WMAP normalization of inflationary cosmologies

Article (Published Version)

WMAP normalization of inflationary cosmologies

Andrew R. Liddle,[1] David Parkinson,[1] Samuel M. Leach,[2] and Pia Mukherjee[1]

[1]*Astronomy Centre, University of Sussex, Brighton BN1 9QH, United Kingdom*

[2]*SISSA-ISAS, Astrophysics Sector, Via Beirut 4, 34014 Trieste, Italy*

(Received 14 July 2006; published 13 October 2006)

We use the three-year WMAP observations to determine the normalization of the matter power spectrum in inflationary cosmologies. In this context, the quantity of interest is not the normalization marginalized over all parameters, but rather the normalization as a function of the inflationary parameters n_{S} and r with marginalization over the remaining cosmological parameters. We compute this normalization and provide an accurate fitting function. The statistical uncertainty in the normalization is 3%, roughly half that achieved by COBE. We use the $k - \ell$ relation for the standard cosmological model to identify the pivot scale for the WMAP normalization. We also quote the inflationary energy scale corresponding to the WMAP normalization.

DOI: 10.1103/PhysRevD.74.083512 PACS numbers: 98.80.−k

I. INTRODUCTION

One of the legacy products of the COBE satellite [1] was a determination of the normalization of the matter power spectrum, which to this day is often referred to as the COBE normalization. Surprisingly, only limited effort has been made so far to extract the equivalent result from Wilkinson Microwave Anisotropy Probe (WMAP) observations [2,3], with the papers to date quoting only the normalization marginalized over all cosmological parameters. For inflationary cosmologies this is not quite the quantity required; inflation models predict the spectral index and tensor-to-scalar ratio, and hence should be normalized assuming these parameters are fixed. Because the normalization does depend significantly on those parameters even across just the region allowed by the data, marginalization over those parameters significantly worsens the apparent uncertainty, as well as losing the information on how the normalization correlates to these parameters.

The COBE normalization of inflationary cosmologies was given, from the four-year data, by Bunn, Liddle and White [4]. They used a data-analysis methodology described by Bunn and White [5]. Their treatment considered both the then-popular critical-density cosmology, and also the flat dark-energy dominated cosmologies favored today. The results were somewhat confusing, as they used a definition of the normalization that depended on the value of Ω_0 (see e.g. Ref. [6]), and also quoted the present-day normalization from which a growth factor correction would have to be subtracted to generate the primordial amplitude predicted by inflation.

Nevertheless, the result had a statistical uncertainty of only 7%, and has been frequently used to constrain inflationary models. In single-field models it simply fixes the overall normalization of the potential; in a one-parameter case such as $V = \frac{1}{2}m^2\phi^2$ this completely fixes the model. In hybrid inflation models the constraint may be less trivial, as the instability point ending inflation may not scale directly with the overall normalization. In models such as the curvaton model [7], where adiabatic perturbations are generated after inflation from isocurvature ones, the normalization constrains a combination of model parameters including the initial value of the curvaton field.

No inflation theory purports to predict the perturbation amplitude at anything like the 7% level already achieved by COBE. Nevertheless, we argue that the WMAP normalization is an important legacy result of the WMAP survey. Further, it is bizarre that the currently quoted normalization uncertainties from WMAP hardly improve on the quoted COBE level at all, due to the marginalization over inflationary parameters. It is also an opportunity to quote the result in more modern model-independent terminology.

II. NORMALIZATION CALCULATION

The normalization calculation is straightforward. The inflationary parameters are n_{S} and r, giving the density perturbation spectral index and the tensor-to-scalar ratio in standard conventions; a typical inflation model would predict values for those and hence should be normalized assuming those fixed values. We carried out a series of runs of the CosmoMC code [8] in which n_{S} and r are fixed at different values across a grid, making 28 evaluations combining $n_{\mathrm{S}} = 0.90$, 0.925, 0.95, 0.975, 1,1. 025, 1.05 and $r = 0$, 0.25, 0.5, 0.75. This comfortably covers the values of these parameters favored by current observations from WMAP combined with other data. In our analysis we consider WMAP data alone, including both temperature and polarization data. A power-law primordial spectrum is assumed throughout and the tensor spectral index fixed by the single-field inflationary consistency equation (see e.g. Ref. [9]). The parameters allowed to vary and subsequently marginalized over are the matter density Ω_{m}, the baryon density Ω_{b}, the Hubble constant h, and the optical depth τ.

The normalization parameter, denoted A_{s} in CosmoMC, is equivalent to $\mathcal{P}_{\mathcal{R}}$ of the textbook by Liddle and Lyth [9], evaluated at the comoving scale 0.05 Mpc^{-1} at some early epoch while the perturbations are still superhorizon. This quantity gives the normalization in a way which is independent of the ultimate value of Ω_0 at the present. We use the related quantity $\delta_{\rm H} \equiv 2\mathcal{P}_{\mathcal{R}}^{1/2}/5$.

The simplest way to define our terminology is to relate it to the perturbations from slow-roll inflation models. There $\delta_{\rm H}$ is given in terms of the Hubble parameter H and scalar field velocity $\dot\phi$ by (see e.g. Ref. [9])

$$\delta_{\rm H}^{\rm inflation} \simeq \frac{H^2}{5\pi|\dot\phi|}. \tag{1}$$

The right-hand side is to be evaluated at the time when the normalization scale crossed outside the horizon during inflation, which can be estimated following Ref. [10]. The spectral index and tensor-to-scalar ratio are given in the slow-roll approximation by

$$n_{\rm S} - 1 \simeq -6\epsilon + 2\eta \tag{2}$$

$$r \simeq 16\epsilon, \tag{3}$$

where the slow-roll parameters are defined from the potential $V(\phi)$ by

$$\epsilon = \frac{m_{\rm Pl}^2}{16\pi}\left(\frac{dV/d\phi}{V}\right)^2; \qquad \eta = \frac{m_{\rm Pl}^2}{8\pi}\frac{d^2V/d\phi^2}{V}. \tag{4}$$

To compare with the COBE normalization of Ref. [4], one should simply look at the critical-density result given in that case,

$$\delta_{\rm H}^{\rm COBE} = 1.91 \times 10^{-5}\frac{\exp[1.01(1-n_{\rm S})]}{\sqrt{1+0.75\hat r}}. \tag{5}$$

Here $\delta_{\rm H}$ is quoted at the present horizon scale $k = aH$, and we have used the notation $\hat r$ to indicate use of an outdated definition of the tensor-to-scalar ratio, connected to the modern one by $r = 1.29\hat r$. The form of the fitting function reflects the fact that the data imparts a pivot scale at which the normalization parameter is independent of $n_{\rm S}$; for COBE that scale is $k_{\rm pivot}^{\rm COBE} = e^{2.02}aH$.

Our result for the normalization is shown in Fig. 1. Generally we find excellent agreement with the COBE normalization; in particular, for $n_{\rm S} = 1$ and $r = 0$ we get $\delta_{\rm H} = 1.91 \times 10^{-5}$, matching the value from the COBE fitting function. Our findings are thus consistent with the conclusion of Ref. [11] that the WMAP measurements of the large angular scale anisotropy show no systematic discrepancy from the COBE measurements.

We find that the results are well fit by the function

$$\delta_{\rm H}^{\rm WMAP} = 1.927 \times 10^{-5}\frac{\exp[(-1.24+1.04r)(1-n_{\rm S})]}{\sqrt{1+0.53r}}, \tag{6}$$

which is the main result of this paper. Here $\delta_{\rm H}^{\rm WMAP}$ is specified at $k = 0.05\ {\rm Mpc}^{-1}$ and hence the coefficients cannot be directly compared with those in Eq. (5). The criteria for obtaining the fit parameters was the minimization of the maximum relative error over the $n_{\rm S}$ and r values studied. Similar results are obtained by minimizing the average deviation or the chi-squared. The fitting function is accurate to within 1% for all the values we considered.

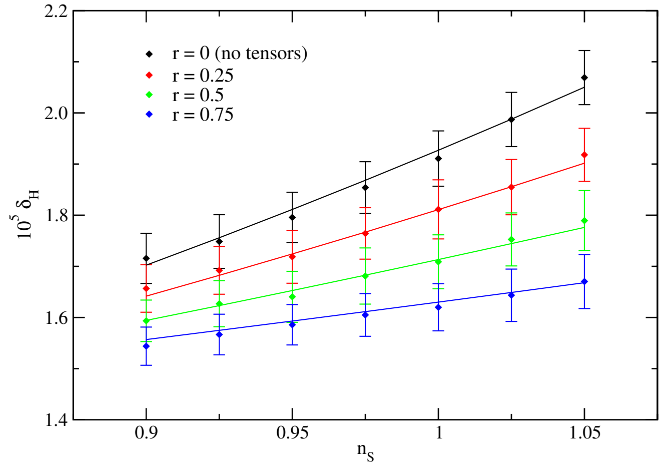

FIG. 1 (color online). The WMAP3 normalization $\delta_{\rm H}$ as a function of $n_{\rm S}$ for the four choices of r, specified at the scale $0.05\ {\rm Mpc}^{-1}$. The uncertainties shown are the standard deviations inferred from the Markov chains. The lines show the fitting function Eq. (6).

The form of the fitting function in Eq. (6) is motivated by the same criteria as for COBE. If there is a pivot scale where the normalization becomes independent of $n_{\rm S}$, the dependence on $n_{\rm S}$ should be exponential to reflect that translation of scales. The introduction of r suppresses the matter power spectrum, the coefficient of order unity required because r is defined by the relative tensor and scalar power spectra rather than precisely by their relative effect on the CMB.

However, with WMAP it turns out that there is not a unique pivot scale independent of r; as r is introduced the curves shown in Fig. 1 become noticeably flatter. The pivot scale when $r = 0$ is $k_{\rm pivot}^{{\rm WMAP},r=0} = e^{-2.48} \times 0.05\ {\rm Mpc}^{-1} = 0.004\ {\rm Mpc}^{-1}$, increasing to $0.02\ {\rm Mpc}^{-1}$ when $r = 0.75$ (when r is large, the low multipoles contain less information about the matter power spectrum normalization). The r-dependence in the exponential is necessary to allow for this in the fitting function.

The statistical uncertainty on $\delta_{\rm H}^{\rm WMAP}$ is independent of $n_{\rm S}$ and r to a good approximation and is 3% (at 68% confidence), corresponding to about twice the accuracy of COBE. The two results are entirely consistent with one another. This uncertainty is also about half of that obtained when marginalizing the amplitude over the posterior distribution of $n_{\rm S}$ and r; the WMAP team quote $\delta_{\rm H} = (1.83 \pm 0.10) \times 10^{-5}$ for the fully-marginalized amplitude in inflationary models [12] (this number corresponding to the scale $0.002\ {\rm Mpc}^{-1}$).

The precision of the normalization is also affected by the overall calibration uncertainty of the WMAP data. This is estimated to be 0.5% in the WMAP3 data [3], the same level reported for the first-year data. That number was obtained [13] by applying an iterative algorithm that simultaneously fits for calibration parameters and sky maps; using detailed simulations of the time-ordered data they arrive at a conservative estimate of 0.5% for the absolute calibration uncertainty. This calibration uncertainty is well below the statistical uncertainty reported above (0.5% is the appropriate number for $\delta_{\rm H}$, and would correspond to a 1% uncertainty in the power spectrum).

Having evaluated the effective pivot scale (for different r values), it is interesting to know what multipole value it corresponds to, in order to see which part of the data drives the result. To make this relation, we ran a series of CMB spectrum computations using spike power spectra ($\Delta \ln k = 0.02$) at different k, using the best-fit WMAP cosmological parameters $\{\Omega_{\rm b}h^2, \Omega_{\rm m}h^2, h, \tau\} = \{0.0223, 0.1263, 0.73, 0.088\}$. Such a sharp input spectrum gets smeared by projection effects, but we can still pick off the corresponding ℓ value where the transfer of power peaks. (If we instead take the mean ℓ, rather than the peak location, the ℓ value corresponding to a given k is reduced by about 15%.)

The correspondence is shown in Fig. 2; ignoring the glitches which coincide with the CMB temperature acoustic troughs, the relation is well fit by

$$\ell = 14\,000 \frac{k}{\rm Mpc^{-1}}, \qquad \ell \gg 1. \tag{7}$$

Using this, we find that the pivot scale for $r = 0$ corresponds to $\ell \simeq 60$, increasing to about 280 for the largest r values considered. We conclude that even with WMAP the majority of the constraining power on the amplitude is at fairly low ℓ, which is why the improvement with respect to COBE is limited. From a physical point of view, cosmic variance limited measurements of the Sachs-Wolfe effect, because of its microphysics independence, carry the most statistical weight in this context.

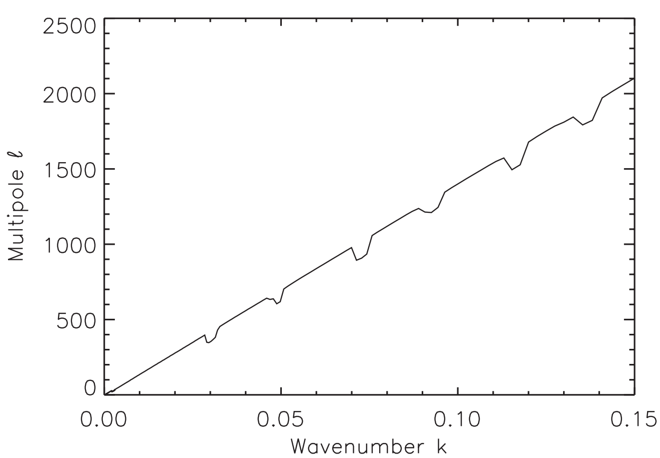

FIG. 2. The relation between multipole ℓ and wavenumber k for the best-fit cosmology to WMAP3 data [2].

On the other hand, the pivot scale for the well-determined combination $\delta_{\rm H} e^{-\tau}$ is found to be 0.033 $\rm Mpc^{-1}$ for $r = 0$, corresponding to an ℓ of about 460, more or less in the middle of the WMAP range. Therein lies the justification for currently using 0.05 $\rm Mpc^{-1}$ for the scalar pivot scale. Currently the CMB acoustic peak region, where the overall level of anisotropy can be best determined, suffers from a perfect degeneracy between the amplitude and the optical depth, broken only towards low ℓ. A better measurement of τ from CMB polarization will help break this degeneracy, allowing a better determination of $\delta_{\rm H}$ and moving its pivot scale up to higher ℓ. $\delta_{\rm H} e^{-\tau}$ is currently known to an accuracy of about 1.3%, so that external constraints on τ could lead to at most a factor two improvement in our knowledge of $\delta_{\rm H}$ with present CMB data.

In single-field slow-roll inflation models, the perturbation normalization directly gives the inflationary energy scale in terms of the slow-roll parameter ϵ or, equivalently, the tensor-to-scalar ratio. Following the usual calculation (see e.g. Refs. [4,9]), we find that the energy scale as a given mode crossed the horizon was

$$\frac{V^{1/4}}{M_{\rm Pl}} = \left(\frac{75\pi^2}{8}\right)^{1/4} r^{1/4} \delta_{\rm H}^{1/2}, \tag{8}$$

where $M_{\rm Pl} = 2.436 \times 10^{18}$ GeV is the reduced Planck mass. Substituting in the WMAP normalization of Eq. (6) gives

$$\frac{V_*^{1/4}}{1\ \rm GeV} = (3.32 \pm 0.05) \times 10^{16} \times r^{1/4} \frac{\exp[(-0.62 + 0.52r)(1 - n_{\rm S})]}{(1 + 0.53r)^{1/4}}, \tag{9}$$

where "*" indicates that this corresponds to when the scale 0.05 $\rm Mpc^{-1}$ crossed outside the horizon during inflation. This agrees with previous results, with a smaller uncertainty in the prefactor. Nevertheless, one cannot expect the other terms to be measured with anything like the precision of the prefactor, the uncertainty on the $r^{1/4}$ being likely to be the limiting term. In particular applications one should also check whether slow-roll corrections are comparable to or greater than the statistical uncertainty.

III. CONCLUSIONS

We have determined the density power spectrum normalization from WMAP, as a function of the inflationary parameters $n_{\rm S}$ and r. This WMAP normalization is the one appropriate to specific inflation models, and its statistical uncertainty improves on the COBE normalization by a factor of approximately two. Explicit examples of how to use the normalization to constrain inflation models are given in Refs. [4,9].

ACKNOWLEDGMENTS

A. R. L., P. M. and D. P. were supported by PPARC. S. M. L. acknowledges a visit to Sussex supported by PPARC. We acknowledge use of the UK National Cosmology Supercomputer (COSMOS) funded by Silicon Graphics, Intel, HEFCE and PPARC, and of the Legacy Archive for Microwave Background Data Analysis (LAMBDA). Support for LAMBDA is provided by the NASA Office of Space Science.

[1] C. L. Bennett *et al.*, Astrophys. J. Lett. **464**, L1 (1996).
[2] D. N. Spergel *et al.* (WMAP Collaboration), astro-ph/0603449; L. Page *et al.* (WMAP Collaboration), astro-ph/0603450.
[3] G. Hinshaw *et al.* (WMAP Collaboration), astro-ph/0603451; N. Jarosik *et al.* (WMAP Collaboration), astro-ph/0603452.
[4] E. F. Bunn, A. R. Liddle, and M. White, Phys. Rev. D **54**, R5917 (1996).
[5] E. F. Bunn and M. White, Astrophys. J. **480**, 6 (1997).
[6] J. Martin, A. Riazuelo, and D. J. Schwarz, Astrophys. J. Lett. **543**, L99 (2000).
[7] D. H. Lyth and D. Wands, Phys. Lett. B **524**, 5 (2002).
[8] A. Lewis and S. Bridle, Phys. Rev. D **66**, 103511 (2002); code available from http://cosmologist.info/cosmomc.
[9] A. R. Liddle and D. H. Lyth, *Cosmological Inflation and Large-Scale Structure* (Cambridge University Press, Cambridge, England, 2000).
[10] A. R. Liddle and S. M. Leach, Phys. Rev. D **68**, 103503 (2003).
[11] C. L. Bennett *et al.*, Astrophys. J. Suppl. Ser. **148**, 1 (2003).
[12] Numbers obtained from http://lambda.gsfc.nasa.gov on September 18th 2006.
[13] G. Hinshaw *et al.* (WMAP Collaboration), Astrophys. J. Suppl. Ser. **148**, 63 (2003).