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The influence of atomic interactions on time-dependent tunneling processes of Bose-Einstein condensates is investigated. In a variety of contexts the relevant condensate dynamics can be described by a Landau-Zener equation modified by the appearance of nonlinear contributions. Based on this equation it is discussed how the interactions modify the tunneling probability. In particular, it is shown that for certain parameter values, due to a nonlinear hysteresis effect, complete adiabatic population transfer is impossible however slowly the resonance is crossed. The results also indicate that the interactions can cause significant increase as well as decrease of tunneling probabilities that should be observable in currently feasible experiments.

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I. INTRODUCTION

Since the first experimental preparation of Bose-Einstein condensates (BECs) in dilute atomic gases the study of their dynamical properties has been a very active field of research. It has become apparent that atomic interactions play a crucial role in the prediction or explanation of a wide range of observable phenomena, including, e.g., free condensate expansion, collective excitations, nonlinear atom optics, solitons, and vortices [1,2]. The recent experiment of Ref. [3] has drawn attention to a further dynamical process, namely time-dependent tunneling. This work investigated the dynamics of a BEC that was accelerated by gravity in the periodic potential formed by two counterpropagating vertical laser beams. In this way, Bloch oscillations of the condensate were induced, and each time the lower turning point of the oscillation was reached a fraction of the atoms tunneled into a continuum Bloch band. The regular output of atom pulses spectacularly proved the macroscopic coherence of the initially prepared condensate.

In the experiment of Ref. [3] the influence of atomic interactions mainly showed up as a degradation in the interference when condensates with high densities were studied. As a further consequence, long-time dephasing of the pulse output was predicted. However, another interesting question arising in this context concerns the problem of how the atomic interactions affect the individual tunneling processes that are fundamental to the dynamics of the system. The purpose of this paper is to work out essential aspects of this question by studying the modification of the tunneling probability in the single process. To this end, the condensate dynamics is modeled in terms of a mean-field description provided by the Gross-Pitaevskii equation. If the periodic potential is sufficiently weak and its period short compared to the condensate extension this equation can be simplified by means of a two-mode expansion. Thereby, the two modes represent the tunneling and the Bragg-scattered components of the condensate wave function. In this way one arrives at a set of equations similar to the familiar Landau-Zener problem [4]. In the present case, however, the equations also contain nonlinear terms that characterize the effect of the interactions. It should be emphasized that these equations, which are central to our investigation, are applicable not only to the description of Bloch band tunneling but on a broader scope. They can also model processes such as population transfer between different hyperfine states by variable external fields [5] or the motion of BECs on coupled potential surfaces (which might be of interest in practical applications). These processes, as well as Bloch band tunneling, are examples of coherent output coupling from BECs [6], so that the present work also relates to this context. Furthermore, it extends recent studies of nonlinear Josephson oscillations that are based on a similar set of equations that contain no explicit time dependence [7–9]. This latter set of equations is relevant not only in the context of Bose condensation but to other areas of physics as well, e.g., polaron dynamics, nonlinear optics, biophysics, and molecular physics [8,10], so that the present study might also be of interest in some of these fields.

The paper is organized as follows. In Sec. II the model is set up and the ''nonlinear Landau-Zener equations'' mentioned above are derived. Section III first gives a brief qualitative discussion of how the nonlinearity affects the tunneling probability. We then discuss the main result of our investigation that shows that for certain parameter values the tunneling probability does not vanish however slowly the system evolves. In other words, a complete adiabatic population transfer is impossible under such circumstances. This effect, which would not be possible in linear two-level crossing models, is a direct consequence of the discrete self-trapping transition [10]. Using a phase-space representation of the problem the tunneling probability for very slow processes can be determined. Subsequently, we discuss the population transfer for fast and slow resonance crossing using numerical methods and simple analytical models. In Sec. IV the predictions of the two-mode system are compared to the numerical solution of the Gross-Pitaevskii equation describing Bloch-band tunneling. These studies also show that significant modifications of the tunneling probability due to nonlinear effects should be observable in currently feasible experiments. A short conclusion is given in Sec. V.

II. THE MODEL

The Gross-Pitaevskii equation for a condensate wave function $\psi(\mathbf{r},t)$ undergoing Bloch band tunneling in a periodic optical potential is given by

$$i\hbar\dot{\psi}=H_{lin}\psi+g|\psi|^2\psi \tag{1}$$

with [11,12]

$$H_{lin}=-\frac{\hbar^2\nabla^2}{2m}+(V_T+V_{opt}-Fz). \tag{2}$$

The atomic mass is denoted m, and $g=4\pi\hbar^2 aN/m$ is the nonlinearity parameter with a the s-wave scattering length and N the number of atoms in the sample. The condensate is assumed to be tightly confined in the radial direction by the trapping potential V_T, the z dependence of which is neglected. The periodic optical potential is given by $V_{opt}=V_0\cos(2k_Lz)$, with the laser wave vector k_L, whereas the potential inducing the Bloch oscillations is characterized by the accelerating force F. Such a potential may be produced by tilting the standing wave [3] or by frequency-shifting the counterpropagating laser fields [13]. We want to discuss the condensate time evolution under the following two conditions. First, the condensate has an initial momentum p_0 well defined on the scale set by the lattice wave vector $2k_L$, i.e., $l_z\gg\lambda=\pi/k_L$ with l_z the axial extension of the BEC. The time evolution starts with the BEC well separated from any tunneling resonances, i.e., avoided crossings between Bloch bands. These resonances occur for condensate momenta around $(2l+1)\hbar k_L$, $l=0,\pm1,\ldots$. For concreteness we take $p_0=0$ in the following [apart from the example of Fig. 5(b)]. The condensate is studied as it passes through the resonance at $\hbar k_L$. The second condition is that the optical potential is sufficiently weak, i.e., $V_0\ll\hbar^2k_L^2/2m$. In this case, the crossing of the resonance can be described in terms of a coupling to a single state with a momentum shifted by $-2\hbar k_L$. The wave function $\psi(\mathbf{r},t)$ is expanded in terms of the original and the Bragg-scattered contribution, i.e.,

$$\psi=e^{i(k_L+Ft/\hbar)z}\phi_+\left(z-\frac{\hbar k_L}{m}t-\frac{Ft^2}{2m}\right)b_+(t)$$
$$+e^{i(-k_L+Ft/\hbar)z}\phi_-\left(z+\frac{\hbar k_L}{m}t-\frac{Ft^2}{2m}\right)b_-(t), \tag{3}$$

with $t=0$ corresponding to the point of exact resonance and $b_+=1$, $b_-=0$, initially. The two envelope functions $\phi_\pm$ (whose radial motion is frozen due to the tight confinement) are normalized to one. It is assumed that ϕ_- is very similar to ϕ_+, the shift in position between the two being negligible at all times t relevant for the tunneling process, i.e., we set $\int d^3r\,\phi_-(z)\phi_+(z-2\hbar k_Lt/m)\approx1$. This approach is valid if the nonlinear effects are not too large and the tunneling process is sufficiently short. Expression (3) is then inserted into Eq. (1) thereby discarding second derivatives of $\phi_\pm$ (slowly varying envelope approximation). By projecting the resulting equation onto the instantaneous wave functions $e^{i(\pm k_L+Ft/\hbar)z}\phi_\pm$ and switching to a rotating frame where $a_\pm(t)=\exp[i\int^t\omega(t')dt']b_\pm(t)$, $\omega(t)=\hbar k_L^2/2m+F^2t^2/2m\hbar-\varepsilon t/2-2\gamma$, we obtain the "nonlinear Landau-Zener equations"

$$i\dot{a}_+=\varepsilon t a_++\Omega a_-+\gamma|a_+|^2a_+,$$
$$i\dot{a}_-=\Omega a_++\gamma|a_-|^2a_-, \tag{4}$$

with

$$\varepsilon=\frac{2Fk_L}{m},\quad \Omega=\frac{V_0}{2\hbar},\quad \gamma=-\frac{g}{\hbar}\int d^3r\,|\phi(\mathbf{r})|^4. \tag{5}$$

In the following, $\Omega>0$ is assumed. As mentioned in the Introduction similar equations can be derived in other contexts, e.g., for population transfer between different hyperfine ground states with variable external fields [5]. In Sec. IV it is shown that Eqs. (4) allow an accurate prediction of tunneling probabilities in realistic situations.

Significant insight into the system behavior can be obtained by noting that Eqs. (4) may be derived from the Hamilton function [7,10]

$$H(N_+,\Theta)=\Delta N_++2\Omega\sqrt{N_+(1-N_+)}\cos\Theta$$
$$+\gamma(N_+^2-N_++1/2), \tag{6}$$

with $N_+=|a_+|^2$ and $\Theta=\arg(a_+a_-^*)$ as canonical variables and $\Delta=\varepsilon t$. The dynamics induced by this Hamilton function for fixed, time-independent Δ is discussed in detail in Refs. [7,8,10]. In the present context the stationary states $S=(N_S,\Theta_S)$ of $H(N_+,\Theta)$ are of particular interest because they take over the role of the adiabatic eigenstates in the linear problem ($\gamma=0$). From $\partial H/\partial\Theta=0$ it follows that they always have $\Theta_S=0$ or π. The condition $\partial H/\partial N_+=0$ then shows that $N_S=n_s+1/2$ is obtained as a real-valued solution of the equation

$$n_s^4+\frac{\Delta}{\gamma}n_s^3+\left(\frac{\Omega^2}{\gamma^2}+\frac{\Delta^2}{4\gamma^2}-\frac{1}{4}\right)n_s^2-\frac{\Delta}{4\gamma}n_s-\frac{\Delta^2}{16\gamma^2}=0. \tag{7}$$

If $|\gamma|/\Omega\leqslant2$, there are exactly two stationary states S_- and S_+ having $\Theta=0$ and π, respectively, for all values of Δ. They correspond to the high- and low-energy eigenstates of the linear problem. For $|\gamma|/\Omega>2$ two further stationary states appear in the vicinity of $\Delta=0$ as discussed in more detail in Sec. III.

III. TUNNELING PROBABILITIES FOR THE NONLINEAR LANDAU-ZENER MODEL

A. Overview and qualitative discussion

In the study of the tunneling probability for the nonlinear Landau-Zener model we are interested in the long-time solution of Eqs. (4), provided the system is prepared at $t\to-\infty$ in the high- or low-energy stationary state. We choose the low-energy state a_+ in the following, i.e., $a_+(t\to-\infty)=1$, and the tunneling probability is thus $P_T=|a_+(t\to\infty)|^2$. In the linear case ($\gamma=0$) the tunneling probability is independent of whether the system starts in the low- or the high-energy state. In the nonlinear case the same still holds true if the sign of the nonlinearity is changed as well, i.e., the tunneling probability for a system starting in "+" is the same as for one initially in "−" and having a nonlinearity param-

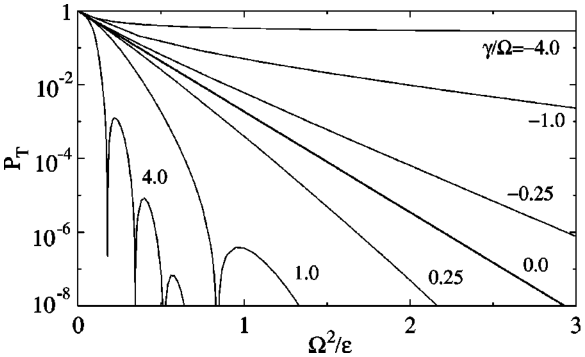

FIG. 1. Tunneling probability P_T as a function of Ω^2/ε for various values of γ/Ω.

eter $-\gamma$. The whole range of dynamical phenomena predicted by Eqs. (4) (for both signs of γ) can thus be observed with a condensate of repulsive interactions by using both initial conditions. Due to the scaling properties of Eqs. (4) P_T can be considered a function of Ω/ε^2 and γ/Ω, only. An overview of the dependence of P_T on these parameters is given in Fig. 1 that shows results obtained from the numerical solution of Eqs. (4). In the linear problem the numerical calculation of P_T is facilitated by transforming from a_+, a_- onto the basis of the instantaneous ''dressed'' eigenstates. In this basis the oscillations of the system trajectory around its asymptotic limit for $t \to \infty$ are suppressed and after a relatively short propagation time the value of P_T can be read off from the population of the eigenstates. This approach is generalized to the nonlinear case by computing the canonical action of the trajectories as determined by the Hamilton function (6). This action plays the role of the adiabatic invariant and, in the linear case, is equivalent to the eigenstate population. Various aspects of the behavior of P_T are discussed below.

For the linear problem the tunneling probability is given exactly by the well-known Landau-Zener formula $P_T = \exp(-2\pi\Omega^2/\varepsilon)$ [4]. A qualitative understanding of how the nonlinear terms influence P_T can be obtained by noting that they give rise to an effective detuning $\Delta_{eff} = \varepsilon t + \gamma(2|a_+|^2 - 1)$ between the states a_+ and a_-. Unless the resonance is traversed too rapidly the system remains in the vicinity of $S_+(\varepsilon t)$ and the population $|a_+(t)|^2$ closely follows its steady-state value $N_{S,+} = n_{s,+}(\varepsilon t) + 1/2$. A rough estimate of the tunneling probability can be obtained from the rate $R = \dot{\Delta}_{eff}(t=0)$ at which the point of zero detuning is crossed. As a first approximation it follows from the linear Landau-Zener formula that $P_T \approx \exp(-2\pi\Omega^2/R)$. As for small Δ,

$$n_{s,+} \approx -\Delta/[2(2\Omega+\gamma)] + \Omega\Delta^3/[2(2\Omega+\gamma)^4], \qquad (8)$$

one thus finds

$$P_T \approx \exp\left\{-2\pi\frac{\Omega^2}{\varepsilon}(1+\gamma/2\Omega)\right\}. \qquad (9)$$

Although not quantitatively accurate this formula indicates two trends confirmed by the detailed investigation: (i) nonlinear effects become significant as soon as γ becomes comparable to Ω (as one might expect) and (ii) P_T is decreased (increased) for $\gamma > 0$ ($\gamma < 0$) as compared to the linear case. Interestingly, the expansion of $n_{s,+}$ also shows that the third-order correction to Δ_{eff} renders the tunneling transition superlinear (sublinear) for $\gamma > 0$ ($\gamma < 0$) in the terminology of Ref. [14], i.e., the resonance is effectively crossed faster (slower) than the linear approximation predicts [15]. The results of Ref. [14] indicate that the approximation (9) should underestimate the nonlinear effects on P_T. This is indeed observed in the present case if Ω/ε^2 is not too small and γ/Ω sufficiently larger than -2.

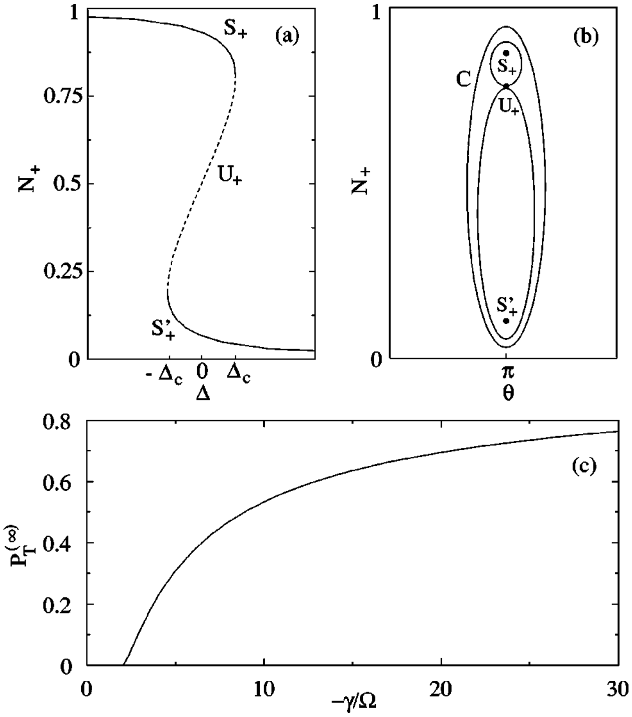

FIG. 2. (a) Coordinate N_+ of stationary states as a function of Δ for $\gamma/\Omega < -2$. (b) Schematic phase-space plot in the vicinity of Δ_c. The system trajectory eventually has to switch onto the orbit *C*. (c) Asymptotic probability $P_T^{(\infty)}$ as a function of $-\gamma/\Omega$.

B. Nonlinear hysteresis effects in the tunneling probability

The above discussion already indicates the significance of the stationary-state behavior for an understanding of the tunneling probability. The breakdown of the expansion for $n_{s,+}$ at $\gamma/\Omega = -2$ now suggests that at this point a qualitative change in the system behavior may occur. Indeed, for $\gamma/\Omega < -2$, as Δ is increased from large negative values, two further stationary states of $H(N_+, \Theta)$, S'_+ and U_+, emerge at a detuning $-\Delta_c < 0$ [see Fig. 2(a)]. They both have the same phase $\Theta = \pi$ as S_+ and populations $N_{S',+} < N_{U,+} < N_{S,+}$. The point S'_+ is stable, whereas U_+ is unstable. With growing Δ, U_+ approaches S_+. At Δ_c they coalesce and only S'_+ remains (besides S_-). We thus observe a typical nonlinear hysteresis phenomenon. How does this scenario affect the tunneling process? As long as $\gamma/\Omega > -2$, for $\varepsilon \to 0$ the system will stay arbitrarily close to S_+ over the whole evolution of the system so that $P_T \to 0$. For $\gamma/\Omega <$

-2, however, the hysteresis effect gives rise to a different behavior that is sketched in Fig. 2(b). For small ε the system will again remain close to S_+, initially. However, as U_+ approaches S_+ it eventually has to switch from its trajectory in the vicinity of S_+ onto a large orbit C encircling S'_+ [16]. Subsequently, the system evolution is quasiadiabatic as $t \to \infty$. The final tunneling probability is thus determined by the canonical action of the orbit C, which is an adiabatic invariant in this case. We thus conclude that in the limit $\Omega^2/\varepsilon \to \infty$ the tunneling probability is given by

$$P_T^{(\infty)} = P_T(\Omega^2/\varepsilon \to \infty) = \frac{1}{2\pi} \oint_{C^\star} \Theta \, dN_+, \qquad (10)$$

with $C^\star$ the orbit passing through $U_+ = S_+$ at $\Delta = \Delta_c$ and encircling S'_+. For small $\varepsilon > 0$ the orbit C, onto which the system switches, completely encloses $C^\star$. Therefore, $P_T^{(\infty)}$ can be expected to be a (at least local) minimum of the tunneling probability. In Fig. 2(c) $P_T^{(\infty)}$ is shown as a function of $-\gamma/\Omega$. It becomes apparent that, even for modest values of $-\gamma/\Omega$, $P_T^{(\infty)}$ is not small compared to 1. The above discussion is confirmed by the numerical simulation of Eqs. (4). In particular, it is found that for fixed $-\gamma/\Omega < -2$, P_T is a monotonically decreasing function of Ω^2/ε (cf. Fig. 1). As $\Omega^2/\varepsilon \to \infty$, it tends to the value given by Eq. (10) which is thus a global lower bound on the tunneling probability. Furthermore, for small ε, the canonical action along the system trajectory abruptly changes around Δ_c, as expected.

C. Rapid passage through resonance

Having discussed the asymptotic limit of the tunneling probability we now study more closely its behavior for small values of Ω^2/ε, i.e., rapid passage through the resonance. In this case some insight may be gained from a perturbative analysis of Eqs. (4) with Ω the small parameter. In a rotating frame with $a_+ = \tilde{a}_+ \exp(-i\varepsilon t^2/2 - i\gamma t)$, $a_- = \tilde{a}_-$, Eqs. (4) read

$$i\dot{\tilde{a}}_+ = \Omega \tilde{a}_- \exp(i\varepsilon t^2/2 + i\gamma t) + \gamma(|\tilde{a}_+|^2 - 1)\tilde{a}_+, \qquad (11)$$

$$i\dot{\tilde{a}}_- = \Omega \tilde{a}_+ \exp(-i\varepsilon t^2/2 - i\gamma t) + \gamma|\tilde{a}_-|^2 \tilde{a}_-. \qquad (12)$$

These equations are now iterated in the standard way with initial conditions $\tilde{a}_+^{(0)}(-\infty) = 1$, $\tilde{a}_-^{(0)}(-\infty) = 0$ up to third order. This yields

$$\begin{aligned}\tilde{a}_-^{(3)}(t) = & -i\Omega \int_{-\infty}^{t} dt_1 \, \mathcal{E}(t_1) \\ & + i\Omega^3 \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 \int_{-\infty}^{t_2} dt_3 \, \mathcal{E}(t_1)\mathcal{E}^*(t_2)\mathcal{E}(t_3) \\ & - \gamma\Omega^3 \int_{-\infty}^{t} dt_1 \left| \int_{-\infty}^{t_1} dt_2 \mathcal{E}(t_2) \right|^2 \int_{-\infty}^{t_1} dt_2 \, \mathcal{E}(t_2) \\ & + O(\Omega^5) \\ \equiv & \, \Omega T_1 + \Omega^3 T_3 + O(\Omega^5), \end{aligned} \qquad (13)$$

with $\mathcal{E}(t) = \exp(-i\varepsilon t^2/2 - i\gamma t)$. The first integral converges for $t \to \infty$, whereas the other two diverge. This behavior is to be expected because, e.g., in the linear problem ($\gamma = 0$), although the moduli of $\tilde{a}_\pm$ converge, their phases contain logarithmically divergent terms. Their influence is reflected in the divergence of the Ω^3 contributions. A convergent approximation of P_T up to order Ω^4 can nevertheless be obtained from $\tilde{a}_-^{(3)}(t \to \infty)$. Writing $P_T \approx 1 - |\tilde{a}_-^{(3)}(t \to \infty)|^2 = 1 - \Omega^2 |T_1|^2 - 2\Omega^4 \, \mathrm{Re}(T_1^* T_3) + O(\Omega^6)$, it is found that the expression $\mathrm{Re}(T_1^* T_3)$ converges; all divergent contributions in the Ω^3 integrals are contained in $\mathrm{Im}(T_1^* T_3)$. Determining the limits $t \to \infty$ of the relevant expressions the approach finally yields

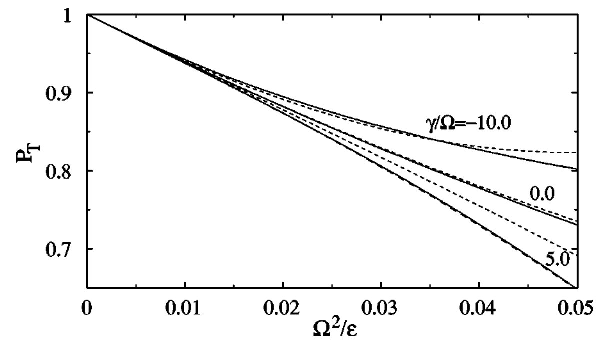

FIG. 3. Numerical calculation (full curves) of P_T and approximation according to Eq. (14) (dashed curves) for $\gamma/\Omega = -10.0, 0.0, 5.0$. The result of Eq. (14) for $\gamma/\Omega = 10.0$ is given by the long-dashed curve that is close to the numerical one for $\gamma/\Omega = 5.0$.

$$P_T = 1 - 2\pi \frac{\Omega^2}{\varepsilon} + 2\pi^2 \frac{\Omega^4}{\varepsilon^2} - \mathcal{N} \frac{\gamma\Omega^4}{\varepsilon^{2.5}} + O(\Omega^6). \qquad (14)$$

The first three terms are familiar from the expansion of the Landau-Zener formula $P_T = \exp(-2\pi\Omega^2/\varepsilon)$ for the linear problem. The fourth term represents the first nonvanishing contribution of the nonlinearity. The numerical coefficient $\mathcal{N}$ is equal to $\mathrm{Re}[e^{-i\pi/4}\sqrt{128\pi}\int_{-\infty}^{\infty} dt |\mathcal{F}(t)|^2 \mathcal{F}(t)] \approx 15.751$ with $\mathcal{F}(t) = \int_{-\infty}^{t} dt_1 \exp(-it_1^2)$.

At this point it has to be emphasized that it is not claimed that Eq. (14) is rigorously valid as an expansion of P_T around $\Omega = \gamma = 0$. Nevertheless, it yields the following useful information. (i) For very rapid passage through resonance the nonlinear interactions do not influence the tunneling probability. Their contributions arise only in higher order in Ω (see Fig. 3). (ii) Equation (14) correctly predicts that for $\gamma > 0$ ($\gamma < 0$) P_T is diminshed (increased). (iii) Quantitatively, Eq. (14) gives a good approximation of P_T, in particular for negative γ, and could be used, e.g., at $\gamma/\Omega = -10$ to estimate P_T for $\Omega^2/\varepsilon < 0.04$ (cf. Fig. 3). For positive γ the agreement is not as good. In the interesting regime $1 < \gamma/\Omega < 10$, $\Omega^2/\varepsilon < 0.04$, the nonlinear effects are underestimated by a factor of about 2 [see result of Eq. (14) for $\gamma/\omega = 10$ in Fig. 3]. This is because the influence of higher-order terms in γ, which are neglected in Eq. (14), is more relevant for positive γ.

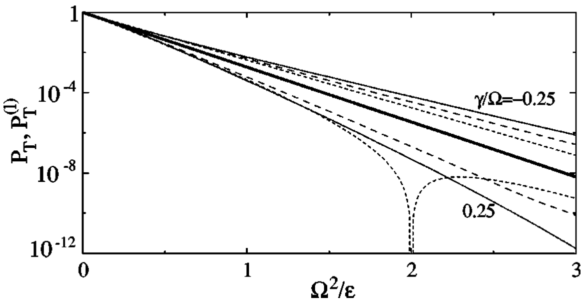

FIG. 4. Tunneling probability for $\gamma/\Omega=\pm 0.25$. Full curves, exact results obtained from Eqs. (4); short-dashed curves, approximation according to Eqs. (15). The long-dashed curves show the tunneling probability with $|a_+|^2$ in Eqs. (4) replaced by the exact stationary-state population as determined from Eq. (7). For larger $|\gamma|/\Omega$ the discrepancy between the approximations and the exact result is even more pronounced. The thick line shows P_T for $\gamma=0$.

D. Slow passage through resonance

To examine the tunneling probability for slow passage through resonance in more detail, we again use the idea that under these circumstances the system stays close to the instantaneous stationary state S_+. A simple analytically tractable model is obtained if in Eqs. (4) $|a_+|^2$ is replaced by the stationary-state population $N_{S,+}(t)$ of the *linear* system, i.e., $|a_+|^2\approx 1/2-\Delta/[4(\Delta^2/4+\Omega^2)^{1/2}]$ with $\Delta=\varepsilon t$. The tunneling probability $P_T^{(l)}$ for this system can be estimated with the help of the semiclassical theory of nonadiabatic transitions [14,17–19], which is based on the study of the quasienergy function $Q(t)=[\Delta_{eff}^2(t)/4+\Omega^2]^{1/2}$. Thereby, $\Delta_{eff}=\varepsilon t+\gamma(2|a_+|^2-1)$. If t_k are the zeroes (or complex crossing points) of $Q(t)$ in the upper complex half plane, the tunneling probability can be approximated as $P_T^{(l)}=|\Sigma_k\Gamma_k\exp(i\mathcal{D}_k)|^2$ with $\mathcal{D}_k=2\int_0^{t_k}Q(t)dt$ and $\Gamma_k=-2i\lim_{t\to t_k}(t-t_k)\Omega\dot{\Delta}_{eff}(t)/Q^2(t)$. As briefly shown in the Appendix, for $|\gamma|/\Omega\lesssim 1$, $P_T^{(l)}$ is well approximated by

$$P_T^{(l)}=\exp(-I')[2\cos(2\pi\gamma\Omega/3\varepsilon)-1]^2,\ \gamma>0,$$

$$P_T^{(l)}=\exp(-I'),\ \gamma<0, \tag{15}$$

with $I'=2\Omega^2\{\pi-\gamma[\frac{2}{3}+2\ln(\sqrt[3]{|\gamma|/\Omega}/2)]\}/\varepsilon$. Comparison to the solution of Eqs. (4) shows that the formula for $\gamma>0$ gives a useful approximation to P_T well before the first minimum of $P_T^{(l)}$, i.e., if $\Omega^2/\varepsilon\lesssim\Omega/4\gamma$ (see Fig. 4). For negative γ Eqs. (15) typically underestimate the nonlinear effects, but they yield a rough first estimate for $-\gamma/\Omega$ less than about 0.25. More interestingly, however, Eqs. (15) show a general behavior that is observed for the tunneling probability of the nonlinear problem as well (see Fig. 1). If $\gamma<0$, $P_T^{(l)}$ is a monotonically decreasing function of Ω^2/ε, whereas for $\gamma>0$ it is oscillating. This behavior is a consequence of the structure of the complex crossing points of the quasienergy. For $\gamma>0$ three crossing points contribute to $P_T^{(l)}$, thus causing interference effects, whereas for $\gamma<0$ there is only one. With a theory similar to the one of Ref. [17] not available for nonlinear systems, the study of this model problem gives an idea of how these features in the behavior of P_T may come about and shows that they appear in a broader class of systems. Numerical studies of the original Eqs. (4) indicate, however, that at fixed $\gamma/\Omega>0$ the minima of P_T approximately occur at positions $kC\Omega^2/\varepsilon$ with $k=1,2,\ldots$, and C a decreasing function of γ/Ω. This is in clear contrast to the behavior of Eqs. (15); furthermore, the first minimum appears at a much larger distance from $\Omega^2/\varepsilon=0$ than indicated by these equations. One might assume that it is possible to improve on the above results by studying Eqs. (4) with $|a_+|^2$ replaced by the exact stationary-state population $N_{S,+}(t)$ as determined from Eq. (7). Unfortunately, the analytical theory of Ref. [17] is not applicable in this case as no complex crossing points appear. The numerically obtained tunneling probability has the same qualitative features as discussed above; quantitatively, however, the approximation to the exact P_T is not as good as one might intuitively expect (cf. Fig. 4).

IV. BLOCH BAND TUNNELING WITH BOSE-EINSTEIN CONDENSATES

In this section the prospects of experimentally observing the nonlinear effects discussed in Sec. III are examined with a focus on Bloch band tunneling. To this end a one-dimensional simulation of Eq. (1) for a sodium condensate is studied. For the calculation, a condensate extension of $l_z=160\ \mu$m and a period of the optical potential $\lambda=0.5\ \mu$m are used. With $\Omega=10^4\ s^{-1}$ higher-order momentum components are detuned by an amount of at least 40Ω from the "$\pm$" modes when the resonance is crossed. The two conditions for the applicability of approximations (3),(4) which are given in Sec. II, are thus well fulfilled. When the wave packet reaches the tunneling resonance at $3\hbar k_L$ that follows the initial one at $\hbar k_L$ the two momentum components + and − have a detuning of 40Ω, i.e., the tunneling process is well defined. For $\Omega^2/\varepsilon\approx 1$ the acceleration is of the order 10 ms^{-2}. The numerical simulations show that on the time scale of passing through the resonance (less than 10^{-2} s in the examples) the spreading of the initial wave packet as well as the spatial shift between the two modes is small. The crucial parameter γ takes on the value $10^{-16}n_c\ m^3s^{-1}$, with n_c the condensate density. In our example a ratio of $\gamma/\Omega=1$ can be reached for a BEC with $N=5\times 10^6$ atoms and a radius of 10 μm. All the parameter values given are well within the realm of currently feasible experiments [2].

The numerical simulations of Eq. (1) are performed with a standard split-step algorithm (see, e.g., [20]). Thereby, the initial state is taken as the ground state of the Gross-Pitaevskii equation for a harmonic potential yielding the required value of l_z. The main results of these calculations are shown in Figs. 5. In these diagrams the fractional population P_+ of the initial momentum component is displayed as a function of $\tau=tm\varepsilon/4\hbar^2k_L^2$, i.e., time measured in units of the temporal distance between two resonance crossings. The point $\tau=0$ corresponds to zero detuning between the two relevant momentum modes. Numerically, $P_+(t)$ is determined from the Fourier transform of $\psi(z,t)$. Experimentally,

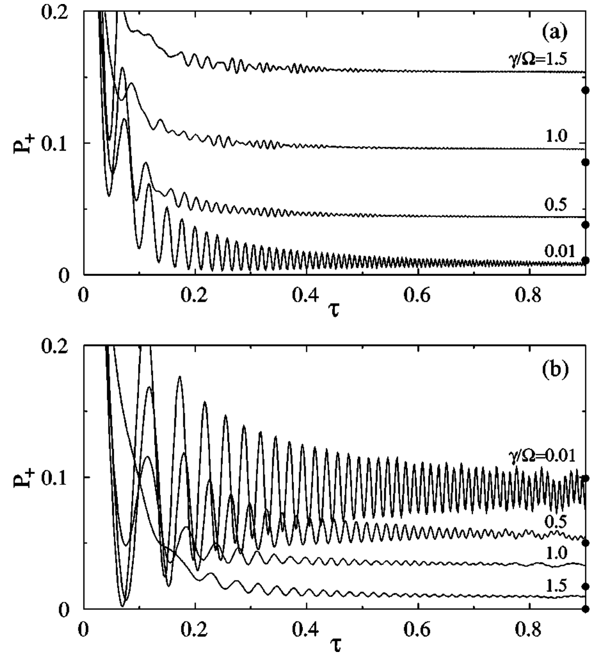

FIG. 5. Fractional population P_+ of the initial momentum component as a function of scaled time $\tau = tm\varepsilon/4\hbar^2 k_L^2$ for various values of γ/Ω. In (a) the initial wave-packet velocity $v_i = 0$ and the acceleration is chosen to obtain a tunneling probability of $P_L = 0.01$ according to the Landau-Zener formula in the absence of nonlinear interactions. In (b) $v_i = -2\hbar/k_L m$ and $P_L = 0.1$. The predictions of Eqs. (4) for P_T at the respective parameter values are indicated by filled circles ●.

it could be obtained by switching the periodic potential off at t and observing the subsequent condensate evolution. In Fig. 5(a) condensates with various values of γ/Ω are started at rest, with the acceleration chosen such that the standard Landau-Zener formula (disregarding the atomic interactions) predicts a tunneling probability of 0.01. Having passed the resonance, P_+ quickly settles down to oscillations around a well-defined mean value that may be regarded as the tunneling probability. These mean values depend somewhat sensitively on the initial conditions. For example, they may vary within an interval of width 0.02 if the initial velocity is changed by several $\pm 0.01\hbar k_L/m$. Nevertheless, they compare reasonably well to the asymptotic tunneling probability P_T (indicated in Fig. 5 with filled circles ●) obtained from Eqs. (4) for the corresponding parameter values. Typically, the values for P_T are smaller than the results obtained from the Gross-Pitaevskii equation by approximately 0.01–0.02. Most importantly, however, it is seen that for $\gamma/\Omega = 1.5$ the tunneling probability is increased by almost an order of magnitude compared to the case $\gamma/\Omega = 0.01$. These results show that P_T is very sensitive to atomic interactions in experimentally relevant situations. Figure 5(b) illustrates that one can also observe a significant decrease in the tunneling probability. As discussed in Sec. III A, this occurs if the condensate starts from the high-energy state; i.e., in the present situation it needs to be launched with a velocity less than $-\hbar k_L/m$. In Fig. 5(b) the initial velocity $-2\hbar k_L/m$ was used that places the wave packet in the middle of two tunneling resonances. The acceleration was chosen such that without nonlinear interactions a tunneling probability of 0.1 is expected. For smaller values of γ/Ω there is again good agreement between the predictions of the two-mode model and the solution of the Gross-Pitaevskii equation. At $\gamma/\Omega = 2.0$, however, Eqs. (4) predict a tunneling probability of about 10^{-4}, whereas the simulation yields a significantly higher value of 0.01.

V. CONCLUSION

In this paper the influence of atomic interactions on time-dependent tunneling processes of Bose-Einstein condensates was investigated. A nonlinear Landau-Zener equation was derived that describes main aspects of processes such as Bloch band tunneling, ground-state population transfer with variable external fields, and condensate motion on coupled potential surfaces. The tunneling probabilities predicted by this model were discussed in detail; in particular, it was shown that for strong enough nonlinearities a complete population transfer by adiabatic following is impossible. This behavior is a consequence of a nonlinear hysteresis effect. To assess the reliability of the nonlinear Landau-Zener model a comparison to simulations of the Gross-Pitaevskii equation was made.

In actual experiments it is only possible to determine the tunneling probability for a certain portion of the $(\Omega^2/\varepsilon, \gamma/\Omega)$ parameter space. For example, the acceleration cannot be made arbitrarily small so that the asymptotic behavior of the tunneling probability shown in Fig. 2(b) may never be verified directly. Furthermore, tunneling probabilities close to 0 or 1 are very difficult to measure accurately due to the long-time oscillations of the mode populations. Nevertheless, as shown in Sec. IV, drastic nonlinear effects already appear in the experimentally accessible parameter space and they are appropriately described with the simple model of Eqs. (4). Its detailed study is thus well justified.

However, to obtain a full understanding of time-dependent tunneling it is necessary to expand the investigation beyond the limits of applicability of Eqs. (4). In the context of Bloch band tunneling new features in the system behavior may arise; for example, if the extension of the condensate becomes comparable to the period of the optical potential. Another potentially interesting question concerns the study of quantum-mechanical effects beyond the mean-field description that was applied here.

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APPENDIX: DERIVATION OF EQS. (15)

The complex crossing points t_k are obtained as solutions of

$$Q^2(t) = \frac{1}{4}\left(\Delta - \frac{\gamma\Delta}{2\sqrt{\Delta^2/4 + \Omega^2}}\right)^2 + \Omega^2 = 0, \qquad \text{(A1)}$$

with $\Delta=\varepsilon t$, which can be converted into a quartic equation. If $\gamma<0$ there is only one root in the upper complex half-plane, which is given by

$$\Delta_1=\varepsilon t_1=2i\Omega[1-(|\gamma|/\Omega)^{2/3}/2+O(|\gamma|^{4/3})].$$

The integral $\mathcal{D}_1$ is approximated by expanding the integrand as

$$Q(t)\approx\sqrt{\Delta^2/4+\Omega^2}-\gamma\Delta^2/[8(\Delta^2/4+\Omega^2)].$$

The integration can then be carried out analytically and the second of Eqs. (15) is obtained with $\Gamma_1=-1$.

In the case $\gamma>0$, Eq. (A1) has three solutions in the upper complex half-plane. They are given by

$$\Delta_{1,2}=2i\Omega\{1+e^{\pm i\pi/3}(\gamma/\Omega)^{2/3}/2+O(\gamma^{4/3})\},$$

$$\Delta_3=2i\Omega\left[1-\frac{1}{32}\frac{\gamma^2}{\Omega^2}+O(\gamma^4)\right],$$

with $\Gamma_{1,2}=-1$ and $\Gamma_3=1$. The integrals $\mathcal{D}_{1,2}$ are determined as in the case $\gamma<0$. The integral $\mathcal{D}_3$ cannot be dealt with in this way. However, a numerical analysis shows that for $\gamma/\Omega<1$ one can set to a good degree of approximation $\mathcal{D}_3=i\,\mathrm{Im}(\mathcal{D}_{1,2})$. This approach yields the first of Eqs. (15). The comparison of these equations to the numerically determined $P_T^{(l)}$ confirms the accuracy of the approximation for $|\gamma|/\Omega\lesssim 1$.

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