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Transport equations for the inflationary trispectrum

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Abstract. We use transport techniques to calculate the trispectrum produced in multiple-field inflationary models with canonical kinetic terms. Our method allows the time evolution of the local trispectrum parameters, τ_{NL} and g_{NL} , to be tracked throughout the inflationary phase. We illustrate our approach using examples. We give a simplified method to calculate the superhorizon part of the relation between field fluctuations on spatially flat hypersurfaces and the curvature perturbation on uniform density slices, ζ , and obtain its third-order part for the first time. We clarify how the ‘backwards’ formalism of Yokoyama et al. relates to our analysis and other recent work. We supply explicit formulae which enable each inflationary observable to be computed in any canonical model of interest, using a suitable first-order ODE solver.

1. Introduction

Cosmological inflation predicts the generation of a primordial perturbation, ζ , believed to have seeded the temperature anisotropy of the cosmic microwave background (“CMB”) and the galaxy density field. This fluctuation is sensitive to the physics that created it, and therefore different models of inflation typically generate perturbations with distinct statistical properties. These properties can be observed by measuring their correlation functions. We expect this approach to provide the most important observational constraints on an era of early-universe inflation.

What information is encoded in these correlation functions? The two-point function is nearly determined by the symmetries of the background, rather than the choice of microphysics, although useful information may be extracted from its scale dependence. The higher n -point functions are much less constrained, but only the three- and four-point functions (the “bispectrum” and “trispectrum”) are likely to be measured in the near future. Canonical single field inflation predicts a bi- and trispectrum which will be undetectable by present-day or near-future experiments [1–6]. But if more than one field is active during inflation, or noncanonical interactions are present, the three- and four-point functions can be measured and their properties can discriminate between these possibilities.

Because of their observational relevance and constraining power, these “nongaussian” effects have received considerable attention. During inflation, each comoving k -mode of a light scalar field receives a perturbation when the corresponding physical scale crosses outside the horizon. Once outside, causality forbids any exchange between neighbouring regions and therefore ζ must be generated by reprocessing the local fluctuations. Where only a single degree of freedom ζ_g is relevant, this gives [2, 7, 8]

$$\zeta(\mathbf{x}) = \zeta_g(\mathbf{x}) + \frac{3}{5}f_{\text{NL}}(\zeta_g^2(\mathbf{x}) - \langle \zeta_g^2 \rangle) + \frac{9}{25}g_{\text{NL}}\zeta_g^3(\mathbf{x}) + \dots, \quad (1)$$

where all quantities are evaluated at the same time, and \mathbf{x} labels a coarse-grained spatial position with sub-horizon details smoothed out. This local character gives each correlation function a very distinctive momentum dependence. At leading-order the bispectrum has only one possibility, generated by the quadratic term in (1). Its amplitude is parametrized by the number f_{NL} [9, 10], which may depend weakly on the smoothing scale. But the trispectrum has two possibilities, generated respectively by the cubic term and the square of the quadratic term. These are conventionally parametrized by the numbers τ_{NL} and g_{NL} [5, 11–14]. In the single-field case, τ_{NL} does not appear in (1) and can be expressed in terms of f_{NL} ; the precise relation is $\tau_{\text{NL}} = (6f_{\text{NL}}/5)^2$. Where more than one light degree of freedom is present, they may all appear in Eq. (1) and this relation is weakened to the Suyama–Yamaguchi inequality $\tau_{\text{NL}} \geq (6f_{\text{NL}}/5)^2$ [15, 16]. The role of such relations in diagnosing the active particle spectrum during inflation was recently emphasized by Assassi et al. [17].

Transport methods.—In this paper we explain how the non-linearity parameters τ_{NL} and g_{NL} can be calculated using “transport” methods.

Such calculations can already be carried out within the “ δN formalism” [2, 7, 8], which requires a Taylor expansion of the background solution in small displacements from a chosen initial condition. An expression for τ_{NL} was given in this formalism by Alabidi & Lyth [13]. A comparable result for g_{NL} was provided by Sasaki, Väiviita & Wands [11] in the context of a curvaton model, and later generalized to an arbitrary number of light fields in Ref. [5]. The “ δN ” Taylor expansion leads to concise and attractive analytic results. But it is not ideally suited to numerical implementation, because it relies on extracting small variations which can easily be swamped by numerical noise.

In Ref. [18] it was explained that the Taylor expansion can be understood as a variational method to compute Jacobi fields for the flow of inflationary trajectories in phase space. These fields can be used to explore local properties of any flow, and were introduced by Jacobi in his reformulation of Hamiltonian mechanics into what is now Hamilton–Jacobi theory. In inflation, they represent the geometrical structure which underlies perturbation theory in the long wavelength limit. They recur in many areas of physics (see, e.g., Refs. [19, 20]), and have been much-studied in WKB approximations to the path integral [21–23].

The Jacobi fields are the necessary ingredient to compute τ_{NL} and g_{NL} , but it is not necessary to use variational techniques to compute them. Their evolution can be determined equally well using an ordinary differential equation—the ‘Jacobi equation’ [24]. The equivalence was emphasized by DeWitt–Morette [21]. The Jacobi equation is usually preferable for numerical implementation. It can be solved using conventional ODE techniques and is usually much more stable against numerical noise. Jacobi methods are widely used in other applications, including gravitational lensing [25, 26].

With this motivation, one can ask whether it is possible to replace the “ δN ” Taylor expansion with an approach based on the Jacobi equation. To do so, one gives an evolution equation for each n -point function. Such equations were introduced in Refs. [27, 28] and were originally framed in real space.‡ Real-space methods are adequate if one wishes to extract only the local part of the three-point function. But if one wishes to include more general momentum-dependence or study n -point functions for $n \geq 4$, where it is necessary to distinguish between “squeezed” and “collapsed” configurations, one must revert to Fourier space. In Ref. [18] it was explained how to formulate evolution equations for the full \mathbf{k} -space correlation functions, which can be integrated using an approach similar to the “line of sight integral” used to simplify solution of the Boltzmann equation. In Ref. [18] this was used to give formal but explicit expressions for the n -point functions in terms of the Jacobi fields and their derivatives, and hence to demonstrate equivalence with the variational “ δN formalism” up to the three-point function.

‡ A similar formalism had been introduced earlier by Yokoyama, Suyama & Tanaka [29, 30], who gave evolution equations for the Taylor coefficients of the δN formalism rather than the n -point functions directly. It was shown in Ref. [18] that these formalisms are equivalent up to the 3-point function. In §§3–5 we extend this equivalence to the 4-point function.

In this paper we specialize this method to the trispectrum. We write a transport equation for the four-point function of field fluctuations $\delta\varphi_\alpha$ defined on spatially flat slices. As in Ref. [18], this can be integrated in terms of Jacobi fields and reproduces the variational formulae discussed above. In a second step, we express the correlation functions of ζ in terms of those of the $\delta\varphi_\alpha$. At this point the required values of τ_{NL} and g_{NL} can be extracted. However, our method is not limited to obtention of the ζ correlation functions and can be deployed to determine the correlation functions of both ζ and any isocurvature modes.

Outline.—In §2 we introduce the transport framework and extend it to third order. In §2.2 we write down the full \mathbf{k} -dependent equation which evolves the four-point function on superhorizon scales. By studying the momentum-dependence of this equation, we can extract (in §2.3) the coefficients of the “squeezed” and “collapsed” configurations. We give separate evolution equations for these.

In §3 we demonstrate that the transport (Jacobi) method is equivalent to the familiar Taylor expansion of the separate universe formalism. We use our evolution equation to derive ordinary differential equations which evolve the separate-universe Taylor coefficients *forward* in time, and which supply the basis of an efficient numerical implementation. In §4 we finish the task of extracting τ_{NL} and g_{NL} by computing the relationship between ζ and the field fluctuations $\delta\varphi_\alpha$. Our final expressions are given in §4.2. We supply explicit expressions which enable each inflationary observable to be computed in any canonical model of interest, using a suitable first-order ODE solver.

In §5 we describe the alternative *backwards* transport method introduced by Yokoyama et al., and extend it to accommodate the trispectrum parameters. We briefly comment on the relative advantages of each formulation. In §6 we discuss some representative numerical results. Finally, we conclude with a short discussion in §7.

Notation and conventions.—We set $c = \hbar = 1$ and work in terms of the reduced Planck mass, $M_{\text{P}}^2 = (8\pi G)^{-1}$ where G is Newton’s constant. The species of light scalar fields are indexed by Greek labels α, β, \dots .

2. Transport Equations

After smoothing on a length scale $k^{-1} \gg (aH)^{-1}$, the field value in each smoothed region of the universe (“patch”) will evolve independently, as though it were in a homogeneous and isotropic *separate universe*. Making use of the slow-roll approximation, and assuming that all fields are canonically normalized and minimally coupled to Einstein gravity, each smoothed field φ_α evolves according to [18, 27, 28]

$$\frac{d\varphi_\alpha}{dN} = -M_{\text{P}}^2 \frac{\partial \ln V(\varphi)}{\partial \varphi_\alpha} \equiv u_\alpha, \quad (2)$$

up to gradient-suppressed corrections. In writing (2) we have used the e-folding number $dN = H dt$ as a time variable, and t is cosmic time. The index α labels the species of light scalar fields and u_α can be interpreted as a flow vector describing the trajectory

of the smoothed field in phase space. In this paper we will take these indices to be contracted using the flat metric $\delta_{\alpha\beta}$, so that index placement is immaterial.

If desired the slow-roll approximation could be abandoned by passing to a Hamiltonian formulation. The resulting transport equations are structurally identical, requiring only specification of suitable initial conditions. This method was described in Refs. [18, 28] and later implemented by Dias, Frazer & Liddle [31] for the purpose of studying D-brane models of inflation. In this paper we will restrict ourselves to the slow-roll approximation, but our evolution equations are unchanged by this choice and can be extended immediately to the full phase space.

2.1. Jacobi equation

The field value varies between coarse-grained patches. Picking a fiducial patch labelled by the spatial position \mathbf{x} , the field in a neighbouring patch at relative position \mathbf{r} will be displaced by a small amount $\delta\varphi_\alpha$,

$$\varphi_\alpha(\mathbf{x} + \mathbf{r}) \approx \varphi_\alpha(\mathbf{x}) + \delta\varphi_\alpha(\mathbf{r}). \quad (3)$$

At a generic position, and provided the region under consideration is not too large, we can expect $|\delta\varphi_\alpha|$ to be small in comparison with $|\varphi_\alpha|$. With these assumptions the evolution of $\delta\varphi_\alpha$ can be obtained by making a Taylor expansion of the velocity u_α in the neighbourhood of the fiducial trajectory. Hence,

$$\begin{aligned} \frac{d\delta\varphi_\alpha(\mathbf{r})}{dN} &= u_{\alpha\beta}[\varphi(\mathbf{x})]\delta\varphi_\beta(\mathbf{r}) + \frac{1}{2!}u_{\alpha\beta\gamma}[\varphi(\mathbf{x})]\delta\varphi_\beta(\mathbf{r})\delta\varphi_\gamma(\mathbf{r}) \\ &+ \frac{1}{3!}u_{\alpha\beta\gamma\delta}[\varphi(\mathbf{x})]\delta\varphi_\beta(\mathbf{r})\delta\varphi_\gamma(\mathbf{r})\delta\varphi_\delta(\mathbf{r}) + \dots \end{aligned} \quad (4)$$

We now exchange \mathbf{r} for a Fourier space description. To keep the resulting equations compact we employ the ‘primed’ DeWitt index convention introduced in Ref. [18]. In this notation, a compound index such as α' includes a field label α and a momentum label \mathbf{k}_α , and also indicates evaluation at some common time of interest N . The summation convention applied to α' implies integration over momentum with measure $d^3\mathbf{k}_\alpha/(2\pi)^3$, and summation over the species α . In this notation we find

$$\begin{aligned} \frac{d\delta\varphi_{\alpha'}}{dN} &= u_{\alpha'\beta'}\delta\varphi_{\beta'} + \frac{1}{2!}u_{\alpha'\beta'\gamma'}\left(\delta\varphi_{\beta'}\delta\varphi_{\gamma'} - \langle\delta\varphi_{\beta'}\delta\varphi_{\gamma'}\rangle\right) \\ &+ \frac{1}{3!}u_{\alpha'\beta'\gamma'\delta'}\left(\delta\varphi_{\beta'}\delta\varphi_{\gamma'}\delta\varphi_{\delta'} - \langle\delta\varphi_{\beta'}\delta\varphi_{\gamma'}\delta\varphi_{\delta'}\rangle\right) + \dots \end{aligned} \quad (5)$$

Eq. (5) is the nonlinear Jacobi equation. We have subtracted a zero-mode, which amounts to discarding disconnected terms in the correlation functions. The u -matrices contained in (5) inherit a dependence on the fiducial region \mathbf{x} through their dependence on the background fields, but the resulting connected correlation functions depend only

on statistical properties of the ensemble of smoothed fields. Explicitly, we find

$$u_{\alpha'\beta'} \equiv (2\pi)^3 \delta(\mathbf{k}_\alpha - \mathbf{k}_\beta) u_{\alpha\beta}[\varphi(\mathbf{x})] \quad (6)$$

$$u_{\alpha'\beta'\gamma'} \equiv (2\pi)^3 \delta(\mathbf{k}_\alpha - \mathbf{k}_\beta - \mathbf{k}_\gamma) u_{\alpha\beta\gamma}[\varphi(\mathbf{x})] \quad (7)$$

$$u_{\alpha'\beta'\gamma'\delta'} \equiv (2\pi)^3 \delta(\mathbf{k}_\alpha - \mathbf{k}_\beta - \mathbf{k}_\gamma - \mathbf{k}_\delta) u_{\alpha\beta\gamma\delta}[\varphi(\mathbf{x})]. \quad (8)$$

2.2. Evolution of correlation functions

The Jacobi equation (5) summarizes evolution in the ensemble of smoothed patches. The u -matrices can be calculated using any suitable method, such as the long-wavelength limit of cosmological perturbation theory or the separate-universe approximation. However they are obtained, they control not only the evolution of physical field fluctuations but also their correlation functions.

To show this we note that for any classical observable O not explicitly depending on time, the time derivative of its expectation value satisfies $d\langle O \rangle/dN = \langle dO/dN \rangle$, provided probability is conserved.§ It also applies quantum-mechanically if O is a Heisenberg picture field. Transport equations for the quantum case, similar to those we will develop here, were given by Andrews & Hall [32] and developed by Ballentine & McRae [33]. The classical limit was studied by Hepp [34].

We define the two-point function $\Sigma_{\alpha'\beta'}$ to satisfy

$$\Sigma_{\alpha'\beta'} \equiv \langle \delta\varphi_{\alpha'} \delta\varphi_{\beta'} \rangle. \quad (9)$$

Recall that our index convention implies that each quantity on the right-hand side is evaluated at the common time of interest, N . Differentiating this expression, and moving the time derivative inside the expectation value as discussed above, we obtain an evolution equation for $\Sigma_{\alpha'\beta'}$,

$$\frac{d\Sigma_{\alpha'\beta'}}{dN} = \left\langle \frac{d\delta\varphi_{\alpha'}}{dN} \delta\varphi_{\beta'} + \delta\varphi_{\alpha'} \frac{d\delta\varphi_{\beta'}}{dN} \right\rangle. \quad (10)$$

Use of Eq. (5) allows the right-hand side to be rewritten in terms of u -matrices and correlation functions. Working to the lowest relevant order,|| we conclude

$$\frac{d\Sigma_{\alpha'\beta'}}{dN} = u_{\alpha'\gamma'} \Sigma_{\gamma'\beta'} + u_{\beta'\gamma'} \Sigma_{\gamma'\alpha'} + [\geq 3\text{p.f.}], \quad (11)$$

where “[$\geq 3\text{p.f.}$]” denotes terms containing higher-order correlation functions which have been omitted, beginning with the three-point function. Eq. (11) will be a good approximation whenever these higher-order correlation functions are negligible, which

§ Technically, the probability distribution P must vanish sufficiently rapidly on the boundary of phase space that $u_\alpha P \rightarrow 0$ there, and therefore integration by parts inside the expectation value does not generate any boundary terms.

|| Retaining higher-order contributions would reproduce the ‘loop corrections’ of the δN formalism; see Refs. [12, 35, 36].

will usually be satisfied during an epoch of quasi-exponential inflation. In that case, the correlation functions typically order their amplitudes in powers of H^2 [37] making the relative error after translation to ζ of order $(H/M_{\text{P}})^2 \ll 1$. A similar procedure gives the evolution of the three-point function. We define

$$\alpha_{\alpha'\beta'\gamma'} \equiv \langle \delta\varphi_{\alpha'}\delta\varphi_{\beta'}\delta\varphi_{\gamma'} \rangle, \quad (12)$$

and the corresponding transport equation is

$$\frac{d\alpha_{\alpha'\beta'\gamma'}}{dN} = u_{\alpha'\lambda'}\alpha_{\lambda'\beta'\gamma'} + u_{\alpha'\lambda'\mu'}\Sigma_{\lambda'\beta'}\Sigma_{\mu'\gamma'} + \text{cyclic} + [\geq 4\text{p.f.}], \quad (13)$$

where ‘‘cyclic’’ denotes the two cyclic permutations of each term, and ‘‘ $[\geq 4\text{p.f.}]$ ’’ again denotes terms involving higher-order correlation functions which have been discarded, beginning with the four-point function. As for the two-point function, Eq. (13) will be a good approximation whenever these are negligible in comparison with the terms which have been retained.

Four-point function.—Eqs. (11) and (13) were given in Ref. [18]. In this section, for the first time, we give the corresponding transport equation for the four-point function. To do so, we must distinguish carefully between the connected and disconnected contributions. The disconnected contributions are always present, even in the case of purely Gaussian statistics, and therefore provide no new information. But if the perturbations develop some *intrinsic* nongaussianity during their evolution, this is encoded in the *connected* part of the four-point function. To obtain it we subtract the disconnected terms from the full four-point function, and define

$$\beta_{\alpha'\beta'\gamma'\delta'} \equiv \langle \delta\varphi_{\alpha'}\delta\varphi_{\beta'}\delta\varphi_{\gamma'}\delta\varphi_{\delta'} \rangle - \Sigma_{\alpha'\beta'}\Sigma_{\gamma'\delta'} - \Sigma_{\alpha'\gamma'}\Sigma_{\beta'\delta'} - \Sigma_{\alpha'\delta'}\Sigma_{\beta'\gamma'}. \quad (14)$$

In statistical language, the four-point function $\langle \delta\varphi_{\alpha'}\delta\varphi_{\beta'}\delta\varphi_{\gamma'}\delta\varphi_{\delta'} \rangle$ is the moment, and the connected part $\beta_{\alpha'\beta'\gamma'\delta'}$ is the cumulant.

The transport equation for $\beta_{\alpha'\beta'\gamma'\delta'}$ is

$$\begin{aligned} \frac{d\beta_{\alpha'\beta'\gamma'\delta'}}{dN} = & \left(u_{\alpha'\lambda'}\beta_{\lambda'\beta'\gamma'\delta'} + 3 \text{ cyclic} \right) + \left(u_{\alpha'\lambda'\mu'}\alpha_{\lambda'\beta'\gamma'}\Sigma_{\mu'\delta'} + 11 \text{ cyclic} \right) \\ & + \left(u_{\alpha'\lambda'\mu'\nu'}\Sigma_{\lambda'\beta'}\Sigma_{\mu'\gamma'}\Sigma_{\nu'\delta'} + 3 \text{ cyclic} \right) + [\geq 5\text{p.f.}]. \end{aligned} \quad (15)$$

It can be obtained by various methods, including the Gauss–Hermite cumulant expansion used in Ref. [27], the method of generating functions used in Ref. [28] and the approach described above.

2.3. Separation of local shapes

The transport equations (11), (13) and (15) evolve each correlation function in its entirety. Although they are first order ordinary differential equations, they are not trivial to solve because they couple the correlation functions associated with different

\mathbf{k} - and species labels.¶ Indeed, the coupled system can be regarded as simply a form of Boltzmann hierarchy. Like the hierarchy used to compute CMB anisotropies it must be truncated—by discarding higher-order correlation functions—if it is to be turned into a practical computational tool. We will see in §3 that it admits a similar kind of formal solution. But if we wish only to track the evolution of the local momentum shapes, then we can extract simpler “flavour” equations which do not involve the continuum of \mathbf{k} -modes. These are ordinary differential equations for a finite number of variables and their numerical solution is straightforward.

Eqs. (11), (13) and (15) show that (at least to this order), each correlation function is sourced by the correlation functions of *lower* order. Hence, we proceed inductively: if the \mathbf{k} -dependence of the two-point function is known, then it can be used to determine the local \mathbf{k} -dependence inherited by the three-point function and subsequently the four-point function.

Two-point function.—Since we anticipate approximate scale-invariance, we write the two-point function as

$$\Sigma_{\alpha'\beta'} = (2\pi)^3 \delta(\mathbf{k}_\alpha + \mathbf{k}_\beta) \frac{\Sigma_{\alpha\beta}}{k_\alpha^3}, \quad (16)$$

where $\Sigma_{\alpha\beta}$ has dimension of [mass]² but is nearly independent of $k_\alpha = k_\beta$. It is this k_α^{-3} dependence which will be inherited by all higher n -point functions. The possible ways in which this inheritance can happen correspond to the possible local (“squeezed” and “collapsed”) momentum shapes.

We first require a transport equation for $\Sigma_{\alpha\beta}$. As described above, this is a flavour-only matrix, carrying indices for the species of scalar fields but not momentum labels. Substituting (16) into (11), we conclude

$$\frac{d\Sigma_{\alpha\beta}}{dN} = u_{\alpha\lambda} \Sigma_{\lambda\beta} + u_{\beta\lambda} \Sigma_{\lambda\alpha}. \quad (17)$$

This is symbolically the same equation as the full \mathbf{k} -space transport equation, Eq. (11), with primed indices exchanged for unprimed ones.

In practice, $\Sigma_{\alpha\beta}$ carries a small dependence on the k -scale at which it is evaluated. This k -dependence, typically characterized by a spectral index, can also be calculated by transport methods; see Dias et al. [39]. Recently Dias, Frazer & Liddle extended this method to obtain the scale-dependence of the spectral index, or “running” [31].

Three-point function.—Examination of the transport equation for the three-point function, Eq. (13), shows that in a small time interval δN , the change to $\alpha_{\alpha'\beta'\gamma'}$ is of the schematic form $\delta\alpha \sim (u'\alpha + u''\Sigma\Sigma)\delta N$, where a prime $'$ applied to u indicates one of the field-space derivatives which generate the index structure for the u -matrices. The $u'\alpha$

¶ For example, the four-point function with momentum labels $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$ and \mathbf{k}_4 couples to other correlation functions with momenta $\mathbf{k}_1 + \mathbf{k}_2$, and so on. Had we retained loop corrections, these would make the hierarchy considerably more complex because each correlation function no longer couples only to a few other isolated \mathbf{k} -modes, but to the whole phase space of soft superhorizon modes. Handling this is a computational challenge. For one approach see, eg., Ref. [38].

terms generate a change $\delta\alpha$ which is proportional to the momentum-dependence already carried by α . Therefore this term can reorganize the amplitudes of these shapes, but introduces no new types of momentum dependence. New shapes are sourced only by the $\Sigma\Sigma$ terms.

Eq. (16) shows that the product $\Sigma\Sigma$ must generate a shape of the form $k_\alpha^{-3}k_\beta^{-3}$, and therefore the most general structure which can be *sourced* during the evolution has the form

$$\alpha_{\alpha'\beta'\gamma'} \supseteq (2\pi)^3 \delta(\mathbf{k}_\alpha + \mathbf{k}_\beta + \mathbf{k}_\gamma) \left(\frac{a_{\alpha|\beta\gamma}}{k_\beta^3 k_\gamma^3} + \frac{a_{\beta|\alpha\gamma}}{k_\alpha^3 k_\gamma^3} + \frac{a_{\gamma|\alpha\beta}}{k_\alpha^3 k_\beta^3} \right), \quad (18)$$

where we use the notation “ \supseteq ” to indicate that the three-point function contains this term together with others which have not been written. The matrices $a_{\alpha|\beta\gamma}$ are symmetric under exchange of $\beta \leftrightarrow \gamma$, but need not possess further symmetries. The full three-point function corresponds to the sourced contribution (18) plus an *unsourced* term appearing as its initial condition. The unsourced piece is generated by quantum interference effects operating around the epoch of horizon exit, and typically has a very complicated momentum dependence [3]. However, its amplitude is small in the canonical models to which we restrict attention in this paper [40, 41].

After substitution of (18) into the transport equation (13), we obtain an evolution equation for $a_{\alpha|\beta\gamma}$,

$$\frac{da_{\alpha|\beta\gamma}}{dN} = u_{\alpha\lambda} a_{\lambda|\beta\gamma} + u_{\beta\lambda} a_{\alpha|\lambda\gamma} + u_{\gamma\lambda} a_{\alpha|\beta\lambda} + u_{\alpha\lambda\mu} \Sigma_{\lambda\beta} \Sigma_{\mu\gamma}. \quad (19)$$

Eq. (19) strictly applies only when the momenta entering the correlation function are not too dissimilar in magnitude. This is usually an acceptable approximation for CMB experiments, but a more refined analysis might be required where larger hierarchies of scale exist. This issue is not confined to the transport framework; it applies to results obtained using any method, including the familiar δN Taylor expansion.

Four-point function.—Eqs. (17) and (19) were given in Ref. [18]. The same analysis applied to the four-point function shows that, in a small time interval δN , the change in the connected part of the correlation function has the schematic form

$$\delta\beta \sim (u'\beta + u''\alpha\Sigma + u'''\Sigma\Sigma\Sigma)\delta N. \quad (20)$$

As for the three-point function, the term $u'\beta$ is simply a shift in the amplitude of shapes already present in β . The sourced contributions are now $u''\alpha\Sigma$ and $u'''\Sigma\Sigma\Sigma$. Of these, the $\Sigma\Sigma\Sigma$ term must generate a shape of the form $k_\alpha^{-3}k_\beta^{-3}k_\gamma^{-3}$, which can be recognized as a g_{NL} -type contribution [14].

The $\alpha\Sigma$ term is more complex, because the momentum δ -function in u'' [see Eq. (7)] reorganizes the momenta appearing in the denominators of the three-point function (18). Written out explicitly, this term is

$$u_{\alpha'\lambda'\mu'} \alpha_{\lambda'\beta'\gamma'} \Sigma_{\mu'\delta'} = (2\pi)^3 \int d^3k_\lambda d^3k_\mu \delta(\mathbf{k}_\alpha - \mathbf{k}_\lambda - \mathbf{k}_\mu) \delta(\mathbf{k}_\lambda + \mathbf{k}_\beta + \mathbf{k}_\gamma) \delta(\mathbf{k}_\mu + \mathbf{k}_\delta) \\ \times u_{\alpha\lambda\mu} \frac{\Sigma_{\mu\delta}}{k_\delta^3} \left(\frac{a_{\lambda|\beta\gamma}}{k_\beta^3 k_\gamma^3} + \frac{a_{\beta|\lambda\gamma}}{k_\lambda^3 k_\gamma^3} + \frac{a_{\gamma|\beta\lambda}}{k_\beta^3 k_\lambda^3} \right), \quad (21)$$

plus the nontrivial permutations of α' , β' , γ' and δ' . The first term in round brackets, $\sim k_\beta^{-3} k_\gamma^{-3}$, has the form of a g_{NL} -type contribution. But the remaining terms involve k_λ^{-3} , and the δ -functions in (21) show that $\mathbf{k}_\lambda = \mathbf{k}_\alpha + \mathbf{k}_\delta$. Therefore this term generates a *different* momentum shape; it is the ‘‘collapsed’’ configuration, which corresponds to a τ_{NL} -type contribution [14]. It follows that the most general structure sourced by time evolution can be written

$$\beta_{\alpha'\beta'\gamma'\delta'} = (2\pi)^3 \delta(\mathbf{k}_\alpha + \mathbf{k}_\beta + \mathbf{k}_\gamma + \mathbf{k}_\delta) \left(\frac{g_{\alpha|\beta\gamma\delta}}{k_\beta^3 k_\gamma^3 k_\delta^3} + 3 \text{ cyclic} + \frac{\tau_{\alpha\beta|\gamma\delta}}{k_\alpha^3 k_\beta^3 |\mathbf{k}_\alpha + \mathbf{k}_\gamma|^3} + 11 \text{ cyclic} \right), \quad (22)$$

where the ‘‘cyclic’’ pieces refer to the cyclic permutations of the preceding terms. The matrix $g_{\alpha|\beta\gamma\delta}$ is symmetric under any exchange of β, γ, δ , but has no symmetries under permutations involving α . The matrix $\tau_{\alpha\beta|\gamma\delta}$ is symmetric under the simultaneous exchanges $\alpha \leftrightarrow \beta$ and $\gamma \leftrightarrow \delta$, giving 12 independent elements.

Substitution of (22) into the transport equation (15) enables us to extract individual evolution equations for $g_{\alpha|\beta\gamma\delta}$ and $\tau_{\alpha\beta|\gamma\delta}$. They are

$$\begin{aligned} \frac{dg_{\alpha|\beta\gamma\delta}}{dN} &= u_{\alpha\lambda} g_{\lambda|\beta\gamma\delta} + u_{\beta\lambda} g_{\alpha|\lambda\gamma\delta} + u_{\gamma\lambda} g_{\alpha|\beta\lambda\delta} + u_{\delta\lambda} g_{\alpha|\beta\gamma\lambda} \\ &\quad + u_{\alpha\lambda\mu} a_{\lambda|\beta\gamma} \Sigma_{\mu\delta} + u_{\alpha\lambda\mu} a_{\lambda|\beta\delta} \Sigma_{\mu\gamma} + u_{\alpha\lambda\mu} a_{\lambda|\gamma\delta} \Sigma_{\mu\beta} + u_{\alpha\lambda\mu\nu} \Sigma_{\lambda\beta} \Sigma_{\mu\gamma} \Sigma_{\nu\delta} \end{aligned} \quad (23)$$

$$\begin{aligned} \frac{d\tau_{\alpha\beta|\gamma\delta}}{dN} &= u_{\alpha\lambda} \tau_{\lambda\beta|\gamma\delta} + u_{\beta\lambda} \tau_{\alpha\lambda|\gamma\delta} + u_{\gamma\lambda} \tau_{\alpha\beta|\lambda\delta} + u_{\delta\lambda} \tau_{\alpha\beta|\gamma\lambda} \\ &\quad + u_{\gamma\lambda\mu} \Sigma_{\mu\alpha} a_{\delta|\lambda\beta} + u_{\delta\lambda\mu} \Sigma_{\mu\beta} a_{\gamma|\lambda\alpha}. \end{aligned} \quad (24)$$

Note that the a -dependent source terms in the second line of (24) preserve the symmetry under simultaneous exchange of the index pairs (α, β) and (γ, δ) . We have dropped the initial value of $\alpha_{\alpha'\beta'\gamma'}$, even though it appears in (15) as a source term and, as a matter of principle, could appear in $\beta_{\alpha'\beta'\gamma'\delta'}$ with a non-negligible coefficient. In Appendix A we show that this will usually be an acceptable approximation in models with canonically normalized scalar fields; the initial value of $\alpha_{\alpha'\beta'\gamma'}$ remains negligible provided $|rf_{\text{NL}}| \lesssim 1$ throughout the evolution where $r < 1$ is the tensor-to-scalar ratio. On the other hand, in non-canonical models where the initial value need *not* be negligible it is important to retain this term [42].

3. Equivalence to Taylor expansion method

Eqs. (23)–(24) enable us to follow the evolution of the sourced, local-mode contributions to the trispectrum. As we will explain in §4, after changing variable to ζ they allow us to calculate the observable quantities τ_{NL} and g_{NL} . However, they are quite different in appearance to the familiar expressions of the ‘‘ δN formalism’’,⁺ which take the form of a Taylor expansion in the initial conditions [2].

⁺ Here and below, we use the term ‘‘ δN formalism’’ to mean a Taylor expansion in the initial conditions, even if the quantity being expanded is not N .

The connexion between these methods was explored in Ref. [18]. By formally integrating the transport equations, in a similar way to the “line of sight” integral used when solving the Boltzmann equation, it is possible to demonstrate equality with the “ δN ” expressions. In Ref. [18] this analysis was given for the two- and three-point functions. Here we extend it to include the four-point function.

Integrating factor.—The “line of sight integral” naturally expresses each correlation function in terms of the underlying Jacobi fields. We briefly recapitulate the argument of Ref. [18]. Without loss of generality, we write the two-point function in the form

$$\Sigma_{\alpha'\beta'} = \Gamma_{\alpha'i'}\Gamma_{\beta'j'}\Sigma_{i'j'}. \quad (25)$$

A suitable choice for $\Gamma_{\alpha'i'}$ means it will function as an integrating factor. In writing (25) we have introduced a new type of primed Latin index (i', j', \dots). This has the same interpretation as the primed Greek indices: i' carries a flavour index i and a momentum label \mathbf{k}_i , which range over the same values as α and \mathbf{k}_α . However, it indicates evaluation at a different time N_0 , as follows. Substitution of Eq. (25) in (11) shows that the terms involving $u_{\alpha'\beta'}$ can be removed if Γ is chosen to satisfy

$$\frac{d\Gamma_{\alpha'i'}}{dN} = u_{\alpha'\beta'}\Gamma_{\beta'i'}. \quad (26)$$

Comparison with Eqs (4)–(5) shows that $\Gamma_{\alpha'i'}$ has an interpretation as a differential coefficient,

$$\Gamma_{\alpha'i'} = \frac{\partial\varphi_{\alpha'}}{\partial\varphi_{i'}} = \frac{\partial\varphi_{\alpha}(\mathbf{k}_{\alpha}, N)}{\partial\varphi_i(\mathbf{k}_i, N_0)} = \delta(\mathbf{k}_{\alpha} - \mathbf{k}_i)\frac{\partial\varphi_{\alpha}(N)}{\partial\varphi_i(N_0)} \quad (27)$$

Eq. (27) is sometimes described as the “Jacobi map”. It has a formal solution in terms of a path-ordered exponential

$$\Gamma_{\alpha'i'} = (2\pi)^3\delta(\mathbf{k}_{\alpha} - \mathbf{k}_i)\mathbf{P}\exp\left(\int_{N_0}^N dN' u_{\alpha i}(N')\right). \quad (28)$$

In this expression, \mathbf{P} denotes the path-ordering operator which rewrites its argument in order of position on the trajectory: objects evaluated early on the trajectory appear to the right of objects evaluated later. This path-ordered exponential is related to the inverse of the van Vleck matrix, which is equivalent to the matrix of Jacobi fields. Reference to Eqs. (13) and (15) shows that, in each transport equation, this choice for Γ will absorb the terms proportional to the n -point function itself. Returning to the two-point function and discarding higher-order contributions, it follows that the “kernel” $\Sigma_{i'j'}$ can be obtained as an integral over the source terms. It is this integral over sources which can be compared to the “line of sight” integral for the Boltzmann equation.

With these choices, and working to leading order, there are *no* sources for the kernel $\Sigma_{i'j'}$. Therefore it is constant, and equal to its initial condition set at horizon crossing. We write this constant value $\mathcal{S}_{i'j'}$.

Three-point function.—When this method is applied to the three-point function, it transpires that the kernel *is* sourced. Again without loss of generality, we write

$$\alpha_{\alpha'\beta'\gamma'} = \Gamma_{\alpha'i'}\Gamma_{\beta'j'}\Gamma_{\gamma'k'}A_{i'j'k'}. \quad (29)$$

We define $\tilde{u}_{i'j'k'} \equiv \Gamma_{i'\alpha'}^{-1}u_{\alpha'\beta'\gamma'}\Gamma_{\beta'j'}\Gamma_{\gamma'k'}$ and obtain

$$A_{i'j'k'} = \mathcal{A}_{i'j'k'} + \left[\int_{N_0}^N dN' \tilde{u}_{i'm'n'}(N')\mathcal{S}_{m'j'}\mathcal{S}_{n'k'} + 2 \text{ cyclic} \right] + \mathcal{O}(H^6). \quad (30)$$

The integration constant $\mathcal{A}_{i'j'k'}$ is the unsourced initial condition which was neglected above, and the estimate $\mathcal{O}(H^6)$ for the terms we have omitted assumes that the correlation functions order themselves in increasing powers of H^2 as described by Jarnhus & Sloth [37]. Defining

$$\begin{aligned} \Gamma_{\alpha'i'j'} &= \Gamma_{\alpha'm'} \int_{N_0}^N \tilde{u}_{m'i'j'}(N') dN' = (2\pi)^3 \delta(\mathbf{k}_\alpha - \mathbf{k}_i - \mathbf{k}_j) \Gamma_{\alpha m} \int_{N_0}^N \tilde{u}_{mij}(N') dN', \\ &= (2\pi)^3 \delta(\mathbf{k}_\alpha - \mathbf{k}_i - \mathbf{k}_j) \Gamma_{\alpha ij}, \end{aligned} \quad (31)$$

we conclude

$$\alpha_{\alpha'\beta'\gamma'} = \Gamma_{\alpha'i'}\Gamma_{\beta'j'}\Gamma_{\gamma'k'}\mathcal{A}_{i'j'k'} + \left(\Gamma_{\alpha'i'j'}\Gamma_{\beta'k'}\Gamma_{\gamma'\ell'}\mathcal{S}_{i'k'}\mathcal{S}_{j'\ell'} + 2 \text{ cyclic} \right) + \text{loops}. \quad (32)$$

It can be shown that the quantity $\Gamma_{\alpha ij}$ appearing on the right-hand side of (31) is equal to*

$$\Gamma_{\alpha ij} = \frac{\partial \Gamma_{\alpha i}}{\partial \varphi_j(N_0)} = \frac{\partial^2 \phi_\alpha(N)}{\partial \varphi_i(N_0) \partial \varphi_j(N_0)}, \quad (33)$$

from which it follows that Eq. (32) is equivalent to the Lyth–Rodríguez Taylor expansion formula for the three-point function [2]. Moreover, differentiation of (31) shows that $\Gamma_{\alpha ij}$ satisfies the evolution equation

$$\frac{d\Gamma_{\alpha ij}}{dN} = u_{\alpha\beta}\Gamma_{\beta ij} + u_{\alpha\beta\gamma}\Gamma_{\beta i}\Gamma_{\gamma j}. \quad (34)$$

Four-point function.—The analysis for the four-point function is similar. We introduce the integrating factor $\Gamma_{\alpha'i'}$,

$$\beta_{\alpha'\beta'\gamma'\delta'} = \Gamma_{\alpha'i'}\Gamma_{\beta'j'}\Gamma_{\gamma'k'}\Gamma_{\delta'\ell'}B_{i'j'k'\ell'}. \quad (35)$$

The kernel $B_{i'j'k'\ell'}$ is given by an integral over sources, as before, which are drawn from the lower-order n -point functions. In this case they are the two- and three-point functions. Keeping only leading-order terms, we find

$$\begin{aligned} B_{i'j'k'\ell'} &= \mathcal{B}_{i'j'k'\ell'} + \left(\int_{N_0}^N dN' \tilde{u}_{i'p'q'}(N')A_{p'j'k'}(N')\mathcal{S}_{q'\ell'} + 11 \text{ cyclic} \right) \\ &+ \left(\int_{N_0}^N dN' \tilde{u}_{i'q'r's'}(N')\mathcal{S}_{q'j'}\mathcal{S}_{r'k'}\mathcal{S}_{s'\ell'} + 3 \text{ cyclic} \right) + \mathcal{O}(H^8), \end{aligned} \quad (36)$$

* Direct differentiation of Eq. (28) is subtle, because of the path-ordered exponential. It is simpler to differentiate the Jacobi equation, Eq. (26), and then solve it using $\Gamma_{\alpha i}$ as an integrating factor.

where we have defined $\tilde{u}_{i'j'k'\ell'} = \Gamma_{i'\alpha'}^{-1} u_{\alpha'\beta'\gamma'\delta'} \Gamma_{\beta'j'} \Gamma_{\gamma'k'} \Gamma_{\delta'\ell'}$. The integration constant $\mathcal{B}_{i'j'k'\ell'}$ is the initial value of the four-point function at time $N = N_0$, as for the three-point function. Taking the four \mathbf{k} -modes entering the four-point function to have a similar time of horizon exit, and the initial time N_0 to be around this epoch, the initial condition was shown in Refs. [4, 6] to be dominated by the correlations induced by decay of gravitational waves into scalar quanta. It is negligible when the amplitude of the four-point function is sufficiently large to be observable. On the other hand, the initial value of the three-point function appears in the kernel $A_{p'j'k'}$ which forms part of the source integral (36), and need not be entirely negligible. However, as discussed below Eqs. (23)–(24), and in more detail in Appendix A, its contribution to τ_{NL} or g_{NL} is likely no more than $\mathcal{O}(1)$ for models with acceptable $|f_{\text{NL}}|$.

To relate (36) to the expressions produced by the Taylor expansion algorithm we must express $A_{p'j'k'}$ purely in terms of correlations at the initial time N_0 . Combining (30) and (36) we find

$$\begin{aligned} & \int_{N_0}^N dN' \tilde{u}_{i'p'q'}(N') A_{p'j'k'}(N') \\ &= \int_{N_0}^N dN' \tilde{u}_{i'p'q'}(N') \left\{ \mathcal{A}_{p'j'k'} + \int_{N_0}^{N'} dN'' \tilde{u}_{p'r's'}(N'') \mathcal{S}_{r'j'} \mathcal{S}_{s'k'} \right. \\ & \quad \left. + \int_{N_0}^{N'} dN'' \tilde{u}_{j'r's'}(N'') \mathcal{S}_{r'p'} \mathcal{S}_{s'k'} + \int_{N_0}^{N'} dN'' \tilde{u}_{k'r's'}(N'') \mathcal{S}_{r'p'} \mathcal{S}_{s'j'} \right\} \end{aligned} \quad (37)$$

The term involving $\mathcal{A}_{p'j'k'}$ presents no difficulties. It makes a contribution to $\beta_{\alpha'\beta'\gamma'\delta'}$ of the form

$$\beta_{\alpha'\beta'\gamma'\delta'} \supseteq \Gamma_{\alpha'p'q'} \Gamma_{\beta'j'} \Gamma_{\gamma'k'} \Gamma_{\delta'\ell'} \mathcal{A}_{p'j'k'} \mathcal{S}_{q'\ell'} + 11 \text{ cyclic}, \quad (38)$$

where, as above, the symbol “ \supseteq ” indicates that the four-point function contains this contribution among others. The other terms in (37) are nested integrals, and divide into two groups. One involves a contraction between the two u -matrices, of the form $\tilde{u}_{i'p'q'} \tilde{u}_{p'r's'}$. We first focus on the other two, which involve no contraction. After summing over perturbations there are twenty-four such terms. Consider the specific choice $\tilde{u}_{i'p'q'} \tilde{u}_{j'r's'}$ which appears in (37). In combination with one of the terms generated by simultaneously exchanging $i' \leftrightarrow j'$ and $k' \leftrightarrow \ell'$ this generates

$$\int_{N_0}^N dN' \int_{N_0}^{N'} dN'' \tilde{u}_{i'p'q'}(N') \tilde{u}_{j'r's'}(N'') \mathcal{S}_{r'p'} \mathcal{S}_{s'k'} \mathcal{S}_{q'\ell'}, \quad (39)$$

in which the integrals are no longer nested. Pairing all such terms in this way generates the 12 cyclic permutations of indices in (39). The corresponding contribution to the four-point function is

$$\beta_{\alpha'\beta'\gamma'\delta'} \supseteq \Gamma_{\alpha'p'q'} \Gamma_{\beta'r's'} \Gamma_{\gamma'k'} \Gamma_{\delta'\ell'} \mathcal{S}_{p'r'} \mathcal{S}_{s'k'} \mathcal{S}_{q'\ell'} + 11 \text{ cyclic}. \quad (40)$$

Now focus on the contracted terms $\tilde{u}_{i'p'q'} \tilde{u}_{p'r's'}$. Summing over the permutations $\ell' \rightarrow \{j', k'\}$ is equivalent to symmetrization over $\{q', r', s'\}$. Therefore this term can be

combined with the $\tilde{u}_{i'q'r's'}$ source in (36), giving a total contribution to the four-point function of the form

$$\beta_{\alpha'\beta'\gamma'\delta'} \supseteq \Gamma_{\alpha'q'r's'}\Gamma_{\beta'j'}\Gamma_{\gamma'k'}\Gamma_{\delta'\ell'}\mathcal{S}_{q'j'}\mathcal{S}_{r'k'}\mathcal{S}_{s'\ell'} + 3 \text{ cyclic}, \quad (41)$$

where we have defined $\Gamma_{\alpha'q'r's'}$ to satisfy

$$\begin{aligned} \Gamma_{\alpha'q'r's'} &= \Gamma_{\alpha'i'} \int_{N_0}^N dN' \tilde{u}_{i'q'r's'}(N') \\ &+ \left(\Gamma_{\alpha'i'} \int_{N_0}^N dN' \tilde{u}_{i'p'q'}(N') \int_{N_0}^{N'} dN'' \tilde{u}_{p'r's'}(N'') + [q' \rightarrow \{r', s'\}] \right). \end{aligned} \quad (42)$$

As with the previous examples of Γ -matrices, the momentum dependence of $\Gamma_{\alpha'q'r's'}$ is a pure δ -function. It can be converted to a pure flavour matrix by the rule

$$\Gamma_{\alpha'q'r's'} = (2\pi)^3 \delta(\mathbf{k}_\alpha - \mathbf{k}_q - \mathbf{k}_r - \mathbf{k}_s) \Gamma_{\alpha qrs}. \quad (43)$$

By explicit differentiation and back-substitution, it can be shown that this flavour matrix satisfies the ordinary differential equation

$$\frac{d\Gamma_{\alpha qrs}}{dN} = u_{\alpha\beta} \Gamma_{\beta qrs} + \left(u_{\alpha\beta\gamma} \Gamma_{\beta q\gamma} \Gamma_{\gamma s} + 2 \text{ cyclic} \right) + u_{\alpha\beta\gamma\delta} \Gamma_{\beta q} \Gamma_{\gamma r} \Gamma_{\delta s}. \quad (44)$$

We have already seen that the lower-order Taylor coefficients $\Gamma_{\alpha i}$ and $\Gamma_{\alpha ij}$ are determined by the evolution equations (26) (with primed indices exchanged for unprimed ones) and (34); for an extended discussion, see Ref. [18]. These equations provide an efficient means to compute the “ δN coefficients” numerically.

Returning to the four-point function, we must also include the initial condition

$$\beta_{\alpha'\beta'\gamma'\delta'} \supseteq \Gamma_{\alpha'i'}\Gamma_{\beta'j'}\Gamma_{\gamma'k'}\Gamma_{\delta'\ell'}\mathcal{B}_{i'j'k'\ell'}. \quad (45)$$

Repeating the steps described above, it can be shown that

$$\Gamma_{\alpha qrs} = \frac{\partial \Gamma_{\alpha qr}}{\partial \varphi_s(N_0)} = \frac{\partial^3 \varphi_\alpha(N)}{\partial \varphi_q(N_0) \partial \varphi_r(N_0) \partial \varphi_s(N_0)}. \quad (46)$$

Therefore we have reproduced the usual Taylor expansion formulae for the trispectrum. Specifically, Eq. (45) matches (8) of Ref. [4], and Eqs. (38), (40) and (41) match (73), (74) and (75) of the same reference. These expressions were later given in slightly more generality by Byrnes, Sasaki & Wands [14]. In the formulation given by these authors, Eqs. (38), (40), (41) and (45) of this paper match (36) of Ref. [14].

4. Transformation to the curvature perturbation

We now have the transport equations which evolve the n -point functions of the scalar field perturbations during inflation, up to and including $n = 4$. These can be obtained either by solving the “shape equations”, Eqs. (23)–(24), or using Eq. (44) to evolve the Γ -matrices. For the latter case, the initial conditions are $\Gamma_{\alpha i} = \delta_{\alpha i}$ at $N = N_0$, with all other Γ -matrices zero there.

4.1. Curvature perturbation at third order

The scalar field fluctuations are not observable by themselves. At present we have observational evidence only for a single primordial fluctuation—the density fluctuation, which is a nonlinear and model-dependent combination of the field fluctuations. The appropriate combination can be deduced from the displacement δN (measured in e-folds) between a fixed spatially-flat hypersurface and an adjacent uniform-density hypersurface with which it coincides on average. This displacement is determined by the field configuration on the spatially flat hypersurface. Therefore $\zeta = \delta N = \delta[N(\varphi_\alpha)]$, yielding

$$\zeta = N_\alpha \delta\varphi_\alpha + \frac{1}{2!} N_{\alpha\beta} (\delta\varphi_\alpha \delta\varphi_\beta - \langle \delta\varphi_\alpha \delta\varphi_\beta \rangle) + \frac{1}{3!} N_{\alpha\beta\gamma} (\delta\varphi_\alpha \delta\varphi_\beta \delta\varphi_\gamma - \langle \delta\varphi_\alpha \delta\varphi_\beta \delta\varphi_\gamma \rangle) + \dots, \quad (47)$$

where $N_\alpha = \partial N / \partial \varphi_\alpha$ and similarly for the higher derivatives. Note that these are ordinary partial derivatives, with all quantities evaluated at the same time: they are not the nonlocal variational derivatives which appear in the Lyth–Rodríguez Taylor expansion. In particular, we are not using the δN formula (47) to account for any time dependence of the correlation functions; this is handled by the transport equations. Eq. (47) is used solely to obtain the relationship between the $\delta\varphi_\alpha$ and ζ . There are various other ways in which this could be obtained. Malik & Wands gave a comprehensive discussion [43] from the viewpoint of traditional cosmological perturbation theory. Another approach was used by Maldacena [1]. Eq. (47) has the advantage that it computes the transformation only in the superhorizon limit $k/aH \rightarrow 0$, which is all we require.

Calculation of the derivatives N_α , $N_{\alpha\beta}$ and $N_{\alpha\beta\gamma}$ is tedious, although straightforward in principle. Ref. [18] used a raytracing method which gave the relation a geometrical meaning. It would be interesting to apply this technique at third order, but it is helpful primarily for analytic and geometric intuition rather than numerical optimization. Ref. [27] exploited the fact that any potential is separable for first order displacements to set up constants of the motion, as originally done by García-Bellido & Wands [41, 44]. However, this method is relatively lengthy even for the second-order coefficient $N_{\alpha\beta}$. Here we employ a simpler alternative.

We first focus on a *single* trajectory and measure the number of e-folds N accumulated along it. During any period where the density decreases monotonically we may measure N as a function of ρ . Consider the number of e-folds ΔN which elapse between some arbitrary point on the trajectory (the “starting point”) and a nearby hypersurface of fixed density ρ_c . Under the slow-roll approximation, the density at the starting point is simply the potential energy evaluated there. Therefore we may express ΔN as a Taylor expansion in the difference $\Delta\rho = \rho_c - V$,

$$\Delta N = N(V + \Delta\rho) - N(V) = \frac{dN}{d\rho} \Delta\rho + \frac{1}{2!} \frac{d^2 N}{d\rho^2} \Delta\rho^2 + \frac{1}{3!} \frac{d^3 N}{d\rho^3} \Delta\rho^3 + \dots. \quad (48)$$

Note that the differential coefficients are *ordinary* derivatives taken along the trajectory. In Eq. (48) they are evaluated at the starting point.

We now perturb the starting point by an amount $\delta\varphi_\alpha$ while keeping the final hypersurface fixed. In general $\delta\varphi_\alpha$ will not be aligned with the inflationary trajectory used to construct the ρ -derivatives in Eq. (48), which therefore vary. The same is true for the displacement $\Delta\rho$. Accounting for both these effects changes the total elapsed e-folds by an amount $\delta(\Delta N)$. Finally, to study fluctuations around the hypersurface $\rho = \rho_c$ we take the limit $\Delta\rho \rightarrow 0$, after which $\delta(\Delta N) \rightarrow \zeta$. The advantage of this method is that it uses the handful of low-order derivatives appearing in Eq. (48) to isolate the limited information we require regarding local properties of the transformation: higher-order information is discarded at the outset. This contrasts with the constants-of-motion approach used in Ref. [27], where high-order information is implicitly kept through the majority of the computation, although it is never used.

Under a shift of the starting point we conclude

$$\delta(\Delta\rho) = -V_\alpha\delta\varphi_\alpha - \frac{1}{2!}V_{\alpha\beta}\delta\varphi_\alpha\delta\varphi_\beta - \frac{1}{3!}V_{\alpha\beta\gamma}\delta\varphi_\alpha\delta\varphi_\beta\delta\varphi_\gamma - \dots \quad (49)$$

By retaining contributions to ρ from the kinetic energy, and evaluating the differential coefficients in (48) without use of the slow-roll approximation, this approach could be extended to provide the transformation from the full phase space variables $\delta\varphi_\alpha, \delta\dot{\varphi}_\alpha$ to ζ . This was done in Ref. [31].

Invoking the slow-roll approximation, we may calculate the derivative $dN/d\rho$,

$$\left.\frac{dN}{d\rho}\right| = \left.\frac{dN}{dt} \frac{dt}{dV}\right|_\gamma = -\frac{3H^2}{V_\alpha V_\alpha} = -\frac{1}{M_{\text{P}}^2} \frac{V}{V_\alpha V_\alpha}, \quad (50)$$

where dV/dt is to be computed along the trajectory γ . Higher derivatives can be obtained in the same way, by repeated differentiation with respect to t and use of the chain rule to convert these into derivatives with respect to ρ . We obtain

$$\begin{aligned} \left.\frac{d^2N}{d\rho^2}\right| &= -\frac{1}{M_{\text{P}}^2} \left(\frac{1}{V_\lambda V_\lambda} - 2\frac{VV_\alpha V_\beta V_{\alpha\beta}}{(V_\lambda V_\lambda)^3} \right), \quad (51) \\ \left.\frac{d^3N}{d\rho^3}\right| &= \frac{1}{M_{\text{P}}^2} \left(4\frac{V_\alpha V_\beta V_{\alpha\beta}}{(V_\lambda V_\lambda)^3} - 12\frac{V(V_\alpha V_\beta V_{\alpha\beta})^2}{(V_\lambda V_\lambda)^5} + 4\frac{VV_\alpha V_{\alpha\beta} V_{\beta\gamma} V_\gamma}{(V_\lambda V_\lambda)^4} + 2\frac{VV_{\alpha\beta\gamma} V_\alpha V_\beta V_\gamma}{(V_\lambda V_\lambda)^4} \right). \quad (52) \end{aligned}$$

The first and second-order variations are

$$N_\alpha = -\left.\frac{dN}{d\rho}\right| V_\alpha, \quad (53)$$

$$N_{\alpha\beta} = -\left.\frac{dN}{d\rho}\right| V_{\alpha\beta} + \left.\frac{d^2N}{d\rho^2}\right| V_\alpha V_\beta + \frac{1}{M_{\text{P}}^2} (V_\alpha A_\beta + V_\beta A_\alpha), \quad (54)$$

which agree with existing expressions in the literature. (See below for the definition of A_α .) At third order we find

$$\begin{aligned} N_{\alpha\beta\gamma} &= -\left.\frac{d^3N}{d\rho^3}\right| V_\alpha V_\beta V_\gamma + \left(\left.\frac{d^2N}{d\rho^2}\right| V_\alpha V_{\beta\gamma} + \text{cyclic}\right) - \left.\frac{dN}{d\rho}\right| V_{\alpha\beta\gamma} \\ &\quad + \frac{1}{M_{\text{P}}^2} (A_\alpha V_{\beta\gamma} + \text{cyclic}) + \frac{1}{M_{\text{P}}^2} (B_{\alpha\beta} V_\gamma + \text{cyclic}) + \frac{1}{M_{\text{P}}^2} (C_\alpha V_\beta V_\gamma + \text{cyclic}). \quad (55) \end{aligned}$$

The tensors A_α , $B_{\alpha\beta}$ and C_α have been defined to satisfy

$$A_\alpha = \frac{V_\alpha}{V_\lambda V_\lambda} - 2 \frac{VV_\kappa V_{\kappa\alpha}}{(V_\lambda V_\lambda)^2} \quad (56)$$

$$B_{\alpha\beta} = \frac{V_{\alpha\beta}}{V_\lambda V_\lambda} - 2 \frac{V_\kappa V_{\kappa\alpha} V_\beta + V_\kappa V_{\kappa\beta} V_\alpha}{(V_\lambda V_\lambda)^2} + 8 \frac{VV_\kappa V_{\kappa\alpha} V_\epsilon V_{\epsilon\beta}}{(V_\lambda V_\lambda)^3} - 2 \frac{VV_{\kappa\alpha} V_{\kappa\beta}}{(V_\lambda V_\lambda)^2} - 2 \frac{VV_\kappa V_{\kappa\alpha\beta}}{(V_\lambda V_\lambda)^2} \quad (57)$$

$$C_\alpha = \frac{V_{\alpha\beta}}{V_\lambda V_\lambda} + 2 \frac{V_\alpha V_\kappa V_{\kappa\epsilon} V_\epsilon}{(V_\lambda V_\lambda)^3} - 12 \frac{VV_\kappa V_{\kappa\epsilon} V_\epsilon V_\rho V_{\rho\alpha}}{(V_\lambda V_\lambda)^3} + 4 \frac{VV_\kappa V_{\kappa\epsilon} V_{\epsilon\alpha}}{(V_\lambda V_\lambda)^3} + 2 \frac{VV_\kappa V_{\kappa\epsilon\alpha} V_\epsilon}{(V_\lambda V_\lambda)^3} \quad (58)$$

4.2. Inflationary observables

Finally, we must assemble all these contributions to obtain expressions for τ_{NL} and g_{NL} . We find

$$P_\zeta = N_\alpha N_\beta \Sigma_{\alpha\beta} \quad (59)$$

$$\frac{6}{5} f_{\text{NL}} = \frac{N_\alpha N_\beta N_\gamma \alpha_{\alpha|\beta\gamma} + N_{\alpha\beta} N_\gamma N_\delta \Sigma_{\alpha\gamma} \Sigma_{\beta\delta}}{(N_\omega N_\zeta \Sigma_{\omega\zeta})^2} \quad (60)$$

$$\tau_{\text{NL}} = \frac{N_\alpha N_\beta N_\gamma N_\delta \tau_{\alpha\beta|\gamma\delta} + 2N_\alpha N_\beta N_\gamma N_{\lambda\mu} a_{\alpha|\beta\lambda} \Sigma_{\gamma\mu} + N_{\alpha\beta} N_\gamma N_\delta N_\lambda N_\mu \Sigma_{\alpha\gamma} \Sigma_{\beta\lambda} \Sigma_{\delta\mu}}{(N_\omega N_\zeta \Sigma_{\omega\zeta})^3} \quad (61)$$

$$\frac{54}{25} g_{\text{NL}} = \frac{N_\alpha N_\beta N_\gamma N_\delta g_{\alpha|\beta\gamma\delta} + 3N_\alpha N_\beta N_\gamma N_{\lambda\mu} a_{\lambda|\alpha\beta} \Sigma_{\gamma\mu} + N_{\alpha\beta\gamma} N_\delta N_\lambda N_\mu \Sigma_{\alpha\delta} \Sigma_{\beta\lambda} \Sigma_{\gamma\mu}}{(N_\omega N_\zeta \Sigma_{\omega\zeta})^3}. \quad (62)$$

5. Alternative approaches

In this paper, our approach to calculating the statistics of the curvature perturbation has been to develop transport equations for objects such as the n -point functions [Eqs. (11), (13) and (15)], or their shape tensors [Eqs. (17), (19) and (23)–(24)]. The results of §3 show that this is equivalent to the Lyth–Rodríguez Taylor expansion

$$\delta\varphi_\alpha = \Gamma_{\alpha i} \delta\varphi_i + \frac{1}{2!} \Gamma_{\alpha ij} \delta\varphi_i \delta\varphi_j + \frac{1}{3!} \Gamma_{\alpha ijk} \delta\varphi_i \delta\varphi_j \delta\varphi_k + \dots, \quad (63)$$

where we recall that objects with Greek indices are evaluated at time N , and those with Latin indices at some earlier time N_0 , which is usually taken as the common time of horizon exit for the \mathbf{k} -modes under consideration.‡

‘Forward’ and ‘backward’ methods.—To solve these equations we must supply a boundary condition at $N = N_0$, but we are free to choose how this is done: we may start either with N_0 at the initial epoch and evolve N *forward* to the time of interest, or fix N at this time and evolve N_0 *backwards*. These approaches are distinct but equally valid, because (at least during inflation) there is no obstacle to computing the relevant initial conditions at any time of our choosing. The transport equations we have described in this paper are of the forwards variety.

‡ Indeed, one can verify that inserting (63) into (5) and equating coefficients order-by-order reproduces the Γ -matrix evolution equations with unprimed indices [Eqs. (26), (34) and (44)].

Eq. (63) shows explicitly what must be computed in order to completely characterize the fluctuations $\delta\varphi_\alpha$ at any given order. At first order in a d -field slow-roll model, we require the d^2 independent components of the Jacobi map $\Gamma_{\alpha i}$. At second order there are d^3 components of $\Gamma_{\alpha ij}$, reduced to $d^2(d+1)/2$ after accounting for symmetries. Finally, at third order there are d^4 components of $\Gamma_{\alpha ijk}$, which reduce to $d^2(d+1)(d+2)/6$ independent components after symmetries. We conclude that to compute all two-point functions in such a model requires solution of $O(d^2)$ differential equations. Likewise, all three-point functions requires $O(d^3)$ equations, and all four-point functions requires $O(d^4)$ equations. This is to be expected, because there are $O(d^m)$ independent m -point functions.

Autocorrelation functions of ζ only.—Sometimes we do not require all correlation functions, but only the autocorrelation functions of ζ . In such cases it would be advantageous if an autonomous set of transport equations could be set up for the Taylor coefficients of ζ rather than $\delta\varphi_\alpha$,

$$\delta N = N_i \delta\varphi_i + \frac{1}{2!} N_{ij} \delta\varphi_i \delta\varphi_j + \frac{1}{3!} N_{ijk} \delta\varphi_i \delta\varphi_j \delta\varphi_k. \quad (64)$$

This would require the solution of only $O(d^{m-1})$ independent equations to obtain the m -point function of ζ . This saving could be helpful in models with a large number of fields. There is currently no *forwards* formulation of this type but a set of *backwards* equations were given by Yokoyama, Suyama & Tanaka [29, 30], and later extended to the trispectrum [45].

The Taylor coefficients for N can be expressed in terms of the Γ -matrices,

$$N_i = N_\alpha \Gamma_{\alpha i} \quad (65)$$

$$N_{ij} = N_\alpha \Gamma_{\alpha ij} + N_{\alpha\beta} \Gamma_{\alpha i} \Gamma_{\beta j} \quad (66)$$

$$N_{ijk} = N_\alpha \Gamma_{\alpha ijk} + N_{\alpha\beta\gamma} \Gamma_{\alpha i} \Gamma_{\beta j} \Gamma_{\gamma k} + (N_{\alpha\beta} \Gamma_{\beta i} \Gamma_{\alpha jk} + 2 \text{ cyclic}). \quad (67)$$

We could attempt to obtain forward transport equations by direct differentiation with respect to time followed by use of the Γ -matrix evolution equations. But this does not generate a closed set of autonomous equations because derivatives of the $N_{\alpha\dots}$ also appear, which obstruct an attempt to eliminate the Γ -matrices in favour of their N counterparts.

Instead, the backwards equations of Yokoyama et al. can be derived as follows. As described above, we fix N to be the late time of interest and aim to evolve N_0 backwards. The backwards evolution of $\Gamma_{\alpha i}$ can be obtained very simply by differentiating (63) while keeping $\delta\varphi_\alpha$ fixed, or alternatively by differentiating (28) with respect to N_0 . Whichever method is chosen, we find $d\Gamma_{\alpha i}/dN_0 = -\Gamma_{\alpha j} u_{ji}$. Subsequently differentiating (65) with respect to N_0 and using this relation, we obtain an autonomous set of equations for N_i ,

$$\frac{dN_i}{dN_0} = -N_j u_{ji}. \quad (68)$$

This technique can be extended to higher orders, giving evolution equations for N_{ij} and N_{ijk} .^{††} We find

$$\frac{dN_{ij}}{dN_0} = -N_k u_{kij} - N_{ik} u_{kj} - N_{jk} u_{ki}, \quad (71)$$

and

$$\frac{dN_{ijk}}{dN_0} = -N_\ell u_{\ellijk} - (N_{i\ell} u_{\elljk} + N_{jk\ell} u_{\ell i} + 2 \text{ cyclic}). \quad (72)$$

The first of these was given in Ref. [18]. Here we have extended the method to include N_{ijk} , which enables trispectrum quantities to be calculated. These equations should be solved with initial conditions chosen so that N_i , N_{ij} and N_{ijk} equal the transformation matrices N_α , $N_{\alpha\beta}$ and $N_{\alpha\beta\gamma}$, respectively, at $N_0 = N$.

If we require only the bispectrum of ζ and are prepared to take the field fluctuations at time N_0 to be Gaussian and uncorrelated with each other, then more is possible. Under these circumstances, Yokoyama et al. showed that the $O(d^2)$ equations for N_{ij} could be replaced by only $O(d)$ equations for an auxiliary quantity $\Theta_\alpha = \Gamma_{\alpha i} N_i$ [29, 30].

Constraint for first-order coefficients.—There is a further simplification which can be made for the N_i system. Using the flow equation $d\phi_i = u_i dN$, it follows that the displacement $d\phi_i = u_i$ precisely tangent to the trajectory generates a change in the e-foldings required to reach the final uniform density slice corresponding to

$$\delta N = u_i N_i = -1. \quad (73)$$

This implies that *one* of the N_i can be determined algebraically in terms of the others, without solving a separate differential equation. Therefore, in a two-field model, the Yokoyama et al. equations (68) can be decoupled,

$$\frac{dN_\phi}{dN_0} = \left(\frac{u_\phi}{u_\chi} u_{\chi\phi} - u_{\phi\phi} \right) N_\phi + \frac{u_{\chi\phi}}{u_\chi}, \quad (74)$$

where we have labelled the fields ϕ and χ . A similar equation can be given for N_χ , but it is unnecessary because (73) can be used to obtain N_χ once N_ϕ is known. Although the possibility of decoupling these equations is interesting, it confers no particular advantages.

A variation of the Yokoyama et al. formulation was recently given by Mazumdar & Wang [46] in which they pointed out the possibility of this decoupling in the two-field case, although without making explicit use of the constraint (73). Their analysis is equivalent to the one presented here, and in Appendix A of Ref. [18]. Mazumdar &

^{††}The evolution equations for $\Gamma_{\alpha ij}$ and $\Gamma_{\alpha ijk}$, which are required to obtain these results, are

$$\frac{d\Gamma_{\alpha ij}}{dN_0} = -\Gamma_{\alpha m} u_{mij} - \Gamma_{\alpha im} u_{mj} - \Gamma_{\alpha mj} u_{mi} \quad (69)$$

$$\frac{d\Gamma_{\alpha ijk}}{dN_0} = -\Gamma_{\alpha m} u_{mijk} - \left(\Gamma_{\alpha im} u_{mj k} + \Gamma_{\alpha mj k} u_{mi} + 2 \text{ cyclic} \right). \quad (70)$$

Wang ascribed the possibility of decoupling to the choice of coordinates used in their derivation. However, the evolution equation (68) can be derived using any convenient method and is independent of such choices. The argument above shows that decoupling is a consequence of the constraint (73), and is a special feature of the two-field system. In a general d -field model, the best that can be obtained is a coupled system of $d - 1$ equations.

6. Numerical results

We now illustrate the transport approach using a number of concrete models. For each model, we numerically solve Eqs. (17), (19) and (23)–(24), and use Eqs. (60)–(62) to determine the values of f_{NL} , τ_{NL} and g_{NL} from horizon crossing onwards. We label the number of e-folds of inflation from $N = 0$ at horizon exit.

6.1. Numerical Examples

D-brane model.—Our first example was studied by Dias, Frazer & Liddle [31]. It is an approximation to inflation driven by the motion of a D-brane in a warped throat, allowing for angular degrees of freedom. In that study, the authors employed the transport approach to calculate the distribution of observable parameters over a large number of realizations of their model. However, they restricted attention to the spectrum and local-type bispectrum. Here we present the evolution of the local-type trispectrum parameters for one typical realization.

The potential is given by

$$V = \alpha_0 + \alpha_1\phi_1 + \alpha_3\phi_1^3 + \beta\phi_2, \quad (75)$$

which contains an inflexion point in the ϕ_1 direction. Inflation occurs close to this inflexion point. We choose $\alpha_0 = 100M^2M_{\text{P}}^2$, $\alpha_1 = M^2M_{\text{P}}$, $\alpha_3 = 5M^2/M_{\text{P}}$, $\beta = 5M^2M_{\text{P}}$, $\phi_{1\text{exit}} = 0.5M_{\text{P}}$, and $\phi_{2\text{exit}} = 0.5M_{\text{P}}$, where the subscript ‘exit’ indicates these are the initial values of the fields at horizon exit. M is an overall normalisation, which can be fixed to match the WMAP normalization of the power spectrum. These initial conditions have been chosen to give 60-e-folds of inflation, taking inflation to end when $\epsilon = 1$. Allowing the system to evolve past this point would lead to erroneous results because we are employing slow-roll equations of motion. As explained in §2 this could be resolved by writing transport equations in the full phase-space. However, for simplicity, we do not do so here. In Fig. 1 we give the evolution of τ_{NL} , g_{NL} , and $(6f_{\text{NL}}/5)^2$ for this choice of parameters and initial conditions.

For single-field models we recall that $\tau_{\text{NL}} = (6f_{\text{NL}}/5)^2$, which is relaxed to an inequality in multiple-field models [15, 16]. The use of the relative magnitude of τ_{NL} and $(6f_{\text{NL}}/5)^2$ as a diagnostic of the spectrum of active fields during inflation was emphasized by Smidt et al. [47], who made a forecast of observational prospects. Very recently, Assassi et al. [17] gave precise formulae in terms of the spectrum of single-particle

states. This signature of multiple active fields is clearly visible in Fig. 1, although in this realization the nongaussian parameters are too small to be observable. (As a point of principle an inflexion point potential may give rise to a large local bispectrum [48] and trispectrum [49], via the hilltop mechanism suggested by Kim et al. [50]. However, an observable signal can usually be obtained only for finely tuned initial conditions and parameter choices.)

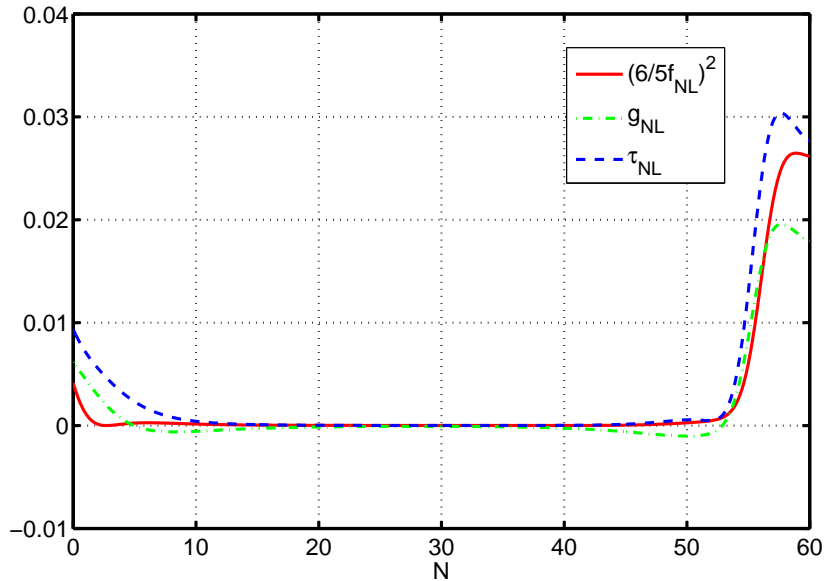


Figure 1. Evolution of τ_{NL} , g_{NL} and $(6/5f_{\text{NL}})^2$ for the inflexion-point potential (75). Initial conditions and parameter choices are described in the main text.

Quadratic-exponential model.—Our second example was constructed by Byrnes et al. [51] as an example of a product-separable model which could give rise to a large f_{NL} for finely-tuned initial conditions. It was later studied by Elliston et al. [52] and Huston et al. [53].

The potential is

$$V = M^4 \phi_1^2 e^{-\lambda \phi_2^2}. \quad (76)$$

We choose the parameter values and initial conditions $\lambda = 0.05/M_{\text{P}}^2$, $\phi_{1\text{exit}} = 16M_{\text{P}}$, and $\phi_{2\text{exit}} = 0.001M_{\text{P}}$, and fix M as before to match the WMAP normalization. These initial values also give 60-efolds of inflation. They have been chosen to select a background trajectory which gives rise to significant nongaussianity. In Fig. 2, we present the evolution of the τ_{NL} and g_{NL} parameters in this model for the first time. We also show the evolution of $(6f_{\text{NL}}/5)^2$. However, although the f_{NL} and τ_{NL} parameters are large at the end of inflation, it is important to note that the fluctuations are still evolving at this time. Therefore the model is not predictive by itself: it must be supplemented by post-inflationary evolution, which tracks the fluctuations until the surface of last scattering, or explains how all isocurvature modes eventually decay.

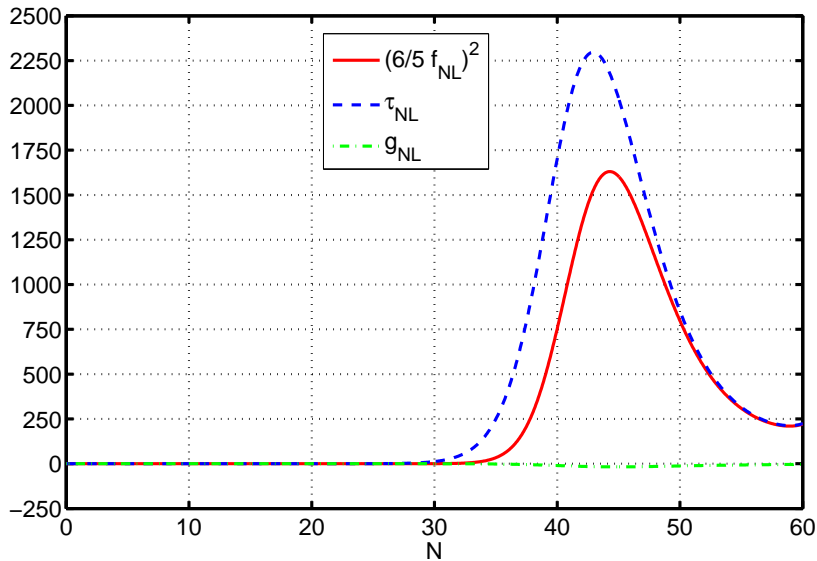


Figure 2. Evolution of τ_{NL} , g_{NL} and $(6/5f_{\text{NL}})^2$ for the potential (76). Initial conditions and parameter choices are described in the main text.

Non-separable hybrid model.—Finally, we present results for a hybrid-type potential in the large field regime studied by Mulryne, Orani & Rajantie [54]. This is an example of a non-separable potential. For general initial conditions, no analytic estimate is known for any of f_{NL} , τ_{NL} or g_{NL} , even assuming slow-roll. Therefore numerical methods, such as our implementation of the transport equations, become essential. The potential contains a hilltop region, and parameter choices and initial conditions can be chosen so that the model is of the type discussed by Kim et al. [50]. This gives rise to large nongaussianity for initial conditions sufficiently close to the hilltop.

The potential satisfies

$$V = M^4 \left[\frac{1}{2} m^2 \phi_1^2 + \frac{1}{2} g^2 \phi_1^2 \phi_2^2 + \frac{\lambda}{4} (\phi_2^2 - v^2)^2 \right], \quad (77)$$

and we choose the parameter values $g^2 = v^2/\phi_{\text{crit}}^2$, $m^2 = v^2$, $v = 0.2M_{\text{P}}$, $\phi_{\text{crit}} = 20M_{\text{P}}$ and $\lambda = 5$. The initial conditions are $\phi_{1\text{exit}} = 15.5M_{\text{P}}$ and $\phi_{2\text{exit}} = 0.0015M_{\text{P}}$. As above, these initial values give 60-e-folds of inflation and have been adjusted to produce significant nongaussianity. M is adjusted as before. In Fig. 3, we present the evolution of the τ_{NL} and g_{NL} parameters in this model for the first time. In contrast to the previous example, the statistics here approach constant values before the end of inflation, reflecting the fact that isocurvature modes decay. We also give the evolution of $(6f_{\text{NL}}/5)^2$.

7. Discussion and Conclusions

In this paper we have provided transport equations to evolve the four-point functions of a collection of light scalar fields during an inflationary phase. The transport system can

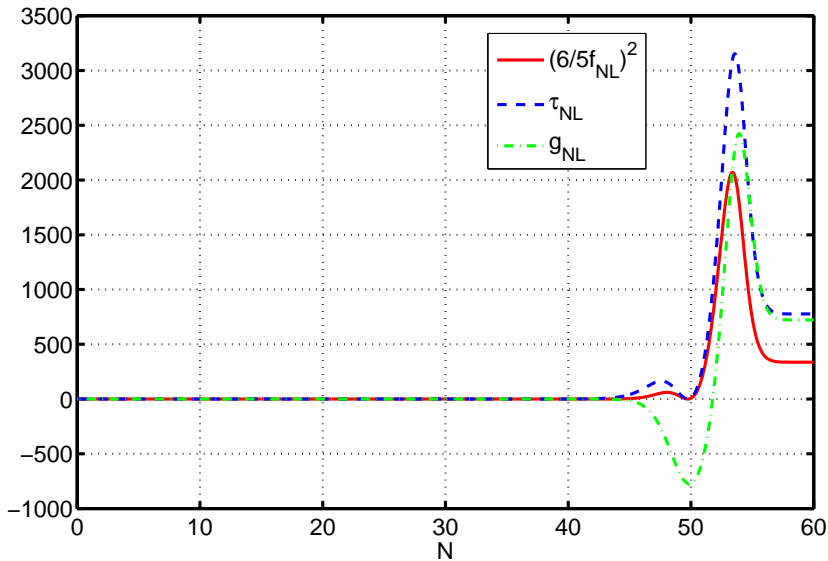


Figure 3. Evolution of τ_{NL} , g_{NL} and $(6/5f_{\text{NL}})^2$ for the potential (77). Initial conditions and parameter choices are described in the main text.

be thought of as a form of Boltzmann hierarchy, and can be solved by similar methods. Since inflationary fluctuations are typically close to Gaussian, connected correlation functions of increasing order are typically decreasing in amplitude. Therefore only a few low-order functions are important in sourcing those of higher order. Truncating the hierarchy to include only these sources generates the local-type “squeezed” and “collapsed” configurations. We parametrize the amplitude of these configurations with “shape tensors” for which we have supplied evolution equations. Expressing the correlation functions of ζ in terms of those of the $\delta\varphi_\alpha$, it is possible to extract τ_{NL} and g_{NL} . This analysis was given in §2.

This method of integrating the transport hierarchy expresses the correlation functions in terms of the Jacobi fields generated by the underlying phase space flow, and their derivatives. One can regard this as a statement of the separate universe approximation. The “Jacobi map” relates these fields to the variation of a general solution of the equations of motion with respect to its constants of integration. Using this equivalence, we have shown that the result reproduces the familiar Taylor expansion used by Lyth & Rodríguez. The procedure can be viewed as an application of classical Hamilton–Jacobi theory.

Our equations supply a toolkit which can be used to study the evolution of inflationary observables in any multi-field model of interest, provided all fields possess canonical kinetic terms. There are two equivalent approaches. First, one can solve Eqs. (17), (19), (23) and (24) for the shape tensors corresponding to the two-, three- and four-point functions, using suitable initial conditions. Eqs. (59)–(61) can then be used to extract observables. Alternatively, one can solve the evolution equations (26) (after exchanging primed for unprimed indices), (34) and (44) for the Taylor coefficients of the

“ δN formalism”, applied to the field fluctuations. Once these are known, Eqs. (65)–(67) can be used to exchange them for the Taylor coefficients of N itself. The usual formulae then allow observables to be computed. If the spectral index or its running are required, they can be extracted using the methods described by Dias et al. [31, 39].

Assuming slow-roll, either method requires the solution of $O(d^m)$ equations to obtain the m -point functions of a d -field model. Since there are $O(d^m)$ independent correlation functions it will not be possible to reduce this asymptotic complexity. But if only the autocorrelation functions of ζ are required, then it may be advantageous to use the ‘backwards’ formalism introduced by Yokoyama, Suyama & Tanaka, in which one can reduce the number of equations to be solved to $O(d^{m-1})$ by forfeiting the possibility of obtaining correlation functions with insertions of isocurvature modes. [For clarity, we emphasize that the formalism of Yokoyama et al. correctly accounts for the influence of these isocurvature modes on the evolution of the ζ correlation functions. But it is not possible to determine mixed correlation functions, such as $\langle \zeta s \rangle$, where s is a field space direction orthogonal to ζ .] Unfortunately, it is often necessary to know something about such correlation functions to determine whether unquenched isocurvature modes remain, which could change the inflationary prediction by transferring their energy to the curvature fluctuation during or after reheating. (We refer to Ref. [18] for a more comprehensive discussion.) But in some cases this may not be a concern, and where this is true our extension of the formalism of Yokoyama et al. allows trispectrum parameters to be obtained.

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Appendix A. Contributions to the four-point function from the initial condition of the three-point function

In this appendix we verify the claim made in §2.3, that the arbitrary initial condition for the three-point function makes a negligible contribution to the sourced component of the four-point function. In the text, this was used to conclude that the initial value need not be retained in Eqs. (23)–(24).

It was first proved by Lyth & Zaballa that the initial condition for the three-point function could be neglected in comparison with the sourced contribution whenever the sum of the two was large enough to be observed [40]. Their argument was later simplified by Vernizzi & Wands [41]. The same result for the four-point function follows from the analysis of Refs. [4, 6]. However, we are unaware of a similar demonstration for the

question addressed in this appendix—the contribution of initial value of the *three*-point function to the sourced component of the *four*-point function.

We work with the variational formulation of the separate universe approximation, as discussed by Lyth & Rodríguez [2]. We write

$$\zeta = N_i \delta\varphi_i + \frac{1}{2!} N_{ij} \delta\varphi_i \delta\varphi_j + \dots, \quad (\text{A.1})$$

where the Latin indices i, j, \dots , have the same meaning as in the main text. We define the trispectrum T_ζ to be the four-point function with its momentum-conservation δ -function stripped away,

$$\langle \zeta(\mathbf{k}_1) \zeta(\mathbf{k}_2) \zeta(\mathbf{k}_3) \zeta(\mathbf{k}_4) \rangle = (2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4) T_\zeta. \quad (\text{A.2})$$

Using the initial value of the three-point function computed in Ref. [3], the corresponding contribution to the sourced part of T_ζ can be written

$$\begin{aligned} T_\zeta \supseteq N_i N_j N_k N_{mn} & \left\{ \delta^{im} \frac{H_*^2}{2k_1^3} \frac{H_*^4}{8k_2^3 k_3^3 k_{14}^3} \sum_{\text{perms}} \frac{\dot{\varphi}_j \delta_{kn}}{2H_*} \left(-3 \frac{k_3^2 k_{14}^2}{k_t} - \frac{k_3^2 k_{14}^2}{k_t^2} (k_2 + 2k_{14}) + \frac{k_2^3}{2} - k_2 k_3^2 \right) \right. \\ & \left. + [\mathbf{k}_1 \rightarrow \{\mathbf{k}_2, \mathbf{k}_3\}] \right\} \\ & + \text{cyclic permutations } \mathbf{k}_4 \rightarrow \{\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3\} \end{aligned} \quad (\text{A.3})$$

where “*” denotes evaluation at horizon exit, $k_t = k_1 + k_2 + k_{14}$, the summation is over all simultaneous permutations of the index set $\{\beta, \gamma, \epsilon\}$ and the momenta $\{\mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_{14}\}$, and we have defined $k_{14} = |\mathbf{k}_1 + \mathbf{k}_4|$.

This contribution can be divided into an effective g_{NL} , an effective τ_{NL} , and an ‘equilateral-type’ term which does not fit naturally into either of the local-type shapes. The effective g_{NL} can be written

$$\Delta g_{\text{NL}} = \frac{25}{72} \frac{N_i N_{ij} \dot{\varphi}_j / H_*}{(N_k N_k)^2} \quad (\text{A.4})$$

(the placement of indices is immaterial in this and other expressions, since contraction occurs under the Kronecker- δ), and the effective τ_{NL} is

$$\Delta \tau_{\text{NL}} = -\frac{1}{2} \frac{N_i N_{ij} N_j}{(N_k N_k)^3}. \quad (\text{A.5})$$

These expressions can be simplified. Introducing the scalar-to-tensor ratio r and the spectral index n_s , we find

$$\Delta g_{\text{NL}} = \frac{25}{1152} r (n_s - 1 + 2\epsilon_*) \ll 1 \quad (\text{A.6})$$

$$\Delta \tau_{\text{NL}} = -\frac{3}{35} r f_{\text{NL}}, \quad (\text{A.7})$$

where f_{NL} is the sourced local-mode contribution to the three-point function. The g_{NL} contribution is clearly negligible. The τ_{NL} contribution is negligible provided $|rf_{\text{NL}}| \lesssim 1$. Taking the bound on r to be roughly $r \lesssim 0.1$, this term can be observationally relevant only if $|f_{\text{NL}}| \gtrsim 50$. This is already on the verge of being ruled out by experiment, so the τ_{NL} contribution is likely to be no more than $\mathcal{O}(1)$ in most acceptable models. It could perhaps be kept if very accurate estimates are required.

Finally, the equilateral-type term is

$$\begin{aligned}
 T_{\zeta} \supseteq & \frac{H_*^6}{8k_1^3 k_2^3 k_3^3 k_{14}^3} \left(N_i N_{ij} \frac{\dot{\varphi}_j}{4H_*} (N_k N_l) \left(-\frac{8k_2^2 k_3^2}{k_t} - k_{14}(k_2^2 + k_3^2) \right) \right. \\
 & \left. - \frac{N_i N_{ij} N_j}{4} \left(-8\frac{k_{14}^2}{k_t}(k_2^2 + k_3^2) - k_{14}^2(k_2 + k_3) - k_2 k_3(k_2 + k_3) \right) \right) \\
 & + [\mathbf{k}_1 \rightarrow \{\mathbf{k}_2, \mathbf{k}_3\}] + (\text{cyclic } \mathbf{k}_4 \rightarrow \{\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3\})
 \end{aligned} \tag{A.8}$$

The coefficients of these contributions are related to those of Eqs. (A.4) and (A.5), and will therefore not typically be large.

References

- [1] J. M. Maldacena, *Non-Gaussian features of primordial fluctuations in single field inflationary models*, *JHEP* **0305** (2003) 013, [arXiv:astro-ph/0210603].
- [2] D. H. Lyth and Y. Rodríguez, *The Inflationary prediction for primordial non-Gaussianity*, *Phys.Rev.Lett.* **95** (2005) 121302, [arXiv:astro-ph/0504045], [doi:10.1103/PhysRevLett.95.121302].
- [3] D. Seery and J. E. Lidsey, *Primordial non-Gaussianities from multiple-field inflation*, *JCAP* **0509** (2005) 011, [arXiv:astro-ph/0506056], [doi:10.1088/1475-7516/2005/09/011].
- [4] D. Seery, J. E. Lidsey, and M. S. Sloth, *The inflationary trispectrum*, *JCAP* **0701** (2007) 027, [arXiv:astro-ph/0610210], [doi:10.1088/1475-7516/2007/01/027].
- [5] D. Seery and J. E. Lidsey, *Non-Gaussianity from the inflationary trispectrum*, *JCAP* **0701** (2007) 008, [arXiv:astro-ph/0611034], [doi:10.1088/1475-7516/2007/01/008].
- [6] D. Seery, M. S. Sloth, and F. Vernizzi, *Inflationary trispectrum from graviton exchange*, *JCAP* **0903** (2009) 018, [arXiv:0811.3934], [doi:10.1088/1475-7516/2009/03/018].
- [7] A. A. Starobinsky, *Multicomponent de Sitter (Inflationary) Stages and the Generation of Perturbations*, *JETP Lett.* **42** (1985) 152–155.
- [8] M. Sasaki and E. D. Stewart, *A General analytic formula for the spectral index of the density perturbations produced during inflation*, *Prog.Theor.Phys.* **95** (1996) 71–78, [arXiv:astro-ph/9507001], [doi:10.1143/PTP.95.71].
- [9] L. Verde, L.-M. Wang, A. Heavens, and M. Kamionkowski, *Large scale structure, the cosmic microwave background, and primordial non-gaussianity*, *Mon.Not.Roy.Astron.Soc.* **313** (2000) L141–L147, [arXiv:astro-ph/9906301].
- [10] E. Komatsu and D. N. Spergel, *Acoustic signatures in the primary microwave background bispectrum*, *Phys.Rev.* **D63** (2001) 063002, [arXiv:astro-ph/0005036], [doi:10.1103/PhysRevD.63.063002].
- [11] M. Sasaki, J. Väliiviita, and D. Wands, *Non-Gaussianity of the primordial perturbation in the curvaton model*, *Phys.Rev.* **D74** (2006) 103003, [arXiv:astro-ph/0607627], [doi:10.1103/PhysRevD.74.103003].

- [12] L. Boubekeur and D. Lyth, *Detecting a small perturbation through its non-Gaussianity*, *Phys.Rev.* **D73** (2006) 021301, [arXiv:astro-ph/0504046], [doi:10.1103/PhysRevD.73.021301].
- [13] L. Alabidi and D. H. Lyth, *Inflation models and observation*, *JCAP* **0605** (2006) 016, [arXiv:astro-ph/0510441], [doi:10.1088/1475-7516/2006/05/016].
- [14] C. T. Byrnes, M. Sasaki, and D. Wands, *The primordial trispectrum from inflation*, *Phys.Rev.* **D74** (2006) 123519, [arXiv:astro-ph/0611075], [doi:10.1103/PhysRevD.74.123519].
- [15] T. Suyama and M. Yamaguchi, *Non-Gaussianity in the modulated reheating scenario*, *Phys.Rev.* **D77** (2008) 023505, [arXiv:0709.2545], [doi:10.1103/PhysRevD.77.023505].
- [16] K. M. Smith, M. LoVerde, and M. Zaldarriaga, *A universal bound on N-point correlations from inflation*, *Phys.Rev.Lett.* **107** (2011) 191301, [arXiv:1108.1805], [doi:10.1103/PhysRevLett.107.191301].
- [17] V. Assassi, D. Baumann, and D. Green, *On Soft Limits of Inflationary Correlation Functions*, arXiv:1204.4207.
- [18] D. Seery, D. J. Mulryne, J. Frazer, and R. H. Ribeiro, *Inflationary perturbation theory is geometrical optics in phase space*, arXiv:1203.2635.
- [19] S. Hawking and G. Ellis, *The Large scale structure of space-time*. Cambridge University Press, 1973.
- [20] M. Visser, *van Vleck determinants: Geodesic focusing and defocusing in Lorentzian space-times*, *Phys.Rev.* **D47** (1993) 2395–2402, [arXiv:hep-th/9303020], [doi:10.1103/PhysRevD.47.2395].
- [21] C. DeWitt-Morette, *The Semiclassical Expansion*, *Annals Phys.* **97** (1976) 367–399, [doi:10.1016/0003-4916(76)90041-5, 10.1016/0003-4916(76)90041-5].
- [22] C. DeWitt-Morette and T. Zhang, *Path integrals and conservation laws*, *Phys.Rev.* **D28** (1983) 2503–2516, [doi:10.1103/PhysRevD.28.2503].
- [23] C. DeWitt-Morette, T. Zhang, and B. Nelson, *Caustic problems in quantum mechanics with applications to scattering theory*, *Phys.Rev.* **D28** (1983) 2526–2546, [doi:10.1103/PhysRevD.28.2526].
- [24] C. J. Jacobi *Journal für die reine und angewandte Mathematik* **17** (1837) 68–82, [doi:10.1515/crll.1837.17.68].
- [25] A. Lewis and A. Challinor, *Weak gravitational lensing of the CMB*, *Phys.Rept.* **429** (2006) 1–65, [arXiv:astro-ph/0601594], [doi:10.1016/j.physrep.2006.03.002].
- [26] A. Lewis, *The full squeezed CMB bispectrum from inflation*, arXiv:1204.5018.
- [27] D. J. Mulryne, D. Seery, and D. Wesley, *Moment transport equations for non-Gaussianity*, *JCAP* **1001** (2010) 024, [arXiv:0909.2256], [doi:10.1088/1475-7516/2010/01/024].
- [28] D. J. Mulryne, D. Seery, and D. Wesley, *Moment transport equations for the primordial curvature perturbation*, *JCAP* **1104** (2011) 030, [arXiv:1008.3159], [doi:10.1088/1475-7516/2011/04/030].
- [29] S. Yokoyama, T. Suyama, and T. Tanaka, *Primordial Non-Gaussianity in Multi-Scalar Slow-Roll Inflation*, *JCAP* **0707** (2007) 013, [arXiv:0705.3178], [doi:10.1088/1475-7516/2007/07/013].
- [30] S. Yokoyama, T. Suyama, and T. Tanaka, *Primordial Non-Gaussianity in Multi-Scalar Inflation*, *Phys.Rev.* **D77** (2008) 083511, [arXiv:0711.2920], [doi:10.1103/PhysRevD.77.083511].
- [31] M. Dias, J. Frazer, and A. R. Liddle, *Multifield consequences for D-brane inflation*, arXiv:1203.3792.
- [32] M. Andrews, *Evolution of moments over quantum wavepackets or classical clusters*, *Journal of Physics A* **18** (1985) 37, [doi:10.1088/0305-4470/18/1/014].
- [33] L. Ballentine, *Moment equations for probability distributions in classical and quantum mechanics*, *Physical Review A* **58** (1998) 1799–1809, [doi:10.1103/PhysRevA.58.1799].
- [34] K. Hepp, *The classical limit for quantum mechanical correlation functions*, *Communications in Mathematical Physics* **35** (Dec., 1974) 265–277.

- [35] D. H. Lyth and D. Seery, *Classicality of the primordial perturbations*, *Phys.Lett.* **B662** (2008) 309–313, [arXiv:astro-ph/0607647], [doi:10.1016/j.physletb.2008.03.010].
- [36] D. Seery, *Infrared effects in inflationary correlation functions*, *Class.Quant.Grav.* **27** (2010) 124005, [arXiv:1005.1649], [doi:10.1088/0264-9381/27/12/124005].
- [37] P. R. Jarnhus and M. S. Sloth, *de Sitter limit of inflation and nonlinear perturbation theory*, *JCAP* **0802** (2008) 013, [arXiv:0709.2708], [doi:10.1088/1475-7516/2008/02/013].
- [38] I. Huston and K. A. Malik, *Second Order Perturbations During Inflation Beyond Slow-roll*, *JCAP* **1110** (2011) 029, [arXiv:1103.0912].
- [39] M. Dias and D. Seery, *Transport equations for the inflationary spectral index*, *Phys.Rev.* **D85** (2012) 043519, [arXiv:1111.6544], [doi:10.1103/PhysRevD.85.043519].
- [40] D. H. Lyth and I. Zaballa, *A Bound concerning primordial non-Gaussianity*, *JCAP* **0510** (2005) 005, [arXiv:astro-ph/0507608], [doi:10.1088/1475-7516/2005/10/005].
- [41] F. Vernizzi and D. Wands, *Non-gaussianities in two-field inflation*, *JCAP* **0605** (2006) 019, [arXiv:astro-ph/0603799], [doi:10.1088/1475-7516/2006/05/019].
- [42] S. Renaux-Petel, *Combined local and equilateral non-Gaussianities from multifield DBI inflation*, *JCAP* **0910** (2009) 012, [arXiv:0907.2476], [doi:10.1088/1475-7516/2009/10/012].
- [43] K. A. Malik and D. Wands, *Cosmological perturbations*, *Phys.Rept.* **475** (2009) 1–51, [arXiv:0809.4944], [doi:10.1016/j.physrep.2009.03.001].
- [44] J. Garcia-Bellido and D. Wands, *Metric perturbations in two field inflation*, *Phys.Rev.* **D53** (1996) 5437–5445, [arXiv:astro-ph/9511029], [doi:10.1103/PhysRevD.53.5437].
- [45] S. Yokoyama, T. Suyama, and T. Tanaka, *Efficient diagrammatic computation method for higher order correlation functions of local type primordial curvature perturbations*, *JCAP* **0902** (2009) 012, [arXiv:0810.3053], [doi:10.1088/1475-7516/2009/02/012].
- [46] A. Mazumdar and L. Wang, *Separable and non-separable multi-field inflation and large non-Gaussianity*, arXiv:1203.3558.
- [47] J. Smidt, A. Amblard, C. T. Byrnes, A. Cooray, A. Heavens, *et al.*, *CMB Constraints on Primordial non-Gaussianity from the Bispectrum (f_{NL}) and Trispectrum (g_{NL} and τ_{NL}) and a New Consistency Test of Single-Field Inflation*, *Phys.Rev.* **D81** (2010) 123007, [arXiv:1004.1409], [doi:10.1103/PhysRevD.81.123007].
- [48] J. Elliston, D. Mulryne, D. Seery, and R. Tavakol, *Evolution of non-Gaussianity in multi-scalar field models*, *Int.J.Mod.Phys.* **A26** (2011) 3821–3832, [arXiv:1107.2270], [doi:10.1142/S0217751X11054280].
- [49] J. Elliston, L. Alabidi, I. Huston, D. J. Mulryne, and R. Tavakol, *Large trispectrum in two-field slow-roll inflation*, arXiv:1203.6844.
- [50] S. A. Kim, A. R. Liddle, and D. Seery, *Non-gaussianity in axion Nflation models*, *Phys.Rev.Lett.* **105** (2010) 181302, [arXiv:1005.4410], [doi:10.1103/PhysRevLett.105.181302].
- [51] C. T. Byrnes, K.-Y. Choi, and L. M. Hall, *Conditions for large non-Gaussianity in two-field slow-roll inflation*, *JCAP* **0810** (2008) 008, [arXiv:0807.1101], [doi:10.1088/1475-7516/2008/10/008].
- [52] J. Elliston, D. J. Mulryne, D. Seery, and R. Tavakol, *Evolution of f_{NL} to the adiabatic limit*, *JCAP* **1111** (2011) 005, [arXiv:1106.2153], [doi:10.1088/1475-7516/2011/11/005].
- [53] I. Huston and A. J. Christopherson, *Calculating Non-adiabatic Pressure Perturbations during Multi-field Inflation*, *Phys.Rev.* **D85** (2012) 063507, [arXiv:1111.6919], [doi:10.1103/PhysRevD.85.063507].
- [54] D. Mulryne, S. Orani, and A. Rajantie, *Non-Gaussianity from the hybrid potential*, *Phys.Rev.* **D84** (2011) 123527, [arXiv:1107.4739]. 9 pages, 7 figures.