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ANALYSIS FOR TIME DISCRETE APPROXIMATIONS OF BLOW-UP SOLUTIONS OF SEMILINEAR PARABOLIC EQUATIONS*

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Abstract. We prove a posteriori error estimates for time discrete approximations, for semilinear parabolic equations with solutions that might blow up in finite time. In particular we consider the backward Euler and the Crank–Nicolson methods. The main tools that are used in the analysis are the reconstruction technique and energy methods combined with appropriate fixed point arguments. The final estimates we derive are conditional and lead to error control near the blow up time.

Key words. semilinear parabolic equations, blow-up solutions and rate, conditional a posteriori estimates, backward Euler method, Crank–Nicolson method, reconstruction technique, energy techniques, fixed point arguments, Duhamel's principle

AMS subject classifications. 65M15, 35K58, 35B44

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1. Introduction.

In this paper we consider semilinear parabolic initial-and-boundary value problems of the form

$$\begin{cases} u_t - \Delta u = f(u) & \text{in } \Omega \times (0,T], \\ u = 0 & \text{on } \partial\Omega \times (0,T], \\ u(\cdot,0) = u_0 & \text{in } \bar{\Omega}, \end{cases} \tag{1.1}$$

with $f(u) = |u|^{p-1}u$, $p \in \mathbb{N} \setminus \{1\}$, and $u_0 : \bar{\Omega} \to \mathbb{R}$ a given initial value that belongs to $L^\infty(\Omega)$. In (1.1), $\Omega \subseteq \mathbb{R}^d$ is a bounded domain with boundary $\partial\Omega$ and $0 < T < \infty$.

Consider now problem (1.1) in $\Omega \times (0,\infty)$. Then, it is well known that its solution might blow up in finite time, even if the initial value u_0 belongs to $L^\infty(\Omega)$; cf., e.g., [37, Chapter II]. That is, there exists a t^*, $0 < t^* < \infty$, such that

$$\lim_{t \to (t^*)^-} \|u(t)\|_{L^\infty(\Omega)} = \infty.$$

On the other hand, it is also known (see, for example, [16]) that, for L^∞ initial data, problem (1.1) admits a unique solution $u \in C^{2,1}\big(\Omega \times (0,t^*)\big) \cap C\big(\bar{\Omega} \times [0,t^*)\big)$. In view of this, in the rest of the paper we assume that $T \leq t^* - \varepsilon$ for some $\varepsilon > 0$, i.e., we assume that the final time T is at most ε close to the blow up time (T is just finite when the solution does not blow up).

Our motivation to consider this problem is that it is the simplest nonlinear PDE with possible blow-up in finite time. Such problems become increasingly important

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in various applications ranging from mathematical biology, to material science, and to optics; see, e.g., [6, 11, 22, 23, 35]. The direct computation of such solutions is possible only through appropriate adaptive methods as illustrated in the pioneering works [2, 36]. The available adaptive techniques are based on *ad hoc* mesh selection criteria that work only under certain structural circumstances related to the nonlinear problem at hand. Our aim is to provide error control based on rigorous analysis for such nonlinear PDEs. The final goal is the design and implementation of appropriate adaptive algorithms. As a first important step, in this paper we prove fully a posteriori error estimates, for backward Euler and Crank–Nicolson time discrete approximations of problem (1.1). Our results lead to error control even near the blow-up time and circumvent long standing issues related to estimation of approximations of this class of nonlinear PDEs, in particular, with respect to "global in time" error control and to exponential constants. The construction of corresponding adaptive algorithms is the subject of an ongoing work and will be reported in a forthcoming paper. Adaptive strategies for other types of parabolic problems with blow-up solutions have been proposed, for example, by Acosta, Durán, and Rossi, [1], and by Groisman [17].

Despite the fact that under reasonable restrictions on the time steps one may prove, in the spirit of [3], that the backward Euler and the Crank–Nicolson approximations for problem (1.1) are well defined, even close to the blow-up time, a priori error estimates for problem (1.1) near the blow-up time do not exist in the literature. Standard analysis yields, in this case, estimates of limited applicability due to the constants involved. In particular, in the final estimates, a constant of the form $\mathrm{e}^{1/\varepsilon}$ appears. One of our tasks in this paper is to address this issue in the a posteriori analysis.

The derivation of the a posteriori error estimates for problem (1.1) follows the reconstruction approach. More specifically, we use the backward Euler reconstruction which is just the piecewise linear interpolant (cf. [33, 28]) and the Crank–Nicolson reconstruction that has been proposed by Akrivis, Makridakis, and Nochetto in [4]. Our analysis is based on energy and semigroup type techniques. A key argument provides the successful passage from "local in time" to "global in time" error control. The final estimates we obtain are *conditional*. Such estimates hold under some a posteriori, and thus, in principle, computationally verifiable conditions. In particular, these conditions are of the form $\mathcal{E} \leq \alpha$, where $\mathcal{E}$ is an a posteriori functional, i.e., it depends on the discrete approximations and the data of the problem, but not on the unknown solution u, and α is a fixed, known number. Conditional estimates have been considered in the past, e.g., by Cuesta and Makridakis, [7], Fierro and Veeser, [9], Kessler, Nochetto, and Schmidt [24], Lakkis and Nochetto, [26], and Makridakis and Nochetto, [29].

Problems of the form (1.1) and their blow-up solutions have been extensively studied by many authors; see, e.g., Giga and Kohn [13, 14, 15], Giga, Matsui, and Sasayama [16], Filippas and Kohn [10], Herrero and Velazquez [19, 20, 21], Merle and Zaag [30, 31, 32], and Groisman, Rossi, and Zaag [18]. In particular, the asymptotic behavior of the blow-up solutions near the blow-up time is studied in depth, yielding the blow-up rates $(t^* - t)^{-\frac{1}{p-1}}$.

To obtain conditional a posteriori error estimates for (1.1) it will be helpful to consider a slightly more general model problem,

$$
(1.2) \qquad \begin{cases} u_t - \nu \Delta u = f(u) & \text{in } \Omega \times (0, T], \\ u = 0 & \text{on } \partial\Omega \times (0, T], \\ u(\cdot, 0) = u_0 & \text{in } \bar{\Omega}, \end{cases}
$$

with $\nu > 0$ and where the nonlinear term $f : \mathbb{R} \to \mathbb{R}$ is a twice continuously differentiable function such that

(i) $|f''(x)| \le g(\rho, R)$ for every $\rho > 0$ and $R > 0$, and for every $x \in \mathbb{R}$, with $|x| \le \rho+R$ and

(ii) $|f''(x_1) - f''(x_2)| \le \frac{C}{R} g(\rho, R)|x_1 - x_2|$ for every $\rho > 0$ and $R > 0$, and for every $x_1, x_2 \in \mathbb{R}$, with $|x_1|$, $|x_2| \le \rho + R$, and $|x_1 - x_2| \le 2R$.

In (i) and (ii), $g : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function and C is a positive constant. Obviously, for $p > 2$, problem (1.1) comprises a special case of problem (1.2). Indeed, in this case, one may easily verify that we can take

$$g(\rho, R) = p(p-1)(\rho + R)^{p-2} \tag{1.3}$$

and $C = p-2$. This is because $|f''(x)| = p(p-1)|x|^{p-2}$ for all $x \in \mathbb{R}$ and $|f''(x_1) - f''(x_2)| \le \frac{1}{R} p(p-1)(p-2)(\rho+R)^{p-2}$, for all x_1, x_2 with $|x_1|, |x_2| \le \rho+R$. We consider problem (1.2) first and then we specify our results for (1.1), instead of handling directly problem (1.1), for two main reasons. The fact that in (1.2) f is general makes the analysis less technical. As a result, it becomes more clear how we apply fixed point arguments to obtain the final estimates. However, because of the more general nature of problem (1.2), the conditions to derive error bounds for problem (1.1) can be relaxed; cf. (3.2), (4.1), and Remark 4.3. The second reason is that working first with (1.2) we obtain a posteriori error bounds for more general cases of interest as well. In particular, problem (1.2) covers many other interesting cases including $f(u) = |u|^p$, $p \in \mathbb{N} \setminus \{1\}$ and $f(u) = \lambda u^p$, $\lambda \in \mathbb{R}$, $p \in \mathbb{N} \setminus \{1\}$. In these cases the solution of problem (1.2) might blow up in finite time and the blow-up rate is the same as in the case of problem (1.1); see, for example, [12, 31, 34, 38].

The paper is organized as follows: In section 2 we briefly present the methods and the corresponding reconstructions for problem (1.2) (and thus for problem (1.1)). In this section we also point out that, in contrast to the linear case, energy techniques do not lead directly to a posteriori error estimates in the $L^\infty(L^2)$ and the $L^2(H^1)$-norm for problem (1.1), Theorem 2.1.

The main results of the paper are presented in sections 3, 4, and 5. In section 3 we prove conditional a posteriori error estimates for problem (1.2) in the $L^\infty(L^\infty)$-norm by using appropriate fixed point arguments. These estimates are then combined with energy techniques and yield upper bounds for problem (1.2) in the $L^\infty(L^2)$- and $L^2(H^1)$-norms as well, Theorem 3.3. In section 4 we deal with problem (1.1). Since now the nonlinear term has a particular form, a slight modification to the analysis of section 3, leads to conditional estimates under relaxed conditions, Theorem 4.1. Finally, in section 5 we discuss how the conditional estimates we have at our disposal might lead to error control near the blow-up time.

2. Discretization methods and error control. In subsections 2.1 and 2.2 we present the backward Euler and the Crank–Nicolson methods and reconstructions for problem (1.2). In the analysis of sections 3 and 4 we use only the error equation (2.8), and not the particular form of the two methods. Since, as we shall see, both methods satisfy (2.8), with different residuals, it is natural to keep the same notation in their definition.

To this end, let $0 = t^0 < t^1 < \cdots < t^N = T$ be a partition of $[0,T]$, $I_n := (t^{n-1}, t^n]$, $k_n := t^n - t^{n-1}$, and $k := \max_{1 \le n \le N} k_n$. For a sequence $\{v^n\}_{n=0}^N$ we use the notation

$$\bar\partial v^n := \frac{v^n - v^{n-1}}{k_n} \quad \text{and} \quad v^{n-\frac{1}{2}} := \frac{v^n + v^{n-1}}{2}, \quad n = 1, \dots, N.$$

In addition the following standard notation will be used in what follows. For $1 \le p \le \infty$, let $L^p := L^p(\Omega)$ and let $\|\cdot\|_{L^p}$ be the corresponding norm. In the special case of $p = 2$, we just write $\|\cdot\|$ instead of $\|\cdot\|_{L^2}$. Let $H^{-1} := H^{-1}(\Omega)$ be the dual space of $H_0^1 := H_0^1(\Omega)$. With $\|\cdot\|_{-1}$ we denote the dual norm in H^{-1}, i.e., for $f \in H^{-1}$,

$$\|f\|_{-1} := \sup_{v \in H_0^1,\, \|\nabla v\|=1} (f, v),$$

where $(\cdot,\cdot)$ is the inner product in L^2. Last, we use the notation $\|\cdot\|_{L^{\infty,\infty}}$ for the $L^\infty(L^\infty)$-norm.

2.1. The backward Euler method and reconstruction. We discretize problem (1.2) only in time by the backward Euler method and we end up with approximations $U^n \in H_0^1$ to the values $u(t^n)$, $n = 0, 1, \dots, N$, defined by

$$\bar\partial U^n - \nu \Delta U^n = f(U^n), \quad n = 1, \dots, N, \tag{2.1}$$

with $U^0 = u_0$.

We define $U : [0, T] \to H_0^1$ to be the following piecewise constant function:

$$U(t) := U^n, \quad t \in I_n.$$

Function U is a first order approximation to u which is defined for every $t \in [0, T]$, but it is discontinuous at the nodes $\{t^n\}_{n=0}^{N-1}$. In order to be able to apply the reconstruction technique, we need to introduce a continuous-in-time function. Thus, we consider the piecewise linear interpolant between the nodal values U^{n-1} and U^n,

$$\hat U(t) := U^n + (t - t^n)\bar\partial U^n, \quad t \in I_n.$$

Then we can easily see that, for $n = 1, \dots, N$,

$$\hat U(t) - U(t) = (t - t^n)\bar\partial U^n, \quad t \in I_n. \tag{2.2}$$

In other words, the difference $\hat U - U$ is of first (and thus of optimal) order of accuracy in time.

The residual $\hat r : I_n \to L^2$, $n = 1, \dots, N$, of $\hat U$ is defined to be the a posteriori quantity $\hat r := \hat U_t - \nu\Delta\hat U - f(\hat U)$. Since for $t \in I_n$, $\hat U_t(t) = \bar\partial U^n$, by using the method (2.1), we see that the residual can be written as

$$\hat r(t) = -\nu\Delta(\hat U - U)(t) + \big(f(\hat U) - f(U)\big)(t), \quad t \in I_n,\ n = 1, \dots, N. \tag{2.3}$$

In view of (2.3), (2.2), and the fact that f is a locally Lipschitz continuous function, we expect that the residual will be of optimal order of accuracy. Therefore, the piecewise linear interpolant $\hat U$ is an appropriate reconstruction in the case of the backward Euler method.

2.2. The Crank–Nicolson method and reconstruction. The Crank–Nicolson method for problem (1.2) produces approximations $\{U^n\}_{n=0}^N$ to the values $u(t^n)$, $n = 0, 1, \dots, N$, defined by

$$\bar\partial U^n - \nu\Delta U^{n-\frac12} = f(U^{n-\frac12}), \quad n = 1, \dots, N, \tag{2.4}$$

with $U^0 = u_0$. In this case we consider the continuous-in-time approximation $U(t)$ to $u(t)$, for $t \in [0, T]$, by linearly interpolating between the nodal values U^{n-1} and U^n:

$$U(t) := U^{n-\frac12} + (t - t^{n-\frac12})\bar\partial U^n, \quad t \in I_n.$$

Since the Crank–Nicolson method is of second order, it is clear that, for $t \in [0, T]$, $u(t) - U(t) = O(k^2)$. However, it is well known (see, for example, [4]) that the direct use of U in the a posteriori error analysis yields, even in linear cases, estimates of first instead of optimal second order of accuracy. This problem can be solved by using a reconstruction of U.

The Crank–Nicolson reconstruction $\hat{U} : [0, T] \to L^2$ of U, that has been proposed in [4], is a piecewise quadratic polynomial and is defined as

$$\hat{U}(t) := U^{n-1} + \nu\Delta \int_{t^{n-1}}^{t} U(s)\, ds + \int_{t^{n-1}}^{t} b(s)\, ds, \quad t \in I_n, \tag{2.5}$$

where $b : I_n \to L^2$ is the linear interpolant of $f(U)$ at the nodes t^{n-1} and $t^{n-\frac{1}{2}}$, i.e.,

$$b(t) := f(U^{n-\frac{1}{2}}) + \frac{2}{k_n}(t - t^{n-\frac{1}{2}})\big[f(U^{n-\frac{1}{2}}) - f(U^{n-1})\big], \quad t \in I_n.$$

From (2.5) we conclude that $\hat{U}$ can be written as

$$\begin{aligned}\hat{U}(t) = U^{n-1} &+ \frac{\nu}{2}(t - t^{n-1})\Delta\big[U(t) + U^{n-1}\big] + (t - t^{n-1}) f(U^{n-\frac{1}{2}}) \\ &+ \frac{1}{k_n}(t - t^{n-1})(t^n - t)\big[f(U^{n-\frac{1}{2}}) - f(U^{n-1})\big], \quad t \in I_n.\end{aligned}$$

Thus $\hat{U}(t^n) = U(t^n) = U^n$, $n = 0, 1, \ldots, N$, i.e., $\hat{U}$ is continuous. From (2.5) it is also easily seen that $\hat{U}$ satisfies

$$\hat{U}_t(t) - \nu\Delta U(t) = b(t), \quad t \in I_n. \tag{2.6}$$

As before, the residual $\hat{r}$ of $\hat{U}$ is defined as $\hat{r}(t) := \big[\hat{U}_t - \nu\Delta\hat{U} - f(\hat{U})\big](t)$, $t \in I_n$, $n = 1, \ldots, N$. From (2.6) we see that the residual can also be written as

$$\hat{r}(t) = -\nu\Delta(\hat{U} - U)(t) + \Big[f\big(U(t)\big) - f\big(\hat{U}(t)\big)\Big] + \Big[b(t) - f\big(U(t)\big)\Big], \quad t \in I_n. \tag{2.7}$$

If $\{U^n\}_{n=0}^{N}$ are second order approximations to u at the nodes t^n, $n = 0, 1, \ldots, N$, then the residual is expected to be of second order as well. This is because, as it has been proven in [4], the difference $\hat{U} - U$ can be expressed as

$$\hat{U}(t) - U(t) = -\frac{1}{2}(t - t^{n-1})(t^n - t)\left[\nu\Delta\bar{\partial}U^n + \frac{2}{k_n}\big(f(U^{n-\frac{1}{2}}) - f(U^{n-1})\big)\right], \quad t \in I_n,$$

and f is a locally Lipschitz continuous function.

Remark 2.1. In the analysis below we assume that $\hat{U} \in H_0^1$. This can easily be proven in the case of problem (1.1) (cf. [25, Chapter 1, Remark 1.2]). For the general cases this is not obvious. A discussion about when $\hat{U}$ indeed belongs to H_0^1 can be found in [5]. However, assuming that $\hat{U} \in H_0^1$ does not comprise loss of generality since this is always true in cases of fully discrete schemes; see, for example, [25, Chapter 7].

2.3. The main error equation. Let the error $\hat{e} : [0, T] \to H_0^1$ be defined by $\hat{e} := u - \hat{U}$, where $\hat{U}$ denotes the backward Euler or the Crank–Nicolson reconstruction that has been introduced in subsection 2.1 or 2.2, respectively. The definition of the residual immediately yields

$$\hat{e}_t(t) - \nu\Delta\hat{e}(t) = \big[f(u) - f(\hat{U})\big](t) - \hat{r}(t), \quad t \in I_n, \tag{2.8}$$

for $n = 1, \dots, N$, with $\hat{e}(0) = 0$. For the backward Euler method the residual is given by (2.3), while for the Crank–Nicolson method it is given by (2.7). A first straightforward result based on the error equation (2.8) is given in subsection 2.4. Afterwards, a motivation and a plan of the forthcoming analysis is provided.

Remark 2.2. Our analysis is general and can be straightforwardly extended to other Crank–Nicolson reconstructions or to any numerical scheme in which a reconstruction function is known. In particular we can alternatively consider the Crank–Nicolson reconstruction proposed by Lozinski, Picasso, and Prachittham in [27], or we can discretize by any other Runge–Kutta method and use the reconstruction discussed in [5].

2.4. A first error estimate: Motivation. In this subsection we apply energy techniques to the error equation (2.8) for the special case $f(u) = |u|u$ and $\nu = 1$. We first observe that energy techniques do not lead directly to fully a posteriori error bounds because of the nature of the nonlinearity. To this end, for $n = 1, \dots, N$, we take in (2.8) the L^2-inner product with $\hat{e}$ to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\hat{e}(t)\|^2 + \|\nabla\hat{e}(t)\|^2 &\le \big([|u|u - |\hat{U}|\hat{U}](t), \hat{e}(t)\big)\\ &\quad + \frac{1}{2}\|\hat{r}(t)\|_{-1}^2 + \frac{1}{2}\|\nabla\hat{e}(t)\|^2, \quad t \in I_n.\end{aligned} \tag{2.9}$$

Note now that

$$\big([|u|u - |\hat{U}|\hat{U}](t), \hat{e}(t)\big) \le \big(2\|\hat{U}(t)\|_{L^\infty} + \|\hat{e}(t)\|_{L^\infty}\big)\|\hat{e}(t)\|^2, \tag{2.10}$$

because

$$|\,|u|u - |\hat{U}|\hat{U}| \le \hat{e}^2 + 2|\hat{U}|\,|\hat{e}|.$$

Invoking (2.10) in (2.9) and applying Gronwall's inequality we conclude to the following theorem.

THEOREM 2.1. *If $f(u) = |u|u$, then the following error estimate holds for the problem* (1.1)

$$\begin{aligned}\max_{0\le t\le T}&\left\{\|\hat{e}(t)\|^2 + \int_0^t e^{2\int_s^t \left[2\|\hat{U}(\tau)\|_{L^\infty} + \|\hat{e}(\tau)\|_{L^\infty}\right] d\tau}\|\nabla\hat{e}(s)\|^2\, ds\right\}\\ &\le \int_0^T e^{2\int_s^T \left[2\|\hat{U}(\tau)\|_{L^\infty} + \|\hat{e}(\tau)\|_{L^\infty}\right] d\tau}\|\hat{r}(s)\|_{-1}^2\, ds.\end{aligned} \tag{2.11}$$

Using similar arguments as above, we can prove estimates of the form (2.11) for the cases $f(u) = |u|^{p-1}u$, $p \in \mathbb{N} \setminus \{1, 2\}$, as well. Obviously, estimate (2.11) is not an a posteriori estimate because of the presence of $e^{2\int_0^T \|\hat{e}(\tau)\|_{L^\infty}\, d\tau}$ in the right-hand side. Naturally, Theorem 2.1 leads to two questions:

- *Is a fully a posteriori result of the form* (2.11) *possible?*
- *What is the behavior of the constant, especially near the blow-up time?*

Our aim in the next section is to show that it is indeed possible to control a posteriori $\int_0^T \|\hat{e}(\tau)\|_{L^\infty}\, d\tau$ and thus to obtain fully a posteriori error control, of course under assumptions of *conditional* type.

In addition, despite the fact that (2.11) is not yet a fully a posteriori estimate, it provides the following important insight: The appearance of the term

$$e^{2\int_0^T \left[2\|\hat{U}(\tau)\|_{L^\infty} + \|\hat{e}(\tau)\|_{L^\infty}\right] d\tau}$$

in the right-hand side relates to the constant of the final estimate. In fact, the main term in the constant of our estimate will be

$$\mathrm{e}^{\int_0^T \|\hat{U}(\tau)\|_{L^\infty}\, d\tau}.$$

Roughly speaking, for reasonably good approximations, we expect that $\hat{U}$ will behave as the exact solution u, $\hat{U} \sim u$. If we further assume now that the exact solution blows up at the finite time t^*, we obtain that $\|u(t)\|_{L^\infty} \sim (t^* - t)^{-\frac{1}{p-1}}$, near the blow-up time (cf. the introduction), where of course $p = 2$ in this special case. Let $T := t^* - \varepsilon$. A simple calculation reveals that

$$\mathrm{e}^{\int_0^T \frac{1}{t^*-t}\, dt} = \frac{t^*}{\varepsilon}.$$

Therefore, we expect that

$$\mathrm{e}^{\int_0^T \|\hat{U}(\tau)\|_{L^\infty}\, d\tau} \sim \frac{C(t^*)}{\varepsilon^q}$$

for some $q > 0$ and a constant $C(t^*)$, which depends only on t^*. In other words, the appearance of $\mathrm{e}^{\int_0^T \|\hat{U}(\tau)\|_{L^\infty}\, d\tau}$ in the estimate (2.11), combined with the blow-up rate of the exact solution, is expected to lead to upper bounds that grow polynomially instead of exponentially, as we approach the blow-up time. We discuss this issue in detail in section 5, see also Remark 4.3.

3. A posteriori error estimates for general f.

3.1. Main ideas. As previously discussed, to prove a posteriori error estimates in the $L^\infty(L^2)$-norm, we first need to estimate a posteriori the term $\int_0^T \|\hat{e}(\tau)\|_{L^\infty}\, d\tau$; cf. Theorem 2.1. So, our goal in the next subsection is to prove conditional a posteriori error estimates in the $L^\infty(L^\infty)$-norm for problem (1.2) using fixed point arguments. In some parts of the proof, we follow arguments from [7]. Furthermore the bootstrap-type argument we use has conceptual similarities to the "thought experiment" of [24, sections 3.3–3.4]. The forthcoming analysis is technically involved and requires careful use of conditional assumptions and new key ideas.

More specifically, in our analysis, the introduction of a new, uniform partition $\{T_m\}_{m=0}^M$ of $[0, T]$ of time step $\delta > 0$ plays a significant role. As we shall see this partition is artificial, i.e., it will be used only for theoretical purposes, and at the end of the analysis it will allow a slight improvement to the error estimates; cf. Remark 3.2 below.

This step-by-step procedure, instead of considering the whole time interval $[0, T]$, to prove the conditional estimates, is required in order to pass from "local in time" to "global in time" estimates. This is because in the analysis the term $\int_0^t \|f'(\hat{U}(s)\|\|_{L^\infty}\, ds$, $t \in [0, T]$, must be controlled; in particular, a condition of the form

$$\int_0^t \|f'(\hat{U})(s)\|_{L^\infty}\, ds < 1, \quad t \in [0, T], \tag{3.1}$$

must be satisfied; cf. (3.12). It is clear that (3.1) can be verified only if t is "sufficiently small." Then, obviously, condition (3.1) leads to local in time estimates. The artificial partition of length δ is chosen as a main tool to overcome this obstacle. On the other hand, one may wonder why not to use the initial partition $\{t^n\}_{n=0}^N$. One of the reasons is that we want to avoid introducing additional restrictions to the time steps.

In fact, using a uniform partition we can guarantee that the whole interval $[0,T]$ will be covered, even if δ is approaching 0^+. This is something we cannot generally do with the (nonuniform in principle) partition $\{t^n\}_{n=0}^N$.

3.2. Estimates in the $L^\infty(L^\infty)$-norm using fixed point arguments. We first assume that $\|f'(\hat U)\|_{L^{\infty,\infty}} < \infty$. Then we choose $0 < \hat R \le 1$ and time steps k_n, $n = 0, 1, \dots, N$, such that

$$Tg(\|\hat U\|_{L^{\infty,\infty}}, \hat R)\hat R \le \frac{1}{4(2+C)} \tag{3.2}$$

and

$$\mathrm{e}^{2\int_0^T \|f'(\hat U)(\tau)\|_{L^\infty}\,d\tau} \int_0^T \|\hat r(\tau)\|_{L^\infty}\,d\tau \le \frac{1}{4}\left(1 - \frac{1}{2(2+C)}\right)\hat R. \tag{3.3}$$

Finally, we choose $\delta > 0$ such that

$$\delta\|f'(\hat U)\|_{L^{\infty,\infty}} \le \frac{3}{8} \tag{3.4}$$

and $M := \frac{T}{\delta} \in \mathbb{N}$. Let $T_m := m\delta$, $m = 0, 1, \dots, M$, be a uniform partition of $[0,T]$.

Remark 3.1 (can conditions (3.2) and (3.3) be satisfied simultaneously?). Let us consider the function $G : \mathbb{R} \to \mathbb{R}$,

$$G(\rho) := \max\left\{\frac{1}{8}, T \max_{0\le R\le 1} g(\rho, R)\right\}.$$

Since the function g is continuous, G is well defined. Also, for every $\rho \in \mathbb{R}$,

$$G(\rho) \ge \frac{1}{8}. \tag{3.5}$$

We choose the time steps k_n, $n = 1, \dots, N$, such that

$$\begin{aligned} &\mathrm{e}^{2\int_0^T \|f'(\hat U)(\tau)\|_{L^\infty}\,d\tau} G(\|\hat U\|_{L^{\infty,\infty}}) \int_0^T \|\hat r(\tau)\|_{L^\infty}\,d\tau \\ &\qquad \le \frac{1}{4}\left(1 - \frac{1}{2(2+C)}\right)\frac{1}{4(2+C)}. \end{aligned} \tag{3.6}$$

We next set

$$\hat R := \frac{1}{4(2+C)} \cdot \frac{1}{G(\|\hat U\|_{L^{\infty,\infty}})}.$$

Relation (3.5) yields that $0 < \hat R \le 1$. Besides that, the definition of G asserts that

$$Tg(\|\hat U\|_{L^{\infty,\infty}}, \hat R)\hat R \le G(\|\hat U\|_{L^{\infty,\infty}})\hat R = \frac{1}{4(2+C)}.$$

In other words, (3.2) holds. Finally, from (3.6), we immediately conclude the validity of (3.3). Now, regarding the question if condition (3.6) (or, equivalently, condition (3.3)) is realistic, this will be the subject of discussion in section 5. At this point we just conclude that conditions (3.2) and (3.3) can be satisfied simultaneously.

3.2.1. Estimate of $\max_{t\in[T_{m-1},T_m]}\|\hat{e}(t)\|_{L^\infty}$. The purpose here is to estimate $\hat{e}$ in the $L^\infty(L^\infty)$-norm in each subinterval $[T_{m-1},T_m]$, $m=1,\dots,M$. Recall that $\{T_m\}_{m=0}^M$ is an artificial partition of $[0,T]$ which is used only in the proof of the final estimates, and must not be confused with the partition $\{t^n\}_{n=0}^N$. In order to avoid any confusion, for $m=1,\dots,M$, we will use the notation $L_m^{\infty,\infty}$ for the space $C\big([T_{m-1},T_m];L^\infty(\Omega)\big)$ and $\|\cdot\|_{L_m^{\infty,\infty}}$ for the norm corresponding to the space $L_m^{\infty,\infty}$.

Since the function f is twice continuously differentiable, we can write

$$f(u)-f(\hat{U})=f'(\hat{U})\hat{e}+\int_0^1(1-\tau)f''(\hat{U}+\tau\hat{e})\,d\tau\,\hat{e}^2. \tag{3.7}$$

Then, in each $[T_{m-1},T_m]$, the error $\hat{e}$ satisfies

$$\begin{cases}\hat{e}_t-\nu\Delta\hat{e}=f'(\hat{U})\hat{e} & \\ \qquad\qquad +\displaystyle\int_0^1(1-\tau)f''(\hat{U}+\tau\hat{e})\,d\tau\,\hat{e}^2-\hat{r} & \text{in } \Omega\times[T_{m-1},T_m],\\ \hat{e}=0 & \text{on } \partial\Omega\times[T_{m-1},T_m],\\ \hat{e}(\cdot,T_{m-1})=\hat{e}(T_{m-1}) & \text{in } \bar{\Omega},\end{cases} \tag{3.8}$$

with $\hat{e}(0)=0$. Through (3.8) a sequence of problems $\{\mathrm{P}_m\}_{m=1}^M$ is defined. Recall that the error $\hat{e}$ is a time-continuous function and thus this sequence of problems is well defined. Assuming that $\hat{e}(T_{m-1})\in L^\infty$, we define the operator $\Phi: L_m^{\infty,\infty}\to L_m^{\infty,\infty}$ as

$$\begin{aligned}\Phi(w)(t)=&\,\mathrm{e}^{(t-T_{m-1})\nu\Delta}\hat{e}(T_{m-1})+\int_{T_{m-1}}^t \mathrm{e}^{(t-s)\nu\Delta}f'(\hat{U})(s)w(s)\,ds\\ &+\int_{T_{m-1}}^t \mathrm{e}^{(t-s)\nu\Delta}\left\{\int_0^1(1-\tau)f''(\hat{U}+\tau w)\,d\tau\,w^2\right\}(s)\,ds\\ &-\int_{T_{m-1}}^t \mathrm{e}^{(t-s)\nu\Delta}\hat{r}(s)\,ds,\quad t\in[T_{m-1},T_m].\end{aligned} \tag{3.9}$$

The aim is to prove that the operator Φ is, in every interval $[T_{m-1},T_m]$, $m=1,\dots,M$, a contraction in the $L^\infty(L^\infty)$-norm, in the closed ball $B(0,\hat{R};L_m^{\infty,\infty})$ of center 0 and radius $\hat{R}>0$. If we manage to achieve this, then by Banach's fixed point theorem, Φ will have a unique fixed point in each interval $[T_{m-1},T_m]$, $m=1,\dots,M$, which, by Duhamel's principle, will also be the unique solution of problem (3.8). Therefore, the problem

$$\begin{cases}\hat{e}_t-\nu\Delta\hat{e}=f(u)-f(\hat{U})-\hat{r} & \text{in } \Omega\times(0,T],\\ \hat{e}=0 & \text{on } \partial\Omega\times(0,T],\\ \hat{e}(0)=0 & \text{in } \bar{\Omega}\end{cases} \tag{3.10}$$

will have a unique solution, the $L^\infty(L^\infty)$-norm of which will be estimated through the analysis below.

To this end, we will derive preliminary estimates for $\|\Phi(w)(t)\|_{L^\infty}$ and $\|\Phi(w_1)(t)-\Phi(w_2)(t)\|_{L^\infty}$ in a generic interval $[T_{m-1},T_m]$ and then we will use induction with respect to m to complete the proof. Next, we shall use the following standard lemma.

LEMMA 3.1 (general form of maximum principle, Thomée [39, page 93]). *Let* $\mathrm{e}^{t\Delta}$

be the solution operator for the problem

$$\begin{cases} w_t - \Delta w = 0 & \text{in } \Omega,\ t > 0, \\ w = 0 & \text{on } \partial\Omega,\ t > 0, \\ w(\cdot, 0) = \upsilon & \text{in } \Omega. \end{cases}$$

In other words, $w(t) = \mathrm{e}^{t\Delta}\upsilon$*. Then, the following estimate is valid:*

$$\|\mathrm{e}^{t\Delta}\upsilon\|_{L^\infty} \le \|\upsilon\|_{L^\infty}, \quad t > 0,\ \upsilon \in L^\infty. \tag{3.11}$$

Estimate (3.11) yields

$$\begin{aligned} \|\Phi(w)(t)\|_{L^\infty} \le \|\hat{e}(T_{m-1})\|_{L_\infty} &+ \int_{T_{m-1}}^{t} \|f'(\hat{U})(s)\|_{L^\infty}\, ds\, \|w\|_{L_m^{\infty,\infty}} \\ &+ \delta \int_0^1 (1-\tau)\|f''(\hat{U} + \tau w)\|_{L_m^{\infty,\infty}}\, d\tau\, \|w\|_{L_m^{\infty,\infty}}^2 \\ &+ \int_{T_{m-1}}^{t} \|\hat{r}(s)\|_{L^\infty}\, ds, \quad t \in [T_{m-1}, T_m]. \end{aligned} \tag{3.12}$$

Let $w \in B(0, \hat{R}; L_m^{\infty,\infty})$. Then, for every $\tau \in [0,1]$, we have that $\tau w \in B(0, \hat{R}; L_m^{\infty,\infty})$. Hence, $|(\hat{U} + \tau w)(x,t)| \le \|\hat{U}\|_{L^{\infty,\infty}} + \hat{R}$, for every $w \in B(0, \hat{R}; L_m^{\infty,\infty})$, $\tau \in [0,1]$, and for every $(x,t) \in \Omega \times [T_{m-1}, T_m]$, $m = 1, \dots, M$. Accordingly, assumption (i) (cf. the introduction) yields, for $m = 1, \dots, M$,

$$\sup_{\tau \in [0,1]} \|f''(\hat{U} + \tau w)\|_{L_m^{\infty,\infty}} \le g(\|\hat{U}\|_{L^{\infty,\infty}}, \hat{R}). \tag{3.13}$$

Combining (3.12) and (3.13) we conclude, for $w \in B(0, \hat{R}; L_m^{\infty,\infty})$, $m = 1, \dots, M$, the estimate

$$\begin{aligned} \|\Phi(w)\|_{L_m^{\infty,\infty}} \le \|\hat{e}(T_{m-1})\|_{L^\infty} &+ \int_{T_{m-1}}^{T_m} \|f'(\hat{U})(s)\|_{L^\infty}\, ds\, \|w\|_{L_m^{\infty,\infty}} \\ &+ \frac{\delta}{2} g(\|\hat{U}\|_{L^{\infty,\infty}}, \hat{R})\hat{R}\, \|w\|_{L_m^{\infty,\infty}} + \int_{T_{m-1}}^{T_m} \|\hat{r}(s)\|_{L^\infty}\, ds. \end{aligned} \tag{3.14}$$

Regarding the contraction property, one calculates for $t \in [T_{m-1}, T_m]$

$$\begin{aligned} \Phi(w_1)(t) - \Phi(w_2)(t) &= \int_{T_{m-1}}^{t} \mathrm{e}^{(t-s)\nu\Delta} f'(\hat{U})(s)\big(w_1(s) - w_2(s)\big)\, ds \\ &+ \int_{T_{m-1}}^{t} \mathrm{e}^{(t-s)\nu\Delta} \bigg\{ (w_1 - w_2)(w_1 + w_2) \int_0^1 (1-\tau) f''(\hat{U} + \tau w_1)\, d\tau \\ &\qquad + w_2^2 \int_0^1 (1-\tau)\big(f''(\hat{U} + \tau w_1) - f''(\hat{U} + \tau w_2)\big)\, d\tau \bigg\}(s)\, ds. \end{aligned}$$

Using again (3.11) we get

$$\begin{aligned} \|\Phi(w_1) - \Phi(w_2)\|_{L_m^{\infty,\infty}} &\le \int_{T_{m-1}}^{T_m} \|f'(\hat{U})(s)\|_{L^\infty}\, ds\, \|w_1 - w_2\|_{L_m^{\infty,\infty}} \\ &+ \frac{\delta}{2}\Big[(\|w_1\|_{L_m^{\infty,\infty}} + \|w_2\|_{L_m^{\infty,\infty}})\|w_1 - w_2\|_{L_m^{\infty,\infty}} \sup_{\tau \in [0,1]} \|f''(\hat{U} + \tau w_1)\|_{L^{\infty,\infty}} \\ &+ \|w_2\|_{L_m^{\infty,\infty}}^2 \sup_{\tau \in [0,1]} \|f''(\hat{U} + \tau w_1) - f''(\hat{U} + \tau w_2)\|_{L_m^{\infty,\infty}} \Big]. \end{aligned} \tag{3.15}$$

Let $w_1, w_2 \in B(0, \hat{R}; L_m^{\infty,\infty})$. Since $\tau \in [0,1]$, we have that $\tau w_1, \tau w_2 \in B(0, \hat{R}; L_m^{\infty,\infty})$. On the other hand, since $w_1, w_2 \in B(0, \hat{R}; L_m^{\infty,\infty})$, it is easily seen that $\|w_1 - w_2\|_{L^{\infty,\infty}} \leq 2\hat{R}$. We obtain, in view of assumption (i), that for $m = 1, \dots, M$,

$$\sup_{\tau\in[0,1]} \|f''(\hat{U} + \tau w_1)\|_{L_m^{\infty,\infty}} \leq g(\|\hat{U}\|_{L^{\infty,\infty}}, \hat{R}) \tag{3.16}$$

and

$$\sup_{\tau\in[0,1]} \|f''(\hat{U}+\tau w_1) - f''(\hat{U}+\tau w_2)\|_{L_m^{\infty,\infty}} \leq \frac{C}{\hat{R}} g(\|\hat{U}\|_{L^{\infty,\infty}}, \hat{R}) \|w_1 - w_2\|_{L_m^{\infty,\infty}}, \tag{3.17}$$

in light of assumption (ii) (cf. the introduction). The combination of (3.15) with (3.16)–(3.17) reveals, for every $w_1, w_2 \in B(0, \hat{R}; L_m^{\infty,\infty})$, that

$$\begin{aligned} \|\Phi(w_1) - \Phi(w_2)\|_{L_m^{\infty,\infty}} &\leq \delta \|f'(\hat{U})\|_{L^{\infty,\infty}} \|w_1 - w_2\|_{L_m^{\infty,\infty}} \\ &\quad + \delta \frac{2+C}{2} g(\|\hat{U}\|_{L^{\infty,\infty}}, \hat{R}) \hat{R}\, \|w_1 - w_2\|_{L_m^{\infty,\infty}}. \end{aligned} \tag{3.18}$$

To complete the analysis we use induction with respect to m.

- *Step* 1. Recall that $\hat{e}(0) = 0$ and let $w \in B(0, \hat{R}; L_1^{\infty,\infty})$. Then the combination of (3.14) with conditions (3.2)–(3.4) ensures that

$$\|\Phi(w)\|_{L_1^{\infty,\infty}} \leq \frac{3}{8}\hat{R} + \frac{1}{8(2+C)}\hat{R} + \frac{1}{4}\left(1 - \frac{1}{2(2+C)}\right)\hat{R} = \frac{5}{8}\hat{R} \leq \hat{R}.$$

In other words, Φ maps the ball $B(0, \hat{R}; L_m^{\infty,\infty})$ on itself. The use of (3.18) gives, in view of (3.2) and (3.4), that

$$\|\Phi(w_1) - \Phi(w_2)\|_{L_1^{\infty,\infty}} \leq \frac{1}{2}\|w_1 - w_2\|_{L_1^{\infty,\infty}}$$

for every $w_1, w_2 \in B(0, \hat{R}; L_1^{\infty,\infty})$. Accordingly, $\Phi : B(0, \hat{R}; L_1^{\infty,\infty}) \to B(0, \hat{R}; L_1^{\infty,\infty})$ is a contraction. Therefore, according to Banach's fixed point theorem, there exists a unique fixed point $\hat{e} \in B(0, \hat{R}; L_1^{\infty,\infty})$ of Φ which is also the unique solution of problem (3.8) for $m = 1$. The following elementary lemma is useful.

LEMMA 3.2. *Let $\beta > 0$ and $\alpha := \frac{\beta}{1+\beta}$. Then for $0 \leq x \leq \alpha$, there holds $\frac{1}{1-x} \leq e^{(1+\beta)x}$.*

Proof. We just note that $0 < \alpha < 1$ and for $0 \leq x \leq \alpha$, $h(x) := e^{(1+\beta)x}(1-x) - 1$ is an increasing function. Since $h(0) = 0$, the proof is complete. ∎

Now, since $\hat{e} \in B(0, \hat{R}; L_1^{\infty,\infty})$ is the unique fixed point of operator Φ, we conclude, in view of (3.14), (3.4), and of Lemma 3.2 (with $\alpha = \frac{1}{2}$), the estimate

$$\max_{t\in[0,T_1]} \|\hat{e}(t)\|_{L^\infty} \leq e^{2\int_0^{T_1} \|f'(\hat{U})(s)\|_{L^\infty}\, ds} e^{\delta\, g(\|\hat{U}\|_{L^{\infty,\infty}}, \hat{R})\hat{R}} \int_0^{T_1} \|\hat{r}(s)\|_{L^\infty}\, ds.$$

- *Step* 2 (*inductive step*). We next assume that for an arbitrary m ($2 \leq m \leq M$),

$$\begin{aligned} \max_{t\in[T_{m-2},T_{m-1}]} \|\hat{e}(t)\|_{L^\infty} &\leq e^{2\int_0^{T_{m-1}} \|f'(\hat{U})(s)\|_{L^\infty}\, ds} \\ &\quad \times e^{(m-1)\delta\, g(\|\hat{U}\|_{L^{\infty,\infty}}, \hat{R})\hat{R}} \int_0^{T_{m-1}} \|\hat{r}(s)\|_{L^\infty}\, ds. \end{aligned} \tag{3.19}$$

Because of (3.2), the estimate

$$\mathrm{e}^{(m-1)\delta\, g(\|\hat U\|_{L^\infty,\infty},\hat R)\hat R} \le \mathrm{e}^{Tg(\|\hat U\|_{L^\infty,\infty},\hat R)\hat R} \le \mathrm{e}^{\frac18} \tag{3.20}$$

is valid. Therefore, by condition (3.3) and (3.19)–(3.20), we obtain

$$\begin{aligned}\|\hat e(T_{m-1})\|_{L^\infty} &\le \mathrm{e}^{2\int_0^{T_{m-1}}\|f'(\hat U)(s)\|_{L^\infty}\,ds}\mathrm{e}^{(m-1)\delta\, g(\|\hat U\|_{L^\infty,\infty},\hat R)\hat R}\int_0^{T_{m-1}}\|\hat r(s)\|_{L^\infty}\,ds\\ &\le \frac{\mathrm{e}^{\frac18}}{4}\hat R \le \frac38 \hat R.\end{aligned} \tag{3.21}$$

Proceeding as in the first step of the induction and combining (3.14) with conditions (3.2)–(3.3) and with (3.21), we see that for $w \in B(0,\hat R; L_m^{\infty,\infty})$,

$$\|\Phi(w)\|_{L_m^{\infty,\infty}} \le \hat R,$$

i.e., operator Φ maps the ball $B(0,\hat R; L_m^{\infty,\infty})$ on itself. Moreover, proceeding as before, we can prove that $\Phi : B(0,\hat R; L_m^{\infty,\infty}) \to B(0,\hat R; L_m^{\infty,\infty})$ is a contraction with constant $\le \frac12$. Thus, Φ has a unique fixed point $\hat e \in B(0,\hat R; L_m^{\infty,\infty})$, which is the unique solution of problem (3.8). Relations (3.14), (3.19), and (3.21), conditions (3.2) and (3.4), and Lemma 3.2 (with $\alpha = \frac12$) give

$$\begin{aligned}\|\hat e\|_{L_m^{\infty,\infty}} \le{}& \mathrm{e}^{2\int_{T_{m-1}}^{T_m}\|f'(\hat U)(s)\|_{L^\infty}\,ds}\mathrm{e}^{\delta\, g(\|\hat U\|_{L^\infty,\infty},\hat R)\hat R}\\ &\Big\{\mathrm{e}^{2\int_0^{T_{m-1}}\|f'(\hat U)(s)\|_{L^\infty}\,ds}\mathrm{e}^{(m-1)\delta\, g(\|\hat U\|_{L^\infty,\infty},\hat R)\hat R}\int_0^{T_{m-1}}\|\hat r(s)\|_{L^\infty}\,ds\\ &\qquad + \int_{T_{m-1}}^{T_m}\|\hat r(s)\|_{L^\infty}\,ds\Big\},\end{aligned}$$

or

$$\|\hat e\|_{L_m^{\infty,\infty}} \le \mathrm{e}^{2\int_0^{T_m}\|f'(\hat U)(s)\|_{L^\infty}\,ds}\mathrm{e}^{m\delta\, g(\|\hat U\|_{L^\infty,\infty},\hat R)\hat R}\int_0^{T_m}\|\hat r(s)\|_{L^\infty}\,ds.$$

In other words, since for $m = 1,\dots,M$, $m\delta \le T$, the following a posteriori error estimate is valid:

$$\begin{aligned}\max_{T_{m-1}\le t\le T_m}\|\hat e(t)\|_{L^\infty} \le{}& \mathrm{e}^{2\int_0^T\|f'(\hat U)(s)\|_{L^\infty}\,ds}\mathrm{e}^{Tg(\|\hat U\|_{L^\infty,\infty},\hat R)\hat R}\\ &\times\int_0^T\|\hat r(s)\|_{L^\infty}\,ds.\end{aligned} \tag{3.22}$$

3.2.2. Estimation of $\|\hat e\|_{L^{\infty,\infty}}$. From the analysis above and the estimate (3.22), it is clear that problem (3.10) has a unique solution for which the following a posteriori estimate is valid:

$$\|\hat e\|_{L^{\infty,\infty}} \le \mathrm{e}^{2\int_0^T\|f'(\hat U)(s)\|_{L^\infty}\,ds}\mathrm{e}^{Tg(\|\hat U\|_{L^\infty,\infty},\hat R)\hat R}\int_0^T\|\hat r(s)\|_{L^\infty}\,ds, \tag{3.23}$$

where $0<\hat R\le 1$ has been chosen such that condition (3.2) is satisfied. Furthermore, $\hat e \in B\big(0,\hat R; L^\infty(L^\infty)\big)$.

Remark 3.2. δ in condition (3.4) is needed only for theoretical purposes. Indeed, as we can see from conditions (3.2)–(3.3), δ does not affect the choice of $\hat{R}$ or the choice of the time steps k_n, $n = 1, \ldots, N$, neither does it appear in estimate (3.23). Since δ is needed only for theoretical purposes, it can become as small as we wish. Actually, we can let $\delta \to 0^+$. But then, according to Lemma 3.2, condition (3.3) can be relaxed and estimate (3.23) can be improved. In particular, condition (3.3) can be written as

$$\mathrm{e}^{(1+\beta)\int_0^T \|f'(\hat{U})(\tau)\|_{L^\infty}\, d\tau} \int_0^T \|\hat{r}(s)\|_{L^\infty}\, ds \le \frac{1}{4}\left(1 - \frac{1}{2(2+C)}\right)\hat{R}, \tag{3.24}$$

and estimate (3.23) can be written as

$$\begin{aligned} \|\hat{e}\|_{L^{\infty,\infty}} \le {} & \mathrm{e}^{(1+\beta)\int_0^T \|f'(\hat{U})(s)\|_{L^\infty}\, ds} \cdot \mathrm{e}^{\frac{(1+\beta)}{2} T g(\|\hat{U}\|_{L^{\infty,\infty}}, \hat{R})\hat{R}} \\ & \times \int_0^T \|\hat{r}(s)\|_{L^\infty}\, ds, \end{aligned} \tag{3.25}$$

where β is any real number with $\beta \ge \frac{1}{15}$.

3.3. The final estimates. Taking in the error equation (2.8) the L^2-inner product with $\hat{e}$ and using relation (3.7), we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\hat{e}(t)\|^2 + \nu\|\nabla\hat{e}(t)\|^2 \le {} & \|f'\big(\hat{U}(t)\big)\|_{L^\infty}\, \|\hat{e}(t)\|^2 \\ & + \frac{1}{2}\sup_{\tau\in[0,1]} \|f''(\hat{U} + \tau\hat{e})\|_{L^{\infty,\infty}} \|\hat{e}\|_{L^{\infty,\infty}}\, \|\hat{e}(t)\|^2 \\ & + \frac{1}{2\nu}\|\hat{r}(t)\|_{-1}^2 + \frac{\nu}{2}\|\nabla\hat{e}(t)\|^2. \end{aligned} \tag{3.26}$$

In (3.26) we have also used the Cauchy–Schwarz and the Young inequalities. Proceeding as in the proof of (3.13) and recalling that $\hat{e} \in B\big(0, \hat{R}; L^\infty(L^\infty)\big)$, we conclude that

$$\sup_{\tau\in[0,1]} \|f''(\hat{U} + \tau\hat{e})\|_{L^{\infty,\infty}} \le g(\|\hat{U}\|_{L^{\infty,\infty}}, \hat{R}).$$

Also, $\|\hat{e}\|_{L^{\infty,\infty}} \le \hat{R}$. Thus, (3.26) now gives

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\hat{e}(t)\|^2 + \frac{\nu}{2}\|\nabla\hat{e}(t)\|^2 \le {} & \|f'\big(\hat{U}(t)\big)\big\|_{L^\infty}\, \|\hat{e}(t)\|^2 + \frac{1}{2} g(\|\hat{U}\|_{L^{\infty,\infty}}, \hat{R})\hat{R}\, \|\hat{e}(t)\|^2 \\ & + \frac{1}{2\nu}\|\hat{r}(t)\|_{-1}^2. \end{aligned}$$

The above relation yields

$$\begin{aligned} & \frac{d}{dt}\Big(\mathrm{e}^{-2\int_0^t \|f'(\hat{U}(\tau))\|_{L^\infty}\, d\tau}\mathrm{e}^{-g(\|\hat{U}\|_{L^{\infty,\infty}}, \hat{R})\hat{R}\, t}\|\hat{e}(t)\|^2\Big) \\ & \qquad + \nu\mathrm{e}^{-2\int_0^t \|f'(\hat{U}(\tau))\|_{L^\infty}\, d\tau}\mathrm{e}^{-g(\|\hat{U}\|_{L^{\infty,\infty}}, \hat{R})\hat{R}\, t}\|\nabla\hat{e}(t)\|^2 \\ & \qquad \le \frac{1}{\nu}\mathrm{e}^{-2\int_0^t \|f'(\hat{U}(\tau))\|_{L^\infty}\, d\tau}\mathrm{e}^{-g(\|\hat{U}\|_{L^{\infty,\infty}}, \hat{R})\hat{R}\, t}\|\hat{r}(t)\|_{-1}^2. \end{aligned}$$

Integrating from 0 to t, and recalling that $\hat{e}(0)=0$, we conclude that

$$\begin{aligned}&\|\hat{e}(t)\|^2+\nu\int_0^t \mathrm{e}^{2\int_s^t\|f'(\hat{U}(\tau))\|_{L^\infty}\,d\tau}\mathrm{e}^{g(\|\hat{U}\|_{L^{\infty,\infty}},\hat{R})\hat{R}\,(t-s)}\|\nabla\hat{e}(s)\|^2\,ds\\&\qquad\le\frac{1}{\nu}\int_0^t \mathrm{e}^{2\int_s^t\|f'(\hat{U}(\tau))\|_{L^\infty}\,d\tau}\mathrm{e}^{g(\|\hat{U}\|_{L^{\infty,\infty}},\hat{R})\hat{R}\,(t-s)}\|\hat{r}(s)\|_{-1}^2\,ds.\end{aligned}$$

Thus we have proved the main result in this section.

THEOREM 3.3. *Let $\hat{e}=u-\hat{U}$ be the error. Then, the following a posteriori error estimates are valid for problem* (1.2)*:*

$$\begin{aligned}&\max_{0\le t\le T}\left\{\|\hat{e}(t)\|^2+\nu\int_0^t \mathrm{e}^{2\int_s^t\|f'(\hat{U}(\tau))\|_{L^\infty}\,d\tau}\mathrm{e}^{g(\|\hat{U}\|_{L^{\infty,\infty}},\hat{R})\hat{R}\,(t-s)}\|\nabla\hat{e}(s)\|^2\,ds\right\}\\&\qquad\le\frac{1}{\nu}\int_0^T \mathrm{e}^{2\int_s^T\|f'(\hat{U}(\tau))\|_{L^\infty}\,d\tau}\mathrm{e}^{g(\|\hat{U}\|_{L^{\infty,\infty}},\hat{R})\hat{R}\,(T-s)}\|\hat{r}(s)\|_{-1}^2\,ds,\end{aligned}\tag{3.27}$$

and

$$\max_{0\le t\le T}\|\hat{e}(t)\|_{L^\infty}\le \mathrm{e}^{(1+\beta)\int_0^T\|f'(\hat{U})(s)\|_{L^\infty}\,ds}\mathrm{e}^{Tg(\|\hat{U}\|_{L^{\infty,\infty}},\hat{R})\hat{R}}\int_0^T\|\hat{r}(s)\|_{L^\infty}\,ds,\tag{3.28}$$

where $\beta\ge\frac{1}{15}$, and $\hat{R}$ and the time steps k_n, $n=1,\dots,N$, have been chosen so that conditions (3.2) *and* (3.24) *are satisfied.*

Remark 3.3. The term $\mathrm{e}^{Tg(\|\hat{U}\|_{L^{\infty,\infty}},\hat{R})\hat{R}}$, which appears in the estimates above, is negligible. Indeed, from (3.2) we have that $\mathrm{e}^{Tg(\|\hat{U}\|_{L^{\infty,\infty}},\hat{R})\hat{R}}\le \mathrm{e}^{\frac{1}{8}}$.

4. A posteriori error estimates for $f(u)=|u|^{p-1}u$.

We are now ready to discuss problem (1.1). Our aim is to show that the results of section 3 can be further improved in the case where $f(u)=|u|^{p-1}u$. In the forthcoming analysis we avoid the repetition of similar arguments; whenever the analysis is modified, we discuss the differences in detail.

4.1. Conditional estimates using fixed point arguments and energy techniques.

As in the general case, we assume that $\|f'(\hat{U})\|_{L^{\infty,\infty}}<\infty$, i.e., $\|\hat{U}\|_{L^{\infty,\infty}}<\infty$. Using the notation of the previous section, we choose again $0<\hat{R}\le 1$ and the time steps k_n, $n=1,\dots,N$, so that

$$\int_0^T\mathcal{E}_p(s;\hat{U},\hat{R})\,ds:=\left[\sum_{j=2}^{p}\binom{p}{j}\int_0^T\|\hat{U}(s)\|_{L^\infty}^{p-j}\,ds\,\hat{R}^{j-2}\right]\hat{R}\le\frac{1}{16}\tag{4.1}$$

and

$$\mathrm{e}^{2\int_0^T\|f'(\hat{U})(\tau)\|_{L^\infty}\,d\tau}\int_0^T\|\hat{r}(s)\|_{L^\infty}\,ds\le\frac{3}{16}\hat{R}\tag{4.2}$$

are satisfied. We finally choose $\delta>0$ so that

$$\delta p\big(\|\hat{U}\|_{L^{\infty,\infty}}+\hat{R}\big)^{p-1}\le\frac{3}{8}\tag{4.3}$$

is satisfied, where $M:=\frac{T}{\delta}\in\mathbb{N}$ and $T_m:=m\delta$, $m=0,1,\dots,M$, is a uniform partition of $[0,T]$.

Remark 4.1. Condition (4.3) implies that

$$\delta\|f'(\hat{U})\|_{L^{\infty,\infty}} = \delta p\|\hat{U}\|^{p-1}_{L^{\infty,\infty}} \le \frac{3}{8} \tag{4.4}$$

is satisfied (compare with (3.4)). Condition (4.3) is more restrictive than (3.4). However, recall that (4.3) will be used for only theoretical purposes and that in practice we only use conditions (4.1) and (4.2). As we will see in Remark 4.3, condition (4.1) is less restrictive compared to (3.2).

We recall that in each time interval $[T_{m-1}, T_m]$, the error satisfies the problem

$$\begin{cases} \hat{e}_t - \Delta\hat{e} = f(u) - f(\hat{U}) - \hat{r} & \text{in } \Omega \times [T_{m-1}, T_m], \\ \hat{e} = 0 & \text{on } \partial\Omega \times [T_{m-1}, T_m], \\ \hat{e}(\cdot, T_{m-1}) = \hat{e}(T_{m-1}) & \text{in } \bar{\Omega}, \end{cases} \tag{4.5}$$

with $\hat{e}(0) = 0$. We assume that $\hat{e}(T_{m-1}) \in L^\infty$ and we define the operator

$$\begin{aligned} \Phi(w)(t) = {} & \mathrm{e}^{(t-T_{m-1})\Delta}\hat{e}(T_{m-1}) + \int_{T_{m-1}}^t \mathrm{e}^{(t-s)\Delta}\big(f(\hat{U}+w) - f(\hat{U})\big)(s)\,ds \\ & - \int_{T_{m-1}}^t \mathrm{e}^{(t-s)\Delta}\hat{r}(s)\,ds, \quad t \in [T_{m-1}, T_m]. \end{aligned} \tag{4.6}$$

If $p > 2$, then the operator Φ is equivalently written (for $\nu = 1$) in the form (3.9).

Assume first that $p > 2$. Then

$$f''(x) = p(p-1)|x|^{p-3}x.$$

Accordingly, in view of (3.9), we have

$$\begin{aligned} \Phi(w)(t) = {} & \mathrm{e}^{(t-T_{m-1})\Delta}\hat{e}(T_{m-1}) + \int_{T_{m-1}}^t \mathrm{e}^{(t-s)\Delta} f'\big(\hat{U}(s)\big)w(s)\,ds \\ & + p(p-1)\int_{T_{m-1}}^t \mathrm{e}^{(t-s)\Delta}\left\{\int_0^1 (1-\tau)|\hat{U}+\tau w|^{p-3}(\hat{U}+\tau w)\,d\tau\, w^2\right\}(s)\,ds \\ & - \int_{T_{m-1}}^t \mathrm{e}^{(t-s)\Delta}\hat{r}(s)\,ds. \end{aligned}$$

Let $w \in B(0, \hat{R}; L^{\infty,\infty}_m)$. Then

$$\begin{aligned} \|\Phi(w)\|_{L^{\infty,\infty}} \le {} & \|\hat{e}(T_{m-1})\|_{L^\infty} + \int_{T_{m-1}}^{T_m} \|f'\big(\hat{U}(s)\big)\|_{L^\infty}\,ds\,\|w\|_{L^{\infty,\infty}_m} \\ & + p(p-1)\int_{Tm-1}^{T_m}\int_0^1 (1-\tau)(\|\hat{U}(s)\|_{L^\infty} + \tau\hat{R})^{p-2}\,d\tau\,ds\,\hat{R}\,\|w\|_{L^{\infty,\infty}_m} \\ & + \int_{T_{m-1}}^{T_m} \|\hat{r}(s)\|_{L^\infty}\,ds. \end{aligned} \tag{4.7}$$

Moreover,

$$(\|\hat{U}(s)\|_{L^\infty} + \tau\hat{R})^{p-2} = \sum_{j=0}^{p-2}\binom{p-2}{j}\|\hat{U}(s)\|^{p-2-j}_{L^\infty}\,\tau^j\,\hat{R}^j. \tag{4.8}$$

Thus,

$$
\begin{aligned}
&p(p-1)\int_0^1 (1-\tau)\big(\|\hat{U}(s)\|_{L^\infty}+\tau\|w\|_{L_m^{\infty,\infty}}\big)^{p-2}\,d\tau\\
&\qquad=\sum_{j=0}^{p-2}\frac{p(p-1)}{(j+1)(j+2)}\binom{p-2}{j}\|\hat{U}(s)\|_{L^\infty}^{p-2-j}\hat{R}^j\\
&\qquad=\sum_{j=0}^{p-2}\binom{p}{j+2}\|\hat{U}(s)\|_{L^\infty}^{p-2-j}\,\hat{R}^j=\sum_{j=2}^{p}\binom{p}{j}\|\hat{U}(s)\|_{L^\infty}^{p-j}\,\hat{R}^{j-2}.
\end{aligned}
\tag{4.9}
$$

Combining (4.7)–(4.9) we obtain, for $w\in B(0,\hat{R};L_m^{\infty,\infty})$, that

$$
\begin{aligned}
&\|\Phi(w)\|_{L^{\infty,\infty}}\le\|\hat{e}(T_{m-1})\|_{L^\infty}+\int_{T_{m-1}}^{T_m}\|f'\big(\hat{U}(s)\big)\|_{L^\infty}\,ds\,\|w\|_{L_m^{\infty,\infty}}\\
&\quad+\left(\sum_{j=2}^{p}\binom{p}{j}\int_{T_{m-1}}^{T_m}\|\hat{U}(s)\|_{L^\infty}^{p-j}\,ds\,\hat{R}^{j-2}\right)\hat{R}\,\|w\|_{L_m^{\infty,\infty}}+\int_{T_{m-1}}^{T_m}\|\hat{r}(s)\|_{L^\infty}\,ds.
\end{aligned}
\tag{4.10}
$$

Now let $p=2$. Then (4.6) yields

$$
\begin{aligned}
\|\Phi(w)\|_{L_m^{\infty,\infty}}\le{}&\|\hat{e}(T_{m-1})\|_{L^\infty}+\int_{T_{m-1}}^{T_m}\|\big(|\hat{U}+w|(\hat{U}+w)-|\hat{U}|\hat{U}\big)(s)\|_{L^\infty}\,ds\\
&+\int_{T_{m-1}}^{T_m}\|\hat{r}(s)\|_{L^\infty}\,ds.
\end{aligned}
$$

It is easily seen that $\|\big(|\hat{U}+w|(\hat{U}+w)-|\hat{U}|\hat{U}\big)(s)\|_{L^\infty}\le(2\|\hat{U}(s)\|_{L^\infty}+\|w(s)\|_{L^\infty})\|w(s)\|_{L^\infty}$. Consequently, for $w\in B(0,\hat{R};L_m^{\infty,\infty})$,

$$
\begin{aligned}
\|\Phi(w)\|_{L_m^{\infty,\infty}}\le{}&\|\hat{e}(T_{m-1})\|_{L^\infty}+\int_{T_{m-1}}^{T_m}\|f'(\hat{U})(s)\|_{L^\infty}\,ds\,\|w\|_{L_m^{\infty,\infty}}+\delta\hat{R}\,\|w\|_{L_m^{\infty,\infty}}\\
&+\int_{T_{m-1}}^{T_m}\|\hat{r}(s)\|_{L^\infty}\,ds.
\end{aligned}
$$

Therefore, (4.10) is valid for $p=2$ as well; hence (4.10) holds for integers $p\ge2$.

On the other hand, for $t\in[T_{m-1},T_m]$,

$$
\Phi(w_1)(t)-\Phi(w_2)(t)=\int_{T_{m-1}}^{t}\mathrm{e}^{(t-s)\Delta}\big(f(\hat{U}+w_1)-f(\hat{U}+w_2)\big)(s)\,ds.
$$

Hence,

$$
\|\Phi(w_1)-\Phi(w_2)\|_{L_m^{\infty,\infty}}\le\int_{T_{m-1}}^{T_m}\|\big(f(\hat{U}+w_1)-f(\hat{U}+w_2)\big)(s)\|_{L^\infty}\,ds.
\tag{4.11}
$$

Notice now that

$$
\begin{aligned}
&|f(\hat{U}+w_1)-f(\hat{U}+w_2)|(s)\\
&\quad=\big|\,|\hat{U}+w_1|^{p-1}(\hat{U}+w_1)-|\hat{U}+w_2|^{p-1}(\hat{U}+w_2)\big|(s)\\
&\quad\le\big[|\hat{U}+w_1|^{p-1}|w_1-w_2|+|(\hat{U}+w_1)^{p-1}-(\hat{U}+w_2)^{p-1}|\,|\hat{U}+w_2|\big](s).
\end{aligned}
\tag{4.12}
$$

In addition,

$$(\hat U + w_1)^{p-1} - (\hat U + w_2)^{p-1} = (w_1 - w_2)\sum_{j=0}^{p-2}(\hat U + w_1)^{p-2-j}(\hat U + w_2)^j. \tag{4.13}$$

Let $w_1, w_2 \in B(0, \hat R; L_m^{\infty,\infty})$. Then, relations (4.12)–(4.13) ensure that

$$\begin{aligned} &\|\big(f(\hat U + w_1) - f(\hat U + w_2)\big)(s)\|_{L^\infty} \\ &\quad\le \Big[(\|\hat U(s)\|_{L^\infty} + \hat R)^{p-1} + (p-1)(\|\hat U(s)\|_{L^\infty} + \hat R)^{p-1}\Big]\|w_1 - w_2\|_{L_m^{\infty,\infty}} \\ &\quad= p(\|\hat U(s)\|_{L^\infty} + \hat R)^{p-1}\|w_1 - w_2\|_{L_m^{\infty,\infty}}. \end{aligned} \tag{4.14}$$

Relation (4.14) yields, in view of (4.11), for every $w_1, w_2 \in B(0, \hat R; L_m^{\infty,\infty})$,

$$\|\Phi(w_1) - \Phi(w_2)\|_{L_m^{\infty,\infty}} \le p\delta\big(\|\hat U\|_{L^{\infty,\infty}} + \hat R\big)^{p-1}\|w_1 - w_2\|_{L_m^{\infty,\infty}}. \tag{4.15}$$

Using now an inductive argument (cf. section 3.2) we can prove, for $m = 1, \dots, M$, that the operator $\Phi : L_m^{\infty,\infty} \to L_m^{\infty,\infty}$ is a contraction and the validity of the following local estimate:

$$\max_{T_{m-1}\le t\le T_m}\|\hat e(t)\|_{L^\infty} \le \mathrm{e}^{2\int_0^{T_m}\|f'(\hat U)(s)\|_{L^\infty}\,ds}\,\mathrm{e}^{2\int_0^{T_m}\mathcal{E}_p(s;\hat U,\hat R)\,ds}\int_0^{T_m}\|\hat r(s)\|_{L^\infty}\,ds.$$

Finally, we can derive the following a posteriori estimate for the error in the $L^\infty(L^\infty)$-norm:

$$\max_{0\le t\le T}\|\hat e(t)\|_{L^\infty} \le \mathrm{e}^{2\int_0^{T}\|f'(\hat U)(s)\|_{L^\infty}\,ds}\,\mathrm{e}^{2\int_0^{T}\mathcal{E}_p(s;\hat U,\hat R)\,ds}\int_0^{T}\|\hat r(s)\|_{L^\infty}\,ds, \tag{4.16}$$

where $f(x) = |x|^{p-1}x$, $p \in \mathbb{N}$, $p > 1$.

As we have seen in Remark 3.2, the coefficient 2 in the exponential terms of estimate (4.16) and of condition (4.2), can be improved to $1+\beta$ with $\beta \ge \frac{1}{15}$. So, condition (4.2) and estimate (4.16) can be written as

$$\mathrm{e}^{(1+\beta)p\int_0^T\|\hat U(s)\|_{L^\infty}^{p-1}\,ds}\int_0^T\|\hat r(s)\|_{L^\infty}\,ds \le \frac{3}{16}\hat R \tag{4.17}$$

and

$$\max_{0\le t\le T}\|\hat e(t)\|_{L^\infty} \le \mathrm{e}^{(1+\beta)p\int_0^T\|\hat U(s)\|_{L^\infty}^{p-1}\,ds}\,\mathrm{e}^{(1+\beta)\int_0^T\mathcal{E}(s;\hat U,\hat R)\,ds}\int_0^T\|\hat r(s)\|_{L^\infty}\,ds, \tag{4.18}$$

respectively.

Remark 4.2. Estimate (4.16) indicates that the error $\hat e$ is uniformly bounded in the $L^\infty(L^\infty)$-norm. This is possible, even when the final time T is near the possible discrete blow-up time. In fact, this result is in complete agreement (for $d = 1$) with Theorem 1 in [8] from Fermanian and Zaag.

Proceeding as in section 3.3, we obtain, in view of conditions (4.1) and (4.17),

$$\begin{aligned} \frac12\frac{d}{dt}\|\hat e(t)\|^2 + \|\nabla\hat e(t)\|^2 &\le \|f'(\hat U)(t)\|_{L^\infty}\|\hat e(t)\|^2 \\ &\quad+ p(p-1)\int_0^1(1-\tau)(\|\hat U(t)\|_{L^\infty} + \tau\hat R)^{p-2}\,d\tau\,\hat R\,\|\hat e(t)\|^2 \\ &\quad+ \frac12\|\hat r(t)\|_{-1}^2 + \frac12\|\nabla\hat e(t)\|^2. \end{aligned} \tag{4.19}$$

Invoking (4.9) in (4.19), we arrive at the estimate

$$\frac{1}{2}\frac{d}{dt}\|\hat{e}(t)\|^2 + \frac{1}{2}\|\nabla \hat{e}(t)\|_{L^2}^2 \leq \|f'(\hat{U})(t)\|_{L^\infty}\|\hat{e}(t)\|^2 + \mathcal{E}_p(t;\hat{U},\hat{R})\|\hat{e}(t)\|^2 + \frac{1}{2}\|\hat{r}(t)\|_{-1}^2.$$

Following the same steps as in section 3.3, we finally conclude the estimate

$$\begin{aligned}\|\hat{e}(t)\|^2 &+ \int_0^t \mathrm{e}^{2p\int_s^t \|\hat{U}(\tau)\|_{L^\infty}^{p-1}\,d\tau}\mathrm{e}^{2\int_s^t \mathcal{E}_p(\tau;\hat{U},\hat{R})\,d\tau}\,\|\nabla \hat{e}(s)\|^2\,ds\\ &\leq \int_0^t \mathrm{e}^{2p\int_s^t \|\hat{U}(\tau)\|_{L^\infty}^{p-1}\,d\tau}\mathrm{e}^{2\int_s^t \mathcal{E}_p(\tau;\hat{U},\hat{R})\,d\tau}\,\|\hat{r}(s)\|_{-1}^2\,ds \quad \forall t\in[0,T].\end{aligned}$$

We collect the results of this section so far in the following theorem.

THEOREM 4.1. *Let $\hat{e} = u - \hat{U}$ denote the error. Then, if we choose $0 < \hat{R} \leq 1$ and the time steps k_n, $n = 1,\dots,N$, so that the conditions* (4.1) *and* (4.17) *are satisfied, the following a posteriori estimates are valid for problem* (1.1)*:*

$$\begin{aligned}\max_{0\leq t\leq T}&\left\{\|\hat{e}(t)\|^2 + \int_0^t \mathrm{e}^{2p\int_s^t \|\hat{U}(\tau)\|_{L^\infty}^{p-1}\,d\tau}\mathrm{e}^{2\int_s^t \mathcal{E}_p(\tau;\hat{U},\hat{R})\,d\tau}\,\|\nabla \hat{e}(s)\|^2\,ds\right\}\\ &\leq \int_0^T \mathrm{e}^{2p\int_s^T \|\hat{U}(\tau)\|_{L^\infty}^{p-1}\,d\tau}\mathrm{e}^{2\int_s^T \mathcal{E}_p(\tau;\hat{U},\hat{R})\,d\tau}\,\|\hat{r}(s)\|_{-1}^2\,ds\end{aligned} \tag{4.20}$$

and

$$\max_{0\leq t\leq T}\|\hat{e}(t)\|_{L^\infty} \leq \mathrm{e}^{(1+\beta)p\int_0^T \|\hat{U}(\tau)\|_{L^\infty}^{p-1}\,d\tau}\mathrm{e}^{(1+\beta)\int_0^T \mathcal{E}_p(\tau;\hat{U},\hat{R})\,d\tau}\int_0^T \|\hat{r}(s)\|_{L^\infty}\,ds, \tag{4.21}$$

where β is any real number greater than $\frac{1}{15}$.

Remark 4.3 (comparison of conditions (4.1) and (3.2)). Let

$$p_s(d) = \begin{cases} \frac{d+2}{d-2}, & d\geq 3,\\ \infty, & d = 1,2,\end{cases}$$

and let $p \in (1, p_s)$. We assume that the data of problem (1.1) are such that its solution blows up at some finite time t^*. Then it is known that the blow-up rate for the quantity $\|u(t)\|_{L^\infty}$ is $\frac{1}{(t^*-t)^{\frac{1}{p-1}}}$. In fact, there exists a constant D, depending only on d, p, and Ω, such that [16, 18, 30]

$$\|u(t)\|_{L^\infty} \leq D\frac{1}{(t^*-t)^{\frac{1}{p-1}}}, \quad t\in[0,t^*). \tag{4.22}$$

Our aim is to obtain reasonable estimates that are valid even close to the blow-up time. Next, we will investigate if conditions (4.1) and (3.2) make sense in that case. To this end, we assume that for some $t_d^* > T$

$$\|\hat{U}(t)\|_{L^\infty} \sim \frac{1}{(t_d^*-t)^{\frac{1}{p-1}}}, \quad t\in[0,T]. \tag{4.23}$$

Note that the ideal situation will be $t^* = t_d^*$. If $\hat{U}$ is a good approximation to u, then t^* and t_d^* should be close. We denote by $\tilde{\varepsilon} = |t^* - t_d^*|$ and let $T^* := \min\{t^*, t_d^*\}$. Since $T^* > T$ we set $T := T^* - \varepsilon$. Next, we investigate the form of our conditions in the case where $\tilde{\varepsilon} \sim \varepsilon$ both being small numbers. Similar conclusions can be drown under alternative reasonable scenarios.

As we have mentioned in the introduction, for $p \geq 3$, problem (1.1) is a special case of problem (1.2). We can easily see that the quantity $g(\|\hat{U}\|_{L^{\infty,\infty}}, \hat{R})$, which appears in condition (3.2), has a polynomial dependence on the quantity $\|\hat{U}\|_{L^{\infty,\infty}}$. In particular, for a given p, there exist constants D_j, $j = 0, 1, \ldots, p-2$, depending only on p, such that

$$g(\|\hat{U}\|_{L^{\infty,\infty}}, \hat{R}) = \sum_{j=0}^{p-2} D_j \|\hat{U}\|^j_{L^{\infty,\infty}} \hat{R}^{p-2-j};$$

cf. (1.3). Since $\|\hat{U}(t)\|_{L^\infty} \sim \frac{1}{(t_d^*-t)^{\frac{1}{p-1}}}$ and t_d^* is an approximation to T^*, we have for $0 \leq t \leq T^* - \varepsilon$ that $\|\hat{U}\|^j_{L^{\infty,\infty}} \sim \frac{1}{\varepsilon^{\frac{j}{p-1}}}$ for $j = 0, 1, \ldots, p-2$. Consequently, condition (3.2) in this special case takes the form

$$\left(\sum_{j=0}^{p-2} E_j \frac{\hat{R}^{p-2-j}}{\varepsilon^{\frac{j}{p-1}}} \right) \hat{R} \leq \frac{1}{4(2+C)}, \tag{4.24}$$

where the constants D_j, $j = 0, 1, \ldots, p-2$, are independent of ε and of $\hat{R}$. Condition (4.24) requires restrictions on $\hat{R}$ as ε tends to zero. On the contrary, notice that in condition (4.1) quantities of the form

$$\int_0^{T^*-\varepsilon} \|\hat{U}(s)\|^{p-j}_{L^\infty} \, ds, \quad j = 2, \ldots, p, \tag{4.25}$$

appear. Since for $j = 2, \ldots, p$,

$$\int_0^{T^*-\varepsilon} \frac{1}{(t^*-t)^{\frac{p-j}{p-1}}} \, dt = \frac{p-1}{j-1} \big((t^*)^{\frac{j-1}{p-1}} - \varepsilon^{\frac{j-1}{p-1}} \big), \tag{4.26}$$

condition (4.1) is expected to yield

$$\left[\sum_{j=2}^{p} F_j \big((t^*)^{\frac{j-1}{p-1}} - \varepsilon^{\frac{j-1}{p-1}} \big) \hat{R}^{j-2} \right] \hat{R} \leq \frac{1}{16}, \tag{4.27}$$

where the constants F_j, $j = 2, \ldots, p$, are independent of ε and of $\hat{R}$. Condition (4.27) is clearly less restrictive than condition (4.24) and does not substantially affect the choice of $\hat{R}$ while ε tends to zero, or, equivalently, while we approach the blow-up time.

5. Time steps close to the blow-up time. In this section we provide arguments that support the claim that our results lead to feasible error control near the blow-up time for the cases $p = 2$ or 3 and $d = 2$. We emphasize again that an important verification of this claim will be done through sophisticated adaptive algorithms. This is the subject of a forthcoming work.

Here we follow the notation and assumptions made in Remark 4.3. We assume that for some $t_d^* > T$, (4.23) holds. We emphasize here that (4.23) is not required for our estimates to hold. We use this assumption only to check the form of the constants and whether the conditional assumptions make sense. Similar conclusions can be drown under alternative reasonable hypotheses. If $\hat{U}$ is a good approximation to u, then t^* and t^*_d should be close; let $\tilde{\varepsilon} = |t^* - t^*_d|$, $T^* = \min\{t^*, t^*_d\}$, and $T = T^* - \varepsilon$. Next, we investigate the form of conditions and constants in the case where $\tilde{\varepsilon} \sim \varepsilon$, both being small numbers.

We start with condition (4.1). Then, it suffices to choose $\hat{R} > 0$ such that

$$\hat{R} \leq \frac{1}{16T}, \quad \text{when } p = 2, \tag{5.1}$$

and

$$\left(3 \int_0^{T^*-\varepsilon} \|\hat{U}(s)\|_{L^\infty} \, ds + (T^* - \varepsilon)\hat{R} \right) \hat{R} \leq \frac{1}{16}, \quad \text{when } p = 3. \tag{5.2}$$

Then according to (4.25), (4.26) in Remark 4.3, we expect that

$$\int_0^{T^*-\varepsilon} \|\hat{U}(s)\|_{L^\infty} \, ds \leq D(\sqrt{T^*} - \sqrt{\varepsilon})$$

for some constant D which depends only on the domain Ω. Consequently, both (5.1) and (5.2) impose realistic restrictions on $\hat{R}$.

Regarding (4.17), we first notice that

$$\int_0^{T^*-\varepsilon} \|\hat{U}(t)\|^{p-1}_{L^\infty} \, dt \leq E_1 \ln \frac{T^*}{\varepsilon},$$

where the constant E_1 depends only on p, d, and Ω. Thus,

$$\mathrm{e}^{p \int_0^{T^*-\varepsilon} \|\hat{U}(t)\|^{p-1}_{L^\infty} \, dt} \leq E_2 \frac{1}{\varepsilon^{q_3(p)}}$$

and

$$\mathrm{e}^{(1+\beta) p \int_0^{T^*-\varepsilon} \|\hat{U}(t)\|^{p-1}_{L^\infty} \, dt} \leq E_3 \frac{1}{\varepsilon^{q_4(p,\beta)}},$$

where the constants E_2, E_3 are independent of ε. Therefore, for the condition (4.17) to be satisfied, it suffices to choose the time steps k_n, $n = 1, \ldots, N$, in such a way that

$$E_3 \frac{1}{\varepsilon^{q_4(p,\beta)}} \int_0^{T^*-\varepsilon} \|\hat{r}(s)\|_{L^\infty} \, ds \leq \frac{3}{16} \hat{R}. \tag{5.3}$$

Let us now consider a fixed ε. Given the rate of convergence of backward Euler and Crank–Nicolson methods, their residual $\hat{r}$ will tend to zero as the size of the time steps decrease. Therefore, with a suitable choice of the time steps, (5.3) will be satisfied. At this point we emphasize the fact that due to the presence of $\frac{1}{\varepsilon^{q_4(p,\beta)}}$ and not of an exponential on $\frac{1}{\varepsilon}$, we anticipate that (5.3) will be satisfied under reasonable choices on the sizes of the time steps. Finally, we observe that as ε becomes smaller, we have to choose smaller time steps. This is something we expect, because as we approach the blow-up time, the choice of extremely small time steps is a necessity; see [36] and [2].

It is important here to notice that the blow-up rate of the solution is such that the term $\mathrm{e}^{p\int_0^{T^*-\varepsilon}\|\hat{U}(\tau)\|_{L^\infty}^{p-1}\,d\tau}$ tends to infinity with polynomial, instead of exponential rate, while ε tends to zero (this is true for every p; not only for $p=2$ or 3). As a consequence, the conditions required in the present analysis are, with a suitable choice of the time steps, realistic. Moreover, choosing again the time steps appropriately, our results provide upper bounds with reasonable constants for the $L^\infty(L^2)$-norm and the $L^\infty(L^\infty)$-norm, respectively. As we have mentioned in the introduction, this is not feasible via standard a priori error analysis, since in the upper bounds of the a priori error estimates a term of the form $\mathrm{e}^{\frac{1}{\varepsilon}}$ appears, i.e., the upper bound tends exponentially to infinity while ε tends to zero. However, via the analysis of this paper, the goal of error control near the blow-up time seems a feasible task.

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