded using the dual norm of the residual of the approximation. In the second part we choose a particular approximation technique, namely a conforming h -adaptive finite element method. We use interpolation estimates to bound the errors in terms of mesh size and polynomial degree.

5.1 Abstract error bounds

In the theorem we assume that $(\tilde{\phi}, \tilde{u}) \in Y$. There is also an assumption on small data which relaxes as the coupling of the equations, measured by the Lipschitz constant of σ denoted C_7 , weakens. In order to get the correct dependency of C_7 we introduce a scaling factor (τ) of the second component of equation (2.5) in the proof of the theorem. Beside the Lipschitz constant C_7 we let C_8 be the embedding constant from H^1 into L^6 , C_9 be a Poincaré-Friedrichs constant and

$$\begin{aligned} C_5(\phi) &= C_7 C_8 (1 + C_9) \|\nabla \phi\|_{L^3(\Omega)} \max(1, g^\circ - g_\circ + \|\nabla g_\phi\|_{L^3(\Omega)}), \\ C_6(\phi) &= \sigma^\circ ((1 + C_9)(C_8 \|\nabla \phi\|_{L^3(\Omega)} + C_8 \|\nabla g_\phi\|_{L^3(\Omega)} + g^\circ - g_\circ), \end{aligned} \quad (5.1)$$

We note that C_5 is directly proportional to C_7 .

Theorem 5.1 *Suppose there is a solution $x = (\tilde{\phi}, \tilde{u}) \in Y$ of equation (3.1) which satisfies*

$$C_5(\phi) \leq \frac{(1 - \delta)^2 \sigma_0}{C_6(\phi) + (1 - \delta) \sigma_0}$$

for a $\delta \in (0, 1)$ with the constants from equation (5.1). Then the solution is unique. Furthermore, if $x_n \in X_n$ is its Galerkin approximation then the following a priori and a posteriori error bounds hold:

$$\begin{aligned} \|x - x_n\|_X &\lesssim \inf_{y_n \in X_n} \|x - y_n\|_X, \\ \|x - x_n\|_X &\lesssim \|Lx_n + Nx_n - b\|_{X^*}. \end{aligned}$$

Proof We introduce for $(\tilde{\varphi}, \tilde{v}) \in X$ the norm:

$$\|(\tilde{\varphi}, \tilde{v})\|_{X^\tau}^2 := \sigma_0 \|\nabla \tilde{\varphi}\|_{L^2(\Omega)}^2 + \tau^2 \|\nabla \tilde{v}\|_{L^2(\Omega)}^2 + \tau^2 \|\sqrt{\kappa} \tilde{v}\|_{L^2(R_u)}^2.$$

We pick a solution $x = (\tilde{\phi}, \tilde{u}) \in Y$. For any function pair $(\psi_n, v_n) = (\tilde{\psi}_n + g_\phi, \tilde{v}_n + g_u)$ with $(\tilde{\psi}_n, \tilde{v}_n) \in X_n$, we have,

$$\begin{aligned} \|(\phi - \phi_n, u - u_n)\|_{X^\tau}^2 &\leq \langle \sigma(u_n) \nabla(\phi - \phi_n), \nabla(\phi - \phi_n) \rangle \\ &\quad + \tau^2 \|\nabla(u - u_n)\|_{L^2(\Omega)}^2 + \tau^2 \|\sqrt{\kappa}(u - u_n)\|_{L^2(R_u)}^2 \\ &\leq \langle \sigma(u_n) \nabla(\phi - \phi_n), \nabla(\phi - \psi_n) \rangle + \tau^2 \langle \nabla(u - u_n), \nabla(u - v_n) \rangle \\ &\quad + \tau^2 \langle \kappa(u - u_n), u - v_n \rangle_{R_u} + \langle \sigma(u_n) \nabla(\phi - \phi_n), \nabla(\psi_n - \phi_n) \rangle \\ &\quad + \tau^2 \langle \nabla(u - u_n), \nabla(v_n - u_n) \rangle + \tau^2 \langle \kappa(u - u_n), v_n - u_n \rangle_{R_u}. \end{aligned}$$

We continue

$$\begin{aligned}
\|(\phi - \phi_n, u - u_n)\|_{X^\tau}^2 &\stackrel{(*)}{\leq} \frac{\sigma^\circ}{\sigma_\circ} \|(\phi - \phi_n, u - u_n)\|_{X^\tau} \|(\phi - \psi_n, u - v_n)\|_{X^\tau} \\
&\quad + C_7 C_8 \|\nabla \phi\|_{L^3(\Omega)} \|u - u_n\|_{H^1(\Omega)} \|\nabla(\phi_n - \psi_n)\|_{L^2(\Omega)} \\
&\quad + \tau^2 \|\sigma(u) [\tilde{\phi}] \nabla \phi - \sigma(u_n) [\tilde{\phi}_n] \nabla \phi_n\|_{L^2(\Omega)} \|\nabla(v_n - u_n)\|_{L^2(\Omega)} \\
&\quad + C_8 \tau^2 \|\nabla g_\phi\|_{L^3(\Omega)} \|\sigma(u) \nabla \phi - \sigma(u_n) \nabla \phi_n\|_{L^2(\Omega)} \|v_n - u_n\|_{H^1(\Omega)} \\
&\leq \frac{\sigma^\circ}{\sigma_\circ} \|(\phi - \phi_n, u - u_n)\|_{X^\tau} \|(\phi - \psi_n, u - v_n)\|_{X^\tau} \\
&\quad + C_6 \tau^2 \|\nabla(\phi - \phi_n)\|_{L^2(\Omega)} \|\nabla(v_n - u_n)\|_{L^2(\Omega)} \\
&\quad + C_5 (\|\nabla(\phi_n - \psi_n)\|_{L^2(\Omega)} + \tau^2 \|\nabla(v_n - u_n)\|_{L^2(\Omega)}) \|\nabla(u - u_n)\|_{L^2(\Omega)},
\end{aligned}$$

where we in (*) use that $\langle \sigma(u) \nabla \phi, \nabla \tilde{\psi} \rangle = \langle \sigma(u_n) \nabla \phi_n, \nabla \tilde{\psi} \rangle = 0$ for any $\tilde{\psi} \in X_n$. We now use the triangle inequality on $\|\nabla(\psi_n - \phi_n)\|_{L^2(\Omega)} \leq \|\nabla(\phi - \phi_n)\|_{L^2(\Omega)} + \|\nabla(\phi - \psi_n)\|_{L^2(\Omega)}$ and $\|\nabla(v_n - u_n)\|_{L^2(\Omega)} \leq \|\nabla(u - u_n)\|_{L^2(\Omega)} + \|\nabla(u - v_n)\|_{L^2(\Omega)}$ and introduce $C_{10} = \sigma^\circ \sigma_\circ^{-1} + C_6 \sigma_\circ^{-1} \tau + C_5 \sigma_\circ^{-1} \tau^{-1} + C_5 \tau^{-1}$ to get,

$$\begin{aligned}
\|(\phi - \phi_n, u - u_n)\|_{X^\tau}^2 &\leq C_{10} \|(\phi - \phi_n, u - u_n)\|_{X^\tau} \|(\phi - \psi_n, u - v_n)\|_{X^\tau} \\
&\quad + C_6 \tau^2 \|\nabla(\phi - \phi_n)\|_{L^2(\Omega)} \|\nabla(u - u_n)\|_{L^2(\Omega)} \\
&\quad + C_5 (\|\nabla(\phi - \phi_n)\|_{L^2(\Omega)} + \tau^2 \|\nabla(u - u_n)\|_{L^2(\Omega)}) \|\nabla(u - u_n)\|_{L^2(\Omega)}, \\
&\leq C_{10} \|(\phi - \phi_n, u - u_n)\|_{X^\tau} \|(\phi - \psi_n, u - v_n)\|_{X^\tau} \\
&\quad + \frac{1}{2\epsilon} (C_5 + C_6 \tau^2) \|\nabla(\phi - \phi_n)\|_{L^2(\Omega)}^2 \\
&\quad + \left(\frac{\epsilon}{2} (C_5 + C_6 \tau^2) + C_5 \tau^2\right) \|\nabla(u - u_n)\|_{L^2(\Omega)}^2.
\end{aligned}$$

for any $\epsilon > 0$. We now want to find the maximum value of C_5 while fulfilling $\frac{1}{2\epsilon} (C_5 + C_6 \tau^2) \leq (1 - \delta) \sigma_0$ and $\left(\frac{\epsilon}{2} (C_5 + C_6 \tau^2) + C_5 \tau^2\right) \leq (1 - \delta) \tau^2$, for some $0 < \delta < 1$, since we then can subtract $(1 - \delta) \|(\phi - \phi_n, u - u_n)\|_{X^\tau}^2$ on both sides of the equality sign. Algebraic manipulation reveals that by choosing ϵ equal to $\epsilon^* = \frac{1 - \delta}{C_6 + (1 - \delta) \sigma_0} > 0$ and $\tau^2 = (\tau^*)^2 := \frac{\epsilon^* (1 - \delta) \sigma_0}{C_6} > 0$ leads to a value $C_5 = \frac{(1 - \delta)^2 \sigma_0}{C_6 + (1 - \delta) \sigma_0}$. Under the assumptions in the statement of the theorem we therefore have,

$$\|(\phi - \phi_n, u - u_n)\|_{X^{\tau^*}} \leq \delta^{-1} C_{10}(\tau^*, \sigma_0, \sigma^0, C_5, C_6) \|(\phi - \psi_n, u - v_n)\|_{X^{\tau^*}}$$

for that fixed $\tau^* > 0$. The *a priori* bound now follows using the Poincaré-Friedrichs inequality and simple algebraic manipulation, since the X^τ -norm and the X -norm are equivalent for fixed τ .

Suppose we have two solutions x_1 and x_2 . The *a priori* bound gives that any Galerkin approximation will eventually get arbitrary close to both solutions which means that they coincide, i.e. $x_1 = x_2$.

Only the *a posteriori* bound now remains to prove. It follows continuing ideas as in the *a priori* bound. We have, following the step (*) and below,

$$\begin{aligned}
\|(\phi - \phi_n, u - u_n)\|_{X\tau}^2 &\leq \langle \sigma(u_n) \nabla(\phi - \phi_n), \nabla(\phi - \phi_n) \rangle \\
&\quad + \tau^2 \langle \nabla(u - u_n), \nabla(u - u_n) \rangle + \tau^2 \langle \kappa(u - u_n), (u - u_n) \rangle_{R_u} \\
&\leq \langle b - L(\tilde{\phi}_n, \tilde{u}_n) - N(\tilde{\phi}_n, \tilde{u}_n), (\phi - \phi_n, \tau^2(u - u_n)) \rangle \\
&\quad + C_6 \tau^2 \|\nabla(\phi - \phi_n)\|_{L^2(\Omega)} \|\nabla(u - u_n)\|_{L^2(\Omega)} \\
&\quad + C_5 \|\nabla(u - u_n)\|_{L^2(\Omega)} (\|\nabla(\phi - \phi_n)\|_{L^2(\Omega)} + \tau^2 \|\nabla(u - u_n)\|_{L^2(\Omega)}),
\end{aligned}$$

Under the assumptions of the theorem we can now repeat the exact same argument as above picking ϵ^* and τ^* in the same way. We get,

$$\begin{aligned}
\|x - x_n\|_X^2 &\lesssim \|x - x_n\|_{X\tau^*}^2 \leq \delta^{-1} \max(1, (\tau^*)^2) \langle b - Lx_n - Nx_n, x - x_n \rangle \\
&\lesssim \|Lx_n + Nx_n - b\|_{X^*} \|x - x_n\|_X,
\end{aligned}$$

completing the proof of the theorem. \square

Remark 5.1 Since $C_5(\phi)$ and $C_6(\phi)$ depend on $\|\nabla\phi\|_{L^3(\Omega)}$ a bound on this norm needs to be computed in order to obtain a computable *a posteriori* error bound. In the setting of creased domains this can in principle be derived from the radius r in Lemma 4.4.

5.2 Error bounds for h-adaptive finite element approximations

The local regularity result presented in Theorem 4.2 indicates that the problem is well suited for adaptive finite element methods. Suppose that \mathcal{T}_ϕ and \mathcal{T}_u are decompositions of Ω into tetrahedrons. We let h_T denote the diameter of element T and assume that \mathcal{T}_ϕ and \mathcal{T}_u are nondegenerate, i.e. there is a mesh-independent constant γ such that

$$\max_T \frac{h_T}{d_T} \leq \gamma,$$

where d_T is the diameter of the largest ball contained in T . We denote the set of elements T' neighbouring T by $S_T = \{\cup T' : T' \cap T \neq \emptyset\}$, assuming that elements are closed. Further let $\mathcal{E}_{I_\phi}, \mathcal{E}_{I_u}$ be the set of all interior facets of the two meshes and let $\mathcal{E}_{N_\phi}, \mathcal{E}_{R_u}$ be the set of boundary facets on N_ϕ and R_u , respectively. It is implicitly supposed here that N_ϕ and R_u are unions of edges. We denote the set of all elements neighboring e by $S_e = \{\cup e : T \cap e \neq \emptyset\}$. The diameter of a facet e is denoted h_e . The spaces of polynomials of total or partial degree less than or equal to k , defined on T , is denoted $P_k(T)$. Let

$$\left. \begin{aligned}
P_n &:= \{\varphi \in C(\Omega) : \varphi|_T \in P_k(T) \text{ and } \varphi|_{D_\phi} = 0\}, \\
U_n &:= \{v \in C(\Omega) : v|_T \in P_\ell(T) \text{ and } v|_{D_u} = 0\},
\end{aligned} \right\} \quad (5.2)$$

and $X_n = P_n \times U_n$. We let $\pi_\phi : H^1(\Omega) \rightarrow P_n$ and $\pi_u : H^1(\Omega) \rightarrow U_n$ be Scott-Zhang interpolants, as defined in [26]. We recall the interpolation bounds

$$\left. \begin{aligned} \|v - \pi_u v\|_{W_2^s(T)} &\lesssim h_T^{r-s} |v|_{W_2^r(S_T)}, \\ \|v - \pi_u v\|_{L^2(e)} &\lesssim h_e^{1/2} |v|_{W_2^1(S_e)}, \end{aligned} \right\} \quad (5.3)$$

for $v \in W_2^r(\Omega; D_u)$, $r \in [1, k+1]$, $s \in [0, r]$ and where $|\cdot|_{W_2^r}$ is the semi-norm only including the derivatives of order r . An analogous bound also holds for π_ϕ . From here on we drop the subscripts of π_u and π_ϕ since it will be clear from the context which operator is meant.

Theorem 5.2 *Let $x = (\tilde{\phi}, \tilde{u}) \in Y$ be a solution of (3.1) and $x_n \in X_n$ be its Galerkin approximation. Then we have the a priori error estimate,*

$$\|x - x_n\|_X^2 \lesssim \sum_{T \in \mathcal{T}_\phi} h_T^{2(r-1)} |\tilde{\phi}|_{W_2^r(S_T)}^2 + \sum_{T \in \mathcal{T}_u} h_T^{2(s-1)} |\tilde{u}|_{W_2^s(S_T)}^2,$$

with $1 \leq r \leq k+1$ and $1 \leq s \leq \ell+1$, and the a posteriori error estimate,

$$\begin{aligned} \|x - x_n\|_X^2 &\lesssim \sum_{T \in \mathcal{T}_\phi} h_T^2 \eta_T^2 + \sum_{e \in \mathcal{E}_{I_\phi}} h_e \eta_{e,I}^2 + \sum_{e \in \mathcal{E}_{N_\phi}} h_e \eta_{e,N}^2 \\ &\quad + \sum_{T \in \mathcal{T}_u} h_T^2 \rho_T^2 + \sum_{e \in \mathcal{E}_{I_u}} h_e \rho_{e,I}^2 + \sum_{e \in \mathcal{E}_{R_u}} h_e \rho_{e,R}^2, \end{aligned}$$

where

$$\begin{aligned} \eta_T(\phi_n, u_n) &= \|\nabla \cdot \sigma(u_n) \nabla \phi_n\|_{L^2(T)}, \\ \eta_{e,I}(\phi_n, u_n) &= \|\nu \cdot [\sigma(u_n) \nabla \phi_n]\|_{L^2(e)}, \\ \eta_{e,N}(\phi_n, u_n) &= \|\nu \cdot \sigma(u_n) \nabla \phi_n\|_{L^2(e)}, \\ \rho_T(\phi_n, u_n) &= \|\nabla \cdot (\nabla u_n + \sigma(u_n) [\tilde{\phi}_n] \nabla \phi_n) + \sigma(u_n) \nabla g_\phi \cdot \nabla \phi_n\|_{L^2(T)}, \\ \rho_{e,I}(\phi_n, u_n) &= \|\nu \cdot ([\nabla u_n] + \sigma(u_n) [\tilde{\phi}_n] [\nabla \phi_n])\|_{L^2(e)}, \\ \rho_{e,R}(\phi_n, u_n) &= \|\nu \cdot (\nabla u_n + \sigma(u_n) [\tilde{\phi}_n] \nabla \phi_n) + \kappa u_n - h\|_{L^2(e)}. \end{aligned}$$

Proof From Theorem 5.1 we have,

$$\|x - x_n\|_X \lesssim \inf_{y_n \in X_n} \|x - y_n\|_X \leq (\|\tilde{\phi} - \pi \tilde{\phi}\|_{H^1}^2 + \|\tilde{u} - \pi \tilde{u}\|_{H^1}^2)^{1/2}.$$

The *a priori* part of the theorem follows from (5.3) with $s = 1$.

We turn to the *a posteriori* bound. For any $y = (\tilde{\varphi}, \tilde{v}) \in X$ we have

$$\begin{aligned} \langle Lx_n + Nx_n - b, y \rangle &= \langle \sigma(u_n) \nabla \phi_n, \nabla(\tilde{\varphi} - \pi \tilde{\varphi}) \rangle + \langle \nabla u_n, \nabla(\tilde{v} - \pi \tilde{v}) \rangle \\ &\quad + \langle \kappa u_n, \tilde{v} - \pi \tilde{v} \rangle + \langle \sigma(u_n) [\tilde{\phi}_n] \nabla \phi_n, \nabla(\tilde{v} - \pi \tilde{v}) \rangle \\ &\quad - \langle \sigma(u_n) \nabla g_\phi \cdot \nabla \phi_n, \tilde{v} - \pi \tilde{v} \rangle - \langle h, \tilde{v} - \pi \tilde{v} \rangle_{R_u}. \end{aligned}$$

We can subtract the interpolants because of Galerkin orthogonality. We apply Green's formula on the elements of the meshes together with the Cauchy-Schwarz inequality to obtain

$$\begin{aligned}
\langle Lx_n + Nx_n - b, y \rangle &\lesssim \sum_{T \in \mathcal{T}_\phi} \|\nabla \cdot (\sigma(u_n) \nabla \phi_n)\|_{L^2(T)} \|\tilde{\varphi} - \pi\tilde{\varphi}\|_{L^2(T)} \\
&+ \sum_{e \in \mathcal{E}_{I_\phi}} \|\nu \cdot [\sigma(u_n) \nabla \phi_n]\|_{L^2(e)} \|\tilde{\varphi} - \pi\tilde{\varphi}\|_{L^2(e)} \\
&+ \sum_{e \in \mathcal{E}_{N_\phi}} \|\nu \cdot \sigma(u_n) \nabla \phi_n\|_{L^2(e)} \|\tilde{\varphi} - \pi\tilde{\varphi}\|_{L^2(e)} \\
&+ \sum_{T \in \mathcal{T}_u} \|\nabla \cdot (\nabla u_n + \sigma(u_n) [\tilde{\phi}_n] \nabla \phi_n) + \sigma(u_n) \nabla g_\phi \cdot \nabla \phi_n\|_{L^2(T)} \|\tilde{v} - \pi\tilde{v}\|_{L^2(T)} \\
&+ \sum_{e \in \mathcal{E}_{I_u}} \|\nu \cdot ((\nabla u_n) + \sigma(u_n) [\tilde{\phi}_n] [\nabla \phi_n])\|_{L^2(e)} \|\tilde{v} - \pi\tilde{v}\|_{L^2(e)} \\
&+ \sum_{e \in \mathcal{E}_{R_u}} \|\nu \cdot (\nabla u_n + \sigma(u_n) [\tilde{\phi}_n] \nabla \phi_n) + \kappa u_n - h\|_{L^2(e)} \|\tilde{v} - \pi\tilde{v}\|_{L^2(e)}
\end{aligned}$$

Use of (5.3), with $s = 0$ and $r = 1$, and the Cauchy-Schwarz inequality gives

$$\begin{aligned}
\frac{\langle Lx_n + Nx_n - b, y \rangle}{\|y\|_X} &\lesssim \left(\sum_{T \in \mathcal{T}_\phi} h_T^2 \eta_T^2 + \sum_{e \in \mathcal{E}_{I_\phi}} h_e \eta_{e,I}^2 + \sum_{e \in \mathcal{E}_{N_\phi}} h_e \eta_{e,N}^2 \right)^{1/2} \\
&+ \left(\sum_{T \in \mathcal{T}_u} h_T^2 \rho_T^2 + \sum_{e \in \mathcal{E}_{I,u}} h_e \rho_{e,I}^2 + \sum_{e \in \mathcal{E}_{R_u}} h_e \rho_{e,R}^2 \right)^{1/2}.
\end{aligned}$$

The theorem follows by taking supremum over all $y \in X$. \square

Remark 5.2 If the Dirichlet data g_ϕ and g_u are not traces of finite element functions then an extra data error term will appear in the error estimates. Similarly if exact quadrature is not used additional error terms have to be considered. We have neglected these standard terms in the analysis to make it more readable. For more details, see [6].

Remark 5.3 For additional flexibility one can consider hp finite element spaces. Theorem 4.2 indicates that even though the choice of the space Y is sharp, the interior regularity of the solution is higher. This is an ideal setting for hp -finite element methods. The above *a priori* and *a posteriori* error bounds can be transferred to hp approximation spaces for which suitable interpolation operators are available. Some ideas for the construction of such operators are collected in [23] and references therein.

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