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Perturbations in cosmologies with a scalar field and a perfect fluid

Nicola Bartolo,[1] Pier-Stefano Corasaniti,[2] Andrew R. Liddle,[1] and Michaël Malquarti[1]
[1]*Astronomy Centre, University of Sussex, Brighton BN1 9QH, United Kingdom*
[2]*ISCAP, Columbia University, Mailcode 5247, New York, New York 10027, USA*
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We study the properties of cosmological density perturbations in a multi-component system consisting of a scalar field and a perfect fluid. We discuss the number of degrees of freedom completely describing the system, introduce a full set of dynamical gauge-invariant equations in terms of the curvature and entropy perturbations, and display an efficient formulation of these equations as a first-order system linked by a fairly sparse matrix. Our formalism includes spatial gradients, extending previous formulations restricted to the large-scale limit, and fully accounts for the evolution of an isocurvature mode intrinsic to the scalar field. We then address the issue of the adiabatic condition, in particular demonstrating its preservation on large scales. Finally, we apply our formalism to the quintessence scenario and clearly underline the importance of initial conditions when considering late-time perturbations. In particular, we show that entropy perturbations can still be present when the quintessence field energy density becomes non-negligible.

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I. INTRODUCTION

The material content of the universe is commonly assumed to be a mixture of fluids, such as radiation or nonrelativistic matter, and scalar fields, either driving a period of early universe inflation [1] or playing the role of dark energy (quintessence) in the present universe [2–4]. The latter possibility has motivated a number of works devoted to the study of cosmological perturbations in a multi-component system consisting of fluids and a scalar field, for instance Refs. [3–12]. Nevertheless, the literature contains some contradictory statements concerning the properties of such perturbations.

In this paper we aim to resolve these discrepancies and will provide a comprehensive analysis of the problem. We will study the role of intrinsic entropy perturbation in the scalar field, and whether the notion of adiabaticity is preserved by the dynamics of the multi-component system when the evolution of such an intrinsic entropy perturbation is explicitly accounted for. In the process of doing so, we will discuss the number of degrees of freedom which completely describe the system and we will find a highly efficient formulation of the perturbation equations including the effects of spatial gradients.

Finally, using our formalism we will specifically discuss the quintessence scenario. We will correct some common misconceptions, discuss the evolution of entropy perturbations, and clearly show the importance of initial conditions when considering late-time perturbations. In particular, we will show that entropy perturbations can be enhanced by the evolution of the field and may still be present when its density is no longer negligible. This is an important result which has generally been overlooked when studying dark energy models.

II. THE DYNAMICAL EQUATIONS

A. Background

Our approach builds on an earlier paper by Malquarti and Liddle [11], and we will largely follow the notation of that article but with some differences in definitions. We assume a flat Friedmann–Robertson–Walker universe throughout, with the background evolution determined by the usual equations

$$3H^2=3\left(\frac{\dot{a}}{a}\right)^2=8\pi G\rho_{\rm tot}, \tag{1}$$

$$2\dot{H}+3H^2=2\frac{\ddot{a}}{a}+\left(\frac{\dot{a}}{a}\right)^2=-8\pi G p_{\rm tot}, \tag{2}$$

where $H\equiv\dot{a}/a$ is the Hubble parameter, a is the scale factor, a dot stands for a derivative with respect to cosmic time t and the subscript "tot" always refers to the sum over all matter components. The fundamental ingredients we consider here are a perfect fluid with constant equation of state $w_{\rm f}\equiv p_{\rm f}/\rho_{\rm f}$ and a minimally coupled scalar field φ with potential $V(\varphi)$. Since we treat the fluid and the scalar field as uncoupled, the conservation of their respective energy-momentum tensors gives

$$\dot{\rho}_{\rm f}=-3H(1+w_{\rm f})\rho_{\rm f} \;\;\Rightarrow\;\; \rho_{\rm f}\propto a^{-3(1+w_{\rm f})}, \tag{3}$$

$$\ddot{\varphi}=-3H\dot{\varphi}-\frac{dV}{d\varphi}. \tag{4}$$

The subscripts "f" and "φ" will always refer to the perfect fluid and the scalar field, respectively.

Useful parameters describing completely the background properties of the scalar field are its equation of state

$$w_\varphi\equiv\frac{p_\varphi}{\rho_\varphi}=\frac{\dot{\varphi}^2/2-V(\varphi)}{\dot{\varphi}^2/2+V(\varphi)}, \tag{5}$$

and its adiabatic sound speed

$$c_{s\varphi}^2\equiv\frac{\dot{p}_\varphi}{\dot{\rho}_\varphi}=w_\varphi-\frac{\dot{w}_\varphi}{3H(1+w_\varphi)}=1+\frac{2}{3}\frac{dV/d\varphi}{H\dot{\varphi}}. \tag{6}$$

We also introduce the total equation of state $w \equiv p_{\rm tot}/\rho_{\rm tot}$ and the total sound speed $c_s^2 \equiv \dot{p}_{\rm tot}/\dot{\rho}_{\rm tot}$, and in order to simplify some expressions we define for each component $\gamma_x \equiv 1 + w_x$ and also $\gamma \equiv 1 + w$.

B. Perturbations

We consider only scalar perturbations and we choose to work in Newtonian gauge [13], where the perturbed metric reads as

$$ds^2 = -(1+2\Phi)dt^2 + a^2(t)(1-2\Psi)d\mathbf{x}^2. \tag{7}$$

Here Φ and Ψ describe the metric perturbation, and in this case are equal to the gauge-invariant potentials defined in Ref. [13]. We work in Fourier space and compute the first-order perturbed Einstein equations. As our system has no anisotropic stress, the $(i-j)$ Einstein equations imply that the metric potentials are equal, $\Psi = \Phi$. The remaining Einstein equations are

$$-3H(H\Phi + \dot{\Phi}) - \frac{k^2}{a^2}\Phi = 4\pi G\,\delta\rho_{\rm tot}, \tag{8}$$

$$\ddot{\Phi} + 4H\dot{\Phi} + (2\dot{H} + 3H^2)\Phi = 4\pi G\,\delta p_{\rm tot}, \tag{9}$$

$$-(H\Phi + \dot{\Phi}) = 4\pi G\,\delta q_{\rm tot}, \tag{10}$$

where $\nabla \delta q_{\rm tot}$ is the total momentum perturbation of the system. Equation (8) comes from the $(0-0)$ Einstein equation, Eq. (9) from the $(i-i)$ Einstein equation, while Eq. (10) is obtained from the $(0-i)$ Einstein equation. The perfect fluid and scalar field perturbation variables are

$$\delta p_{\rm f} = w_{\rm f}\delta\rho_{\rm f}, \tag{11}$$

$$\delta q_{\rm f} = \rho_{\rm f}\gamma_{\rm f}\mathcal{V}_{\rm f}, \tag{12}$$

$$\delta\rho_\varphi = \dot{\varphi}\,\delta\dot{\varphi} - \dot{\varphi}^2\Phi + \frac{dV}{d\varphi}\,\delta\varphi, \tag{13}$$

$$\delta p_\varphi = \dot{\varphi}\,\delta\dot{\varphi} - \dot{\varphi}^2\Phi - \frac{dV}{d\varphi}\,\delta\varphi, \tag{14}$$

$$\delta q_\varphi = -\dot{\varphi}\,\delta\varphi, \tag{15}$$

where $\mathcal{V}_{\rm f}$ is the fluid velocity potential defined so that the fluid velocity is given by $\delta\mathbf{u} \equiv \nabla\mathcal{V}_{\rm f}$—this is possible since for scalar perturbations the flow is irrotational. Note that this definition is slightly different from the one used in Ref. [11]. The conservation of the energy-momentum tensors for each component provides the equations

$$\delta\ddot{\varphi} + 3H\delta\dot{\varphi} + \frac{k^2}{a^2}\delta\varphi + \frac{d^2V}{d\varphi^2}\delta\varphi = 4\dot{\varphi}\dot{\Phi} - 2\frac{dV}{d\varphi}\Phi, \tag{16}$$

$$\dot{\delta}_{\rm f} - 3\gamma_{\rm f}\dot{\Phi} = \gamma_{\rm f}\frac{k^2}{a^2}\mathcal{V}_{\rm f}, \tag{17}$$

$$\dot{\mathcal{V}}_{\rm f} = 3Hw_{\rm f}\mathcal{V}_{\rm f} - \frac{w_{\rm f}}{\gamma_{\rm f}}\delta_{\rm f} - \Phi, \tag{18}$$

where $\delta_{\rm f} \equiv \delta\rho_{\rm f}/\rho_{\rm f}$. We also define $\delta_\varphi \equiv \delta\rho_\varphi/\rho_\varphi$ and $\delta_\pi \equiv \delta p_\varphi/\rho_\varphi$. These equations are the perturbed Euler–Lagrange equation for the scalar field and the continuity and Euler equations for the fluid.

As we will see later, it is useful to introduce the comoving density perturbation for each component [14]

$$\epsilon_x \equiv \delta\rho_x - 3H\delta q_x, \tag{19}$$

which is a gauge-invariant quantity. We also introduce the gauge-invariant entropy perturbation variables [15–17], namely the relative entropy perturbation between the fluid and the scalar field

$$S \equiv \frac{3H\gamma_{\rm f}\gamma_\varphi\Omega_{\rm f}}{\gamma}\left(\frac{\delta\rho_\varphi}{\dot{\rho}_\varphi} - \frac{\delta\rho_{\rm f}}{\dot{\rho}_{\rm f}}\right) = \Omega_{\rm f}\frac{\gamma_\varphi\delta_{\rm f} - \gamma_{\rm f}\delta_\varphi}{\gamma}, \tag{20}$$

and the intrinsic entropy perturbation of the scalar field

$$\Gamma \equiv \frac{3H\gamma_\varphi c_{s\varphi}^2}{1-c_{s\varphi}^2}\left(\frac{\delta\rho_\varphi}{\dot{\rho}_\varphi} - \frac{\delta p_\varphi}{\dot{p}_\varphi}\right) = \frac{\delta_\pi - c_{s\varphi}^2\delta_\varphi}{1-c_{s\varphi}^2}, \tag{21}$$

where $\delta_\pi \equiv \delta p_\varphi/\rho_\varphi$. The normalizations have been chosen in order to simplify some later expressions. Note that Γ is well defined even if $c_{s\varphi}^2 \simeq 1$ since, as can easily be shown, we have $\Gamma = \epsilon_\varphi/\rho_\varphi$. By definition the perfect fluid does not have an intrinsic entropy perturbation. Adiabaticity is defined by the condition $S = \Gamma = 0$, since in this case it is possible to define a slicing for which all matter component perturbations vanish.

C. Degrees of freedom

The system of Eqs. (8), (9), (10), (16), (17) and (18) describes the evolution of four variables, namely Φ, $\rho_{\rm f}$, $\mathcal{V}_{\rm f}$ and $\delta\varphi$. Equations (9) and (16) are second order and if we introduce two new variables for $\dot{\Phi}$ and $\delta\dot{\varphi}$ (and therefore two new equations) we end up with six variables describing the perturbations, six first-order dynamical equations and two constraint equations [Eqs. (8) and (10)]. The two constraint equations reduce the number of degrees of freedom to 4 and as a result two dynamical equations must be redundant. This comes from the fact that the conservation of the total energy-momentum tensor is a consequence of the Einstein equations and therefore the conservation equations for one matter component implies the ones for the other. As a result, it is possible to write this system as four differential equations for the four dynamical degrees of freedom which completely describe the perturbations, e.g., Φ, $\delta_{\rm f}$, δ_φ and δ_π, which are the basic variables studied in Ref. [11] in the large-scale limit $k/aH \ll 1$. A general solution to those equations permits both an isocurvature perturbation between the scalar field density contrast and the fluid density contrast, and an isocurvature perturbation intrinsic to the scalar field, i.e., between its density and pressure perturbations.

For our discussion, it is useful to combine Eqs. (8) and (10) and find the constraint equation

$$\frac{k^2}{a^2}\Phi = -4\pi G \epsilon_{\rm tot}. \tag{22}$$

Note that this is a gauge-invariant equation, though had we included an anisotropic stress then Φ must be replaced by the second metric potential Ψ [18]. If the fluid is completely absent, so that we simply have a single scalar field, the constraint equation, Eq. (22), reduces to

$$\frac{k^2}{a^2H^2}\Phi = -\frac{4\pi G}{H^2}(\dot{\varphi}\delta\dot{\varphi} - \dot{\varphi}^2\Phi - \ddot{\varphi}\delta\varphi) = -\frac{3}{2}\Gamma. \tag{23}$$

The system is completely described by two dynamical degrees of freedom and this equation implies that one of the scalar field degrees of freedom is removed, e.g., $\delta\dot{\varphi}$. The right-hand side of Eq. (23) is simply proportional to the intrinsic entropy perturbation of the scalar field Γ, hence in the large-scale limit this is forced to vanish if Φ is to remain small. This is a known result already shown in Refs. [18,19].

By contrast, once a fluid is added we have

$$\frac{k^2}{a^2H^2}\Phi = -\frac{3}{2}[\Omega_\varphi\Gamma + \Omega_{\rm f}(\delta_{\rm f} - 3H\gamma_{\rm f}\mathcal{V}_{\rm f})]. \tag{24}$$

This equation shows that the fluid comoving density perturbation can compensate the scalar field intrinsic entropy perturbation, and, as a result, in the presence of a fluid it is possible to have a non-vanishing scalar field intrinsic entropy perturbation even on large scales. Note that the presence of the fluid changes the structure of the equations even if it is a sub-dominant component of the total energy density. This is because the fluid creates a new set of hypersurfaces, those on which its density is uniform, which need not align with hypersurfaces of uniform scalar field density.

Since we are interested in studying the evolution of isocurvature and adiabatic modes, we find it useful to use the gauge-invariant comoving curvature perturbation [13]

$$\mathcal{R} \equiv \frac{2(H\Phi + \dot{\Phi})}{3\gamma H} + \Phi. \tag{25}$$

The equation of motion for $\mathcal{R}$ is given by [13]

$$\dot{\mathcal{R}} = \frac{2}{3H\gamma}\left[-c_{\rm s}^2\frac{k^2}{a^2}\Phi + 4\pi G\,\delta p_{\rm nad}\right], \tag{26}$$

where $\delta p_{\rm nad} \equiv \delta p_{\rm tot} - c_{\rm s}^2\delta\rho_{\rm tot}$ is the non-adiabatic pressure perturbation. Note that even on large scales $\mathcal{R}$ can evolve due to the presence of a non-vanishing non-adiabatic pressure perturbation, as recently stressed in different works [16,15]. The non-adiabatic pressure perturbation depends on the intrinsic and relative entropy perturbations [17,16], and in our case we find

$$\frac{\delta p_{\rm nad}}{\rho_{\rm tot}} = \Omega_\varphi[(w_{\rm f} - c_{{\rm s}\varphi}^2)S + (1 - c_{{\rm s}\varphi}^2)\Gamma]. \tag{27}$$

From now on we find it convenient to describe the system in terms of the gauge-invariant variables Φ, $\mathcal{R}$, S and Γ, rather than the set of variables Φ, $\delta_{\rm f}$, δ_φ, and δ_π. Note that such a change of variables is completely determined by Eqs. (20), (21) and the expression

$$\mathcal{R} = \Phi - \frac{1}{3\gamma}(\Omega_\varphi\delta_\varphi + \Omega_{\rm f}\delta_{\rm f}) - \frac{2}{9\gamma}\frac{k^2}{a^2H^2}\Phi, \tag{28}$$

obtained from Eqs. (8) and (25). In the next section we find a first-order system of dynamical equations expressed in these variables.

III. MATRIX FORMULATION

A. Evolution equation

In the long-wavelength limit ($k/aH \ll 1$) Malquarti and Liddle [11] were able to express the dynamical equations in a first-order matrix formulation, using $N \equiv \log(a/a_0)$ as a time variable. They took as basic variables Φ, $\delta_{\rm f}$, δ_φ, and δ_π. Here we show that our set of variables can bring the matrix into an even more efficient form. Moreover, we compute the general equations without the long-wavelength approximation.

We define the vector $\mathbf{v} \equiv (\Phi, \mathcal{R}, S, \Gamma)^T$ and use a prime to denote a derivative with respect to N. Lengthy but straightforward algebra leads to the expression

$$\mathbf{v}' = \left[\mathcal{M}_0 + \mathcal{M}_1\frac{k^2}{a^2H^2} + \mathcal{M}_2\frac{k^4}{a^4H^4}\right] \times \mathbf{v}, \tag{29}$$

where the only relevant matrix for the long-wavelength approximation ($k/aH = 0$) is given by

$$
\mathcal{M}_0=\begin{pmatrix} -(1+3\gamma/2) & 3\gamma/2 & 0 & 0 \\ 0 & 0 & \Omega_\varphi(w_{\rm f}-c_{s\varphi}^2)/\gamma & \Omega_\varphi(1-c_{s\varphi}^2)/\gamma \\ 0 & 0 & 3(w_\varphi-w_{\rm f})+3\gamma_{\rm f}\Omega_{\rm f}(w_{\rm f}-c_{s\varphi}^2)/\gamma & 3\gamma_{\rm f}\Omega_{\rm f}(1-c_{s\varphi}^2)/\gamma \\ 0 & 0 & -3\gamma/2 & 3(w_\varphi-\gamma/2) \end{pmatrix}, \tag{30}
$$

and the two matrices incorporating spatial gradients are

$$
\mathcal{M}_1=\begin{pmatrix} 0 & 0 & 0 & 0 \\ -2c_s^2/3\gamma & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 1/3 \\ 0 & -\gamma_\varphi & -1/3 & -1/3 \end{pmatrix}, \tag{31}
$$

and

$$
\mathcal{M}_2=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2\gamma_\varphi/9\gamma & 0 & 0 & 0 \\ -2\gamma_\varphi/9\gamma & 0 & 0 & 0 \end{pmatrix}. \tag{32}
$$

The different non-vanishing entries clearly show the couplings between adiabatic and relative/intrinsic entropy perturbations on large scales ($\mathcal{M}_0$) and on small scales ($\mathcal{M}_1$ and $\mathcal{M}_2$). The first two lines of the matrices (dynamical equations for Φ and $\mathcal{R}$) are straightforward from Eqs. (26) and (25). The equation for the relative entropy, here expressed as

$$
S'=\left[3(w_\varphi-w_{\rm f})+\frac{3\gamma_{\rm f}\Omega_{\rm f}(w_{\rm f}-c_{s\varphi}^2)}{\gamma}\right]S+\frac{3\gamma_{\rm f}\Omega_{\rm f}(1-c_{s\varphi}^2)}{\gamma}\Gamma + \frac{k^2}{a^2H^2}\left[\frac{1}{3}S+\frac{1}{3}\Gamma\right]+\frac{k^4}{a^4H^4}\left[\frac{2}{9}\frac{\gamma_\varphi}{\gamma}\Phi\right], \tag{33}
$$

has been obtained both in the context of multiple interacting fluids [16,17,20], and in the framework of inflation when several interacting scalar fields are present [18], Eq. (33) being a particular case. However, in general it is not possible to find a dynamical equation for the intrinsic entropy perturbation of a given component without knowing its underlying physics. In the case under study we are able to fully specify the evolution of the system through the equation for the intrinsic entropy perturbation of the scalar field as

$$
\Gamma'=-\frac{3}{2}\gamma S+3\left(w_\varphi-\frac{\gamma}{2}\right)\Gamma+\frac{k^2}{a^2H^2}\left[-\gamma_\varphi\mathcal{R}-\frac{1}{3}S-\frac{1}{3}\Gamma\right] + \frac{k^4}{a^4H^4}\left[-\frac{2}{9}\frac{\gamma_\varphi}{\gamma}\Phi\right]. \tag{34}
$$

Equations (33) and (34) show that, on large scales, the relative entropy perturbation and the intrinsic entropy perturbation of the scalar field are mutually sourced and evolve independently of the curvature perturbations. In particular, Eq. (34) confirms the conclusions drawn from the constraint equation, Eq. (24), namely that in the presence of the fluid it is possible to have an intrinsic entropy perturbation relative to the scalar field even on large scales. When the fluid is very sub-dominant ($\Omega_{\rm f}\simeq 0$), we have $S\simeq 0$ and therefore on large scales Γ decays exponentially[1] with decay rate $-3+3\gamma_\varphi/2$, dynamically recovering the single scalar field case for which the intrinsic entropy perturbation vanishes (cf. Sec. II C).

When the fluid is completely absent, the matrices in Eq. (29) reduce to 3×3 matrices, in the variables Φ, $\mathcal{R}$ and Γ, but the constraint in Eq. (23) allows one to eliminate one more degree of freedom. For example, using Eq. (23) one can find 2×2 matrices for $\mathcal{R}$ and Γ, or if one additionally goes to the large-scale limit, the constraint equation, Eq. (23), forces Γ to vanish and gives $\mathcal{M}_0$ as a 2×2 matrix for Φ and $\mathcal{R}$.

B. Adiabatic condition

The adiabatic condition requires that the relative entropy perturbation S and the intrinsic entropy perturbation Γ vanish. From our equations it is immediately clear that on large scales ($k/aH \ll 1$, so that only $\mathcal{M}_0$ need be considered) if the perturbations are initially adiabatic then S and Γ remain zero for all times. In this case $\mathcal{R}$ is constant and Φ rapidly approaches its asymptotic value $\Phi=3\gamma\mathcal{R}/(2+3\gamma)$ (for constant or sufficiently slowly varying γ). This demonstrates that adiabaticity on large scales holds regardless of any time-dependence of the background variables such as w_φ and $c_{s\varphi}^2$ (this was already pointed out in Ref. [11]). In fact, preservation of adiabaticity is implied by the separate universe approach to large-scale perturbations [15]. However adiabaticity will be broken once the perturbations move out of the large-scale regime, with the matrices $\mathcal{M}_1$ and $\mathcal{M}_2$ sourcing S and Γ through the curvature perturbations $\mathcal{R}$ and Φ. In particular, note that this is also true for the single scalar field case, as is evident from looking at Eq. (23). On the other hand, as we will see, if an isocurvature perturbation is initially present it can be wiped out on large scales by the scalar field dynamics.

Aspects of these results have appeared in previous works [7,9,10,12], but without noting that adiabaticity is always preserved on large scales. Our set of variables makes unam-

[1]Actually, Γ remains constant in the special case $w_\varphi=1$, but when the field is dominant this equation of state is usually not considered and anyway would rapidly evolve toward $w_\varphi<1$.

TABLE I. Values of the three parameters Ω_φ, γ_φ, and $c^2_{s\varphi}$ during the four different possible regimes of a quintessence scenario (a particular scenario would feature only one type of tracker regime) until the field starts dominating.

	Ω_φ	γ_φ	$c^2_{s\varphi}$
Kinetic	0	2	1
Potential I	0	0	1
Potential II	0	0	$-2-w_f$
Usual tracker	0	γ_φ	w_φ
Perfect tracker	Ω_φ	γ_f	w_φ

TABLE II. Eigenvectors and corresponding eigenvalues of the matrix $\mathcal{M}_0$ of Eq. (29) according to the different regimes.

Kinetic regime	
$\mathbf{v}_3=(0,0,0,1)$	$n_3=3-3\gamma_f/2$
$\mathbf{v}_4=(0,0,2-\gamma_f,\gamma_f)$	$n_4=0$
Potential regime I	
$\mathbf{v}_3=(0,0,0,1)$	$n_3=-3-3\gamma_f/2$
$\mathbf{v}_4=(0,0,2-\gamma_f,\gamma_f)$	$n_4=-6$
Potential regime II	
$\mathbf{v}_3=(0,0,-2-\gamma_f,\gamma_f)$	$n_3=0$
$\mathbf{v}_4=(0,0,-2,1)$	$n_4=-3+3\gamma_f/2$
Tracker regime ($\gamma_\varphi\leqslant\gamma_f$)	
$\mathbf{v}_3=\ \dots$	$\mathrm{Re}(n_3)<0$
$\mathbf{v}_4=\ \dots$	$\mathrm{Re}(n_4)<0$

biguously clear the fact that the adiabatic condition is not an instantaneous notion on large scales, and holds independently of the evolution of the background.

IV. APPLICATION TO THE QUINTESSENCE SCENARIO

A. Analytical description

In this section, we discuss the large-scale evolution of perturbations in quintessence scenarios. As described in the Appendix, before the quintessence field starts dominating the evolution of the universe its dynamics can feature up to four different regimes during which the coefficients of the matrix $\mathcal{M}_0$ are constant. These are summarized in Table I. Note that, as compared to Refs. [8] and [11], we altered the names of two regimes (potential I and II) in order to make our explanations clearer. Now, following Ref. [11], for each regime it is possible to perform an eigenvector decomposition of the matrix $\mathcal{M}_0$ and therefore compute analytically the large-scale evolution of the perturbations (i.e., during each regime $\mathbf{v}$ can be written as a sum of four terms proportional to $\mathbf{v}_i\exp(n_iN)=\mathbf{v}_i a^{n_i}$ for $i=1$ to 4, respectively). However, the matching conditions between the different regimes are not obvious as S and Γ contain non-trivial functions of the background. In that respect, the formulation in Ref. [11] is more appropriate when following the modes over different regimes, since to a first approximation Φ, δ_f, δ_φ, and δ_π can be taken as conserved through the transitions between regimes.

First of all, it is easy to find that for any regime $\mathcal{M}_0$ possesses two eigenvectors

$$\mathbf{v}_1=(3\gamma,2+3\gamma,0,0),\qquad n_1=0,$$
$$\mathbf{v}_2=(1,0,0,0),\qquad n_2=-1-3\gamma/2, \tag{35}$$

where n_x is the eigenvalue of $\mathbf{v}_x$. These two vectors correspond to the two well-known adiabatic modes, the first one being constant, and the second one rapidly decaying.

Now, it is possible to find the two remaining entropy modes for a general case, but they cannot be expressed in a simple form. Nevertheless, it is straightforward and more clear to perform an eigenmode decomposition by considering each regime separately. The modes are given for each regime in Table II. For simplicity, we do not display complicated expressions; for a detailed analysis the reader should refer to Ref. [11]. Most of these results have already been discussed in that paper, but here we would like to comment further in the light of our new set of variables and new findings.

As is well known, we see that entropy perturbations decay during the tracker regime [9,11]. This is due to the scaling and attractor properties of that regime. More striking is the fast-growing mode during the kinetic regime and the constant mode during the second potential regime. This contradicts the claim by Brax *et al.* [8] that the final value of the quintessence perturbations is insensitive to the initial conditions. The difference comes from the fact that these authors considered the basic variable $\delta\varphi$ and its time derivative. They found that there are two decaying modes for every regime. However, this does not mean that observationally relevant variables (such as δ_φ) are decaying, since one has to take into account the evolution of background quantities (such as ρ_φ) as well. Our set of variables is therefore more appropriate.

Since in general we expect that there could be an initial relative entropy perturbation S (for example in the case of a quintessence field present during inflation [23]) and since S sources Γ, we can expect a non-zero intrinsic entropy on large scales which would then evolve according to our set of equations, Eqs. (33) and (34). Now, using the results given in the Appendix, Eqs. (A4) and (A5), we can show that the growth of Γ during the kinetic regime—$\exp(\Delta N_k n_{3(k)})=\exp(3\gamma_f\Delta N_p/2)$—is exactly compensated by its subsequent decay during the potential regime I—$\exp(\Delta N_{pI} n_{3(pI)})=\exp(-3\gamma_f\Delta N_p/2)$. As a result, after the three regimes preceding the tracker regime, entropy modes are neither enhanced nor suppressed. However, as shown in the Appendix, according to the initial conditions, the potential energy of the field can undergo a very large drop before it reaches a constant value and the kinetic regime starts. In general, this transition phase could last a non-negligible number of e-foldings and would feature the same eigenmodes as the kinetic regime, in particular the same growing mode. Since this first phase of growth would not be compensated by the decay during the first potential regime, there remains the possibility that entropy modes may be enhanced at the beginning of the tracker regime. Now, if the tracker regime is not long enough to erase completely these entropy perturbations by the time the field becomes non-negligible, we may be able to see an imprint of these initial perturbations in observations [9,11]. As a result, we can see that in a quintessence scenario the initial conditions, as well as the history of the evolution of the background, are relevant when considering the late-time value of the perturbations.

In this respect many different assumptions can be made. In Ref. [21], Kneller and Strigari assumed equipartition as an initial value for the field, and in most cases this led to a field dynamics featuring a very long kinetic regime followed by the potential regimes and no tracker regime. In Ref. [22], de la Macorra studied an actual physical model in which the quintessence field is a dark condensate which arises after a phase transition. Its evolution starts in the kinetic regime and again does not feature a tracker regime. In both scenarios entropy perturbations would still be present today. Alternatively, Malquarti and Liddle [23] studied the evolution of a quintessence model during inflation in order to investigate the initial conditions of the quintessence field at the beginning of the radiation-dominated era. They found that typically the tracker starts at low redshift after a long period of potential regime II, but again, as a result, entropy perturbations generated during inflation could still lead to observable consequences. Finally, note that in general the initial entropy perturbations do not need to be of the order Φ and may be much larger.

B. Numerical examples

In order to illustrate our results, we carried out simulations for two quintessence models in a realistic universe (with radiation, dark matter and dark energy). Note that the initial relative amplitude between S and Γ does not play an important role, since, as expected, due to the fact that they are coupled, they quickly become of the same order (this has been numerically checked by choosing many different initial conditions).

The first example is an inverse power-law model [3] $V(\varphi)=V_0(\varphi/m_{\rm Pl})^{-\alpha}$, with $\alpha=1$ and $V_0=10^{-123}m_{\rm Pl}^4$, starting with the initial conditions at $N=-50$ given by $\rho_\varphi=10^{-20}\rho_{\rm f}$, $T=V$ and $S=-\Gamma=E_{\rm i}$. The results are shown in Fig. 1 where we display the evolution of some background variables and of the two entropy variables S and Γ. After a very sharp transition toward the kinetic regime, the field undergoes the four regimes described in Table I before starting dominating. First, we can clearly identify these regimes and see the growing, constant and decaying behavior of S and Γ corresponding to the modes displayed in Table II, except for the fact that S is constant during the potential regime I. This is because Γ is many orders of magnitude larger than S, and $1-c_{s\varphi}^2$ is not exactly 0, hence the last entry on the third line of the matrix $\mathcal{M}_0$ is not zero. We checked that if S and Γ are of the same order of magnitude at the beginning of the first potential regime, then S decays according to the modes displayed in Table II. We note the oscillations during the tracker regime and the non-trivial evolution through the transitions

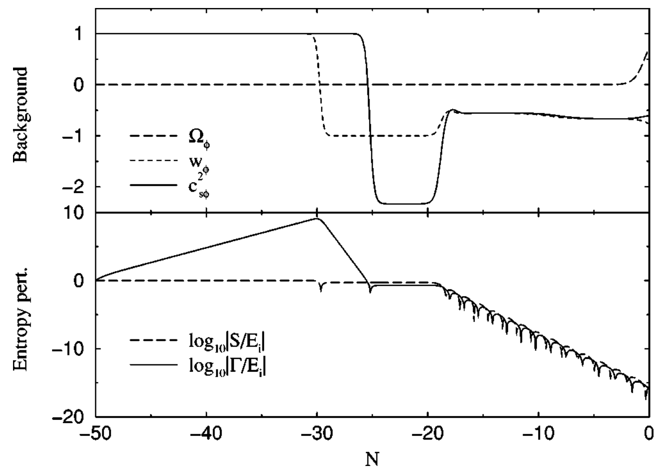

FIG. 1. Evolution in a realistic universe of background quantities (top) and entropy perturbation variables (bottom) of a quintessence field undergoing the four different possible regimes before its domination. We use an inverse power-law potential $V(\varphi)=V_0(\varphi/m_{\rm Pl})^{-\alpha}$ (see Sec. IV B for parameters). Note the transition between radiation domination and matter domination at $N\simeq-9$.

(yet keeping the same order of magnitude). We also observe that in this particular case the transition phase toward the kinetic regime is too short to enhance the entropy mode significantly, leading to the result that at the beginning of the tracker regime S and Γ have about the same amplitude as at the initial stage. In other words they are neither enhanced nor suppressed by the dynamics of the field, until tracking begins.

The second example is a double-exponential model [24] $V(Q)=V_0[\exp(-\alpha\kappa\varphi)+\exp(-\beta\kappa\varphi)]$, with $\alpha=1000$, $\beta=1$, $V_0=10^{-122}m_{\rm Pl}^4$ and $\kappa=\sqrt{8\pi/3m_{\rm Pl}^2}$, starting with the initial conditions at $N=-50$ given by $\rho_\varphi=5\times10^{-3}\rho_{\rm f}$, $T=V$ and $S=E_{\rm i}$ and $\Gamma=0$. The results are shown in Fig. 2. We display the same variables as in Fig. 1. In addition, in order to observe the transition phase preceding the kinetic regime,

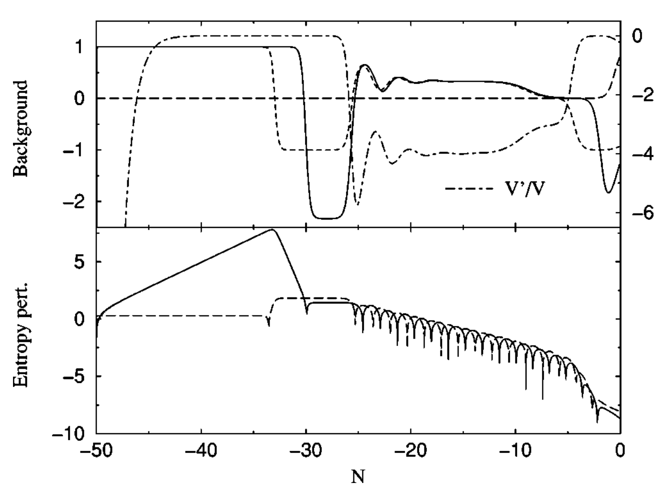

FIG. 2. As in Fig. 1, but now for a double-exponential potential $V(Q)=V_0[\exp(-\alpha\kappa\varphi)+\exp(-\beta\kappa\varphi)]$ (see Sec. IV B for parameters). We also display V'/V (to be read on the right-hand side of the graph).

we also display V'/V, where V is the potential energy density of the field—as described in the Appendix, the kinetic regime starts when $|V'/V|$ drops under ~ 1. First, we note that although $\Gamma=0$ initially, it evolves very quickly to be of the same order as S; as previously explained, this is due to the coupling between S and Γ. Then, we observe the same behavior as for the first example, but this time we can see that the transition phase toward the kinetic regime lasts a few e-foldings. As a result, the entropy modes at the beginning of the tracker regime are nearly a factor 100 larger than initially. This shows that, as discussed in Sec. IV A, entropy modes can be enhanced before the field reaches the tracker regime.

We have shown that, in general, one must take into account the initial conditions for the quintessence field and its perturbations in order to make any prediction.

V. DISCUSSION

We have explored the nature of scalar perturbations for a universe filled with both a scalar field and a perfect fluid. We have introduced a useful set of variables and have provided a full analysis including spatial gradients. In particular, we have focused on the isocurvature perturbation modes and on the degrees of freedom which completely characterize the system. While for the case where only the scalar field is present its intrinsic entropy perturbation is forced to vanish at linear order for superhorizon scales, the presence of a fluid—even if sub-dominant—allows the possibility for such an intrinsic contribution to be present on large scales. However, in the case of a very sub-dominant fluid the intrinsic entropy of the scalar field decays, dynamically recovering the single scalar field situation.

We have recast the basic evolution equations in a rather simple matrix formalism in terms of the gauge-invariant variables for the adiabatic and isocurvature perturbations, taking into account the dynamics of the perturbations when a given wavelength re-enters the horizon. In particular, we have obtained an equation for the intrinsic entropy perturbation which shows that, on large scales, an initial adiabatic condition is indeed preserved, regardless of the evolution. Only when the perturbations approach the horizon are the adiabatic and entropy perturbations fully coupled together.

Finally, we have applied our formalism to the quintessence scenario. In this case we have analyzed the large-scale evolution of the adiabatic and entropy perturbations in the different regimes which the quintessence scalar field dynamics may feature. As is well known, entropy perturbations are suppressed during the tracking regime, but, as already shown in Ref. [11], during the kinetic regime one entropy mode undergoes an exponential growth. We have shown that it is then exactly compensated by an exponential decay during the first potential regime and then remains constant during the second potential regime. Therefore, after the three regimes preceding the tracker regime, entropy modes are neither enhanced nor suppressed. However, we discussed the remaining possibility of an enhancement during the short transition phase preceding the kinetic regime. We have studied two special cases numerically and have confirmed our analytical analysis. Moreover, we have observed that, in one of the cases, at the beginning of the tracker regime, entropy perturbations are larger than initially, and therefore we have concluded that entropy mode enhancement is possible.

To summarize, we have shown that in general it is incorrect to assume that the observational imprint of quintessence perturbations will be independent of the initial conditions, because entropy perturbations can still be present when the quintessence energy density is no longer negligible. Note that this can happen as soon as the tracker starts, long before quintessence domination. In this case, entropy perturbations would feed curvature perturbations, but then would slowly decay to become negligible today.

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APPENDIX: DYNAMICAL REGIMES OF A QUINTESSENCE FIELD

As described in Ref. [8], a tracking quintessence field can feature up to four different dynamical regimes when in presence of a dominant fluid with constant equation of state. Here we clearly demonstrate the existence of these regimes and compute some relevant parameters. We will use the same notation and definitions as in the main body of the article.

We assume that the quintessence field features a tracking solution $\rho_{\rm tr}(N)$—note that it does not need to have a constant equation of state. This means that at each time (i.e., for each value of H) there exists a stable field configuration for which its kinetic energy density $T\equiv\dot{\varphi}^2/2$ and its potential energy density V are of the same order, and hence in Eq. (4) the "friction" term due to the Hubble expansion and the slope of the potential balance each other. We will show that, according to the initial conditions, the scalar field can feature up to three different regimes before it reaches the tracker. We assume that the field is always subdominant and therefore does not influence the evolution of the universe, especially the evolution of H. As a result we have $H\propto\exp(-3\gamma_{\rm f}N/2)$. Using Eq. (4) it is easy to see that

$$\frac{T'}{T}=-\frac{V'}{T}-6, \tag{A1}$$

$$\frac{V'}{V}=\frac{dV/d\varphi}{V}\frac{\sqrt{2T}}{H}. \tag{A2}$$

In order to help the reader to follow the explanation, an example of a quintessence field evolution featuring the four possible regimes before domination is shown in Fig. 3.

Let us start by looking at the initial condition $T\lesssim V\gg\rho_{\rm tr}$ at time $N_{\rm i}$. The field, and hence the slope of the potential, has the same value as for the tracker configuration at an

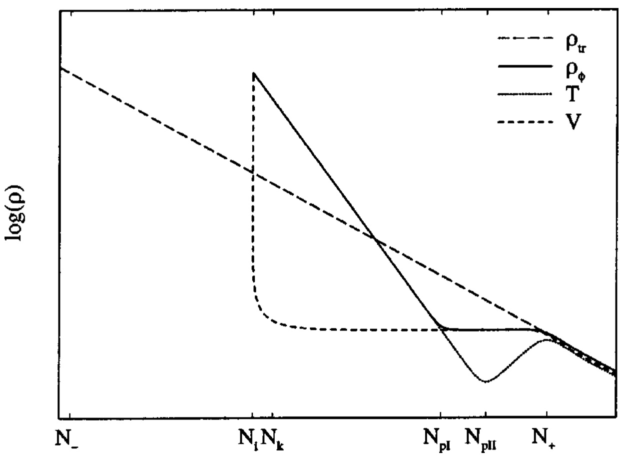

FIG. 3. Example of a quintessence field evolution featuring the four possible regimes before domination: kinetic, potential I, potential II and tracker. Note the transition phase toward the kinetic regime. The different parameters are described in the Appendix.

earlier time $N_- < N_i$, but because $H(N_i) \ll H(N_-)$ the friction term is actually negligible and the field fast-rolls down its potential ($V' \ll T'$) and its kinetic energy almost instantaneously dominates its energy density. At some time N_k shortly after N_i (at the latest when $T \sim \rho_{tr}$), the potential freezes at some value $V_+ \ll \rho_{tr}$ corresponding to the tracker configuration at a later time $N_+ > N_k$, and, since $H(N) \gg H(N_+)$ for $N < N_+$, it remains frozen until N_+. In this case, the evolution of T can easily be computed analytically. We assume that at N_+ the tracker solution has an equation of state γ_{tr}^+, express $dV/d\varphi$ as a function of V_+ and γ_{tr}^+, and for $N_k < N < N_+$ we find

$$T(N) = \frac{\gamma_{tr}^+ (2-\gamma_{tr}^+) V_+}{(\gamma_f + 2)^2} [C e^{-3(N-N_+)} + e^{(3/2)\gamma_f (N-N_+)}]^2, \tag{A3}$$

where C is a constant of integration depending on the initial conditions.

Let us explain this behavior. Starting from the time N_k the field is in the kinetic regime: $\rho_\varphi = T \propto \exp(-6N)$ and V = constant. At some time N_{pI} the field reaches the configuration $T \sim V$, and since the friction term is still extremely large [$H(N_{pI}) \gg H(N_+)$] the kinetic term keeps on decaying and the field enters the first potential regime: $\rho_\varphi = V$ = constant and $T \propto \exp(-6N)$. At some time N_{pII} the term V'/T can balance the friction term and T starts growing again. Here begins the second potential regime: $\rho_\varphi = V$ = const and $T \propto \exp(3\gamma_f N)$. Finally, at N_+ the field enters the tracker regime: $T \sim V \sim \rho_{tr}$. Note that the whole evolution described above goes through all the possible initial conditions.

Now we can compute a few parameters. First, using the solution for T given in Eq. (A3) and noting that $c_{s\varphi}^2 = -1 - T'/3T$, it is straightforward to recover the values displayed in Table I. In addition, let us define ΔN_r as the number of e-foldings that the field spends in the regime "r" and also $\Delta N_p \equiv \Delta N_{pI} + \Delta N_{pII}$. Using Eq. (A2), the fact that at N_k and N_+ we have $V'/V = -3\gamma_{tr}^+ \sim -1$ and the evolution for T and H described above, we find

$$\Delta N_k = \frac{\gamma_f}{2-\gamma_f} \Delta N_p, \tag{A4}$$

$$\Delta N_{pI} = \frac{\gamma_f}{2} \Delta N_{pII} = \frac{\gamma_f}{2+\gamma_f} \Delta N_p. \tag{A5}$$

These last two results are used in Sec. IV A to show that the growth of one of the entropy modes during the kinetic regime is exactly compensated by its subsequent decay during the first potential regime.

[1] For an overview of inflation see A. R. Liddle and D. H. Lyth, *Cosmological Inflation and Large-scale Structure* (Cambridge University Press, Cambridge, England, 2000).
[2] C. Wetterich, Nucl. Phys. **B302**, 668 (1988); E.J. Copeland, A.R. Liddle, and D. Wands, Ann. N.Y. Acad. Sci. **688**, 647 (1993); P.G. Ferreira and M. Joyce, Phys. Rev. D **58**, 023503 (1998); E.J. Copeland, A.R. Liddle, and D. Wands, *ibid.* **57**, 4686 (1998); I. Zlatev, L. Wang, and P.J. Steinhardt, Phys. Rev. Lett. **82**, 896 (1999); A.R. Liddle and R.J. Scherrer, Phys. Rev. D **59**, 023509 (1999); V. Sahni and A. Starobinsky, Int. J. Mod. Phys. D **9**, 373 (2000).
[3] B. Ratra and P.J.E. Peebles, Phys. Rev. D **37**, 3406 (1988).
[4] R.R. Caldwell, R. Dave, and P.J. Steinhardt, Phys. Rev. Lett. **80**, 1582 (1998).
[5] P.T.P. Viana and A.R. Liddle, Phys. Rev. D **57**, 674 (1998); J.C. Hwang and H. Noh, *ibid.* **64**, 103509 (2001); M. Kawasaki, T. Moroi, and T. Takahashi, Phys. Lett. B **533**, 294 (2002); R. Dave, R.R. Caldwell, and Paul J. Steinhardt, Phys. Rev. D **66**, 023516 (2002).
[6] F. Perrotta, C. Baccigalupi, and S. Matarrese, Phys. Rev. D **61**, 023507 (2000); L. Amendola, Mon. Not. R. Astron. Soc. **312**, 521 (2000); astro-ph/0311175.
[7] F. Perrotta and C. Baccigalupi, Phys. Rev. D **59**, 123508 (1999).
[8] P. Brax, J. Martin, and A. Riazuelo, Phys. Rev. D **62**, 103505 (2000).
[9] L.R. Abramo and F. Finelli, Phys. Rev. D **64**, 083513 (2001).
[10] C. Skordis and A. Albrecht, Phys. Rev. D **66**, 043523 (2002).
[11] M. Malquarti and A.R. Liddle, Phys. Rev. D **66**, 123506 (2002).
[12] M. Doran, C.M. Müller, G. Schäfer, and C. Wetterich, Phys. Rev. D **68**, 063505 (2003).
[13] V. Mukhanov, H. Feldman, and R. Brandenberger, Phys. Rep. **215**, 203 (1992).
[14] J.M. Bardeen, Phys. Rev. D **22**, 1882 (1980).
[15] D. Wands, K.A. Malik, D.H. Lyth, and A.R. Liddle, Phys. Rev. D **62**, 043527 (2000).

[16] K.A. Malik, D. Wands, and C. Ungarelli, Phys. Rev. D **67**, 063516 (2003).
[17] H. Kodama and M. Sasaki, Prog. Theor. Phys. Suppl. **78**, 1 (1984).
[18] C. Gordon, D. Wands, B.A. Bassett, and R. Maartens, Phys. Rev. D **63**, 023506 (2001).
[19] B.A. Bassett, F. Tamburini, D.I. Kaiser, and R. Maartens, Nucl. Phys. **B561**, 188 (1999).
[20] J. Hwang and H. Noh, Class. Quantum Grav. **19**, 527 (2002).
[21] J.P. Kneller and L.E. Strigari, Phys. Rev. D **68**, 083517 (2003).
[22] A. de la Macorra, astro-ph/0212275.
[23] M. Malquarti and A.R. Liddle, Phys. Rev. D **66**, 023524 (2002).
[24] T. Barreiro, E.J. Copeland, and N.J. Nunes, Phys. Rev. D **61**, 127301 (2000).