

## Non-perturbative thermal flows and resummations

Article (Published Version)

Litim, Daniel F and Pawlowski, Jan M (2006) Non-perturbative thermal flows and resummations. *Journal of High Energy Physics*, 2006 (11). p. 26. ISSN 1126-6708

This version is available from Sussex Research Online: <http://sro.sussex.ac.uk/id/eprint/27778/>

This document is made available in accordance with publisher policies and may differ from the published version or from the version of record. If you wish to cite this item you are advised to consult the publisher's version. Please see the URL above for details on accessing the published version.

### **Copyright and reuse:**

Sussex Research Online is a digital repository of the research output of the University.

Copyright and all moral rights to the version of the paper presented here belong to the individual author(s) and/or other copyright owners. To the extent reasonable and practicable, the material made available in SRO has been checked for eligibility before being made available.

Copies of full text items generally can be reproduced, displayed or performed and given to third parties in any format or medium for personal research or study, educational, or not-for-profit purposes without prior permission or charge, provided that the authors, title and full bibliographic details are credited, a hyperlink and/or URL is given for the original metadata page and the content is not changed in any way.

## Non-perturbative thermal flows and resummations

This content has been downloaded from IOPscience. Please scroll down to see the full text.

JHEP11(2006)026

(<http://iopscience.iop.org/1126-6708/2006/11/026>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 84.92.41.220

This content was downloaded on 19/06/2014 at 09:01

Please note that [terms and conditions apply](#).

## Non-perturbative thermal flows and resummations

---

**Daniel F. Litim**

*School of Physics and Astronomy, U Southampton  
Highfield, SO17 1BJ, U.K., and  
Physics Department, CERN, Theory Division  
1211 Geneva 23, Switzerland  
E-mail: [daniel.litim@cern.ch](mailto:daniel.litim@cern.ch)*

**Jan M. Pawłowski**

*Institut für Theoretische Physik, Universität Heidelberg  
Philosophenweg 16, D-69120 Heidelberg, Germany  
E-mail: [J.Pawłowski@ThPhys.Uni-Heidelberg.de](mailto:J.Pawłowski@ThPhys.Uni-Heidelberg.de)*

**ABSTRACT:** We construct a functional renormalisation group for thermal fluctuations. Thermal resummations are naturally built in, and the infrared problem of thermal fluctuations is well under control. The viability of the approach is exemplified for thermal scalar field theories. In gauge theories the present setting allows for the construction of a gauge-invariant thermal renormalisation group.

**KEYWORDS:** Renormalization Group, Nonperturbative Effects, Field Theories in Lower Dimensions.

---

## Contents

|  |           |
|--|-----------|
| <b>1. Introduction</b>                     | <b>1</b>  |
| <b>2. Functional flow equations</b>        | <b>2</b>  |
| <b>3. Quantum vs. thermal fluctuations</b> | <b>4</b>  |
| <b>4. Thermal resummations</b>             | <b>6</b>  |
| <b>5. Thermal derivative expansion</b>     | <b>7</b>  |
| <b>6. Optimised thermal flows</b>          | <b>9</b>  |
| <b>7. Thermal gauge theories</b>           | <b>12</b> |
| <b>8. Discussion and conclusions</b>       | <b>14</b> |

---

## 1. Introduction

An important goal in thermal field theory is the computation of physical quantities like the pressure or screening masses of quantum fields. This is of interest for on-going and future experiments at RHIC, LHC and the FAIR facility, which are sensitive to the quark-gluon-plasma in the core region of heavy ion collisions. Recent RHIC data even suggests that a thermal quark gluon plasma is strongly coupled, see [1]. On the theoretical side, one observes that standard thermal perturbation theory is plagued by infrared divergences due to massless excitations, which at least requires resummations. Resummed thermal perturbation theory often displays a weak convergence behaviour even deeply in its domain of validity, see, e.g. [2–4]. In gauge theories, the situation is additionally complicated by the magnetic sector. In our opinion, this asks for a method able to incorporate both the weakly and the strongly coupled regimes. Moreover, such a method would benefit from a clear separation of quantum fluctuations and thermal effects.

In the past decade much progress in the description of quantum fluctuations has been achieved within the functional renormalisation group, for reviews see [5–11]. The functional RG, applicable both at weak and strong coupling, has also been used for the computation of thermal effects, e.g. [12–17, 6, 7, 19, 18, 20, 21, 8, 22–26]. A direct implementation of thermal fluctuations has been put forward within the real time formalism [14–16] and the imaginary time formalism [7].

In the present paper, we extend and detail the proposal [7]. We construct thermal flows that are well-defined both in the infrared and in the ultraviolet, and provide ideal

starting points for numerical and analytical studies. In scalar theories, we work out the scheme-independent part of the integrated flow, which contains the one loop thermally resummed perturbation theory. We equally provide thermal flows within a thermal derivative expansion, and reproduce explicitly the leading order results for the pressure and the thermal self energy.

For gauge theories, and for specific momentum cutoff and gauges, the thermal flow even respects gauge invariance for arbitrary scale  $k$ . This follows from two observations. A mass-like momentum cutoff – even though it is not applicable for integrating-out quantum fluctuations [7] – is a viable cutoff for thermal fluctuations. The second observation concerns Wilsonian flows for gauge theories in axial gauges [27, 28]. Gauge invariance for physical Green functions is controlled via modified Ward-Takahashi identities. In an axial gauge they reduce to the standard Ward-Takahashi identities for any scale  $k$ , if a mass-like regulator is employed. Although of limited use in the generic case, this is precisely the missing piece to construct a gauge invariant flow for thermal fluctuations.

The outline of the paper is as follows. In section 2, we review functional flows in the imaginary time formalism. In section 3, we discuss our projection method for thermal fluctuations. Scheme independence of the full one-loop thermal resummation is worked out in section 4. Explicit thermal flows within a thermal derivative expansion and results for the thermal pressure and the self-energy are obtained in section 5 and section 6. The extension to thermal gauge theories is detailed in section 7, including a gauge-invariant thermal flow and results for the thermal mass and pressure. Our conclusions are contained in section 8.

## 2. Functional flow equations

To start with, we discuss the standard Wilsonian approach to quantum field theories. For simplicity, we focus on a bosonic degree of freedom. A generalisation to fermions and gauge fields is straightforward. All the physically relevant information can be obtained from the (regularised) partition function. It reads

$$\exp W_{k,T}[J] = \int \mathcal{D}\phi \exp \left( - S_{k,T}[\phi] + \text{Tr } J\phi \right) \tag{2.1}$$

In  $(d+1)$  dimensions and using the imaginary time formalism at temperature  $T$ , the trace stands for

$$\text{Tr} = T \sum_n \int \frac{d^d q}{(2\pi)^d}, \tag{2.2}$$

and the implicit replacements  $q_0 \rightarrow 2\pi nT$  for bosonic fields are understood,  $n$  labelling the Matsubara frequencies and  $J$  stands for the corresponding sources. The term  $S_{k,T}[\phi] = S + \Delta S_{k,T}$  contains the (gauge-fixed) classical action  $S[\phi]$  and a quadratic regulator term  $\Delta S_{k,T}[\phi]$ , given by

$$\Delta S_{k,T}[\phi] = \frac{1}{2} \text{Tr} \left[ \phi(-q) R_k(q^2) \phi(q) \right]. \tag{2.3}$$

eq. (2.3) introduces a coarse-graining via the operator  $R_k(q)$ . The flow of (2.1) related to an infinitesimal change of  $t = \ln k/\Lambda$  (with  $\Lambda$  being some fixed UV scale) is

$$\frac{\partial}{\partial t} e^{W_{k,T}[J]} = \frac{1}{2} \int \mathcal{D}\phi \operatorname{Tr} \left( \phi \frac{\partial R_k}{\partial t} \phi \right) e^{-S_{k,T}[\phi] + \operatorname{Tr} J\phi} . \quad (2.4)$$

Performing a Legendre transformation leads to the coarse-grained effective action  $\Gamma_{k,T}[\phi]$ ,

$$\Gamma_{k,T}[\phi] = \operatorname{Tr} J\phi - W_{k,T}[J] - \Delta S_{k,T}[\phi] - C_{k,T} , \quad (2.5)$$

Note that the constant  $C_k$  is usually not mentioned when one is interested in field independent quantities. However as we shall see later in the discussion of the thermal pressure we have to take it into account. It is straightforward to obtain the flow equation for  $\Gamma_{k,T}[\phi]$  by using (2.5):

$$\partial_t \Gamma_{k,T}[\phi] = \frac{1}{2} \operatorname{Tr} \left[ G_{k,T}[\phi] \frac{\partial R_k}{\partial t} \right] - \partial_t C_{k,T} , \quad (2.6a)$$

with

$$G_{k,T}[\phi] = \left( \frac{\delta^2 \Gamma_{k,T}[\phi]}{\delta\phi\delta\phi} + R_k \right)^{-1} , \quad (2.6b)$$

denoting the full (field-dependent) regularised propagator of  $\phi$ . Let us be more specific about the regulator function  $R_k(q)$ . The general requirements are

- (i) it has a non-vanishing limit for  $q^2 \rightarrow 0$ , typically  $R_k \rightarrow k^2$  for bosons, and  $R_k \rightarrow k$  for fermions. This precisely ensures the IR finiteness of the propagator at non-vanishing  $k$  even for vanishing momentum  $q$ .
- (ii) it vanishes in the limit  $k \rightarrow 0$ , and for  $q^2 \gg k^2$ . The latter condition ensures that large momentum fluctuation have efficiently been integrated-out whereas the first condition guarantees that any dependence on  $R_k$  drops out in the limit  $k \rightarrow 0$ .
- (iii)  $R_k$  diverges like  $\Lambda^2$  for bosons, and like  $R_k \rightarrow \Lambda$  for fermions, when  $k \rightarrow \infty$  (or  $k \rightarrow \Lambda$  with  $\Lambda$  being some UV scale much larger than the relevant physical scales). Thus, the saddle point approximation to (2.1) becomes exact and  $\Gamma_{k \rightarrow \Lambda}$  reduces to the (gauge-fixed) classical action  $S$ .

These conditions guarantee that  $\Gamma_k$  has the limits

$$\lim_{k \rightarrow \infty} \Gamma_{k,T}[\phi] = S[\phi] \quad (2.7)$$

$$\lim_{k \rightarrow 0} \Gamma_{k,T}[\phi] = \Gamma_T[\phi] . \quad (2.8)$$

For any given scale  $k$  the main contributions to the running of  $\Gamma_{k,T}$  in (2.6) come from momenta about  $q^2 \approx k^2$ , if the regulator function obeys the conditions (i)-(iii). This is so because  $\partial_t R_k$  is peaked about  $q^2 \approx k^2$ , and sufficiently suppressed elsewhere. The physics behind this is that a change of  $\Gamma_{k,T}$  due to a further coarse graining (i.e. the integrating-out

of a thin momentum shell about  $k$ ) is dominated by the fluctuations with momenta about  $k$ . Contributions from fluctuations with momenta much smaller/larger than  $k$  should be negligible. The flow equation (2.6) connects the classical action  $S[\phi]$  with the full quantum effective action  $\Gamma_T[\phi]$  at temperature  $T$ . The full quantum effective action is hence defined by the flow equation and an initial action at some UV scale. Note that the limits (2.7) and (2.8) are strictly speaking only valid with a suitable choice of  $C_k$ .

### 3. Quantum vs. thermal fluctuations

At finite temperature the flow (2.6) constitutes the integrating-out of quantum as well as thermal fluctuations. eq. (2.6) can be projected on thermal fluctuations: instead of computing the flow for  $\Gamma_{k,T}[\phi]$  as in (2.6), we propose to study the flow for the difference

$$\bar{\Gamma}_{k,T}[\phi] = \Gamma_{k,T}[\phi] - \Gamma_{k,0}[\phi] \tag{3.1}$$

between the effective action at vanishing and non-vanishing temperature [7]. eq. (3.1) is evaluated for fields periodic in time.<sup>1</sup> It entails the thermal effects in the effective action as we have effectively removed the quantum fluctuations. The flow of (3.1) reads

$$\partial_t \bar{\Gamma}_{k,T}[\phi] = \frac{1}{2} \int \frac{d^3q}{(2\pi)^3} \left[ T \sum_n G_{k,T}[\phi] \partial_t R_k - \int \frac{dq_0}{2\pi} G_{k,0}[\phi] \partial_t R_k \right] - \partial_t \bar{C}_{k,T}, \tag{3.2}$$

where  $\bar{C}_{k,T} = C_{k,T} - C_{k,0}$ . The flow (3.2) vanishes at  $T = 0$ , and hence is a thermal flow. It can be used within a two-step procedure: first, one computes the flow (2.6) at vanishing temperature within a truncation that is adapted to the zero temperature theory. Then the result is used as an input for the thermal flow (3.2). We emphasise that the truncation at finite temperature may differ even qualitatively from that at  $T = 0$ .

The thermal flow (3.2) has another important advantage that is particularly interesting for its application to gauge theories. We can relax the conditions (i)–(iii) on the regulator. The aim is to find a reliable, but still sufficiently simple and manageable formulation of the flow equation. The first observation is that the flow equations (2.6) and (3.2) indeed are simplified for a mass-like regulator given by

$$\begin{aligned} \text{bosons:} & \quad R_k(q) \sim k^2 \\ \text{fermions:} & \quad R_k(q) \sim k \end{aligned} \tag{3.3}$$

or variants of it.<sup>2</sup> The key characteristic of  $R_k$  in (3.3) is that it does not depend on momenta.

Formally speaking, (3.3) is a viable IR regulator in the sense of condition (i), and it allows as well to reach the UV initial condition, due to condition (iii). However, the choice (3.3) violates condition (ii), which is one of the basic requirements for a Wilsonian

---

<sup>1</sup>For  $\Gamma_{k,0}$ , this means that Greens functions  $\Gamma_{k,0}^{(n)}$  are multiplied with periodic fields, and integrated over the compact time interval.

<sup>2</sup>Sometimes it is convenient to multiply the mass term with a (momentum-independent) wave function renormalisation,  $R_k = Z_k k^2$  (and analogous for fermions). The same reasoning applies.

cut-off. Indeed, the operator  $\partial_t R_k$  appearing in (2.6) is neither peaked about  $p^2 \approx k^2$ , nor does it lead to a sufficient suppression of high momentum modes. The flow equation (2.6) would then receive contributions from the high momentum region for any value of  $k$ . An immediate consequence of this is that an additional UV regularisation is required, as the flow equation (2.6) is no longer well-defined for large loop momenta. Stated differently, one might say that a mass term regulator leads to a break-down of the Wilsonian picture, since it is no longer related to an integrating-out of momentum degrees of freedom. Rather, it corresponds to a flow within the space of massive theories.

Apart from these more formal objections one should mention that numerical solutions of (2.6) are much more involved than for regulators satisfying (ii). At every iterative step a  $d$ -dimensional momentum integral has to be performed in (2.6) over a non-trivial function which is not strongly peaked about some momentum region. This is, numerically, a quite tedious problem. For this reason most of the sophisticated numerical investigations are based on non-local regulator functions (like the sharp cut-off, exponential or algebraic ones). It has also been observed that approximate solutions to the flow equation (expansions in powers of the field, derivative expansions) show a rather poor convergence behaviour, when (3.3) is used. In gauge theories, this has been seen from explicit perturbative computations [29–31]. Consequently at  $T = 0$  it is not advisable to directly resort to a mass-like regulator.

While (3.3) is not viable for quantum fluctuations being present in (2.6), we will now argue that it is perfectly viable for thermal ones in (3.2). The main point is that the large momentum fluctuations (not sufficiently controlled by (3.3), introducing UV divergences to the flow equation) have nothing to do with the heat bath. Therefore, subtracting the zero temperature quantities will render the flow equation (3.2) finite and well-defined, even in the presence of a mass-like regulator  $R_k = k^2$ . Then the flow (3.2) boils down to

$$\partial_t \bar{\Gamma}_{k,T}[\phi] = k^2 \int \frac{d^3q}{(2\pi)^3} \left[ T \sum_n G_{k,T}[\phi] - \int \frac{dp_0}{2\pi} G_{k,0}[\phi] \right] - \partial_t \bar{C}_{k,T} . \quad (3.4)$$

Here, the momentum-independent regulator  $\partial_t R_k \sim R_k$  acts only as a multiplicative constant because of (3.3). In this case, the suppression of large momenta does not originate from  $\partial_t R_k$ , but from the cancellation between the propagator terms. For large internal momenta, the Matsubara sum can be replaced by an integral, thereby cancelling the  $T = 0$  contribution. Therefore, one may read (3.4) as a Wilsonian flow for thermal fluctuations: At the starting point  $k = \Lambda$  ( $\Lambda$  being some large UV scale) all fluctuations are suppressed and (3.4) vanishes. For any  $k < \Lambda$ , the flow of  $\Gamma_{k,T}[\phi]$  would receive contributions for all momenta. In contrast, the difference  $\bar{\Gamma}_{k,T}[\phi]$  is sensitive only to thermal fluctuations, which are peaked in the infrared region and naturally decay in the UV region. It follows that the integrand in (3.4) is peaked about  $q^2 \approx k^2$ . In other words, condition (ii) is effectively guaranteed even in the case of a mass-like regulator by the very nature of the temperature fluctuations. This amounts to the fact that the mass-like regulator seems to be a reasonable choice even for numerical applications in thermal field theories.

It is worth mentioning that the cancellation of UV divergences in (3.4) is reminiscent of the BPHZ-procedure. There, the subtraction of possibly divergent terms takes place on



the level of the integrand rather than on the level of the regularised full expressions.

Let us comment on the initial condition to (3.4) [and (3.2)]. In contrast to the flow (2.6) with the limits (2.7) and (2.8), the flow equation (3.4) [resp. (3.2)] has the limits

$$\lim_{k \rightarrow \infty} \bar{\Gamma}_{k,T}[\phi] = 0 \tag{3.5a}$$

$$\lim_{k \rightarrow 0} \bar{\Gamma}_{k,T}[\phi] = \Gamma_T[\phi] - \Gamma_0[\phi] . \tag{3.5b}$$

The boundary condition (3.5a) looks rather simple. The flow equation (3.4) needs in addition the knowledge of the massive  $T = 0$  quantum theories. This point is qualitatively shared by the proposals [14, 38]. It seems likely to find a good approximation for the issues under investigation, since (3.4) is eventually projecting-out thermal fluctuations. Those should not be too sensitive to the details of the quantum effective action at  $T = 0$ . Moreover we deal with a situation where the original fields are still sensible degrees of freedom. Thus, a perturbatively resummed quantum effective action should be a good starting point. Here we are actually taking advantage of the fact that for a mass-like regulator the flow is describing a path in the set of massive vector boson theories rather than a Wilsonian integrating-out.

#### 4. Thermal resummations

The present formulation naturally incorporates thermal resummations within general truncation schemes. Moreover, an important, direct, consequence of the flow (3.2) is scheme independence of lowest order resummed perturbation theory: at this order the effective action at  $k = 0$  does neither depend on the regulator nor on the truncation. This is proven below, and put to work within the derivative expansion in the next section. We discuss a scalar  $\phi^4$ -theory with a massless (at  $T = 0$ ), neutral scalar field and coupling  $\lambda$ ,

$$S_{\text{cl},T}[\phi] = -\frac{1}{2} \int_x \phi(x) \partial^2 \phi(x) + \frac{\lambda}{4!} \int_x \phi^4(x), \tag{4.1}$$

where  $\int_x = \int_0^{1/T} d\tau \int d^3x$ . We are interested in the  $\lambda$ -dependence of thermal corrections. It is well-known that naive perturbation theory breaks down beyond one-loop, and we are left with an expansion in  $\lambda^{1/2}$  rather than in  $\lambda$ . For the proof of scheme independence we rewrite the flow (3.2) as

$$\partial_t \bar{\Gamma}_{k,T} = \frac{1}{2} \text{Tr} \left[ \frac{1}{\Gamma_{k,T}^{(2)} + R_k} - \frac{1}{\Gamma_{0,T}^{(2)} + R_k} \right] \partial_t R_k + \frac{1}{2} \text{Tr} \frac{1}{\Gamma_{0,T}^{(2)} + R_k} \partial_t R_k - \frac{1}{2} \text{Tr} \frac{1}{\Gamma_{k,0}^{(2)} + R_k} \partial_t R_k \tag{4.2}$$

where we have dropped the constant contribution. The traces  $\text{Tr}$  in (4.2) have to be taken at temperature  $T$  (first terms), and  $T = 0$  (last term). For computing the one loop effective action from (4.2) we have to insert the classical action on the rhs of (4.2). Then the first term on the rhs of (4.2) vanishes. The remaining terms are total  $t$ -derivatives and can be

integrated trivially. After a reordering we end up with

$$\Gamma_{k,T}^{1\text{-loop}} = \left[ \frac{1}{2} \text{Tr} \ln \frac{S_{\text{cl},T}^{(2)} + R_k}{S_{\text{cl},T}^{(2)} + R_\Lambda} + \Gamma_{\Lambda,T}^{1\text{-loop}} \right] - \left[ \frac{1}{2} \text{Tr} \ln \frac{S_{\text{cl},0}^{(2)} + R_k}{S_{\text{cl},0}^{(2)} + R_\Lambda} - (\Gamma_{k,0}^{1\text{-loop}} - \Gamma_{\Lambda,0}^{1\text{-loop}}) \right]. \quad (4.3)$$

The second bracket in (4.3) vanishes identically due to (2.6). The  $\Lambda, T$ -dependent terms in (4.3) just arrange for the appropriate renormalisation; indeed their sum has to be  $\Lambda$ -independent. This results in

$$\Gamma_{k,T}^{1\text{-loop}} = \frac{1}{2} \int_x \phi(x) \left( \partial^2 + \Pi_{k,T}^{1\text{-loop}} \right) \phi(x) + \frac{\lambda}{4!} \int_x \phi^4(x) + \mathcal{O}(\lambda^2), \quad (4.4)$$

where  $\Pi_{k,T}^{1\text{-loop}} = m_T^2(1 + \mathcal{O}(k/T))$  is the momentum-independent self-energy at one loop, and  $m_T^2 \propto \lambda$  is the standard thermal mass at  $k = 0$ . Its scheme independence follows from (4.3). At  $k = 0$  we are left with the renormalised one-loop effective action at finite temperature  $\Gamma_{0,T}^{1\text{-loop}} = \Gamma_{0,0}^{1\text{-loop}} + \Delta\Gamma_{0,T}^{1\text{-loop}}$ . Now we re-insert  $\Gamma_{k,T}^{1\text{-loop}}$ , (4.4) on the rhs of (4.2), and project the flow (4.2) on the leading resummed term of order  $\lambda^{3/2}$ . This term only receives contributions from the vanishing Matsubara frequency  $n = 0$ . Due to the subtraction the first term on the rhs of (4.2) has an additional (dimensionless) power of  $\lambda k/T$ . If integrating the zero frequency flow from  $\Lambda \rightarrow \infty$  to  $k = 0$ , we can rescale all expressions with the one loop self energy  $\Pi_{0,T} = m_T^2$ . The first term in (4.2) is of order  $\lambda^2$  due to the additional power  $\lambda k/m_T \propto \lambda^{1/2}$ , whereas the second term receives  $\lambda^{3/2} T^2$ -contributions from the thermal trace at  $k = 0$ . This term is a total  $t$ -derivative and we arrive at

$$\Gamma_{0,T} = \frac{1}{2} \text{Tr} \ln \frac{\Gamma_{0,T}^{(2)}}{\Gamma_{0,T}^{(2)} + R_\Lambda} + \Gamma_{\Lambda,T} + \mathcal{O}(\lambda^2), \quad (4.5)$$

where the  $\mathcal{O}(\lambda^2)$ -terms also include additional subtractions. eq. (4.5) includes the full 1-loop resummed perturbation theory, but also applies to more general truncations of  $\Gamma_{0,T}$ . We emphasise that (4.5) is valid for general regulators and truncations and hence is scheme independent. Due to the scheme independence of the one loop effective action  $\Gamma_{0,T}$ , (4.5) scheme-independently contains the full bubble resummation that leads to the lowest order in the thermal resummation. We conclude that the flow (3.2) leads to the full thermal resummation at lowest order for any regulator and any truncation. This includes in particular the thermal mass contribution at order  $\lambda^{3/2}$  as well as the pressure contributions at order  $\lambda$  and  $\lambda^{3/2}$ . In scalar theories this is a direct consequence of the fact that these terms are only related to bubble diagrams.

## 5. Thermal derivative expansion

It is useful to discuss the above results in a very simple truncation, the leading order in the derivative expansion at finite temperature. In this approximation, the effective action for a real scalar field  $\phi$  reads

$$\Gamma_{k,T}[\phi] = \frac{1}{2} \text{Tr} \phi(-p) p^2 \phi(p) + \int_x V_{k,T}(\phi), \quad (5.1)$$

with the effective potential  $V_{k,T}(\phi)$ . We follow the reasoning of section 3 and consider the flow for the thermal difference  $U_{k,T}(\phi) = V_{k,T}(\phi) - V_{k,0}(\phi)$ , explicitly given by

$$\partial_t U_{k,T} = \frac{1}{2} \int \frac{d^3 q}{(2\pi)^3} \left[ T \sum_n \frac{\partial_t R_k}{(2\pi nT)^2 + \mathbf{q}^2 + R_k + V''_{k,T}} - \int \frac{dq_0}{2\pi} \frac{\partial_t R_k}{q^2 + R_k + V''_{k,0}} \right]. \quad (5.2)$$

We have dropped a field-independent constant, and  $q^2 = q_0^2 + \mathbf{q}^2$ . For lowest order resummed perturbation theory we can concentrate on the vanishing Matsubara frequency. The flow (5.2) then reads

$$\partial_t U_{k,T} = \frac{1}{2} T \int \frac{d^3 q}{(2\pi)^3} \frac{\partial_t R_k}{\mathbf{q}^2 + R_k + V''_{k,T}} + \mathcal{O}(\lambda^2) = \frac{1}{2} T \int \frac{d^3 q}{(2\pi)^3} \frac{\partial_t R_k}{\mathbf{q}^2 + R_k + V''_{0,T}} + \mathcal{O}(\lambda^2), \quad (5.3)$$

and (4.5) follows.

At finite temperature a regularisation of the theory is already achieved for cutoffs depending only on spatial momenta,  $R_k = R_k(\mathbf{q})$ . These cutoffs are well-adapted to the present situation: then the thermal flows (3.2) successively sum up thermal fluctuation at momenta  $\mathbf{q}^2 \approx k^2$ , and the thermal properties of the theory are unchanged. Technically this has the benefit that the Matsubara sum is unchanged. In the present case this allows us to perform the Matsubara sum and the  $q_0$ -integration in (5.2) analytically,

$$\partial_t U_{k,T} = \frac{1}{2} \int \frac{d^3 q}{(2\pi)^3} \partial_t R(\mathbf{q}) \left[ \frac{1}{2} \left( \frac{1}{\omega_{k,T}} - \frac{1}{\omega_{k,0}} \right) + \frac{n(\omega_{k,T})}{\omega_{k,T}} \right], \quad (5.4)$$

where  $\omega_{k,T} = \left( \mathbf{q}^2 + R_k(\mathbf{q}) + V''_{k,T} \right)^{1/2}$ . We note the appearance of the Bose-Einstein distribution

$$n(\omega) = \frac{1}{e^{\omega/T} - 1} \quad (5.5)$$

in the flow, a direct consequence of the thermal sum. Every single term on the rhs in (5.4) remains well-defined even for large spatial momenta, due to  $\partial_t R$ . Along the flow, (5.4) uses the zero temperature running for  $V_{k,0}$ . While the last term in (5.4) is clearly of a thermal origin, the first two terms display a non-trivial cancellation of zero temperature and thermal contributions.

In the high temperature limit, the Bose-Einstein distribution is strongly enhanced, with  $n(\omega) \rightarrow T/\omega$ . Consequently, the flow (5.4) is parametrically dominated by the last term for  $T \rightarrow \infty$ , and fixed  $k$ . In this limit, we introduce rescaled fields as  $\varphi = \phi/\sqrt{T}$  and the rescaled potential as  $V_k(\varphi) = U_k(\phi)/T$ . Then

$$\partial_t V_k = \frac{1}{2} \int \frac{d^3 q}{(2\pi)^3} \frac{\partial_t R(\mathbf{q})}{\mathbf{q}^2 + R_k(\mathbf{q}) + V''_k}. \quad (5.6)$$

All temperature dependence has disappeared and we are left with a standard flow in three dimensions. We emphasise that (5.6) equals the zero mode flow in (5.3), as all higher modes decouple due to their masses  $2\pi T \rightarrow \infty$ . We also point out that (5.6), unlike (5.4), is closed since contributions from  $\omega_{k,0}$  are absent in this limit.

In the low temperature limit  $T \rightarrow 0$ , the thermal flow (5.4) vanishes identically. At finite temperature, and in a one-loop approximation,  $V_{k,T}$  is substituted by the classical potential  $V_{\text{cl}}$  and the first two terms in (5.4) vanish. Then the thermal corrections reduce to a total  $t$ -derivative, and we obtain the cutoff-independent result

$$V_{0,T}^{1\text{-loop}} = V_{0,0}^{1\text{-loop}} + T \int \frac{d^3q}{(2\pi)^3} \ln \left[ 1 - \exp(-\sqrt{\mathbf{q}^2 + V_{\text{cl}}''/T}) \right] \quad (5.7)$$

upon integration. This equals (4.3) in the present truncation. From (5.7), we derive the free pressure for a single bosonic degree of freedom  $P[T] = \pi^2 T^4/90$  (where  $P = -V$ ). At this order, the self-energy correction to the propagator  $\Gamma_{k,T}^{(2)}(\mathbf{q}) = \Pi_{k,T}^{1\text{-loop}}(\mathbf{q})$  is momentum-independent and given by a thermal mass  $\Pi_{0,T}(\mathbf{q}^2 = 0) = m_T^2 = \lambda T^2/24$ . We have proven in section 4 that the next-to-leading order of the self-energy is also scheme independent. This is most directly seen by studying (5.3) for the self-energy. The  $n = 0$  Matsubara mode generates a  $\lambda^{3/2}$ -term from the leading order thermal mass. The scheme independent coefficient follows from integrating the total  $t$ -derivative in (5.3), leading to  $\delta m_T^2 = -\lambda T m_T/(8\pi)$ . Hence,

$$\Pi_{0,T} = \frac{\lambda T^2}{24} \left[ 1 - 3 \left( \frac{\lambda}{24\pi^2} \right)^{1/2} + \mathcal{O}(\lambda) \right]. \quad (5.8)$$

The thermal pressure can be obtained by re-inserting the self-energy  $\Pi_{k,T}$  into the flow (5.2):  $\Gamma_{k,T}^2[\phi = 0] = (2\pi nT)^2 + \mathbf{q}^2 + \Pi_{k,T}$ . This produces  $k$ -dependent bubble diagrams with regulator insertions. The lowest order (two loop vacuum bubbles) are total  $t$ -derivatives, and the field-independent part  $\Gamma_{0,T}[0]$  reads

$$P = \frac{\pi^2 T^4}{90} \left[ 1 - \frac{15}{8} \left( \frac{\lambda}{24\pi^2} \right) + \frac{15}{2} \left( \frac{\lambda}{24\pi^2} \right)^{3/2} + \mathcal{O}(\lambda^2) \right]. \quad (5.9)$$

In the above discussion, we have focused on scheme independent aspects of thermal flows. Now we specify explicit momentum cutoffs, which should be useful for numerical or analytical studies beyond the present order. Following the reasoning of section 3, we consider first a mass term momentum cutoff  $R_k(\mathbf{q}) = k^2$ . Then (5.4) turns into

$$\partial_t U_{k,T} = \frac{k^2}{2} \int \frac{d^3q}{(2\pi)^3} \left[ \frac{1}{\omega_{k,T}} \coth \frac{\omega_{k,T}}{2T} - \frac{1}{\omega_{k,0}} \right], \quad (5.10)$$

where  $\omega_{k,T} = (\mathbf{q}^2 + k^2 + V_{k,T}'')^{1/2}$  depends on spatial momenta  $\mathbf{q}$ . At fixed momentum cutoff  $k$ , we note that both terms in the integrand diverge quadratically in the UV, while the sum remains well-defined even for large loop momenta. Furthermore, terms sensitive to  $V_{k,T}''$ ,  $V_{k,0}''$  or their difference are suppressed by additional powers in  $1/\mathbf{q}^2$  and are, hence, finite. This is an explicit example of the cancellation emphasised above.

## 6. Optimised thermal flows

The physical content of a given truncation and its stability and convergence towards the physical theory is enhanced by suitably optimised choices for the Wilsonian cutoff [32, 33,

11]. Here, the thermal flows (5.4) and (5.10) still require a spatial momentum integration due to the non-trivial  $\mathbf{q}$ -dependence of the integrand. We take advantage of a simple optimised momentum cutoff [34]

$$R_k(\mathbf{q}) = (k^2 - \mathbf{q}^2)\theta(k^2 - \mathbf{q}^2), \quad (6.1)$$

which is known to display remarkable stability properties. This regulator cuts off the propagating momentum modes in (5.4) homogeneously, since  $\mathbf{q}^2 + R(\mathbf{q}^2) = k^2$  becomes momentum-independent for all infrared modes with  $\mathbf{q}^2 < k^2$ . Furthermore, the remaining loop integration in (5.4) is performed analytically, which is a crucial advantage in view of numerical studies. The result is

$$\partial_t U_{k,T} = \frac{k^4}{6\pi^2} \left[ \frac{1}{2} \left( \frac{k}{\omega_{k,T}} - \frac{k}{\omega_{k,0}} \right) + \frac{k}{\omega_{k,T}} n(\omega_{k,T}) \right], \quad (6.2)$$

where  $\omega_{k,T} = \left( k^2 + V_{k,T}'' \right)^{1/2}$ . The truncated flow (6.2) clearly displays the thermal flow structure: the last term is proportional to the thermal distribution  $n(\omega)$  which vanishes for  $T \rightarrow 0$ , and requires no UV renormalisation. The first term has the subtraction at  $T = 0$  which removes the quantum fluctuation. It is reminiscent of the UV-subtraction in thermal perturbation theory. Here, however, the subtraction is not necessary to get a finite flow. Without the subtraction, the flow (6.2) has been provided as a proper-time flow in [23]. As shown in [35], such a proper-time flow derives from (6.1) to leading order in the derivative expansion. Beyond this order, it carries inherent approximations discussed in [35].

Within the flow (6.2) the study of the scheme independent leading order is most convenient due to its analytic structure. We solve (6.2) iteratively, starting at  $U_{\Lambda,T} = V_{\Lambda,T} - V_{\Lambda,0} = 0$  and vanishing mass at  $T = 0$ . To leading order in  $\lambda$ , corrections in  $U_{k,T}$  and its derivatives originate solely from the thermal (last) term in (6.2). Performing also the  $k$ -integration we find the free pressure  $P = \pi^2 T^4 / 90$ . For the running thermal mass, we find

$$\partial_t m_{k,T}^2 = \frac{k^5}{12\pi^2} \left[ \left( \frac{n'(\omega_{k,T})}{\omega_{k,T}^2} - \frac{n(\omega_{k,T})}{\omega_{k,T}^3} \right) \lambda_{k,T} - \frac{1}{2} \left( \frac{\lambda_{k,T}}{\omega_{k,T}^3} - \frac{\lambda_{k,0}}{\omega_{k,0}^3} \right) \right]. \quad (6.3)$$

To leading order in the coupling, we have  $\lambda_{k,T} = \lambda_{k,0} = \lambda$ , and only the first term in (6.3) contributes. To this order, the running thermal mass reads

$$m_{k,1\text{-loop}}^2 = \frac{\lambda}{12\pi^2} \int_k^\Lambda dx [x n(x) - x^2 n'(x)] + m_{\Lambda,1\text{-loop}}^2. \quad (6.4)$$

The integral can be performed in terms of poly-logarithms. For  $\Lambda \rightarrow \infty$  we have  $m_{\Lambda,T}^2 \rightarrow 0$ , and at  $k = 0$  we find  $m_T^2 = m_{0,1\text{-loop}}^2 = \lambda T^2 / 24$ . While the  $k$ -running in (6.4) is scheme-dependent, the endpoint is not, and coincides with the standard expression for the one-loop thermal correction. Beyond the leading order, the running thermal masses have to be taken into account. In lowest order resummed perturbation theory the Matsubara zero mode leads to a  $\lambda^{3/2}$ -correction to the self-energy. This term is produced in the flow from scales

$$k \ll 2\pi T, \quad (6.5)$$

also assuming  $\lambda \ll 1$ . In this region the flow effectively reduces to the high temperature limit (5.6) with  $n(\omega) \rightarrow T/\omega$ , as  $T$  is larger than all other scales. We derive from (6.2)

$$\partial_t U_{k,T} = \frac{k^4}{6\pi^2} \left( \frac{kT}{\omega_{k,T}^2} + \mathcal{O}(k/\omega_{k,T}) \right), \quad (6.6)$$

leading to

$$\partial_t m_{k,T}^2 = -\lambda_{k,T} \frac{k^4}{6\pi^2} \left( \frac{kT}{\omega_{k,T}^4} + \mathcal{O}(k/\omega_{k,T}^3) \right). \quad (6.7)$$

for the mass. At scales (6.5) the running of the one loop mass is sub-leading and we can substitute  $m_{k,1\text{-loop}}^2 \rightarrow m_T^2$  in (6.6),(6.7). Moreover we are only interested in the  $\lambda^{3/2}$ -term  $m_{3/2}^2$ . For dimensional reasons this term is proportional to  $\lambda_{k,T} T m_T = \lambda T m_T + \mathcal{O}(\lambda^2)$ . After rescaling  $k$  with  $m_T$  we arrive at the integrated flow

$$m_{3/2}^2 = \lambda \frac{T m_T}{6\pi^2} \int_0^\infty dk \left( \frac{k^4}{(k^2 + 1)^2} - 1 \right) = -\lambda \frac{T m_T}{8\pi}, \quad (6.8)$$

which leads to (5.8). The subtraction of 1 removes the infrared one loop contribution. We proceed with the  $\lambda^{3/2}$ -contribution  $P_{3/2}$  to the thermal pressure  $P$ , which follows directly from (6.6),

$$P_{3/2} = \frac{T m_T^3}{6\pi^2} \int_0^\infty dk \left( \frac{k^4}{(k^2 + 1)} - k^2 + 1 \right) = \frac{T m_T^3}{12\pi}. \quad (6.9)$$

The subtraction of  $k^2 - 1$  removes the tree level and one loop infrared contributions. eq. (6.9) provides the  $\lambda^{3/2}$ -order in (5.9).

In section 4 we have shown that the results (6.8), (6.9) are scheme-independent. Therefore it should be possible to turn the corresponding flows into total  $t$ -derivatives. We exemplify this structure with an independent derivation of the thermal pressure up to order  $\lambda^{3/2}$ . The flow  $\partial_t P_{k,T}$  of the pressure is given by (minus) (6.2) at vanishing field. It depends on the running mass via  $\omega_{k,T}(\phi) = (k^2 + m_{k,T}^2 + \frac{1}{2}\lambda\phi^2)^{1/2}$  at  $\phi = 0$ . Note that we keep a classical  $\lambda$  for technical reasons. The pressure does not depend on this choice. Now we rescale the mass in  $\omega_{k,T}$  with a parameter  $\alpha$ ,  $m_{k,T}^2 \rightarrow \alpha m_{k,T}^2$ . With this modification we get

$$P(\alpha) = P_{\text{free}} + \alpha P_1 + \alpha^{3/2} P_{3/2} + \mathcal{O}(\lambda^2), \quad (6.10)$$

since  $m_{3/2}^2$  does not contribute to  $P_{3/2}$ , and the  $\lambda$ -dependence only originates in  $m_{k,1\text{-loop}}$ . Now we take the  $\alpha$ -derivative,

$$\partial_\alpha \partial_t P_{k,T} = -m_{k,T}^2 \frac{1}{\lambda} \partial_\phi^2 \partial_t U_{k,T}|_{\phi=0} + \mathcal{O}(\lambda^2) = -\frac{1}{\lambda} m_{k,T}^2 \partial_t m_{k,T}^2 + \mathcal{O}(\lambda^2), \quad (6.11)$$

where we have used that  $\partial_\alpha \omega_{k,T} = \lambda^{-1} m_{k,T}^2 \partial_\phi^2 \omega_{k,T}$  at vanishing field. Note also that  $\partial_t m_{k,T}^2$  depends on  $\lambda_{k,T} - \lambda$  only at the order  $\mathcal{O}(\lambda^2)$ , as already indicated below (6.7). Taking derivatives as in (6.11) is a standard computational trick used in thermal perturbation theory where mass-derivatives help to analytically perform thermal sums and momentum

integrals. Eq (6.11) carries an  $\alpha$ -dependence since  $\partial_t m_{k,T}^2(\alpha)$  is a function of  $\alpha$  via  $\omega_{k,T}$ . At  $\alpha = 1$  the flow (6.11) can be expressed in terms of a total  $t$ -derivative,

$$m_{k,T}^2 \partial_t m_{k,T}^2 = \frac{1}{2} \partial_t (m_{k,T}^2)^2. \quad (6.12)$$

Eq. 6.12 relates to the  $t$ -derivative of the two loop vacuum bubble, and contains the full contribution to the pressure of order  $\lambda$ . This reduction leads to the diagrammatic structure of resummed perturbation theory: the flow can be written as a  $t$ -derivative of the infrared regularised resummed diagrams. Now we reconstitute the  $\alpha$ -dependence in (6.12) with the help of (6.10). By taking the  $\alpha$ -derivative of (6.10) we conclude that the one loop term in (6.12) is  $\alpha$ -independent, and the resummed contributions of order  $\lambda^{3/2}$  are proportional to  $\alpha^{1/2}$ . Performing the  $t$ -integration we arrive at

$$\partial_\alpha P_{0,T} = -\frac{1}{2\lambda} m_T^4 + \frac{T}{8\pi} \alpha^{1/2} m_T^3 + \mathcal{O}(\lambda^2) = -\frac{T^2 m_T^2}{48} \left( 1 - \frac{6}{\pi} \alpha^{1/2} \frac{m_T}{T} \right) + \mathcal{O}(\lambda^2), \quad (6.13)$$

where we have used (5.8) for substituting the  $\lambda$ -dependence with the appropriate powers of  $m_T$ . We integrate (6.13) over  $\alpha$  from 0 to 1, and arrive at

$$P_{0,T} = \frac{\pi^2 T^4}{90} - \frac{T^2}{48} \left( m_T^2 - \frac{4}{\pi} \frac{m_T^3}{T} \right) + \mathcal{O}(\lambda^2), \quad (6.14)$$

where we have used that  $P_{0,T}|_{\alpha=0}$  is the tree-level pressure. Eq. (6.14) agrees with (5.9).

## 7. Thermal gauge theories

We now turn to thermal flows for gauge theories. We aim at deriving well-defined flows for gauge theories in terms of the fundamental degrees of freedom, the gluons. The inclusion of matter fields is discussed elsewhere. For specific regulators and gauges this flow is gauge invariant. The classical Yang-Mills action is given by

$$S_A[A] = \frac{1}{2} \int_T d^4x \operatorname{tr} F^2(A), \quad (7.1)$$

with field strength  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g[A_\mu, A_\nu]$ , and  $\operatorname{tr} t^a t^b = -\delta^{ab}/2$  in the fundamental representation. The regulator term for gauge fields is given by

$$\Delta S_k[A] = \frac{1}{2} T \sum_n \int \frac{d^3q}{(2\pi)^4} A_\mu^a(-q) R_{k,ab}^{\mu\nu}(q) A_\nu^b(q) \quad (7.2)$$

where the implicit replacement  $q_0 \rightarrow 2\pi nT$  is understood. With the regularisation (7.2) and using the definition (3.1), the flow (3.2) reads

$$\partial_t \bar{\Gamma}_{k,T}[A] = \frac{1}{2} \int \frac{d^3q}{(2\pi)^3} \left[ T \sum_n G_{k,T\mu\nu}^{ab}[A] \partial_t R_{k,ba}^{\nu\mu} - \int \frac{dq_0}{2\pi} G_{k,T\mu\nu}^{ab}[A] \partial_t R_{k,ba}^{\nu\mu} \right] - \partial_t \bar{C}_{k,T}, \quad (7.3)$$

In the presence of matter fields  $\phi$  and corresponding regulators  $R_\phi$  additional terms proportional to  $\partial_t R_\phi$  will be present in (7.3). For mass-like regulators the flow (7.3) reduces to

$$\partial_t \bar{\Gamma}_{k,T}[A] = k^2 \int \frac{d^3 q}{(2\pi)^3} \text{tr} \left[ T \sum_n G_{k,T}[A] \partial_t R_k - \int \frac{dq_0}{2\pi} G_{k,T}[A] \partial_t R_k \right] - \partial_t \bar{C}_{k,T}, \quad (7.4)$$

where the trace  $\text{tr}$  sums over Lorentz and gauge group indices. So far we have not specified the gauge fixing procedure. In general, gauge fixing leads to additional ghost fields  $C$  as well as a regularisations  $R_C$ . Since the thermal bath singles out a preferred rest frame, we have an additional Lorentz vector  $n_\mu$  at our disposal. Therefore, it is natural to employ an axial gauge fixing. An axial gauge has the further advantage that ghost fields decouple completely from the theory, and possible Gribov copies are absent. We employ

$$S_{\text{gf}}[A] = \frac{1}{2} T \sum_n \int \frac{d^d q}{(2\pi)^4} n_\mu A_\mu^a \frac{1}{\xi n^2} n_\nu A_\nu^a. \quad (7.5)$$

In [27, 7, 28] we discussed the various aspects of an axial gauge fixing (7.5). In particular we showed that the spurious propagator singularities of perturbation theory are naturally absent in a Wilsonian approach. Furthermore, the gauge fixing parameter  $\xi$  with mass dimension  $-2$  has a non-perturbative fixed point at  $\xi = 0$ . This singles out the  $nA = 0$  gauges and tremendously simplifies the problem of gauge invariance, because it allows for a momentum independent choice of  $\xi$ .

Gauge invariance for physical Green functions corresponds to the requirement of a modified Ward Identity (mWI) to hold. Gauge transformations on the fields are generated by  $\delta_\alpha$  with action

$$\delta_\alpha A = D(A)\alpha. \quad (7.6)$$

For momentum independent gauge fixing parameter  $\xi$ , the mWI in the presence of the cut-off term (7.2) reads

$$\delta_\alpha \Gamma_{k,T}[A] = T \sum_n \int \frac{d^3 q}{(2\pi)^3} \text{tr} \left[ \frac{1}{n^2 \xi} n_\mu \partial_\mu \alpha n_\nu A_\nu + \frac{1}{2} [\alpha, R_k^{\mu\nu}] G_{k,T\nu\mu}[A] \right]. \quad (7.7)$$

The two terms on the r.h.s. are remnants from the gauge fixing and the coarse-graining, respectively. The mWI (7.7) is compatible with (2.6) [36, 27, 28, 37, 11], i.e. a solution to (7.7) at some scale  $k = \Lambda$  remains a solution for  $k < \Lambda$  if  $\Gamma_{k,T}[\phi]$  is integrated according to the flow equation. In particular, the terms proportional to  $R_k$  vanish for  $k \rightarrow 0$ , thereby ensuring gauge invariance for physical Green functions. The mWI related to the thermal flow (7.3) follows from (7.7) as

$$\delta_\alpha \bar{\Gamma}_{k,T}[A] = \frac{g}{2} \int \frac{d^3 q}{(2\pi)^3} \text{tr} \left[ \int \frac{dq_0}{2\pi} [\alpha, R_k] G_{k,0}[A] - T \sum_n [\alpha, R_k] G_{k,T}[A] \right] \quad (7.8)$$

The compatibility of (7.8) with (7.3) is a direct consequence of the compatibility of (7.7) with (2.6). The linear term related to the gauge fixing [the first term in (7.7)] has cancelled



out, since, as emphasized in section 3, we are only looking at fields  $\phi$  at temperature  $T$  and corresponding gauge transformations. This also implies — up to modifications for topologically non-trivial configurations — that  $\alpha$  has to be periodic. Apart from this simplification, the same reasoning as for (7.7) above applies.

With a mass-like regulator, however, we can go a step further. For a regulator as in (3.3), the right hand side in (7.8) vanishes since then  $[\alpha, R_k] = 0$ . All coarse-graining dependence of (7.8) drops out for arbitrary scale  $k$ , and not only in the limit  $k \rightarrow 0$ . This is an immediate consequence of  $R_k$  being momentum independent and the axial gauge fixing [27–29], and reduces (7.7) to the standard Ward Identity in the presence of an axial gauge fixing. It follows, that

$$\delta_\alpha \bar{\Gamma}_{k,T}[A] = 0 \tag{7.9}$$

and we end up with the statement that (3.4) corresponds to a gauge invariant thermal Wilsonian renormalisation group for  $\bar{\Gamma}_{k,T}[\phi]$ , valid at any scale  $k$ .

The above flow can be used for computing the self-energy and the thermal pressure for one-loop resummed perturbation theory. Following the computations in section 4 and 5 we arrive at the well-known result for the thermal mass

$$m_{\text{gluon}}^2 = \frac{1}{6} g^2 T^2 N. \tag{7.10}$$

The thermal mass (7.10) can be re-inserted into the flow. However, full resummations require also the one-loop running of propagator and classical vertices. This is straightforward but tedious, and the results shall be presented elsewhere. Finally we want to comment on the computation of the thermal pressure. It is well-known that the computation requires the correct normalisation of the path integral, only physical degrees of freedom contribute to the pressure. In the present formalism this is the correct choice of  $\bar{C}_{k,T}$ . This amounts to the projection of the field independent part of the flow in momentum space onto the transversal parts of momenta orthogonal to the gauge fixing vector  $n$ . Such a procedure immediately results in

$$P = (N^2 - 1) \frac{\pi^2 T^4}{45}. \tag{7.11}$$

The computation of higher contributions, in particular the resummation, requires the full one loop running of vertices and propagators. We also emphasise that (4.5) already provides the closed expression for the resummation. We hope to report on this matter in near future.

## 8. Discussion and conclusions

In this paper, we have developed a thermal renormalisation group in terms of the functional  $\bar{\Gamma}_T[\phi] = \Gamma_T[\phi] - \Gamma_0[\phi]$ . The approach implements a thermal projection at every integration step of the Wilsonian flow. From  $\bar{\Gamma}_T[\phi]$  all thermal observables — including the thermal pressure and the thermal self-energy — can be obtained. Our flow for  $\bar{\Gamma}_T$  is free of infrared and ultraviolet divergences and therefore well-suited for numerical and analytical studies.

The running zero-temperature effective action serves as a boundary condition for the flow. The flow for  $\bar{\Gamma}_T[\phi]$  can be solved with standard techniques including expansions in derivatives or vertex functions, and is not confined to the weakly coupled domain. Moreover, the thermal flow derives from a path integral representation of the theory. This ensures that no double counting occurs in the flow for  $\bar{\Gamma}_T[\phi]$  and systematic solutions thereof. While the present construction is based on the imaginary time formalism, it is straightforward to implement these ideas even in a real-time formulation.

For scalar theories, we have shown that the leading-order resummed perturbation theory follows independently of the scheme, and independently of the truncation. Quantitatively, this has been worked out to leading order within a thermal derivative expansion. Furthermore, an optimised thermal flow for the effective potential has been provided. Its simple analytic form allows for stable numerical integrations, in particular at strong coupling. As an aside, we point out how [38] – where an infrared regulated resummation for the thermal pressure of scalar theories has been proposed – is related to our work. If we restrict the full thermal flow (3.4) to scalar fields, and to leading order in the derivative expansion of the effective action at vanishing field, then (3.4) corresponds to the imaginary-time analogue of [38]. Hence, [38] has the interpretation of a leading-order Wilsonian flow with mass term cutoff. Since (3.4) defines the field-dependent effective action, it allows for an extension of [38] to higher order operators as well as to fermions and gauge fields.

For gauge theories, the following picture has emerged: the standard Wilsonian flow equation (2.6), equipped with the modified Ward identity (7.7), allows for the consistent computation of infrared quantities starting with an initial action in the ultraviolet. Gauge invariance is ensured in the physical limit. A gauge invariant implementation for all scales  $k$  – in the Wilsonian sense – is problematic due to the poor performance of a mass-term regulator for quantum fluctuations in the ultraviolet. The important new result is that the thermal flow (3.2), instead, stays well-defined even for a mass-like regulator (3.4) since the zero temperature contribution renders the flow finite. A mass term regulator fully pays off when combined with the axial gauge fixing, as it implies a gauge-invariant thermal flow for all scales  $k$ . The difference to the flow (2.6) stems now from the initial condition, which is no longer the bare action, but the running  $T = 0$  quantum effective action, or some approximation to it.

## Acknowledgments

DFL is supported by an EPSRC Advanced Fellowship. JMP is supported by the DFG under contract GI328/1-2.

## References

- [1] U.W. Heinz, *Hydrodynamics at RHIC: how well does it work, where and how does it break down?*, *J. Phys.* **G31** (2005) S717–S724 [[nuc1-th/0412094](#)].
- [2] U. Kraemmer and A. Rebhan, *Advances in perturbative thermal field theory*, *Rept. Prog. Phys.* **67** (2004) 351 [[hep-ph/0310337](#)].

- [3] J.O. Andersen and M. Strickland, *Resummation in hot field theories*, *Ann. Phys. (NY)* **317** (2005) 281 [[hep-ph/0404164](#)].
- [4] J.P. Blaizot, E. Iancu and A. Rebhan, *On the apparent convergence of perturbative QCD at high temperature*, *Phys. Rev. D* **68** (2003) 025011 [[hep-ph/0303045](#)].
- [5] T.R. Morris, *Elements of the continuous renormalization group*, *Prog. Theor. Phys. Suppl.* **131** (1998) 395–414 [[hep-th/9802039](#)].
- [6] D.F. Litim, *Wilsonian flow equation and thermal field theory*, [hep-ph/9811272](#).
- [7] D.F. Litim and J.M. Pawłowski, *On gauge invariant Wilsonian flows*, [hep-th/9901063](#).
- [8] J. Berges, N. Tetradis and C. Wetterich, *Non-perturbative renormalization flow in quantum field theory and statistical physics*, *Phys. Rept.* **363** (2002) 223 [[hep-ph/0005122](#)].
- [9] J. Polonyi, *Lectures on the functional renormalization group method*, *Central Eur. J. Phys.* **1** (2003) 1 [[hep-th/0110026](#)].
- [10] M. Salmhofer and C. Honerkamp, *Fermionic renormalization group flows: technique and theory*, *Prog. Theor. Phys.* **105** (2001) 1.
- [11] J.M. Pawłowski, *Aspects of the functional renormalisation group*, [hep-th/0512261](#).
- [12] N. Tetradis and C. Wetterich, *The high temperature phase transition for  $\phi^4$  theories*, *Nucl. Phys. B* **398** (1993) 659.
- [13] S.-B. Liao and M. Strickland, *Renormalization group approach to field theory at finite temperature*, *Phys. Rev. D* **52** (1995) 3653 [[hep-th/9501137](#)].
- [14] M. D’Attanasio and M. Pietroni, *Wilson renormalization group formulation of real time thermal field theories*, *Nucl. Phys. B* **472** (1996) 711 [[hep-ph/9601375](#)].
- [15] M. D’Attanasio and M. Pietroni, *Gauge-invariant renormalization group at finite temperature*, *Nucl. Phys. B* **498** (1997) 443 [[hep-th/9611038](#)].
- [16] D. Comelli and M. Pietroni, *Screening masses in  $SU(N)$  from Wilson renormalization group*, *Phys. Lett. B* **417** (1998) 337 [[hep-ph/9708489](#)].
- [17] B. Bergerhoff and J. Reingruber, *Thermal renormalization group-equations and the phase-transition of scalar  $O(N)$ -theories*, *Phys. Rev. D* **60** (1999) 105036 [[hep-ph/9809251](#)].
- [18] J.O. Andersen and M. Strickland, *Application of renormalization group techniques to a homogeneous bose gas at finite temperature*, *Phys. Rev. A* **60** (1999) 1442 [[cond-mat/9811096](#)].
- [19] B.-J. Schaefer and H.-J. Pirner, *The equation of state of quarks and mesons in a renormalization group flow picture*, *Nucl. Phys. A* **660** (1999) 439 [[nucl-th/9903003](#)].
- [20] F. Freire and D.F. Litim, *Charge cross-over at the  $U(1)$ -higgs phase transition*, *Phys. Rev. D* **64** (2001) 045014 [[hep-ph/0002153](#)].
- [21] C. Honerkamp and M. Salmhofer, *The temperature-flow renormalization group and the competition between superconductivity and ferromagnetism*, [[cond-mat/0105218](#)].
- [22] T. Baier, E. Bick and C. Wetterich, *Temperature dependence of antiferromagnetic order in the hubbard model*, [cond-mat/0309715](#).
- [23] J. Braun, K. Schwenzer and H.-J. Pirner, *Linking the quark meson model with QCD at high temperature*, *Phys. Rev. D* **70** (2004) 085016 [[hep-ph/0312277](#)].

- [24] J.-P. Blaizot, R. Mendez Galain and N. Wschebor, *Non perturbative renormalization group, momentum dependence of  $n$ -point functions and the transition temperature of the weakly interacting bose gas*, cond-mat/0412481.
- [25] J. Braun and H. Gies, *Chiral phase boundary of QCD at finite temperature*, *JHEP* **06** (2006) 024 [hep-ph/0602226]; *Running coupling at finite temperature and chiral symmetry restoration in QCD*, hep-ph/0512085.
- [26] B.-J. Schaefer and J. Wambach, *Susceptibilities near the QCD (tri)critical point*, hep-ph/0603256.
- [27] D.F. Litim and J.M. Pawłowski, *Flow equations for Yang-Mills theories in general axial gauges*, *Phys. Lett.* **B 435** (1998) 181 [hep-th/9802064].
- [28] D.F. Litim and J.M. Pawłowski, *Renormalisation group flows for gauge theories in axial gauges*, *JHEP* **09** (2002) 049 [hep-th/0203005].
- [29] M. Simionato, *Gauge consistent wilson renormalization group. II: non- abelian case*, *Int. J. Mod. Phys.* **A 15** (2000) 2153 [hep-th/9810117].
- [30] M. Simionato, *On the consistency of the exact renormalization group approach applied to gauge theories in algebraic non-covariant gauges*, *Int. J. Mod. Phys.* **A 15** (2000) 4811 [hep-th/0005083].
- [31] A. Panza and R. Soldati, *Infrared singularities in the renormalization group flow of Yang-Mills theories in the axial gauge*, *Phys. Lett.* **B 493** (2000) 197 [hep-th/0006170].
- [32] D.F. Litim, *Optimisation of the exact renormalisation group*, *Phys. Lett.* **B 486** (2000) 92 [hep-th/0005245]; *Critical exponents from optimised renormalisation group flows*, *Nucl. Phys.* **B 631** (2002) 128 [hep-th/0203006].
- [33] D.F. Litim, *Universality and the renormalisation group*, *JHEP* **07** (2005) 005 [hep-th/0503096].
- [34] D.F. Litim, *Optimised renormalisation group flows*, *Phys. Rev.* **D 64** (2001) 105007 [hep-th/0103195].
- [35] D.F. Litim and J.M. Pawłowski, *Completeness and consistency of renormalisation group flows*, *Phys. Rev.* **D 66** (2002) 025030 [hep-th/0202188]; *Wilsonian flows and background fields*, *Phys. Lett.* **B 546** (2002) 279 [hep-th/0208216].
- [36] M. D'Atanasio and T.R. Morris, *Gauge invariance, the quantum action principle and the renormalization group*, *Phys. Lett.* **B 378** (1996) 213 [hep-th/9602156].
- [37] F. Freire, D.F. Litim and J.M. Pawłowski, *Gauge invariance and background field formalism in the exact renormalisation group*, *Phys. Lett.* **B 495** (2000) 256 [hep-th/0009110].
- [38] I.T. Drummond, R.R. Horgan, P.V. Landshoff and A. Rebhan, *Eliminating infrared divergences in the pressure*, *Phys. Lett.* **B 398** (1997) 326 [hep-th/9610189].