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A new view of k -essence

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k -essence models, relying on scalar fields with noncanonical kinetic terms, have been proposed as an alternative to quintessence in explaining the observed acceleration of the Universe. We consider the use of field redefinitions to cast k -essence in a more familiar form. While k -essence models cannot in general be rewritten in the form of quintessence models, we show that in certain dynamical regimes an equivalence can be made, which in particular can shed light on the tracking behavior of k -essence. In several cases, k -essence cannot be observationally distinguished from quintessence using the homogeneous evolution, though there may be small effects on the perturbation spectrum. We make a detailed analysis of two k -essence models from the literature and comment on the nature of the fine-tuning arising in the models.

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I. INTRODUCTION

One of the greatest challenges in modern cosmology is understanding the nature of the dark energy responsible for the observed acceleration of the present Universe [1]. A popular framework, known as quintessence, involves scalar field models in which the field slow-rolls down a potential, with its potential energy acting analogously to that of early Universe inflation [2,3] models. However, recently a second possibility, that of an effective scalar field theory described by a Lagrangian with a noncanonical kinetic term, has also been proposed. Such a model may lead to early time acceleration, where it is named k -inflation [4,5], or acceleration in the present Universe under the name k -essence [6–8]. It is worth noting that tachyon dark energy models [9] may be seen as special cases of k -essence.

Allowing the dark energy to be dynamical provides an opportunity to study the so-called coincidence problem, which asks why dark energy domination begins just at the epoch when we cosmologists exist to observe it. Traditional quintessence models appear promising in this regard, as they can support scaling or tracking solutions, in which the scalar field energy density follows that of the dominant source of matter [3]. Unfortunately, these models require a fine-tuning of parameters, making them not particularly more attractive than a pure cosmological constant in the cases considered so far. Alternatively, a class of k -essence models [7,8] has been claimed to solve the coincidence problem in a generic way; after a long period of perfect tracking, the domination of dark energy is triggered by the transition to matter domination, a time period during which structures—and cosmologists—can form. Therefore, it is important to understand the extent to which these models differ from quintessence models. In particular, one would like to understand whether these models suffer from the same sort of fine-tuning issues as quintessence, and also whether observations can differentiate between the two models. We will address this question by employing field redefinitions which allow k -essence models to be recast in a form similar to quintessence, though we stress immediately that in general the two

ideas are distinct and it is not possible to write an arbitrary k -essence model in quintessence form.

Throughout this article a prime denotes a derivative with respect to the argument of the function to which it is applied, and a dot denotes a derivative with respect to time.

II. k -ESSENCE

To begin with, we wish to be absolutely clear in our terminology. Although the literature contains different usages, particularly of the word quintessence, it is important that our own usage be unambiguous. We will use the word “quintessence” exclusively to refer to models which feature a single scalar field with a canonical kinetic term (whereas in some other papers the definition of quintessence is so general as to include k -essence within it).

We now introduce the k -essence model. Neglecting for now the part of the Lagrangian containing ordinary matter, the action for a k -essence field ϕ is given by

$$S = \int d^4x \sqrt{-g} \left[-\frac{m_{\text{Pl}}^2}{16\pi} R + K(\phi) \tilde{p}(X) \right], \quad (1)$$

where we assume $K(\phi) > 0$ and $X = \frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi$. The name of the model suggests that the field should be driven only by its kinetic energy. For that to be strictly true one should impose $\tilde{p}(0) = 0$, otherwise that term could be separated as a potential term independent of $\nabla_\mu \phi$. Indeed, for sufficiently small X , one could even write $\tilde{p}(X) \approx \tilde{p}(0) + \tilde{p}'(0)X$ and, after field redefinition, obtain a canonical scalar field with a potential. However this condition is not imposed in most k -essence papers, e.g. Refs. [7,8], and we will not impose it either.

In order to be even more general, one may include a separate potential term $\bar{U}(\phi)$ in the Lagrangian, now setting $\tilde{p}(0) = 0$ [10]. Obviously, this model includes the one given by Eq. (1). We give an extension of our results to this more general possibility in the Appendix.

Now, we recall some properties summarized in Ref. [8]. Using the perfect fluid analogy, the pressure and the energy density are given by

$$p = K(\phi)\tilde{p}(X), \quad (2)$$

$$\varepsilon = K(\phi)\tilde{\varepsilon}(X), \quad (3)$$

where

$$\tilde{\varepsilon}(X) = 2X\tilde{p}'(X) - \tilde{p}(X). \quad (4)$$

The equation of state parameter is given by

$$w_k = \frac{\tilde{p}(X)}{\tilde{\varepsilon}(X)} = \frac{\tilde{p}(X)}{2X\tilde{p}'(X) - \tilde{p}(X)}, \quad (5)$$

while the effective sound speed is given by

$$c_{\text{sk}}^2 = \frac{\tilde{p}'(X)}{\tilde{\varepsilon}'(X)} = \frac{\tilde{p}'(X)}{\tilde{p}'(X) + 2X\tilde{p}''(X)}. \quad (6)$$

This definition comes from the equation describing the evolution of linear perturbations in a k -essence dominated Universe [5], and therefore is relevant when studying the stability of the theory. We note, however, that, as shown in Ref. [11], $c_{\text{sk}}^2 > 0$ is not a sufficient condition for the theory to be stable. It is important to notice that the effective sound speed is not always equal to 1, as it is in quintessence models. Therefore, the behavior of perturbations in this case is genuinely different from that in the case of canonical scalar fields and this difference may be observable [12]. Also, note that the definition given in Eq. (6) is different from the thermodynamic definition of the isentropic sound speed,

$$c_s^2 = \left(\frac{\partial p}{\partial \varepsilon} \right)_s = \frac{\dot{p}}{\dot{\varepsilon}}. \quad (7)$$

As explained in Ref. [13], the difference arises from the fact that “a well-defined concept of sound speed does not exist for classical scalar fields” and therefore their density perturbations behave quite differently from the usual hydrodynamic case.

Although models with negative energy density, super-negative or diverging equation of state and/or imaginary or diverging sound speed have been studied, e.g. Refs. [6–8], it is possible to restrict the class of models by imposing one or more of the following independent constraints:

$$\varepsilon > 0 \Rightarrow 2X\tilde{p}'(X) > \tilde{p}(X),$$

$$w_k > -1 \Rightarrow \tilde{\varepsilon}(X)\tilde{p}'(X) > 0, \quad (8)$$

$$c_{\text{sk}}^2 > 0 \Rightarrow 2X\tilde{p}''(X)\tilde{p}'(X) > -\tilde{p}'^2(X).$$

From Eq. (1), in the case of a flat Robertson-Walker metric

$$ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2, \quad (9)$$

the Euler-Lagrange equation for the k -essence field is

$$\tilde{\varepsilon}'(X)\ddot{\phi} + 3H\tilde{p}'(X)\dot{\phi} + \frac{K'(\phi)}{K(\phi)}\tilde{\varepsilon}(X) = 0, \quad (10)$$

where $H = \dot{a}/a$. We see that if $\tilde{\varepsilon}'(X) = 0$ at some X_c , so that c_{sk}^2 diverges, the equation is singular and reduces to a first-order equation which gives a constraint on ϕ . Some problems may arise at this singularity, but we leave this issue for future investigation. In any case, regions separated by a diverging sound speed are disconnected and may be considered as different models. In each such region the Euler-Lagrange equation may be rewritten as

$$\ddot{\phi} + 3Hc_{\text{sk}}^2(X)\dot{\phi} + \frac{K'(\phi)}{K(\phi)}\frac{\tilde{\varepsilon}(X)}{\tilde{\varepsilon}'(X)} = 0. \quad (11)$$

Finally, note that, for models in which $\tilde{p}(X)$ is analytic and equal to 0 at the origin, then if $\tilde{p}'(X=0) < 0$ the function $\tilde{\varepsilon}(X)$ must be negative for some X . Given that $\tilde{p}(X)$ is analytic at the origin, we then have

$$\tilde{p}(X) = -p_n X^n + \mathcal{O}(X^{n+1}), \quad (12)$$

which implies

$$\tilde{\varepsilon}(X) = -p_n(2n-1)X^n + \mathcal{O}(X^{n+1}), \quad (13)$$

with n a positive integer and where $p_n > 0$ is the first non-zero coefficient. Hence $\tilde{\varepsilon} < 0$ for some range of X .

In the models described in Ref. [8], $\tilde{p}(X)$ is negative for small X and therefore, if $\tilde{p}(0) = 0$ then $\tilde{p}'(X)$ must be negative close to the origin. Hence $\tilde{\varepsilon}(X) < 0$ over some range of X . As a result, for these models, if one imposes $\varepsilon(X) > 0$ and the analyticity of $\tilde{p}(X)$ at the origin, then $\tilde{p}(0)$ must be non-zero and, as explained above, this implies that one has, in effect, introduced a potential term.

III. DYNAMICAL EQUIVALENCE

In this section we present a model which is dynamically equivalent to the k -essence model. Consider a new action

$$S = \int d^4x \sqrt{-g} \left\{ -\frac{m_{\text{Pl}}^2}{16\pi} R + K(\phi)[\tilde{p}(\chi) + (\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - \chi)\tilde{p}'(\chi)] \right\}, \quad (14)$$

in which we have introduced a field χ , which acts like a Lagrange multiplier. The variational principle with respect to χ gives

$$\tilde{p}''(\chi) \left(\chi - \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi \right) = 0, \quad (15)$$

and therefore—as long as $\tilde{p}''(\chi) \neq 0$ —Eq. (14) is dynamically equivalent to Eq. (1). Note, however, that, when quantized, the theories will no longer be equivalent. Now perform a field transformation

$$Q \equiv \int_{\phi_0}^{\phi} \sqrt{K(\sigma)} d\sigma, \quad (16)$$

which leads to

$$S = \int d^4x \sqrt{-g} \left\{ -\frac{m_{\text{Pl}}^2}{16\pi} R + \frac{\tilde{p}'(\chi)}{2} \nabla_{\mu} Q \nabla^{\mu} Q + V(Q) [\tilde{p}(\chi) - \chi \tilde{p}'(\chi)] \right\}, \quad (17)$$

where $V(Q) \equiv K[\phi(Q)]$. From now on, for simplicity, we assume that $\tilde{p}''(X)$ has a constant sign, although this is not a particularly strong assumption. This allows us to define a new field by computing a modified Legendre transformation

$$\psi \equiv \tilde{p}'(\chi), \quad (18)$$

$$W(\psi) \equiv \chi \tilde{p}'(\chi) - \tilde{p}(\chi), \quad (19)$$

which finally gives

$$S = \int d^4x \sqrt{-g} \left[-\frac{m_{\text{Pl}}^2}{16\pi} R + \frac{\psi}{2} \nabla_{\mu} Q \nabla^{\mu} Q - V(Q) W(\psi) \right]. \quad (20)$$

This action is very simple, but contains a non-dynamical field ψ coupled noncanonically to a canonical scalar field Q . Clearly, if we impose $\varepsilon > 0$, then the condition $w_k > -1$ implies that ψ is positive, and therefore that the kinetic term has the canonical sign. In the opposite case, the kinetic term has the sign of the phantom model introduced by Caldwell [14].

For the flat Robertson-Walker metric (9), the energy density and pressure of this two-field component of matter are

$$\rho_{Q\psi} = \frac{\psi}{2} \dot{Q}^2 + V(Q) W(\psi), \quad (21)$$

$$p_{Q\psi} = \frac{\psi}{2} \dot{Q}^2 - V(Q) W(\psi). \quad (22)$$

The Euler-Lagrange equation for Q is given by

$$\psi \ddot{Q} + (3H\psi + \dot{\psi}) \dot{Q} + V'(Q) W(\psi) = 0, \quad (23)$$

and the constraint equation for ψ is given by

$$W'(\psi) = \frac{\dot{Q}^2}{2V(Q)}. \quad (24)$$

If ψ is positive and remains almost constant, the field Q plays the role of a canonical scalar field. Indeed, by renormalizing Q as $\bar{Q} = \sqrt{\psi} Q$ we obtain an (almost) equivalent quintessence model which should mimic the k -essence field

during the time period for which the assumption of approximately constant ψ holds. If this is the case when k -essence starts dominating, it will not be possible to distinguish it observationally from a quintessence field, unless one takes into account perturbations. As described in Refs. [6–8] the k -essence field can undergo several attractor regimes during which its kinetic energy remains constant. Hence, ψ is indeed constant over those periods of time. Thus, during some time periods, the homogeneous part of the k -essence field will behave exactly like a quintessence field.

IV. EXAMPLES

In this section we study two examples of k -essence models. We first study the model described in Ref. [6] and given by

$$K(\phi) = \left(\frac{\phi}{m_{\text{Pl}}} \right)^{-\alpha}, \quad (25)$$

$$\tilde{p}(X) = P_0 \left[-\frac{X}{m_{\text{Pl}}^4} + \left(\frac{X}{m_{\text{Pl}}^4} \right)^2 \right]. \quad (26)$$

It is easy to check that this is a rather unconventional model; for some ranges of X we can have $\varepsilon < 0$, $w_k < -1$ or $c_{\text{sk}}^2 < 0$, and w_k and c_{sk}^2 diverging at $X_w = m_{\text{Pl}}^4/3$ and $X_c = m_{\text{Pl}}^4/6$, respectively. Nevertheless, this model features an interesting behavior—in the presence of a dominating fluid with equation of state parameter w_f there exists a stable scaling solution for which X is constant and

$$w_k = \frac{(1+w_f)\alpha}{2} - 1. \quad (27)$$

Using the transformation described in the last section, we find

$$V(Q) = (\beta/\alpha)^{\beta} \left(\frac{Q}{m_{\text{Pl}}} \right)^{-\beta}, \quad (28)$$

where

$$\beta = \frac{2\alpha}{2-\alpha}, \quad (29)$$

$$\psi = \frac{P_0}{m_{\text{Pl}}^4} \left[-1 + \frac{2\chi}{m_{\text{Pl}}^4} \right], \quad (30)$$

$$W(\psi) = \frac{(\psi m_{\text{Pl}}^4 + P_0)^2}{4P_0}. \quad (31)$$

Provided that X is constant during the scaling regime (so that ψ is constant), the k -essence field behaves like an inverse power-law quintessence field in its well-known tracker regime. The transformation described in Sec. III allows us to find the appropriate power. We normalize the kinetic term and obtain

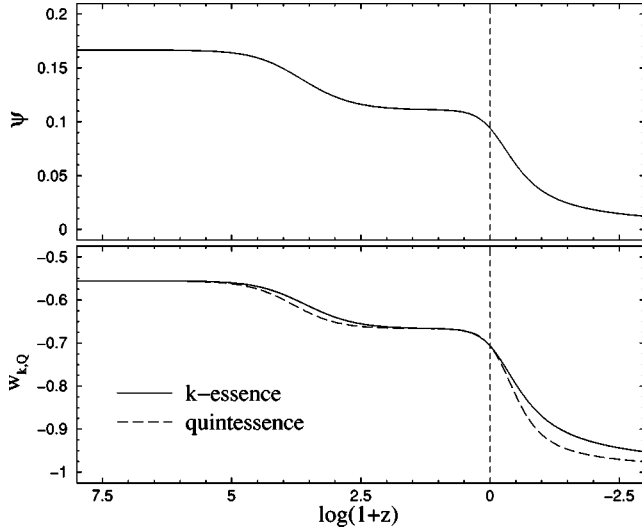


FIG. 1. Top panel: Evolution of ψ for the first example of a k -essence field described in Sec. IV in a cosmologically realistic case and with $\alpha = 2/3$. Note the transitions from radiation domination to matter domination and from matter domination to k -essence domination. Bottom panel: Evolution of the equation of state parameter w for this field and for the almost equivalent quintessence field, that is to say an inverse power-law model with $\beta = 1$.

$$\tilde{V}(\tilde{Q}) = \tilde{V}_0 \left(\frac{\tilde{Q}}{m_{\text{pl}}} \right)^{-\beta}, \quad (32)$$

where $\tilde{V}_0 = \psi^{\beta/2} (\beta/\alpha)^\beta (\psi m_{\text{pl}}^4 + P_0)^2 / 4P_0$. Obviously, as soon as the scaling solution is modified (for instance at matter-radiation equality) ψ evolves until the k -essence field enters a new scaling solution with a new quintessence-like behavior. In Fig. 1 we show the evolution of ψ for a cosmologically realistic model, and also the evolution of the equation of state parameter w for the k -essence field and for the almost equivalent quintessence model. The value of \tilde{V}_0 has been chosen so that both models yield the same value of the equation of state parameter in the present epoch. Clearly, as long as ψ does not remain exactly constant, the equivalence is not perfect and therefore the two models may in principle be distinguished from one another. However in practice we see in Fig. 1 that the evolution of w is almost exactly the same in the two cases out to high redshift.

As a second example, consider the model described in Ref. [7] given by

$$K(\phi) = \left(\frac{\phi}{M_0} \right)^{-2}, \quad (33)$$

and

$$\tilde{p}(X) = M_0^4 [-2.01 + 2\sqrt{1 + X/M_0^4} + 3 \times 10^{-17} (X/M_0^4)^3 - 10^{-24} (X/M_0^4)^4], \quad (34)$$

where $M_0 \equiv \sqrt{3/8\pi} m_{\text{pl}}$. The constants appearing in this expression are very specific, and it may be that there are simpler versions, but so far this is the best example that we

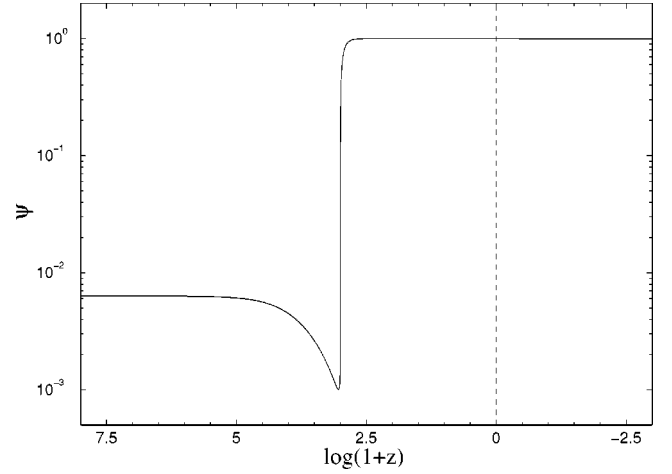


FIG. 2. Evolution of ψ for the second example of a k -essence field described in Sec. IV in a cosmologically realistic case. The dramatic change in ψ is triggered by the transition from radiation domination to matter domination.

know of in the literature that features an interesting property: the transition between an exact tracker regime and the domination of the k -essence field is triggered by matter-radiation equality and therefore, in a sense, the coincidence problem is solved. Again, for some ranges of X , contrary to what is assumed in the first part of Ref. [7], we can have $\varepsilon < 0$, $w_k < -1$ or $c_{\text{sk}}^2 < 0$ and w_k and c_{sk}^2 diverging at $X = X_w \approx 2.1 \times 10^7 M_0^4$ and $X = X_c \approx 1.6 \times 10^7 M_0^4$, respectively. Nevertheless, as long as we take $X \leq X_c$, we have $\varepsilon > 0$, $w_k > -1$ and $c_{\text{sk}}^2 > 0$ and, as explained in Sec. II, we know that the field cannot cross this boundary.

Using the transformation described in Sec. III we find

$$V(Q) = e^{-2Q}. \quad (35)$$

Note that, since $\tilde{p}''(X)$ changes sign at $X \approx 1.6 \times 10^6 M_0^4$ and $X \approx 1.5 \times 10^7 M_0^4$, the Legendre transformation described in Sec. III cannot be computed. Nevertheless we still define ψ as $\tilde{p}'(X)$, though W can no longer be expressed as a function of ψ .

We have performed a numerical simulation in order to reproduce the result given in Ref. [7], that is to say a stable scaling solution during radiation domination which then evolves to k -essence domination after matter-radiation equality. As an aside, we find that the basin of attraction of this solution does not seem to be very large. Indeed, random initial conditions, even with a subdominant k -essence field, almost never lead to the desired solution, but instead to an early period of k -essence domination, or in some cases the solution even ceases to exist when, after a finite time, it reaches the singularity X_c . However we will not pursue this further here, but leave it for a future investigation.

In Fig. 2 we show the evolution of ψ for this model. Clearly ψ is almost constant during radiation domination, then evolves to another constant soon after matter-radiation equality, remaining nearly constant even during the transition to k -essence domination. Consequently, we know that during

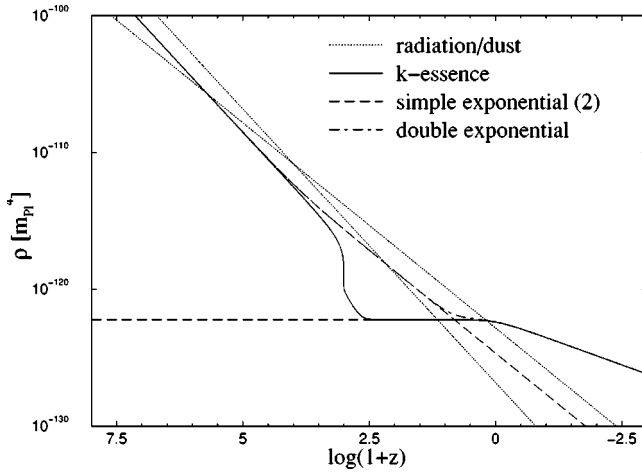


FIG. 3. Evolution of the energy density of radiation, dust-like matter and the second example of a k -essence model studied in Sec. IV. We also show the evolution of the energy density for two exponential quintessence models ($\alpha_1 \approx 5.8 m_{\text{pl}}^{-1}$ and $\alpha_2 \approx 72.8 m_{\text{pl}}^{-1}$) and for the double exponential quintessence model obtained by adding both potentials.

these two time periods the k -essence field behaves like a quintessence field. Renormalizing Q in order to have a canonical kinetic term, we find two exponential potential quintessence models given by

$$\tilde{V}(\tilde{Q}) \propto \exp(-\alpha_i \tilde{Q}) \quad (i=1 \text{ or } 2), \quad (36)$$

with $\alpha_1 \approx 5.8 m_{\text{pl}}^{-1}$ and $\alpha_2 \approx 73 m_{\text{pl}}^{-1}$, which mimic the k -essence field during the two time periods described above, respectively.

In order to mimic the k -essence field during these two time periods with only one quintessence model one may add both potentials to obtain a double exponential quintessence model [15]. In this case we have normalized the potential to fit the late-time behavior of the k -essence field (note that the relative normalization of the two exponentials can always be set to 1 through a field translation). The evolution of the energy density for all these models is shown in Fig. 3 and it is evident that the quintessence models closely mimic the k -essence field while ψ is constant. Moreover, as expected, the double exponential model allows us to mimic the k -essence field for early times as well as for the late-time evolution of the Universe.

In Fig. 4 we reproduce the plots shown in Ref. [7], adding the quintessence models described here. Again, the quintessence models very closely mimic the k -essence model during the period for which they are equivalent; for instance the second exponential potential model gives indistinguishable evolution of ρ_k/ρ_m and w back to the redshifts of several hundred (though note that this potential does not exhibit tracker behavior). Accordingly, observations studying the homogeneous evolution, such as supernovae observations, are not able to distinguish between such models. For the double exponential model, the mimicking is not perfect during the recent past and therefore one might be able to tell the difference.

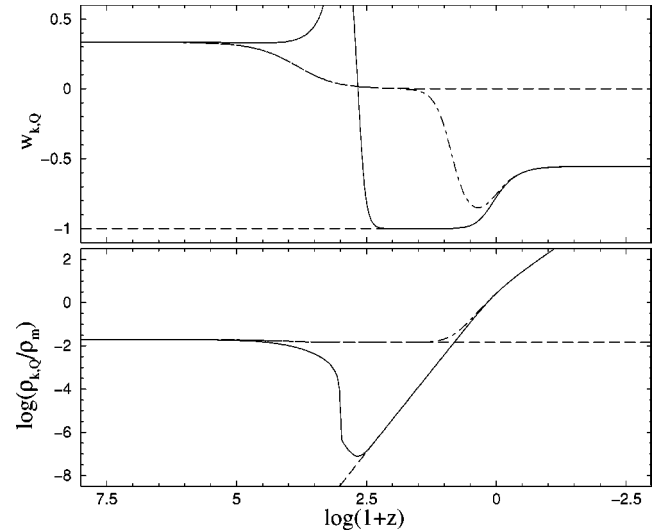


FIG. 4. Evolution of the equation of state w_k (top) and the ratio ρ_k/ρ_m (bottom) of the k -essence field as shown in Ref. [7]. The quintessence fields plotted in Fig. 3 are also shown, with the line styles having the same meaning.

V. DISCUSSION

Quintessence and k -essence are two attempts to explain, in terms of scalar fields, the current observation of an accelerating Universe. Although they are similar in that they both involve the dynamics of light scalar fields, they differ in that quintessence relies on precise functional forms for the potential of the field, whereas k -essence derives its particular behavior from the presence of noncanonical kinetic terms associated with the field. Given that both models attempt to explain the same observations, and that both models involve evolving scalar fields, a natural question that arises is whether it is possible to write one model in terms of the parameters of the other. In regimes where the effective equation of state parameter for the k -essence field becomes less than -1 , such a relationship is not possible while maintaining conventional canonical kinetic terms for the quintessence fields, but for w_k greater than -1 such equivalences may exist.

In particular, in this paper we have addressed this question by attempting to rewrite k -essence models in terms of quintessence potentials, relating the two sets of fields through field redefinitions. Our results are intriguing; we have found a dynamically equivalent action which has similarities with a canonical scalar field action and may be easier to study. We have examined two cases from the literature, and shown that during some regimes the homogeneous part of the k -essence field can behave exactly like a quintessence field, and have obtained exact equivalences in those cases.

It could well prove impossible to differentiate between the two models, quintessence and k -essence, by measuring the evolution of the equation of state parameter. To distinguish between the models, it appears necessary to combine such studies with searches for subtle effects on the perturbations from the different sound speed in the two models [12].

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APPENDIX: EXTENSION TO k -ESSENCE MODELS WITH A GENERAL POTENTIAL

In this section, we briefly extend our discussion to a more general model [10] given by

$$S = \int d^4x \sqrt{-g} \left[\frac{m_{\text{Pl}}^2}{16\pi} R + K(\phi) \tilde{p}(X) - \bar{U}(\phi) \right], \quad (\text{A1})$$

where we assume $K(\phi) > 0$, $\tilde{p}(0) = 0$ and $X = \frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi$. The pressure and the energy density are given by

$$p = K(\phi) \tilde{p}(X) - \bar{U}(\phi), \quad (\text{A2})$$

$$\varepsilon = K(\phi) \tilde{\varepsilon}(X) + \bar{U}(\phi), \quad (\text{A3})$$

where $\tilde{\varepsilon}(X)$ is still defined by Eq. (4). As usual, the equation of state parameter is given by $w_k = p/\varepsilon$ and the sound speed is still defined by Eq. (6).

Now, we consider the action

$$S = \int d^4x \sqrt{-g} \left\{ \frac{m_{\text{Pl}}^2}{16\pi} R + K(\phi) [\tilde{p}(X) + (\frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi - \chi) \tilde{p}'(X)] - \bar{U}(\phi) \right\}, \quad (\text{A4})$$

and, using a method similar to that in Sec. III, and defining $U(Q) \equiv \bar{U}[\phi(Q)]$, it is possible to find an equivalent model described by the action

$$S = \int d^4x \sqrt{-g} \left[-\frac{m_{\text{Pl}}^2}{16\pi} R + \frac{\psi}{2} \nabla_\mu Q \nabla^\mu Q - V(Q) W(\psi) - U(Q) \right]. \quad (\text{A5})$$

The dynamical equations resulting from this action are

$$\rho_{Q\psi} = \frac{\psi}{2} \dot{Q}^2 + V(Q) W(\psi) + U(Q), \quad (\text{A6})$$

$$p_{Q\psi} = \frac{\psi}{2} \dot{Q}^2 - V(Q) W(\psi) - U(Q). \quad (\text{A7})$$

The Euler-Lagrange equation for Q is given by

$$\psi \ddot{Q} + (3H\psi + \dot{\psi}) \dot{Q} + V'(Q) W(\psi) + U'(Q) = 0, \quad (\text{A8})$$

and the constraint equation for ψ is still given by Eq. (24).

Canonically normalizing Q via $\tilde{Q} \equiv \sqrt{|\psi|} Q$, the equation of motion for the field \tilde{Q} is then

$$\begin{aligned} \ddot{\tilde{Q}} + 3H\dot{\tilde{Q}} \pm \tilde{V}'(\tilde{Q}) W(\psi) \pm \tilde{U}'(\tilde{Q}) - \frac{3H\tilde{Q}\dot{\psi}}{2\psi} \\ + \frac{\tilde{Q}\dot{\psi}^2}{4\psi^2} - \frac{\tilde{Q}\ddot{\psi}}{2\psi} = 0, \end{aligned} \quad (\text{A9})$$

where “ \pm ” stands for the sign of ψ and

$$\tilde{V}(\tilde{Q}) \equiv V(\tilde{Q}/\sqrt{|\psi|}), \quad (\text{A10})$$

$$\tilde{U}(\tilde{Q}) \equiv U(\tilde{Q}/\sqrt{|\psi|}). \quad (\text{A11})$$

In addition, the constraint equation transforms to

$$\dot{\tilde{Q}}^2 - 2|\psi| W'(\psi) \tilde{V}(\tilde{Q}) - \frac{\tilde{Q}\dot{\tilde{Q}}\dot{\psi}}{\psi} + \frac{\tilde{Q}^2\dot{\psi}^2}{4\psi^2} = 0. \quad (\text{A12})$$

In the limit in which the approximation $\psi \approx \text{const}$ applies, the equation of motion becomes

$$\ddot{\tilde{Q}} + 3H\dot{\tilde{Q}} \pm \tilde{V}'(\tilde{Q}) W(\psi) \pm \tilde{U}'(\tilde{Q}) = 0, \quad (\text{A13})$$

and the constraint equation simplifies significantly to become

$$\dot{\tilde{Q}}^2 - 2|\psi| W'(\psi) \tilde{V}(\tilde{Q}) = 0. \quad (\text{A14})$$

Thus, even in this extended class of models, if ψ remains constant during some time period, the dynamics will be equivalent to that of a quintessence model or a phantom model for this period.

As a simple example, consider the k -essence example of Ref. [10], in which the equation of state $w < -1$ is obtained. This has $p(X) = P_0 [\exp(-\alpha X/m_{\text{Pl}}^4) - 1]$, where $\alpha > 0$ and $P_0 > 0$, and unspecified functions for $K(\phi)$ and $\bar{U}(\phi)$. This means that the translation to our new action is affected by

$$\psi = -\frac{\alpha P_0}{m_{\text{Pl}}^4} \exp(-\alpha \chi/m_{\text{Pl}}^4), \quad (\text{A15})$$

$$W(\psi) = P_0 + \frac{\psi m_{\text{Pl}}^4}{\alpha} \left[1 - \ln \left(-\frac{\psi m_{\text{Pl}}^4}{\alpha P_0} \right) \right]. \quad (\text{A16})$$

Note that $-\alpha P_0/m_{\text{Pl}}^4 < \psi < 0$ so that this model does indeed correspond to $w < -1$. The condition that the energy density be positive becomes

$$\frac{U(Q)}{V(Q)} > \frac{\psi m_{\text{pl}}^4}{\alpha} \left[2 \ln \left(-\frac{\psi m_{\text{pl}}^4}{\alpha P_0} \right) - 1 \right] - P_0. \quad (\text{A17})$$

Further, a necessary condition for the stability of the theory is that the sound speed be positive, $c_s^2 > 0$. This condition becomes

$$\psi < -\frac{\alpha P_0}{m_{\text{pl}}^4} \exp(-1/2), \quad (\text{A18})$$

which is satisfied for $X < X_c = m_{\text{pl}}^4/2\alpha$. Therefore we assume the theory to be valid until some cutoff below X_c .

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