Density Bounds and Tangent Measures

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Declaration

I hereby declare that this thesis has not been, and will not be, submitted in whole or in part to another University for the award of any other degree.

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ADRIAN MARTIN - PhD MATHEMATICS

DENSITY BOUNDS AND TANGENT MEASURES

SUMMARY

A major theme in geometric measure theory is establishing global properties, such as rectifiability, of sets or measures from local ones, such as densities or tangent measures. In establishing sufficient conditions for rectifiability it is useful to know what local properties are possible in a given setting, and this is the theme of this thesis.

It is known, for 1-dimensional subsets of the plane with positive lower density, that the tangent measures being concentrated on a line is sufficient to imply rectifiability. It is shown here that this cannot be relaxed too much by demonstrating the existence of a 1-dimensional subset of the plane with positive lower density whose tangent measures are concentrated on the union of two halflines, and yet the set is unrectifiable.

A class of metrics are also defined on $\mathbb{R}$, which are functions of the Euclidean metric, to give spaces of dimension s ($s > 1$), where the lower density is strictly greater than 2^{1-s}, and a method for gaining an explicit lower bound for a given dimension is developed. The results are related to the generalised Besicovitch $\frac{1}{2}$ conjecture.

Set functions are defined that measure how easily the subsets of a set can be covered by balls (of any radius) with centres in the subset. These set functions are studied and used to give lower bounds on the upper density of subsets of a normed space, in particular Euclidean spaces. Further attention is paid to subsets of $\mathbb{R}$, where more explicit bounds are given.

Dedicated to my niece

Olivia Anne Daynes

born 25 March 2012.

Contents

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Chapter 1

Introduction and Background

1.1 Introduction

This thesis is concerned with the small-scale measure theoretic properties of various sets and spaces. Also of interest is how these local properties relate to global properties, such as rectifiability. In this chapter we develop this and introduce the fundamental concepts that we will use throughout the rest of thesis.

The remaining three chapters are all connected with this theme but are fairly independent of each other. In Chapter 2 we give an example of an unrectifiable set with fairly regular tangent measures, which serves to give a limit on how far rectifiability theorems involving tangent measures can be improved. In Chapter 3 we give an example of an unrectifiable metric space with particularly high lower density, again this limits how far rectifiability theorems can be improved, but in this case with reference to lower densities. In Chapter 4 we develop actual bounds on what values for upper densities are possible.

Most of the results in this chapter are based on those in Mattila's book, [19], with some further references to Federer's book, [14]. For brevity, only results from other sources will be individually referenced.

1.2 Measures

In this section, we will reproduce some basic definitions and results from measure theory that we will use throughout this thesis. We begin with σ-algebras.

Definition 1.2.1 *Let X be some set. Then we say that $\mathcal{A} \subseteq \mathcal{P}(X)$ is a σ-algebra if and only if*

(i) $\emptyset \in \mathcal{A}$

(ii) $A \in \mathcal{A} \Rightarrow X \backslash A \in \mathcal{A}$

(iii) $\{A_i\}_{i \in \mathbb{N}} \subseteq \mathcal{A} \Rightarrow \bigcup_{i \in \mathbb{N}} A_i \in \mathcal{A}$

Proposition 1.2.2 *If $\mathcal{A}, \mathcal{A}_i \subseteq \mathcal{P}(X)$ are σ-algebras then*

(i) $X \in \mathcal{A}$

(ii) $\{A_i\}_{i \in \mathbb{N}} \subseteq \mathcal{A} \Rightarrow \bigcap_{i \in \mathbb{N}} A_i \in \mathcal{A}$

(iii) $\mathcal{P}(X)$ *is a σ-algebra*

(iv) $\bigcap_i \mathcal{A}_i$*, where i ranges over some (possibly uncountable) set, is a σ-algebra.*

Properties (iii) and (iv) above ensure that following is well defined.

Definition 1.2.3 *If X is a topological space, then its Borel sets are the smallest σ-algebra of X containing all of its open sets.*

For a definition of a measure, we will use what is sometimes referred to as an outer measure. This will be be useful as we will often only require subadditivity and the restriction to a σ-algebra of sets would be inconvenient.

Definition 1.2.4 *Let X be some set. Then we say $\mu : \mathcal{P}(X) \to [0, \infty]$ is a measure if and only if*

(i) $\mu(\emptyset) = 0$ *(null empty set)*

(ii) $A \subseteq B \Rightarrow \mu(A) \leqslant \mu(B)$ *(monotone)*

(iii) $\mu\left(\bigcup_{i\in\mathbb{N}} A_i\right) \leqslant \sum_{i\in\mathbb{N}} \mu(A_i)$ *(countably subadditive)*

In which case we call (X,μ) *a measure space and, if* (X,d) *is a metric space, then we call* (X,μ,d) *a metric measure space.*

Furthemore, we say that μ *is finite if and only if* $\mu(X) < \infty$ *and* σ*-finite if and only if there exist* $\{A_i\}_{i\in\mathbb{N}} \subseteq \mathcal{P}(X)$ *such that, for all* $i \in \mathbb{N}$, $\mu(A_i) < \infty$ *and* $\bigcup_{i\in\mathbb{N}} A_i = X$.

Definition 1.2.5 *We say that* $A \subseteq X$ *is* μ*-measurable if and only if for every* $E \subseteq X$

$$\mu(E) = \mu(E \cap A) + \mu(E \backslash A).$$

We say a function $f : (X,\mu) \to (Y,\nu)$ *is measurable if and only if for every set* $A \subseteq Y$ *which is* ν*-measurable we have that* $f^{-1}(A)$ *is* μ*-measurable.*

Proposition 1.2.6 *Let* (X,μ) *be a measure space and* $\mathcal{M}$ *the collection of* μ*-measurable sets.*

(i) $\mathcal{M}$ *is a* σ*-algebra,*

(ii) $\mu(A) = 0 \Rightarrow A \in \mathcal{M}$,

(iii) if $\{A_i\}_{i\in\mathbb{N}} \subseteq \mathcal{M}$ *is such that* $i \neq j \Rightarrow A_i \cap A_i = \emptyset$ *then* $\mu\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \sum_{i\in\mathbb{N}} \mu(A_i)$,

(iv) if $\{A_i\}_{i\in\mathbb{N}} \subseteq \mathcal{M}$ *is such that, for all* $i \in \mathbb{N}$, $A_i \subseteq A_{i+1}$, *then*

$$\mu\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \lim_{i\to\infty} \mu(A_i),$$

(v) if $\{A_i\}_{i\in\mathbb{N}} \subseteq \mathcal{M}$ *is such that, for all* $i \in \mathbb{N}$, $A_i \supseteq A_{i+1}$ *and* $\mu(A_1) < \infty$, *then*

$$\mu\left(\bigcap_{i\in\mathbb{N}} A_i\right) = \lim_{i\to\infty} \mu(A_i),$$

(vi) a function $f : X \to [-\infty,\infty]$ *is measurable if and only if for any* $a \in \mathbb{R}$ *we have* $f^{-1}([-\infty,a]) \in \mathcal{M}$.

Statement (iii) in Proposition 1.2.6 indicates that our chosen definition of a measure is not much of a restriction, especially when one considers that an additive set function defined on a σ-algebra can be extended to a measure simply by taking the infimum of the value of that function for supersets in the σ-algebra.

In particular, Lebesgue measure, which we denote $\mathcal{L}$ on $\mathbb{R}$ and $\mathcal{L}^n$ on $\mathbb{R}^n$, extends in this fashion and retains its usual properties, except countable additivity, which it retains on Lebesgue measurable sets.

Furthermore, if we restrict ourselves to measurable functions integrated over measurable sets, then integration retains all of its usual properties when applied to measures as defined in this thesis.

The notion of a measure is often too general for many of the standard results that will be needed, as well as for those proved in this thesis. We therefore define the following classes of measures.

Definition 1.2.7 *Let X be a topological space, μ a measure on it, and $\mathcal{M}$ the collection of its measurable sets. Then,*

(i) if $\mathcal{M}$ contains all of the Borel sets of X then μ is called Borel,

(ii) if μ is Borel and for every $A \subseteq X$ there exists a Borel set B such that $A \subseteq B$ and $\mu(A) = \mu(B)$, then μ is called Borel regular,

(iii) if X is Hausdorff and locally compact, μ is Borel, and

(a) $K \subseteq X$ is compact $\Rightarrow \mu(K) < \infty$,

(b) for every open set $U \subseteq X$ we have $\mu(U) = \sup\{\mu(K) : K \subseteq U, K \text{ is compact}\}$, and

(c) for every $A \subseteq X$ we have $\mu(A) = \inf\{\mu(U) : U \supseteq A, U \text{ is open}\}$,

then μ is called a Radon measure,

(iv) if X is a metric space and for every $x \in X$ there exists $r > 0$ such that $\mu(B(x,r)) < \infty$ then μ is called locally finite.

We note that a Radon measure is also Borel regular and that $\mathcal{L}$ and $\mathcal{L}^n$ are examples of Radon measures.

Since this thesis is in the field of geometric measure theory, it will, of course, be using measure theory to study the properties of certain sets. Thus the concept of restricting a measure to a set, and the closely related notion of the support of a measure, are central.

Definition 1.2.8 *Given a measure space (X, μ) where X is a topological space, we define the support of μ, $\operatorname{supp}(\mu)$ to be $X \backslash \bigcup \{U : U \text{ is open}, \mu(U) = 0\}$*

Proposition 1.2.9 *If (X, μ) is a measure space where X is a separable metric space and μ is Borel, then $\operatorname{supp}(\mu)$ is a closed set with $\mu(X \backslash \operatorname{supp}(\mu)) = 0$, furthermore it is the intersection of all such sets.*

Definition 1.2.10 *Given a measure space (X, μ) and a set $A \subseteq X$ then we define the restriction of μ to A, written $\mu \llcorner A$, by*

$$\mu \llcorner A(E) = \mu(E \cap A)$$

Proposition 1.2.11 *Let (X, μ) be a measure space and $A \subseteq X$. Then,*

(i) $\mu \llcorner A$ *is a measure,*

(ii) *if X is a topological space, $\mu(A) < \infty$ and μ is Borel regular, then $\mu \llcorner A$ is also Borel regular, and*

(iii) *if $E \subseteq X$ is μ-measurable, then it also $\mu \llcorner A$-measurable.*

The following are extremely useful tools and will be applied in many of the proofs in this thesis.

Theorem 1.2.12 *Let μ be a Borel regular measure on a metric space X, $A \subset X$ a μ-measurable set, and $\varepsilon > 0$. Then,*

(i) *if $\mu(A) < \infty$ then there exists a closed set $C \subseteq A$ such that $\mu(A \backslash C) < \varepsilon$,*

(ii) *if there exists $\{U_i\}_{i \in \mathbb{N}} \subseteq \mathcal{P}(X)$ such that, for all $i \in \mathbb{N}$, U_i is open and $\mu(U_i) < \infty$, and $A \subseteq \bigcup_{i \in \mathbb{N}} U_i$ then there exists an open set $U \supseteq A$ such that $\mu(U \backslash A) < \varepsilon$.*

Corollary 1.2.13 *Let μ be a measure on a $\mathbb{R}^n$. Then,*

(i) if μ is Borel regular, $A \subset X$ is a μ-measurable set with $\mu(A) < \infty$, then for any $\varepsilon > 0$ there exists a compact set $K \subseteq A$ such that $\mu(A \backslash K) < \varepsilon$,

(ii) μ is a Radon measure if and only if it is Borel regular and locally finite.

1.3 Hausdorff Measure and Dimension

We are now ready to define Hausdorff measure, the predominant measure of study in this thesis. All of the results that follow in this section are fairly standard and are either explicitly stated or immediate consequences of results in [19].

Definition 1.3.1 *Let X be a separable metric space, $s \in [0, \infty)$, $\delta \in (0, \infty]$ and $E \subseteq X$. Then we define*

$$\mathcal{H}_\delta^s(E) = \inf \left\{ \sum_{i \in \mathbb{N}} (\mathrm{diam} E_i)^s : E \subseteq \bigcup_{i \in \mathbb{N}} E_i, \mathrm{diam} E_i < \delta \right\},$$

interpreting $0^0 = 1$ with the exception $\mathrm{diam}(\emptyset)^0 = 0$.

We may now define the s-dimensional Hausdorff measure by

$$\mathcal{H}^s(E) = \lim_{\delta \downarrow 0} \mathcal{H}_\delta^s(E).$$

Proposition 1.3.2 *With the notation above,*

(i) $\mathcal{H}_\delta^s$ is a well defined measure on X, and the covering sets used in Definition 1.3.1 may be restricted to open sets, closed sets, or, if X is a normed space, convex sets without changing the resultant set function,

(ii) if $0 < \delta_1 < \delta$, then $\mathcal{H}_{\delta_1}^s(E) \geqslant \mathcal{H}_\delta^s(E)$,

(iii) $\mathcal{H}^s$ is a well defined Borel regular measure on X,

(iv) if X is a normed space, $x \in X$ and $E \subseteq X$ then $\mathcal{H}_\delta^s(E + x) = \mathcal{H}_\delta^s(E)$ and $\mathcal{H}^s(E + x) = \mathcal{H}^s(E)$, that is $\mathcal{H}_\delta^s$ and $\mathcal{H}^s$ are translation invariant,

(v) *if X is a normed space and $\lambda \in (0,\infty)$ then $\mathcal{H}^s_\delta(\lambda E) = \lambda^s \mathcal{H}^s_\delta(E)$ and $\mathcal{H}^s(\lambda E) = \lambda^s \mathcal{H}^s(E)$,*

(vi) *for any $E \subseteq X$, $\mathcal{H}^0(E) = \#(E)$, and*

(vii) *if $X = \mathbb{R}^n$, then $\mathcal{H}^n = 2^{-n}\alpha(n)\mathcal{L}^n$, where $\alpha(n)$ is the volume of a unit ball in $\mathbb{R}^n$.*

Hausdorff measure is an example of Carathéordory's construction of a measure, which is used to define a whole class of Borel regular measures using coverings of decreasing diameter. The reason these measures are Borel is due to Theorem 1.3.3, below; they are regular simply because of the way that they are defined using covering sets.

Theorem 1.3.3 *Let (X,μ,d) be a metric measure space. Then, μ is a Borel measure if and only if for every $A, B \subseteq X$ such that $d(A,B) > 0$ we have*

$$\mu(A \cup B) = \mu(A) + \mu(B).$$

Hausdorff measures are used to give a notion of dimension to a set or space. There are many different types of dimension with differing definitions, but Hausdorff is probably the most used in geometric measure theory, and the only one that will be used in this thesis.

Definition 1.3.4 *Let X be a separable metric space and $E \subseteq X$. Then we define the Hausdorff dimension of E to be*

$$\inf\{s \in [0,\infty) : \mathcal{H}^s(E) < \infty\}.$$

Definition 1.3.5 *We call $A \subseteq X$ an s-set if and only if it is $\mathcal{H}^s$-measurable and $0 < \mathcal{H}^s(A) < \infty$.*

Proposition 1.3.6 *Let X be a separable metric space, and $s \in [0,\infty)$. Then,*

(i) *if $E \subseteq X$ has a Hausdorff dimension of s then $\mathcal{H}^t(E) = \infty$ for all $t < s$ and $\mathcal{H}^t(E) = 0$ for all $t > s$,*

(ii) *an s-set has Hausdorff dimension s, and*

(iii) *if $E \subseteq X$ is an s-set, then $\mathcal{H}^s \llcorner E$ is a finite Radon measure on X.*

We now have all that we need to formulate a suitable definition of rectifiability. Rectifiability will not so often be used in the content of this thesis, indeed most of the objects discussed are not rectifiable. But it does, as we shall see, provide a major motivation for the material.

Definition 1.3.7 *Let X be a separable metric space and $n \in \mathbb{N}$. Then, we say that a set $E \subseteq X$ is $\mathcal{H}^n$-countably n-rectifiable (henceforth n-rectifiable or just rectifiable) if and only if there exists $\{f_k\}_{k\in\mathbb{N}}$ such that*

(i) $f_k : \mathbb{R}^n \to X$ *is a Lipschitz function, for every* $k \in \mathbb{N}$, *and*

(ii) $\mathcal{H}^n \left(E \backslash \bigcup_{k\in\mathbb{N}} f_k \left(\mathbb{R}^n\right)\right) = 0.$

We say that a measure μ on $\mathbb{R}^n$ is n-rectifiable if and only if there exist a Borel function f such that

$$\mu\left(A\right) = \int_A f \, d\mathcal{H}^n \llcorner E,$$

where E is an n-rectifiable set.

On the other hand we say that a set (or measure) is unrectifiable if it is not rectifiable, and purely unrectifiable if it intersects any n-rectifiable set in an $\mathcal{H}^n$-negligible set.

Lipschitz functions are the most convenient for use in geometric measure theory, but it is worth noting that an equivalent definition may be given using continuously differentiable functions instead. This helps to emphasise why the concept is so useful, as it provides a notion that is applicable to measure theory which represents what may be thought of as a physically meaningful solution to a problem. Areas in which such problems occur include material science, liquid crystals and image analysis; a good source for more examples with a fuller treatment is [1].

1.4 Densities

Along with tangent measures, densities are one of the fundamental objects of study in this thesis. We only present here the definitions that we will be using and a few basic results, which will be used in the later chapters.

Definition 1.4.1 *Given a measure, μ, and $s \geqslant 0$, we define the s-dimensional upper and lower densities of μ at the point x by*

$$\underline{D}^s(\mu, x) = \liminf_{r \downarrow 0} \frac{\mu(B_r(x))}{(2r)^s}$$

and

$$\overline{D}^s(\mu, x) = \limsup_{r \downarrow 0} \frac{\mu(B_r(x))}{(2r)^s}$$

respectively. Where these values coincide, we may define the density μ at x by

$$D^s(\mu, x) = \lim_{r \downarrow 0} \frac{\mu(B_r(x))}{(2r)^s}.$$

Since we are primarily concerned with Hausdorff measures restricted to a certain set, it will be convenient to also define the density of a set at a point.

Definition 1.4.2 *We define the s-dimensional upper and lower densities of a set E at a point x by*

$$\underline{D}^s(E, x) = \liminf_{r \downarrow 0} \frac{\mathcal{H}^s \llcorner E(B_r(x))}{(2r)^s}$$

and

$$\overline{D}^s(E, x) = \limsup_{r \downarrow 0} \frac{\mathcal{H}^s \llcorner E(B_r(x))}{(2r)^s}$$

respectively. Again, where these values coincide, we define the density of E at x by

$$D^s(E, x) = \lim_{r \downarrow 0} \frac{\mathcal{H}^s(B_r(x))}{(2r)^s}.$$

The above definition would work equally well if we were to use closed balls instead of open. We will use these interchangeably, as is most convenient, throughout the thesis.

Proposition 1.4.3 *With the notation above, we have*

$$\underline{D}^s(\mu, x) = \liminf_{r \downarrow 0} \frac{\mu(\overline{B}_r(x))}{(2r)^s},$$

$$\overline{D}^s(\mu, x) = \limsup_{r \downarrow 0} \frac{\mu\left(\overline{B}_r(x)\right)}{(2r)^s},$$

$$\underline{D}_s(E, x) = \liminf_{r \downarrow 0} \frac{\mathcal{H}^s \llcorner E\left(\overline{B}_r(x)\right)}{(2r)^s},$$

and

$$\overline{D}_s(E, x) = \limsup_{r \downarrow 0} \frac{\mathcal{H}^s \llcorner E\left(\overline{B}_r(x)\right)}{(2r)^s}.$$

We give an upper bound to the upper density of sets. We will look at lower bounds on upper density in Chapter 4 and lower densities in Chapter 3.

Theorem 1.4.4 *Let X be a separable metric space and $E \subseteq X$ be an s-set. Then,*

$$\overline{D}^s(E, x) \leqslant 1,$$

for $\mathcal{H}^s$-almost every $x \in X$.

One of the reasons that densities are of so much interest is because of their relationship to rectifiability. The following theorem is an example of this.

Theorem 1.4.5 *Let $E \subseteq \mathbb{R}^n$ be an k-set for some integer $0 \leqslant k \leqslant n$. Then E is k-rectifiable if and only if $D^k(E, x)$ exists and has equality with 1 for $\mathcal{H}^k$-almost every $x \in E$. Furthermore, $D^s(E, x)$ exists and takes a positive, finite value for $\mathcal{H}^s$-almost every $x \in E$ only if s is an integer.*

Theorem 1.4.5 follows from the main result in [9]. The proof of the above was the result of a long string of results culminating with a stronger result proved in a paper by Preiss, [21]. Earlier work was completed by Besicovitch, Marstrand and Mattila.

Theorem 1.4.5 does not generalise completely to metric spaces, and this is discussed further in Chapter 3.

Theorem 1.4.6 below is a form of the Vitali covering theorem, as is presented in [19]. Although this does not relate specifically to densities, it is presented in this section as its utility to this area is quite clear, and this is the only context in which it will be used in this thesis.

Theorem 1.4.6 *Let μ be a Radon measure on $\mathbb{R}^n$, $E \subseteq \mathbb{R}^n$ and*

$$\mathcal{E} \subseteq \left\{\overline{B}\left(x,r\right) : x \in \mathbb{R}^n, r \in (0,\infty)\right\}$$

be such that

$$\inf\left\{r \in (0,\infty) : \overline{B}\left(x,r\right) \in \mathcal{E}\right\} = 0$$

for every $x \in E$. Then there exists a countable collection of disjoint sets, $\mathcal{A} \subseteq \mathcal{E}$, such that

$$\mu\left(E \backslash \bigcup_{A \in \mathcal{A}} A\right) = 0.$$

Furthermore, if $\mu = \mathcal{L}^n$, then, for any $\varepsilon > 0$, $\mathcal{A}$ can be chosen such that

$$\sum_{A \in \mathcal{A}} \mathcal{L}^n\left(A\right) \leqslant \mathcal{L}^n\left(E\right) + \varepsilon.$$

1.5 Tangent Measures

Tangent measures were first defined by Preiss in [21]. They represent what a measure looks like on a small scale in much the same way that a classical tangent does for a smooth curve. They do this by zooming in on a particular point and taking a limit. Therefore, before we can give a formal definition of a tangent measure, we must first define a suitable form of convergence.

Definition 1.5.1 *A sequence of Radon measures, $\{\mu_k\}_{k\in\mathbb{N}}$, on a metric space converges weakly to a measure, μ, and we write $\mu_k \stackrel{w}{\to} \mu$ if and only if, for every $\phi \in C_c\left(\mathbb{R}^n\right)$, we have $\int \phi \, d\mu_n \to \int \phi \, d\mu$.*

We mean by $C_c\left(\mathbb{R}^n\right)$ the space of all compactly supported continuous functions mapping $\mathbb{R}^n$ to $\mathbb{R}$.

Lemma 1.5.2 *Let $\{\mu_i\}_{i\in\mathbb{N}}$ be a sequence of Radon measures and μ a Radon measure, all on a locally compact metric space X, such that $\mu_i \stackrel{w}{\to} \mu$. Then, for any open set $U \subseteq X$*

and compact set $K \subseteq X$, we have

$$\mu(U) \leqslant \liminf_{i\to\infty} \mu_i(U),$$

and

$$\mu(K) \geqslant \liminf_{i\to\infty} \mu_i(K).$$

We are now ready to formulate our definition of tangent measures. We assume from now on that we are working in Euclidean space, $\mathbb{R}^n$.

Definition 1.5.3 *We let μ be a Radon measure on $\mathbb{R}^n$ and $s \geqslant 0$. We define the rescaled measures $\mu_{x,r}$ by*

$$\mu_{x,r}(E) = \frac{\mu(rE+x)}{r^s}.$$

Then we say that a Radon measure ν is a tangent measure to μ at x if and only if there exists a sequence $\{r_i\}_{i\in\mathbb{N}} \downarrow 0$ such that $\mu_{x,r_i} \stackrel{w}{\to} \nu$ as $i \to \infty$.

We denote the set of all such tangents of μ at x as $\mathrm{Tan}^s(\mu, x)$.

Actually, the original definition of tangent measures was slightly more general in that the normalisation is not restricted to be some power of the scaling ratio, but is an arbitrary sequence converging to zero. This will not prove to be too much of a restriction for the type of measures we will be looking at - that is, those with positive and finite upper and lower densities - as the more general definition would only add scalar multiples (including zero) to our tangent set.

As was the case with densities, tangent measures provide a characterisation of rectifiability. The following theorem is based on a result presented in [9].

Theorem 1.5.4 *Let $E \subseteq \mathbb{R}^n$ and $0 \leqslant k \leqslant n$ be an integer. Then E is k rectifiable if and only if there exists, for $\mathcal{H}^k$-almost every $x \in E$, a k-dimensional plane L, such that*

$$\mathrm{Tan}^k\left(\mathcal{H}^k \llcorner E, x\right) = \left\{\mathcal{H}^k \llcorner L\right\}.$$

The conditions on the tangent measures Theorem 1.5.4 can be relaxed significantly. The extent to which this can be done is the topic of Chapter 2.

The following two results were proved originally proved in [21] and are presented for the definition used here in [20].

Theorem 1.5.5 *We let* μ *be a Radon measure on* $\mathbb{R}^n$ *and* $s \geqslant 0$. *Then* $\operatorname{Tan}^s(\mu, x)$ *is closed.*

Theorem 1.5.6 *We let* μ *be a Radon measure on* $\mathbb{R}^n$ *and* $s \geqslant 0$. *Then, for* μ*-almost every* $x \in \mathbb{R}^n$, *if* $\nu \in \operatorname{Tan}^s(\mu, x)$ *and* $a \in \operatorname{supp}(\nu)$ *then*

$$\nu_{a,1} \in \operatorname{Tan}^s(\mu, x).$$

Since, for Hausdorff measures restricted to s-sets, the measure of a ball of radius r scales with r^s, we get the following corollary to Theorem 1.5.6.

Corollary 1.5.7 *Let* $s \geqslant 0$, $E \subseteq \mathbb{R}^n$ *be an* s*-set. Then, for* $\mathcal{H}^s$*-almost every* $x \in \mathbb{R}^n$, *if* A *is an* s*-set such that* $\mathcal{H}^s \llcorner A \in \operatorname{Tan}^s(\mathcal{H}^s \llcorner E, x)$, $a \in A$ *and* $r > 0$ *then*

$$\mathcal{H}^s \llcorner r(A - a) \in \operatorname{Tan}^s(\mathcal{H}^s \llcorner E, x).$$

The following lemma follows from the more general Lemma 14.7 in [19], or immediately from a result given in [20]. However, a standalone proof is given here for convenience and to highlight some differences between the definition of tangent measures as defined here and the originals. Indeed, this result would be false under Preiss's original definition.

Proposition 1.5.8 *Let* $\nu \in \operatorname{Tan}^s(\mu, x)$. *Then*

$$\underline{D}^s(\nu, 0) \geqslant \underline{D}^s(\mu, x)$$

and

$$\overline{D}^s(\nu, 0) \leqslant \overline{D}^s(\mu, x).$$

Proof Pick (r_i) such that $\mu_{x,r_i} \xrightarrow{w} \nu$ and then choose $r_0 > 0$. Now

$$\begin{aligned}
\underline{D}^s(\mu, x) &= \liminf_{r \downarrow 0} \frac{\mu\left(\overline{B}_r(x)\right)}{(2r)^s} \\
&\leqslant \liminf_{i \to \infty} \frac{\mu\left(\overline{B}_{r_0 r_i}(x)\right)}{(2r_0 r_i)^s} \\
&= \liminf_{i \to \infty} \frac{\mu_{x,r_i}\left(\overline{B}_{r_0}(0)\right)}{(2r_0)^s} \\
&\leqslant \frac{\nu\left(\overline{B}_{r_0}(0)\right)}{(2r_0)^s} \quad \text{by Lemma 1.5.2.}
\end{aligned}$$

But, since r_0 is arbitrary, we conclude that $\underline{D}^s(\nu, 0) \geqslant \underline{D}^s(\mu, x)$. A similar argument using open balls gives $\overline{D}^s(\nu, 0) \leqslant \overline{D}^s(\mu, x)$. □

Corollary 1.5.9 *Let* $\nu \in \operatorname{Tan}^s(\mu, x)$. *Then*

$$\underline{D}^s(\mu, x) > 0 \Rightarrow 0 \in \operatorname{supp}(\nu).$$

Chapter 2

Tangent Measures of a One Dimensional Subset of $\mathbb{R}^2$

2.1 Introduction

In Chapter 1 we stated that rectifiability in $\mathbb{R}^n$ is essentially equivalent to all of the set's only tangent measure being a m-dimensional plane. Thus tangent measures can be used as a means of establishing rectifiability. A natural question then follows: How far can the conditions on the tangent measures be relaxed, whilst still ensuring rectifiability?

Since, in a rectifiable set, the tangents are planes anyway, the question essentially becomes one on the existence of sets with certain kinds of tangents. We will be imposing a limit on how far the result can be extended by demonstrating the existence of an unrectifiable set with tangents that are somehow quite close to meeting known conditions for implying rectifiability.

In this chapter we will only be looking at 1-dimensional subsets of $\mathbb{R}^2$, and so, for convenience, we write $\operatorname{Tan}(\mu, x)$ for $\operatorname{Tan}^1(\mu, x)$.

We first state a result by O'Neil, presented in [20], which gives a positive result in this matter.

Theorem 2.1.1 *Let $m, n \in \mathbb{N}$ such that $m \leqslant n$. Then, if μ is a Radon measure on $\mathbb{R}^n$*

such that, for μ-almost every $x \in \mathbb{R}^n$,

(i) $0 < \underline{D}^s(\mu, x) \leqslant \overline{D}^s(\mu, x) < \infty$, *and*

(ii) the projection of any $\nu \in \operatorname{Tan}^s(\mu, x)$ onto any m-dimensional subspace of $\mathbb{R}^n$ is a convex set,

then μ is m-rectifiable.

The condition that μ be Radon is slightly stricter than in the original statement but is sufficient for our purposes and avoids introducing new terminology.

In this chapter we will demonstrate, by an example based on a set originally defined by Dickinson in [11], that the condition on the projections of tangent measures cannot be weakened to the union of two convex sets. In particular the tangents of this unrectifiable set consist of lines, halflines and the union of two halflines. This would seem to indicate that the above theorem is somehow quite close to being optimal.

An example was given by De Lellis and Otto in [10] of a set whose tangents consist entirely lines, halflines, segments, lines with a segment removed or the empty set, however the set in question did not meet the other criterion as its lower density was zero at almost every point.

2.2 Approximately Constant Functions

A convenient way of expressing the 1-dimensional subsets of $\mathbb{R}^2$ that we will be looking at is expressing them as graphs of functions, and restricting 1-dimensional Hausdorff measure to their graphs.

We will be using functions whose codomain is the set $\mathbb{R} \cup \{\infty\}$. We are doing this as we are only concerned as to whether a sequence of functions is unbounded, not whether it limits to ∞, $-\infty$ or both. It will soon become clear why this is the case. We use the obvious arithmetic on this set, apart from $\infty - \infty$ which we define to be zero; this is because the sum will only ever be used to indicate that a function taking the value ∞ has zero distance from another function taking the same value, which is not unreasonable.

Definition 2.2.1 *Let $f : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ and $S \subseteq \mathbb{R}^2$ its graph, that is*

$$S = \{(x, f(x)) : x, f(x) \in \mathbb{R}\}.$$

Then we call $\mathcal{H}^1 \llcorner S$ the graph measure of f and denote it γ_f.

The types of functions that we are interested in are those that, despite having a fractal structure, are in some sense quite flat locally. We formalise with the notion of approximately constant functions.

Definition 2.2.2 *Let $f : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$. If for any given $\varepsilon > 0$ there exist $-\infty = a_0 < a_1 < \ldots < a_n = \infty$ such that, for all $1 \leqslant i \leqslant n$,*

$$\sup_{x \in [a_{i-1}, a_i]} f(x) - \inf_{x \in [a_{i-1}, a_i]} f(x) < \varepsilon \min\{a_i - a_{i-1}, 1\},$$

then we say that f is an approximately constant function. We also define

$$w_\varepsilon(f) = \sup\{\min\{a_i - a_{i-1} : 1 < i < n\}\},$$

where the supremum is taken over all partitions that satisfy the above condition.

Since we will be using these to look at tangent measures, we will want to rescale these functions in the same way.

Definition 2.2.3 *Let $f : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$, $r \in (0, \infty)$ and $x \in \mathbb{R}$. Then we write*

$$f_{x,r}(t) = \frac{f(rt + x) - f(x)}{r}.$$

Proposition 2.2.4 *Let f be approximately constant. Then for any $r \in (0, \infty)$ and $x_0 \in \mathbb{R}$ we have that $f_{x_0,r}$ is also approximately constant.*

Proof We fix $\varepsilon > 0$. Then we can find $-\infty = a_0 < a_1 < \ldots < a_n = \infty$ such that

$$\sup_{x \in [a_{i-1}, a_i]} f(x) - \inf_{x \in [a_{i-1}, a_i]} f(x) < r\varepsilon \min\{a_i - a_{i-1}, 1\}.$$

Now, taking $\tilde{a}_i = \frac{a_i - x_0}{r}$ for each $0 \leqslant i \leqslant n$ gives us suitable choices to show that $f_{x_0,r}$ is also approximately constant. $\square$

Because these functions are so flat, there is a very simple characterisation of their graph measures.

Proposition 2.2.5 *Let $f : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ be approximately constant. Then for any $A \in \mathbb{R}^2$*

$$\gamma_f(A) = \mathcal{L}\{x \in \mathbb{R} : (x, f(x)) \in A\}.$$

Proof Put $P = \{x \in \mathbb{R} : (x, f(x)) \in A\}$.

Since the mapping $(x, y) \mapsto (x, 0)$ is Lipschitz with constant at most 1, we have

$$\begin{aligned} \gamma_f(A) &= \mathcal{H}^1(A \cap f(\mathbb{R})) \\ &\geqslant \mathcal{H}^1\{(x, 0) : (x, f(x)) \in A\} \\ &= \mathcal{L}(P). \end{aligned} \tag{2.1}$$

On the other hand, since f is approximately constant, for any $\delta > 0$ and $-\infty < a < b < \infty$, we can find $a = \tilde{a}_0 < \ldots < \tilde{a}_{\tilde{n}} = b$ such that, for every $1 < i < \tilde{n}$,

$$\sup_{x \in [\tilde{a}_{i-1}, \tilde{a}_i]} f(x) - \inf_{x \in [\tilde{a}_{i-1}, \tilde{a}_i]} f(x) < \frac{\delta}{4(b-a)}(\tilde{a}_i - \tilde{a}_{i-1}).$$

And thus, splitting the rectangles if necessary, we can find $a = a_0 < a_1 < \ldots < a_n = b$ such that for all $1 < i < n$

$$\sup_{x \in [a_{i-1}, a_i]} f(x) - \inf_{x \in [a_{i-1}, a_i]} f(x) < \frac{\delta}{2}(a_i - a_{i-1})$$

and

$$a_i - a_{i-1} < \frac{\delta}{2}.$$

This gives that for any $0 < \delta < 1$ the diameter of the rectangle is less than δ, and so

$$\begin{aligned}
\gamma_f\left([a,b]\times\mathbb{R}\right) &= \mathcal{H}^1\left(\left([a,b]\times\mathbb{R}\right)\cap f\left(\mathbb{R}\right)\right)\\
&= \lim_{\delta\downarrow 0}\mathcal{H}^1_\delta\left(\left([a,b]\times\mathbb{R}\right)\cap f\left(\mathbb{R}\right)\right)\\
&\leqslant \lim_{\delta\downarrow 0}\sum_{i=1}^{n}\sqrt{\left(a_i-a_{i-1}\right)^2+\frac{\delta^2}{4}\left(a_i-a_{i-1}\right)^2}\\
&= \lim_{\delta\downarrow 0}\sqrt{1+\frac{\delta^2}{4}}\sum_{i=1}^{n}a_i-a_{i-1}\\
&= \lim_{\delta\downarrow 0}\sqrt{1+\frac{\delta^2}{4}}\left(b-a\right)\\
&= b-a.
\end{aligned}$$

But, by the definition of Lebesgue measure, there exist, for any $\varepsilon > 0$, intervals, I_k, where $k \in K$ and K is finite or countable, each of length $l_k \in (0,\infty)$, such that $P \subseteq \bigcup_{k\in K} I_k$ but $\sum_{k\in K} l_k \leqslant \mathcal{L}(P) + \varepsilon$, and so

$$\begin{aligned}
\gamma_f\left(A\right) &\leqslant \gamma_f\left(P\times\mathbb{R}\right)\\
&\leqslant \sum_{k\in K}\gamma_f\left(I_k\times\mathbb{R}\right)\\
&= \sum_{k\in K} l_k\\
&\leqslant \mathcal{L}\left(P\right)+\varepsilon. \qquad (2.2)
\end{aligned}$$

The inequalities (2.1) and (2.2), along with the arbitrariness of ε, give the result. □

An immediate question we may ask is whether approximately constant functions are the only ones with the above property. The answer is no, as the following example demonstrates.

Example 2.2.6 *Let*

$$A_n = \bigcup_{k=0}^{3^{n-1}-1}\left(3^{-n}\left[1,2\right]+k3^{1-n}\right)$$

and $f = \sum_{n\in\mathbb{N}} 3^{-(n+1)}\chi_{A_n}$. *Then f has a graph measure as in Proposition 2.2.5 but is not approximately constant.*

The above example has the projection property since it can be covered by a series of arbitrarily small equilateral triangles whose projections do not overlap, and whose diameter is equal to the length of their projection. They cannot, however, be covered by arbitrarily thin strips as any strip would have a projection with length in the range $\left(3^{-n}, 3^{-(n-1)}\right]$, for some $n \in \mathbb{N}$, and such a strip would have to contain a jump with height at least $3^{-(n+1)}$; it cannot, therefore, be an approximately constant function.

We now give a definition for convergence that is appropriate to finding the tangents to the graphs. We call it local convergence in measure, and it is very closely based on the standard definition, but slightly tailored to our current needs, in particular in how it allows unbounded sequences of functions to converge to a value of ∞.

Definition 2.2.7 *Let $f_n, f : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$. Then we say that f_n converges locally in measure to f, and we write $f_n \overset{m}{\to} f$, if and only if given any $a, b \in \mathbb{R}$ and $\varepsilon > 0$*

$$\lim_{n\to\infty} \mathcal{L}\left\{x \in [a,b] : |f_n(x) - f(x)| > \varepsilon \text{ and } \max\{|f_n(x)|, |f(x)|\} < \frac{1}{\varepsilon}\right\} = 0.$$

In order to use this to calculate tangent measures, we would hope that this kind of convergence of the functions was equivalent to weak convergence of their graph measures. Actually, we cannot quite achieve this, but one implication is true and, in the other direction, we can get a partial result, which will be sufficient for our purposes. We begin with the direction that allows us to show the existence of tangent measures.

Theorem 2.2.8 *Let f_n, f be approximately constant functions, then*

$$f_n \overset{m}{\to} f \Rightarrow \gamma_{f_n} \overset{w}{\to} \gamma_f.$$

Proof Fix $\phi \in C_c\left(\mathbb{R}^2\right)$ and $\varepsilon > 0$.

Let $X = \{x \in \mathbb{R} \colon (\{x\} \times \mathbb{R}) \cap \operatorname{supp}(\phi) \neq \emptyset\}$, that is the projection of $\operatorname{supp}(\phi)$ onto the x-axis, then let $a = \inf X$ and $b = \sup X$. Similarly let $Y = \{y \in \mathbb{R} \colon (\mathbb{R} \times \{y\}) \cap \operatorname{supp}(\phi) \neq \emptyset\}$ and $s = \sup Y$.

Since ϕ is uniformly continuous there exists some $\delta > 0$ such that $|f_n(x) - f(x)| < \delta \Rightarrow |\phi(x, f_n(x)) - \phi(x, f(x))| < \frac{\varepsilon}{2(b-a)}$. And since $f_n \overset{m}{\to} f$ we have, for sufficiently large n and for all $x \in [a,b] \setminus E$ where $\mathcal{L}(E) < \frac{\varepsilon}{4 \max|\phi|}$, either $|f_n(x) - f(x)| < \delta$ or $f_n(x), f(x) > s$ and thus $\phi(x, f_n(x)) = \phi(x, f(x)) = 0$ (defining $\phi(x, \infty) = 0$). So,

$$\begin{aligned}
\left|\int \phi\, d\mu - \int \phi\, d\mu_n\right| &= \left|\int_a^b \phi(x, f(x))\, dx - \int_a^b \phi(x, f_n(x))\, dx\right| \\
&= \left|\int_a^b \phi(x, f(x)) - \phi(x, f_n(x))\, dx\right| \\
&\leqslant \int_a^b |\phi(x, f(x)) - \phi(x, f_n(x))|\, dx \\
&< \int_{[a,b]\setminus E} \frac{\varepsilon}{2(b-a)}\, dx + \int_E 2\max|\phi|\, dx \\
&< \int_a^b \frac{\varepsilon}{2(b-a)}\, dx + \frac{\varepsilon}{4\max|\phi|} 2\max|\phi| \\
&= \varepsilon,
\end{aligned}$$

and hence $\gamma_{f_n} \overset{w}{\to} \gamma_f$.

□

We now begin work on the opposite direction.

Lemma 2.2.9 *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of approximately constant functions and $c \in \mathbb{R} \cup \{\infty\}$. Then,*

$$\gamma_{f_n} \overset{w}{\to} \mathcal{H}^1 \llcorner (\mathbb{R} \times \{c\}) \Rightarrow f_n \overset{m}{\to} c,$$

where we interpret $\mathcal{H}^1 \llcorner (\mathbb{R} \times \{\infty\})$ as the zero measure.

Proof Fix $a < b$ and $0 < \varepsilon < b - a$.

Then for $c \in \mathbb{R}$ we define $\phi \in C_c(\mathbb{R}^2)$ by

$$\phi(x) = \max\left\{1 - \frac{2}{\varepsilon} d\left(x, \left[a + \frac{\varepsilon}{2}, b - \frac{\varepsilon}{2}\right] \times \left[c - \frac{\varepsilon}{2}, c + \frac{\varepsilon}{2}\right]\right), 0\right\}.$$

This gives us

$$\int \phi\, d\mathcal{H}^1 \llcorner \{c\} = b - a - \frac{\varepsilon}{2}$$

and

$$\begin{aligned}\int \phi \, d\gamma_{f_n} &= \int_{a-\varepsilon}^{b+\varepsilon} \phi\left(x, f_n(x)\right) \, dx \\ &\leqslant \int_a^b \phi\left(x, f_n(x)\right) \, dx + \varepsilon \\ &\leqslant \mathcal{L}\left\{x \in [a,b] : \left|f_n(x) - c\right| \leqslant \varepsilon\right\} \\ &= b - a - \mathcal{L}\left\{x \in [a,b] : \left|f_n(x) - c\right| > \varepsilon\right\}.\end{aligned}$$

But then for sufficiently large n, since $\gamma_{f_n} \xrightarrow{w} \mathcal{H}^1 \llcorner \{c\}$,

$$\begin{aligned}\mathcal{L}\left\{x \in [a,b] : \left|f_n(x)\right| > \varepsilon\right\} &\leqslant b - a - \int \phi \, d\gamma_{f_n} \\ &< b - a - \int \phi \, d\mathcal{H}^1 \llcorner \{c\} + \frac{\varepsilon}{2} \\ &= \varepsilon.\end{aligned}$$

If, on the other hand, $c = \infty$ then we define $\psi \in C_c\left(\mathbb{R}^2\right)$ by

$$\psi(x) = \max\left\{1 - d\left(x, [a,b] \times \left[\frac{-1}{\varepsilon}, \frac{1}{\varepsilon}\right]\right), 0\right\}.$$

Then we have

$$\begin{aligned}\int \psi \, d\gamma_{f_n} &\geqslant \int_a^b \psi\left(x, f_n(x)\right) \, dx \\ &\geqslant \mathcal{L}\left\{x \in [a,b] : \left|f_n(x)\right| < \frac{1}{\varepsilon}\right\}.\end{aligned}$$

But, since $\gamma_{f_n} \xrightarrow{w} 0$, we have, for sufficiently large n, $\mathcal{L}\left\{x \in [a,b] : \left|f_n(x)\right| < \frac{1}{\varepsilon}\right\} < \varepsilon$.

In either case, the arbitrariness of a, b and ε give $f_n \xrightarrow{m} c$. $\square$

Lemma 2.2.10 *Let f_n be approximately constant functions, $-\infty = a_0 < \ldots < a_k = \infty$ be extended real numbers and $\mu_1, \ldots, \mu_k$ be measures on $\mathbb{R}^2$, then*

$$\gamma_{f_n} \llcorner ([a_{j-1}, a_j] \times \mathbb{R}) \xrightarrow{w} \mu_j \llcorner ([a_{j-1}, a_j] \times \mathbb{R}) \ \forall j \in \{1, \ldots, k\}$$
$$\Leftrightarrow \gamma_{f_n} \xrightarrow{w} \mu := \sum_{j=1}^{k} \mu_j \llcorner ([a_{j-1}, a_j] \times \mathbb{R}).$$

Proof Since for any $1 \leqslant j < k$ there exists $\delta > 0$ such that $\gamma_{f_n}([a_j - \delta, a_j])$ is arbitrarily small, we have, for any $1 \leqslant j < k$ and $1 \leqslant i \leqslant k$,

$$\mu(\{a_j\} \times \mathbb{R}) = \mu_i \llcorner ([a_{i-1}, a_i] \times \mathbb{R})(\{a_j\} \times \mathbb{R}) = 0.$$

Thus we may fix $\phi \in C_c(\mathbb{R}^2)$ and write, assuming the left hand side of the equivalence,

$$\begin{aligned} \int \phi \, d\gamma_{f_n} &= \sum_{j=1}^{k} \int_{[a_{j-1}, a_j] \times \mathbb{R}} \phi \, d\gamma_{f_n} \\ &= \sum_{j=1}^{k} \int \phi \, d\gamma_{f_n} \llcorner ([a_{j-1}, a_j] \times \mathbb{R}) \\ &\to \sum_{j=1}^{k} \int \phi \, d\mu_j \llcorner ([a_{j-1}, a_j] \times \mathbb{R}) \\ &= \int \phi \, d\mu. \end{aligned}$$

Conversely, we can assume the right hand side, define $\psi_n(x) = \max\{1 - nd([a_{j-1}, a_j], x), 0\}$ (noting $\psi_n \in C_c(\mathbb{R}^2)$), and write

$$\begin{aligned} \int \phi \, d\gamma_{f_n} \llcorner ([a_{j-1}, a_j] \times \mathbb{R}) &= \int_{[a_{j-1}, a_j] \times \mathbb{R}} \phi \, d\gamma_{f_n} \\ &= \int \phi \psi_n \, d\gamma_{f_n} + o(1) \\ &= \int \phi \psi_n \, d\mu + o(1) + o(1) \\ &\to \int_{[a_{j-1}, a_j] \times \mathbb{R}} \phi \, d\mu \\ &= \int \phi \, d\mu \llcorner ([a_{j-1}, a_j] \times \mathbb{R}) \\ &= \int \phi \, d\mu_j \llcorner ([a_{j-1}, a_j] \times \mathbb{R}). \end{aligned}$$

□

We now have the tools we need to prove the following theorem, which will enable us to restrict the set of possible measures. Indeed, the measure defined in the next chapter will have, at almost every point, all of the allowed measures in its tangent set, and so we will have determined it completely.

Theorem 2.2.11 *Let (f_n) be a sequence of approximately constant functions and μ be* *some measure. If for each* $\varepsilon > 0$ *we have* $\gamma_{f_n} \xrightarrow{w} \mu$ *and* $\lim_{n\to\infty} w_\varepsilon(f_n) = \infty$ *then there exists* $f : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ *and* $j \in \mathbb{R}$ *such that* f *is constant on* $(-\infty, j]$ *and* (j, ∞)*,* $f_n \xrightarrow{m} f$ *and* $\mu = \gamma_f$*.*

Proof Fix $\phi \in C_c(\mathbb{R}^2)$, and let $X = \{x \in \mathbb{R} \colon (\{x\} \times \mathbb{R}) \cap \operatorname{supp}(\phi) \neq \emptyset\}$, $a = \min\{\inf X, j\}$ and $b = \max\{\sup X, j\}$.

We take a subsequence, $\{f_{i(n)}\}$, such that $w_{\frac{1}{n}}(f_{i(n)}) \geqslant n$. Then there exist $(l_{i(n)}), (r_{i(n)}) \subseteq \mathbb{R}$ and $(j_{i(n)})$ such that

$$f(\mathbb{R}) \cap \operatorname{supp}(\phi) \subseteq L_{i(n)} \cup R_{i(n)},$$

where

$$L_{i(n)} = \left\{(x, y) \in \mathbb{R}^2 : x \in \left(-\infty, j_{i(n)}\right], y \in \left[l_{i(n)}, l_{i(n)} + \frac{1}{n}\right]\right\}$$

and

$$R_{i(n)} = \left\{(x, y) \in \mathbb{R}^2 : x \in \left(j_{i(n)}, \infty\right), y \in \left[r_{i(n)}, r_{i(n)} + \frac{1}{n}\right]\right\}.$$

We may now take a further subsequence, $f_{k(n)}$, such that $j_{k(n)} \to j \in [a, b]$.

Now, focussing on $(-\infty, j]$ we can take a still further subsequence, $f_{m(n)}$, to ensure either $l_{m(n)} \to l \in \mathbb{R}$ or $|l_{m(n)}| \to \infty$, in which case we set $l = \infty$. This convergence of $(j_{m(n)})$ and $(l_{m(n)})$ give us $f_{m(n)} \xrightarrow{m} l$ on $(-\infty, j]$. So, using Theorem 2.2.8 and Lemma 2.2.10, we can write

$$\gamma_{f_{m(n)}} \llcorner ((-\infty, j] \times \mathbb{R}) \xrightarrow{w} \mathcal{H}^1 \llcorner ((-\infty, j] \times \{l\}).$$

But, since we assumed weak convergence of the whole sequence, we have

$$\gamma_{f_n} \llcorner ((-\infty, j] \times \mathbb{R}) \xrightarrow{w} \mathcal{H}^1 \llcorner ((-\infty, j] \times \{l\}).$$

And, by Lemma 2.2.9, we have $f_n \xrightarrow{m} l$ on $(-\infty, j]$.

We similarly get

$$\gamma_{f_n} \llcorner ((j, \infty) \times \mathbb{R}) \xrightarrow{w} \mathcal{H}^1 \llcorner ((j, \infty) \times \{r\})$$

and $f_n \xrightarrow{m} r$ on (j, ∞).

Thus we define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} l & x \in (-\infty, j] \\ r & x \in (j, \infty), \end{cases}$$

where $f_n \overset{m}{\to} f$ and, by Lemma 2.2.10, $\gamma_{f_n} \overset{w}{\to} \gamma_f$. □

Corollary 2.2.12 *Let f be an approximately constant function such that, for any decreasing sequence, $\{r_i\}$, with $r_i \to 0$ as $i \to \infty$, any $x \in \mathbb{R}$ and any $\varepsilon > 0$, we have*

$$\lim_{i\to\infty} w_\varepsilon (f_{x,r_i}) = \infty.$$

Then, for all $x \in \mathbb{R}^2$, we have

$$\mu \in \operatorname{Tan}(\gamma_f, x) \Rightarrow \mu = \gamma_g,$$

where $g : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ is constant on $(-\infty, j]$ and (j, ∞), for some $j \in \mathbb{R}$.

Proof By Proposition 2.2.4 f_{x,r_i} are approximately constant and so the result follows immediately from Theorem 2.2.11. □

2.3 A Set from Dickinson

We are now ready to introduce the main object of study in this chapter: the set defined by Dickinson in [11]. Actually, the following definition is a slight variation of the one in [11], but there are no essential differences.

Definition 2.3.1 *Let*

$$R_m = \bigcup_{k=1}^{2^{m^2-1}} 2^{-m^2} (2k-1, 2k],$$

then

$$\sigma(x) := \begin{cases} \sum_{m\in\mathbb{N}} \frac{1}{m} 2^{-m^2} \chi_{R_m}(x) & x \in (0,1] \\ \infty & \textit{otherwise} \end{cases}$$

shall be called the Dickinson function; we put $R := \sigma((0,1])$ and $\rho := \gamma_\sigma$.

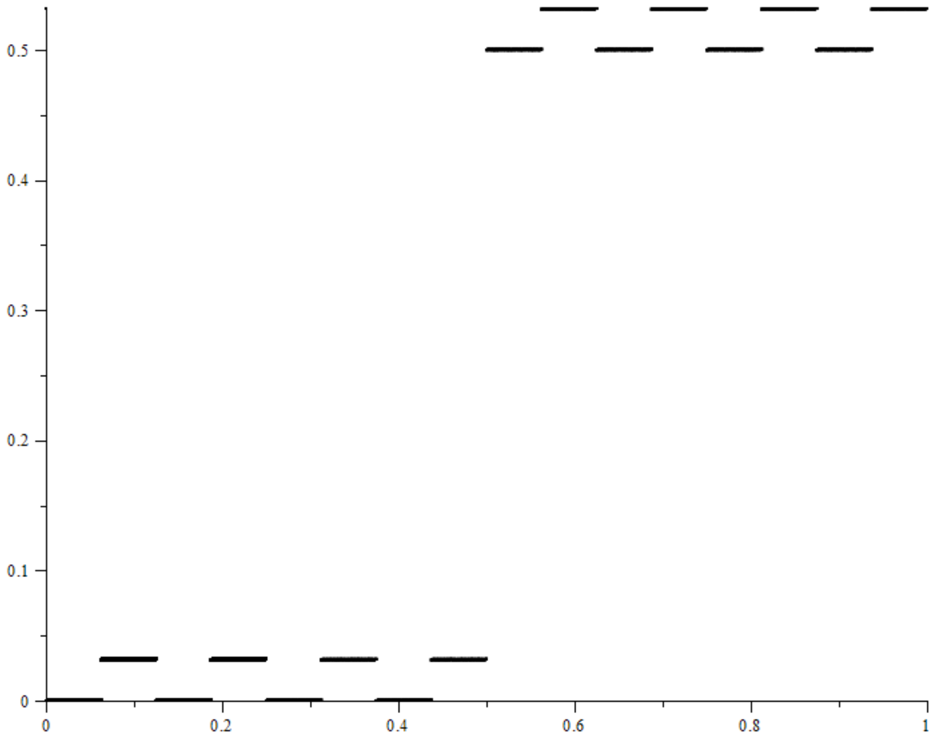

Figure 2.1: The Dickinson set, R

We begin by showing that this set can indeed be represented by approximately constant functions.

Lemma 2.3.2

$$\sum_{n=m+1}^{\infty} \frac{1}{n} 2^{-n^2} \leqslant \frac{1}{m+1} 2^{-2m} 2^{-m^2}.$$

Proof

$$\begin{aligned}
\sum_{n=m+1}^{\infty} \frac{1}{n} 2^{-n^2} &\leqslant \frac{1}{m+1} \sum_{n=m+1}^{\infty} 2^{-n^2} \\
&= \frac{1}{m+1} \sum_{n=1}^{\infty} 2^{-(m+n)^2} \\
&= \frac{1}{m+1} 2^{-m^2} \sum_{n=1}^{\infty} 2^{-n(n+2m)} \\
&\leqslant \frac{1}{m+1} 2^{-2m} 2^{-m^2} \sum_{n=1}^{\infty} 2^{-n} \\
&= \frac{1}{m+1} 2^{-2m} 2^{-m^2}.
\end{aligned}$$

□

Corollary 2.3.3 *The Dickinson function is approximately constant.*

Proof From Lemma 2.3.2 R can be contained in 2^{m^2} strips of height $\frac{1}{m+1} 2^{-2m} 2^{-m^2}$ and the result follows. □

The following lemma gives us the remaining condition we require in order to be able to apply Corollary 2.2.12.

Lemma 2.3.4 *Let $\{r_i\}_{i \in \mathbb{N}} \subseteq \mathbb{R}$ be a decreasing sequence with $r_i \to 0$ as $i \to \infty$. Then, for any $x \in \mathbb{R}^2$ and $\varepsilon > 0$,*

$$\lim_{i \to \infty} w_\varepsilon \left(\sigma_{x, r_i}\right) = \infty.$$

Proof Using the coverings implied in Lemma 2.3.2, it is sufficient to show that for a given $\varepsilon > 0$ and $n \in \mathbb{N}$ we can find, for any sufficiently small r, an $m \in \mathbb{N}$ such that

(i) $\frac{1}{r} 2^{-m^2} > n$,

(ii) $\frac{1}{r} \frac{2}{m+1} 2^{-2m} 2^{-m^2} < \varepsilon$, and

(iii) $\frac{2}{m+1}2^{-2m} < \varepsilon$.

Above, (i) gives us the increasing width required for $w_\varepsilon\left(\sigma_{x,r_i}\right) = \infty$ whilst (ii) and (iii) give us the two required restrictions on height.

We now pick $m \in \mathbb{N}$ to be such that

$$\frac{1}{\sqrt{m+1}}2^{-(m+1)^2} < r \leqslant \frac{1}{\sqrt{m}}2^{-m^2},$$

and so we have

$$\begin{aligned} \frac{1}{r}2^{-m^2} &\geqslant \sqrt{m} \\ &\to \infty \text{ as } m \to \infty, \end{aligned}$$

$$\begin{aligned} \frac{1}{r}\frac{1}{m+1}2^{-2m}2^{-m^2} &= \frac{1}{r}\frac{2}{m+1}2^{-(m+1)^2} \\ &= \frac{1}{r}\frac{1}{\sqrt{m+1}}2^{-(m+1)^2}\frac{2}{\sqrt{m+1}} \\ &< \frac{2}{\sqrt{m+1}} \\ &\to 0 \text{ as } m \to \infty, \end{aligned}$$

and

$$\frac{1}{m+1}2^{-2m} \to 0 \text{ as } m \to \infty.$$

But $m \to \infty$ as $r \to 0$, and thus the result follows. $\square$

We now give a lower bound on the lower density of this set, which we shall need later. Actually, this is the exact value for almost every point, but it will be more convenient to prove the reverse inequality later.

Lemma 2.3.5 *For every $x \in R$,*

$$\underline{D}^1\left(\rho, x\right) \geqslant \frac{1}{2}.$$

Proof Fix $x \in (0,1]$ and $r \in \left(0, \frac{1}{2}\right)$. Then there exist $m \in \mathbb{N}$ and $0 \leqslant k < 2^{m^2}$ such that

$r \in \left(2^{-(m+1)^2}, 2^{-m^2}\right]$ and $x \in \left(k2^{-m^2}, (k+1)\, 2^{-m^2}\right]$.

Now let

$$\begin{aligned} I &= B_r(x) \cap \left(k2^{-m^2}, (k+1)\, 2^{-m^2}\right) \\ &= \left(\max\left\{x - r, k2^{-m^2}\right\}, \min\left\{x + r, (k+1)\, 2^{-m^2}\right\}\right), \end{aligned}$$

so that $\sup I - \inf I \geqslant r$ (since $r \leqslant 2^{-m^2}$).

But, from Lemma 2.3.2, we have that, for all $x_1, x_2 \in \left(k2^{-m^2}, (k+1)\, 2^{-m^2}\right]$,

$$|\sigma(x_1) - \sigma(x_2)| \leqslant \frac{1}{m+1} 2^{-2m} 2^{-m^2}.$$

And so we may use Pythagoras's Theorem to give

$$\begin{aligned} \frac{\rho\left(B_r\left((x, \sigma(x))\right)\right)}{2r} &\geqslant \frac{\sqrt{r^2 - \frac{1}{m+1} 2^{-2m} 2^{-m^2}}}{2r} \\ &= \frac{\sqrt{r^2 - \frac{1}{2(m+1)} 2^{-(m+1)^2}}}{2r} \\ &\geqslant \frac{\sqrt{r^2 - \frac{1}{2(m+1)} r^2}}{2r} \\ &= \frac{1}{2}\sqrt{1 - \frac{1}{2(m+1)}} \\ &\to \frac{1}{2} \text{ as } m \to \infty. \end{aligned}$$

But, since $m \to \infty$ as $r \to 0$, we have $\underline{D}^1\left(\rho, (x, \sigma(x))\right) \geqslant \frac{1}{2}$. □

We now come to the part of section where we give the main thrust of how we are going to prove the existence of certain measures in the tangent set. The language of probability will be useful here. If we pick a point at random, and its tangent set contains a measure with probability 1, then almost all of the points must contain that measure.

As we rescale, different levels of iteration in the definition of the Dickinson function are going to dominate. We can use this to make the problem essentially discrete, by checking how the rescaled measure looks at scales where each level is dominant. To formalise this, we will use the notion of R sequences, which we define below.

Definition 2.3.6 *We say that a set $A \subseteq \mathbb{R}$ is an R_m set if and only if there exists $K \subseteq \left\{k \in \mathbb{N} \ : 1 \leqslant k \leqslant 2^{2m+1}\right\}$ such that*

$$A = \left(\bigcup_{k \in K} \left((k-1)\, 2^{-(m+1)^2}, k 2^{-(m+1)^2} \right] \right) + \left\{ j 2^{-m^2} : 0 \leqslant j < 2^{m^2} \right\}.$$

And $\{A_m\}$ is an R sequence if and only if A_m is an R_m set for every $m \in \mathbb{N}$.

We say that $\{A_m\}$ is a joint R sequence if and only if, for every $m \in \mathbb{N}$, $A_m = F_{2m-1} \cap G_{2m}$ where $\{F_m\}$ and $\{G_m\}$ are R sequences.

We can think of R_m as an event, which occurs if x occurs, in a certain collection of the $(m+1)$th iteration. It is repeated over the sections of the mth iteration, and so the probability of x being in R_m does not depend on which section of the mth iteration x was in. We formalise this notion of independence below.

Lemma 2.3.7 *Let (A_m) be an R sequence or a joint R sequence and J a finite subset of $\mathbb{N}$. Then $\mathcal{L}\left(\bigcap_{j \in J} A_j\right) = \prod_{j \in J} \mathcal{L}(A_j)$.*

Proof Take (A_m) to be an R sequence.

If we assume that the property holds for all J where $\#(J) = n$ and we are given a set $K \subseteq \mathbb{N}$ where $\#(K) = n+1$, then we can pick $k_0 = \min(K)$ so that

$$\mathcal{L}\left(\bigcap_{k \in K \setminus \{k_0\}} A_k \right) = \prod_{k \in K \setminus \{k_0\}} \mathcal{L}(A_k).$$

But each of the sets A_k where $k \in K \setminus \{k_0\}$ repeat over the intervals that make up A_{k_0} and thus so does their intersection. Thus

$$\mathcal{L}\left(\bigcap_{k \in K} A_k \right) = \prod_{k \in K} \mathcal{L}(A_k)$$

and, since the assumption is clearly true where J is a singleton set, the lemma is true for all R sequences by induction.

If $\{F_m\}$ and $\{G_m\}$ are R sequences then $F_1, G_2, F_3, G_4, \ldots$ is itself an R sequence and so the result extends to joint R sequences by the associativity of multiplication and intersection.

□

Below we state one of the classic Borel-Cantelli lemmas, and then relate it to R sequences.

Lemma 2.3.8 *Let (X, μ) be a probability space and $A_n \subseteq X$ where $\{A_n\}$ is independent and $\sum_{n\in\mathbb{N}} \mu(A_n) = \infty$, then*

$$\mu\left(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} A_n\right) = 1.$$

Corollary 2.3.9 *Let (A_n) be an R sequence or a joint R sequence such that $\sum_{n\in\mathbb{N}} \mathcal{L}(A_n) = \infty$. Then for almost every $x \in (0,1]$ there exists a subsequence $A_{k(n)}$ such that $x \in A_{k(n)}$ for every $n \in \mathbb{N}$.*

Proof If we randomly pick $x \in (0,1]$ using a uniform distribution, then Lemma 2.3.7 gives us that the events $x \in A_n$ are independent. So we can apply Lemma 2.3.8 to give

$$\mathcal{L}\left(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} A_n\right) = 1.$$

The above is the set of all $x \in (0,1)$ for which a suitable subsequence exists, and so one exists almost everywhere. □

So we now have all we need to begin the proof of our main theorem: if we can find an R sequence, or a joint R sequence, where, when $x \in R_m$ we can produce a rescaled measure that looks, with increasing m, more and more like one of our target tangent measures, then Corollary 2.3.9 is sufficient to show that that measure must be part of the tangent set at almost every point.

Theorem 2.3.10 *For $\mathcal{H}^1$ almost every $x \in R$ we have that $\operatorname{Tan}(\rho, x)$ consists exactly of the measures γ_f, where*

$$f(x) = \begin{cases} h_1 & x \in (-\infty, j] \\ h_2 & x \in (j, \infty), \end{cases}$$

for any $h_1, h_2 \in \mathbb{R} \cup \{\infty\}$ and $j \in \mathbb{R}$ such that $f(0) = 0$ or $f(\varepsilon) = 0$ for any $\varepsilon > 0$ (in case the jump is at 0).

Proof Lemma 2.3.4 allows us to apply Corollary 2.2.12 to give that any tangent measure of ρ is of the form γ_f where

$$f(x) = \begin{cases} h_1 & x \in (-\infty, j] \\ h_2 & x \in (j, \infty), \end{cases}$$

with $h_1, h_2 \in \mathbb{R} \cup \{\infty\}$ and $j \in \mathbb{R}$. On the other hand, Lemma 2.3.5 and Corollary 1.5.9 give us the further condition on the value of f near 0.

This means that $\operatorname{Tan}(\rho, x)$ (for any $x \in (0,1]$) may only consist of measures of the above form. It remains to show that, for almost every $x \in (0,1]$, $\operatorname{Tan}(\rho, x)$ does include all of the above possible measures.

Suppose that we have a sequence, (r_m), with $r_m \downarrow 0$ and an R sequence or a joint R sequence, (A_m), with $\sum_{m\in\mathbb{N}} \mathcal{L}(A_m) = \infty$ such that given any $\varepsilon > 0$ and $a, b \in \mathbb{R}$ where $a < b$ we have

$$\begin{aligned} &x \in A_m \Rightarrow \\ &\qquad \mathcal{L}\left\{t \in [a,b] : |\sigma_{x,r_m}(t) - f(t)| > \varepsilon \text{ and } \max\{|\sigma_{x,r_m}(t)|, |f(t)|\} < \frac{1}{\varepsilon}\right\} < a_m \end{aligned} \tag{2.3}$$

where $a_m \downarrow 0$. But, by Corollary 2.3.9, for almost every $x \in (0,1)$ there is a subsequence, $(A)_{l(m)}$, such that $x \in A_{l(m)}$ for every $m \in \mathbb{N}$. Thus we have $\sigma_{x,r_{l(m)}} \overset{m}{\to} f$ and so, by Theorem 2.2.8, $\gamma_f \in \operatorname{Tan}(\rho, x)$.

To avoid the cases at the very edge of R, we define $M_0 \in \mathbb{N}$ such that $x \in \left(2^{-M_0^2}, 1 - 2^{-M_0^2}\right]$ and $M_0 \geqslant 3$ (to ensure $\log(m) > 1$) and examine the sequences after this point.

We let, for the time being, $j = 0$, $h_1 = 2s - 1$ and $h_2 = 0$, where $s \in \{0,1\}$, and fix $\varepsilon > 0$ and $a, b \in \mathbb{R}$ where $a < b$. Next we find sequences (r_m), (A_m) and (a_m).

We let $r_m = \frac{1}{m} 2^{-m^2}$ and

$$K_m = \left\{ i \in \mathbb{N} : i \leqslant \left\lfloor \frac{1}{m \log m} 2^{2m+1} \right\rfloor \right\}.$$

Then

$$F_m = \left(\bigcup_{k \in K_m} \left((k-1)\, 2^{-(m+1)^2}, k2^{-(m+1)^2} \right] \right) + \left\{ i2^{-m^2} : 0 \leqslant i < 2^{m^2} \right\}$$

and $G_m = R_m - s2^{-m^2}$, where R_m is as it is in Definition 2.3.1), are R sequences; and thus $A_m = F_{2m} \cap G_{2m-1}$ forms a joint R sequence.

$\mathcal{L}(F_m) = 2^{-(2m+1)} \# (K_m)$ and $\mathcal{L}(G_m) = \frac{1}{2}$ for all $m \in \mathbb{N}$, so

$$\begin{aligned}
\sum_{m \in \mathbb{N}} \mathcal{L}(A_m) &\geqslant \sum_{m=M_0}^{\infty} \frac{1}{2} 2^{-(4m+1)} \# (K_{2m}) \\
&\geqslant \frac{1}{2} \sum_{m=M_0}^{\infty} 2^{-(2m+1)} \left(\frac{1}{m \log m} 2^{2m+1} - 1 \right) \\
&= \frac{1}{2} \left(\sum_{m=M_0}^{\infty} \frac{1}{m \log m} - \sum_{m=M_0}^{\infty} 2^{-(2m+1)} \right) \\
&\geqslant \frac{1}{2} \left(\infty - \frac{1}{6} \right) \\
&= \infty.
\end{aligned}$$

Now we take $1 \leqslant n \leqslant 2^{(2m)^2}$ such that $(n-1)\, 2^{-(2m)^2} < x \leqslant n2^{-(2m)^2}$, and $1 \leqslant k \leqslant 2^{(2m)+1}$ such that $(k-1)\, 2^{-(2m+1)^2} < x - (n-1)\, 2^{-(2m)^2} \leqslant k2^{-(2m+1)^2}$. That is, $x \in A_m$ implies that $k \in K_{2m}$ (the converse is not necessarily true). So, if we assume $x \in A_m$, then

$$(n-1)\, 2^{-m^2} < x \leqslant (n-1)\, 2^{-m^2} + \left(\frac{1}{m \log m} \right) 2^{2m+1} 2^{-(m+1)^2}.$$

So, for any $t \in [0, \sqrt{m}]$, we have

$$x + \frac{1}{m} 2^{-m^2} t \geqslant x > (n-1)\, 2^{-m^2}$$

and

$$\begin{aligned}
x + \frac{1}{m}2^{-m^2}t &\leqslant x + \frac{1}{\sqrt{m}}2^{-m^2} \\
&\leqslant (n-1)\,2^{-m^2} + \left(\frac{1}{m\log m}\right)2^{2m+1}2^{-(m+1)^2} + \frac{1}{\sqrt{m}}2^{-m^2} \\
&= n2^{-m^2} + \frac{-m\log m + 1 + \sqrt{m}\log m}{m\log m}2^{-m^2} \\
&< n2^{-m^2}.
\end{aligned}$$

We use the above two inequalities to give us

$$\begin{aligned}
\left|\sigma_{x,r_m}(t)\right| &= \frac{\left|\sigma\left(r_m t + x\right) - \sigma(x)\right|}{r_m} \\
&= m2^{m^2}\left|\sigma\left(\frac{1}{m}2^{-m^2}t + x\right) - \sigma(x)\right| \\
&\leqslant m2^{m^2}\left|\sum_{i=m+1}^{\infty}\frac{1}{i}2^{-i^2}\right| \\
&\leqslant m2^{m^2}\frac{1}{m+1}2^{-2m}2^{-m^2} \quad \text{by Lemma 2.3.2} \\
&= \frac{m}{m+1}2^{-2m} \\
&\to 0.
\end{aligned}$$

We also get that for any $t \in [-\sqrt{m}, 0]$

$$\begin{aligned}
x + \frac{1}{m}2^{-m^2}t &\geqslant x - \frac{1}{\sqrt{m}}2^{-m^2} \\
&> (n-1)\,2^{-m^2} - \frac{1}{\sqrt{m}}2^{-m^2} \\
&= (n-2)\,2^{-m^2} + \frac{m-\sqrt{m}}{m}2^{-m^2} \\
&> (n-2)\,2^{-m^2}.
\end{aligned}$$

Furthermore, for $t \in \left[-m, 0 - \frac{1}{\sqrt{\log m}}\right]$ we have

$$\begin{aligned} x + \frac{1}{m} 2^{-m^2} t &\leqslant x - \frac{1}{m} 2^{-m^2} \left(\frac{1}{\sqrt{\log m}} \right) \\ &\leqslant (n-1)\, 2^{-m^2} + \left(\frac{1}{m \log m} \right) 2^{-m^2} - \frac{1}{m} 2^{-m^2} \left(\frac{1}{\sqrt{\log m}} \right) \\ &= (n-1)\, 2^{-m^2} \frac{1 - \sqrt{\log m}}{m \log m} \\ &< (n-1)\, 2^{-m^2}. \end{aligned}$$

Exactly one of $\left((n-2)\, 2^{-m^2}, (n-1)\, 2^{-m^2}\right]$ and $\left((n-1)\, 2^{-m^2}, n 2^{-m^2}\right]$ belong to R_m, but, from our choice of G_m, which one is determined by the value of s, being the former if $s = 1$ and the latter if $s = 0$. So,

$$\begin{aligned} \left|\sigma_{x,r_m}(t) - (2s-1)\right| &= \left| \frac{\sigma(r_m t + x) - \sigma(x)}{r_m} - (2s-1) \right| \\ &= \left| m 2^{m^2} \left(\sigma\left(\frac{1}{m} t + x \right) - \sigma(x) \right) - h_1 \right| \\ &= \left| m 2^{m^2} \left(s \frac{1}{m} 2^{-m^2} + \sum_{i=m+1}^{\infty} \left(\frac{1}{i} 2^{-i^2} \chi_{R_i} \left(\frac{1}{m} t + x \right) \right) \right. \right. \\ &\qquad \left. \left. - (1-s) \frac{1}{m} 2^{-m^2} - \sum_{i=m+1}^{\infty} \left(\frac{1}{i} 2^{-i^2} \chi_{R_i}(x) \right) \right) - (2s-1) \right| \\ &\leqslant \left| m 2^{m^2} \left(s \frac{1}{m} 2^{-m^2} - (1-s) \frac{1}{m} 2^{-m^2} \right) - (2s-1) \right| \\ &\qquad + 2m 2^{m^2} \sum_{i=m+1}^{\infty} \frac{1}{i} 2^{-i^2} \\ &= 2m 2^{m^2} \sum_{i=m+1}^{\infty} \frac{1}{i} 2^{-i^2} \\ &\leqslant 2m 2^{m^2} \frac{1}{m+1} 2^{-2m} 2^{-m^2} \quad \text{by Lemma 2.3.2} \\ &= \frac{2m}{m+1} 2^{-m} \\ &\to 0. \end{aligned}$$

And so there exists $M \geqslant M_0$ such that for $m \geqslant M$ we have $[a, b] \subseteq [-\sqrt{m}, \sqrt{m}]$, $\left|\sigma_{x,r_m}(t)\right| < \varepsilon$ for $t \in [0, b]$ and $\left|\sigma_{x,r_m}(t) - (2s-1)\right| < \varepsilon$ for $t \in \left[a, 0 - \frac{1}{\sqrt{\log m}}\right]$. Thus (2.3) is satisfied taking

$$a_m = \begin{cases} b - a & m < M \\ \frac{1}{\sqrt{\log m}} & m \geqslant M. \end{cases}$$

We have now established that $\gamma_f \in \operatorname{Tan}(\rho, x)$ for $j = 0$, $h_1 \in \{1, -1\}$ and $h_2 = 0$; a similar argument will also give the same result when $j = 0$, $h_1 = 0$ and $h_2 \in \{-1, 1\}$.

We can use Corollary 1.5.7 to extend this, for $\mathcal{H}^1$-almost every x, to the cases where $h_1, h_2, j \in \mathbb{R}$, $h_1 h_2 = 0$ and $|h_1| + |h_2| > 0$.

If we take $j = 0$, $h_1 = \frac{1}{k}$ and $h_2 = 0$ we get a sequence of functions that converge locally in measure to the zero function as $k \to \infty$, and so by Theorems 2.2.8 and 1.5.5 we have also obtain the case where $h_1 = h_2 = 0$. Similarly we get the last case where $\{h_1, h_2\} = \{0, \infty\}$ by fixing $j \in \mathbb{R}$ and taking a sequence where $h_1 = k$ and $h_2 = 0$ or $h_1 = 0$ and $h_2 = k$.
$\square$

Finally, we use our knowledge of the tangent set in deriving the following properties of our set.

Theorem 2.3.11 *We have*

(i) $\underline{D}^1(\rho, x) = \frac{1}{2}$ *almost everywhere,*

(ii) $\overline{D}^1(\rho, x) = 1$ *almost everywhere, and*

(iii) ρ *is purely unrectifiable.*

Proof (i) From Theorem 2.3.10 (ii) with $t = 0$ we know that, for almost every $x \in R$, $\mathcal{H}^1 \llcorner (0, \infty) \times \{0\} \in \operatorname{Tan}(\rho, x)$. So, we may use Proposition 1.5.8 to get

$$\underline{D}^1(\rho, x) \leqslant \underline{D}^1\left(\mathcal{H}^1 \llcorner (0, \infty) \times \{0\}, x\right) = \frac{1}{2}.$$

Combining the above with Lemma 2.3.5 gives $\underline{D}^1(\rho, x) = \frac{1}{2}$.

(ii) We can similarly use Theorem 2.3.10 (i) to get that, for almost every $x \in R$,

$\mathcal{H}^1 \llcorner \mathbb{R} \times \{0\} \in \operatorname{Tan}(\rho, x)$ and thus

$$\overline{D}^1(\rho, x) \geqslant \overline{D}^1\left(\mathcal{H}^1 \llcorner \mathbb{R} \times \{0\}, x\right) = 1.$$

But, since for any $x \in \mathbb{R}^2$ and $r > 0$ we have

$$\rho(B_r(x)) \leqslant \rho((x - r, x + r) \times \mathbb{R}) = 2r,$$

we get $\overline{D}^1(\rho, x) \leqslant 1$. Hence $\overline{D}^1(\rho, x) = 1$.

(iii) Any 1-rectifiable set, E, has $\underline{D}^1\left(\mathcal{H}^1 \llcorner E, x\right) = 1$ for $\mathcal{H}^1$ almost every x (see, for example, Theorem 2.63 in [1]) but by (i) we have $\underline{D}^1(\rho, x) = \frac{1}{2}$ for $\mathcal{H}^1$ almost every x. Thus $\mathcal{H}^1(R \cap E) = 0$ and R is purely unrectifiable.

$\square$

Chapter 3

Unrectifiable Metric Spaces with High Lower Hausdorff s-Densities

3.1 Introduction

In this chapter we will be looking at what range of lower densities are possible in a metric space. We begin by stating the famous Besicovitch $\frac{1}{2}$-conjecture: If $E \subseteq \mathbb{R}^2$ is a 1-set and $\underline{D}^1(E, x) > \frac{1}{2}$, then E is rectifiable. We can generalise this problem using the following notation from [22].

Definition 3.1.1 *Let X be a separable metric space and $k \in \mathbb{N}$. We denote by $\sigma_k(X, d)$ the smallest number such that, for all $E \subseteq X$ satisfying $0 < \mathcal{H}^k(E) < \infty$, we have that*

$$\underline{D}_k(E) > \sigma_k \Rightarrow E \text{ is } k\text{-rectifiable},$$

for almost every $x \in E$.

In this language, the Besicovitch $\frac{1}{2}$-conjecture can be stated as $\sigma_1(\mathbb{R}^2) \leqslant \frac{1}{2}$.

Now, we know from Theorem 1.4.4 that $\sigma_k(X) \leqslant 1$, for any separable metric space X. Theorem 1.4.5 says that a rectifiable subset of $\mathbb{R}^n$ has an upper density of 1 almost everywhere and, thanks to Kirchheim's result in [15] and [16], this can be extended to metric spaces. In light of this, the problem can be thought of as one of the existence of unrectifiable subsets of a particular space with lower densities between a certain value and 1. For instance, we know that $\sigma_1\left(\mathbb{R}^2\right) \geqslant \frac{1}{2}$ because of the set defined in Chapter 2, and many more like it, which are unrectifiable but have a lower density of $\frac{1}{2}$. This will be our approach in putting a lower bound on $\sup\{\sigma_k(X) : X \text{ is a metric space}\}$.

Before we continue, we shall look briefly at the positive results in this field, to put the work into context. We begin with Besicovitch showing that $\sigma_1\left(\mathbb{R}^2\right) \leqslant 1 - 10^{-2576}$ in [2] and then $\sigma_1\left(\mathbb{R}^2\right) \leqslant \frac{3}{4}$ in [4]. Then Marstrand showing $\sigma_2\left(\mathbb{R}^3\right) < 1$ in [17], Mattila showing $\sigma_k(\mathbb{R}^n) < 1$ in [18], and Chlebík showing $\sup\{\sigma_k(\mathbb{R}^n) : n \in \mathbb{N}\} < 1$ in [6]. Finally Preiss and Tišer showing in [22] that $\sigma_1(X) \leqslant \frac{2+\sqrt{46}}{12} < \frac{3}{4}$, for any metric space X.

On the other hand it was shown by Schechter in [23], using a different, but similar, construction to the one used here, that there exists a metric space such that $\sigma_2(X) > \frac{1}{2}$.

In our construction we produce, for any $s > 1$, an unrectifiable metric space X of σ-finite $\mathcal{H}^s$ measure and a number $\xi(s)$ such that $\underline{D}^s(\mathcal{H}^s, x) \geqslant \xi(s) > 2^{\lceil s\rceil - s - 1}$, for any $x \in X$. And, consequently,

$$\sup\{\sigma_k(X) : X \text{ is a metric space}\} \geqslant \xi(s).$$

3.2 Construction of the Space

The idea of this construction is to take $\mathbb{R}$, with the Euclidean metric, and form a new metric space by specifying the distance between two points as a function of their Euclidean distance. It is clear that the resultant "distance" function will not, in general, be a bona fide metric, given an arbitrary function. There has been some study of the functions that do indeed produce a metric; we shall not go into too much depth, as we will need only the basics. Our references for the general results will be a paper of Doboš, [12], dealing with metric preserving functions on the Euclidean space $\mathbb{R}$, and a more general survey paper by Corazza, [8].

Definition 3.2.1 *We say that* $f : [0, \infty) \to [0, \infty)$ *is metric preserving if and only if for any metric space,* (X, d)*, we have that* $(X, f \circ d)$ *is also a metric space; we say that it is Euclidean metric preserving if and only if* $(\mathbb{R}, (x, y) \mapsto f(|x - y|))$ *is a metric space.*

We also set $\mathcal{E} = \left\{ f : [0, \infty) \to [0, \infty) : f \in C^0, f \text{ is Euclidean metric preserving} \right\}$.

Clearly every metric preserving function is also Euclidean metric preserving, but the converse is not true, as is demonstrated by an example given in [12]. Although we will not use the result here, it is interesting to note that a function that is metric preserving on the Euclidean space $\mathbb{R}^2$ will be metric preserving on any metric space.

Proposition 3.2.2 *Let* $f \in \mathcal{E}$*. Then* $(X, (x, y) \mapsto f(|x - y|))$ *is a separable, locally compact metric space with the same topology as the Euclidean space* $\mathbb{R}$*.*

Proof Since f is continuous, $\mathbb{Q}$ is still dense and a set in $\mathbb{R}$ is open if and only if it is open in $(X, (x, y) \mapsto f(|x - y|))$. It thus follows that the same sets are compact in each space and so $(X, (x, y) \mapsto f(|x - y|))$ is also locally compact. $\square$

Proposition 3.2.3 *Let* $f : [0, \infty) \to [0, \infty)$ *and* $0 \leqslant y \leqslant x < \infty$*, then* $f \in \mathcal{E}$ *if and only if*

(i) $f^{-1}(\{0\}) = \{0\}$,

(ii) f *is continuous at* 0*, and*

(iii) $\max\{f(x + y), f(x - y)\} \leqslant f(x) + f(y)$.

Proof We note that $f^{-1}(\{0\}) = \{0\}$ if and only if $f(|x - y|) = 0 \Leftrightarrow x = y$. Since the metric is translation invariant, we can consider the intermediate point in the triangle inequality to be 0, thus the triangle inequality reduces to condition (iii), with $f(x + y)$ being the maximum if the intermediate point is between the other two points, and $f(x - y)$ being the maximum otherwise. Symmetry is clear from the definition. Thus f being metric preserving is equivalent to conditions (i) and (iii) above.

It therefore remains to show that (ii) is equivalent to f being continuous. If f is discontinuous at 0 then f is obviously not continuous. On the other hand, suppose that f is continuous at zero and discontinuous at some point a, but then there would exist some $\varepsilon > 0$ such that $x \in (a - \varepsilon, a)$ and $y \in (a - x, \varepsilon)$ would provide a contradiction to (iii).

□

We now define the class of functions, $f_{s,\tau}$, that we shall use to produce our metrics.

Definition 3.2.4 *Given* $s \in (1,2)$ *and* $\tau \in \left(0, \frac{1}{3}\right)$*, we define the functions* $h_\tau, g_{s,\tau}, f_{s,\tau} : [0,\infty) \to [0,\infty)$ *as follows*

$$h_\tau(x) = \begin{cases} x & x \in [1, 1+\tau) \\ 3 + 3\tau - 2x & x \in \left[1+\tau, 1 + \frac{3\tau}{2} + \frac{\tau^2}{2}\right) \\ 2x - 2\tau^2 - 3\tau - 1 & x \in \left[1 + \frac{3\tau}{2} + \frac{\tau^2}{2}, (1+\tau)^2\right) \\ 0 & \textit{otherwise}, \end{cases}$$

$$g_{s,\tau} = h_\tau\left(\lambda_{s,\tau}(x-1) + 1\right),$$

and

$$f_{s,\tau}(x) = \sum_{k \in \mathbb{Z}} (1+\tau)^{-k} g_{s,\tau}\left((1+\tau)^{sk} x\right),$$

where $\lambda_{s,\tau} = \frac{2\tau+\tau^2}{(1+\tau)^s - 1}$.

Furthermore, we set

$$c_{s,\tau} = \frac{1-\tau^2}{\left(\frac{(\tau^2+3\tau)((1+\tau)^s-1)}{2(\tau^2+2\tau)} + 1\right)^{\frac{1}{s}}} \qquad d_{s,\tau} = \frac{1+\tau}{\left(\frac{\tau((1+\tau)^s-1)}{\tau^2+2\tau} + 1\right)^{\frac{1}{s}}}$$
$$c_\tau = c_{2,\tau} = \frac{1-\tau^2}{\sqrt{1 + \frac{3\tau}{2} + \frac{\tau^2}{2}}} \qquad d_\tau = d_{1,\tau} = 1 + \frac{\tau}{2}.$$

Not all of these functions will be Euclidean metric preserving, and most of this section is dedicated to investigating which of them are.

The upper bound of $\frac{1}{3}$ on τ is not too significant; it is just a simple value chosen to be small enough to avoid some complications involved with higher values, and small enough that we can be confident we are not missing any genuine metric preserving functions.

As we can see from the definition, and from Figure 3.2, $f_{s,\tau}$ is closely related to the function $x \mapsto x^{\frac{1}{s}}$. To make that relationship more explicit, we introduce the notion of geometrically

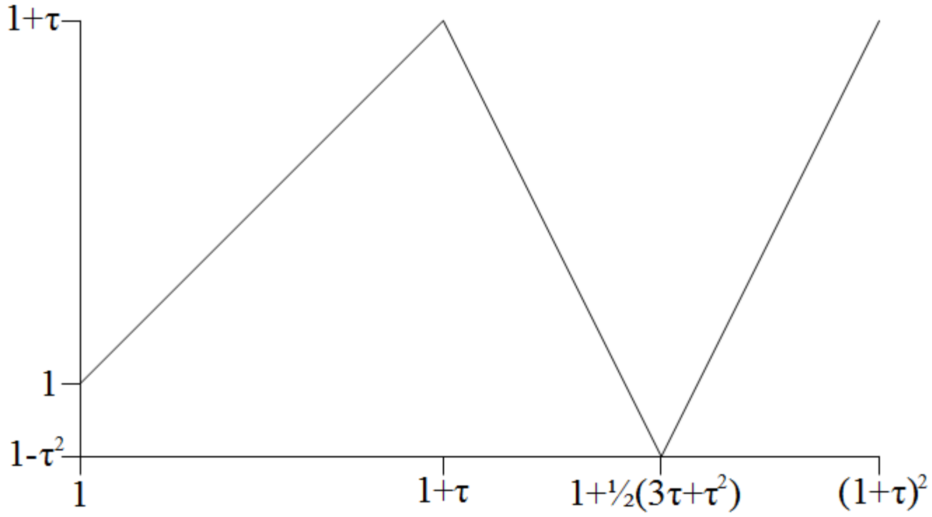

Figure 3.1: A graph of h_τ

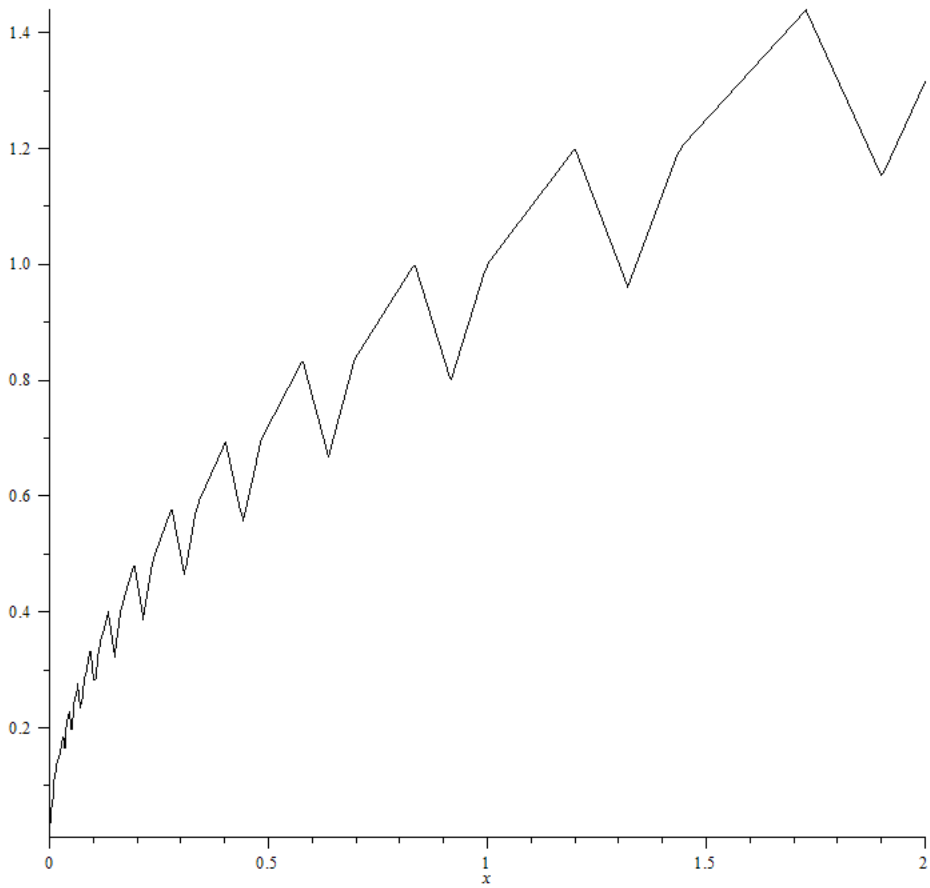

Figure 3.2: A graph of $f_{2,\frac{1}{5}}$

periodic functions, which will also come in very useful later.

Definition 3.2.5 *We shall call a function, $f:(0,\infty)\to\mathbb{R}$, geometrically periodic with period $p>1$ if*

$$f(x)=f(p^m x),$$

for all $x\in(0,\infty)$ and $m\in\mathbb{Z}$.

We now give some of the easier to obtain properties of $f_{s,\tau}$.

Proposition 3.2.6 *For the above functions, we have*

(i) for each $x\in(0,\infty)$,

$$\left\{k\in\mathbb{Z}:g_{s,\tau}\left((1+\tau)^{sk}x\right)>0\right\}=\left\{\left\lceil\frac{-\log x}{s\log(1+\tau)}\right\rceil\right\},$$

and thus, for each $x\in(0,\infty)$, there is exactly one non-zero term in the sum defining $f_{s,\tau}$,

(ii) $f_{s,\tau}^{-1}(\{0\})=\{0\}$,

(iii) $x^{-\frac{1}{s}}f_{s,\tau}(x)$ is geometrically periodic with period $(1+\tau)^s$ for $(x\in(0,\infty))$,

(iv) $c_\tau x^{\frac{1}{s}}\leqslant c_{s,\tau}x^{\frac{1}{s}}\leqslant f_{s,\tau}(x)\leqslant d_{s,\tau}x^{\frac{1}{s}}\leqslant d_\tau x^{\frac{1}{s}}$ for $(x\in(0,\infty))$,

(v) $\frac{c_{s,\tau}}{d_{s,\tau}}\geqslant 1-\frac{3\tau}{2}$, and

(vi) $f_{s,\tau}$ is continuous and piecewise affine.

Proof (i) From the definition of h we have that

$$\{x:h_\tau(x)>0\}=\left[1,(1+\tau)^2\right).$$

Thus

$$\{x:g_{s,\tau}(x)>0\}=[1,(1+\tau)^s)$$

and

$$\left\{x:g_{s,\tau}\left((1+\tau)^{sk}x\right)>0\right\}=\left[(1+\tau)^{-sk},(1+\tau)^{s(1-k)}\right),$$

which forms a partition of $(0,\infty)$ when k ranges over $\mathbb{Z}$.

(ii) This follows immediately from (i) and the definition of $f_{s,\tau}$.

(iii) Fixing $x\in(0,\infty)$, there exists $k\in\mathbb{Z}$ such that

$$\begin{aligned}
f_{s,\tau}\left(\left(1+\tau\right)^{s}x\right)\left(\left(1+\tau\right)^{s}x\right)^{-\frac{1}{s}} &= \left(1+\tau\right)^{-k}h_{s,\tau}\left(\left(1+\tau\right)^{s(k+1)}x\right)\left(\left(1+\tau\right)^{s}x\right)^{-\frac{1}{s}}\\
&= \left(1+\tau\right)^{-(k+1)}h_{s,\tau}\left(\left(1+\tau\right)^{s(k+1)}x\right)x^{-\frac{1}{s}}\\
&= f_{s,\tau}\left(x\right)x^{-\frac{1}{s}}.
\end{aligned}$$

The result follows.

(iv) We examine $f_{s,\tau}$ on the interval $[1,(1+\tau)^s)$. We note that the maxima and minimum of h_τ are $h_\tau\left(1+\tau\right)=h_\tau\left(\left(1+\tau\right)^2\right)=1+\tau$ and $h_\tau\left(1+\frac{3\tau}{2}+\frac{\tau^2}{2}\right)=1-\tau^2$ respectively. Thus the maxima and minimum of $f_{s,\tau}$ are

$$f_{s,\tau}\left(1+\frac{\left(1+\tau\right)^{s}-1}{\tau^{3}+2\tau^{2}}\right)=f_{s,\tau}\left(\left(1+\tau\right)^{s}\right)=1+\tau$$

and

$$f_{s,\tau}\left(1+\frac{2\tau+\tau^{2}}{\left(1+\tau^{s}\right)-1}\left(\frac{3\tau}{2}+\frac{\tau^{2}}{2}\right)\right)=1-\tau^{2}$$

respectively. Now,

$$\frac{d}{dx}f_{s,\tau}\left(x\right)x^{-\frac{1}{s}}=f_{s,\tau}^{'}\left(x\right)x^{-\frac{1}{s}}-\frac{1}{s}f_{s,\tau}\left(x\right)x^{-\frac{s+1}{s}}.$$

When $\frac{d}{dx}f_{s,\tau}\left(x\right)$ is negative the above expression is clearly negative. On the other hand, when $g_{s,\tau}^{'}\left(x\right)$ is positive, we have that

$$\begin{aligned}
f_{s,\tau}^{'}\left(x\right) &= \frac{2\tau+\tau^{2}}{\left(1+\tau\right)^{s}-1}h_{\tau}^{'}\left(\frac{2\tau+\tau^{2}}{\left(1+\tau\right)^{s}-1}\left(x-1\right)+1\right)\\
&\geqslant \frac{2\tau+\tau^{2}}{\left(1+\tau\right)^{s}-1}\\
&\geqslant \frac{2\tau+\tau^{2}}{s\tau\left(1+\frac{\tau}{2}\right)}\\
&= \frac{2}{s},
\end{aligned}$$

and so

$$\begin{aligned}
\frac{d}{dx} f_{s,\tau}(x)\, x^{-\frac{1}{2}} &\geqslant \frac{2}{s} x^{-\frac{1}{s}} - \frac{1}{s} f_{s,\tau}(x)\, x^{-\frac{s+1}{s}} \\
&\geqslant \frac{2}{s} x^{-\frac{1}{s}} - \frac{1}{s} f_{s,\tau}(x)\, x^{-\frac{s+1}{s}} \\
&\geqslant \frac{2}{s} x^{-\frac{1}{s}} - \frac{1}{s} (1+\tau)(x)\, x^{-\frac{s+1}{s}} \\
&\geqslant \frac{2}{s} x^{-\frac{1}{s}} - \frac{1}{s} (1+\tau)(x)\, x^{-\frac{1}{s}} \\
&= \frac{x^{-\frac{1}{s}}}{s} (1-\tau) \\
&> 0.
\end{aligned}$$

Using this and the fact that, by (ii) and (iii), the endpoints are equal, we get that the maximum and minimum of $f_{s,\tau}(x)\, x^{\frac{1}{s}}$ over the range $[1, (1+\tau)^s)$ are

$$\frac{f_{s,\tau}\left(1 + \frac{(1+\tau)^s - 1}{2+\tau}\right)}{\sqrt[s]{1 + \frac{(1+\tau)^s - 1}{2+\tau}}} \quad \text{and} \quad \frac{f_{s,\tau}\left(1 + \frac{(1+\tau^s)-1}{2\tau+\tau^2}\left(\frac{3\tau}{2} + \frac{\tau^2}{2}\right)\right)}{\sqrt[s]{1 + \frac{(1+\tau^s)-1}{2\tau+\tau^2}\left(\frac{3\tau}{2} + \frac{\tau^2}{2}\right)}}$$

respectively. These extend to $(0, \infty)$ by (iv) and evaluate to the inner bounds on $f_{s,\tau}(x)\, x^{-\frac{1}{s}}$.

For the outer bounds, we note that

$$\frac{\partial}{\partial s} \left(\alpha (1+\tau)^s + \beta\right)^{-\frac{1}{s}} = -\frac{1}{s} \left(\alpha (1+\tau)^s + \beta\right)^{-\frac{s+1}{s}} \alpha \log(1+\tau)(1+\tau)^s$$

and, since both $c_{s,\tau}$ and $d_{s,\tau}$ are of this form with $\alpha > 0$, they both decrease with s and so we get the required bounds by substituting $s = 2$ and $s = 1$ respectively.

(v) Noting $c_\tau \geqslant \frac{1-\tau^2}{\sqrt{(1+\tau)^2}} = 1 - \tau$ we get

$$\begin{aligned}
\frac{c_{s,\tau}}{d_{s,\tau}} &\geqslant \frac{c_\tau}{d_\tau} \\
&\geqslant \frac{1-\tau}{1+\frac{\tau}{2}} \\
&= \frac{(1-\tau)\left(1-\frac{\tau}{2}\right)}{1-\frac{\tau^2}{4}} \\
&\geqslant (1-\tau)\left(1-\frac{\tau}{2}\right) \\
&\geqslant 1-\frac{3\tau}{2}.
\end{aligned}$$

(vi) h_τ is piecewise affine by definition and thus, using (i), so is $f_{s,\tau}$. Similarly, it is easily checked that h_τ is continuous on $\left[1,(1+\tau)^2\right)$, hence $g_{s,\tau}$ is on $[1,(1+\tau)^s)$, and so we get that $f_{s,\tau}$ is continuous on $(0,\infty)$ by noting that for any $k \in \mathbb{Z}$

$$f_{s,\tau}\,(1+\tau)^{sk} = 1\,(1+\tau)^{-k} = (1+\tau)\,(1+\tau)^{k-1} = \lim_{x\uparrow(1+\tau)^{sk}} f_{s_\tau}\,(x)\,.$$

Finally, it is an immediate consequence of (ii) and (v) that $f_{s,\tau}$ is continuous at 0.

$\square$

We can see from Propositions 3.2.3 and 3.2.6 that the real stumbling block for the resulting function to be a metric is the triangle inequality. This is what we shall be looking at next. We attack this from two directions: If x and y are similar in size, then the proofs, where $f_{s,\tau}\,(|x+y|)$ and $f_{s,\tau}\,(|x-y|)$ are handled separately, are based on the large scale behaviour of the function being like $x^{\frac{1}{s}}$, the former on the fact that $x^{\frac{1}{s}}$ is concave and the latter on the fact that it is increasing; on the other hand, if x and y are quite different in size, it is based on the fact that $f_{s,\tau}$ at the larger value grows relatively slowly compared with $x^{\frac{1}{s}}$ near the smaller value. We begin with the case where the values are very different.

Lemma 3.2.7

$$f_{s,\tau}\,(x \pm y) \leqslant f_{s,\tau}\,(x) + 2\lambda_{s,\tau}\,(1+\tau)\,x^{\frac{1}{s}-1}y.$$

Proof From its definition $h_\tau(x \pm y) \leqslant h_\tau(x) + 2y$, and so

$$g_{s,\tau}(x \pm y) \leqslant g_{s,\tau}(x) + 2\lambda_{s,\tau} y.$$

Thus

$$f_{s,\tau}(x + y) \leqslant f_{s,\tau}(x) + 2\lambda_{s,\tau}(1+\tau)^{(s-1)k} y, \tag{3.1}$$

where

$$k = \left\lceil -\frac{\log x}{s \log(1+\tau)} \right\rceil.$$

But we have

$$\begin{aligned}
(1+\tau)^k &\leqslant (1+\tau)^{1-\frac{\log x}{s\log(1+\tau)}} \\
&= (1+\tau)\left((1+\tau)^{\frac{\log x}{\log(1+\tau)}}\right)^{-\frac{1}{s}} \\
&= (1+\tau)\left((1+\tau)^{\frac{\log x}{\log(1+\tau)}}\right)^{-\frac{1}{s}} \\
&= (1+\tau)\left((1+\tau)^{\log_{1+\tau}(x)}\right)^{-\frac{1}{s}} \\
&= (1+\tau)\, x^{-\frac{1}{s}},
\end{aligned}$$

and so

$$\begin{aligned}
(1+\tau)^{(s-1)k} &\leqslant \left((1+\tau)\, x^{-\frac{1}{s}}\right)^{s-1} \\
&= (1+\tau)^{s-1} x^{\frac{1}{s}-1} \\
&\leqslant (1+\tau)\, x^{\frac{1}{s}-1}.
\end{aligned}$$

Substituting the above into (3.1) gives the required result. $\square$

Lemma 3.2.8 *Suppose that* $0 < y \leqslant \left(\frac{c_\tau}{2\lambda_{s,\tau}(1+\tau)}\right)^{\frac{s}{s-1}} x$, *then*

$$f_{s,\tau}(x \pm y) \leqslant f_{s,\tau}(x) + f_{s,\tau}(y).$$

Proof Using Lemma 3.2.7 we get

$$\begin{aligned}
f_{s,\tau}(x \pm y) &\leqslant f_{s,\tau}(x) + 2\lambda_{s,\tau}(1+\tau)\, x^{\frac{1}{s}-1} y \\
&= f_{s,\tau}(x) + \frac{2\lambda_{s,\tau}(1+\tau)}{c_\tau} x^{\frac{1}{s}-1} c_\tau y \\
&= f_{s,\tau}(x) + \left(\left(\frac{c_\tau}{2\lambda_{s,\tau}(1+\tau)} \right)^{\frac{s}{s-1}} x \right)^{\frac{1}{s}-1} c_\tau y \\
&\leqslant f_{s,\tau}(x) + y^{\frac{1}{s}-1} c_\tau y \\
&= f_{s,\tau}(x) + c_\tau y^{\frac{1}{s}} \\
&\leqslant f_{s,\tau}(x) + f_{s,\tau}(y).
\end{aligned}$$

□

We are now ready to look at the cases when x and y are comparable in size.

Lemma 3.2.9 *Suppose that* $0 < \frac{3\tau}{2\left(1-\frac{1}{s}\right)-3\tau} x \leqslant y \leqslant x$, *then*

$$f_{s,\tau}(x+y) \leqslant f_{s,\tau}(x) + f_{s,\tau}(y).$$

Proof We suppose that assumptions of the lemma are true and set $t = \frac{y}{x}$, then

$$t \geqslant \frac{3\tau}{2\left(1-\frac{1}{s}\right)-3\tau}. \tag{3.2}$$

Since $t \mapsto (1+t)^{\frac{1}{s}}$ is a concave function that has value 1 and gradient $\frac{1}{s}$ when $t = 0$, we have

$$(1+t)^{\frac{1}{s}} \leqslant 1 + \frac{t}{s}.$$

We also note that, since $t \in [0,1]$,

$$1 + t^{\frac{1}{s}} \geqslant 1 + t.$$

So, we are now able to write

$$
\begin{aligned}
f_{s,\tau}(x+y) &\leqslant d_{s,\tau}x^{\frac{1}{s}}(1+t)^{\frac{1}{s}} \\
&\leqslant d_{s,\tau}x^{\frac{1}{s}}\left(1+\frac{t}{s}\right) \\
&= d_{s,\tau}x^{\frac{1}{s}}\left(1+\left(-1+\frac{1}{s}+\frac{3\tau}{2}\right)t+\left(1+\frac{3\tau}{2}\right)t\right) \\
&\leqslant d_{s,\tau}x^{\frac{1}{s}}\left(1+\frac{3\tau}{2}+\left(1+\frac{3\tau}{2}\right)t\right) \quad \text{by (3.2)} \\
&= d_{s,\tau}\left(1+\frac{3\tau}{2}\right)x^{\frac{1}{s}}(1+t) \\
&\leqslant c_{s,\tau}x^{\frac{1}{s}}(1+t) \quad \text{by Proposition 3.2.6} \\
&\leqslant c_{s,\tau}x^{\frac{1}{s}}\left(1+t^{\frac{1}{s}}\right) \\
&\leqslant f_{s,\tau}(x)+f_{s,\tau}(y).
\end{aligned}
$$

□

Lemma 3.2.10 *Suppose that* $0<\left(\frac{d_{s,\tau}}{c_{s,\tau}}-1\right)^{s}x\leqslant y\leqslant x$. *Then*

$$f_{s,\tau}(x-y)\leqslant f_{s,\tau}(x)+f_{s,\tau}(y).$$

Proof We suppose that $0<\left(\frac{d_{s,\tau}}{c_{s,\tau}}-1\right)^{s}x\leqslant y\leqslant x$ and take $\alpha\in[c,d]$ such that $f_{s,\tau}(x)=\alpha x^{\frac{1}{s}}$. Then

$$
\begin{aligned}
f_{s,\tau}(x-y) &\leqslant \frac{d_{s,\tau}}{\alpha}f_{s,\tau}(x) \\
&= f_{s,\tau}(x)+\left(\frac{d_{s,\tau}}{c_{s,\tau}}-\frac{\alpha}{c_{s,\tau}}\right)\frac{c_{s,\tau}}{\alpha}f_{s,\tau}(x) \\
&\leqslant f_{s,\tau}(x)+\left(\frac{d_{s,\tau}}{c_{s,\tau}}-1\right)\frac{c_{s,\tau}}{\alpha}f_{s,\tau} \\
&\leqslant f_{s,\tau}(x)+\left(\frac{y}{x}\right)^{\frac{1}{s}}\frac{c_{s,\tau}}{\alpha}f_{s,\tau} \\
&\leqslant f_{s,\tau}(x)+\left(\frac{y}{x}\right)^{\frac{1}{s}}c_{s,\tau}x^{\frac{1}{s}} \\
&= f_{s,\tau}(x)+c_{s,\tau}y^{\frac{1}{s}} \\
&\leqslant f_{s,\tau}(x)+f_{s,\tau}(y).
\end{aligned}
$$

□

Combining these approaches allows us to give our first estimate on which τ do indeed lead to metric space.

Theorem 3.2.11 *If for any given $s \in (1,2]$ we define*

$$\tau_0(s) = \inf\left\{\tau > 0 : \max\left\{\frac{3\tau}{2\left(1-\frac{1}{s}\right)-3\tau}, \left(\frac{d_{s,\tau}}{c_{s,\tau}}-1\right)^s\right\} > \left(\frac{c_{s,\tau}}{2\lambda_{s,\tau}(1+\tau)}\right)^{\frac{s}{s-1}}\right\},$$

then $\tau_0(s) > 0$ and, for any $\tau \in (0, \tau_0(s)]$, $f_{s,\tau} \in \mathcal{E}$.

Proof Recall from Proposition 3.2.6 that $f_{s,\tau}(\{0\})$ and $f_{s,\tau}$ is continuous at 0. And, by combining Lemma 3.2.8 with Lemmas 3.2 and 3.2.10, we have that

$$\max\{f_{s,\tau}(x+y), f_{s,\tau}(x-y)\} \leqslant f_{s,\tau}(x) + f_{s,\tau}(y)$$

for any $0 < y < x < \infty$, whenever the conditions of the theorem are met.

It remains to show that $\tau_0(s) > 0$. However,

$$\frac{3\tau}{2\left(1-\frac{1}{s}\right)-3\tau} \to 0 \quad \text{as } \tau \to 0$$

and

$$\begin{aligned}\left(\frac{c_\tau}{2\lambda_{s,\tau}(1+\tau)}\right)^{\frac{s}{s-1}} &\to \left(\frac{1}{2^{\frac{2}{s}}(1)}\right)^{\frac{s}{s-1}} \quad \text{as } \tau \to 0 \\ &> 0,\end{aligned}$$

and the result follows. □

The above proves the existence of metric spaces of the form $(\mathbb{R}, (x,y) \mapsto f_{s,\tau}(|x-y|))$ for any $s \in (1,2]$, but does not, on its own, give a very good estimate for which values of τ will work. It does, however, lead on to a method for establishing much better estimates numerically.

Lemma 3.2.12 *Suppose that $f_{s,\tau}(x+y) > f_{s,\tau}(x) + f_{s,\tau}(y)$ for some $0 < y \leqslant x$. Then there exists $k \in \mathbb{N} \cup \{0\}$ such that*

$$f_{s,\tau}(\hat{x}+\hat{y}) > f_{s,\tau}(\hat{x}) + f_{s,\tau}(\hat{y}),$$

where

$$\hat{x} = \frac{1}{\lambda_{s,\tau}}\left(\frac{3\tau}{2} + \frac{\tau^2}{2}\right) \quad \textit{and} \quad \hat{y} = \hat{x}\,(1+\tau)^{-sk}.$$

Proof We suppose that $f_{s,\tau}(x+y) > f_{s,\tau}(x) + f_{s,\tau}(y)$, that is $\varphi_{s,\tau}(x,y) > 0$, where we define

$$\phi_{s,\tau}(x,y) = f_{s,\tau}(x+y) - f_{s,\tau}(x) - f_{s,\tau}(y).$$

Using the fact that for any $k \in \mathbb{Z}$

$$\varphi_{s,\tau}\left((1+\tau)^{sk}x, (1+\tau)^{sk}y\right) = (1+\tau)^k \varphi_{s,\tau}(x,y), \tag{3.3}$$

we can find x_1 and y_1 such that $\varphi_{s,\tau}(x_1,y_1) > 0$ and $x_1 \in [a,d)$, where

$$\begin{aligned} a &= (1+\tau)^{-s}\left(1 + \lambda_{s,\tau}\left(\frac{3\tau}{2} + \frac{\tau^2}{2}\right)\right) \\ b &= 1 \\ c &= 1 + \frac{1}{\lambda_{s,\tau}}\tau \\ d &= 1 + \frac{1}{\lambda_{s,\tau}}\left(\frac{3\tau}{2} + \frac{\tau^2}{2}\right). \end{aligned}$$

Now, for some $j \in \mathbb{N} \cup \{0\}$, we have that $x_1 + y_1 \in (1+\tau)^{sj}[a,d)$ and we set $\eta = (1+\tau)^{-sj}(x_1+y_1)$.

We now need to look at different cases.

(i) If $x_1 \in [a,b]$, or if $x_1 \in (b,c)$ and $\eta \geqslant x_1$, then (ignoring the corners, as we may since they have zero length), for $t \in [0, x_1 - a]$,

$$\frac{d}{dt}f_{s,\tau}(x_1 - t) \leqslant \frac{d}{dt}f_{s,\tau}(x_1 + y_1 - t)$$

and so we may take $x_2 = a$ and have $\varphi_{s,\tau}(x_2,y_1) \geqslant \varphi_{s,\tau}(x_1,y_1) > 0$.

(ii) If $x_1 \in [c,d)$ and $\eta \leqslant x_1$, then, for $t \in [0, c - x_1]$,

$$\frac{d}{dt}f_{s,\tau}(x_1 + t) \leqslant \frac{d}{dt}f_{s,\tau}(x_1 + y_1 + t)$$

and so we may take $x_2 = c$ and have $\varphi_{s,\tau}(x_2,y_1) \geqslant \varphi_{s,\tau}(x_1,y_1) > 0$.

(iii) If, on the other hand, $x_1 \in (b, c)$ and $\eta \leqslant x_1$. Then we take $\tilde{t} = \min\left\{(1+\tau)^{sj}(c-\eta), d - x_1\right\}$.

If the former of the above terms is the minimum, then

$$f_{s,\tau}\left(x_1 + y_1 + \tilde{t}\right) = (1+\tau)^j (1+\tau) \geqslant (1+\tau)^j f_{s,\tau}\left(x_1 + \tilde{t}\right),$$

but

$$(1+\tau)^j f_{s,\tau}(x_1) \geqslant (1+\tau)^j f_{s,\tau}(\eta) = f_{s,\tau}(x_1 + y_1),$$

and thus, $\varphi\left(x_1 + \tilde{t}, y_1\right) \geqslant \varphi(x_1, y_1) > 0$. But $x_1 + \tilde{t} \in [b, c]$ and $(1+\tau)^{-sj}\left(x_1 + y_1 + \tilde{t}\right) = b$ and so we can use (ii) to get $x_2 = d$ with $\varphi_{s,\tau}(x_2, y_1) > 0$.

If, however, the latter of the above terms is the minimum then we can take $x_2 = x_1 + \tilde{t} = c$. And, since for $t \in \left[0, \tilde{t}\right]$

$$\frac{d}{dt} f_{s,\tau}(x_1 + t) > 0,$$

we have that $f_{s,\tau}(x_2 + y_1) \geqslant f_{s,\tau}(x_1 + y_1)$

Since this covers every case we now have

$$x_2 = \frac{1}{\lambda_{s,\tau}}\left(\frac{3\tau}{2} + \frac{\tau^2}{2}\right)(1+\tau)^{sk},$$

for some $k \in \mathbb{Z}$, and $y_2 = y_1$ such that $\varphi_{s,\tau}(x_2, y_2) > 0$.

We may now repeat this process with y_2 to give

$$\begin{aligned} y_3 &= \frac{1}{\lambda_{s,\tau}}\left(\frac{3\tau}{2} + \frac{\tau^2}{2}\right)(1+\tau)^{sl_1} \\ x_3 &= x_2 (1+\tau)^{sl_2} \\ &= \frac{1}{\lambda_{s,\tau}}\left(\frac{3\tau}{2} + \frac{\tau^2}{2}\right)(1+\tau)^{sl_3}, \end{aligned}$$

for some $l_1, l_2, l_3 \in \mathbb{Z}$. Finally, we can use (3.3) to find $l_4 \in \mathbb{Z}$ so that when $\hat{x} = (1+\tau)^{sl_4} \max\{x_3, y_3\}$ and $\hat{y} = (1+\tau)^{sl_4} \min\{x_3, y_3\}$ we have $\varphi_{s,\tau}(x_3, y_3) > 0$. $\square$

Lemma 3.2.13 *Suppose that* $f_{s,\tau}(x-y) > f_{s,\tau}(x) + f_{s,\tau}(y)$ *for some* $x > y > 0$. *Then* *either there exists $k \in \mathbb{N} \cup \{0\}$ such that*

$$f_{s,t}\left(\hat{x}-\hat{y}\right) > f_{s,t}\left(\hat{x}\right) + f_{s,t}\left(\hat{y}\right)$$

where

$$\hat{x} = \frac{1}{\lambda_{s,\tau}}\left(\frac{3\tau}{2}+\frac{\tau^2}{2}\right) \quad \textit{and} \quad \hat{y} = \hat{x}\left(1+\tau\right)^{-sk},$$

or

$$f_{s,t}\left(1+\frac{1}{\lambda_{s,\tau}}\left(1+\tau\right)\right) > f_{s,t}\left(\hat{x}\right) + f_{s,t}\left(\hat{x}-\left(1+\frac{1}{\lambda_{s,\tau}}\left(1+\tau\right)\right)\right).$$

Proof We suppose that $f_{s,\tau}\left(x+y\right) > f_{s,\tau}\left(x\right)+f_{s,\tau}\left(y\right)$, that is $\varphi_{s,\tau}\left(x,y\right) > 0$, where we define

$$\varphi_{s,\tau}\left(x,y\right) = f_{s,\tau}\left(x-y\right) - f_{s,\tau}\left(x\right) - f_{s,\tau}\left(y\right).$$

Using the fact that for any $k \in \mathbb{Z}$

$$\varphi_{s,\tau}\left(\left(1+\tau\right)^{sk}x, \left(1+\tau\right)^{sk}x\right) = \left(1+\tau\right)^{k}\varphi_{s,\tau}\left(x,y\right), \tag{3.4}$$

we can find x_1 and y_1 such that $\varphi_{s,\tau}\left(x_1,y_1\right) > 0$ and $x_1 \in [a,d)$, where

$$\begin{aligned} a &= 1 \\ b &= 1+\frac{1}{\lambda_{s,\tau}}\tau \\ c &= 1+\frac{1}{\lambda_{s,\tau}}\left(\frac{3\tau}{2}+\frac{\tau^2}{2}\right) \\ d &= \left(1+\tau\right)^{s}. \end{aligned}$$

We will first show that $x_1 - y_1 \in (a,d)$, and so we assume that this is not the case and $x_1 - y_1 \leqslant a$. Then, from our assumptions and the construction of $f_{s,\tau}$, we can see that

$$f_{s,\tau}\left(y_1\right) < f_{s,\tau}\left(x_1-y_1\right) - f_{s,\tau}\left(x_1\right) \leqslant \tau^2.$$

But it must also be the case that

$$x_1 \geqslant 1+\frac{1}{\lambda_{s,\tau}}\frac{3\tau}{2}$$

and thus

$$y_1 \geqslant \frac{1}{\lambda_{s,\tau}} \frac{3\tau}{2}.$$

So, we have

$$\begin{aligned}
\frac{1}{9} &\geqslant \tau^2 \\
&> f_{s,\tau}(y_1) \\
&\geqslant \frac{1-\tau}{1+\tau} \frac{1}{\lambda_{s,\tau}} \frac{3\tau}{2} \\
&\geqslant \frac{1-\tau}{1+\tau} \frac{1}{2+\tau^2} \frac{3\tau}{2} \\
&\geqslant \frac{9}{76},
\end{aligned}$$

which is a contradiction. Thus $x_1 - y_1 \in [a, d)$.

We note that if $x_1 \in [a, c]$

$$\frac{d}{dt} f_{s,\tau}(x_1 + t) \leqslant \frac{d}{dt} f_{s,\tau}(x_1 - y_1 + t)$$

for $t \in [0, c - x_1]$, and if $x_1 \in (c, d)$

$$\frac{d}{dt} f_{s,\tau}(x_1 - t) \leqslant \frac{d}{dt} f_{s,\tau}(x_1 - y_1 - t)$$

for $t \in [0, \min\{x_1 - c, x_1 - y_1 - a\}]$. So, with $\hat{x} = c$, we have $\varphi_{s,\tau}(\hat{x}, y_1) > 0$, as if $x_1 - y_1 - a < x_1 - c$ it would lead to a contradiction with the first part of this proof.

Now, for some $j \in \mathbb{N} \cup \{0\}$, we have that $y_1 \in (1+\tau)^{-sj}[a, d]$.

Again, we need to look at different cases.

(i) If $(1+\tau)^{sj} y_1 \in [c, d)$, we have, for $t \in \left[(1+\tau)^{-sj} c, y_1\right]$,

$$\frac{d}{dt} f_{s,\tau}(y_1 - t) \leqslant \frac{d}{dt} f_{s,\tau}(\hat{x} - y_1 + t)$$

and so we may take $\hat{y} = (1+\tau)^{-sj} c$ and have $\varphi_{s,\tau}(\hat{x}, \hat{y}) \geqslant \varphi_{s,\tau}(\hat{x}, y_1) > 0$.

(ii) If $(1+\tau)^{sj}\, y_1 \in [b,c)$, we have, for $t \in \left[y_1, (1+\tau)^{-sj}\, c\right]$,

$$\frac{d}{dt} f_{s,\tau}\left(y_1+t\right) \leqslant \frac{d}{dt} f_{s,\tau}\left(\hat{x}-y_1-t\right)$$

and so we may take $\hat{y} = (1+\tau)^{-sj}\, c$ and have $\varphi_{s,\tau}\left(\hat{x}, \hat{y}\right) \geqslant \varphi_{s,\tau}\left(\hat{x}, y_1\right) > 0$.

(iii) If $(1+\tau)^{sj}\, y_1 \in [a,b)$ and

$$f'_{s,\tau}\left(y_1\right) \leqslant -f'_{s,\tau}\left(\hat{x}-y_1\right),$$

then, necessarily, $\hat{x}-y_1 \in [b,c]$ and we have, for $t \in \left[0, \min\left\{(1+\tau)^{-sj}\, b - y_1, \hat{x}-y_1-b\right\}\right]$,

$$\frac{d}{dt} f_{s,\tau}\left(y_1+t\right) \leqslant \frac{d}{dt} f_{s,\tau}\left(\hat{x}-y_1-t\right).$$

So, when $(1+\tau)^{-sj}\, b - y_1 \leqslant \hat{x}-y_1-b$ we have $\varphi_{s\tau}\left(\hat{x}, (1+\tau)^{-sj}\, b\right) \geqslant \varphi_{s,\tau}\left(\hat{x}, y_1\right) > 0$, in which case we can use (ii) to give us $\varphi_{s,\tau}\left(\hat{x}, \hat{y}\right) > 0$, where $\hat{y} = (1+\tau)^{-sj}\, b$. And when $(1+\tau)^{-sj}\, b - y_1 > \hat{x}-y_1-b$ we have $\varphi_{s,\tau}\left(\hat{x}-b\right) \geqslant \varphi_{s,\tau}\left(\hat{x}, y_1\right) > 0$, and so

$$f_{s,\tau}\left(1+\frac{1}{\lambda_{s,\tau}}(1+\tau)\right) > f_{s,\tau}\left(\hat{x}\right) + f_{s,\tau}\left(\hat{x} - \left(1+\frac{1}{\lambda_{s,\tau}}(1+\tau)\right)\right).$$

(iv) If $(1+\tau)^{sj}\, y_1 \in [a,b)$ and

$$f'_{s,\tau}\left(y_1\right) \geqslant -f'_{s,\tau}\left(\hat{x}-y_1\right),$$

then there are two cases for us to consider.

Firstly, if $y_1 - (1+\tau)^{-sj}\, a \leqslant b - \left(\hat{x}-y_1\right)$, we have that, for $t \in \left[0, y_1 - (1+\tau)^{-sj}\, a\right]$,

$$\frac{d}{dt} f_{s,\tau}\left(y_1-t\right) \leqslant \frac{d}{dt} f_{s,\tau}\left(\hat{x}-y_1+t\right).$$

So $\varphi_{s,\tau}\left(\hat{x}, (1+\tau)^{-sj}\, a\right) \geqslant \varphi_{s,\tau}\left(\hat{x}, y_1\right) > 0$, and we can use (i) to give us $\varphi_{s,\tau}\left(\hat{x}, \hat{y}\right) > 0$, where $\hat{y} = (1+\tau)^{-s(j+1)}\, c$.

On the other hand, if $y_1 - (1+\tau)^{-sj}\, a > b - (\hat{x} - y_1)$, we have

$$\frac{d}{dt} f_{s,\tau}\left(y_1 - t\right) \leqslant \frac{d}{dt} f_{s,\tau}\left(\hat{x} - y_1 + t\right)$$

for $t \in [0, b - (\hat{x} - y_1)]$. So $\varphi_{s,\tau}\left(\hat{x}, \hat{x} - b\right) \geqslant \varphi_{s,\tau}\left(\hat{x}, y_1\right) > 0$, and so

$$f_{s,\tau}\left(1 + \frac{1}{\lambda_{s,\tau}}\left(1+\tau\right)\right) > f_{s,\tau}\left(\hat{x}\right) + f_{s,\tau}\left(\hat{x} - \left(1 + \frac{1}{\lambda_{s,\tau}}\left(1+\tau\right)\right)\right).$$

So, in any case, we either have that

$$f_{s,\tau}\left(1 + \frac{1}{\lambda_{s,\tau}}\left(1+\tau\right)\right) > f_{s,\tau}\left(\hat{x}\right) + f_{s,\tau}\left(\hat{x} - \left(1 + \frac{1}{\lambda_{s,\tau}}\left(1+\tau\right)\right)\right),$$

or

$$f_{s,\tau}\left(\hat{x} - \hat{y}\right) > f_{s,\tau}\left(\hat{x}\right) + f_{s,\tau}\left(\hat{y}\right),$$

where y has equality with either $\hat{x}\left(1+\tau\right)^{-sj}$ or $\hat{x}\left(1+\tau\right)^{-s(j+1)}$. $\square$

Using Lemmas 3.2.12 and 3.2.13 in conjunction with Theorem 3.2.11 allows us to extend the value of τ for which we can guarantee a metric space.

Theorem 3.2.14 *Let*

$$K = \left\lfloor \frac{\log\left(2\lambda_{s,\tau}\left(1+\tau\right)\right) - \log\left(c_{s,\tau}\right)}{\left(s-1\right)\log\left(1+\tau\right)} \right\rfloor,$$

$$\Phi_s\left(\tau\right) = \min\left\{ f_{s,\tau}\left(\hat{x}\left(1 + \left(1+\tau\right)^{-sk}\right)\right) - f_{s,\tau}\left(\hat{x}\right) - f_{s,\tau}\left(\hat{x}\left(1+\tau\right)^{-sk}\right) > 0 : 0 \leqslant k \leqslant K \right\},$$

$$\tilde{\Psi}_s\left(\tau\right) = \min\left\{ f_{s,\tau}\left(\hat{x}\left(1 - \left(1+\tau\right)^{-sk}\right)\right) - f_{s,\tau}\left(\hat{x}\right) - f_{s,\tau}\left(\hat{x}\left(1+\tau\right)^{-sk}\right) > 0 : 0 \leqslant k \leqslant K \right\},$$

and

$$\Psi_s\left(\tau\right) = \min\left\{ \tilde{\Psi}_s\left(\tau\right), f_{s,\tau}\left(1 + \frac{1}{\lambda_{s,\tau}}\left(1+\tau\right)\right) - f_{s,\tau}\left(\hat{x}\right) - f_{s,\tau}\left(\hat{x} - \left(1 + \frac{1}{\lambda_{s,\tau}}\left(1+\tau\right)\right)\right) \right\},$$

where $\hat{x}$ are defined as in Lemmas 3.2.12 and 3.2.13, and $\tau_0\left(s\right)$ is defined as in Theorem 3.2.11. Then, for

$$\tau < \tau_1\left(s\right) = \inf\left\{ \tau \in \left(\tau_0, \frac{1}{3}\right] : \min \Phi_{s,\tau}, \Psi_{s,\tau} > 0 \right\},$$

we have that $f_{s,\tau} \in \mathcal{E}$.

Proof We suppose that $f_{s,\tau} \notin \mathcal{E}$.

By Theorem 3.2.11 we have that $\tau > \tau_0$. And by Proposition 3.2.6 $f_{s,\tau}$ satisfies all of the properties to be in $\mathcal{E}$ apart from the conditions on $f_{s,\tau}(x+y)$ and $f_{s,\tau}(x-y)$. So, using Lemmas 3.2.12 and 3.2.13 and the conditions of this theorem, we obtain some $j > K$ such that either

$$f_{s,\tau}\left(\hat{x}\left(1+(1+\tau)^{-sj}\right)\right) > f_{s,\tau}(\hat{x}) + f_{s,\tau}\left(\hat{x}(1+\tau)^{-sj}\right)$$

or

$$f_{s,\tau}\left(\hat{x}\left(1-(1+\tau)^{-sj}\right)\right) > f_{s,\tau}(\hat{x}) + f_{s,\tau}\left(\hat{x}(1+\tau)^{-sj}\right).$$

But,

$$(1+\tau)^{-sj} \leqslant (1+\tau)^{-s(K+1)} < \left(\frac{c_{s,\tau}}{2\lambda_{s,\tau}(1+\tau)}\right)^{\frac{s}{s-1}},$$

which contradicts Lemma 3.2.8. $\square$

It is clear from the statement of the above theorem that τ_1 is the optimal value for which all $\tau \in (0, \tau_1]$ will give us a metric space. But, since the functions of τ used are not monotonically decreasing, there is no guarantee that there are not higher values that have been missed; experimental checking with a variety of values of s, however, suggests this is not the case, and certainly not when $s = 2$.

Since the above functions over the stated range consist of a finite number of smooth functions, it is possible to generate a bound to any precision required. In the case $s = 2$ the functions are polynomials (quintics) so an exact solution is possible. However, since there is no reason to believe the value to be of any deep significance, we shall content ourselves with the numerical estimate $\tau_1(2) > 0.2478$.

Since we now have a good idea of when our functions do indeed generate a metric space, we give some notation for the resultant space, before going on to study the density properties of this space in the next section.

Definition 3.2.15 *Let $s \in [0,1)$ and $\tau \in \left(0, \frac{1}{3}\right)$ be such that $f_{s,\tau} \in \mathcal{E}$. Then we define the* *metric space*

$$\mathbb{R}_{s,\tau} = \left(\mathbb{R}, (x, y) \mapsto f_{s,\tau}\left(|x - y|\right)\right).$$

3.3 Density Properties

In this section we will be calculating the upper and lower densities of $\mathcal{H}^s$ in $\mathbb{R}_{s,\tau}$. In doing so we shall also speak of the Lebesgue measure of sets; clearly this is not defined in $\mathbb{R}_{s,\tau}$ and, when we do mention them, we are referring to the Lebesgue measure of the same set of points, but considered as a subset of the Euclidean space $\mathbb{R}$. This should not be confusing, as Lebesgue measure will always be meant in this way, whilst Hausdorff measure and diameter will always refer to the metric in $\mathbb{R}_{s,\tau}$.

Definition 3.3.1 *We define $m_\tau^o, m_\tau^i : (0, \infty) \to \mathbb{Z}$ by*

$$\begin{aligned} m_\tau^0(r) &= \max\left\{k \in \mathbb{Z} : r \in f_{s,\tau}\left((1+\tau)^{-sk}\left[1, (1+\tau)^s\right)\right)\right\}, \\ m_\tau^1(r) &= \min\left\{k \in \mathbb{Z} : r \in f_{s,\tau}\left((1+\tau)^{-sk}\left[1, (1+\tau)^s\right)\right)\right\}, \end{aligned}$$

and $l_{s,\tau}^i, l_{s,\tau} : (0, \infty) \to (0, \infty)$ by

$$\begin{aligned} l_{s,\tau}^0(r) &= \inf\left\{x \in (0, \infty) : f_{s,\tau}(x) > r\right\}, \\ l_{s,\tau}^1(r) &= \inf\left\{x \in \left(l_{s,\tau}^0, \infty\right) : f_{s,\tau} < r\right\}, \\ l_{s,\tau}^2(r) &= \inf\left\{x \in \left(l_{s,\tau}^1, \infty\right) : f_{s,\tau} > r\right\}, \\ l_{s,\tau}^3(r) &= \inf\left\{x \in \left(l_{s,\tau}^2, \infty\right) : f_{s,\tau} < r\right\}, \\ l_{s,\tau}^4(r) &= \inf\left\{x \in \left(l_{s,\tau}^3, \infty\right) : f_{s,\tau} > r\right\}, \ \textit{and} \\ l_{s,\tau}(r) &= \sup\left\{x \in (0, \infty) : f_{s,\tau}(x) < r\right\}, \end{aligned}$$

unless the set in the definition of $l_{s,\tau}^i$ is empty, in which case we may take $l_{s,\tau}^i = l_{s,\tau}$. Finally $\delta_{s,\tau} : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ by

$$\delta_{s,\tau}(E) = \sup\left\{f_{s,\tau}\left(|x - y|\right) : x, y \in E\right\},$$

that is $\delta_{s,\tau}(E) = \operatorname{diam}(E)$ where a metric space is defined.

Proposition 3.3.2

$$m_\tau^0(r) - m_\tau^1(r) = \begin{cases} 0 & \textit{if } r \in \bigcup_{k\in\mathbb{Z}} (1+\tau)^k \left[1, (1+\tau)\left(1-\tau^2\right)\right) \\ 1 & \textit{if } r \in \bigcup_{k\in\mathbb{Z}} (1+\tau)^k \left[(1+\tau)\left(1-\tau^2\right), 1+\tau\right), \end{cases}$$

$$l_{s,\tau}(r) = \begin{cases} l_{s,\tau}^2(r) & \textit{if } m_\tau^0(r) = m_\tau^1(r) \\ l_{s,\tau}^4(r) & \textit{if } m_\tau^0(r) = m_\tau^1(r) + 1, \end{cases}$$

and

$$\mathcal{L}\left(f_{s,\tau}^{-1}([0,r])\right) = \begin{cases} l_{s,\tau}^2(r) - l_{s,\tau}^1(r) + l_{s,\tau}^0(r) & \textit{if } m_\tau^0(r) = m_\tau^1(r) \\ l_{s,\tau}^4(r) - l_{s,\tau}^3(r) + l_{s,\tau}^2(r) = l_{s,\tau}^1(r) + l^0_{s,\tau(r)} & \textit{if } m_\tau^0(r) = m_\tau^1(r) + 1. \end{cases}$$

Proof The first statement follows from the fact that $(1+\tau)\left(1-\tau^2\right) < (1+\tau)$ but $(1+\tau)^2\left(1-\tau^2\right) > (1+\tau)$ for $\tau \in \left(0, \frac{1}{3}\right)$. The statements follow immediately from this and the definition of $f_{s,\tau}$. □

We again use the concept of geometric periodicity, this will mean that we will only need to study the functions on a finite interval, and not the whole domain or range.

Lemma 3.3.3 $l_{s,\tau}^o(r)\, r^s$, $l_{s,\tau}^i r^s$ *and* $l_{s,\tau}(r)\, r^s$ *are geometrically periodic with period* $(1+\tau)$.

Proof We have that $r = f_{s,\tau}(x)$ if and only if $(1+\tau) r = (1+\tau) f_{s,\tau}(x) = f_{s,\tau}\left((1+\tau)^s x\right)$. The result now follows from the definitions of the functions and from Proposition 3.2.6 (iii). □

Before we can link the Hausdorff and Lebesgue measures, it will be useful to bound Lebesgue measure in $\mathbb{R}$ with the diameter of the set in $\mathbb{R}_{s,\tau}$.

Lemma 3.3.4 *Suppose that* $E \subseteq \mathbb{R}_{s,\tau}$ *has* $\operatorname{diam}(E) = r \in (0,\infty)$, *then* $\mathcal{L}(E) \leqslant r^s$.

Proof We assume without loss of generality that $\inf(E) = 0$, and let $a = \sup(E)$. And we take $\overline{E}$ to be the closure of E. We note that

$$\overline{E} \subseteq f_{s,\tau}^{-1}([0,r]) \cap \left(a - f_{s,\tau}^{-1}([0,r])\right)$$

(where $a - f_{s,\tau}^{-1}([0,r]) = \left\{x \in \mathbb{R}_{s,\tau} : a - x \in f_{s,\tau}^{-1}([0,r])\right\}$ and subtraction is defined in the usual way).

Thus we have three possibilities:

(i) $a \in \left[0, l_{s,\tau}^{0}(r)\right]$, in which case

$$\mathcal{L}(E) \leqslant \mathcal{L}\left(\overline{E}\right) \leqslant l_{s,\tau}^{0}(r),$$

(ii) $a \in \left[l_{s,\tau}^{1}(r), l_{s,\tau}^{2}(r)\right]$, in which case

$$\mathcal{L}(E) \leqslant \mathcal{L}\left(\overline{E}\right) \leqslant l_{s,\tau}^{2}(r) - 2\left(l_{s,\tau}^{1}(r) - l_{s,\tau}^{0}(r)\right),$$

or

(iii) $a \in \left[l_{s,\tau}^{3}(r), l_{s,\tau}^{4}(r)\right]$, in which case

$$\mathcal{L}(E) \leqslant \mathcal{L}\left(\overline{E}\right) \leqslant l_{s,\tau}^{4}(r) - 2\left(l_{s,\tau}^{3}(r) - l_{s,\tau}^{2}(r)\right) - 2(r) - 2\left(l_{s,\tau}^{1}(r) - l_{s,\tau}^{0}(r)\right).$$

We now need to show that each of the above expressions is bounded above by r^s for all $r \in (0, \infty)$. By Lemma 3.3.3 it is enough to show this is the case for $r \in [1, 1+\tau)$. We also note that, for $i \in \{0, 1, 2\}$,

$$l_{s,\tau}^{i}(r) - 1 = \frac{1}{\lambda_{s,\tau}}\left(l_{2,\tau}^{1}(r) - 1\right)$$

whilst

$$r^s - 1 = \frac{r^s - 1}{r^2 - 1}\left(r^2 - 1\right);$$

the above coefficients are, by definition, equal when $r = 1 + \tau$. But, while the upper is constant, the lower is decreasing with r, and does not take a higher value for $r \in [1, 1+\tau)$. It is therefore sufficient to show (i) and (ii) in the case where $s = 2$.

(i) We note that r^2 and $l_{2,\tau}^{0}(r)$ take the same value when $r = 1$, but their gradients are 2 and 1 respectively. Since r^2 is convex, the result follows.

(ii) We note that

$$l_{2,\tau}^{2}(r) = (1+\tau)^2 - \frac{1}{2}(1+\tau-y)$$

and

$$l_{2,\tau}^{1}(r) - l_{s,\tau}^{0}(r) = \frac{3}{2}(1+\tau-y).$$

Thus

$$\mathcal{L}(E) \leqslant (1+\tau)^2 - \frac{7}{2}(1+\tau-y).$$

This takes the value $(1+\tau)^2$ when $r=(1+\tau)$ and has gradient $\frac{7}{2}$, whilst r^2 takes the same value but has gradient $2+2\tau < \frac{7}{2}$. The result again follows from the convexity of r^2.

(iii) We note, using that $\tau \leqslant \frac{1}{3}$, that

$$l_{s,\tau}^{4}(r) - l_{s,\tau}^{3}(r) \leqslant \lim_{r \uparrow 1+\tau}\left(l_{s,\tau}^{4}(r) - l_{s,\tau}^{3}(r)\right) = \frac{1}{\lambda_{s,\tau}}(1+\tau)^s \tau^2 \leqslant \frac{1}{\lambda_{s,\tau}}(1+\tau)^s \frac{1}{9}$$

and

$$l_{s,\tau}^{3}(r) - l_{s,\tau}^{2}(r) \geqslant \lim_{r \uparrow 1+\tau}\left(l_{s,\tau}^{3}(r) - l_{s,\tau}^{2}(r)\right) = \frac{1}{\lambda_{s,\tau}}\frac{3}{2}(1+\tau)^{s+1} \geqslant \frac{1}{\lambda_{s,\tau}}(1+\tau)^s \frac{5}{2}.$$

But now,

$$\begin{aligned}
\mathcal{L}(E) &\leqslant l_{s,\tau}^{4}(r) - 2\left(l_{s,\tau}^{3}(r) - l_{s,\tau}^{2}(r)\right) - 2(r) - 2\left(l_{s,\tau}^{1}(r) - l_{s,\tau}^{0}(r)\right) \\
&= \left(l_{s,\tau}^{4}(r) - l_{s,\tau}^{3}(r)\right) - \left(l_{s,\tau}^{3}(r) - l_{s,\tau}^{2}(r)\right) + l_{s,\tau}^{2}(r) - 2\left(l_{s,\tau}^{1}(r) - l_{s,\tau}^{0}(r)\right) \\
&\leqslant l_{s,\tau}^{2}(r) - 2\left(l_{s,\tau}^{1}(r) - l_{s,\tau}^{0}(r)\right),
\end{aligned}$$

and the result follows from (ii).

We have the result in any case. □

We show the equivalence of the two measures in stages.

Lemma 3.3.5 *Let $E \subseteq \mathbb{R}_{s,\tau}$, then*

$$\mathcal{H}^s(E) \geqslant \mathcal{L}(E).$$

Proof

$$\begin{aligned}
\mathcal{H}^s(E) &\geqslant \inf\left\{\sum_{i\in\mathbb{N}}(\operatorname{diam}(A_i))^s : E\subseteq\bigcup_{i\in\mathbb{N}}A_i\right\}\\
&\geqslant \inf\left\{\sum_{i\in\mathbb{N}}(\operatorname{diam}(A_i))^s : \sum_{i\in\mathbb{N}}\mathcal{L}(A_i)\geqslant\mathcal{L}(E)\right\}\\
&\geqslant \inf\left\{\sum_{i\in\mathbb{N}}(\operatorname{diam}(A_i))^s : \sum_{i\in\mathbb{N}}(\operatorname{diam}(A_i))^s\geqslant\mathcal{L}(E)\right\} \quad \text{by Lemma 3.3.4}\\
&\geqslant \mathcal{L}(E).
\end{aligned}$$

□

Lemma 3.3.6 *Let $E\subseteq\mathbb{R}_{s,\tau}$, then*

$$\mathcal{H}^s(E)\leqslant(d_{s,\tau})^s\,\mathcal{L}(E).$$

Proof If $\mathcal{L}(E)=\infty$ then there is nothing to prove, so we assume that $\mathcal{L}(E)<\infty$. We now fix $\varepsilon>0$ and find $\delta>0$ such that

$$\min\left\{\mathcal{H}^s(E),\frac{1}{\varepsilon}\right\}<\mathcal{H}^s_\delta(E)+\varepsilon$$

(we do not yet know that $\mathcal{H}^s(E)$ is finite).

Then there exist $\{(a_i,b_i)\}_{i\in\mathbb{N}}\subseteq\mathbb{R}^2$ such that

$$E\subseteq\bigcup_{i\in\mathbb{N}}[a_i,b_i],$$

$$\sum_{i\in\mathbb{N}}(b_i-a_i)<\mathcal{L}(E)+\varepsilon$$

and, for every $i\in\mathbb{N}$,

$$0<b_i-a_i<\delta.$$

Now,

$$\begin{aligned}(\operatorname{diam}([a_i,b_i]))^s &= (\sup\{f_{s,\tau}(|x-y|) : x,y\in[a_i,b_i]\})^s\\ &\leqslant \left(\sup\left\{d_{s,\tau}(|x-y|)^{\frac{1}{s}} : x,y\in[a_i,b_i]\right\}\right)^s\\ &= (d_{s,\tau})^s(b_i-a_i).\end{aligned}$$

Therefore,

$$\begin{aligned}\min\left\{\mathcal{H}^s(E),\frac{1}{\varepsilon}\right\} &< \mathcal{H}^s_\delta(E)+\varepsilon\\ &\leqslant \sum_{i\in\mathbb{N}}(\operatorname{diam}([a_i,b_i]))^s+\varepsilon\\ &\leqslant (\delta_{s,\tau})^s\sum_{i\in\mathbb{N}}(b_i-a_i)+\varepsilon\\ &\leqslant (\delta_{s,\tau})^s(\mathcal{L}(E)+\varepsilon)+\varepsilon.\end{aligned}$$

Since $\mathcal{L}(E)$ is finite, for sufficiently small ε we get

$$\mathcal{H}^s(E) < (\delta_{s,\tau})^s(\mathcal{L}(E)+\varepsilon)+\varepsilon,$$

and letting $\varepsilon\downarrow 0$ gives the result. □

Theorem 3.3.7 *Let $E\subseteq\mathbb{R}_{s,\tau}$, then*

$$\mathcal{H}^s(E)=\mathcal{L}(E).$$

Furthermore, it does not affect the value of the measure if we restrict the class of covering sets in the definition of $\mathcal{H}^s$ to be closed intervals of the form I such that

$$(\operatorname{diam}(I))^s=\mathcal{L}(I).$$

Proof If $\mathcal{H}^s(E)=\infty$ then we have $\mathcal{H}^s(E)=\mathcal{L}(E)$ immediately from Lemma 3.3.6, so we need only consider E with $\mathcal{H}^s(E)<\infty$. We fix $\varepsilon>0$. We may now find $\delta>0$ such that

$$\mathcal{H}^s(E)<\mathcal{H}^s_\delta(E)+\varepsilon,$$

and then $K \in \mathbb{N}$ such that

$$(1+\tau)^{-K} < \delta.$$

We now define $\mathcal{E} \subseteq \mathcal{P}(\mathbb{R}_{s,\tau})$ by

$$\mathcal{E} = \left\{ \left[x - \frac{1}{2}(1+\tau)^{-sk}, x + \frac{1}{2}(1+\tau)^{-sk} \right] : x \in E, k \geqslant K \right\}.$$

So, we can use Theorem 1.4.6 to give us a countable collection of disjoint sets, $\mathcal{A} \subseteq \mathcal{E}$, such that

$$\mathcal{L}\left(E \backslash \bigcup_{A \in \mathcal{A}} A \right) = 0$$

and

$$\sum_{A \in \mathcal{A}} \mathcal{L}(A) < \mathcal{L}(E) + \varepsilon.$$

Furthermore, by Lemma 3.3.6, it follows that

$$\mathcal{H}^s\left(E \backslash \bigcup_{A \in \mathcal{A}} A \right) = 0.$$

We note that

$$\left[x - \frac{1}{2}(1+\tau)^{-sk}, x + \frac{1}{2}(1+\tau)^{-sk} \right] = f_{s,\tau}^{-1}\left(\left[0, (1+\tau)^{-k} \right] \right) + x - \frac{1}{2}(1+\tau)^{-sk}.$$

Thus, for any $A \in \mathcal{A}$, we have

$$\mathcal{L}(A) = (\operatorname{diam}(A))^s.$$

So,

$$
\begin{aligned}
\mathcal{H}^s(E) &< \mathcal{H}^s_\delta(E) + \varepsilon \\
&\leqslant \mathcal{H}^s_\delta\left(\bigcup_{A\in\mathcal{A}} A\right) + \mathcal{H}^s_\delta\left(E \backslash \bigcup_{A\in\mathcal{A}} A\right) + \varepsilon \\
&= \mathcal{H}^s_\delta\left(\bigcup_{A\in\mathcal{A}} A\right) + \varepsilon \\
&\leqslant \sum_{A\in\mathcal{A}} (\operatorname{diam}(A))^s + \varepsilon \\
&= \sum_{A\in\mathcal{A}} \mathcal{L}(A) + \varepsilon \\
&< \mathcal{L}(E) + 2\varepsilon.
\end{aligned}
$$

Letting $\varepsilon \downarrow 0$ and combining with Lemma 3.3.5 gives the main result. The final statement follows from the construction used in this proof and noting that restricting the class of covering sets cannot decrease the infimum used in the definition. □

With this equivalence established, it is now easy to calculate the upper and lower densities of the space.

Theorem 3.3.8 *For any $x \in \mathbb{R}_{s,\tau}$,*

$$\underline{D}^s(\mathcal{H}^s, x) = 2^{1-s} \inf_{r\in[1,1+\tau)} \left\{\frac{l_{s,\tau}(r)}{r^s}\right\}$$

and

$$\overline{D}^s(\mathcal{H}^s, x) = 2^{1-s} \sup_{r\in[1,1+\tau)} \left\{\frac{l_{s,\tau}(r)}{r^s}\right\}.$$

Proof Since $\mathcal{L}(B(x,r)) = 2l_{s,\tau}(r)$, we may use Theorem 3.3.7 and Lemma 3.3.3 to give

$$
\begin{aligned}
\underline{D}^s(\mathcal{H}^s, x) &= \liminf_{r\downarrow 0} (2r)^{-s}\, 2l_{s,\tau}(r) \\
&= 2^{1-s} \inf\left\{\frac{l_{s,\tau}(r)}{r^s} : r \in [1, 1+\tau)\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
\overline{D}^s(\mathcal{H}^s, x) &= \limsup_{r\downarrow 0} (2r)^{-s}\, 2l_{s,\tau}(r) \\
&= 2^{1-s} \sup\left\{\frac{l_{s,\tau}(r)}{r^s} : r \in [1, 1+\tau)\right\}.
\end{aligned}
$$

□

We can use this along with our calculated value of $\tau_1(2)$ to give us $\underline{D}^2(\mathcal{H}^2, x) \approx 0.5117$ and $\overline{D}^2(\mathcal{H}^2, x) \approx 0.5307$, for every $x \in \mathbb{R}_{2,\mathcal{T}}$. In particular $\underline{D}^2(\mathcal{H}^2, x) > 0.5117$, and so

$$\sup\{\sigma_2(X) : X \text{ is a metric space}\} > 0.5117.$$

3.4 Extension to $s > 2$

We now extend our density results to higher dimensions by forming some new spaces. We do this by taking Cartesian products of $\mathbb{R}_{s,\mathcal{T}}$ with a Euclidean space.

Definition 3.4.1 *Let $s > 2$. Then we define the metric space,*

$$\mathbb{R}_{s,\mathcal{T}} = (\mathbb{R}_{s_0,\mathcal{T}} \times \mathbb{R}^n, d),$$

where $n = \lceil s-2 \rceil$, $s_0 = s - n$, and $d : \mathbb{R}_{s,\mathcal{T}}^2 \to [0,\infty)$ is defined by

$$d((x, a_1, \ldots, a_n), (y, b_1, \ldots, b_n)) = \max\{f_{s,\mathcal{T}}(|x-y|), |a_1 - b_1|, \ldots, |a_1 - b_1|\}.$$

For simplicity, we shall only calculate the Hausdorff measure of a ball in this resultant space. This is, of course, all we shall need to calculate the densities of $\mathcal{H}^s$ in these spaces.

Lemma 3.4.2 *Let $s > 2$, $x \in \mathbb{R}_{s,\mathcal{T}}$ and $y \in \mathbb{R}_{s_0,\mathcal{T}}$. Then,*

$$\mathcal{H}^s(B(x,r)) = (2r)^n \mathcal{H}^{s_0}(B(y,r)).$$

Proof Since the metrics are translation invariant, we may assume that $x = (y, a)$, for some $a \in \mathbb{R}^n$. We prove each inequality separately.

(i) We fix $\varepsilon > 0$, and pick $\delta \in (0, \varepsilon)$ such that

$$\mathcal{H}^s(B(x,r)) < \mathcal{H}^s_\delta(B(x,r)) + \varepsilon.$$

We may now find a collection of sets, $\{A_i\}_{i\in\mathbb{N}} \subseteq \mathcal{P}(\mathbb{R}_{s_0,\tau})$, such that

(1) $B(y,r) \subseteq \bigcup_{i\in\mathbb{N}} A_i$,

(2) $\operatorname{diam}(A_i) < \delta$ for any $1 \leqslant i \leqslant n$, and

(3) $\sum_{i\in\mathbb{N}} \operatorname{diam}(A_i) < \mathcal{H}^{s_0}(B(y,r)) + \varepsilon$.

We note that

$$B(x,r) \subseteq \bigcup_{i\in\mathbb{N}} A_i \times \left(\bigcup_{j=1}^{m_i} I_{i,j}\right),$$

where

$$m_i = \left\lceil \frac{2r}{\operatorname{diam}(A_i)} \right\rceil$$

and $\{I_{i,j}\} \subseteq \mathbb{R}^n$ is a collection of closed cubes with $\operatorname{diam}(I_{i,j}) = \operatorname{diam}(A_i)$, for any $i \in \mathbb{N}$ and $j \in \{1,\ldots,m_i\}$.

Thus,

$$\begin{aligned}
\mathcal{H}^s(B(x,r)) &< \mathcal{H}^s_\delta(B(x,r)) + \varepsilon \\
&\leqslant \sum_{i\in\mathbb{N}} \sum_{j=1}^{m_i} \operatorname{diam}(A_i) + \varepsilon \\
&\leqslant \sum_{i\in\mathbb{N}} (2r+\delta) \operatorname{diam}(A_i) + \varepsilon \\
&< (2r+\varepsilon)\, \mathcal{H}^{s_0}(B(y,r)) + \varepsilon.
\end{aligned}$$

Letting $\varepsilon \downarrow 0$ gives us

$$\mathcal{H}^s(B(x,r)) \leqslant 2r\mathcal{H}^{s_0}(B(y,r)). \tag{3.5}$$

(ii) For the other inequality, we note that there exists $\{E_i\}_{i\in\mathbb{N}} \subseteq \mathcal{P}(\mathbb{R}_{s,\tau})$ such that

(1) $B(x,r) \subseteq \bigcup_{i\in\mathbb{N}} E_i$,

(2) $\sum_{i\in\mathbb{N}} (\operatorname{diam}(E_i))^s < \mathcal{H}^s(B(x,r)) + \varepsilon$, and

(3) for each $i \in \mathbb{N}$, $E_i = \tilde{E}_i \times R_i$, where $\tilde{E}_i \subseteq \mathbb{R}_{s_0,\tau}$ is the projection of E_i onto $\mathbb{R}_{s_0,\tau}$, and $R_i \subseteq \mathbb{R}^n$ is a closed cube with $\operatorname{diam}(R_i) = \operatorname{diam}(E_i)$.

The qualification (3) may be made since any set in $\mathbb{R}_{s,\tau}$ is the subset of a set of the same diameter that has the properties mentioned in (3).

So, we can use Theorem 3.3.7 to write

$$\begin{aligned}
\mathcal{H}^{s_0}\left(B\left(y,r\right)\right)\left(2r\right)^n &= \mathcal{L}^{n+1}\left(B\left(x,r\right)\right) \\
&\leqslant \sum_{i\in\mathbb{N}} \mathcal{L}^{n+1}\left(E_i\right) \\
&= \sum_{i\in\mathbb{N}} \mathcal{L}\left(\tilde{E}_i\right)\left(\operatorname{diam}\left(E_i\right)\right)^n \\
&\leqslant \sum_{i\in\mathbb{N}} \left(\operatorname{diam}\left(\tilde{E}_i\right)\right)^{s_0}\left(\operatorname{diam}\left(E_i\right)\right)^n \\
&\leqslant \sum_{i\in\mathbb{N}} \left(\operatorname{diam}\left(\tilde{E}_i\right)\right)^{s} \\
&< \mathcal{H}^s\left(B\left(x,r\right)\right)+\varepsilon.
\end{aligned}$$

Letting $\varepsilon \downarrow 0$ in (3.5) and (3.6) gives us the result. □

It is worth noting at this point that the above theorem implies that $\mathbb{R}_{s,\tau}$ is of σ-finite $\mathcal{H}^s$ measure.

We are now ready to extend our density results.

Theorem 3.4.3 *For any* $x \in \mathbb{R}_{s,\tau}$*,*

$$\underline{D}^s\left(\mathcal{H}^s, x\right) = 2^{1-s_0} \inf_{r\in[1,1+\tau)} \left\{\frac{l_{s_0,\tau}\left(r\right)}{r^{s_0}}\right\}$$

and

$$\overline{D}^s\left(\mathcal{H}^s, x\right) = 2^{1-s_0} \sup_{r\in[1,1+\tau)} \left\{\frac{l_{s_0,\tau}\left(r\right)}{r^{s_0}}\right\}.$$

Proof By Lemma 3.4.2 we have that

$$\frac{\mathcal{H}^s\left(B\left(x,r\right)\right)}{\left(2r\right)^s} = \frac{\left(2r\right)^n\mathcal{H}^{s_0}\left(B\left(y,r\right)\right)}{\left(2r\right)^s} = \frac{\mathcal{H}^{s_0}\left(B\left(y,r\right)\right)}{\left(2r\right)^{s_0}},$$

where $y \in \mathbb{R}_{s_0,\tau}$. The result now follows from Theorem 3.3.8. □

With the estimate we made at the end of the last section, Theorem 3.4.3 gives us that

$$\sup \{\sigma_k (X) : X \text{ is a metric space}\} > 0.5117,$$

for any $k \in \mathbb{N}$.

At this point all that remains for us to do is to demonstrate that $\mathbb{R}_{k,\tau}$ is purely unrectifiable for any integer $k \geqslant 2$. This is easy given the following Lemma, which is an immediate consequence of Kirchheim's result in [16].

Lemma 3.4.4 *Let X be a k-rectifiable separable metric space with σ-finite $\mathcal{H}^k$ measure. Then*

$$\underline{D}^s (X, x) = \overline{D}^s (X, x)$$

for $\mathcal{H}^k$-almost every $x \in X$.

Theorem 3.4.5 *Let $k \in \mathbb{N} \backslash \{1\}$. Then $\mathbb{R}_{k,\tau}$ is purely unrectifiable.*

Proof The result follows immediately from Lemma 3.4.4 and Theorem 3.4.3. $\square$

Chapter 4

Lower Bounds on Upper Hausdorff s-Densities

4.1 Introduction

In Chapter 3 we were concerned with the question of how large lower densities can be for purely unrectifiable sets. In this chapter we are concerned with how small upper densities can be. Whilst in Chapter 3, and in a sense Chapter 2, we were using extreme examples to place limits on how far bounds could be extended, we will, in this chapter, be working on improving the bounds themselves.

Besicovitch showed in [2] that if $E \subset \mathbb{R}^2$ is a 1-set then $\overline{D}^s(E, x) > \frac{1}{2}$ almost everywhere. The proof generalises to separable metric spaces and general s with the bound 2^{-s}, as is shown in, for example, [14].

The bound appears in the proof because a covering, as per the definition of Hausdorff measure, uses general sets, but, in order to use the definition of density, we require balls centred in the set. The way this is achieved in the proof is taking an arbitrary point and putting a ball whose radius is the diameter of the set that needs covering. The bound 2^{-s} is from the normalisation term in density being $(2r)^s$, where r is the radius of the ball.

We use the same idea, but we use more efficient coverings with balls to improve the bound.

We do this using the notion of centred contents.

4.2 Centred Content

Centred contents are defined by Chlebík in [6]. They are a means of measuring how difficult it is to cover an arbitrary subset of a given set with ball centred in that subset. They are used in [6] to define a lower bound for upper densities in a normed space. For our bound, we will actually be using a modified definition, which we define in the next section. But, since that definition is more convoluted, it is useful to first examine centred contents here.

Definition 4.2.1 *Let X be a metric space, $s \in [0, \infty)$. Then we define the s-dimensional centred content, $\mathcal{C}_{\infty}^{s} : \mathcal{P}(X) \to [0, \infty]$, by*

$$\mathcal{C}_{\infty}^{s}(E) = \sup \left\{ \tilde{\mathcal{C}}_{\infty}^{s}(A) : A \subseteq E \right\},$$

where

$$\tilde{\mathcal{C}}_{\infty}^{s}(E) = \inf \left\{ \sum_{i \in \mathbb{N}} r_i^s : E \subseteq \bigcup_{i \in \mathbb{N}} B(x_i, r_i), x_i \in E, r_i \in (0, \infty) \right\},$$

defining $\tilde{\mathcal{C}}_{\infty}^{s}(\emptyset) = 0$.

We now give some elementary properties of centred contents.

Proposition 4.2.2 *Let X be a metric space, $s \in [0, \infty)$, and $E \subseteq X$. Then $\tilde{\mathcal{C}}_{\infty}^{s}$ is subadditive, and $\mathcal{C}_{\infty}^{s}$ is a measure.*

Proof Suppose that $\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{P}(X)$ are coverings of E_1 and E_2 respectively, comprising balls centred in E_1 and E_2 respectively. Then the union of these coverings is a cover of $E_1 \cup E_2$ comprising balls centred in $E_1 \cup E_2$. It follows that $\tilde{\mathcal{C}}_{\infty}^{s}$ is subadditive.

It is immediate from the definition that $\mathcal{C}_{\infty}^{s}$ is monotonic and $\mathcal{C}_{\infty}^{s}(\emptyset) = 0$. For subadditivity we note that for any $\varepsilon > 0$ there exists $\tilde{E} \subseteq E_1 \cup E_2$ such that

$$\begin{aligned}
\mathcal{C}_{\infty}^{s}\left(E_1 \cup E_2\right) &< \tilde{\mathcal{C}}_{\infty}^{s}\left(\tilde{E}\right) + \varepsilon \\
&\leqslant \tilde{\mathcal{C}}_{\infty}^{s}\left(E_1 \cap \tilde{E}\right) + \tilde{\mathcal{C}}_{\infty}^{s}\left(E_2 \cap \tilde{E}\right) + \varepsilon \\
&\leqslant \mathcal{C}_{\infty}^{s}\left(E_1 \cap \tilde{E}\right) + \mathcal{C}_{\infty}^{s}\left(E_2 \cap \tilde{E}\right) + \varepsilon \\
&\leqslant \mathcal{C}_{\infty}^{s}\left(E_1\right) + \mathcal{C}_{\infty}^{s}\left(E_2\right) + \varepsilon,
\end{aligned}$$

and let $\varepsilon \downarrow 0$. $\square$

Proposition 4.2.3 *Let X be a metric space, $s \in [0, \infty)$, and $E \subseteq X$. Then,*

$$2^{-s}\mathcal{H}^{s}(E) \leqslant \mathcal{C}_{\infty}^{s}(E) \leqslant \mathcal{H}_{\infty}^{s}(E) \leqslant \min\left\{\mathcal{H}^{s}(E), (\operatorname{diam}(E))^{s}\right\}.$$

Proof As an immediate consequence of Definition 4.2.1, we have that, for any $\varepsilon > 0$, there exists $\{(x_i, r_i)\}_{i\in\mathbb{N}} \subseteq E \times (0, \infty)$ such that

$$E \subseteq \bigcup_{i\in\mathbb{N}} B(x_i, r_i)$$

and

$$\mathcal{C}_{\infty}^{s}(E) \geqslant \tilde{\mathcal{C}}_{\infty}^{s}(E) \geqslant \sum_{i\in\mathbb{N}} r_i^s - \varepsilon.$$

But, $\operatorname{diam}(B(x_i, r_i)) \leqslant 2r_i$ and so

$$2^{-s}\mathcal{H}_{\infty}^{s}(E) \leqslant 2^{-s}\sum_{i\in\mathbb{N}} (2r_i)^s \leqslant \mathcal{C}_{\infty}^{s}(E) + \varepsilon. \tag{4.1}$$

On the other hand, there exists $A \subseteq E$ such that $\mathcal{C}_{\infty}^{s}(E) < \tilde{\mathcal{C}}_{\infty}^{s}(A) + \varepsilon$. Now, from the definition of $\mathcal{H}_{\infty}^{s}$, there exists a collection of non-empty sets $\{A_i\}_{i\in\mathbb{N}} \subseteq \mathcal{P}(X)$ such that

$$A \subseteq \bigcup_{i\in\mathbb{N}} A_i \quad \text{and} \quad \sum_{i\in\mathbb{N}} (\operatorname{diam}(A_i))^s \leqslant \mathcal{H}_{\infty}^{s}(E) + \varepsilon.$$

Thus, picking any $x_i \in A_i$, we have

$$A \subseteq \bigcup_{i\in\mathbb{N}} B\left(x_i, \operatorname{diam}(A_i) + \frac{\varepsilon}{2^i}\right).$$

So,

$$\begin{aligned}
\mathcal{C}^s_\infty(E) &< \tilde{\mathcal{C}}^s_\infty(A)+\varepsilon \\
&\leqslant \sum_{i\in\mathbb{N}}(\mathrm{diam}A_i)^s+2\varepsilon \\
&\leqslant \mathcal{H}^s_\infty(A)+3\varepsilon \\
&\leqslant \mathcal{H}^s_\infty(E)+3\varepsilon. \qquad (4.2)
\end{aligned}$$

Letting $\varepsilon\downarrow 0$ in (4.1) and (4.2) gives the first two inequalities, $\mathcal{H}^s_\infty(E)\leqslant\mathcal{H}^s(E)$ follows from Proposition 1.3.2, and $\mathcal{H}^s_\infty(E)\leqslant(\mathrm{diam}(E))^s$ follows immediately from the definition. □

Proposition 4.2.4 *Let X be a normed space, $E\subseteq X$, $a\in X$ and $\lambda\in(0,\infty)$. Then,*

$$\tilde{\mathcal{C}}^s_\infty(\lambda E+a)=\lambda^s\tilde{\mathcal{C}}^s_\infty(E)$$

and

$$\mathcal{C}^s_\infty(\lambda E+a)=\lambda^s\mathcal{C}^s_\infty(E).$$

Proof The first equality comes from the fact that $\{E_i\}_{i\in\mathbb{N}}\supseteq E$ if and only if $\{\lambda E_i+a\}_{i\in\mathbb{N}}\supseteq \lambda E+a$ and that $\mathrm{diam}(\lambda E_i+a)=\lambda\mathrm{diam}(E_i)$.

The second equality follows from the first, noting that $A\subseteq E$ if and only if $\lambda A+a\subseteq\lambda E+a$. □

4.3 Measured Centred Content

We will now define a variation of centred content. This is significantly more difficult to work with and, consequently, we will have to reduce the generality of our bound from normed spaces to finite dimensional normed spaces. I do not know whether the bound is actually false in the more general case but, if it were true, then it would require a proof significantly different from the one presented here. The advantage of this definition is, as we shall see in the next section, that the density bounds produced are, in some sense, the best possible.

In the proof of the bound on upper density, the sets we are trying to cover with centred balls are those which make up an efficient small scale covering as per the definition of Hausdorff measure. As such, it is possible to restrict our attention to the kind of sets we will necessarily encounter there: Borel sets of finite s-dimensional measure where the measure is somehow evenly spread out. We formalise this in the following definition.

Definition 4.3.1 *Let X be a separable metric space, $s \in [0, \infty)$, and $E \subseteq X$. Then we define the s-dimensional measured centred content, $\mathcal{M}^s_\infty : \mathcal{P}(X) \to [0, \infty]$, by*

$$\mathcal{M}^s_\infty (E) = \begin{cases} \lim_{\rho \downarrow 1} \mathcal{M}^s_\rho (E) & E \textit{ is Borel} \\ \inf \{\mathcal{M}^s_\infty (A) : E \subseteq A, A \textit{ is Borel}\} & \textit{otherwise} \end{cases}$$

where

$$\mathcal{M}^s_\rho (E) = \sup \left\{ \tilde{\mathcal{C}}^s_\infty (A) : A \subseteq E, A \textit{ is Borel}, \mathcal{H}^s (A) \leqslant \rho \mathcal{H}^s_\infty (A) \right\},$$

defining $\mathcal{M}^s_\rho (\emptyset) = 0$.

We define $\mathcal{M}^s_\infty$ separately for Borel sets as it is will be useful, in showing that $\mathcal{M}^s_\infty$ is a measure, to have that every set has a Borel superset of equal $\mathcal{M}^s_\infty$ content, by analogy with the Borel regularity property. Of course, $\mathcal{M}^s_\infty$ is actually neither Borel nor regular, as these sets are not in general $\mathcal{M}^s_\infty$-measurable.

We now attempt to show that $\mathcal{M}^s_\infty$ has many of the same properties as $\mathcal{C}^s_\infty$ does.

Proposition 4.3.2 *Let X be a separable metric space, $s \in [0, \infty)$ and $\rho \in (1, \infty)$. Then, the supremum in the definition of $\mathcal{M}^s_\rho$ is taken over a non-empty set. Furthermore, for any $\tilde{\rho} \in (1, \rho)$, $\mathcal{M}^s_{\tilde{\rho}} \leqslant \mathcal{M}^s_\rho$. Thus $\mathcal{M}^s_\infty$ is well defined.*

Proof For any non-empty set E, $\{x\} \subseteq E$ for some x and

$$\begin{aligned} \mathcal{H}^s (\{x\}) &= \begin{cases} 0 & s > 0 \\ 1 & s = 0 \end{cases} \\ &= \mathcal{H}^s_\infty (\{x\}) \\ &\leqslant \rho \mathcal{H}^s_\infty (\{x\}), \end{aligned}$$

so the set the supremum is taken over is non-empty.

Since

$$\left\{\tilde{\mathcal{C}}^s_\infty(A) : A \subseteq E, A \text{ is Borel}, \mathcal{H}^s(A) \leqslant \tilde{\rho}\mathcal{H}^s_\infty(A)\right\}$$
$$\subseteq \left\{\tilde{\mathcal{C}}^s_\infty(A) : A \subseteq E, A \text{ is Borel}, \mathcal{H}^s(A) \leqslant \rho\mathcal{H}^s_\infty(A)\right\},$$

we have $\mathcal{M}^s_{\tilde{\rho}}(E) \leqslant \mathcal{M}^s_\rho(E)$. □

Proposition 4.3.3 *Let X be a separable metric space, $s \in [0,\infty)$ and $E \subseteq X$ be Borel. Then,*

$$\mathcal{M}^s_\infty(E) \leqslant \mathcal{C}^s_\infty(E) \leqslant \mathcal{H}^s_\infty(E) \leqslant \min\left\{\mathcal{H}^s(E), (\operatorname{diam}(E))^s\right\}.$$

Proof It is immediate from the definitions that $\mathcal{M}^s_\infty(E) \leqslant \mathcal{C}^s_\infty(E)$, and the other inequalities are shown in Proposition 4.2.3. □

We note that we do not have the same lower bound on $\mathcal{M}^s_\infty$ that we did on $\mathcal{C}^s_\infty$, the reason for this is that, in the case of $\mathcal{C}^s_\infty$, we can consider a covering of the whole set, whilst in $\mathcal{M}^s_\infty$ we could only do this if the original set met the conditions on the subsets used in Definition 4.3.1. Actually, it would be enough for the set to contain a suitable subset of equal $\mathcal{H}^s_\infty$ measure, but even this is hard to guarantee for an arbitrary set.

Proposition 4.3.4 *Let X be a separable metric space, $s \in [0,\infty)$ and $\rho \in (1,\infty)$. Then, $\mathcal{M}^s_\rho$ and $\mathcal{M}^s_\infty$ are monotonic set functions.*

Proof It follows immediately from the definition that $\mathcal{M}^s_\rho$ is monotonic. Taking limits then gives that $\mathcal{M}^s_\infty$ is monotonic when restricted to the Borel sets. This means that any set, including Borel sets, equals the infimum of $\mathcal{M}^s_\infty$ on Borel supersets, and thus $\mathcal{M}^s_\infty$ is monotonic. □

Proving the final property required for $\mathcal{M}^s_\infty$ to be a measure is somewhat more difficult than it was for $\mathcal{C}^s_\infty$, and we proceed in stages.

Lemma 4.3.5 *Let X be a separable metric space, $s \in [0,\infty)$, and $E \subseteq X$ be a Borel set such that $\mathcal{H}^s_\infty(E) < \infty$. Furthermore, let $E_1, E_2 \subseteq X$ be disjoint Borel sets such that $E \subseteq E_1 \cup E_2$. Then $\mathcal{M}^s_\infty(E) \leqslant \mathcal{M}^s_\infty(E_1) + \mathcal{M}^s_\infty(E_2)$.*

Proof We fix $s \in [0,\infty)$ and $\rho \in (1,\infty)$. Since $\mathcal{H}^s_\infty(E) < \infty$, $\mathcal{M}^s_\infty(E) < \infty$ by Proposition 4.3.3. Thus there exists $\rho_1 \in (1, \rho)$ such that

$$\max \left\{ \frac{\mathcal{M}_{\rho_1}^s (E_1)}{\mathcal{M}_\infty^s (E_1)}, \frac{\mathcal{M}_{\rho_1}^s (E_2)}{\mathcal{M}_\infty^s (E_2)} \right\} < \rho.$$

We may then find $\rho_2 \in (1, \rho_1)$ such that

$$\rho_2 \leqslant \rho_1 \left(1 - \frac{\rho_2 - 1}{\rho_1 - 1} \right) \tag{4.3}$$

and

$$\frac{\rho_2 - 1}{\rho_1 - 1} \mathcal{H}_\infty^s (E) \leqslant \rho - 1. \tag{4.4}$$

We now take a Borel set, $A \subseteq E$, such that $\mathcal{H}^s (A) \leqslant \rho_2 \mathcal{H}_\infty^s (A)$ and $\rho \tilde{\mathcal{C}}_\infty^s (A) > \mathcal{M}_{\rho_2}^s (E) \geqslant \mathcal{M}_\infty^s (E)$, and assume without loss of generality that $\mathcal{H}_\infty^s (A \cap E_1) \leqslant \mathcal{H}_\infty^s (A \cap E_2)$.

But then

$$\begin{aligned} \mathcal{H}^s (E_1 \cap A) &= \mathcal{H}^s (A) - \mathcal{H}^s (A \cap E_2) \\ &\leqslant \rho_2 \mathcal{H}_\infty^s (A) - \mathcal{H}_\infty^s (A \cap E_2) \\ &= \mathcal{H}_\infty^s (A) - \mathcal{H}_\infty^s (A \cap E_2) + (\rho_2 - 1) \mathcal{H}_\infty^s (A) \\ &\leqslant \mathcal{H}_\infty^s (A \cap E_1) + (\rho_2 - 1) \mathcal{H}_\infty^s (A) . \end{aligned}$$

We first look at the case where $(\rho_2 - 1) \mathcal{H}_\infty^s (A) \leqslant (\rho_1 - 1) \mathcal{H}_\infty^s (A \cap E_1)$. Then

$$\begin{aligned} \mathcal{H}^s (A \cap E_1) &\leqslant \mathcal{H}_\infty^s (A \cap E_1) + (\rho_1 - 1) \mathcal{H}_\infty^s (A \cap E_1) \\ &= \rho_1 \mathcal{H}_\infty^s (A \cap E_1) , \end{aligned}$$

and hence $\mathcal{M}_{\rho_1}^s (A \cap E_1) \geqslant \tilde{\mathcal{C}}_\infty^s (A \cap E_1)$. We can similarly get $\mathcal{M}_{\rho_1}^s (A \cap E_2) \geqslant \tilde{\mathcal{C}}_\infty^s (A \cap E_2)$.

So,

$$\begin{aligned}
\mathcal{M}^s_\infty(E) &\leqslant \mathcal{M}^s_{\rho_1}(E) \\
&< \rho\tilde{\mathcal{C}}^s_\infty(A) \\
&\leqslant \rho\left(\tilde{\mathcal{C}}^s_\infty(A\cap E_1)+\tilde{\mathcal{C}}^s_\infty(A\cap E_2)\right) \\
&\leqslant \rho\left(\mathcal{M}^s_{\rho_1}(A\cap E_1)+\mathcal{M}^s_{\rho_1}(A\cap E_2)\right) \\
&\leqslant \rho\left(\rho\mathcal{M}^s_\infty(A\cap E_1)+\rho\mathcal{M}^s_\infty(A\cap E_2)\right) \\
&= \rho^2\left(\mathcal{M}^s_\infty(E_1)+\mathcal{M}^s_\infty(E_2)\right).
\end{aligned}\tag{4.5}$$

If, however, $(\rho_2-1)\,\mathcal{H}^s_\infty(A)\geqslant(\rho_1-1)\,\mathcal{H}^s_\infty(A\cap E_1)$, then

$$\begin{aligned}
\mathcal{H}^s(A\cap E_2) &\leqslant \mathcal{H}^s(A) \\
&\leqslant \rho_2\mathcal{H}^s_\infty(A) \\
&\leqslant \rho_1\left(\mathcal{H}^s_\infty(A)-\frac{\rho_2-1}{\rho_1-1}\mathcal{H}^s_\infty(A)\right) \quad \text{by (4.3)} \\
&\leqslant \rho_1\left(\mathcal{H}^s_\infty(A)-\mathcal{H}^s_\infty(A\cap E_1)\right) \\
&\leqslant \rho_1\mathcal{H}^s_\infty(A\cap E_2).
\end{aligned}$$

So,

$$\begin{aligned}
\mathcal{M}^s_\infty(E) &\leqslant \mathcal{M}^s_{\rho_1}(E) \\
&< \rho\tilde{\mathcal{C}}^s_\infty(A) \\
&\leqslant \rho\left(\tilde{\mathcal{C}}^s_\infty(A\cap E_2)+\tilde{\mathcal{C}}^s_\infty(A\cap E_1)\right) \\
&\leqslant \rho\left(\tilde{\mathcal{C}}^s_\infty(A\cap E_2)+\mathcal{H}^s_\infty(A\cap E_1)\right) \\
&\leqslant \rho\left(\tilde{\mathcal{C}}^s_\infty(A\cap E_2)+\frac{\rho_2-1}{\rho_1-1}\mathcal{H}^s_\infty(A)\right) \\
&\leqslant \rho\left(\tilde{\mathcal{C}}^s_\infty(A\cap E_2)+\frac{\rho_2-1}{\rho_1-1}\mathcal{H}^s_\infty(E)\right) \\
&\leqslant \rho\left(\tilde{\mathcal{C}}^s_\infty(A\cap E_2)+(\rho-1)\right) \quad \text{by (4.4)} \\
&\leqslant \rho\left(\mathcal{M}^s_{\rho_1}(A\cap E_2)+(\rho-1)\right) \\
&\leqslant \rho\left(\rho\mathcal{M}^s_\infty(A\cap E_2)+(\rho-1)\right) \\
&\leqslant \mathcal{M}^s_\infty(A\cap E_1)+\rho^2\mathcal{M}^s_\infty(A\cap E_2)+\rho(\rho-1).
\end{aligned}\tag{4.6}$$

Letting $\rho \downarrow 1$ in (4.5) and (4.6) gives us the result in either case. $\square$

Lemma 4.3.6 *Let X be a separable metric space, $s \in [0, \infty)$, $E \subseteq X$ a Borel set, and $\{E_i\} \subseteq \mathcal{P}(X)$ disjoint Borel sets such that $E \subseteq \bigcup_{i \in \mathbb{N}} E_i$. Then*

$$\mathcal{M}^s_\infty (E) \leqslant \sum_{i \in \mathbb{N}} \mathcal{M}^s_\infty (E_i) .$$

Proof Fixing $\rho > 1$, we find a Borel set $A \subseteq E$ such that

$$\mathcal{H}^s (A) < \rho \mathcal{H}^s_\infty (A) < \infty$$

and

$$\rho \mathcal{M}^s_\infty (A) \geqslant \rho \tilde{\mathcal{C}}^s_\infty (A) > \min \left\{ \mathcal{M}^s_\infty (E) , \frac{1}{\rho - 1} \right\} ;$$

the second term in the above minimum being introduced to cover the case where $\mathcal{M}^s_\infty (E) = \infty$.

We note that

$$\sum_{i \in \mathbb{N}} \mathcal{H}^s (A \cap E_i) = \mathcal{H}^s (A) < \infty,$$

and so there exists $n \in \mathbb{N}$ such that

$$\sum_{i=n+1}^{\infty} \mathcal{H}^s (E_i) < \rho - 1.$$

Thus, we have

$$\begin{aligned}
\min\left\{\mathcal{M}^s_\infty(E), \frac{1}{\rho-1}\right\} &< \rho\mathcal{M}^s_\infty(A) \\
&\leqslant \rho\left(\sum_{i=1}^n \mathcal{M}^s_\infty(A\cap E_i) + \mathcal{M}^s_\infty\left(\bigcup_{i=n+1}^\infty A\cap E_i\right)\right) \quad \text{by Lemma 4.3.5} \\
&\leqslant \rho\left(\sum_{i=1}^n \mathcal{M}^s_\infty(A\cap E_i) + \mathcal{H}^s\left(\bigcup_{i=n+1}^\infty A\cap E_i\right)\right) \\
&\leqslant \rho\left(\sum_{i=1}^n \mathcal{M}^s_\infty(A\cap E_i) + \sum_{i=n+1}^\infty \mathcal{H}^s(A\cap E_i)\right) \\
&\leqslant \rho\left(\sum_{i=1}^n \mathcal{M}^s_\infty(A\cap E_i) + (\rho-1)\right) \\
&\leqslant \rho\left(\sum_{i=1}^n \mathcal{M}^s_\infty(E_i) + (\rho-1)\right).
\end{aligned}$$

Letting $\rho\downarrow 1$ gives us the result whether or not $\mathcal{M}^s_\infty(E)$ is finite. □

Theorem 4.3.7 *Let X be a separable metric space. Then $\mathcal{M}^s_\infty$ is a measure on X.*

Proof $\mathcal{M}^s_\infty(\emptyset) = \tilde{\mathcal{C}}^s_\infty(\emptyset) = 0$ as an immediate consequence of the definition and $\mathcal{M}^s_\infty$ is monotonic by Proposition 4.3.4, so it only remains to show that $\mathcal{M}^s_\infty$ is subadditive.

We let $E\subseteq X$ and $\{E_i\}_{i\in\mathbb{N}}\subseteq\mathcal{P}(X)$ such that $E\subseteq\bigcup_{i\in\mathbb{N}}E_i$, and we fix $\rho>1$.

For each $i\in\mathbb{N}$ we can define

$$\tilde{E}_i = E_i\setminus\bigcup_{j\in\mathbb{N},j<i}E_j$$

and take Borel sets $A_i\supseteq\tilde{E}_i$ such that $\mathcal{M}^s_\infty(A_i)=\mathcal{M}^s_\infty\left(\tilde{E}_i\right)$. Then $\{A_i\}_{i\in\mathbb{N}}$ is a disjoint Borel cover of E.

So,

$$\begin{aligned}
\mathcal{M}_{\infty}^{s}(E) &\leqslant \mathcal{M}_{\infty}^{s}\left(\bigcup_{i\in\mathbb{N}} E_i\right) \\
&= \mathcal{M}_{\infty}^{s}\left(\bigcup_{i\in\mathbb{N}} \tilde{E}_i\right) \\
&\leqslant \mathcal{M}_{\infty}^{s}\left(\bigcup_{i\in\mathbb{N}} A_i\right) \\
&\leqslant \sum_{i\in\mathbb{N}} \mathcal{M}_{\infty}^{s}(A_i) \quad \text{by Lemma 4.3.6} \\
&= \sum_{i\in\mathbb{N}} \mathcal{M}_{\infty}^{s}\left(\tilde{E}_i\right) \\
&\leqslant \sum_{i\in\mathbb{N}} \mathcal{M}_{\infty}^{s}(E_i).
\end{aligned}$$

□

Fortunately, the scaling and translation properties of $\mathcal{M}_{\infty}^{s}$ are not much more difficult to prove than those of $\mathcal{C}_{\infty}^{s}$.

Proposition 4.3.8 *Let X be a normed space, $s \in [0,\infty)$, $\rho \in (1,\infty)$, $a \in X$ and $\lambda \in (0,\infty)$. Then,*

$$\mathcal{M}_{\rho}^{s}(\lambda E + a) = \lambda^s \mathcal{M}_{\rho}^{s}(E)$$

and

$$\mathcal{M}_{\infty}^{s}(\lambda E + a) = \lambda^s \mathcal{M}_{\infty}^{s}(E).$$

Proof The above equality for $\mathcal{M}_{\rho}^{s}$ can be shown to hold in the same way as was shown for $\mathcal{C}_{\infty}^{s}$ in Proposition 4.2.4, provided that

$$\mathcal{H}^s(A) \leqslant \rho\mathcal{H}_{\infty}^{s}(A) \Leftrightarrow \mathcal{H}^s(\lambda A + a) \leqslant \rho\mathcal{H}_{\infty}^{s}(\lambda A + a).$$

But, if $\mathcal{H}^s(A) \leqslant \rho\mathcal{H}_{\infty}^{s}(A)$,

$$\begin{aligned}
\mathcal{H}^s(\lambda A + a) &= \lambda^s\mathcal{H}^s(A) \\
&\leqslant \rho\lambda^s\mathcal{H}_{\infty}^{s}(A) \\
&= \rho\mathcal{H}_{\infty}^{s}(\lambda A + a).
\end{aligned}$$

We can achieve the reverse implication in a similar fashion.

Letting $\rho \downarrow 1$ then gives the equality for $\mathcal{M}_\infty^s$. $\square$

4.4 Density Bounds

In this section we prove the central result of the chapter, the general lower bound on the upper density of sets. We will also show that this bound is a characterisation in Euclidean spaces, that is a set exists with an upper density that precludes any stronger bound.

Before we defined $\mathcal{M}_\infty^s$, we mentioned that the class of sets used were all that we were required to consider when dealing with efficient small scale coverings used in the definition of Hausdorff measure. This is formalised in Lemma 4.4.2, and we use Lemma 4.4.1 as a stepping stone.

Lemma 4.4.1 *Let X be a separable metric space and $E \subseteq X$ be an s-set. Then, for any $\rho \in (1, \infty)$ and $\delta \in (0, \infty)$, there exist Borel sets $\{A_i\}_{i\in\mathbb{N}} \subseteq \mathcal{P}(E)$ such that*

$$\rho^2 \mathcal{H}^s \left(\bigcup_{i\in\mathbb{N}} A_i \right) > \rho \mathcal{H}^s (E) > \sum_{i\in\mathbb{N}} (\operatorname{diam}(A_i))^s ,$$

and for every $i \in \mathbb{N}$

$$\operatorname{diam}(A_i) < \delta \;\; \textit{and} \;\; \rho \mathcal{H}_\infty^s (A_i) \geqslant \mathcal{H}^s (A_i) .$$

Proof Fixing $\rho \in (1, \infty)$, $\delta \in (0, \infty)$ and $\tilde{\rho} \in (1, \rho)$, we can find $\tilde{\delta} \in (0, \delta)$ such that

$$\tilde{\rho} \mathcal{H}_{\tilde{\delta}}^s (E) > \mathcal{H}^s (E) .$$

We can now find a collection of disjoint Borel sets, $\{A_i\} \subseteq \mathcal{P}(E)$, such that

$$E \subseteq \bigcup_{i\in\mathbb{N}} A_i \;\; \text{and} \;\; \rho \mathcal{H}_{\tilde{\delta}}^s (E) > \sum_{i\in\mathbb{N}} (\operatorname{diam}(A_i))^s$$

and, for every $i \in \mathbb{N}$, $\operatorname{diam}(A_i) < \tilde{\delta} < \delta$.

Now we let

$$\mathcal{I} = \{i \in \mathbb{N} : \rho \mathcal{H}^s_\infty (A_i) \leqslant \mathcal{H}^s (A_i)\},$$

and write

$$\begin{aligned}
\sum_{i\in\mathbb{N}} \mathcal{H}^s (A_i) &= \mathcal{H}^s (E) \\
&< \tilde{\rho}\mathcal{H}^s_{\tilde{\delta}} (E) \\
&\leqslant \tilde{\rho} \sum_{i\in\mathbb{N}} \mathcal{H}^s_\infty (A_i) \\
&\leqslant \tilde{\rho} \left(\sum_{i\in\mathcal{I}} \frac{1}{\rho} \mathcal{H}^s (A_i) + \sum_{i\in\mathbb{N}\backslash\mathcal{I}} \mathcal{H}^s (A_i) \right).
\end{aligned}$$

We can rearrange the above to give

$$\left(1 - \frac{\tilde{\rho}}{\rho}\right) \sum_{i\in\mathcal{I}} \mathcal{H}^s (A_i) \leqslant (\tilde{\rho} - 1) \sum_{i\in\mathbb{N}\backslash\mathcal{I}} \mathcal{H}^s (A_i).$$

By choosing $\tilde{\rho}$ to be sufficiently small, we may ensure that those intervals identified by $\mathcal{I}$ make up an arbitrarily small proportion of the whole, in particular

$$\sum_{i\in\mathcal{I}} \mathcal{H}^s (A_i) < \left(1 - \frac{1}{\rho}\right) \mathcal{H}^s (E).$$

Thus

$$\rho\mathcal{H}^s \left(\bigcup_{i\in\mathbb{N}\backslash\mathcal{I}} A_i \right) > \mathcal{H}^s (E).$$

Finally, we note

$$\sum_{i\in\mathbb{N}\backslash\mathcal{I}} (\operatorname{diam}(A_i))^s \leqslant \sum_{i\in\mathbb{N}} (\operatorname{diam}(A_i))^s < \rho\mathcal{H}^s_\delta (E) \leqslant \rho\mathcal{H}^s (E).$$

We now have that $\{A_i\}_{i\in\mathbb{N}\backslash\mathcal{I}}$ satisfies all of required properties (if $\mathbb{N}\backslash\mathcal{I}$ is finite then we can make up the rest of the collection with empty sets). $\square$

Lemma 4.4.2 *Let X be a separable metric space, $E \subseteq X$ be an s-set. Then, for any*

$\rho \in (1,\infty)$ and $\delta \in (0,\infty)$, there exists a Borel set $A \subseteq E$ such that $\operatorname{diam}(A) < \delta$ *and*

$$(\operatorname{diam}(A))^s < \rho\mathcal{H}^s(A) < \rho^2\mathcal{H}^s_\infty(A).$$

Proof We fix $\rho \in (1,\infty)$ and $\delta \in (0,\infty)$. We can now use Lemma 4.4.1 to give us Borel sets $\{A_i\}_{i\in\mathbb{N}} \subseteq \mathcal{P}(E)$ such that

$$\rho\mathcal{H}^s\left(\bigcup_{i\in\mathbb{N}} A_i\right) > \sqrt{\rho}\mathcal{H}^s(E) > \sum_{i\in\mathbb{N}} (\operatorname{diam}(A_i))^s,$$

and for every $i \in \mathbb{N}$

$$\operatorname{diam}(A_i) < \delta \quad \text{and} \quad \rho\mathcal{H}^s_\infty(A_i) > \sqrt{\rho}\mathcal{H}^s_\infty(A_i) \geqslant \mathcal{H}^s(A_i).$$

It is now sufficient to show that, for some $i \in \mathbb{N}$, $(\operatorname{diam}(A_i))^s < \rho\mathcal{H}^s(A_i)$. We assume that there is no such i, then

$$\begin{aligned}
\mathcal{H}^s(E) &< \sqrt{\rho}\mathcal{H}^s\left(\bigcup_{i\in\mathbb{N}} A_i\right) \\
&\leqslant \sqrt{\rho}\sum_{i\in\mathbb{N}} \mathcal{H}^s(A_i) \\
&\leqslant \frac{1}{\sqrt{\rho}}\sum_{i\in\mathbb{N}} (\operatorname{diam}(A_i))^s \\
&< \mathcal{H}^s(E),
\end{aligned}$$

which is a contradiction. □

In the proof of our density bound, we will be required to use $\mathcal{M}^s_\rho$. However, it is desirable to have the bound itself purely in terms of $\mathcal{M}^s_\infty$. We will be rescaling so that the sets we form the bound from will be of the same diameter, but we will have no information on the shape; we thus need something akin to a compactness result to ensure convergence when taking the limit $\rho \downarrow 1$. In doing this, we will use the Hausdorff metric on sets.

Definition 4.4.3 *Let X be a metric space. Then we define $\delta_H : \mathcal{K}^2 \to [0,\infty)$, where $\mathcal{K}$ is the set of non-empty compact subsets of X, by*

$$\delta_H(E_1, E_2) = \max\left\{\sup_{x\in E_1} d(x, E_2), \sup_{x\in E_2} d(x, E_1)\right\},$$

and call it the Hausdorff metric.

Proposition 4.4.4 *With the notation above, $(\mathcal{K}, \delta_H)$ is a metric space.*

We are now ready to proceed with our convergence result. The proof of which is inspired by, and partly based on, the proof given in [13] of the Blaschke Selection Theorem, originally proved in [5].

It is the need for this lemma that restricts our bound to finite dimensional normed spaces.

Lemma 4.4.5 *Let X be a finite dimensional normed space. Then,*

$$\lim_{\rho \downarrow 1} \left(\sup \left\{ \mathcal{M}^s_\rho (E) : E \subseteq X, \operatorname{diam}(E) = 2 \right\} \right) = \sup \left\{ \mathcal{M}^s_\infty (E) : E \subseteq X, \operatorname{diam}(E) = 2 \right\}.$$

Proof For convenience we write

$$\eta = \sup \left\{ \mathcal{M}^s_\infty (E) : E \subseteq X, \operatorname{diam}(E) = 2 \right\}.$$

We suppose that

$$\lim_{\rho \downarrow 1} \left(\sup \left\{ \mathcal{M}^s_\rho (E) : E \subseteq X, \operatorname{diam}(E) = 2 \right\} \right) > \eta.$$

Then for some $\varepsilon > 0$ there exists

$$\{E_{0,i}\}_{i \in \mathbb{N}} \subseteq \mathcal{P}\left(\overline{B}(0,2)\right)$$

such that, for each $i \in \mathbb{N}$, $E_{0,i}$ is a compact set with diameter 2 and

$$\mathcal{M}^s_{\frac{i+1}{i}} (E_{0,i}) > \eta + \varepsilon.$$

We may take the above sets to be compact as, if they were otherwise, we could take their closure; we may assume that they are subsets of $\overline{B}(0,2)$ since $\mathcal{M}^s_\infty$ is translation invariant.

Now, for each $k \in \mathbb{N}$, we can find $\{x_{k,j}\}_{j=1}^{n_k}$ such that

$$\overline{B}(0,2) \subseteq \bigcup_{j=1}^{n_k} \overline{B}\left(x_{k,j}, \frac{1}{4k}\right).$$

So, for any $k \in \mathbb{N}$, we can define $\mathcal{J}_k : \mathcal{P}\left(\overline{B}(0,2)\right) \to \mathcal{P}\left(\{j \in \mathbb{N} : j \leqslant n_k\}\right)$ by

$$\mathcal{J}_k(A) = \left\{ j \in \mathbb{N} : A \cap \overline{B}\left(x_{k,j}, \frac{1}{4k}\right) \neq \emptyset \right\}.$$

Now, since the codomain of $\mathcal{J}_k$ is finite, we can use the pigeon hole principle to recursively define subsequences $\{E_{k,i}\}_{i\in\mathbb{N}} \subseteq \{E_{k-1,i}\}_{i\in\mathbb{N}} \subseteq \{E_{0,i}\}_{i\in\mathbb{N}}$ such that $\mathcal{J}_k(E_{k,i}) = \mathcal{J}_k(E_{k,j})$ for all $i, j, k \in \mathbb{N}$.

We now define

$$F_k = \bigcup_{j \in \mathcal{J}_k(E_{k,1})} \overline{B}\left(x_{k,j}, \frac{1}{4k}\right)$$

and note that, for any $i, k \in \mathbb{N}$,

$$E_{k,i} \subseteq F_k \subseteq \left\{ x \in X : d(x, E_{k,i}) \leqslant \frac{1}{2k} \right\}.$$

It then follows that, for any $i, j, k \in \mathbb{N}$,

$$\begin{aligned} \delta_H(E_{k,i}, E_{k,j}) &\leqslant \delta_H(E_{k,i}, F_k) + \delta_H(E_{k,j}, F_k) \\ &\leqslant \frac{1}{2k} + \frac{1}{2k} \\ &= \frac{1}{k}. \end{aligned}$$

In particular we note that

$$\delta_H(E_{i,i}, E_{j,j}) \leqslant \frac{1}{\min\{i, j\}}$$

and define, for any $m \in \mathbb{N}$, the set

$$E_m = \bigcup_{i=m}^{\infty} E_{i,i}.$$

So,

$$\begin{aligned}
\operatorname{diam}(E_m) &= \sup\{\|x-y\| : x, y \in E_m\} \\
&\leqslant \sup\{\operatorname{diam}(E_{i,i}) + \delta_H(E_{i,i}, E_{j,j}) : i, j \in \mathbb{N}, i, j \geqslant m\} \\
&= 2 + \sup\{\delta_H(E_{i,i}, E_{j,j}) : i, j \in \mathbb{N}, i, j \geqslant m\} \\
&\leqslant 2 + \frac{1}{m}.
\end{aligned}$$

And, defining

$$\tilde{E}_m = \frac{2E_m}{\operatorname{diam}(E_m)},$$

we get that $\operatorname{diam}\left(\tilde{E}_m\right) = 2$ and

$$\begin{aligned}
\mathcal{M}^s_\infty\left(\tilde{E}_m\right) &= \left(\frac{2}{\operatorname{diam}(E_m)}\right)^s \mathcal{M}^s_\infty(E_m) \\
&= \left(\frac{2}{\operatorname{diam}(E_m)}\right)^s \lim_{j\to\infty} \mathcal{M}^s_{\frac{j+1}{j}}(E_m) \\
&\geqslant \left(\frac{2}{\operatorname{diam}(E_m)}\right)^s \lim_{j\to\infty} \mathcal{M}^s_{\frac{j+1}{j}}(E_{j,j}) \\
&\geqslant \left(\frac{2}{\operatorname{diam}(E_m)}\right)^s (\eta + \varepsilon) \\
&\geqslant \left(\frac{2}{2+\frac{1}{m}}\right)^s (\eta + \varepsilon) \\
&> \eta
\end{aligned}$$

for sufficiently large m. But this is a contradiction and so

$$\lim_{\rho\downarrow 1}\left(\sup\left\{\mathcal{M}^s_\rho(E) : E \subseteq X, \operatorname{diam}(E) = 2\right\}\right) \leqslant \eta.$$

However, if

$$\lim_{\rho\downarrow 1}\left(\sup\left\{\mathcal{M}^s_\rho(E) : E \subseteq X, \operatorname{diam}(E) = 2\right\}\right) < \eta$$

then there exists $A \subseteq X$ with $\operatorname{diam}(A) = 2$ and

$$\mathcal{M}^s_\infty(A) > \lim_{\rho\downarrow 1}\left(\sup\left\{\mathcal{M}^s_\rho(E) : E \subseteq X, \operatorname{diam}(E) = 2\right\}\right).$$

But,

$$\begin{aligned}\lim_{\rho\downarrow 1}\left(\sup\left\{\mathcal{M}_{\rho}^{s}(E) : E\subseteq X, \operatorname{diam}(E)=2\right\}\right) &\leqslant \lim_{\rho\downarrow 1}\mathcal{M}_{\rho}^{s}(A)\\ &= \mathcal{M}_{\infty}^{s}(A),\end{aligned}$$

which is also a contradiction. □

We are, at last, ready to prove our density bound. With the preceding lemmas in place, the proof is fairly similar to the bound using $\mathcal{C}_{\infty}^{s}$, which is presented in [7].

Theorem 4.4.6 *Let X be a finite dimensional normed space and $E \subseteq X$ be an s-set. Then,*

$$\overline{D}^{s}(E,x) \geqslant \frac{1}{\sup\left\{\mathcal{M}_{\infty}^{s}(A) : A\subseteq X, \operatorname{diam}(A)=2\right\}}$$

for $\mathcal{H}^{s}$ almost every $x \in E$.

Proof For convenience we write

$$\eta = \sup\left\{\mathcal{M}_{\infty}^{s}(A) : A\subseteq X, \operatorname{diam}(A)=2\right\}.$$

Since $\overline{D}^{s}(E,\cdot)$ is an $\mathcal{H}^{s}$ measurable function,

$$E_1 = \left\{x\in X : \overline{D}^{s}(E,x) < \frac{1}{\eta}\right\}$$

is also $\mathcal{H}^{s}$ measurable. We assume that $\mathcal{H}^{s}(E_1) > 0$.

We use the fact that

$$E_1 = \bigcup_{k\in\mathbb{N}}\left\{x\in E : r<\frac{1}{k} \Rightarrow \frac{k+1}{k}\mathcal{H}^{s}\llcorner E\,(B(x,r)) < \frac{1}{\eta}(2r)^{s}\right\}$$

to find $\rho\in(1,\infty)$, $\delta\in(0,\infty)$ and $E_2\subseteq E_1$, with $\mathcal{H}^{s}(E_2)>0$, such that, for every $x\in E_2$ and $r\in(0,\delta)$,

$$\rho^{3}\mathcal{H}^{s}\llcorner E\,(B(x,r)) < \frac{1}{\eta}(2r)^{s}. \tag{4.7}$$

By Lemma 4.4.5 we may find $\tilde{\rho} \in (0, \rho)$ such that

$$\sup \left\{ \mathcal{M}^s_{\tilde{\rho}}(A) : A \subseteq X, \operatorname{diam}(A) = 2 \right\} < \rho\eta.$$

Now we use Lemma 4.4.2 to give us $A \subseteq E_2$ such that $\operatorname{diam}(A) < \delta$ and

$$(\operatorname{diam}(A))^s < \tilde{\rho}\mathcal{H}^s(A) < \tilde{\rho}^2 \mathcal{H}^s_\infty(A). \tag{4.8}$$

So,

$$\begin{aligned} \tilde{\mathcal{C}}^s_\infty(A) &\leqslant \mathcal{M}^s_{\tilde{\rho}}(A) \\ &= \left(\frac{\operatorname{diam}(A)}{2} \right)^s \mathcal{M}^s_{\tilde{\rho}} \left(\frac{2}{\operatorname{diam}(A)} A \right) \\ &< \left(\frac{\operatorname{diam}(A)}{2} \right)^s \rho\eta. \end{aligned}$$

Thus, there exists $\{(x_i, r_i)\}_{i \in \mathbb{N}} \subseteq A \times (0, \delta)$ such that

$$A \subseteq \bigcup_{i \in \mathbb{N}} B(x_i, r_i)$$

and

$$\sum_{i \in \mathbb{N}} r_i^s < \rho^2 \eta \left(\frac{\operatorname{diam}(A)}{2} \right)^s. \tag{4.9}$$

We may now write

$$\begin{aligned} (\operatorname{diam}(A))^s &< \tilde{\rho}\mathcal{H}^s(A) \quad (\text{by } (4.8)) \\ &< \rho\mathcal{H}^s(A) \\ &\leqslant \rho \sum_{i \in \mathbb{N}} \mathcal{H}^s \llcorner E(B(x_i, r_i)) \\ &< \frac{2^s}{\rho^2 \eta} \sum_{i \in \mathbb{N}} r_i^s \quad (\text{by } (4.7)) \\ &< (\operatorname{diam}(A))^s \quad (\text{by } (4.9)). \end{aligned}$$

But this is a contradiction and so $\mathcal{H}^s(E_1) = 0$ after all. $\square$

Corollary 4.4.7 *Let X be a finite dimensional normed space and $E \subseteq X$. Then*

$$\overline{D}^s(E,x) \geqslant \frac{1}{\sup\{\mathcal{C}^s_\infty(A) : A \subseteq X, \operatorname{diam}(A) = 2\}}$$

for $\mathcal{H}^s$ almost every $x \in E$.

We now come on to the result that justifies defining $\mathcal{M}^s_\infty$ in addition to $\mathcal{C}^s_\infty$.

Theorem 4.4.8

$$\begin{aligned}\sup\left\{\kappa : E \subseteq \mathbb{R}^n \textit{ and } 0 < \mathcal{H}^s(E) < \infty \Rightarrow \overline{D}^s(E,x) \geqslant \kappa \textit{ for } \mathcal{H}^s \textit{ almost every } x \in E\right\} \\ = \frac{1}{\sup\{\mathcal{M}^s_\infty(A) : A \subseteq \mathbb{R}^n, \operatorname{diam}(A) = 2\}}.\end{aligned}$$

Proof We fix $s \in [0,\infty)$ and $\rho \in (1,\infty)$, and pick $E \subseteq \mathbb{R}^n$ such that $\operatorname{diam}(E) = 2$, $\rho\tilde{\mathcal{C}}^s_\infty(E) > \sup\{\mathcal{M}^s_\infty(A) : \operatorname{diam}(A) = 2\}$ and $\mathcal{H}^s(E) < \rho\mathcal{H}^s_\infty(E) \leqslant \rho 2^s$.

We define

$$\kappa = \sup\left\{t : \overline{D}^s(E,x) \geqslant t \text{ for } \mathcal{H}^s \text{ almost every } x \in E\right\},$$

and

$$E_1 = \left\{x \in E : \overline{D}^s(E,x) \geqslant \kappa\right\}.$$

We note that $\mathcal{H}^s(E \backslash E_1) = 0$. Furthermore, for each $x \in E_1$, there exists $\{r_{x,j}\}_{j\in\mathbb{N}}$ such that $r_{x,j} \to 0$ as $j \to \infty$ and, for each $j \in \mathbb{N}$, we have

$$\rho\mathcal{H}^s \llcorner E_1\left(\overline{B}(x,r_j)\right) \geqslant \kappa\,(2r_{x,j})^s.$$

Since, by Proposition 1.3.6, $\mathcal{H}^s \llcorner E_1$ is a Radon measure, we may use Theorem 1.4.6 to give us $\{(x_i,r_i)\}_{i\in\mathbb{N}} \subseteq E_1 \times \{r_{x_i,j}\}_{j\in\mathbb{N}}$ such that $\mathcal{H}^s(E_1 \setminus E_2) = 0$, where

$$E_2 = \bigcup_{i\in\mathbb{N}} \overline{B}(x_i,r_i).$$

Now,

$$\begin{aligned}
\sup\left\{\mathcal{M}_{\infty}^{s}(A) : A \subseteq X, \operatorname{diam}(A) = 2\right\} &< \rho\tilde{\mathcal{C}}_{\infty}^{s}(E) \\
&\leqslant \rho\left(\tilde{\mathcal{C}}_{\infty}^{s}(E_2) + \tilde{\mathcal{C}}_{\infty}^{s}(E_1 \setminus E_2) + \tilde{\mathcal{C}}_{\infty}^{s}(E \setminus E_1)\right) \\
&= \rho\tilde{\mathcal{C}}_{\infty}^{s}(E_2) \\
&\leqslant \rho\sum_{i\in\mathbb{N}} r_i^s \\
&\leqslant \rho\sum_{i\in\mathbb{N}} \frac{\rho}{\kappa} 2^{-s}\mathcal{H}^s \llcorner E_1\left(\overline{B}(x_i, r_i)\right) \\
&\leqslant \frac{\rho^2}{\kappa} 2^{-s}\mathcal{H}^s(E) \\
&< \frac{\rho^3}{\kappa}.
\end{aligned}$$

Thus $\kappa < \frac{\rho^3}{\sup\{\mathcal{M}_{\infty}^{s}(A):A\subseteq X,\operatorname{diam}(A)=2\}}$. So, letting $\rho \to 1$, and combining this with Theorem 4.4.6, gives the result. □

4.5 Subsets of $\mathbb{R}$

We now apply our generic bound on upper densities to produce some more explicit bounds in the case of the Euclidean space $\mathbb{R}$. This is a particularly easy case as we do not need to take a supremum over sets of diameter two, it is sufficient to consider the set $[-1, 1]$. Actually, these bounds turned out to be pre-existing, but our generic bound from the last section does allow for new, and significantly simpler, proofs. Theorem 4.5.2 was shown by Besicovitch in [3], and Theorem 4.5.3 was shown by Walker in [24].

Both theorems rely on the fact that we are able to pick a point in the set that requires covering that is fairly close to the centre. We make this explicit in the following lemma.

Lemma 4.5.1 *Let $s \in (0, 1]$, $\rho \in (1, \infty)$ and $E \subseteq [-1, 1]$ such that $\rho\tilde{\mathcal{C}}_{\infty}^{s}(E) > \mathcal{C}_{\infty}^{s}([-1, 1])$. Then, there exists $x \in E$ such that*

$$|x| < \rho - 2(2\rho)^{-\frac{1}{s}}.$$

Proof Letting

$$\lambda = \inf \{t \in [0,1] : E \cap (t-1, 1-t) = \emptyset\},$$

we have that

$$\begin{aligned}
\mathcal{C}^s_\infty([-1,1]) &< \rho \tilde{\mathcal{C}}^s_\infty(E) \\
&\leqslant \rho \left(\tilde{\mathcal{C}}^s_\infty(E \cap [-1, \lambda - 1]) + \tilde{\mathcal{C}}^s_\infty(E \cap [1-\lambda, 1]) \right) \\
&\leqslant \rho \left(\mathcal{C}^s_\infty([-1, \lambda - 1]) + \mathcal{C}^s_\infty([1-\lambda, 1]) \right) \\
&= 2\rho \left(\frac{\lambda}{2} \right)^s \mathcal{C}^s_\infty([-1,1]).
\end{aligned}$$

Rearranging the above gives us that $\lambda > 2(2\rho)^{-\frac{1}{s}}$. The result then follows from the definition of λ. □

Theorem 4.5.2 *Let $s \in (0,1]$ and $E \subseteq \mathbb{R}$ such that $0 < \mathcal{H}^s(E) < \infty$. Then*

$$\overline{D}^s(E,x) \geqslant \left(2 \left(1 - 2^{-\frac{1}{s}} \right) \right)^{-s}$$

for $\mathcal{H}^s$ almost every $x \in E$.

Proof If we take $\rho \in (1, \infty)$ then we can find $E \subseteq [-1,1]$ such that $\rho \tilde{\mathcal{C}}^s_\infty(E) > \mathcal{C}^s_\infty([-1,1])$. So, by Lemma 4.5.1,

$$E \subseteq [-1,1] \subseteq B\left(x, 1 + \rho - 2(2\rho)^{-\frac{1}{s}}\right).$$

Thus

$$\begin{aligned}
\mathcal{C}^s_\infty([-1,1]) &< \rho \tilde{\mathcal{C}}^s_\infty(E) \\
&\leqslant \rho \left(1 + \rho - 2(2\rho)^{-\frac{1}{s}} \right)^s.
\end{aligned}$$

Letting $\rho \downarrow 1$ gives $\mathcal{C}^s_\infty([-1,1]) \leqslant \left(2 \left(1 - 2^{-\frac{1}{s}} \right) \right)^s$, and the result then follows from Corollary 4.4.7. □

It is proved in [24] that the above bound is the best possible for subsets of $\mathbb{R}$ and $s \in [0, s_0]$, where $s_0 \approx 0.6635$. This bound matches the upper density of the s-dimensional Cantor set for $s \in [0, s_0]$.

Theorem 4.5.3 *Let $E \subseteq \mathbb{R}$ have $0 < \mathcal{H}^s(E) < \infty$. Then*

$$\overline{D}^s(E, x) \geqslant 2^{1-s}\left(1 - \left(1 - 2^{1-\frac{1}{s}}\right)^s\right)$$

for $\mathcal{H}^s$ almost every $x \in E$.

Proof If we take $\rho \in (1, \infty)$ then we can find $E \subseteq [-1, 1]$ such that $\rho\tilde{\mathcal{C}}^s_\infty(E) > \mathcal{C}^s_\infty([-1, 1])$. Then we use Lemma 4.5.1 to give us $x \in E$, which we assume without loss of generality to be positive, such that

$$E \subseteq [-1, 1] \subseteq \overline{B}(x, 1 - x) \cup (E \cap [-1, 2x - 1]).$$

Thus

$$\begin{aligned} \mathcal{C}^s_\infty([-1, 1]) &< \rho\tilde{\mathcal{C}}^s_\infty(E) \\ &\leqslant \rho\left((1 - x)^s + \tilde{\mathcal{C}}^s_\infty(E \cap [-1, 2x - 1])\right) \\ &\leqslant \rho\left((1 - x)^s + \mathcal{C}^s_\infty([-1, 2x - 1])\right) \\ &= \rho\left((1 - x)^s + x^s\mathcal{C}^s_\infty([-1, 1])\right). \end{aligned}$$

Provided ρ is sufficiently small, $1 - \rho(2x)^s$ is positive and so we may rearrange the above to give

$$\mathcal{C}^s_\infty([-1, 1]) \leqslant \frac{\rho(1 - x)^s}{1 - \rho(2x)^s}.$$

Since the above function is increasing in x, we may substitute in the bound from Lemma 4.5.1 and let $\rho \downarrow 1$ to give

$$\mathcal{C}^s_\infty([-1, 1]) \leqslant \frac{2^{s-1}}{1 - \left(1 - 2^{1-\frac{1}{s}}\right)^s}.$$

The result then follows from Corollary 4.4.7. $\square$

Neither of these theorems required the additional information incorporated into the definition $\mathcal{M}^s_\infty$ as compared to $\mathcal{C}^s_\infty$. It does seem that improvements could be made to the above, possibly using $\mathcal{C}^s_\infty$, but maybe requiring $\mathcal{M}^s_\infty$. The obvious conjecture is that the optimal bound remains equal to the upper density of the Cantor set for all $s \in [0,1]$. If it could be shown that $\mathcal{M}^s_\infty([0,1]) = \mathcal{M}^s_\infty(C^s)$, where C^s is the s-dimensional Cantor set scaled to fill $[-1,1]$, or even the intermediary sets that are used to define it iteratively, then it may be shown that the conjecture holds true for s slightly larger than s_0.

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