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Samuel M. Leach,[1] Misao Sasaki,[2] David Wands,[3] and Andrew R. Liddle[1]
[1]Astronomy Centre, University of Sussex, Brighton, BN1 9QJ, United Kingdom
[2]Department of Earth and Space Science, Graduate School of Science, Osaka University, Toyonaka 560-0043, Japan
[3]Relativity and Cosmology Group, School of Computer Science and Mathematics, University of Portsmouth, Portsmouth PO1 2EG, United Kingdom

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We show that there exists a simple mechanism which can enhance the amplitude of curvature perturbations on superhorizon scales, relative to their amplitude at horizon crossing, even in some single-field inflation models. We give a criterion for this enhancement in general single-field inflation models; the condition for a significant effect is that the quantity $a\dot{\phi}/H$ become sufficiently small, as compared to its value at horizon crossing, for some time interval during inflation.

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I. INTRODUCTION

The standard, single-field, slow-roll inflation model predicts that the curvature perturbation on comoving hypersurfaces, $\mathcal{R}_c$, remains constant from soon after the scale crosses the Hubble horizon, giving the formula [1,2]

$$\mathcal{R}_c \approx \mathcal{R}_c(t_k) \approx \left(\frac{H^2}{2\pi\dot{\phi}} \right)_{k=aH} \tag{1}$$

where H is the Hubble parameter, $\dot{\phi}$ is the time derivative of the inflation field ϕ, and t_k is a time shortly after horizon crossing. However, one may consider a model in which slow-roll is violated during inflation. Recently, Leach and Liddle [3] studied the behavior of the curvature perturbation in a model in which inflation is temporarily suspended, finding a large amplification of the curvature perturbation relative to its value at horizon crossing for a range of scales extending significantly beyond the Hubble horizon.

In this short paper, we consider single-field inflation models and analyze the general behavior of the curvature perturbation on superhorizon scales. We show analytically when and how this large enhancement occurs. We find that a necessary condition is that the quantity $z \equiv a\dot{\phi}/H$ becomes smaller than its value at the time of horizon crossing. We then present a couple of integrals which involve the above quantity and which give a criterion for enhancement.

II. ENHANCEMENT OF THE CURVATURE PERTURBATION

We assume a background metric of the form

$$\begin{aligned} ds^2 &= -dt^2 + a^2(t)\delta_{ij}dx^i dx^j \\ &= a^2(\eta)(-d\eta^2 + \delta_{ij}dx^i dx^j). \end{aligned} \tag{2}$$

On this background the growing mode solution of the curvature perturbation on comoving hypersurfaces is known to stay constant in time on superhorizon scales in the absence of any entropy perturbation [1,2,4–6]. This follows from the equation for $\mathcal{R}_c$:

$$\mathcal{R}_c'' + 2\frac{z'}{z}\mathcal{R}_c' + k^2\mathcal{R}_c = 0, \tag{3}$$

where the prime denotes the conformal time derivative, $d/d\eta$, and $z = a\dot{\phi}/H$. One readily sees that on superhorizon scales, when the last term can be neglected, there exists a solution with $\mathcal{R}_c$ constant, which corresponds to the growing adiabatic mode.

However, this does not necessarily mean that $\mathcal{R}_c$ must stay constant in time after its scale crosses the Hubble horizon. In fact, if the contribution of the other independent mode (i.e. the decaying mode) to $\mathcal{R}_c$ is large at horizon crossing, $\mathcal{R}_c$ will not become constant until the decaying mode dies out. The important point here is that the decaying mode is, by definition, the mode that decays asymptotically in the future, but it does not necessarily start to decay right after horizon crossing. In what follows, we show that there indeed exists a situation in which the decaying mode can stay almost constant for a while after the horizon crossing before it starts to decay. In such a case, the contribution of the two modes to the curvature perturbation is found to almost cancel at horizon crossing. This gives a small initial amplitude of $\mathcal{R}_c$, but results in a large final amplitude for $\mathcal{R}_c$ after the decaying mode becomes negligible.

Let $u(\eta)$ be a solution of Eq. (3) for any given k. For much of the following discussion it is not necessary to specify the nature of the solution u, but for clarity let us identify it straightaway as the late-time asymptotic solution at η_* (taking η_* for instance as the end of inflation). For any other solution, $v(\eta)$, independent of $u(\eta)$, it is easy to show from Eq. (3) that the Wronskian $W = v'u - u'v$ obeys

$$W' = -2\frac{z'}{z}W, \tag{4}$$

and hence $W \propto 1/z^2$. Therefore we have

$$\left(\frac{v}{u}\right)' = \frac{W}{u^2} \propto \frac{1}{z^2u^2}. \tag{5}$$

Hence the decaying mode, v, which vanishes as $\eta\rightarrow\eta_*$, may be expressed in terms of the growing mode, u, as

$$v(\eta)\propto u(\eta)\int_{\eta_*}^{\eta}\frac{d\eta'}{z^2(\eta')u^2(\eta')}.$$

Without loss of generality, we may assume $v=u$ at some initial epoch, which we take to be shortly after horizon crossing, $\eta=\eta_k$ $(<\eta_*)$. Then v is expressed as

$$v(\eta)=u(\eta)\frac{D(\eta)}{D(\eta_k)}, \tag{6}$$

where

$$D(\eta)=3\mathcal{H}_k\int_{\eta}^{\eta_*}d\eta'\frac{z^2(\eta_k)u^2(\eta_k)}{z^2(\eta')u^2(\eta')}, \tag{7}$$

and, for convenience, the conformal Hubble parameter $\mathcal{H}_k=(a'/a)_k$ at $\eta=\eta_k$ is inserted to make D dimensionless. In terms of u and v, the general solution of $\mathcal{R}_c$ may be expressed as

$$\mathcal{R}_c(\eta)=\alpha u(\eta)+\beta v(\eta), \tag{8}$$

where α and β are constants and we assume $\alpha+\beta=1$ without loss of generality. Thus, if the amplitude of $\mathcal{R}_c$ at horizon crossing differs significantly from that of the growing mode, $\alpha u(\eta_k)$, it can only be because $|\beta|\gg1$.

Using Eq. (6) and noting $\alpha+\beta=1$, $\mathcal{R}_c$ and $\mathcal{R}_c'$ at the initial epoch $\eta=\eta_k$ are given by

$$\mathcal{R}_c(\eta_k)=u(\eta_k),$$
$$\mathcal{R}_c'(\eta_k)=u'(\eta_k)-\frac{3(1-\alpha)\mathcal{H}_k u(\eta_k)}{D_k}, \tag{9}$$

where $D_k=D(\eta_k)$. Then α can be expressed in terms of the initial conditions as

$$\alpha=1+D_k\frac{1}{3\mathcal{H}_k}\left[\frac{\mathcal{R}_c'}{\mathcal{R}_c}-\frac{u'}{u}\right]_{\eta=\eta_k}. \tag{10}$$

If we assume $\mathcal{R}_c(\eta_k)$ to be a complex amplitude determined by an initial vacuum state for quantum fluctuations, then $\mathcal{R}_c'/(\mathcal{H}_k\mathcal{R}_c)$ at the time of horizon crossing will be at most of order unity. This implies that $|\alpha|$, and hence $|\beta|$, can become large if $D_k\gg1$ or $(D_k/\mathcal{H}_k)|u'/u|\gg1$.

III. LONG-WAVELENGTH APPROXIMATION

Equation (3) can be written in terms of the canonical field perturbation, $Q=z\mathcal{R}$, as

$$Q''+\left(k^2-\frac{z''}{z}\right)Q=0. \tag{11}$$

From this we see that the general solution for $k^2\ll|z''/z|$ is given approximately by

$$\mathcal{R}_c\approx A+B\int_{\eta_*}^{\eta}\frac{d\eta'}{z^2(\eta')}, \tag{12}$$

where A and B are constants.

The requirement that $v\rightarrow0$ as $\eta\rightarrow\eta_*$ uniquely identifies the decaying mode as proportional to $\int_{\eta}^{\eta_*}d\eta'/z^2(\eta')$ in Eq. (12), but one is always free to include arbitrary contributions from the decaying mode in the growing mode. Nonetheless, it is convenient to identify the constant A in Eq. (12) as an approximate solution for the growing mode, u, on sufficiently large scales. Thus we put the lowest order solutions for u and v as

$$u_0=\text{const},\quad v_0=u_0\frac{D(\eta)}{D_k}, \tag{13}$$

where, and in the rest of the paper, $D(\eta)$ is the integral given by Eq. (7) but with u approximated by u_0:

$$D(\eta)\approx3\mathcal{H}_k\int_{\eta}^{\eta_*}d\eta'\frac{z^2(\eta_k)}{z^2(\eta')}. \tag{14}$$

As long as slow roll is satisfied, the long-wavelength solutions u_0 and v_0 used above are accurate enough for super-horizon modes. However, corrections to the growing mode u due to the effect of finite wave number k may become substantial if there is an epoch at which slow roll is violated [7] and it becomes important to include these in the definition of the growing mode.

In order to include the effect of a finite wave number, k, the growing mode solution can be rewritten in the form

$$u(\eta)=\sum_{n=0}^{\infty}u_n(\eta)k^{2n}, \tag{15}$$

where Eq. (3) requires

$$u_{n+1}''+2\frac{z'}{z}u_{n+1}'=-u_n. \tag{16}$$

Note that, starting from $u_0=\text{const}$, each successive correction obtained as a solution of the second-order equation for u_{n+1} has two arbitrary constants of integration. In particular the $O(k^2)$ correction to u_0 can be written as

$$u\approx u_0+[C_1+C_2D(\eta)+F(\eta)]u_0, \tag{17}$$

where

$$F(\eta)=k^2\int_{\eta}^{\eta_*}\frac{d\eta'}{z^2(\eta')}\int_{\eta_k}^{\eta'}z^2(\eta'')d\eta''. \tag{18}$$

The $O(k^2)$ effect cannot be neglected if this integral becomes larger than unity. As may be guessed from the form of the integral, such a situation appears if there is an epoch during which $z^2(\eta)\ll z^2(\eta_k)$. To be specific, let us assume $z(\eta)\ll z_k=z(\eta_k)$ for $\eta>\eta_0(>\eta_k)$. Then $F(\eta)$ will become large and approximately constant for $\eta_k<\eta<\eta_0$ and will decay when $\eta > \eta_0$. This behavior is quite similar to the behavior of the lowest order decaying mode $v_0(\eta)$ given in Eq. (13). In other words, the growing mode can be substantially contaminated by a component that behaves like the decaying mode, and it can no longer be assumed as being constant on large scales.

However, we can use our freedom to pick the two arbitrary constants of integration $C_1 = 0$ and $C_2 = -F_k/D_k$, where $F_k = F(\eta_k)$, in the solution for u so that

$$u \approx \left[1 - F_k \frac{D(\eta)}{D_k} + F(\eta)\right] u_0, \tag{19}$$

and then $u \to u_0$ when $\eta \to \eta_k$ and again when $\eta \to \eta_*$. Thus, as far as is possible, the growing mode solution may still be considered approximately constant on superhorizon scales.[1]

In order to evaluate the enhancement coefficient α in Eq. (10) we require u'/u at $\eta = \eta_k$ which will be non-zero. We find

$$\left[\frac{u'}{u}\right]_{\eta = \eta_k} \approx \frac{3\mathcal{H}_k}{D_k} F_k. \tag{20}$$

Then Eq. (10) for α may be approximated as

$$\alpha \approx 1 + \frac{D_k}{3\mathcal{H}_k} \frac{\mathcal{R}_c'}{\mathcal{R}_c} - F_k, \tag{21}$$

where D_k and F_k are those given in the long-wavelength approximation, Eqs. (14) and (18), and for definiteness we will take $(k/\mathcal{H}_k)^2 = 0.1$.

In slow-roll inflation, the time variation of $\dot{\phi}$ is small and z increases rapidly, approximately proportional to the scale factor a. Hence neither the integral D_k nor F_k can become large. Soon after horizon crossing $\mathcal{R}_c'/\mathcal{R}_c \ll \mathcal{H}$, so that $\alpha \approx 1$ and the standard result $\mathcal{R}_c(\eta) \approx \mathcal{R}_c(\eta_k)$ holds. However, if the slow-roll condition is violated, $\dot{\phi}$ may become very small and z may decrease substantially to give a large value of D_k and F_k. (The case where z actually crosses zero is treated separately in the Appendix.) Then at late times, we have

$$\mathcal{R}_c(\eta_*) = \alpha u(\eta_*) \approx \alpha u(\eta_k) = \alpha \mathcal{R}_c(\eta_k). \tag{22}$$

Thus the final amplitude will be enhanced by a factor $|\alpha|$, which can be large if $D_k \gg 1$ or $F_k \gg 1$.

IV. STAROBINSKY'S MODEL

As an example we consider the model discussed by Starobinsky [8], where the potential has a sudden change in its slope at $\phi = \phi_0$ such that

$$V(\phi) = \begin{cases} V_0 + A_+(\phi - \phi_0) & \text{for } \phi > \phi_0, \\ V_0 + A_-(\phi - \phi_0) & \text{for } \phi < \phi_0. \end{cases} \tag{23}$$

If the change in the slope is sufficiently abrupt [8], then the slow-roll can be violated and for $A_+ > A_- > 0$ the field enters a friction-dominated transient (or ''fast-roll'') solution with $\ddot{\phi} \approx -3H\dot{\phi}$ [3] until the slow-roll conditions are once again satisfied:

$$3H_0\dot{\phi} = \begin{cases} -A_+ & \text{for } \phi > \phi_0, \\ -A_- - (A_+ - A_-)e^{-3H_0\Delta t} & \text{for } \phi < \phi_0. \end{cases} \tag{24}$$

For $\phi < \phi_0$ we have

$$z \simeq -a_0 \frac{A_- e^{H_0\Delta t} + (A_+ - A_-)e^{-2H_0\Delta t}}{3H_0^2}. \tag{25}$$

This decreases rapidly to a minimum value $z_{\min} \approx (A_-/A_+)^{2/3} z_0$ for $A_+ \gg A_-$, which can cause a significant change in $\mathcal{R}_c$ on superhorizon scales.

First let us discuss the behavior of $D(\eta)$. For a mode that leaves the horizon in the slow-roll regime z grows proportional to a while $\phi > \phi_0$, so that the integrand of $D(\eta)$ remains small. Hence $D(\eta) \approx D_k$, which implies $\mathcal{R}_c(\eta) \approx \mathcal{R}_c(\eta_k)$ until $\eta = \eta_0$. Even after the slow-roll condition is violated, $\mathcal{R}_c(\eta)$ still remains constant until z becomes smaller than z_k and the integrand of $D(\eta)$ becomes large again. Then $D(\eta)$ may decrease rapidly, until $\mathcal{R}_c$ approaches the asymptotic value for $\eta \to \eta_*$, given by Eq. (22). Substituting the above solution for z in Eq. (25) into Eq. (14) we obtain

$$D_k \approx \begin{cases} 1 + \dfrac{A_+}{A_-}\left(\dfrac{k}{\mathcal{H}_k}\dfrac{\mathcal{H}_0}{k}\right)^3 & \text{for } k > (k/\mathcal{H}_k)\mathcal{H}_0, \\ 1 + \dfrac{A_+}{A_-}\left(\dfrac{\mathcal{H}_k}{k}\dfrac{k}{\mathcal{H}_0}\right)^3 & \text{for } k < (k/\mathcal{H}_k)\mathcal{H}_0, \end{cases} \tag{26}$$

which shows that for $A_+/A_- \gg 1$, we have $D_k \gg 1$ on scales $(A_-/A_+)^{1/3}\mathcal{H}_0 \lesssim k \lesssim (A_+/A_-)^{1/3}\mathcal{H}_0$.

A similar behavior is expected for $F(\eta)$. Using again the solution for z in Eq. (25), the double integral in Eq. (18) is evaluated to give

$$F_k \approx \begin{cases} \dfrac{1}{15}\dfrac{A_+}{A_-}\left(\dfrac{k}{\mathcal{H}_k}\dfrac{\mathcal{H}_0}{k}\right)^5 & \text{for } k > (k/\mathcal{H}_k)\mathcal{H}_0, \\ \dfrac{2}{5}\dfrac{A_+}{A_-}\left(\dfrac{k}{\mathcal{H}_0}\right)^2 & \text{for } k < (k/\mathcal{H}_k)\mathcal{H}_0. \end{cases} \tag{27}$$

Thus $F_k \gg 1$ for $(A_-/A_+)^{1/2}\mathcal{H}_0 \lesssim k \lesssim (A_+/A_-)^{1/5}\mathcal{H}_0$.

Combining the effects of D_k and F_k, we see that the correction due to F_k dominates on scales $k < \mathcal{H}_0$ and D_k on scales $k > \mathcal{H}_0$. In particular the spiky dip in the spectrum seen in Fig. 1 at $k \sim (A_-/A_+)^{1/2}\mathcal{H}_0$ is caused by F_k; i.e., it is the $O(k^2)$ effect in the perturbation equation (3). To summarize, the curvature perturbation is significantly affected by the dis-

[1] Equivalently, one can view this as exploiting the ambiguity in the definition of the growing mode u_0 to include part of the decaying mode v_0 which as far as possible cancels out the variation in u_1.

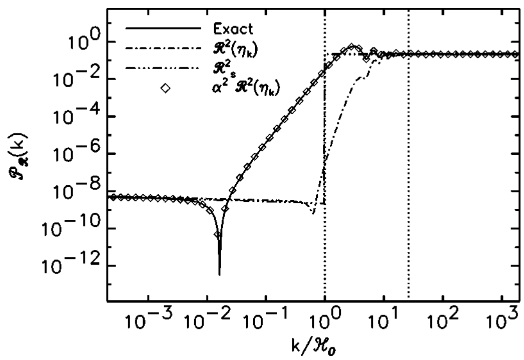

FIG. 1. The power spectrum for the Starobinsky model [8] with $A_+/A_-=10^4$. Plotted are the exact asymptotic value of the curvature perturbation $\mathcal{R}_c^2(\eta_*)$, the horizon-crossing value $\mathcal{R}_c^2(\eta_k)$, and the enhanced horizon-crossing amplitude $\alpha^2\mathcal{R}_c^2(\eta_k)$ using the long-wavelength approximation. The range of scales between the dotted lines corresponds to modes leaving the horizon during the transient epoch, defined as the region where $z'/z<0$. Also plotted is the slow-roll amplitude $\mathcal{R}_s^2$ given by Eq. (31).

continuity at $\phi\sim\phi_0$ even on superhorizon scales from $k\sim(A_-/A_+)^{1/2}\mathcal{H}_0$ up to $k\sim(A_+/A_-)^{1/3}\mathcal{H}_0$.

Similar behavior was observed in the model studied by Leach and Liddle [3] for false-vacuum inflation with a quartic self-interaction potential [9], whose power spectrum is shown in Fig. 2. In this model there is no discontinuity in the potential, so the oscillations seen in Starobinsky's model are washed out.

In both cases our analytic estimate of the enhancement on superhorizon scales is in excellent agreement with the numerical results on all scales. Thus our approximate formula for α given by Eq. (21) will be very useful for estimation of the curvature perturbation spectrum in general models of single-field inflation.

It may be noted that in the Leach and Liddle model the long-wavelength condition, $k^2\ll|z''/z|$, is violated for modes $k<\mathcal{H}_0$. It is rather surprising that our long-wavelength approximation still works very well for this model.

V. INVARIANT SPECTRA

A striking feature of these results is that the modes which leave the horizon during the transient regime share the same underlying spectrum as that produced during the subsequent slow-roll era. This is a manifestation of the ''duality invariance'' of perturbation spectra produced in apparently different inflationary scenarios [10].

Starting from a particular asymptotic background solution, $z(\eta)$, one finds a two parameter family of solutions

$$\tilde{z}(\eta)=C_1 z(\eta)+C_2 z(\eta)\int_\eta^{\eta_*}\frac{d\eta'}{z^2(\eta')}, \tag{28}$$

which leave z''/z unchanged in the perturbation equation (11) and thus generate the same perturbation spectrum from vacuum fluctuations [10] (up to the overall normalization C_1). The variable z itself obeys the second-order equation

$$z''+\left(a^2\frac{d^2V}{d\phi^2}-5\mathcal{H}^2+\mathcal{H}'+2\frac{\mathcal{H}''}{\mathcal{H}}-2\frac{\mathcal{H}'^2}{\mathcal{H}^2}\right)z=0. \tag{29}$$

Thus for a weakly interacting field ($d^2V/d\phi^2\approx$const) in a quasi–de Sitter background ($H\approx$const) the equation can be approximated by the linear equation of motion

$$z''+\left(a^2\frac{d^2V}{d\phi^2}-2\mathcal{H}^2\right)z\approx0. \tag{30}$$

The general solution $\tilde{z}(\eta)$ is related to the asymptotic late-time solution $z(\eta)$ by the expression given in Eq. (28).

This means that the usual slow-roll result [taking $\dot{\phi}\approx-(dV/d\phi)/3H$] for the amplitude of the curvature perturbations in Eq. (1),

$$\mathcal{R}_c\approx-\left(\frac{3H^3}{2\pi(dV/d\phi)}\right)_{k=\mathcal{H}}, \tag{31}$$

may continue to be a useful approximation even when the actual background scalar field solution at horizon crossing is no longer described by slow roll, as was noted previously by Seto, Yokoyama and Kodama [11] and seen in our figures.

VI. SUMMARY

We have found that in single-field models of inflation the curvature perturbation can be enhanced on superhorizon scales, provided that $a\dot{\phi}/H$ becomes small compared to its value at horizon crossing. Violation of slow roll is a necessary, but not sufficient, condition for this to take place. We have presented a quantitative criterion for this enhancement, namely that either of the integrals D_k and F_k defined by Eqs. (7) and (18), respectively, become larger than unity. In the long-wavelength approximation ($k^2\ll|z''/z|$) these integrals are expressed in terms of the background quantity $z=a\dot{\phi}/H$, as given by Eqs. (14) and (18), so an analytical formula for the final curvature perturbation amplitude may be derived without assuming slow roll inflation. In the case of a weakly self-interacting field in de Sitter inflation we recover the usual slow-roll formula for the amplitude of the scalar perturbations even when the background solution is far from slow-roll at horizon crossing.

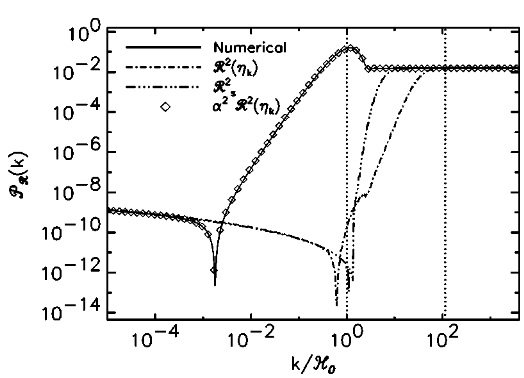

FIG. 2. Power spectrum for the false-vacuum quartic model as in Fig. 1.

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APPENDIX: IF $\dot{\phi}$ CROSSES ZERO

The case when $\dot{\phi}$ and hence z changes its sign can be treated as follows. For simplicity, let us assume z changes the sign only once at $\eta=\eta_0$. Since the integral F_k is still well defined in this case, we focus on the integral D_k.

In the vicinity of $\eta=\eta_0$, z can be expressed as $z=z_0'(\eta-\eta_0)$ where $z_0'=z'(\eta_0)$. Hence the equation for $\mathcal{R}_c$ becomes

$$\left[\frac{d^2}{d\eta^2}+\frac{2}{\eta-\eta_0}\frac{d}{d\eta}+k^2\right]\mathcal{R}_c=0. \tag{A1}$$

The two independent solutions can be found as

$$u\approx C\left(1-\frac{1}{6}k^2(\eta-\eta_0)^2+\cdots\right), \tag{A2}$$

$$v\approx D\left(\frac{1}{\eta-\eta_0}-\frac{1}{2}k^2(\eta-\eta_0)+\cdots\right). \tag{A3}$$

It is apparent that u should be chosen as the growing mode, and it remains constant across the epoch $\eta=\eta_0$.

We require v to describe the decaying mode. As before, we consider an integral expression of v in terms of z^2 and u. Then

$$v=u\int_\eta^{\eta_*}\frac{d\eta'}{z^2u^2}\approx u\int_\eta^{\eta_*}\frac{d\eta'}{z^2C^2} \tag{A4}$$

for $\eta>\eta_0$. This v behaves in the limit $\eta\to\eta_0+0$ as

$$v\sim\frac{1}{z_0'^{\,2}C^2(\eta-\eta_0)}. \tag{A5}$$

This should be extended to the region $\eta<\eta_0$ as the solution (A3), which implies

$$v=u\lim_{\epsilon\to 0}\left(\int_\eta^{\eta_0-\epsilon}\frac{d\eta'}{z^2u^2}+\int_{\eta_0+\epsilon}^{\eta_*}\frac{d\eta'}{z^2u^2}-\frac{2}{z_0'^{\,2}C^2\epsilon}\right), \tag{A6}$$

for $\eta<\eta_0$. Thus introducing the function $\tilde{D}(\eta)$ by

$$\frac{\tilde{D}(\eta)}{3\mathcal{H}_k}=\lim_{\epsilon\to 0}\left(\int_\eta^{\eta_0-\epsilon}d\eta'\frac{z_k^2u_k^2}{z^2u^2}+\int_{\eta_0+\epsilon}^{\eta_*}d\eta'\frac{z_k^2u_k^2}{z^2u^2}-\frac{2}{\epsilon}\frac{z_k^2u_k^2}{z_0'^{\,2}u_0^2}\right), \tag{A7}$$

and $\tilde{D}_k=\tilde{D}(\eta_k)$, where $u_0=u(\eta_0)$, the decaying mode v normalized to u at $\eta=\eta_k$ is given by

$$v(\eta)=u(\eta)\frac{\tilde{D}(\eta)}{\tilde{D}_k}. \tag{A8}$$

Thus exactly the same argument applies to this case, by replacing the original D_k by the above $\tilde{D}_k$.

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