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We explore the stability properties of multifield solutions of assisted inflation type, where several fields collectively evolve to the same configuration. In the case of noninteracting fields, we show that the condition for such solutions to be stable is less restrictive than that required for tracking in quintessence models. Our results, which do not rely on the slow-roll approximation, further indicate that to linear order in homogeneous perturbations the fields are in fact unaware of each other's existence. We end by generalizing our results to some cases of interacting fields and to other background solutions and dynamics, including the high-energy braneworld.

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I. INTRODUCTION

The dynamical possibilities in multifield inflationary scenarios are considerably richer than in single-field models. An example is assisted inflation [1,2], where a collection of fields evolve together, perhaps driving inflation that would not be possible if only one field were present. Configurations with multiple scalar fields have been considered in the case of exponential potentials with equal or differing slopes. Later extensions of these models include the effect of curvature and a barotropic perfect fluid [3], Bianchi backgrounds [4], and polynomial potentials [5]. The stability of solutions as critical points in phase space was studied for decoupled [2,6,7] and coupled [8–10] exponential potentials, and for decoupled inverse power-law potentials [7]. Late-time dark-energy scenarios can be found in Refs. [7,11]. Multifield scenarios naturally emerge in Kaluza-Klein compactifications [5] and in M theory, via the nonperturbative dynamics of a stack of M5-branes distributed along an $S^1/\mathbb{Z}_2$ orbifold [12]. In the latter case, the scalar fields and their exponential potentials are related, respectively, to the distances and instanton interactions between the branes. A model motivated by string theory is N-flation, where the scalar fields are axions [13].

Thus far, general conditions for the existence of classically stable assisted inflation solutions have not been given in the literature.[1] This is in contrast to the related case of tracking quintessence models, featuring a single field and a barotropic fluid, where the tracking condition was found to be [15]

$$\Gamma \equiv \frac{VV''}{V'^2} > 1 - \frac{2-\gamma}{4+2\gamma}, \tag{1}$$

where $V(\phi)$ is the scalar field potential, γ the fluid barotropic index defined by $p = (\gamma - 1)\rho$, and prime a derivative with respect to the field value ϕ. The condition is often simplified to $\Gamma > 1$, as otherwise the field energy density does not grow relative to the fluid [15].

The aim of this article is to derive those stability conditions. In the first instance we are interested in the case of n scalar fields ϕ_i each with an identical potential $V(\phi_i)$. In that case Kim *et al.* [7] showed that, provided the fields all have equal value, the system was dynamically equivalent to a single-field model with field χ and potential $W(\chi) \equiv nV(\chi/\sqrt{n})$. This generalizes the correspondence for exponential potentials derived in Ref. [1], and also shows there is no assisted inflation phenomenon for inverse power-law potentials [7]. However, while such solutions obviously exist, that paper did not address under what circumstances they are stable, which we do here. We then also consider some generalizations of this basic setup. None of our results require the Universe to accelerate, so they are valid also beyond the slow-roll approximation.

II. BACKGROUND EVOLUTION

The background equations of motion with an Einstein-Hilbert gravitational action and n Klein-Gordon homogeneous scalars read

$$H^2 = \frac{\kappa^2}{3}\left(W + \frac{1}{2}\sum_{i=0}^{n-1}\dot{\phi}_i^2\right), \tag{2}$$

$$\ddot{\phi}_i + 3H\dot{\phi}_i + \partial_i W = 0, \tag{3}$$

where H is the Hubble parameter, κ is the gravitational coupling constant, dots represent derivatives with respect to synchronous time t, $W \equiv W(\phi_0, \phi_1, \ldots, \phi_{n-1})$ is the potential with both self-interaction and interaction terms, and $\partial_i \equiv \partial/\partial\phi_i$. For simplicity, we set $\kappa^2 = 1$. The two equations can be combined to give

$$\dot{H} = -\frac{1}{2}\sum_i \dot{\phi}_i^2. \tag{4}$$

We are going to discuss the classical stability (that is, against homogeneous perturbations) of solutions of the form

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[1] Inhomogeneous perturbations of multifield configurations were considered in Ref. [14].

$$\phi_i(t) = \phi(t), \qquad i = 0, \ldots, n-1, \tag{5}$$

i.e. where the fields all evolve together. This class of solutions includes multifield fixed point solutions, e.g. as in Refs. [3,7], but is much more general—the solutions need not be of scaling type (commonly defined as the scalar field kinetic and potential energies maintaining a fixed ratio, e.g. Ref. [16]), and indeed need not even be inflationary.

We begin by determining what forms of potential support such solutions. Denoting derivatives with respect to ϕ with primes, this solution is possible only if

$$\partial_i W|_{\phi_j=\phi} = V'(\phi) \quad \forall \ i, j, \tag{6}$$

where in the left-hand side $\phi_j = \phi$ for all $j = 0, \ldots, n-1$, and V' is defined by this equation. If the fields are mutually decoupled, then each can be written with its own potential V_i obeying

$$V_i(x) + \Lambda_i = V(x) \tag{7}$$

for all i. Here the Λ_i are constants, which can all be absorbed into some $\Lambda = \sum_i \Lambda_i$ acting as a cosmological constant, so that $W = \sum_i V_i(\phi_i) + \Lambda$. We will see that this constant does not affect whether the solution is stable or not, though it can alter the qualitative behavior in the vicinity of the solution. Ordinarily we will consider Λ to be zero or negligibly small, so that all the fields have the same potential.

If one allows for a coupling between the fields, it must still satisfy Eq. (6). This happens, for instance, when the interaction term is symmetric in all the fields, so that one can write the potential as

$$W = \sum_i V_i(\phi_i) + f\left[\prod_k g(\phi_k)\right], \tag{8}$$

for some functions f and g. Also, there can be more general situations, as the following three-field system shows: $W = \phi_0^a + a(2b-a)^{-1}\phi_0^{a-b}\phi_1^b + b(2b-a)^{-1}\phi_2^a$.

III. STABILITY

Let us first consider the simple case of an expanding universe ($H > 0$) and no interaction terms, $V_i = V$. We do so using two different formalisms, the latter of which will prove the more powerful.

A. Synchronous time formalism

Perturbing the above equations of motion around the solution Eq. (5),

$$H(t) \to H(t) + \delta H(t), \qquad \phi_i(t) \to \phi(t) + \delta\phi_i(t), \tag{9}$$

and defining the $(2n+1)$-vector (T denotes transposition)

$$X \equiv (H, \phi_i, \dot{\phi}_i)^T, \tag{10}$$

the linearized equations of motion can be written in a matrix form $\delta\dot{X} = M\delta X$, where the entries $m_{ij} = (M)_{ij}$ are

$$\begin{aligned} m_{ij} &= 0, \qquad i, j = 0, \ldots, n, \\ m_{0j} &= -\dot{\phi}, \qquad j = n+1, \ldots, 2n, \\ m_{ij} &= \delta_{ij}, \qquad j = i+n = n+1, \ldots, 2n, \\ m_{i0} &= -3\dot{\phi}, \qquad i = n+1, \ldots, 2n, \\ m_{ij} &= -V''\delta_{ij}, \qquad i = j+n = n+1, \ldots, 2n, \\ m_{ij} &= -3H\delta_{ij}, \qquad i, j = n+1, \ldots, 2n, \end{aligned} \tag{11}$$

where

$$V''(\phi) \equiv \partial_i^2 W|_{\phi_j=\phi}. \tag{12}$$

The characteristic equation $\det(M - \lambda I_{2n+1}) = 0$ determines the eigenvalues $\lambda = 0$ and

$$\lambda_\pm^* = -\frac{3H}{2}\left(1 \pm \sqrt{1 - \frac{4}{9}\frac{V''}{H^2}}\right), \tag{13}$$

$$\tilde{\lambda}_\pm = -\frac{3H}{2}\left[1 \pm \sqrt{1 - \frac{4(V'' - 3n\dot{\phi}^2)}{9H^2}}\right], \tag{14}$$

where $\lambda_\pm^*$ is $n-1$ times degenerate.

The solution is stable provided no eigenvalue has a positive real part. Equation (13) imposes the simple stability condition

$$V'' \geq 0, \tag{15}$$

while the real part of $\tilde{\lambda}_-$ is negative definite only if a stronger condition is valid:

$$V'' \geq 3n\dot{\phi}^2. \tag{16}$$

In the slow-roll approximation $3H^2 \approx nV$, $-3H\dot{\phi} \approx V'$, this is nothing but the tracking condition

$$\Gamma \equiv \frac{VV''}{V'^2} \geq 1, \tag{17}$$

but note that Eq. (16) is valid also outside inflation.

This looks like a natural conclusion exactly matching the expectation inherited from the known quintessence case. However in fact the interpretation is quite different, as can be seen by considering the implications in the simple situation of the power-law potential

$$V = \phi^p, \qquad \Gamma = 1 - \frac{1}{p}. \tag{18}$$

Clearly, this potential with $p > 0$ (in particular, $p = 2$) does not satisfy the condition (16), implying Eq. (5) is not a slow-roll attractor, contradicting the known result [17] that solutions for this potential do approach radial trajectories which, by field rotation, can be described as all the fields rolling together (see also Ref. [18]).

The reason for this discrepancy is that this formalism fails to recognize that trajectories distinguished only by a shift in time, $t \to t + \delta t$, are physically equivalent and should be identified, even if the apparent perturbation *at fixed time* appears to grow. This differs from the case of a tracking quintessence model, where the existence of a barotropic fluid acts as a "clock," with reference to which time shifts in the scalar field evolution can be physically distinguished.

The time-shift mode can be removed by defining quantities which are gauge invariant with respect to the gauge group of gravitation (group of infinitesimal coordinate transformations) [19]. In the particular case of multiple scalar fields and homogeneous perturbations, this is $\delta\tilde{\phi}_i = \delta\phi_i(t) - \dot{\phi}\delta t$ [14]. Rather than writing down gauge-ready equations using the synchronous time formalism, we now employ a simpler alternative.

B. Hamilton-Jacobi formalism

In the Hamilton-Jacobi (HJ) formalism [20,21], the role of the clock is invested in a matter field, so that one can get rid of the unphysical degree of freedom represented by shifts in the time coordinate t. When many matter fields are present, one can choose one and study homogeneous perturbations on constant time slices. More specifically, we can take $\phi_0 = \phi$ as the new time coordinate and perturb only the other $n - 1$ fields $\phi_i(\phi)$. The equation of motion for ϕ is a constraint determining the relation between different time coordinates. The first HJ equation is given by Eq. (4),

$$\dot{\phi} = -\frac{2H'(\phi)}{A}, \tag{19}$$

where

$$A \equiv 1 + \sum_i \phi_i'^2; \tag{20}$$

here and in what follows i runs from 1 to $n - 1$.

Since $\partial_t^2 = \dot{\phi}^2\partial_\phi^2 - (3H\dot{\phi} + W')\partial_\phi$, one gets $n - 1$ equations of the form

$$\dot{\phi}^2\phi_i'' - W'\phi_i' + \partial_i W = 0. \tag{21}$$

The expansion of the Universe is encoded in the friction term and in $\dot{\phi}^2$ via Eq. (19). Plugging Eq. (19) in the Friedmann equation one gets

$$2H'^2 - 3AH^2 + AW = 0. \tag{22}$$

When perturbing Eqs. (21) and (22) around Eq. (5), some simplifications occur, since $\phi_i'' = 0$ and $A = n$ on the background ($\delta A = 2\sum_i \delta\phi_i'$). The final result is

$$\dot{\phi}^2\delta\phi_i'' - V'\delta\phi_i' + \sum_j(\partial_j\partial_i W - \partial_j W')\delta\phi_j = 0, \tag{23}$$

$$\frac{4}{n}H'\delta H' - 6H\delta H + \sum_i(V'\delta\phi_i - \dot{\phi}^2\delta\phi_i') = 0, \tag{24}$$

where all background quantities are functions of ϕ. In general, the matrix $\partial_j\partial_i W$ is symmetric with all different entries.

When the fields are decoupled, $\partial_j\partial_i W(\phi) = V''(\phi)\delta_{ij}$, $\partial_j W' = 0$, and $j = i$ in Eq. (23). For simplicity we shall concentrate on this case. Again, the system can be written as $\delta X' = M\delta X$, where X is now a $(2n - 1)$-vector and the coefficients of the matrix M are given by Eqs. (23) and (24). Define the objects

$$\lambda_0 = -\frac{3H}{\dot{\phi}}, \tag{25}$$

$$\lambda_\pm = \frac{V'}{2\dot{\phi}^2}\left(1 \pm \sqrt{1 - \frac{4\dot{\phi}^2 V''}{V'^2}}\right). \tag{26}$$

The eigenvalues of the decoupled system are λ_0 and the $(n - 1)$-times degenerate pair $\lambda_\pm$. Concerning the first eigenvalue, it is normally convenient to take $\dot{\phi} > 0$ (arranged if necessary by sending $\phi \to -\phi$) so that increasing ϕ corresponds to increasing time. The first eigenvalue is then negative definite, as required for stability. If $\dot{\phi}$ were negative, then the two time choices would flow in opposite directions and the stability condition would become a positive-definite eigenvalue, the physics being unchanged. Therefore Eq. (25) holds in either case.

With the convention $\dot{\phi} > 0$ one has that $V' < 0$, as the situations of physical interest always have the field rolling down the potential and hence $\dot{\phi}$ should have the opposite sign to V'. If $V'' > 0$ [Eq. (15)] holds there are two decaying modes related to $\lambda_\pm$ (with the equality, one direction is flat) and Eq. (5) is a classical attractor. If $V'' > V'^2/4\dot{\phi}^2$, the perturbations periodically overshoot the attractor.

Notice that

$$\dot{\phi}^2\lambda_+\lambda_- = \lambda_+^*\lambda_-^*, \tag{27}$$

meaning that the perturbations of the ith field ($i \neq 0$) are related in both formalisms. However, they are not equal except in the slow-roll regime. To show this, let us denote as $\delta\phi_i^{(t)}$ the perturbation of the ith field in the t-formalism. Using the diagonalized linear equations and Eq. (27), one finds

$$\frac{\delta\ddot{\phi}_i^{(t)}}{\delta\phi_i^{(t)}} - \frac{\delta\ddot{\phi}_i}{\delta\phi_i} = -\frac{\ddot{\phi}}{\dot{\phi}}\frac{\delta\dot{\phi}_i}{\delta\phi_i}. \tag{28}$$

The right-hand side is nonvanishing because $\delta\phi_i^{(t)}$ is not gauge invariant. However, it is negligible in the slow-roll approximation. This is expected since, in this case, the eigenvalues would be almost constant and perturbations would be exponential: $\delta\phi_i \sim e^{\lambda_\pm^* t}$ in the t-formalism, while $\delta\phi_i \sim e^{\lambda_\pm \phi}$ in the HJ formalism. Then the eigenvalues would be individually related, $\lambda^*_\pm \approx \dot{\phi}\lambda_\pm$.

C. Interpretation

The conclusion from these calculations is that the stability condition for multifield tracking in identical potentials is simply $V'' > 0$, rather than a condition similar to the quintessence tracking condition. The interpretation is in fact straightforward, and most easily seen by studying Eq. (23) for the perturbed scalar field evolution. This equation is *independent* of the perturbed metric δH. Accordingly, a particular field perturbation is insensitive, at linear order, to the perturbation in the metric caused by any other field perturbation. The fields evolve independently of each other. Whether or not the field perturbation dies away, i.e. the fields approach one another, simply depends on whether those fields which happen to be further down the potential are evolving more slowly, which is the case provided the potential is less steep there. Assisted inflation does take place, but not because the fields are actually aware of each other's value and are drawn towards one another, but rather just as a generic property of trajectories on convex potentials.

Multifield systems which do not satisfy Eq. (15) will never attain the configuration equation (5). On the other hand, Eq. (15) guarantees the stability of the equal-field solution Eq. (5) to small perturbations, but does not tell us whether such a solution acts as a *global* attractor for the system. In the simple case of shallow exponential potentials, this global attractor property is known to hold [3]. However, it is easy to find counterexamples. Consider the case of steep exponential potentials $V_i(\phi_i) \propto \exp(\lambda\phi_i)$ for $\lambda^2 > 6$. These potentials are so steep that there are no scaling solutions, as the potential energy falls off faster than the kinetic energy. Our particular solution is stable to small perturbations ($V'' = \lambda^2 V > 0$), but if the fields are initially set well apart the rapid decay of their velocity will prevent them asymptoting to equal field values.

Note also that our condition $V'' > 0$ is a local stability condition referring to the position on the potential that the fields happen to be at during a given epoch. Some potentials may have V'' with the same sign everywhere, and the stability or otherwise is then a global property of the potential. But potentials where V'' changes sign may experience sequences of epochs where the fields alternately converge or diverge from each other, the ultimate stability being determined by the sign of V'' after its final sign change.

We mention briefly the effect of lifting the assumption $\Lambda = 0$, which changes the value of H corresponding to a given location of the scalar fields (we assume we stay on the expanding branch). Looking at the eigenvalue equation (13), we see that changing the value of H can modify (or create) an imaginary part to the eigenvalues but can change the real part only by an overall multiplier, leaving its sign intact. Hence introducing Λ can change the type of attractor (oscillatory or monotonic), but not whether the potential gives stable solutions or not. A positive Λ can also impact on whether or not the solution Eq. (5) is a global attractor, in the sense specified above. Such cosmologies will be asymptotically de Sitter, with rapidly decaying field velocities that may prevent approach to the equal-field solution.

The impact of Λ is somewhat less transparent in the Hamilton-Jacobi formalism, Eq. (26), where at first sight $\lambda_\pm$ seems independent of H even though we have verified that within slow-roll $\lambda^*_\pm \approx \dot{\phi}\lambda_\pm$. The resolution is that at a given location on the potential, the introduction of Λ will modify the value of $\dot{\phi}$ in Eq. (26), again potentially introducing or modifying an imaginary part to the eigenvalue.

One might ask how our conclusions would change if a fluid component with equation of state $p = (\gamma - 1)\rho$ were added, to give a multifield quintessence scenario. Adding a new component changes the nature of the dynamical system, and the new degree of freedom may well induce an instability, even if the fluid component is taken as initially subdominant. E.g., for sufficiently steep exponential potentials the late-time global attractor has the field scaling with the fluid [3]. Nevertheless, one would expect that within the field sector the stability condition remains the same, i.e. provided $V'' > 0$, the stable solution will still maintain equal-field values. It is easy to see that this is true if $\rho = 0$ in the background solution, provided $\gamma > 0$ (the extra degree of freedom would decouple from the scalar fields with eigenvalue $\propto -\gamma H/\dot{\phi}$).

Notice also that when $\gamma = 2$ (stiff matter, $p = \rho$) Eq. (1) reduces to the pseudotracking condition Eq. (17) found in synchronous time for slowly rolling fields. Stiff matter decays as $\rho \sim a^{-6}$, this being the fastest rate at which a scalar field density can be diluted (kinetic regime, $\dot{\phi}^2 \gg V$). Since slow rolling prevents this condition, for all purposes the fluid is negligible relative to the scalars, which is precisely the configuration leading to Eqs. (16) and (17). However, this implies that one cannot use the barotropic fluid as a reliable clock throughout the whole evolution of the system, and the calculation in synchronous time becomes nonphysical. In other words, classical stability in multifield cosmologies with a barotropic fluid can be consistently studied in synchronous time for $\gamma < 2$ but not in the limit $\gamma \to 2$ (no fluid), where the HJ formalism is more appropriate.

IV. GENERALIZATIONS

A. With field couplings

The next simplest case is that of a symmetric interaction, Eq. (8), so that the nondiagonal entries of $\partial_j\partial_i W$ are all equal, and we can define the useful quantity

$$y(\phi) \equiv \partial_j\partial_i W = \partial_j\partial_i f, \qquad j \neq i. \tag{29}$$

Defining

$$\lambda_{\pm}^{(n)} = \frac{V'}{2\dot{\phi}^2}\left[1 \pm \sqrt{1 - 4\dot{\phi}^2 \frac{V'' + (n-2)y}{V'^2}}\right], \qquad (30)$$

one can show that the HJ eigenvalues are λ_0 and the $(n-1)$-times degenerate pair $\lambda_{\pm}^{(1)}$. Stability is achieved if

$$V'' \geq y. \qquad (31)$$

We can also consider the case where the coupling is symmetric in all the fields except one which remains uncoupled ($\partial_j W' = 0$). It is natural to choose the decoupled field as the clock field $\phi = \phi_0$ (though one would be free to choose another). Then the eigenvalues are λ_0, $\lambda_{\pm}^{(n)}$, and the $(n-2)$-times degenerate pair $\lambda_{\pm}^{(1)}$, while the stability condition is

$$V'' \geq \max\{y, -(n-2)y\}. \qquad (32)$$

With interactions included, stability now requires steeper self-interaction potentials, which is a confirmation of the claim made in Refs. [4,5] that multifield inflation with cross couplings is less likely to happen. If $y < 0$, the larger the number of fields the steeper the coupling term. As an application of Eq. (32), consider n scalars with potentials $V_0 = C\phi^p$, $V_i = \phi_i^p$, and power-law interaction $f = B(\phi_1 \cdots \phi_{n-1})^{p/(n-1)}$, where B, p, and $C = 1 + B/(n-1)$ are constants. Let also $B, p > 0$. Then $y = Bp^2\phi^{p-2}/(n-1)^2$ and the solution is stable if $p \geq C$.

If the cross coupling is asymmetric, the symmetric matrix $\partial_j\partial_i W(\phi)$ and the eigenvalues of the linearized equation are more complicated; however, for a given model it is straightforward to check its classical stability via Eqs. (23) and (24).

B. More general backgrounds and the braneworld

We can generalize all the above results to other background equations and solutions. It is instructive to consider background solutions of the form

$$\phi_i(t) = \alpha_i \phi^{\beta_i}(t), \qquad (33)$$

for some real α_i and β_i. Equation (33) includes the solution Eq. (16) ($\alpha_i = 1 = \beta_i$), scaling solutions [1,2], the power-law inflationary attractor studied in Ref. [7], and other situations where, for instance, some fields are subdominant with respect to others ($\alpha_i \approx 0$ for some i). The condition on the potentials for Eq. (33) to be a solution is

$$\partial_i W|_{\phi_j=\phi} = \alpha_i\beta_i\phi^{\beta_i-1}\left[V'(\phi) + (1-\beta_i)\frac{\dot{\phi}^2}{\phi}\right]. \qquad (34)$$

Again, V' is defined to be the potential of ϕ_0, differentiated with respect to ϕ_0, where all fields are set to be equal to ϕ_0; in other words, Eq. (6) with $i = 0$. This condition is fulfilled, e.g., by slowly rolling fields with different polynomial potentials. On this background, $A(\phi) = 1 + \sum_j(\alpha_j\beta_j)^2\phi^{2(\beta_j-1)}$ in Eq. (19).

In high-energy braneworld scenarios inflation can be sustained by steeper potentials [22] and it would be interesting to see how the stability conditions are affected. Neglecting brane-bulk energy exchange and the contribution of the Weyl tensor, the Klein-Gordon equation (3) [as well as Eq. (34)] is unchanged, while the effective Friedmann equation on the brane at high energies reads

$$H^{2-\theta} = \frac{\kappa_\theta^2}{3}\rho. \qquad (35)$$

Here, $\theta = 0$ in the pure 4D regime, $\theta = 1$ in the high-energy Randall-Sundrum braneworld, and $\theta = -1$ in the high-energy Gauss-Bonnet scenario; κ_θ is the effective gravitational coupling felt by the matter (with energy density ρ) on the brane. See Refs. [23] for a review of this formalism and references to the original results.

We continue to work with the more general background solution given by Eq. (33). In units $\kappa_\theta^2 = 1 - \theta/2$, the HJ equations become

$$\dot{\phi} = -\frac{2H'}{H^\theta A}, \qquad (36)$$

$$H'^2 = \frac{AH^{2\theta}}{2}\left[\frac{6H^{2-\theta}}{2-\theta} - W\right]. \qquad (37)$$

Linearizing Eq. (37) one gets

$$\delta H' = -\frac{H}{\dot{\phi}}(3 + \theta\epsilon)\delta H + \frac{H^\theta}{4\dot{\phi}}\left(2\sum_i \partial_i W\delta\phi_i - \dot{\phi}^2\delta A\right), \qquad (38)$$

with

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{A\dot{\phi}^2}{2H^{2-\theta}}, \qquad (39)$$

and

$$\delta A = 2\sum_i \alpha_i\beta_i\phi^{\beta_i-1}\delta\phi_i'. \qquad (40)$$

Also, since $\phi_i'' = \alpha_i\beta_i(\beta_i-1)\phi^{\beta_i-2} \neq 0$, the perturbed equation (21) now contains the metric perturbation:

$$\delta\phi_i'' = -\phi_i''\frac{12H^{1-\theta}}{A\dot{\phi}^2}\delta H + \sum_j\left(\frac{V'}{\dot{\phi}^2}\delta_{ij} + \frac{2\phi_i''\phi_j'}{A}\right)\delta\phi_j' + \frac{1}{\dot{\phi}^2}\sum_j\left(\frac{2\phi_i''}{A}\partial_j W - \partial_j\partial_i W + \phi_i'\partial_j W'\right)\delta\phi_j. \qquad (41)$$

The eigenvalue equations can be solved numerically or analytically for particular cases. A simple possibility is $\beta_i = 1$ for all i, whereupon $\phi_i'' = 0$ and many of the terms in Eq. (41) vanish. The background condition on the total potential becomes $\partial_i W = \alpha_i V'$, and the perturbation of the Hubble parameter decouples from the scalar ones. In this case, the form of the effective Friedmann equation affects only Eq. (38). The eigenvalue λ_0 is positive definite only if [see Eq. (38)]

$$-\theta\epsilon \leq 3, \tag{42}$$

which is automatically satisfied for $\theta \geq 0$ (4D and Randall-Sundrum braneworld). In the Gauss-Bonnet scenario, it must be $\epsilon \leq 3$, which is true during accelerating expansion.

When there is no coupling between fields ($\partial_j \partial_i W = 0$), the other stability condition Eq. (15) becomes

$$\alpha_i V'' \geq 0. \tag{43}$$

For quadratic potentials this states that none of the scalars is tachyonic. In the presence of interactions and for $\alpha_i \neq \alpha_j$, the degeneracy of the eigenvalues is broken and more involved relations arise.

V. CONCLUSIONS

We have derived a set of stability conditions against classical perturbations for multifield cosmological solutions. These can be summarized as follows.

(i) For the solution equation (5), if the fields are noninteracting the only condition is $V'' \geq 0$, while for symmetric interactions with positive coupling the self-interacting potentials must be steeper.

(ii) For the solution equation (33) with $\beta_i = 1$ and no cross coupling, the stability condition reads $\alpha_i V'' \geq 0$ for all i.

(iii) These results are valid for both four-dimensional cosmology and high-energy Randall-Sundrum braneworld. On a Gauss-Bonnet braneworld, the extra condition $\epsilon \leq 3$ is required.

(iv) In all other cases, the linearized dynamical system can be studied numerically via Eqs. (38) and (41).

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