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Inadequacy of Perfect-Reflector Models in Cavity QED for Systems with Low-Frequency Excitations

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It is shown that the model of perfectly reflecting boundaries widely employed in cavity QED is unsuitable for systems that have long-wavelength excitations. A prime example is a free charged particle near a reflecting wall. Modeling the wall as perfectly reflecting from the outset ignores evanescent waves that couple to the particle through virtual excitations at low energies, which can lead to errors in order of magnitude and even sign. The example of a free electron near an imperfectly reflecting wall characterized by a constant frequency-independent refractive index n is investigated in detail by determining its energy shift relative to an electron in vacuum through both nonrelativistic and relativistic calculations.

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Quantum electrodynamics (QED) has been spectacularly successful in describing the interaction of charges and electromagnetic fields. Arguably its two most famous tests are the Lamb shift in hydrogenlike atoms and the $g - 2$ anomalous magnetic moment of the electron. However, high-precision experiments do not and cannot measure these quantities in isolation, but are subject to a variety of apparatus-dependent corrections. One type of corrections comes about, for example, because of the presence of polarizable, magnetizable, or conducting materials in the apparatus, which alter the electromagnetic surroundings of the system under investigation compared to the same system in free space. In another strand of research developments, optical and microwave cavities have become invaluable tools for manipulating quantum systems, admitting a whole range of unconventional states and processes (cf., e.g., Ref. [1]). Both apparatus-dependent corrections in high-precision experiments and systems purposely confined in cavities are covered by what is now commonly known as the theory of cavity QED. The success of QED has been carried forward to cavity QED, for example, by the measurement of the Casimir-Polder force between an atom and a surface [2,3]. For $g - 2$ experiments the task of cavity QED has been the estimation of small corrections [4].

The model that is most frequently employed for describing the cavity walls is that of a perfectly reflecting surface; it appears to capture the essence of the impact that the cavity has on the electromagnetic field. The fact that real materials do not reflect perfectly is commonly ignored because physical intuition suggests that cavity-QED effects reach a maximum for perfectly reflecting walls, and for systems with imperfectly reflecting walls they differ merely by a minor numerical factor. This is, indeed, confirmed by explicit analytical calculation for an atom interacting with a single imperfectly reflecting wall

that is described by a constant refractive index n allowed to vary continuously between 1 and infinity [5]. For $n = 1$ the wall is electromagnetically transparent and the energy level shift in the atom vanishes, for $n \rightarrow \infty$ one recovers the result of Casimir and Polder for the level shift of an atom in the vicinity of a perfectly reflecting wall [2], and for $1 < n < \infty$ the result for the level shift smoothly interpolates between zero and the Casimir-Polder result. If one chooses to quantize the electromagnetic field by using photon annihilation and creation operators and normal modes, then a corroborating explanation of the underlying physical processes in the perfect-reflector approximation readily offers itself. Let us consider a setup where a medium is located at $z > 0$ and the space $z < 0$ is just vacuum. If the medium has a finite refractivity and reflectivity one has to include both left- and right-incident modes. Some of the right-incident modes suffer total internal reflection at the interface at $z = 0$, and their components on the vacuum side $z < 0$ are evanescent rather than traveling waves [6]. As the refractive index n increases, a larger and larger fraction of the right-incident modes becomes evanescent for $z < 0$, and simultaneously the decay lengths of the evanescent modes become shorter and shorter. In the limit $n \rightarrow \infty$ all right-incident modes are evanescent on the vacuum side, and their decay lengths are tending to zero. Thus, an atom that is a little distance away from the interface on the vacuum side feels nothing of these right-incident modes, and one might as well ignore them. This is exactly what the perfect-reflector model does from the outset; for a single wall it consists only of left-incident modes that satisfy the boundary conditions of a perfect reflector at $z = 0$: the electric field parallel to the interface and the magnetic field perpendicular to it must vanish. Comparing perfect-reflector calculations (cf., e.g., Sec. I.2 of [2]) with those for the same system but imperfectly reflecting boundaries (cf., e.g., Ref. [5]) one cannot fail to notice that

the perfect-reflector model is also most attractive because of its great technical simplicity. What is almost a back-of-the-envelope calculation in the perfect-reflector model becomes a major undertaking for weeks or months for imperfect reflectors.

It is the purpose of this Letter to point out that, apparently unbeknown to previous workers in the field, there are a variety of systems that cannot sensibly be described in cavity QED by using the perfect-reflector model. Indeed, if one compares, for example, the analogous calculation to that of Casimir and Polder [2] for a free electron near a perfectly reflecting wall [7] with the results of a calculation that models the wall as a non-dispersive dielectric of finite reflectivity, one finds disagreement of the first with the limit of unit reflectivity in the results of the latter, as we shall show in the following.

To make the argument we concentrate on the simplest possible system: a free electron a distance a away from a perfectly reflecting wall. Also for simplicity we shall model the imperfectly reflecting wall by a nondispersive and nonabsorbing dielectric half-space in the region $z > 0$. Thus, the dielectric is characterized by a single number, its refractive index, which is real, the same for all frequencies, and can be anything between 1 and infinity. The interaction between the electron and the electromagnetic field, which in turn couples to the reflecting wall, can be described by the minimal-coupling interaction

Hamiltonian

$$H_{\text{int}} = -\frac{e}{m} \mathbf{p} \cdot \mathbf{A} + V_{\text{image}}(\mathbf{r}), \quad (1)$$

where \mathbf{p} is the electron momentum operator, \mathbf{A} is the quantized electromagnetic field, $V_{\text{image}}(\mathbf{r})$ is the potential due to the Coulomb interaction of the electron with its image on the other side of the interface, and e and m are electron charge and mass. The energy shift due to the electrostatic interaction of the electron with its image is straightforward to work out and agrees with the classical value,

$$\Delta E_{\text{image}} = -\frac{e^2}{16\pi\epsilon_0 a} \frac{n^2 - 1}{n^2 + 1}. \quad (2)$$

In the limit $n \rightarrow \infty$ this agrees, of course, with the electrostatic interaction energy of the electron and its image on the other side of a perfectly reflecting boundary.

To evaluate the radiative energy shift due to the $\mathbf{p} \cdot \mathbf{A}$ term in H_{int} one needs to quantize the electromagnetic vector field. This is done most simply by an expansion in terms of photon annihilation and creation operators and normal modes, which each consist of an incident, a reflected, and a transmitted wave. For the dielectric we dispense with writing down the normal mode expansion, as it is lengthy but straightforward to derive [6], and refer the reader to, e.g., the Appendix of Ref. [8]. For a perfectly reflecting wall the mode expansion reads in Coulomb gauge

$$\begin{aligned} \mathbf{A}_{\text{TE}} &= \frac{1}{(2\pi)^{3/2}} \int \frac{d^3\mathbf{k}}{\sqrt{2\epsilon_0\omega}} [a_{\mathbf{k},\text{TE}} e^{-i\omega t + i\mathbf{k}_{\parallel}\cdot\mathbf{r}_{\parallel}} 2i(\hat{\mathbf{k}}_{\parallel} \wedge \hat{\mathbf{z}}) \sin(k_z z) + \text{H.c.}], \\ \mathbf{A}_{\text{TM}} &= \frac{1}{(2\pi)^{3/2}} \int \frac{d^3\mathbf{k}}{\sqrt{2\epsilon_0\omega}} \left[a_{\mathbf{k},\text{TM}} e^{-i\omega t + i\mathbf{k}_{\parallel}\cdot\mathbf{r}_{\parallel}} \left(2i \frac{k_z}{k} \hat{\mathbf{k}}_{\parallel} \sin(k_z z) - 2 \frac{k_{\parallel}}{k} \hat{\mathbf{z}} \cos(k_z z) \right) + \text{H.c.} \right], \end{aligned} \quad (3)$$

which is also what one gets if one takes the limit $n \rightarrow \infty$ in the mode expansion for the dielectric case.

The lowest-order contribution to the radiative energy shift is second order in e ,

$$\Delta E_{\text{rad}} = \frac{e^2}{m^2} \sum_{\psi', (\mathbf{k}, \sigma)} \int \frac{|\langle \psi' | 1_{\mathbf{k}, \sigma} | \mathbf{p} \cdot \mathbf{A} | \psi; 0 \rangle|^2}{\frac{\mathbf{p}^2}{2m} - [\frac{(\mathbf{p}-\mathbf{k})^2}{2m} - \hbar\omega]}. \quad (4)$$

To be able to evaluate this expression we make the no-recoil approximation; i.e., we neglect the change in the momentum of the electron after a collision with a photon,

so that the denominator of the above equation becomes simply $-\hbar\omega$. Then the calculation is straightforward. For the perfectly reflecting case we substitute the modes (3) and obtain

$$\Delta E_{\text{rad}}^{(\text{perf})} = \frac{e^2}{32\pi\epsilon_0 a m^2} \langle p_{\parallel}^2 \rangle - \frac{e^2}{16\pi\epsilon_0 a m^2} \langle p_z^2 \rangle, \quad (5)$$

which agrees with previous results for this model [7]. For the electron near the dielectric wall the calculation is a little longer but still straightforward. We get

$$\begin{aligned} \Delta E_{\text{rad}}^{(\text{diel})} &= (\Delta_{\parallel}^{\text{trav}} + \Delta_{\parallel}^{\text{ev}}) \langle p_{\parallel}^2 \rangle + (\Delta_z^{\text{trav}} + \Delta_z^{\text{ev}}) \langle p_z^2 \rangle \\ \text{with } \Delta_{\parallel}^{\text{trav}} &= \frac{e^2}{8\pi^2 \epsilon_0 m^2} \int_0^{\infty} d\omega \int_0^1 d\beta \cos(2\omega a \beta) \left(\frac{\sqrt{n^2 - 1 + \beta^2} - \beta}{\sqrt{n^2 - 1 + \beta^2} + \beta} - \beta^2 \frac{\sqrt{n^2 - 1 + \beta^2} - n\beta}{\sqrt{n^2 - 1 + \beta^2} + n\beta} \right), \\ \Delta_z^{\text{trav}} &= \frac{e^2}{4\pi^2 \epsilon_0 m^2} \int_0^{\infty} d\omega \int_0^1 d\beta \cos(2\omega a \beta) (1 - \beta^2) \frac{\sqrt{n^2 - 1 + \beta^2} - n\beta}{\sqrt{n^2 - 1 + \beta^2} + n\beta}, \\ \Delta_{\parallel}^{\text{ev}} &= \frac{e^2}{4\pi^2 \epsilon_0 m^2} \int_0^{\infty} d\omega \int_0^1 d\gamma \sqrt{n^2 - 1} \gamma \sqrt{1 - \gamma^2} e^{-2\omega a \sqrt{n^2 - 1} \gamma} \left[\frac{n^2(n^2 - 1)\gamma^2}{\gamma^2 - 1 - n^4\gamma^2} - 1 \right], \\ \Delta_z^{\text{ev}} &= \frac{e^2}{2\pi^2 \epsilon_0 m^2} \int_0^{\infty} d\omega \int_0^1 d\gamma \sqrt{n^2 - 1} \gamma \sqrt{1 - \gamma^2} e^{-2\omega a \sqrt{n^2 - 1} \gamma} \frac{n^2[(n^2 - 1)\gamma^2 + 1]}{\gamma^2 - 1 - n^4\gamma^2}. \end{aligned} \quad (6)$$

Here the ω integral is over the frequency of the (virtual) photon exchanged between atom and wall, and the β and γ integrals stem from integrations over angles of incidence. Doing the ω integrals first one can carry out all integrations analytically and finds

$$\begin{aligned} \Delta_{\parallel}^{\text{trav}} &= \frac{e^2}{32\pi\epsilon_0 m^2 a}, & \Delta_z^{\text{trav}} &= \frac{e^2}{16\pi\epsilon_0 m^2 a}, & \Delta_{\parallel}^{\text{ev}} &= -\frac{e^2}{32\pi\epsilon_0 m^2 a} \frac{2n^4 + n^2 + 1}{(n^2 + 1)^2}, \\ \Delta_z^{\text{ev}} &= -\frac{e^2}{16\pi\epsilon_0 m^2 a} \frac{(3n^2 + 1)n^2}{(n^2 + 1)^2}, & \Delta E_{\text{rad}}^{(\text{diel})} &= -\frac{e^2}{32\pi\epsilon_0 m^2 a} \frac{n^2(n^2 - 1)}{(n^2 + 1)^2} \langle p_{\parallel}^2 \rangle - \frac{e^2}{16\pi\epsilon_0 m^2 a} \frac{2n^4 - n^2 - 1}{(n^2 + 1)^2} \langle p_z^2 \rangle. \end{aligned} \quad (7)$$

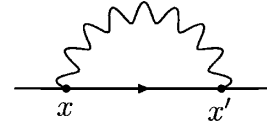
The limit $n \rightarrow \infty$ of this result obviously does not reproduce the radiative shift (5) calculated in the perfect-reflector model: the coefficient of $\langle p_{\parallel}^2 \rangle$ has a different sign and that of $\langle p_z^2 \rangle$ differs by a factor of 2.

One can compare the two calculations step by step in an effort to try and find the reason for this discrepancy. In the perfect-reflector model right-incident modes are neglected from the outset, and thus the shifts due to evanescent modes do not arise. That this is a serious error is obvious from the fact that $\Delta_{\parallel}^{\text{ev}}$ and Δ_z^{ev} do not vanish in the limit $n \rightarrow \infty$, although in the identical calculation for the energy level shift of an atom, which gives the same integrals as in Eq. (6) but with an additional factor $\omega/(\omega + \omega_{ji})$ incorporating the atomic transition frequency ω_{ji} , they do vanish in this limit [5]. One could believe that, having ignored the evanescent modes, one should at least find agreement between the shifts due to the traveling modes $\Delta_{\parallel}^{\text{trav}}$ and Δ_z^{trav} for $n \rightarrow \infty$ and the perfect-reflector result (5). This is, indeed, the case for the parallel direction but not for the z direction for which the shifts disagree in sign. One can see why by inspecting (6): Calculating the shift from the perfect-reflector modes (3) leads to identical expressions but with the limit $n \rightarrow \infty$ taken in the integrand. For $\Delta_{\parallel}^{\text{trav}}$ this goes well, but for Δ_z^{trav} the limit $n \rightarrow \infty$ does not commute with the limit $\beta \rightarrow 0$ of the angular integration, which corresponds to grazing incidence. Thus, the perfect-reflector model does not just go wrong by ignoring the evanescent modes but also mishandles traveling modes at grazing incidence.

One should, of course, be careful and examine all possible sources of error in the calculation for the dielectric wall. By placing a cutoff on the ω integration, one quickly convinces oneself that the delta functions arising from the ω integration in $\Delta_{\parallel}^{\text{trav}}$ and Δ_z^{trav} cause no problems. Likewise, one can restore the denominator of (4) and see that the no-recoil approximation neglects terms of order $1/am$ (the ratio of the electron's Compton wavelength to the distance from the wall) and $\ln(am)/am$ but otherwise does not affect the result and, in particular, not the limit $n \rightarrow \infty$.

Since the nonrelativistic calculation offers no clue as to the origin of the discrepancy between the perfect-reflector and dielectric models, it makes sense to calculate the energy shift in relativistic quantum field theory. There the shift manifests itself in boundary-dependent corrections to the self-energy operator. The Feynman diagram representing the self-energy in the lowest non-

vanishing order is



The electron line (\longrightarrow) is unaffected by the presence of the boundary and represents just the standard free electron propagator $S^C(x - x')/i$. The photon line ($\sim\sim\sim$) stands for the photon propagator $iD_{\mu\nu}^C(x, x')$. The vertex is the standard QED vertex $e\gamma^\mu$. The expression for the electron self-energy is thus

$$\Sigma(x, x') = -ie^2 \gamma^\mu S^C(x - x') \gamma^\nu D_{\mu\nu}^C(x, x'). \quad (8)$$

The photon propagator is the only part that is sensitive to the presence of the cavity wall; it can be assembled from the normal modes for the electromagnetic field [5,6] by taking the vacuum expectation value of the time-ordered product of two field operators,

$$D^{\mu\nu C}(x, x') = -i \langle 0 | T A^\mu(x) A^\nu(x') | 0 \rangle.$$

If both points x and x' are outside the dielectric (i.e., for $x_3, x'_3 < 0$) the photon propagator can, just like in the case of a perfectly conducting cavity [9], be written as a sum of the propagator of the free photon field $D^{C(0)}(x - x')$ and a boundary-dependent correction $\bar{D}(x, x')$. Thus, to calculate the boundary-dependent part of the electron self-energy we need to substitute $\bar{D}(x, x')$ for $D^C(x, x')$ in (8). The resulting expression is translation invariant in all but the z (or 3) direction, so that we have the Fourier representation

$$\Sigma(x, x') = \int \frac{d^4 q}{(2\pi)^4} e^{-iq(x-x')} \tilde{\Sigma}(q, x_3 + x'_3). \quad (9)$$

We make a few basic assumptions and simplifications for evaluating $\tilde{\Sigma}(q, x_3 + x'_3)$: (i) since the electron is localized a distance a away from the wall, we approximate $x_3 + x'_3 \approx -2a$, but we do not approximate $x_3 - x'_3$; (ii) we calculate on the average mass shell of the external electron line, which is equivalent to the no-recoil approximation in the nonrelativistic calculation; (iii) we evaluate the expressions asymptotically for a distance a that is much larger than the Compton wavelength of the electron; and (iv) in scalar denominators we neglect the momentum of the electron parallel to the wall compared to its rest mass.

The calculation is quite lengthy. Furthermore, it is complicated by the fact that the convergence of the integrals is rather fragile and does not tolerate approximation in the integrands. The trick is to resort to methods of asymptotic analysis that are specially suited to two-dimensional Fourier integrals [10]. We shall report the details of the calculation elsewhere [11] and here just cite the results. To lowest order in $1/aq_0$ (the ratio of the electron's Compton wavelength to the distance a from the wall), we find

$$\tilde{\Sigma}(q, -2a) = -\frac{e^2}{32\pi\epsilon_0 q^0 a} \left[\frac{n^2(n^2-1)}{(n^2+1)^2} \mathbf{q}^{\parallel} \cdot \boldsymbol{\gamma}^{\parallel} + 2 \frac{2n^4 - n^2 - 1}{(n^2+1)^2} q^3 \gamma^3 + 2 \frac{n^2-1}{n^2+1} q^0 \gamma^0 \right]. \quad (10)$$

The first term in the brackets is the radiative shift for motion parallel to the wall, the second is the one for motion perpendicular to it, and the third is the shift due to electrostatic interactions of the electron with its image. Substituting the self-energy correction (10) into the Dirac equation for the electron and taking the nonrelativistic limit, one sees immediately that the above result reproduces the energy shifts (7) and (2) of the nonrelativistic calculation.

The field-theoretical calculation gives important insight into the limit $n \rightarrow \infty$, which should correspond to perfect reflectivity, because it reveals that the nature of the point $k_0 = 0$ subtly changes in this limit. On closer examination it turns out that the limits $k_0 \rightarrow 0$ and $n \rightarrow \infty$ do not commute. In other words, it is low-frequency photons that cause the problem, and whenever a system has low-frequency excitations the “perfect-reflector” model that assumes $n \rightarrow \infty$ from the outset is physically inapplicable as it ignores a part of the photon phase space that is physically significant, namely, evanescent waves. We shall demonstrate that this is, indeed, the case by examining, for example, the evanescent part of the shift contributed by TE (transverse electric or s polarized) waves in the nonrelativistic calculation. Putting a lower cutoff μ on the integration over the photon frequency ω we have

$$\Delta_{\parallel, \text{TE}}^{\text{ev}} = -\frac{e^2}{4\pi^2 \epsilon_0 m^2} \times \int_{\mu}^{\infty} d\omega \int_0^1 d\gamma \sqrt{n^2-1} \gamma \sqrt{1-\gamma^2} e^{-2\omega a \sqrt{n^2-1} \gamma}. \quad (11)$$

The integration over frequency is elementary, and the subsequent integration over γ can be expressed as the difference between a Bessel and a Struve function [12]

$$\Delta_{\parallel, \text{TE}}^{\text{ev}} = -\frac{e^2}{16\pi m^2 a} \left(\frac{1}{\xi} [I_1(\xi) - L_1(\xi)] \right)_{\xi=2a\mu\sqrt{n^2-1}}. \quad (12)$$

Using the asymptotic properties of $I_1(\xi)$ and $L_1(\xi)$ [13], we obtain the two different results

$$\lim_{\mu \rightarrow 0} \Delta_{\parallel, \text{TE}}^{\text{ev}} = -\frac{e^2}{32\pi m^2 a}, \quad (13)$$

$$\lim_{n \rightarrow \infty} \Delta_{\parallel, \text{TE}}^{\text{ev}} = -\lim_{n \rightarrow \infty} \frac{e^2}{16\pi^2 m^2 a^2 \mu \sqrt{n^2-1}} \rightarrow 0.$$

This exemplifies our observation that the perfect-reflector limit $n \rightarrow \infty$ is not interchangeable with the lower limit $\mu \rightarrow 0$ of the integration over photon frequencies.

In summary, we have shown that the “perfect-reflector” model is inappropriate for systems with a gapless excitation spectrum. As regards experiments, the most important consequences of this finding concern cavity-QED corrections to the anomalous magnetic moment, resulting from vertex corrections, which hitherto have been calculated only in the “perfect-reflector” model [4]. In order to obtain an estimate of the $g-2$ corrections for unbound particles in physically realistic systems one must calculate the vertex corrections for an electron near an imperfect boundary [11].

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