The trispectrum of the cosmic microwave background on subdegree angular scales: an analysis of the BOOMERanG data


This version is available from Sussex Research Online: http://sro.sussex.ac.uk/id/eprint/28301/

This document is made available in accordance with publisher policies and may differ from the published version or from the version of record. If you wish to cite this item you are advised to consult the publisher’s version. Please see the URL above for details on accessing the published version.

Copyright and reuse:
Sussex Research Online is a digital repository of the research output of the University.

Copyright and all moral rights to the version of the paper presented here belong to the individual author(s) and/or other copyright owners. To the extent reasonable and practicable, the material made available in SRO has been checked for eligibility before being made available.

Copies of full text items generally can be reproduced, displayed or performed and given to third parties in any format or medium for personal research or study, educational, or not-for-profit purposes without prior permission or charge, provided that the authors, title and full bibliographic details are credited, a hyperlink and/or URL is given for the original metadata page and the content is not changed in any way.

http://sro.sussex.ac.uk
The trispectrum of the cosmic microwave background on subdegree angular scales: an analysis of the BOOMERanG data

G. De Troia,1,7⋆ P. A. R. Ade,2 J. J. Bock,3 J. R. Bond,4 A. Boscaleri,5 C. R. Contaldi,4 B. P. Crill,6 P. de Bernardis,1 P. G. Ferreira,7 M. Giacometti,1 E. Hivon,8 V. V. Hristov,6 M. Kunz,7,9 A. E. Lange,6 S. Masi,1 P. D. Mauskopf,2 T. Montroy,10 P. Natoli,11 C. B. Netterfield,12 E. Pascale,5 F. Piacentini,1 G. Polenta,1 G. Romeo13 and J. E. Ruhl10

1Dipartimento di Fisica, Università La Sapienza, Piazzale A. Moro 2, I-00185 Rome, Italy
2Department of Physics and Astronomy, Cardiff University, Cardiff CF24 3YB
3Jet Propulsion Laboratory, Pasadena, CA, USA
4Canadian Institute for Theoretical Astrophysics, University of Toronto, Canada
5IROE-CNR, Firenze, Italy
6California Institute of Technology, Pasadena, CA, USA
7Astrophysics, University of Oxford, Keble Road, Oxford OX13RH
8IPAC, California Institute of Technology, Pasadena, CA, USA
9Astronomy Centre, University of Sussex, Brighton BN1 9QJ
10Department of Physics, Case Western Reserve Uni, Cleveland, OH, USA
11Dipartimento di Fisica, Università Tor Vergata, Via della Ricerca Scientifica, 1 I-00133 Rome, Italy
12Departments of Physics and Astronomy, University of Toronto, Canada
13Istituto Nazionale di Geofisica, Rome, Italy

Accepted 2003 March 27. Received 2003 March 26; in original form 2003 January 20

ABSTRACT

The trispectrum of the cosmic microwave background can be used to assess the level of non-Gaussianity on cosmological scales. It probes the fourth-order moment, as a function of angular scale, of the probability distribution function of fluctuations and has been shown to be sensitive to primordial non-Gaussianity, secondary anisotropies (such as the Ostriker–Vishniac effect) and systematic effects (such as astrophysical foregrounds). In this paper we develop a formalism for estimating the trispectrum from high-resolution sky maps that incorporates the impact of finite sky coverage. This leads to a series of operations applied to the data set to minimize the effects of contamination due to the Gaussian component and correlations between estimates at different scales. To illustrate the effect of the estimation process, we apply our procedure to the BOOMERanG data set and show that it is consistent with Gaussianity. This work presents the first estimation of the cosmic microwave background trispectrum on subdegree scales.

Key words: methods: statistical – cosmic microwave background.

1 INTRODUCTION

The cosmic microwave background (CMB) has become the observational tool par excellence for probing the statistical nature of inhomogeneities in the Universe. The small deviations from homogeneity which have been detected by over two dozen different experiments can be directly related to the primordial origin of perturbations in the early Universe and therefore to fundamental physics at very high energies. A new threshold was crossed in the experimental forum with the high-resolution, high-sensitivity mapping of significant fractions of the CMB sky by the BOOMERanG (de Bernardis et al. 2000) and MAXIMA (Hanany et al. 2000) experiments. A careful analysis of the variance of fluctuations in these maps has led to accurate estimates of the angular power spectrum, far surpassing previous experimental analyses on small angular scales. More recent results from BOOMERanG (Netterfield et al. 2002; Ruhl et al. 2002), MAXIMA (Lee et al. 2001) and from other experiments such as DASI (Halverson et al. 2002), CBI (Pearson et al. 2002), VSA (Grainge et al. 2002), ACBAR (Kuo et al. 2002) and ARCHEOPS (Benoit et al. 2003) have put our knowledge of the CMB angular power spectrum on even more solid ground. However, there is more information in the CMB fluctuations than is provided by its power

*E-mail: Grazia.DeTroia@roma1.infn.it

© 2003 RAS
and Castro (2003) showed it to be a powerful probe of the OV effect. Aghanim et al. (2003) found it to be very sensitive to point sources. Different kinds of non-Gaussianity compared with the bispectrum; the case of statistically homogeneous and isotropic fluctuations is in terms of all its higher-order moments. In the cubic moment of the field, we consider in this study, we probe a rather non-trivial function, which we define as

\[ \langle a_{\ell_1} a_{\ell_2} a_{\ell_3} \rangle = \mathcal{C}_\ell \delta_{\ell_1} \delta_{\ell_2} \delta_{\ell_3}. \]

If we consider the three-point function of the temperature field, we obtain the bispectrum, defined as

\[ \langle a_{\ell_1} a_{\ell_2} a_{\ell_3} \rangle = \left( \frac{\ell_1}{m_1}, \frac{\ell_2}{m_2}, \frac{\ell_3}{m_3} \right) \mathcal{B}_{\ell_1 \ell_2 \ell_3}. \]

The term \((\cdot \cdot \cdot)\) is a Wigner 3J symbol, which arises due to the ‘selection rules’ of the moments.

Following the same steps, we can construct the four-point function and the associated trispectrum. We represent the rotationally invariant solution for the trispectrum as in Hu (2001):

\[ \langle a_{\ell_1} a_{\ell_2} a_{\ell_3} a_{\ell_4} \rangle = \sum_{L M} \left( \frac{\ell_1}{m_1}, \frac{\ell_2}{m_2}, \frac{\ell_3}{m_3}, \frac{\ell_4}{m_4} \right) \mathcal{B}_{\ell_1 \ell_2 \ell_3 \ell_4} (L). \]

Using the orthogonality properties of the Wigner 3J symbols and the relation \(Q = T + G\), we can invert equation (4) to obtain the estimator

\[ \mathcal{T}_{\ell_1 \ell_2 \ell_3 \ell_4} (L) = (2L + 1) \sum_{m_1 m_2 m_3 m_4} \left( \frac{\ell_1}{m_1}, \frac{\ell_2}{m_2}, \frac{\ell_3}{m_3}, \frac{\ell_4}{m_4} \right) \mathcal{B}_{\ell_1 \ell_2 \ell_3 \ell_4} (L). \]

The term \(G_{ij}^{\ell_1 \ell_2} (L)\) represents the unconnected Gaussian contribution and it is given in Hu (2001) as

\[ G_{ij}^{\ell_1 \ell_2} (L) = (-1)^{i+j+1} \sqrt{2(2L+1)(2L+1)} \times C_{\ell_1} C_{\ell_2} \delta_{\ell_1 \ell_2} \delta_{\ell_3 \ell_4} + (2L+1) \delta_{\ell_1 \ell_2} \]

\[ \times \left[ (-1)^{i+j+1+L} \delta_{\ell_1 \ell_2 + \delta_{\ell_3 \ell_4}} \right]. \]

The term \(T_{ij}^{\ell_1 \ell_2} (L)\) is the connected part of the angular trispectrum and its expectation value is exactly zero for a Gaussian field. This means that the connected part is sensitive to the presence of non-Gaussianities. The unconnected term is non-zero only for \(L = 0\) or \(l_1 = l_2 = l_3 = l_4\), but only with full sky coverage. In the case of incomplete sky coverage the unconnected terms can contaminate all other modes. We will discuss this situation in Section 3.
For the purpose of this work we have not computed the possible trispectrum components. We concentrated only on the simpler case \(\ell_1 = \ell_2 = \ell_3 = \ell_4 = \ell\), i.e. the diagonal component. Recent papers have investigated in detail the power of the diagonal trispectrum in the presence of some non-Gaussian signals mentioned in the introduction. In particular, Aghanim et al. (2003) has shown that for simulated point-source maps the diagonal trispectrum is much more powerful than the nearly diagonal estimator \((\ell, \ell + 1, \ell + 2, \ell + 3)\), even though the latter does not contain a Gaussian contribution. Also, in Castro (2003) it is discussed how the O v effect generates a signature on the diagonal trispectrum, which could easily be detected on the arcmin scales probed by the Planck\(^2\) mission.

Finally, it should be noticed that computing all components of the trispectrum is a serious computational challenge. Many modes are also correlated due to the limited sky coverage. For these reasons we have decided to restrict this analysis to the case of \(\ell_1 = \ell_2 = \ell_3 = \ell_4 = \ell\).

We start with the method described in Kunz et al. (2001). We define

\[
(\hat{a}_{m_1,a_{m_2,m_3,m_4}}) = \sum_{a=0}^{N} T_{\ell a} \bar{T}_{\ell m_1,m_2,m_3,m_4},
\]

where \(T_{\ell a}\) are the components of the trispectrum which we wish to estimate and \(\tau\) is a tensor, which we have to determine in order to construct an estimator for \(T_{\ell a}\). The geometrical considerations stated above, together with the required symmetries with respect to the interchange of \(\ell, m\) pairs suggest

\[
\bar{T}_{\ell m_1,m_2,m_3,m_4} = \sum_{a=0}^{2} \frac{(-1)^{a}}{2a} \left( \frac{\ell}{m_1} \frac{\ell}{m_2} \frac{2a}{M} \right) \left( \frac{-M}{m_3} \frac{m_4}{m_5} \right) \nonumber
\]

\[+ \text{ ineq. permut.} \]

Although the \(\bar{\tau}\) values satisfy all the correct symmetries, they define an incomplete basis. To correct for this deficiency we define an orthonormalized set of tensors

\[
T_{\ell m_1,m_2,m_3,m_4} = \sum_{a=0}^{2} \frac{(-1)^{a}}{2a} \bar{T}_{\ell m_1,m_2,m_3,m_4},
\]

where the matrix \(\bar{T}_{\ell m_1,m_2,m_3,m_4}\) is derived from the required property that the \(\tau\) be orthogonal with respect to the product given in equation (5) of Kunz et al. (2001). The estimator of the trispectrum is then given by

\[
\hat{T}_{\ell a} = \sum_{m_1,m_2,m_3,m_4} T_{\ell m_1,m_2,m_3,m_4} a_{m_1} a_{m_2} a_{m_3} a_{m_4}
\]

\[= \sum_{m_1,m_2,m_3,m_4} \bar{T}_{\ell m_1,m_2,m_3,m_4} a_{m_1} a_{m_2} a_{m_3} a_{m_4}
\]

\[= \sum_{a=0}^{2} \frac{(-1)^{a}}{2a} \bar{T}_{\ell m_1,m_2,m_3,m_4} a_{m_1} a_{m_2} a_{m_3} a_{m_4}.
\]

Note that there are only int(\(\ell/3\)) independent estimators due to the symmetry properties of \(a_{m}\). In this paper we will consider the ‘normalized’ trispectrum used in Kunz et al. (2001), where we divide each estimate of the trispectrum by \((\bar{C}_{\ell})^2\), where \(\bar{C}_{\ell} = \frac{1}{2\ell+1} \bar{L}_{\ell} \). Its statistical properties are equivalent to those of the unnormalized estimator, and it is more robust with respect to fluctuations in the power spectrum (Aghanim et al. 2003).

3 APPLICATION TO HIGH-RESOLUTION MAPS WITH INCOMPLETE SKY COVERAGE

In this paper we will be focusing on a high-resolution map with incomplete sky coverage, in particular the BOOMERanG data set. This leads to a set of algorithmic problems which did not have to be addressed in Kunz et al. (2001). The three problems we wish to highlight are as follows.

Speed. The numerical evaluation of Wigner 3J coefficients for large values of \(\ell\) becomes time consuming and practically infeasible. Indeed, beyond the COBE resolution of a maximum \(\ell\) of approximately 25 it is not possible to estimate the \(T_{\ell a}\) sufficiently rapidly for a robust Monte Carlo assessment of the statistics.

Gaussian contamination. The finite sky coverage will induce correlations between the estimators with different values of \(\ell\) and \(\alpha\) (or \(\alpha\)). Consequently, all estimators may be heavily contaminated by the Gaussian (or disconnected) contributions to the maps.

Correlations. The correlations between modes in the cut sky mean that the \(T_{\ell a}\) will be even more correlated than in the full-sky case.

We shall now focus on the solutions we propose to these three problems.

3.1 Speed

We have opted to use the method described in Hu (2001) and Spergel & Goldberg (1999) for calculating \(T_{\ell a}\); we define a new set of sky maps weighted in rings centred around a point \(\hat{q}\):

\[
e_{\ell}^{(1)}(\hat{q}) = \sqrt{\frac{2\ell + 1}{4\pi}} \int \delta(\hat{n}) P_{\ell}(\hat{n} \cdot \hat{q}).
\]

To implement this method we start with the relation (11) and we use the relation (1) to express the temperature \(T\) as a function of spherical harmonics and the relation

\[
P_{\ell}(\hat{n} \cdot \hat{q}) = \frac{4\pi}{2\ell + 1} \sum_{m} Y_{\ell m}^{*}(\hat{n}) Y_{\ell m}(\hat{q}).
\]

and to also express the Legendre polynomials as a function of spherical harmonics. Combining them with equation (11) we obtain

\[
e_{\ell}^{(1)}(\hat{q}) = \sqrt{\frac{4\pi}{2\ell + 1}} \sum_{m} a_{\ell m} Y_{\ell m}(\hat{q}).
\]

The \(e_{\ell}\) calculation is quite fast because we can use the fast Fourier transform on rings of equal latitude (Mucciaccia, Natoli & Vittorio 1997).

We can then rewrite equation (5) in terms of this new set of sky maps (Komatsu 2002):

\[
T_{\ell a} \equiv \frac{1}{4\ell + 1} \sum_{M=-2a}^{2a} \sum_{2\ell+1}^{2\ell+1} \tilde{T}_{\ell m_1,m_2,m_3,m_4}
\]

where

\[
\tilde{T}_{\ell m_1,m_2,m_3,m_4} = \frac{1}{4\ell + 1} \sum_{M=-2a}^{2a} \sum_{2\ell+1}^{2\ell+1} \tilde{T}_{\ell m_1,m_2,m_3,m_4}\]

\[\times \int \delta(\hat{n}) e_{\ell}(\hat{n}) e_{\ell}(\hat{q}) Y_{\ell m}(\hat{n}).
\]

If we expand the Wigner 3J symbols in terms of spherical harmonics and use the addition theorem we obtain

\[
\tilde{T}_{\ell a} = N_{a}^{-1} \int d^{2}\hat{n} \int d^{2}\hat{q} e_{\ell}(\hat{n}) e_{\ell}(\hat{q}) e_{\ell}(\hat{q}) P_{\ell a}(\hat{n} \cdot \hat{q}).
\]
where

\[ N_{\ell L} = \frac{1}{3} \left( \begin{array}{ccc} \ell & \ell & L \\ 0 & 0 & 0 \end{array} \right)^2. \] (17)

We have thus computed a set of \( T^{\alpha \ell} \) that we can orthonormalize to obtain the estimator for the trispectrum \( \bar{T}^{\alpha \ell} \). This method is very fast, especially when estimating the trispectrum at high values of \( \ell \).

Note that direct evaluation of the Wigner 3J coefficients, e.g. by recurrence relations, would result in an \( O(\ell^3) \) problem, requiring \( \sim 10^{10} \) operations for \( \ell \sim 1000 \).

### 3.2 Gaussian contamination

Kunz et al. (2001) found that the purely Gaussian contribution to the trispectrum (the disconnected part) corresponds to the \( \alpha = 0 \) term. By orthonormalizing all other estimators with respect to this tensor it is possible to remove the Gaussian contribution exactly on a map by map basis. The resulting estimators are only sensitive to non-Gaussian contributions, i.e. in the case of Gaussian skies they would have a zero expectation value. Additionally, the variance of this estimator is shown to be minimal (Kunz et al. 2001). In the cut sky case, there are strong cross-correlations between components of the trispectrum with different values of \( \ell \) and \( \alpha \). In this case, the orthonormalization method fails. Given that the Gaussian contribution to the trispectrum may be much larger than the non-Gaussian contribution, it is essential that we remove it as completely as possible none the less.

To overcome these problems we have chosen to employ the following Monte Carlo scheme: we generate an ensemble of maps with the same angular power spectrum, sky coverage and noise as the data maps we want to analyse. We estimate the \( T^{\alpha \ell} \) from each map and calculate the mean of these quantities over the whole ensemble. Let us denote this mean by \( \bar{T}^{\alpha \ell} \). We then use this quantity to correct for the Gaussian contamination in the estimate of the trispectrum from the data by defining \( T_{GC}^{\alpha \ell} = \bar{T}^{\alpha \ell} - T_{G}^{\alpha \ell} \). Note that, by so doing, we are removing the Gaussian contamination before performing the full-sky orthonormalization, i.e. before multiplying by \( L \).

### 3.3 Correlations

The fact that there is only limited sky coverage also implies that there will be correlations between values of the trispectrum at different \( \ell \) values. This is shown in Section 5 below. To strongly suppress the correlations and to end up with a simple covariance matrix, we consider band-averaged values of the trispectrum. Since there is no a priori given bandwidth, we study the cases of \( \Delta \ell = 40, 50 \) and 60 and \( \Delta \alpha = 10 \) and 15. The choice \( \Delta \ell \simeq 50 \) is consistent with previous analyses of the BOOMERanG power spectrum (see, e.g., Ruhl et al. 2002). In this way we can check the sensitivity of the results to the chosen band size.

### 4 NUMERICAL IMPLEMENTATION AND CONSISTENCY TESTS

The process we use is basically the same as in a number of previous analyses (Kogut et al. 1996; Ferreira et al. 1998; Kunz et al. 2001; Santos et al. 2002). We generate an ensemble of Gaussian maps with the same angular power spectrum and noise property as estimated from the BOOMERanG data and the same sky coverage. We then apply our estimators to the set of maps to obtain a distribution for each estimator in the Gaussian case. In particular, we characterize the full distribution in terms of the mean values of the estimators and the covariance matrix between them. These quantities are used to define a standard multivariate \( \chi^2 \) as a goodness of fit. The estimators of the trispectrum are then evaluated from the BOOMERanG data.

In Section 5 we will discuss their behaviour. The goodness of fit of these estimators is compared against its distribution for a new ensemble of Gaussian maps. From this comparison we can quantify the confidence with which the data can be said to be Gaussian from the point of view of our estimator. It is clear that the numerical details of this process must be well understood if we are to believe in our results. We focus on the particularities of the analysis in this paper, which differ from previous analyses.

It is important to compare the results using this hybrid pixel/harmonic analysis with the standard methods that have been used before. We do so by looking at the two lower-order statistics, i.e. we have calculated the power spectrum \( C_\ell \) and the bispectrum \( B_{eff} \) using this new approach as well as summing up the 3J symbols.

The relevant expressions are:

\[ C_\ell = \frac{1}{4\pi} \int dq \left| e_\ell(q) \right|^2 \] (18)

and (Spergel & Goldberg 1999)

\[ \left( \begin{array}{ccc} \ell & \ell & \ell \\ 0 & 0 & 0 \end{array} \right) B_{eff} = \int dq e_\ell(q) e_\ell(q) e_\ell(q). \] (19)

We have compared \( C_\ell \) and \( B_{eff} \) using these expressions with the standard results obtained using \( a_{cm} \) and the Wigner 3J symbols, for a maximum \( \ell \) value of 1000. Using a set of CMB Gaussian maps with the best-fitting power spectrum measured by BOOMERanG (Netterfield et al. 2002) and a pixel resolution of 7 arcmin we have found that the bispectrum obtained with \( e_\ell \) is affected by a pixelization effect for high values of \( \ell \) (while the power spectrum shows no difference). To check for this, we have performed the same analysis at higher resolution (\( \simeq 3 \) arcmin) and have found that the pixelization effect vanishes. Given that we are restricted to the pixelization level of the data, we can use the comparison of the two estimates of the bispectrum to define a maximum \( \ell \) out to which we can trust the new estimate of the trispectrum. Note that it is computationally intractable to perform such a comparison in terms of the trispectrum, although this would be preferable. From Fig. 1 one can see that discrepancies arise for \( \ell > 800 \) and we chose not to estimate the trispectrum beyond \( \ell = 700 \), leaving a conservative margin as

![Figure 1](http://mnras.oxfordjournals.org/)

Figure 1. In this figure we represent the normalized bispectrum \( I^L_\ell = B_{eff}/C_\ell^{3/2} \) calculated for a full-sky Gaussian map with a pixel size of 7 arcmin. The crosses show \( I^L_\ell \) obtained with the \( e_\ell \) method, the boxes \( I^L_\ell \) with \( a_{cm} \) and the 3J symbols. We can see that at \( \ell \approx 800 \) that the difference between the two plots is clearly evident, due to a pixelization effect. We limit therefore our analysis to \( \ell \leq 700 \).
we did not test the trispectrum itself. Furthermore, we chose not to consider any \( \ell \) below 100 because the BOOMERanG data are not very sensitive to these modes, due to limited sky coverage and data filtering.

Another novelty in our analysis (as compared with that of Kunz et al. 2001) is the method for constructing \( \mathcal{L}^{\alpha, \beta}_{\ell} \). There, a Gram–Schmidt (GS) procedure was used to calculate the orthonormal transformation matrix \( \mathcal{L}^{\alpha, \beta}_{\ell} \). As a result of the inherent instability of the GS procedure, it is not applicable to large matrices, i.e. for large \( \ell \). We have therefore opted to use an alternative orthonormalization method. We subtract the \( \alpha = 0 \) part and use a Jacobi routine to obtain a spectral decomposition (SD) of the remaining matrix. The eigenvectors of the non-vanishing eigenvalues then form the transformation matrices. This method is robust and, moreover, gives us an unambiguous procedure for ordering the estimators through the different eigenvalues. As a strong consistency test we have applied our trispectrum code to the COBE data and compared the results with those of Kunz et al. (2001). The results of the SD method lie, up to a possible sign change, very close to the original ones (see Fig. 2). In any case, the statistical significance of the results (and the conclusions one can extract) are the same as in Kunz et al. (2001). We advocate the use of the SD method from now on, even in the case of analyses limited to low \( \ell \) values.

For our analysis we have used the best four of the six 150-GHz channels of the BOOMERanG 1998 flight; we naively coadd the data taken at the scan speed of 1 deg s\(^{-1}\) (1 dps). We simulate three sets of 1000 Gaussian maps each. In fact, we need three statistically independent ensembles of simulated maps for our analysis: one to estimate the Gaussian contribution described above, another to estimate the covariance matrix of our estimator and a third one to obtain a spectral decomposition (SD) of the remaining matrix. The transformation matrix \( \mathcal{L}^{\alpha, \beta}_{\ell} \) is calculated through the orthonormalization method. We subtract the \( \alpha = 0 \) part and use a Jacobi routine to obtain a spectral decomposition (SD) of the remaining matrix. This method is robust and, moreover, gives us an unambiguous procedure for ordering the estimators through the different eigenvalues. As a strong consistency test we have applied our trispectrum code to the COBE data and compared the results with those of Kunz et al. (2001). The results of the SD method lie, up to a possible sign change, very close to the original ones (see Fig. 2). In any case, the statistical significance of the results (and the conclusions one can extract) are the same as in Kunz et al. (2001). We advocate the use of the SD method from now on, even in the case of analyses limited to low \( \ell \) values.

5 RESULTS AND APPLICATION TO THE BOOMERanG DATA
To show in detail the method proposed in Sections 2 and 3 we are going to discuss the results obtained at each step from both the data and the Monte Carlo simulations. We start with the estimate of the \( \tilde{T}_{\alpha, \beta}^{\ell} \) without Gaussian corrections. In the top panel of Fig. 3 we plot \( \tilde{T}_{\alpha, \beta}^{\ell} \) as a function of \( \ell \) for selected values of \( \alpha \). We can highlight two features. First, the \( \tilde{T}_{\alpha, \beta}^{\ell} \) are highly correlated for adjacent values of \( \ell \) due to the finite sky coverage, as expected. Secondly, and because of the finite sky coverage, there is a strong contamination from the disconnected component of the trispectrum. This is evident in the fact that the values of \( \tilde{T}_{\alpha, \beta}^{\ell} \) scatter about the \( (C_{\ell})^{2} \) and that the 95 per cent confidence limits are not centred about zero. As one would expect the lower the value of \( \alpha \), the more contaminated the

Figure 2. In this figure we reproduce the COBE results for the trispectrum with the GS and SD orthonormalization methods (see the text) and compare them with Kunz et al. (2001). The crosses are the results of Kunz et al. (2001), squares are GS method results and triangles are SD method results.
estimate is by the disconnected part. As advocated in Section 3, we correct for the contamination due to the disconnected component by using a Monte Carlo ensemble (of 1000 realizations) to generate a correction. This can be seen as a bias that must be subtracted off all estimates of $\bar{T}_\alpha$ with $\alpha > 0$. In the bottom panel of Fig. 3 we plot the 'Gaussian-corrected' estimate of $\bar{T}_\alpha$ with corresponding 95 per cent confidence limits. As expected the estimates now scatter around zero, while the confidence limits, although not necessarily symmetric around the $\ell$-axis are much more centred. The remaining asymmetry is merely a manifestation that for low $\alpha$ the distribution of the $\bar{T}_\alpha$ is slightly skewed.

Let us now proceed to the orthonormalized estimators, $\hat{T}_{a\ell}$; a selection of estimators are plotted for a choice of $a$ in Fig. 5. As noted above, $a$ are limited to $a \leq \text{int}(l/3)$, and we see a clear suppression of the high $a$ values for each $\ell$ (or, in the case of the figure, of the low $\ell$ values for fixed $a$), as the maps with limited sky coverage contain less information than full-sky maps.

Once we have calculated the trispectrum estimators both for the BOOMERanG data and for the Monte Carlo simulations, we can proceed to obtain the $\chi^2$ distribution for the simulated Gaussian maps and compare it with the data. We used two different approaches: one taking as estimator $\hat{T}_{a\ell}$ (the orthonormalized trispectrum corrected for Gaussian contamination and normalized to $C_\ell^2$) and the other one taking its absolute value $|\hat{T}_{a\ell}|$.

We construct a standard multivariate $\chi^2$ as

$$\chi^2 = \sum_{\ell,\ell',a,a'} ((\hat{T}_{a\ell})_G - \hat{T}_{a\ell}) C_{\ell,a,a'}^{-1} ((\hat{T}_{a'\ell'})_G - \hat{T}_{a'\ell'})$$

deriving the expectation values $\langle \hat{T}_{a\ell} \rangle_G$ and the covariance matrix $C_{\ell,a,a'} = \langle \hat{T}_{a\ell} \hat{T}_{a'\ell'} \rangle_G - \langle \hat{T}_{a\ell} \rangle_G \langle \hat{T}_{a'\ell'} \rangle_G$ from one of the two remaining Monte Carlo ensembles. As discussed earlier, we do not

Figure 3. Top panel: an estimate of the non-orthogonalized trispectrum, $\hat{T}_\alpha$ (multiplied by $\ell^4$) for $\alpha = 5, 100, 400$ from the BOOMERanG data (crosses) and the corresponding 95 per cent confidence limits from the 1000 Monte Carlo simulations. Bottom panel: an estimate of the non-orthogonalized trispectrum corrected for Gaussian contamination, $\bar{T}_\alpha$ (multiplied by $\ell^4$) for $\alpha = 5, 100, 400$ from the BOOMERanG data (crosses) and the corresponding 95 per cent confidence limits from the 1000 Monte Carlo simulations.

Figure 4. An estimate of the non-orthogonalized trispectrum corrected for Gaussian contamination and normalized $\bar{T}_\alpha / \langle \hat{C}_\ell \rangle^2$ for $\alpha = 5, 100, 400$ from the BOOMERanG data (crosses) and the corresponding 95 per cent confidence limits from the 1000 Monte Carlo simulations.
Figure 5. Top panel: an estimate of the orthogonalized trispectrum corrected for Gaussian contamination and normalized $\hat{T}_{\alpha\ell}/(\hat{C}_\ell)^2$ for $\alpha = 5, 100, 200$ from the BOOMERanG data (crosses) and the corresponding 95 per cent confidence limits from the 1000 Monte Carlo simulations. Bottom panel: the absolute value of the estimate of the orthogonalized trispectrum corrected for Gaussian contamination and normalized $\hat{T}_{\alpha\ell}/(\hat{C}_\ell)^2$ for $\alpha = 5, 100, 200$ from the BOOMERanG data (crosses) and the corresponding 95 per cent confidence limits from the 1000 Monte Carlo simulations.

Figure 6. The $\chi^2$ distribution of Monte Carlo simulated maps (histogram) and data value (vertical line) for the trispectrum estimator $|\hat{T}_{\alpha\ell}|$ in the case of $\Delta \alpha = 10$ and $\Delta \ell = 40, 50, 60$ (top) and in the case of $\Delta \alpha = 15$ and $\Delta \ell = 40, 50, 60$ (bottom).

sum over all $\ell$ and $\alpha$, but bin both $\alpha$ and $\ell$, varying the size of the bins. Finally, we calculate the $\chi^2$ distribution from the last ensemble of Gaussian maps.

In Figs 6 and 7 we show the $\chi^2$ distribution derived from 1000 Gaussian realizations compared with the BOOMERanG data for both estimators and for different bin widths. The probability $P(\chi^2 > \chi^2_R)$ that a Gaussian map has a larger $\chi^2$ than the BOOMERanG map is given in Table 1. Although the values vary considerably with the choice of bin-widths, none of them is below 5 per cent or 2$\sigma$. We conclude that the trispectrum does not detect any non-Gaussianity in the co-added BOOMERanG 150-GHz maps.

6 CONCLUSIONS

We have applied an improved version of the method of Kunz et al. (2001) for measuring the trispectrum to the four best 150-GHz maps.
Figure 7. The $\chi^2$ distribution of Monte Carlo simulated maps (histogram) and data value (vertical line) for the trispectrum estimator $\hat{T}_{a,l}$ in the case of $\Delta a = 10$ and $\Delta l = 40, 50, 60$ (top) and in the case of $\Delta a = 15$ and $\Delta l = 40, 50, 60$ (bottom).

Table 1. Probability that the Gaussian models have a $\chi^2$ greater than the data value for both the trispectrum estimators and for different bin widths in $\ell$ and $a$.

| Bin width | $|\hat{T}_{a,l}|$ (per cent) | $\hat{T}_{a,l}$ (per cent) |
|-----------|-------------------------------|-----------------------------|
| $\Delta \ell = 40$ | 21.8 | 10.6 |
| $\Delta a = 10$ | 13.7 | 19.3 |
| $\Delta \ell = 50$ | 76 | 23.2 |
| $\Delta a = 10$ | 15.7 | 26 |
| $\Delta \ell = 60$ | 8.7 | 8.5 |
| $\Delta a = 15$ | 87.9 | 53.8 |

BOOMERanG maps. To this end, we used maps containing only one multipole each to avoid computing the Wigner $3J$ symbols and subtracted the average Gaussian contribution using an ensemble of simulated maps. We then orthogonalized these maps and normalized them to $C_\ell$. We then binned the resulting trispectrum values with a variety of different bin sizes, and computed the $\chi^2$ value, using a full covariance matrix estimated from a second ensemble of simulated Gaussian maps. When comparing the data $\chi^2$ value to the Gaussian realizations (obtained from a third ensemble of simulated Gaussian maps) we concluded that the trispectrum does not detect any deviations from Gaussianity.

This work complements the pixel-space analysis (Polenta et al. 2002) of the BOOMERanG data.

In this paper we have studied for the first time the trispectrum of real CMB data with subdegree resolution. The main problem encountered was the limited sky coverage, which introduces strong correlations, and prevents the use of orthogonalization to remove the Gaussian (connected) part of the trispectrum. We expect therefore that the MAP and Planck satellites will be able to improve on this analysis considerably, but this study provides a proof of feasibility for measuring the trispectrum of full-sky high-resolution maps as well as first results on small angular scales.

ACKNOWLEDGMENTS

GDT acknowledges financial support from the Dottorato in Astronomia dell’Università La Sapienza and from a Marie Curie pre-doctoral fellowship. MK acknowledges financial support from the Swiss National Science foundation. PGF acknowledges the support of the Royal Society. The BOOMERanG project has been supported by Programma Nazionale di Ricerche in Antartide, Università di Roma ‘La Sapienza’, and Agenzia Spaziale Italiana in Italy, by NASA and by NSF OPP in the US. We acknowledge the use of the HEALPix package and of the Oxford Beowulf cluster for our computations. We thank Andrew Jaffe and Alessandro Melchiorri for their helpful comments and Jonathan Patterson for his help.

REFERENCES

Coles P., Barrow J.D., 1987, MNRS, 228, 407
de Bernardis P. et al., 2000, Nat, 404, 995
Ferreira P.G., Jaffe A.H., 2000, MNRS, 312, 89
Ferreira P.G., Magueijo J., Gorski K.M., 1998, Apj, 503, L1

© 2003 RAS, MNRAS 343, 284–292
Hu W., 2001, Phys. Rev. D, 64, 083005

This paper has been typeset from a \TeX/\LaTeX file prepared by the author.