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Consistency relation in general braneworld inflation

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We study gravitational perturbations in general braneworlds carrying a perturbation of the de Sitter universe. We derive our results from a full five-dimensional quantum field theory, and exhibit explicit formulas for the scalar and tensor power spectra. We show by explicit calculation that the dark radiation induced by our perturbation is zero, which implies that the deformation arises from the matter theory only and enables proper comparison with four-dimensional results. We argue that for the consistency relation to hold in the braneworld, significant fine tuning of the matter theory is necessary.

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I. INTRODUCTION

Recent experimental results from the WMAP project [1] have lent strong support to the common view that the observed homogeneity, isotropy and large-scale structure of the Universe arises from an early period of accelerated expansion known as inflation [2]. This expansion is commonly supposed to be driven by a light quantum scalar field, or inflaton, that violates the weak energy condition and dominates the energy density of the Universe. During inflation all massless or sufficiently light degrees of freedom (any with mass m less than $3/2$ of the Hubble rate H during inflation) are quantum mechanically excited, and pick up a nearly scale invariant fluctuation. The characteristics of this fluctuation are controlled by the expansion rate, and therefore depend on the inflaton potential via the Friedmann equation. After the inflaton decays and reheats the Universe, the inflaton fluctuation is communicated to the curvature of spatial slices [3]. This process seeds primordial structure formation, meaning that characteristics of the primordial inflaton fluctuation are observable today in the large-scale distribution of the galaxies [4] and fluctuations in the cosmic microwave background (CMB) [5].

The curvature fluctuation need not be the only fossil from the early Universe. Since the graviton is massless, one would expect it to acquire a similar fluctuation in which small tensor perturbations would have been excited. It is significant that the subsequent evolution of these tensor perturbations differs from that of the inflaton, because they do not decay: the relative weakness of the gravitational coupling implies, in fact, that tensor perturbations would essentially not interact with other constituents of the Universe on their journey towards us. Therefore such perturbations would almost certainly still be in their primordial state and could offer great insight into the conditions and physics of the early Universe. Tensor perturbations of this type are in principle observable today as a stochastic background of gravity waves, and could be measured (for example) either directly with gravity wave observatories such as LIGO or GEO [6,7] (in an appropriate part of parameter space), or via their imprint in the polarization field of the CMB [8–11]. In particular, should the gravity wave power spectrum be observed in the near future, then in addition to the already observed amplitude A_S^2 and spectral index n_S of the scalar spectrum, one would also have similar information A_T^2, n_T available for tensor perturbations. Such extra information would be of great importance for cosmology, and cosmological parameter estimation [12].

The usual scalar field inflationary paradigm supposes that whatever the vacuum expectation value (vev) of the inflaton field, it is rolling only very slowly during inflation. Under these circumstances, in order to provide a sufficient amount of expansion, the inflaton potential must be extraordinarily flat. The particles associated with quantum fluctuations around such a flat potential must be almost massless. To quantify this, it is conventional to characterize the flatness of the potential using the two lowest-order slow-roll parameters ε and η,

$$\varepsilon = -\frac{\dot{H}}{H^2} = \frac{2}{2}\left(\frac{H'}{H}\right)^2, \quad \text{and } \eta = \frac{2}{2}\frac{H''}{H}, \tag{1}$$

where an overdot denotes a time derivative, a prime $'$ denotes a derivative with respect to the scalar field, and κ_4^2 is the four-dimensional gravitational coupling. These quantities are part of an infinite sequence of slow-roll parameters which characterize gradients and curvatures of the inflaton potential. If inflation is of the slow-roll variety, then ε and η and all other slow-roll parameters must be very small.

In the context of scalar field inflation, all details of the power spectrum, including amplitudes and spectral indices, are determined by properties of the scalar potential. In the slow-roll formalism this means they can be expressed at lowest order in terms of the characteristic numbers ε and η. This dependency on a single source implies that one might expect to find some relations between the various measurable quantities. For example, in the special case of the standard cosmology, one finds [13], to lowest order in

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the slow-roll approximation [14],

$$n_T \approx -2\frac{A_T^2}{A_S^2}. \tag{2}$$

This is not an exact relationship. It is proved by expanding both sides in terms of the slow-roll parameters, and noticing that they agree to lowest order. We would like to stress that, in the context of scalar field inflation, whatever exact relation exists between observables is not known. Indeed, we do not even have an exact expression for A_S^2 or A_T^2; all that is known is a perturbation expansion in the slow-roll parameters. For this reason, if inflation was not of the slow-roll variety, then higher-order terms in the expansion could be important, and Eq. (2) might not apply.

Equation (2) is at present only a theoretical prediction. However, it is one of only a handful of testable predictions made by the inflationary paradigm, and for this reason has the potential to be a powerful discriminant between competing models. Over and above the general current evidence in favor of an inflationarylike epoch, an observation of this relation in the real Universe would provide extremely strong support for a minimal scalar field model. On the other hand, more complex models weaken Eq. (2) to an inequality; for example, this occurs in models containing isocurvature modes. Therefore observations of gravity waves at a lower level than predicted by Eq. (2) can be consistent with inflation, whereas observing an excess of primordial gravitational power would be a severe blow to the inflationary program. Equation (2) is of considerable observational importance.

One can calculate a similar equation that is exact to next-order in the slow-roll expansion [15]. This next-order term does not preserve the functional form of Eq. (2). Instead, one has

$$n_T = -2\frac{A_T^2}{A_S^2}\left[1 - \frac{A_T^2}{A_S^2} + (1 - n_S)\right]. \tag{3}$$

This is proved by expanding both sides in terms of the slow-roll parameters to next-order. Equation (2) is informally called the inflationary consistency relation, whereas Eq. (3) is sometimes known as the next-order consistency relation. We will return to this equation later, giving a brief derivation in Sec. III and considering first-order perturbations in Secs. IV and V. The analysis of circumstances in which Eq. (2) may fail to hold is the principal concern of this paper.

Over the last few years there has been considerable interest in cosmological models which involve large extra dimensions in an essential way [16–23]. Extra dimensions have been a common ingredient in models of high energy physics beyond the standard model from the time of Kaluza and Klein to the present day, but recent developments in string theory and M theory [24–28] have suggested the possibility that these extra dimensions may have important cosmological implications, opening up one or more large dimensions to gravity while keeping matter fixed firmly in $3 + 1$ dimensions. Such models are often called brane universes, or simply braneworlds. It is natural to ask both how inflation is implemented in these scenarios, and what possible modifications arise in its predictions for late-universe observables [29–34]. Here one discovers a remarkable surprise. Although predictions for the tensor and scalar amplitudes and spectral indices are modified, as a result of their sensitivity to the behavior of gravity in the large extra dimensions, the lowest-order consistency relation survives [35,36].

This is a nontrivial feature of the model, and at the time of writing we are not aware of any simple argument which demonstrates why it should be true. Because of its potential observational importance, this is both an immediate and pressing observational difficulty, and a challenging theoretical puzzle. The continued appearance of Eq. (2) in the braneworld potentially jeopardizes the long-standing hope of observationally reconstructing the inflaton potential [15,37]. An understanding of the origin of the braneworld degeneracy is essential to the reconstruction program, for if such degeneracies apply to an open set of models then it may be difficult or impossible to place confidence in inflaton potentials reconstructed from minimal scenarios. One should notice, however, that this consistency relation is derived assuming that the bulk is empty. This need not be the case. If the bulk contains a scalar field, then there is typically a tachyonic instability [38]. Requiring that this instability is stabilized can only be achieved in a regime where fluctuations of the bulk scalar field dominate the scalar perturbation as seen on the brane. In this case, the scalar perturbation spectrum does not coincide with the spectrum assumed in this paper, and the consistency relation Eq. (2) is destroyed.

In this paper, we attempt to clarify the circumstances under which one expects degeneracies between brane cosmology and conventional cosmology to persist. We carry out this program by marginally perturbing the cosmology which gives rise to Eq. (2), and asking if the consistency relation is still satisfied in the perturbed cosmology. It is possible to solve both the gravitational and scalar field equations for the power spectra and spectral indices. This is fairly straightforward in the scalar case, but gravitational perturbations cannot be handled so easily and our technique requires a considerable extension of existing methods. We do not seek to provide a mechanism from which the perturbation may originate. In five dimensions one can appeal to possible brane-bulk interactions, but it is also possible to regard the cosmology as simply a model for a universe which is close to the de Sitter state, but does not exactly coincide with it.

This paper is organized as follows. In Sec. II we briefly review the calculation of the four-dimensional amplitudes and spectral indices, and present a new derivation of the same quantities in the braneworld. We use the same framework of quantum field theory that is applied in the 4D case [30]. In Sec. III we discuss the consistency relation in the unperturbed four- and five-dimensional cases, and show how it arises. In Secs. IV and V we calculate the effect of an arbitrary perturbation of the Hubble rate, ΔH, on the power spectra of scalar and tensor quantities, and study the effect on the consistency relation. This is done in both the four- and five-dimensional cases. We begin with an exact de Sitter cosmology fixed by $H =$ constant, and assume it is still valid to treat the field fluctuations such as ϕ as free, massless fields propagating over the background. The two-point correlation functions can then be calculated. We demonstrate explicitly that the perturbation comes entirely from the matter sector on the brane, and does not involve dark radiation or other unknown physics impinging from the bulk which might reasonably be expected to trivially alter four-dimensional physics. Finally, we state our conclusions (Sec. VI). Some material extraneous to the main text, involving normalization of the graviton zero mode and the mathematical details concerning the various integral solutions we employ, is presented for reference in an appendix. We begin by reviewing the quantum field theory (QFT) calculation of the power spectra in four dimensions and presenting a new calculation in the braneworld.

II. THE FOUR- AND FIVE-DIMENSIONAL LOWEST-ORDER RESULTS FROM QUANTUM FIELD THEORY

Scalar field inflation is based on free, massless field theory. If ϕ is the inflaton subject to some potential $V(\phi)$, then one treats the gross evolution of ϕ classically with the addition of a fluctuating part $\delta\phi$ which is to be treated quantum mechanically. It is a good approximation to take $\delta\phi$ to be a free, massless field. This approximation leads to a central equation governing the behavior of the inflaton, known as the Mukhanov equation [3]. The inflaton field ϕ itself will not appear, so in order to reduce clutter in equations, in the remainder of this paper we simply drop the prefix δ from the fluctuating field $\delta\phi$.

A. Four-dimensional scalar power spectrum

Let ϕ be a free, massless scalar field. Its correlation functions are controlled by the functional integral,

$$\phi(x_1)\cdots\phi(x_n) = \int[\mathcal{D}\phi]\phi(x_1)\cdots\phi(x_n) \times \exp\left(-\frac{i}{2}\int_{\mathcal{M}} dx\,\phi\Box\phi\right), \quad (4)$$

where $[\mathcal{D}\phi]$ is the functional measure, $\mathcal{M}$ is the background spacetime with metric g_{ab} and invariant volume measure dx, and we have chosen units in which $\hbar = 1$. The operator $\Box$ is defined by $\Box = \nabla^a\nabla_a$, where ∇_a is the covariant derivative compatible with g_{ab}. In particular, the two-point function satisfies $\langle\phi(x_1)\phi(x_2)\rangle = -i\Box^{-1}(x_1, x_2)$ [39].

Now let $\mathcal{M}$ be de Sitter space. We choose local coordinates in which the metric takes the form [40]

$$ds^2 = \frac{1}{H^2\tau^2}(-d\tau^2 + \delta_{ij}dx^i dx^j). \quad (5)$$

This form of the metric with flat spatial slices is particularly common and convenient when discussing inflation. The infinite past corresponds to $\tau \to -\infty$. It is well known that if $\tau_1 > \tau_2$ the propagator satisfies [41]

$$\langle\phi(x_1)\phi(x_2)\rangle = \int_{\mathbf{R}^3}\frac{d^3k}{(2\pi)^3}\frac{\pi i}{4k}H^2\tau_1\tau_2 L^{(1)}(-k\tau_1) \times L^{(2)}(-k\tau_2)e^{-ik\cdot(x_1-x_2)}, \quad (6)$$

where $L^{(n)}(z)$ is a useful abbreviation,

$$L^{(n)}(z) = z^{1/2}H^{(n)}_{3/2}(z) \quad \text{for } n = 1, 2, \quad (7)$$

and where $H^{(n)}_\nu$ is a Hankel function of the nth kind of order ν. The boundary conditions are chosen to correspond with the Bunch-Davies vacuum [41]. This prescription demands that Eq. (6) is close to the flat space limit $\sim e^{-ik|\tau|}$, with an appropriate normalization, whenever the wave vector k is small ($k \to \infty$) compared to the curvature of spacetime or when approaching the asymptotically early or late times ($\tau \to \pm\infty$).

Since ϕ is free there are no singularities requiring renormalization in the operator product expansion, although the propagator is logarithmically divergent in both the ultraviolet and infrared. One can take the $x_1 \to x_2$ limit to find an effective variance, $\sigma^2_\phi(x) = \langle\phi(x)\phi(x)\rangle$, which satisfies

$$\sigma^2_\phi(\tau) = (H\tau)^2\int_0^\infty \frac{kdk}{8\pi}(-k\tau)L^{(2)}(-k\tau), \quad (8)$$

where $\tau_1 = \tau_2 = \tau$. This is independent of the spatial coordinates $\boldsymbol{x}$, and gives a direct measure of the strength of fluctuations present in ϕ. One writes this as a power spectrum, $\Delta^2_\phi(k) = d\sigma^2_\phi(k)/d\ln k$. On sufficiently large scales ($k \to 0$) the power spectrum of a free, massless scalar field such as ϕ approaches a well-known finite limit $\Delta^2_\phi = (H/2\pi)^2$ [14,42].

The description of matter fluctuations is achieved via the intrinsic curvature perturbation, ζ [3], for which our conventions coincide with Wands *et al.* [43]. We define ζ by setting

$$\zeta = -\psi - \frac{H}{\dot\phi_{\rm cl}}\delta\phi, \quad (9)$$

where $\phi_{\rm cl}$ is the classical background evolution, and for clarity we add a δ to ϕ. ζ is a gauge-invariant quantity conserved on large scales in the absence of isocurvature perturbations. The quantity ψ is the Newtonian gravitational potential. One evaluates ζ during inflation with $\delta\phi$ defined on slices $\psi = 0$. After inflaton decay, there is no field ϕ, so $\zeta = -\psi$ is the perturbation on spatial slices. Therefore,

$$A_S^2 = \frac{4}{25}\left(\frac{H^2}{\dot{\phi}_{\rm cl}^2}\right)\Delta_\phi^2, \tag{10}$$

where rather than deal directly with the power spectrum Δ^2, it is conventional to rescale the power spectra of scalar and tensor perturbations by introducing new quantities $A_S^2 = (4/25)\Delta_S^2$ and $A_T^2 = (1/100)\Delta_T^2$ [14,15], where S and T represent the scalar and tensor spectra, respectively.

B. Four-dimensional tensor power spectrum

Linear gravitational waves consist of small perturbations E_{ab} to the metric,

$$ds^2 = (\eta_{ab} + E_{ab})dx^a dx^b, \tag{11}$$

subject to the irreducibility requirements $\nabla^a E_{ab} = 0$, ${\rm tr}E_{ab} = 0$ with respect to the Lorentz metric η_{ab}. The gravitational action is

$$S = -\frac{1}{2\kappa_4^2}\int_{\mathcal{M}} dxR, \tag{12}$$

where $R = {\rm tr}R_{ab}$ is the trace of the Ricci tensor. To find an action for E_{ab} one expands R and dx to second order in E_{ab}. The result is

$$S_2 = -\frac{1}{8\kappa_4^2}\int_{\mathcal{M}} d^n x E^{ab}\Box E_{ab}, \tag{13}$$

where $\Box = \nabla^a\nabla_a$ is still the scalar, not tensor, d'Alembertian. Thus E_{ab} behaves like some number of fields in the trivial (scalar) representation of the Lorentz group, one for each polarization state of the graviton. This means that the action for gravitational perturbations is the same as two copies of the scalar action, Eq. (4), except that the overall normalization is changed by a factor $(4\kappa_4^2)^{-1}$. As a result, the gravitational power spectrum satisfies

$$A_T^2 = 2\cdot\frac{\kappa_4^2}{25}\cdot A_\phi^2 = \frac{\kappa_4^2}{50}\frac{H^2}{\pi^2}. \tag{14}$$

C. The braneworld cosmology

In the braneworld, one works on a five-dimensional (Schwarzschild-) anti–de Sitter space, often abbreviated (S)AdS [16,21–23,44]. We denote the total space $\mathcal{M}$. Throughout this paper, we consider only the pure anti–de Sitter case where there is no Schwarzschild-like mass. There is an embedded four-dimensional hypersurface Σ which supports the various matter and gauge fields which comprise our cosmology. The metric is taken to be [18,19]

$$ds^2 = -n^2(t, y)dt^2 + a^2(t, y)\delta_{ij}dx^i dx^j + dy^2. \tag{15}$$

The brane is considered to be embedded at $y = 0$. The coordinate y measures distances along the extra dimension, and there is a $\mathbf{Z}_2$ symmetry which acts via $y \mapsto y$. The metric functions $a(t, y)$ and $n(t, y)$ depend on the four-dimensional brane geometry.

The $\mathbf{Z}_2$ symmetry is motivated from heterotic M theory [25,26,45–47], and we loosely refer to this construction as an orbifold. These coordinates do not cover the full AdS space, unless the brane is empty [16,21]. Therefore, the coordinate y does not take on unboundedly large values but instead only assumes values in some interval $y \in [-y_h, y_h]$ [21,22]. In general we will work on the $y > 0$ branch, which usually makes no difference to computations except that factors of 2 must sometimes be inserted by hand to account for the other half. The location of the coordinate horizon at $y = y_h$ depends on the brane tension and matter theory [30]; it is defined in terms of H in Eq. (20) below.

The effective Einstein equations on the brane have been found previously [20]. They are, with κ_4^2 and κ_5^2 the four- and five-dimensional gravitational couplings, respectively,

$$G_{ab} = \kappa_4^2 T_{ab} + \kappa_5^2\pi_{ab} + F_{ab}, \tag{16}$$

where G_{ab} is the effective four-dimensional Einstein tensor; T_{ab} is the energy-momentum tensor of whatever matter and gauge degrees of the freedom reside on the brane; π_{ab} is a tensor quadratic in T_{ab}; and F_{ab} is the limit as one approaches the brane of the so-called electric part of the Weyl tensor in the bulk [20]. The hierarchy of Planck scales is controlled by a parameter $\mu = \kappa_4^2/\kappa_5^2 = M_5^3/M_4^2$. If there is no four-dimensional cosmological constant, then the hierarchy parameter μ is the AdS curvature scale, and is related to the brane tension λ by $\lambda = 6\mu/\kappa_5^2$. When applied to cosmological models, Eq. (16) implies that the Friedmann equation receives corrections quadratic in the density and pressure, which arise from the term π_{ab} [19],

$$H^2 = \frac{\kappa_4^2}{3}\rho\left(1 + \frac{\rho}{2\lambda}\right) + \frac{C}{R^4}, \tag{17}$$

where $R = a(t, y = 0)$ refers to the scale factor on the brane, and C is a constant of integration arising from the F_{ab} term in the Einstein equation. The "dark radiation" C behaves as a noninteracting matter component with a radiation equation of state. We set C to zero, but the role of dark radiation must be reconsidered when describing perturbed universes in Sec. V B. For future use, we note that the $\lambda \to 0$ limit can be identified with $\mu \to 0$ at fixed κ_5, and similarly as $\lambda \to \infty$. The $\lambda \to 0$ limit sends κ_4 to zero and so switches off four-dimensional gravity on the brane, whereas $\lambda \to \infty$ suppresses the corrections in Eq. (17) and so one recovers conventional cosmological evolution. We will make use of these limits later.

We shall need explicit forms for $a(t, y)$ and $n(t, y)$. The general solution for $a(t, y)$, in the absence of dark radiation, is [19],

$$\left(\frac{a}{R}\right)^2 = \frac{H^2}{2\mu^2}[\cosh 2\mu(y_h - y) - 1]. \tag{18}$$

There is a gauge freedom in choosing n that corresponds to a reparametrization of the time. The conventional choice is to set t equal to cosmic time on the brane, which forces $n(t, y = 0) = 1$ and implies

$$\frac{R}{a}n = 1 - \left(\frac{R}{a}\right)^2 \frac{\dot{H}}{2\mu^2}\left[\frac{H^2}{\mu^2 + H^2}\cosh 2\mu(y_\infty - y) - 1\right]. \tag{19}$$

We believe this expression for n is new. The quantities $y_h(t)$ and $y_\infty(t)$ are defined by

$$\tanh 2\mu y_h = \frac{(1 + H^2/\mu^2)^{1/2}}{1 + H^2/2\mu^2}, \tag{20}$$

$$\tanh 2y_\infty = \frac{1}{(1 + H^2/\mu^2)^{1/2}}. \tag{21}$$

These are time dependent. Clearly, in virtue of Eq. (18), $y = y_h$ is always a zero of a and defines a Cauchy horizon or coordinate singularity, where the Gaussian normal coordinates used to write Eq. (15) break down. There is an analytic extension beyond this horizon [21]. The location $y = y_h$ is a global minimum for a. Although a always goes to zero at the Cauchy horizon, in general n does not: indeed, it is typically discontinuous there. However, the values of a and n for $y > y_h$ are not meaningful, so this discontinuity is not seen by observers in the spacetime. There is not such a simple geometric interpretation for y_∞.

D. Five-dimensional braneworld scalar power spectrum

The scalar power spectrum is trivial if we do not couple ϕ to gravitational perturbations off the brane. Let ϕ be a free massless scalar field propagating over Σ. Then the propagator for ϕ is still defined by Eq. (4) (with integration over spacetime $\mathcal{M}$ replaced by integration over the slice Σ which corresponds to our Universe) and is exactly the same as the four-dimensional case, Eq. (10) [43]. This approximation was also made in [48]. When off-brane gravitational perturbations are included, the scalar spectrum should be expected to change, but the details of how this happens are somewhat calculationally inaccessible. In this paper, we ignore such effects.

E. Five-dimensional braneworld tensor power spectrum

The situation for gravitational perturbations is more complicated, and was first analyzed by Langlois *et al.* [30] for the case of a de Sitter brane; see also Gorbunov *et al.* [34] for a more detailed treatment. The simpler case of tensor perturbations around flat branes was studied in detail by Giudice *et al.* [32]. An alternative approach is based on the AdS/CFT correspondence, in the special case that the brane carries a large N conformal field theory (CFT). This was first done by Nojiri *et al.* [49,50], and later by Hawking *et al.* [51].

Langlois *et al.* worked in the Schrödinger picture. Here we repeat the calculation in a QFT. We will use this approach to generalize the calculation to an arbitrarily perturbed de Sitter brane in Sec. V. Let E_{ij} be a small perturbation of the metric, Eq. (15):

$$ds^2 = -n^2(t, y)dt^2 + a^2(t, y)(\delta_{ij} + E_{ij})dx^i dx^j + dy^2, \tag{22}$$

where E_{ij} is transverse and traceless with respect to the three-dimensional spatial metric δ_{ij}. Just as in four dimensions E_{ij} behaves like two copies of a scalar field. As in any Kaluza-Klein type decomposition, in order to make up the full SO(3, 1) graviton, one should include contributions from a graviscalar φ and graviphoton A_i which are the other components in a decomposition of metric perturbations under the isometry group of Eq. (15). We ignore φ and A_i, because they can be set to zero by a gauge transformation and do not contribute to the vacuum fluctuation during inflation [33].

The two-point function for E_{ij} satisfies

$$\langle E^{ij}(x_1)E_{rs}(x_2)\rangle = \int [\mathcal{D}E_{mn}]E^{ij}(x_1)E_{rs}(x_2) \times \exp\left[-\frac{i}{8\kappa_5^2}\int_{\mathcal{M}} dx E^{mn}\left(\frac{\Box_\parallel}{n^2} + \Box_\perp\right) \times E_{mn}\right], \tag{23}$$

where we have decomposed the five-dimensional braneworld d'Alembertian, $\Box_{\rm BW} = \nabla^a\nabla_a = \Box_\parallel/n^2 + \Box_\perp$, into two terms $\Box_\parallel$ and $\Box_\perp$, defined by

$$\Box_\parallel = \frac{\partial^2}{\partial t^2} - \left(3\frac{\dot{a}}{a} - \frac{\dot{n}}{n}\right)\frac{\partial}{\partial t} + \frac{n^2}{a^2}\Delta, \tag{24}$$

$$\Box_\perp = \frac{\partial^2}{\partial y^2} + \left(3\frac{a'}{a} + \frac{n'}{n}\right)\frac{\partial}{\partial y}. \tag{25}$$

Because Eq. (15) is not a product metric $\Box_\parallel$ and $\Box_\perp$ are not the on- and off-brane d'Alembertians, but in the important special case that the brane is endowed with a de Sitter geometry $\dot{H} = 0$ these operators separate [30]. In this case $\Box_\perp$ is an honest Sturm-Liouville operator and one can write E_{ij} as a sum over its eigenfunctions. This allows us to reexpress the path integral measure as a product of four-dimensional path integrals, essentially rewriting the whole theory as an effective four-dimensional theory with an infinite tower of increasingly massive fields. This strategy is key to the solubility of the de Sitter model. When considering the more general, perturbed theory later we will have to largely abandon this approach, although we retain some of its aspects.

Following [30] we define a set of weighted eigenfunctions, $\mathcal{E}_\alpha(y)$, of $\Box_\perp$ by

$$\Box_\perp \mathcal{E}_\alpha(y) = -\frac{\alpha^2}{n^2}\mathcal{E}_\alpha(y). \tag{26}$$

A standard argument [52,53] shows that the $\mathcal{E}_\alpha$ can be chosen to be orthonormal,

$$2\int_0^{y_h} n^2 dy \mathcal{E}^*_\alpha \mathcal{E}_\beta = \delta_{\alpha\beta}, \tag{27}$$

provided the $\mathcal{E}_\alpha$ obey suitable boundary conditions at $y = 0$ and $y = y_h$. We have added a factor 2 by hand in the normalization, to take account of the other branch of the orbifold.

One should pay attention to which combinations of boundary conditions are possible [54,55]. The case of gravitational waves is slightly more restrictive than either spin-0 or spin-1/2, since in order to disallow any anisotropic stress on the brane [30,34], one must choose the derivatives of the $\mathcal{E}_\alpha$ to vanish at $y = 0$ and $y = y_h$. With these boundary conditions, the allowed values of α consist of a discrete zero-mode bound state at $\alpha = 0$ and a continuum of massive modes for $\alpha > 3H/2$. A standard argument from Sturm-Liouville theory shows that the eigenfunctions $\mathcal{E}_\alpha$ form a complete set [56,57]. Performing this decomposition for E_{ij}, expressing the path integral measure in the same terms, and integrating over the transverse dimension in the action gives, for coordinates x_1, x_2 on the brane,

$$\begin{aligned}\langle E^{ij}(x_1)E_{rs}(x_2)\rangle \simeq \prod_\alpha \mathcal{E}^2_\alpha(0)\int[\mathcal{D}E^{mn}_\alpha]E^{ij}_\alpha(x_1)E_{rs,\alpha}(x_2) \\ \times \exp\bigg[-\frac{i}{8\kappa_5^2}\int_\Sigma dx E_{mn,\alpha}(\Box - \alpha^2) \\ \times E^{mn}_\alpha\bigg],\end{aligned} \tag{28}$$

where $E^{ij}(x, y) = \sum_\alpha E^{ij}_\alpha(x)\mathcal{E}_\alpha(y)$, dx is the volume measure on the de Sitter slice Σ, $\Box$ is the de Sitter Laplacian, and there are off-diagonal terms proportional to $\mathcal{E}_\alpha\mathcal{E}_{\alpha'}$ ($\alpha \neq \alpha'$) which we have neglected. Thus the field E_{ij} behaves like a collection of four-dimensional Klein-Gordon fields in de Sitter space, with masses described by the allowed values of α. At low energies, or during inflation, only the $\alpha = 0$ zero mode will be excited, so since $\mathcal{E}_0$ is independent of y, one has

$$\begin{aligned}\langle E^{ij}(x_1)E_{rs}(x_2)\rangle = \mathcal{E}^2_0\int[\mathcal{D}E^{mn}_0]E^{ij}_0(x_1)E_{rs,0}(x_2) \\ \times \exp\bigg(-\frac{i}{8\kappa_5^2}\int_\Sigma dx E_{mn,0}\Box E^{mn}_0\bigg).\end{aligned} \tag{29}$$

Therefore, the dominant contribution coming from the zero mode is no more than the standard result, with an extra factor of $\mathcal{E}^2_0$ in the normalization. This is still a free theory, so there is no obstruction to taking the coincidence limit $x_1 \to x_2$ as in four dimensions, and the power spectrum follows. Eliminating the five-dimensional coupling κ_5^2 in favor of its four-dimensional counterpart gives a final result

$$A_T^2 = \frac{\kappa_4^2 F^2}{50}\frac{H^2}{\pi^2}, \tag{30}$$

where $\mathcal{E}^2_0 = \mu F^2$. The quantity F expresses a renormalization of the amplitude A_T^2 in comparison with the four-dimensional result. This can be understood as a volume term arising from integrating out the extra dimension: the zero mode is a collective excitation which couples in the same way as the four-dimensional metric [23]. We derive an explicit formula for F in Appendix A; here, we merely quote the result,

$$F^2\left(\sqrt{1+\frac{H^2}{\mu^2}} - \frac{H^2}{\mu^2}\,\text{arc sinh}\frac{\mu}{H}\right) = 1. \tag{31}$$

III. THE CONSISTENCY RELATION

We now briefly describe how the consistency relation Eq. (2) arises. Before proceeding to formal expressions, it is useful to indicate why one should expect a consistency relation on general grounds. This is sometimes explained by saying that in the slow-roll formalism one has fewer parameters than observables, so some kind of relationship is inevitable. Although this is true, the relationship has to be developed order-by-order in perturbation theory and it is not clear why the final result is what it is. We cannot give a first principles derivation, but the following analysis is suggestive.

The amplitude of the scalar power spectrum must be related to the density perturbation δ. On purely geometrical grounds, this is $A_S^2 \sim \delta^2 \sim (H^2/\dot\phi)^2\delta\phi^2$. On the other hand, since the graviton is an effective massless free field its spectrum should be $A_T^2 \sim \delta\phi^2$. The ratio A_S^2/A_T^2 therefore involves only geometrical quantities. Writing $H' = dH/d\phi$, the tensor spectral index is $n_T \sim d\ln\delta\phi^2/d\ln k \sim \dot\phi H'/H^2$, again only using geometrical considerations and enough QFT to compute the spectrum of free, massless field in de Sitter space.

So far we have not used the Einstein field equations. These are seemingly necessary to relate A_S^2/A_T^2 and n_T, since one must know how to express $\dot\phi$ in terms of H. Knowledge of this relationship is equivalent to the Friedmann equation, which is a constraint in Einstein gravity. Although the Einstein field equations are sufficient, they are in fact stronger than necessary, because any theory of gravity which gives an action for the metric $ds^2 = -dt^2 + e^{3\lambda(t)}dx^2$ which is proportional to $\dot\lambda^2$ necessarily implies the Hamilton-Jacobi relation $\dot{\phi} \propto H'$ [15]. The exact constants of proportionality are fixed by the details of the theory. In the case of Einstein gravity, one obtains

$$H' = -\frac{\kappa_4^2}{2}\dot{\phi}. \tag{32}$$

A. Four dimensions

Consider four-dimensional inflation driven by a scalar field. The matter and gravitational power spectra satisfy Eqs. (10) and (14), respectively. As noted above, their ratio is a purely geometrical quantity

$$\frac{T}{S} = \frac{2}{2}\frac{\dot{\phi}^2}{H^2}. \tag{33}$$

The value of this quantity is set by the Friedmann equation and the classical field equation for ϕ, so one can consider it to be a function of the type of matter under discussion and the theory of gravity being employed. In particular, if one assumes that the scalar field ϕ is the only constituent of the Universe, then

$$\frac{A_T^2}{A_S^2} = \frac{2}{\kappa_4^2}\left(\frac{H'}{H}\right)^2 = \varepsilon. \tag{34}$$

We define a tensor spectral index n_T by $n_T = d\ln A_T^2/d\ln k$. To evaluate this one endows H with some extremely slow time dependence owing to motion of ϕ on very long time scales. One can then show that n_T is given, to first order in the slow-roll expansion, by

$$n_T = -\frac{4}{\kappa_4^2}\left(\frac{H'}{H}\right)^2 = -2\varepsilon. \tag{35}$$

Substituting this expression into Eq. (34) immediately reproduces Eq. (2).

B. The braneworld

In the braneworld, since the entire effect of the large extra dimensions appears as a renormalization F of the tensor amplitude, the ratio A_T^2/A_S^2 is still independent of A_ϕ^2.

In order to give an explicit expression, one needs to know the relationship between H and ϕ. As before, this depends only on the evolution of ϕ and the theory of gravity under discussion. Taking into account the relevant modifications, one finds

$$\frac{A_T^2}{A_S^2} = \frac{2F^2}{\kappa_4^2}\frac{\mu^2}{H^2+\mu^2}\left(\frac{H'}{H}\right)^2. \tag{36}$$

The tensor spectral index $n_T = d\ln A_T^2/d\ln k$ no longer depends merely on the functional form of A_ϕ^2, which is unchanged from the four-dimensional case, but instead receives nontrivial corrections from the renormalization F^2. In particular, $d\ln A_T^2 = 2d\ln HF$. At first sight, this would appear to break any hope of retaining the consistency relation. However, HF satisfies a particular differential equation [35]:

$$d\log HF = \frac{\mu F^2}{(H^2+\mu^2)^{1/2}}d\log H. \tag{37}$$

Combining n_T with Eqs. (36) and (37) gives back the consistency relation Eq. (2). The relation Eq. (37) is derived, together with an explicit expression for F, in Appendix A.

Although we have presented this result only for the case of a pure anti–de Sitter bulk with $\mathbf{Z}_2$ symmetry, the appearance of the consistency relation holds rather more generally. In particular, Huey and Lidsey [35] have argued that it persists if one allows different anti–de Sitter curvatures $\mu_<$, $\mu_>$ on the $y < 0$ and $y > 0$ branches.

IV. FLUCTUATIONS IN A PERTURBED FOUR-DIMENSIONAL DE SITTER SPACE

A. Introduction

In this section, we aim to formally calculate the power spectrum of a scalar field propagating over a background de Sitter cosmology with some first-order perturbation, restricting the calculation to purely four dimensions at this stage. Let $\mathcal{M}$ be four-dimensional de Sitter space with fixed, time-independent Hubble parameter H_0. Consider a small perturbation ΔH of the Hubble rate, with arbitrary time dependence, where ΔH is supposed to be sufficiently small that terms quadratic or higher in ΔH can be ignored. We wish to calculate the power spectrum of a free, massless scalar field propagating on this fixed geometry.

The study of inflationary fluctuations is quite mature and a number of effects are routinely included in calculations [15,58,59]. An important example is the coupling of metric perturbations to ϕ, since observations are approaching the precision at which one should include next-order effects. It is well known that in pure de Sitter space there is, in fact, no coupling: any metric fluctuations are pure gauge. To obtain coupling between scalar field fluctuations and bulk metric perturbations, one must have some measure of tilt away from de Sitter space. This tilt is precisely what is measured by the slow-roll parameters ε and η, and their higher-order relatives. If $\varepsilon = 0$, then one is in exact de Sitter space, and there are no metric fluctuations, whereas if $\varepsilon \neq 0$, then one should take account of the coupling. In the present case, bearing in mind the order to which we carry perturbation theory, there is no coupling between the scalar field perturbation and gravitational fluctuations.

B. Scalar and tensor power spectra

The two-point function $G(x_1, x_2) = i\langle\phi(x_1)\phi(x_2)\rangle$ is given by Eq. (4), so its Fourier transform $G(x_1, x_2) = (2\pi)^{-3}\int d^3k\, G(t_1, t_2; k)e^{i\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x}_2)}$ should satisfy

$$\left(\frac{\partial^2}{\partial t_1{}^2} + 3H\frac{\partial}{\partial t_1} + \frac{k^2}{R^2}\right)G(t_1, t_2; k) = \frac{\delta(t_1 - t_2)}{R^3}. \quad (38)$$

This can be solved to first order in ΔH. Thus, one is trying to build a Green's function $G = G_0 + G_1$, where G_0 is the background Green's function and G_1 is a perturbation. It is convenient to change variable to the conformal time, $d\tau = R_0^{-1}dt$, and set $G_n = u_n/R_0$ for $n = 0, 1$. One separates Eq. (38) into zero- and first-order parts. The zero-order part is Mukhanov's equation for the background [60,61],

$$\left[\frac{\partial^2}{\partial \tau_1{}^2} + k^2 - 2(R_0H_0)^2\right]u_0 = \frac{\delta(\tau_1 - \tau_2)}{R_0} \quad (39)$$

and the first-order part is a sourced Mukhanov equation,

$$\left[\frac{\partial^2}{\partial \tau_1{}^2} + k^2 - 2(R_0H_0)^2\right]u_1 = 2k^2\frac{\Delta R}{R_0}u_0 - 3\Delta H(R_0u_0' - u_0H_0R_0^2) - 3\frac{\Delta R}{R_0} \times \frac{\delta(\tau_1 - \tau_2)}{R_0}, \quad (40)$$

at first order, where δ is the Dirac delta function. The unperturbed Green's function G_0 can be thought of as the Feynman propagator for the free theory defined by the de Sitter background, and G_1 as the first term in a Feynman series for interactions introduced by the departure of the background from exact de Sitter. Mukhanov's equation has a well-known solution [41],

$$G_0 = \frac{i\pi}{4k}\frac{1}{R_0(\tau_1)R_0(\tau_2)}L^{(1)}(-k\tau_1)L^{(2)}(-k\tau_2) \quad (41)$$

if $\tau_2 > \tau_1$, and the same expression with τ_1 and τ_2 exchanged if not. The functions $L^{(n)}$ are defined in Eq. (7), and the result has been written in terms of G in order to exhibit the symmetry between τ_1 and τ_2. The boundary conditions as $|k\tau| \to \infty$ are fixed by the Bunch-Davies vacuum.

The remaining obstacle is the first-order sourced Mukhanov equation, Eq. (40). To solve this, one can choose either to employ some generally applicable technology, such as standard Sturm-Liouville theory [52], or seek direct solutions. For technical reasons we prefer the Sturm-Liouville approach, although a direct integral solution can also be given, which we briefly summarize in Appendix C. Returning to the Sturm-Liouville method, we define eigenfunctions for the background equation, of weight $-m^2$,

$$\left[\frac{\partial^2}{\partial \tau^2} + k^2 - \frac{2}{\tau^2}\right]\Omega_m = -m^2\Omega_m. \quad (42)$$

One must impose sufficient boundary conditions to make the Ω_m behave well, after which the background field equation and the Ω_m form a self-adjoint set. The boundary condition on the Ω_m at $\tau = -\infty$ is expected to be immaterial, provided the Ω_m decay sufficiently fast there. At $\tau = 0$, we demand that the Ω_m be regular. This selects the Bessel function,

$$\Omega_m(k, \tau) = \sqrt{-K\tau}J_{3/2}(-K\tau), \quad (43)$$

where $K^2 = k^2 + m^2$ as before. The Ω_m obey an orthonormality relation on $\tau \in (-\infty, 0]$

$$\int_{-\infty}^{0} d\tau\, \Omega_m(\tau)\Omega_n(\tau) = \delta(m - n) \quad (44)$$

or, equivalently, $\int_{-\infty}^{\infty} dm\, \Omega_m(\tau)\Omega_m(\sigma) = \delta(\tau - \sigma)$. The background equation on $y \in [0, y_h]$ is a singular in the Sturm-Liouville sense, so there is a continuum of eigenvalues and not simply a discrete set. Using the Ω_m and the completeness relation, the part of u_1 given by the driving term in Eq. (40) can be solved

$$u_1^{\mathrm{drive}}(\tau_1, \tau_2) = -\int_{-\infty}^{0} d\sigma\Gamma(\tau_1, \sigma)U(\sigma, \tau_2), \quad (45)$$

where the kernel or solution operator $\Gamma(\tau_1, \tau_2)$ is

$$\Gamma(\tau_1, \tau_2) = \int_{-\infty}^{\infty} \frac{dm}{m^2}\Omega_m(\tau_1)\Omega_m(\tau_2), \quad (46)$$

and $U(\sigma, \tau_2)$ satisfies

$$U(\sigma, \tau_2) = -2k^2\frac{\Delta R}{R_0}(\sigma)u_0(\sigma, \tau_2) - 3\Delta H(\sigma) \times [R_0(\sigma)u_0'(\sigma, \tau_2) - R_0^2(\sigma)H_0u_0(\sigma, \tau_2)]. \quad (47)$$

From Eq. (43) it is clear that Eq. (45) is no more than solution via a Fourier-Bessel transform. There is also an impulsive contribution to u_1 arising from the δ function. In order to preserve the asymptotic vacuum we must make the same choice of boundary conditions which governed the zero-order term, so this must be just proportional to u_0, giving

$$u_1^{\mathrm{impulse}}(\tau_1, \tau_2) = -3\frac{\Delta R}{R_0}(\tau_2)u_0(\tau_1, \tau_2). \quad (48)$$

One can now assemble G_0 and G_1 to construct the full two-point function, restoring the necessary factors of $\exp[i\boldsymbol{k} \cdot (\boldsymbol{x}_1 - \boldsymbol{x}_2)]$ and integrations over $\boldsymbol{k}$. We find

$$G(x_1, x_2) = \int \frac{d^3k}{(2\pi)^3}W(\tau_1, \tau_2; k)\exp[i\boldsymbol{k} \cdot (\boldsymbol{x}_1 - \boldsymbol{x}_2)], \quad (49)$$

where W satisfies

$$W(\tau_1, \tau_2; k) = \frac{i\pi}{4k}\frac{1}{R(\tau_1)R(\tau_2)}\left[1 - 3\frac{\Delta R}{R_0}(\tau_2)\right] \times L^{(1)}(-k\tau_2)L^{(2)}(-k\tau_1) - \frac{1}{R_0(\tau_1)}\int_{-\infty}^{0} d\sigma\Gamma(\tau_1, \sigma)U(\sigma, \tau_2), \quad (50)$$

if $\tau_1 > \tau_2$, and the same expression with τ_1 and τ_2 exchanged in the first term otherwise. To find the power spectrum one lets $\boldsymbol{x}_1$ and τ_1 approach $\boldsymbol{x}_2$ and τ_2, respectively, and takes a logarithmic derivative with respect to k. The result is

$$A_\phi^2 = -i\frac{k^3}{2\pi^2}W(\tau, \tau; k). \tag{51}$$

We take the limit $k/RH \to 0$ to give the asymptotic behavior on large scales [15]. One must divide the σ integral into two regions, where $\sigma < \tau_2$ and $\sigma > \tau_2$, respectively. We find

$$A_\phi^2 \xrightarrow{k/RH\to 0} \frac{H_0^2}{4\pi^2}\left[1 - 3\frac{\Delta R}{R_0} - \sqrt{\frac{\pi}{2}}k\tau Q(\tau)\right], \tag{52}$$

where $Q(\tau)$ is defined by a convolution with the kernel Γ,

$$Q(\tau) = \lim_{k/RH\to 0}\int_{-\infty}^0 d\sigma\, \Gamma(\sigma, \tau)\mathcal{L}(\sigma, \tau) \times \left\{2k^2\frac{\Delta R}{R_0} + 3\frac{\Delta H}{H_0}\left[\frac{3}{2\sigma^2} - \frac{3k}{\sigma}\mathcal{H}(\sigma, \tau)\right]\right\}. \tag{53}$$

The auxiliary quantities $\mathcal{L}$ and $\mathcal{H}$ satisfy

$$\mathcal{L}(\sigma, \tau) = \begin{cases} iL^{(1)}(-k\sigma), & \tau > \sigma, \\ -iL^{(2)}(-k\sigma), & \tau < \sigma, \end{cases} \tag{54}$$

and

$$\mathcal{H}(\sigma, \tau) = \begin{cases} H_{3/2}^{\prime(1)}(-k\sigma)/H_{3/2}^{(1)}(-k\sigma), & \tau > \sigma, \\ H_{3/2}^{\prime(2)}(-k\sigma)/H_{3/2}^{(2)}(-k\sigma), & \tau < \sigma. \end{cases} \tag{55}$$

The function Q will appear frequently. It is nonlocal and represents the effect of time dependence in our perturbation.

When comparing Eq. (52) with similar models in the literature, one should bear in mind that this expression constitutes a global solution, which does not involve a power series expansion around any particular point on the potential. To recover a standard result, one can expand in Taylor series around a value $H_0 = H(\phi_{\text{cl},0})$. By doing so, one can recover only the lowest order (de Sitter space) result from Eq. (52), since only first-order departures from de Sitter space are included. For comparison, the leading order term in the standard analysis is $\sim\varepsilon \approx (H'/H)^2$, which would be at second order in our calculation scheme. Therefore, although the motivation is the same in each case, the present result is not directly related to the Stewart-Lyth next-order calculation [58]. In particular, the Stewart-Lyth procedure involves a power series expansion of the potential, which is locally matched onto an exact solution, and explicitly quantizes the matter fluctuations. This does not happen in our approach: there is no coupling to matter fluctuations. This compromise does not represent the calculation one would ideally wish to carry out, but it does represent the first meaningful generalization which can practically be calculated in the braneworld. The main virtue of our approach is the nonperturbative treatment of the time dependence. Of course, the result is still perturbative in the amplitude.

Using Eq. (10) gives a final expression for the power spectrum of scalar curvature perturbations. When evaluated on the horizon scale, $-k\tau = 1$, we obtain an expression for the asymptotic amplitude, in terms of the values quantities had at horizon crossing:

$$A_S^2 = \frac{1}{25\pi^2}\frac{H_0^4}{\dot{\phi}_{\text{cl}}^2}\left[1 + 2\frac{\Delta H}{H_0} - 3\frac{\Delta R}{R_0} + \sqrt{\frac{\pi}{2}}Q(-k^{-1})\right], \tag{56}$$

where the extra term involving $\Delta H/H_0$ arises from the use of the full perturbed Hubble rate, rather than H_0, in Eq. (10).

The case of gravitational waves is similar, and in fact the reasoning applied in Sec. II B is still relevant: the action for each polarization of the graviton is the same as the free, massless scalar field action except that the relative normalization differs by a factor $(4\kappa_4^2)^{-1}$. There are two polarization modes, so Eq. (14) governs the tensor power spectrum,

$$A_T^2 = \frac{\kappa_4^2}{50}\frac{H_0^2}{\pi^2}\left[1 - 3\frac{\Delta R}{R_0} + \sqrt{\frac{\pi}{2}}Q(-k^{-1})\right]. \tag{57}$$

The ratio of tensor to scalar amplitudes satisfies

$$\frac{A_T^2}{A_S^2} = \frac{\kappa_4^2}{2}\frac{\dot{\phi}_{\text{cl}}^2}{H_0^4}\left(1 - 2\frac{\Delta H}{H_0}\right) \simeq \varepsilon\left(1 - 2\frac{\Delta H}{H_0}\right), \tag{58}$$

and the tensor spectral index is

$$n_T = -2\varepsilon\left(1 - \frac{\Delta H}{H_0}\right) - 3\frac{\Delta H}{H_0} + \sqrt{\frac{\pi}{2}}\frac{d}{d\ln k}Q(-k^{-1}). \tag{59}$$

V. FLUCTUATIONS IN A PERTURBED DE SITTER BRANEWORLD

We now repeat the same calculation in the braneworld. Consider a de Sitter brane with Hubble parameter H_0 immersed in anti–de Sitter space and allow small fluctuations $\Delta\rho$ in the matter density. These fluctuations are taken to vary with λ in such a way as to keep ΔH the same, regardless of the value of λ. We define this to be our notion of the "same" perturbation in the braneworld and in four dimensions. As one sends $\lambda \to \infty$ one should recover the four-dimensional result with this choice of ΔH, which is an expectation we will explicitly verify later.

A. Scalar power spectrum

First, consider some free, massless scalar field ϕ propagating over the brane Σ. This theory is just the same as one would find in four dimensions, provided scalar fluctuations coming from the bulk are ignored. (As discussed above, there are circumstances under which this may not be a good approximation.) The resulting fluctuation equation will coincide with Eq. (38) and the power spectrum one derives will equal Eq. (56), provided that R is taken to satisfy the expansion law for the on-brane cosmological scale factor.

This is equivalent to supposing that the Mukhanov equation remains a valid description of the perturbation on the brane [48].

B. Tensor power spectrum

The case of gravitational waves is not the same. In a general geometry, the graviton wave operator $\Box_{\mathrm{BW}}$ couples the t and y dependence of the graviton $\boldsymbol{k}$-modes, so that an explicit solution is extremely difficult. One can always work on the brane universe in black hole coordinates [21,22], where the metric is explicitly stationary, and one recovers ordinary differential equations. Unfortunately, the boundary conditions are nontrivial to apply, because the brane appears as a quite arbitrarily curved figure. In this section we make progress by a different route.

We begin by rewriting the general formula Eq. (19) for $n(t, y)$ in terms of H', where

$$\dot{H} = \dot{\phi}H' = -\frac{2\mu}{\kappa_4^2}\frac{H'^2}{\sqrt{H^2+\mu^2}}. \tag{60}$$

Since $\dot{H} \propto H'^2$, if we perturb around the de Sitter solution then the term $H' = \Delta H$ is small and we may neglect its square. Hence, for a perturbed de Sitter brane, we still retain $n = a/R$. We emphasize that this is only true for perturbations around de Sitter space supported by a single scalar field where the background H satisfies $H' = 0$. When calculating spectral indices we will again endow H with very weak time dependence, but for the purposes of the calculation presented in this section the background H is to be regarded as fixed, in analogy with the four-dimensional calculation of A_ϕ^2.

Let us write the metric functions a and n as in Eqs. (18) and (19),

$$n(y) = \frac{H}{\sqrt{2}}[\cosh 2\mu(y_h - y) - 1]^{1/2}, \tag{61}$$

where H is a constant. The other function a satisfies $a(t, y) = R(t)n(y)$ where $R(t)$ is the scale factor on the brane. Under a variation $H \mapsto H_0 + \Delta H$, the function n acquires a time dependence, and $n(t, y)$ becomes

$$n(t, y) \mapsto \left(1 + \frac{\Delta H}{H_0}\right)\frac{H_0}{\sqrt{2}}[\cosh 2\mu(y_h + \Delta y_h - y) - 1]^{1/2}, \tag{62}$$

since the horizon location y_h depends on time. The perturbed n, Eq. (62), has the same values at each end point as the unperturbed n_0, so it satisfies $n(t, 0) = 1$ and $n(t, y_h) = 0$. This may appear surprising, because one would typically expect a perturbation to disturb these values. The condition $n(t, y = 0) = 1$ arises because of the gauge condition which fixes t, and as a result the perturbation in the cosh term exactly cancels the perturbation in the prefactor at $y = 0$. The second occurs because $y = y_h$ is a minimum of n_0, so it is not displaced to first order. If more terms were retained in the perturbation expansion, or any dark radiation were to be present, then $n(t, y_h)$ would change.

Any perturbation of H in a four-dimensional cosmology is necessarily sourced by a corresponding change in the matter density ρ, in virtue of the four-dimensional Friedmann equation. This simplicity does not carry over to the braneworld. Instead, the possible existence of a dark radiation component allows a one-parameter family of choices, all of which can source any given ΔH. This allows us to identify two distinct perturbation modes, which we designate type I and type II: the first corresponding to a perturbation of the density ρ which leaves the dark radiation C intact, and the second corresponding to the opposite arrangement. A general perturbation will be some admixture of the two. Recall that in our geometry, the dark radiation component is initially absent. To proceed it is necessary to decide how ΔH is to be split between ρ and C.

The presence of a dark radiation component C will presumably change the physics, since it involves the introduction of an extra tunable parameter in the description. For this reason we would like it to be absent, because in the four-dimensional model the perturbation came entirely from the matter sector. In order to achieve a proper comparison with the braneworld result, the perturbation here should also arise entirely from density perturbations and not from the introduction of dark radiation. In Appendix B we show that the model described above does not contain a dark radiation component. Therefore, although there is no reason of principle why type II perturbations should not be present, our future considerations will be restricted to cases where they are not. We should like to observe that there appears to be no known analytic solution, either perturbative or nonperturbative, for the form of the gravitational wave function in the presence of dark radiation.

C. The tensor zero mode

We now solve for the graviton zero mode. The method of analysis applied in the unperturbed case, based on a standard decomposition of the path integral action into harmonics of the transverse dimension, no longer makes sense here because the metric functions, such as Eq. (62), no longer separate. Our analysis is based on a specific ansatz: we suppose that the graviton zero mode remains distinct, and carries no dependence on the transverse dimension. To understand why this supposition works, consider the classical field equation for the graviton, $\Box_{\mathrm{BW}}\Psi = 0$,

$$\left(-\frac{1}{n^2}\frac{\partial^2}{\partial t^2} - \frac{\omega}{n^2}\frac{\partial}{\partial t} + \frac{\Delta}{a^2} + \frac{\partial^2}{\partial y^2} + \sigma\frac{\partial}{\partial y}\right)\Psi = 0, \tag{63}$$

where the coefficient functions ω and σ are given by

$$\omega = 3\frac{\dot{a}}{a} - \frac{\dot{n}}{n} \quad \text{and} \quad \sigma = 3\frac{a'}{a} + \frac{n'}{n}. \tag{64}$$

This is to be expanded to first order in the perturbations Δa and Δn. The explicit solution to the background equation has previously appeared in the literature [30,34]. At first order one obtains, writing $\Psi = \Psi_0 + \Psi_1$ for the expansion of the field,

$$\begin{aligned}\Psi_1'' + 4\frac{n_0'}{n_0}\Psi_1' - \frac{1}{n_0^2}\ddot{\Psi}_1 - \frac{3H_0}{n_0^2}\dot{\Psi}_1 - \frac{k^2}{a_0^2}\Psi_1 \\ = \frac{\Delta\omega}{n_0^2}\dot{\Psi}_0 - \frac{2}{n_0^2}\frac{\Delta n}{n_0}(\ddot{\Psi}_0 + 3H_0\dot{\Psi}_0) \\ - \frac{2k^2}{a_0^2}\frac{\Delta a}{a_0}\Psi_0 - \Delta\sigma\Psi_0'.\end{aligned} \tag{65}$$

One can show that $\Delta\omega = 3\Delta H$. Restricting attention to the perturbation of the zero mode and making use of the background field equation, this becomes

$$\begin{aligned}\Psi_1'' + 4\frac{n_0'}{n_0}\Psi_1' - \frac{1}{n_0^2}\ddot{\Psi}_1 - \frac{3H_0}{n_0^2}\dot{\Psi}_1 - \frac{k^2}{a_0^2}\Psi_1 \\ = \frac{3\Delta H}{n_0^2}\dot{\Psi}_0 + \frac{2k^2}{R_0^2 n_0^3}\Psi_0\left(\Delta n - \frac{\Delta a}{R_0}\right).\end{aligned} \tag{66}$$

The right-hand side appears to be a complicated function of t and y. This is true in general, but the special relationship $a = Rn$ reduces the term in brackets to $n_0\Delta R/R_0$; this is a consequence of the assumption that $\dot{H}$ vanishes. In virtue of this simplification, we can separate the field equation into a y-derivative piece

$$\Psi_1'' + 4\frac{n_0'}{n_0}\Psi_1' = 0 \tag{67}$$

and a t-derivative piece

$$\ddot{\Psi}_1 + \frac{3H_0}{n_0^2}\dot{\Psi}_1 + \frac{k^2}{R_0^2}\Psi_1 = \frac{2k^2}{R_0^2}\Psi_0\frac{\Delta R}{R_0} - 3\Delta H\dot{\Psi}_0. \tag{68}$$

The y equation has solutions $\Psi_1 \propto$ constant or $\Psi_1 \propto n_0^{-4}$. Since $n_0 \rightarrow 0$ at the Cauchy horizon, the correct solution is to take the y dependence of Ψ_1 to be constant, preventing an unwanted divergence at $y = y_h$. This choice is a necessary consequence of the boundary conditions on Ψ, which enforce $\Psi' = 0$ at the horizon in order to keep anisotropic stress absent.

On its own, this calculation is insufficient to obtain the two-point function for the zero mode, which should properly be obtained from a functional integral like Eq. (4). Consider the two-point function for a polarization mode ϕ of the graviton, and suppose we can split ϕ into a zero-mode piece ϕ_0, or collective excitation, which has no transverse dependence, and an unimportant remainder which encodes the details of heavy Kaluza-Klein modes. We assume it is permissible to ignore these heavy modes. In order for this procedure to make sense, we must suppose that the zero mode is stable under small perturbations. It is not guaranteed that this happens, but if it does then the two-point function becomes

$$\begin{aligned}\langle\phi(x_1)\phi(x_2)\rangle = \int[\mathcal{D}\phi_0]\phi(x_1)\phi(x_2) \\ \times \exp\left(-\frac{i}{8\kappa_5^2}\int dx\,\phi_0\Box_{\mathrm{BW}}\phi_0\right).\end{aligned} \tag{69}$$

Since ϕ_0 has no transverse dependence by assumption, the action of the braneworld Laplacian $\Box_{\mathrm{BW}}$ on ϕ_0 is the same as the de Sitter Laplacian $\Box_{\mathrm{dS}}$, where

$$\Box_{\mathrm{dS}} = -\frac{\partial^2}{\partial t^2} - 3H\frac{\partial}{\partial t} + \frac{\Delta}{R^2}. \tag{70}$$

Dropping the 0 subscript on ϕ_0, the action for ϕ must be

$$S = \int dx\,\phi\Box_{\mathrm{BW}}\phi = \int d^3x\,dt\,dy\,R^3n^2\phi\Box_{\mathrm{dS}}\phi. \tag{71}$$

One now integrates over y to obtain an effective four-dimensional action, which, since the only y dependence occurs in n, must be of the form

$$S = \int d^3x\,dt\,dy\left(n_0^2R^3 + 2n_0\frac{\Delta n_0}{\Delta H}R_0^3\Delta H\right)\phi\Box_{\mathrm{dS}}\phi. \tag{72}$$

This can be split in two, and each integral performed separately. The integral $\int dy\,n_0^2$ is just the familiar normalization factor $(\mu F^2)^{-1}$. The new contribution from the perturbed piece is, explicitly,

$$\begin{aligned}Y^2 &= 4\int_0^{y_h} dy\,n_0\frac{\Delta n}{\Delta H} \\ &= \frac{H_0}{\mu^3}\left(\frac{2\mu(1+\cosh\mu y_h)}{\sqrt{H_0^2+\mu^2}} + \pi + 4\arctan e^{\mu y_h}\right. \\ &\quad \left. - 2\sinh\mu y_h\right).\end{aligned} \tag{73}$$

Y^2 has a simple geometrical interpretation. In the background geometry, the brane and the horizon are parallel. Integrating over the volume between them, with the correct AdS measure, gives the normalization μF^2. When the perturbation is introduced, the volume of the AdS slice between the brane and the horizon is changed, because of the deformation suffered by the metric function n. The extra normalization piece Y^2 takes account of this change in volume. A subtle feature is that the background normalization should be integrated between $y = 0$ and the real horizon at $y = y_h + \Delta y_h$, but in fact it is easy to see that this introduces no new terms, because $n_0(t, y = y_h) = 0$. Therefore, we are entitled to carry all volume integrals only to the unperturbed horizon, at $y = y_h$.

This understanding of the origin of Y^2 provides a useful physical characterization of the approximation that $\dot{H} = 0$, whose ramifications are not obvious merely from inspection of the formulas for n and a. The physical content of this approximation is that we are including only the "-breathing-mode" of the perturbation. In particular, couplings of the wave zero mode to curvature fluctuations in the bulk are neglected; a more sensitive analysis will be needed to decide if such couplings play an important role.

Combining the two integrals appearing here gives the four-dimensional effective action correct to first order,

$$\langle\phi(x_1)\phi(x_2)\rangle = \int[\mathcal{D}\phi]\phi(x_1)\phi(x_2) \times \exp\left[-\frac{i}{8\kappa_5^2}\frac{1}{\mu F^2}\int_\Sigma d^3x\,dtR^3 \times (1+\mu F^2Y^2\Delta H)\phi\Box_{\rm dS}\phi\right], \tag{74}$$

where x_1 and x_2 are taken to lie on the brane. This is now amenable to solution using the four-dimensional methods of the preceding section. The two-point function satisfies

$$\langle\phi(x_1)\phi(x_2)\rangle = -4i\kappa_4^2F^2G(x_1, x_2), \tag{75}$$

where $G(x_1, x_2)$ is the Green's function for $\Box_{\rm dS}$ in the measure $R^3(1 + \mu F^2Y^2\Delta H)$. Therefore $G(t_1, t_2; k)$ should solve [cf. Eq. (38)]

$$\left(\frac{\partial^2}{\partial {t_1}^2} + 3H\frac{\partial}{\partial t_1} + \frac{k^2}{R^2}\right)G_0(t_1, t_2; k) = (1 - F^2Y^2\Delta H)\frac{\delta(t_1 - t_2)}{R^3}. \tag{76}$$

It is now immediate that the tensor power spectrum satisfies

$$A_T^2 = \frac{\kappa_4^2F^2H_0^2}{50\pi^2}\left[1 - 3\frac{\Delta R}{R_0} - \mu H_0F^2Y^2\frac{\Delta H}{H_0} + \sqrt{\frac{\pi}{2}}Q(-k^{-1})\right], \tag{77}$$

where F involves the background Hubble rate H_0 only. One should check that this expression has the correct form in the decoupling limit $\lambda \to \infty$. In the present case, this is equivalent to $\mu \to \infty$ (see Sec. II C), so it is easy to verify that $Y^2 \to 0$, and A_T^2 reverts smoothly to its four-dimensional equivalent. In showing this, it is essential that $\sinh\mu y_h$ and $\cosh\mu y_h$ diverge only linearly as $\mu \to \infty$ (recalling that y_h is a function of μ).

D. Braneworld consistency relation

In five dimensions there an obstruction to any attempt to reestablish the consistency relation. This obstruction arises from the change in normalization of the graviton zero mode, and, in particular, its dependence on the brane tension λ. To see how this works in detail, we make the approximation that to calculate the tensor spectral index one takes H_0 to be a slowly rolling function of ϕ,

$$H_0^2 + 2H_0\Delta H_0 = \frac{\kappa_4^2}{3}\rho_\phi\left(1 + \frac{\rho_\phi}{2\lambda}\right) + \frac{\kappa_4^2}{3}\Delta\rho\left(1 + \frac{\rho_\phi}{\lambda}\right). \tag{78}$$

Therefore the perturbation ΔH satisfies

$$\Delta H = \frac{\kappa_4}{2\sqrt{3}}\frac{\Delta\rho}{\rho_\phi^{1/2}}\left(1 + \frac{\rho_\phi}{\lambda}\right)\left(1 + \frac{\rho_\phi}{2\lambda}\right)^{-1/2}, \tag{79}$$

where ρ_ϕ is independent of λ. One must now ask what sort of perturbation $\Delta\rho$ is to be expected. Generically $\Delta\rho$ should not depend on λ either, otherwise one has to know the brane tension in advance in order to tune the perturbation correctly. This is manifestly undesirable, so the only λ dependence in Eq. (79) is as explicitly written. For example, one cannot produce a $\Delta\rho$ which depends on λ from a generic scalar field theory.

The ratio A_T^2/A_S^2 in the braneworld satisfies

$$\frac{A_T^2}{A_S^2} = \varepsilon\left(1 - \mu F^2Y^2\Delta H - 2\frac{\Delta H}{H_0}\right), \tag{80}$$

where the slow-roll parameter ε is defined conventionally,

$$\varepsilon = \frac{2}{\kappa_4^2}F^2\left(\frac{H_0'}{H_0}\right)^2\frac{\mu^2}{H_0^2 + \mu^2}. \tag{81}$$

To complete the analysis, one only needs an expression for the tensor index n_T. By replacing $d\ln k \sim H\,dt$ and replacing t with the background evolution of $\phi_{\rm cl}$, one obtains

$$n_T = -2\varepsilon\left(1 - \frac{\Delta H}{H_0}\right) - 3\frac{\Delta H}{H_0} - \frac{d}{d\ln k}\left[\mu F^2Y^2\Delta H + \sqrt{\frac{\pi}{2}}Q(-k^{-1})\right]. \tag{82}$$

The appropriate minimal consistency relation, in this case, should be the first-order relation $n_T = -2A_T^2/A_S^2$. One should use the first-order relation and not the next-order relation [15], because we do not include the next-order effects which lead to Eq. (3).

If one demands that the first-order consistency relation holds, then F^2, Y^2 Q and their derivatives must be related in a particular way. Of course, one can expect to find general solutions ΔH which make this consistency relation true. But if one demands in addition that the appropriate $\Delta\rho$ is independent of λ, as we have argued above that a general matter theory should obey, then it is no longer so clear that solutions exist. Indeed, by power expanding in λ, which should be good at least in a local neighborhood of $\lambda = 0$, one can show in the background limit where $\rho_\phi =$ constant that a solution with $\Delta\rho$ independent of λ is not possible.

It is conceivable that solutions with $\Delta\rho$ a function of λ exist, but such solutions are fine tuned. In other words, it may be possible to recover the consistency relation for some choices of the matter theory, but this is no longer generic. This is the principal result of this paper: our breathing-mode approximation suggests that the low-order consistency relation is broken in the braneworld, in a generic manner, when perturbations away from the de Sitter background are considered.

VI. CONCLUSIONS

In this paper, we apply the apparatus of five-dimensional quantum field theory to the question of gravitational perturbations in Randall-Sundrum type cosmologies. We have developed a perturbation expansion for the gravitational wave modes around the pure de Sitter case $H =$ constant which applies in the braneworld and in four dimensions, and introduced a calculational approximation for the tensor spectrum which consists of including only the ''breathing-mode'' of the bulk. We use this technology to calculate the power spectrum of scalars and gravitational waves as seen on the brane, or in four dimensions, and write a consistency relation in the four-dimensional case. We also suggest, within the context of our approximation, that a general perturbation ΔH may not have the correct λ dependence to preserve the consistency relation. This happens because we acquire an extra normalization term Y^2 in the four-dimensional effective action which accounts for changes in the volume of the bulk AdS slice which lies between the brane and the Cauchy horizon.

Moreover, we are genuinely comparing like with like when we contrast this result in the braneworld with a four-dimensional reference geometry: in each case, the perturbation is solely to the matter component. It is important to be specific about how the perturbation occurs in the braneworld, where a perturbation to H can be partitioned between ordinary matter and the dark radiation. Therefore, the extra physics which we see does genuinely arise from a bulk effect, namely, the change in volume between the brane and the horizon, but it is certainly not a back-reaction effect caused, for example, by scattering off Weyl curvature in the bulk. We anticipate that such back-reaction corrections would enter at a higher order in perturbation theory, but at present such refinements are out of reach of analytical treatment.

This analysis addresses a troubling feature of the braneworld model: it predicts an identical observational degeneracy in comparison with the conventional four-dimensional cosmology. This is important; a complete degeneracy would hinder any attempt to observationally reconstruct the inflaton potential [37]. We have shown, by an explicit calculation, that degeneracies of this type may not be generic. Indeed, the degeneracy could be broken for an open neighborhood of models close to the de Sitter solution. Our methods do not say much about models which are distant from de Sitter space. In order to reach a more general set of models, one could attempt to work in the slow-roll expansion, rather like the calculation of Stewart and Lyth [58] or Kosowsky and Turner [62] in four dimensions. This analysis has not yet been done, and there are rather formidable calculation obstacles to carrying it out. Our approach represents an alternative that at least allows us to probe some speculative aspects of the physics away from de Sitter space.

Our calculation relies on exploiting a technical device to calculate the tensor power spectrum in a model perturbed around a de Sitter brane carrying a single scalar field. This extends the range of models in which one knows how to solve (at least rather approximately) for the spectrum of gravitational waves produced during an inflationary epoch. This is a hard problem, whose complete solution is not yet understood. Our method relies on the presence of a distinct, stable zero mode which has trivial dependence on the transverse dimension, and will not easily generalize to full case of arbitrary time evolution on the brane, and for which a stable zero mode may not exist, but may suggest future directions in which to proceed. One such possibility is to study the brane universe in explicitly static SAdS coordinates, where there is a holonomic timelike Killing vector $\partial/\partial T$. The graviton field equation is then independent of T and becomes an ordinary differential equation similar to the Regge-Wheeler equation of black hole perturbation theory. The brane appears as a Neumann boundary condition applied to what is effectively a moving mirror, and it is possible that this framework is accessible to analytic attack. Our calculation does not yet include back-reaction from other fields on the brane, so it not general enough (for example) to include other types of matter, or to generalize to a second order result.

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APPENDIX A: THE NORMALIZATION FUNCTION F

In this section we sketch how the normalization function F and the central differential equation Eq. (37) are obtained.

One defines F to satisfy $2\mu F^2 = \mathcal{E}_0^2$, where $\mathcal{E}_0$ is the zero mode of $\Box_\perp$. The $\mathcal{E}_\alpha$ are normalized in the Sturm-Liouville measure arising from $\Box_\perp$, that is, $2\int dy\, n^2\mathcal{E}_\alpha\mathcal{E}_\beta = \delta_{\alpha\beta}$. The factor of 2 has been added to take account of the other branch of the orbifold, since we work on $y \in [0, y_h]$. Because the $\mathcal{E}_0$ are independent of y, this just says $2\mu F^2 \int n^2\, dy = 1$.

It is easy to evaluate this integral directly, but for the purposes of obtaining Eq. (37), it is convenient to employ the relation $n' = -\sqrt{H^2 + \mu^2 n^2}$ which arises from the Einstein field equation. In that case, the normalization requirement depends only on the integral of the purely geometrical quantity n,

$$2\mu F^2 \int_0^1 \frac{n^2\, dn}{\sqrt{H^2 + \ell^2 n^2}} = 1. \tag{A1}$$

This does not depend on a detailed knowledge of the form of n, except through n'. Here μ is the ratio κ_4^2/κ_5^2 of the four- and five-dimensional gravitational couplings and ℓ is the AdS radius, which in the case of vanishing four-dimensional cosmological constant equals μ. This is the case throughout the main body of this paper. One now makes a trigonometric substitution to evaluate the integral. The result is

$$\frac{\mu}{\ell} F^2 \left(\sqrt{1 + \frac{H^2}{\ell^2}} - \frac{H^2}{\ell^2} \text{arc sinh} \frac{\ell}{H} \right) = 1. \tag{A2}$$

If $\mu = \ell$ then this result agrees with Refs. [30,34]. To derive Eq. (37), one can differentiate this result directly, but it is easier to proceed as follows. Multiply Eq. (A1) by H^2 and differentiate logarithmically. One finds,

$$\frac{d\log HF}{d\log H} + \mu F^2 \frac{d}{d\log H} \int_0^1 \frac{n^2 dn}{\sqrt{H^2 + \mu^2 n^2}} = 1. \tag{A3}$$

It is now easy to differentiate under the integral sign, and integrate the resulting expression which gives

$$\frac{d}{d\log H} \int_0^1 \frac{n^2\, dn}{\sqrt{H^2 + \mu^2 n^2}} = \frac{1}{\mu F^2} - \frac{1}{\sqrt{H^2 + \mu^2}}. \tag{A4}$$

Substituting this into Eq. (A3) gives the result Eq. (37). This relation was first noticed by Huey and Lidsey [35].

Because the normalization integral does not involve integrating over a solution to the field equation $\Box\phi = 0$, one can interpret this result as a statement about the dependence of the metric g_{ab} on the initial conditions prescribed on the brane.

APPENDIX B: DARK RADIATION

Dark radiation is equivalent to Weyl curvature in the bulk spacetime [19]. One can measure such curvature by any suitable invariant formed from the Weyl tensor C_{abcd}, which is the part of the Riemann tensor not determined by the Ricci curvature. It can be described as the "free" part of the gravitational field. For example, one can choose the square of the Weyl tensor, $\Psi = C^{abcd}C_{abcd}$. By allowing H and y_h to vary with time, while keeping the general form of the solution (18) and (19), one finds a Weyl invariant of the form

$$\begin{aligned}\Psi = \frac{2}{H^8}[\mu^4 \text{cosech}\mu(y_h - y)]^8((4\dot{H}^2 - 2H\ddot{H}) \\ \times \sinh^2\mu(y_h - y) + H(2\mu\dot{H}\dot{y}_h + \mu H^2\dot{y}_h - \mu H\ddot{y}_h) \\ \times \sinh 2\mu(y_h - y) \\ + \mu^2 H^2[3 + \dot{y}_h{}^2 \cosh 2\mu(y_h - y)])^2.\end{aligned} \tag{B1}$$

Ψ has a leading contributions proportional to $\dot{y}_h{}^2$. Since Ψ is quadratic in C_{abcd}, this means that C_{abcd} itself is quadratic in ΔH and therefore zero at first order. Alternatively, one can see that since C is zero in the unperturbed geometry, it can enter only at $\Delta\dot{H}$ in the perturbed cosmology. Since we are ignoring time variation in $\Delta\dot{H}$ as a second order effect, no Weyl curvature, or dark radiation, is induced to leading order-by the perturbation.

APPENDIX C: DIRECT SOLUTION OF THE MUKHANOV EQUATION

In this appendix, we briefly outline a direct integral solution of the sourced Mukhanov equation without using the Fourier-Bessel transform technology which is employed in the main text. One is trying to solve the equation

$$\left(\frac{d^2}{d\tau^2} + k^2 - \frac{2}{\tau^2}\right)u(\tau) = f(\tau) \tag{C1}$$

for some source function $f(\tau)$. One can integrate to find

$$\begin{aligned}u(\tau) = -\frac{1}{\tau k^{3/2}}\bigg[\alpha S^{(1)}(k\tau) + \beta S^{(2)}(k\tau) \\ + \int_{-\infty}^{\tau} \frac{f(x)\, dx}{x k^{3/2}} (S^{(1)}(k\tau)S^{(2)}(kx) \\ - S^{(1)}(kx)S^{(2)}(k\tau)) \bigg],\end{aligned} \tag{C2}$$

where the functions $S^{(1)}(z)$ and $S^{(2)}(z)$ are related to the Hankel functions of order $3/2$,

$$S^{(1)}(z) = \cos z + z \sin z, \qquad S^{(2)}(z) = \sin z - z \cos z. \tag{C3}$$

For calculating the perturbed power spectrum, the appropriate source function is

$$f = 2k^2 \frac{\Delta R}{R_0} u_0 - 3\Delta H(R_0 u_0' - R_0^2 H_0 u_0). \tag{C4}$$

By proceeding with the standard argument by which one calculates the power spectrum, one arrives at the expression

$$A_\phi^2 = \frac{H_0^2}{4\pi^2}\left[1 - 3\frac{\Delta R}{R_0} - \frac{i}{k^2}\sqrt{\frac{\pi}{2}}\int_{-\infty}^{\tau}\frac{d\sigma}{\sigma}\bar{S}(\tau,\sigma)C(\sigma)\right], \tag{C5}$$

where

$$\bar{S} = (1 - k^2\tau\sigma)\sin k\sigma - k(\tau+\sigma)\cos k\sigma \tag{C6}$$

and

$$C = 2k^2\frac{\Delta R}{R_0}(-k\sigma)^{1/2}H_{3/2}^{(1)}(-k\sigma) + 3\Delta H\left(\frac{3}{2H_0\sigma^2}H_{3/2}^{(1)} - \frac{k}{H_0\sigma}H_{3/2}^{(1)\prime}\right), \tag{C7}$$

where the argument of each Hankel function is $-k\sigma$.

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