

## Ising exponents from the functional renormalisation group

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We study the 3d Ising universality class using the functional renormalization group. With the help of background fields and a derivative expansion up to fourth order we compute the leading index, the subleading symmetric and antisymmetric corrections to scaling, the anomalous dimension, the scaling solution, and the eigenperturbations at criticality. We also study the cross correlations of scaling exponents, their dependence on dimensionality, and the numerical convergence of the derivative expansion. Collecting all available data from functional renormalization group studies to date, we estimate that systematic errors are in good agreement with findings from Monte Carlo simulations,  $\epsilon$ -expansion techniques, and resummed perturbation theory.

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**I. INTRODUCTION**

Continuous phase transitions of numerous systems in statistical and particle physics belong to the Ising universality class, characterized by the short range nature of the interaction, a scalar order parameter, and the dimension. The absence of a physical length scale at the phase transition implies scale invariance. Many fluids, magnets, or particle physics models thus share the same behavior at criticality described by universal numbers such as the scaling exponent for the correlation length  $\nu$ , its subleading correction  $\omega$ , and the anomalous dimension of the order parameter at criticality  $\eta$ . Further critical exponents, for example the specific heat  $\alpha$ , the spontaneous magnetisation  $\beta$ , the magnetic susceptibility  $\gamma$ , and the magnetization at criticality as a function of the external field  $\delta$ , are linked to  $\nu$  and  $\eta$  by scaling relations [1].

The computation of universal indices—in a quantum field theoretical or statistical physics setting—has become a benchmark test for perturbative and nonperturbative methods in field theory. Accurate predictions for exponents, scaling functions or amplitude ratios are available based on the renormalization group, resummations of perturbation theory, and numerical simulations on the lattice (see [2] for an overview). An important continuum method in the above is the functional renormalization group [3,4], based on the infinitesimal integrating-out of momentum modes from a path integral representation of the theory with the help of a Wilsonian momentum cutoff [5]. By construction, functional flow equations continuously interpolate between the microscopic theory at short distances and the full quantum effective theory at large distances. Powerful optimization techniques are available to maximize the physics content in given approximation, and to minimize cutoff artifacts along the flow [6–10]. A particular strength of the functional renormalization group is its flexibility, allowing for perturbative as well as nonperturbative

approximations even in the presence of strong correlations or couplings [10–15].

Fixed point studies for Ising-like theories have been performed within various realizations of the functional renormalization group including Polchinski's formulation [11], Wetterich's equation [12], exact background field flows [16–19], the proper-time approximation [20–24], and discretized (hierarchical) transformations [25,26]. The derivative expansion [27,28] and variations thereof [12,29] are the expansion schemes of choice based on a small anomalous dimension. Scaling behavior in more complicated theories, e.g. thermal field theory [30], gauge theories [31–33] and gravity [34], can equally be accessed using thermal or gauge-covariant derivative and vertex expansions.

In this paper, we study the Wilson-Fisher fixed point in three dimensions within a background field formulation. In the past, background field methods have mostly been employed for gauge theories and gravity, where they allow for a gauge-invariant implementation of the cutoff [14,35,36], also offering new expansion schemes [37]. The motivation for using this technique for nongauge systems is twofold. First, the presence of a background field allows for a reorganization of the flow equation. While this is of no relevance for the full flow, it does make a difference once approximations are invoked. In particular, derivative expansions of standard and background field flows are different. This allows for complementary measurements of universal scaling exponents. Second, background field flows have provided very good numerical results to lower orders in the derivative expansion. Therefore, it is important to understand whether this pattern carries over to higher order.

In addition, we discuss numerical evidence for the convergence of the derivative expansion and provide an estimate for systematic uncertainties. Error estimates are obtained by probing the dependence on the shape of the Wilsonian cutoff function [38,39]—which vanishes for the

physical theory and hence should become small with increasing order in the expansion—and by checking the numerical convergence of successive orders, which we extend up to fourth order. We estimate the systematic error by comparing different projections of the Wilson-Fisher fixed point onto the flow equation, using a weighted average over the available data. We find a coherent picture and good agreement with the mean values and error estimates from Monte Carlo and perturbative studies. We also evaluate the cross dependences of scaling exponents, and find an interesting link between the expected error in an observable and its sensitivity to tiny variations of the dimensionality.

The format of the paper is as follows. We recall the basic setup (Sec. II) and our approximations (Sec. III), followed by a discussion of results (Sec. IV) and their optimization (Sec. V). Two sections deal with the dependence on dimensionality (Sec. VI) and the cross correlation of exponents (Sec. VII). We evaluate the convergence of the derivative expansion (Sec. VIII) as well as systematic uncertainties (Sec. IX), and close with a brief discussion (Sec. X).

## II. RENORMALIZATION GROUP

Wilson's renormalization group (RG) is based on the integrating-out of momentum degrees of freedom from a path integral representation of the theory. Modern, functional, implementations of this idea employ an infrared momentum cutoff term  $\Delta S_k = \frac{1}{2} \int_q \varphi(q) R_k(q) \varphi(-q)$  for the propagating modes  $\varphi(q)$ , added to the Schwinger functional with classical action  $S$  and external current  $J$ ,

$$\ln Z_k[J] = \ln \int [d\varphi] \exp\left(-S - \Delta S_k + \int \varphi \cdot J\right). \quad (1)$$

The cutoff function  $R_k(q)$  can be viewed as a momentum-dependent mass term with  $k$  denoting the RG momentum scale. It obeys  $R_k(q^2) \rightarrow 0$  for  $k^2/q^2 \rightarrow 0$  to ensure that the large momentum modes  $q^2 \gtrsim k^2$  can propagate freely, and  $R_k(q^2) > 0$  for  $q^2/k^2 \rightarrow 0$  which ensures that the low momentum modes  $q^2 \lesssim k^2$  are suppressed in the functional integral. This makes  $R_k$  an infrared cutoff. The change of (1) with the RG scale  $k$  ( $t = \ln k$ ) reads  $\partial_t Z_k = -\langle \partial_t \Delta S_k \rangle_J$ . In terms of the effective action  $\Gamma_k[\phi] = \sup_J (-\ln Z_k[J] + \phi \cdot J) + \Delta S_k$  it is given by Wetterich's flow equation [4]

$$\partial_t \Gamma_k[\phi] = \frac{1}{2} \text{Tr} \frac{1}{\Gamma_k^{(2)}[\phi] + R_k}, \quad (2)$$

an exact, functional differential equation which links the scale dependence of  $\Gamma_k[\phi]$  with its second functional derivative  $\Gamma_k^{(2)}[\phi] \equiv \frac{\delta^2 \Gamma_k[\phi]}{\delta \phi \delta \phi}$  and (the scale dependence of) the regulator function  $R_k$ . The trace denotes a momentum integration, and  $\phi = \langle \varphi \rangle_J$  denotes the expectation value of the field  $\varphi$  at fixed external current  $J$ . By construction, the flow (2) interpolates between an initial

microscopic action  $\Gamma_\Lambda \approx S$  at  $k = \Lambda$  and the full quantum effective action at  $k = 0$ .

Next, we discuss background field flows following [16–19] where a nonpropagating background field  $\bar{\phi}$  is introduced into the effective action  $\Gamma_k[\phi] \rightarrow \Gamma_k[\phi, \bar{\phi}]$  by coupling the fluctuation field  $(\phi - \bar{\phi})$  to the regulator and the external current. For the derivation of the flow, the background field acts as a spectator, and we obtain (2) with the replacement  $\Gamma_k^{(2)}[\phi] \rightarrow \frac{\delta^2 \Gamma_k[\phi, \bar{\phi}]}{\delta \phi \delta \phi}$ . The background field dependence of  $\Gamma_k[\phi, \bar{\phi}]$  is governed by

$$\frac{\delta}{\delta \bar{\phi}} \Gamma_k[\phi, \bar{\phi}] = \frac{1}{2} \text{Tr} \frac{1}{\Gamma_k^{(2)}[\phi, \bar{\phi}] + R_k} \frac{\delta R_k[\bar{\phi}]}{\delta \bar{\phi}} \quad (3)$$

and vanishes in the infrared limit  $k \rightarrow 0$  where  $R \rightarrow 0$ . Subsequently, the background field will be identified with the physical mean  $\bar{\phi} = \phi$ , leading to a background field flow for an effective action  $\Gamma_k[\phi] \equiv \Gamma_k[\phi, \phi]$ . This technique is standard practice in the study of gauge theories and gravity leading to gauge-invariant flows within the background field method. In gauge and nongauge theories, this procedure can simplify the evaluation of the operator trace in (2), which makes it attractive for our purposes [18,24]. The key difference between standard and background fields stems from the fact that the presence of the background field, at an intermediate stage of the computation, corresponds to a reorganization of the flow. This aspect is exploited below.

To be specific, we introduce the background field by substituting  $q^2 \rightarrow \Gamma_k^{(2)}[\bar{\phi}, \bar{\phi}](q^2)$  in the regulator function  $R_k(q^2)$  (other choices such as  $R_k \rightarrow R_k[\bar{\phi}]$  can be used as well). For some class of  $R_k$ -functions [18], the flow (2) takes a very convenient form,

$$\partial_t \Gamma_k = \text{Tr} \left( \frac{k^2}{\Gamma_k^{(2)}/m + k^2} \right)^m + \mathcal{O}(\partial_t \Gamma_k^{(2)}). \quad (4)$$

Here,  $m \in [1, \infty]$  parametrizes a remaining freedom in the choice for the cutoff function which we fix later. The term  $\propto \partial_t \Gamma_k^{(2)}$  originates from the implicit  $k$ -dependence introduced in  $R_k$  via  $\Gamma_k^{(2)}$ , and reflects the reorganization of the flow through background fields. The term  $\partial_t \Gamma_k^{(2)}$  on the right-hand side (rhs) of (4) can be replaced through a series—starting off with the leading term in (4) and functional derivatives thereof—by making repeated use of (4). Closed forms for  $\partial_t \Gamma_k$ , or the flow for a few relevant couplings are available under certain approximations [9,18,32]. Below, we need the flow for several field-dependent functions and therefore limit ourselves to the leading term [17,18].

The first term on the rhs of (4), or linear combinations for various  $m$ , is equivalent to Liao's proper-time flow equation [17,19,20]

$$\partial_t \Gamma_k = -\frac{1}{2} \text{Tr} \int_0^\infty \frac{ds}{s} (\partial_t f_k) \exp\left(-s \frac{\delta^2 \Gamma_k}{\delta \phi \delta \phi}\right), \quad (5)$$

originally derived from a proper-time regularization of the one-loop effective action. Equations (4) and (5) are linked via  $f_k \equiv f_{\text{PT}}(s\Lambda^2) - f_{\text{PT}}(sk^2)$  with  $f_{\text{PT}}(x) = \Gamma(m, x)/\Gamma(m)$ . The flow (4) can equally be obtained from generalized Callan-Symanzik flows [17] without the necessity for background fields. The flow equation (4) in the approximation (5) has previously been used for studies of phase transitions [16,21–24], tunneling phenomena [40], spontaneous and chiral symmetry breaking [41–43], gravity [44], and a general proof of convexity [19]. Here, we will use it to analyze the infrared scaling at the Wilson-Fisher fixed point to fourth order in the derivative expansion.

### III. APPROXIMATIONS

In this section, we detail our ansatz for the effective action for a real scalar field based on the derivative expansion and the relevant renormalization group equations. With up to fourth-order derivative operators, the effective action reads

$$\Gamma_k = \int d^D x \left[ V_k(\phi) + \frac{1}{2} Z_k(\phi) \partial_\mu \phi \partial_\mu \phi + W_k(\phi) (\partial^2 \phi)^2 \right]. \quad (6)$$

The ansatz (6) should capture the relevant infrared physics provided the anomalous dimension of the fields stay small. Note that the derivative expansion has no small parameter directly associated with it, because the integrand of (2) receives dominant contributions for  $q^2/k^2 \lesssim 1$  [28]. In addition, a proof of convergence is not available. Hence, the numerical convergence has to be checked *a posteriori*.

The three functions  $V$ ,  $Z$  and  $W$  in our ansatz (6) are symmetric under reflection in field space  $\phi \leftrightarrow -\phi$ . In principle, there are three independent tensor structures available to fourth order in the derivative expansion,

$$W_k(\phi) (\partial^2 \phi)^2, \quad H_k(\phi) \partial_\mu \phi \partial_\mu \phi (\partial^2 \phi), \quad J_k(\phi) (\partial_\mu \phi \partial_\mu \phi)^2 \quad (7)$$

with  $J$  ( $H$ ) symmetric (antisymmetric) under reflection in field space. In the free theory limit, the operators (7) scale identically. At an interacting fixed point this degeneracy is lifted, and the higher derivative operators contribute with different strengths to the flow (4). We expect that the term  $\sim W$  is the most relevant one, for reasons detailed in Sec. IV. Therefore we neglect  $H$  and  $J$ . Then the initial conditions for the flow at momentum scale  $k = \Lambda$  are

$$V_\Lambda(\phi) = \frac{1}{2} m_\Lambda^2 \phi^2 + \frac{1}{4} \lambda_\Lambda \phi^4, \quad Z_\Lambda(\phi) = 1, \quad W_\Lambda(\phi) = 0. \quad (8)$$

For  $k < \Lambda$ , higher-order couplings are switched on due to the renormalization group running (4), and the functions

$V$ ,  $Z$  and  $W$  develop a nontrivial field dependence. The Wilson-Fisher scaling solution for  $k \rightarrow 0$  corresponds to critical initial conditions  $m_{\Lambda,c}^2$  and  $\lambda_{\Lambda,c}$ . The renormalization group equations for the functions  $V$ ,  $Z$  and  $W$  are obtained by inserting (6) into (5) and expanding the exponential by making use of the Baker-Campbell-Hausdorff formula. The partial differential equations for  $V$ ,  $Z$  and  $W$  are of the form

$$\partial_t X = -\frac{1}{2} \int_0^\infty \frac{ds}{s} \int \frac{d^D p}{(2\pi)^D} (\partial_t f_k) e^{-sA_0} K_X \quad (9)$$

where  $X = V, Z$  or  $W$ , and  $A_0 = V'' + Zp^2 + 2Wp^4$ , with primes on functions denoting derivatives with respect to (w.r.t.) the fields. The equations (9) encode the central physics of our setup. The kernels  $K_X$  encode the interactions amongst the operators in the ansatz (6) under the renormalization group. We have  $K_V = 1$ . The kernels  $K_Z$  ( $K_W$ ) are polynomials in the loop momentum variable  $p$  up to order  $p^{14}$  ( $p^{20}$ ) with coefficient functions depending polynomially on  $V, Z, W$  and their derivatives, and the proper time-integration parameter  $s$ . The expressions are very long and not given explicitly.

For a fixed point study, it is convenient to introduce dimensionless variables and a specific cutoff. We use the parameter  $1/m = 0$  which is equivalent to the step function  $f_{\text{PT}}(y) = \theta(1 - y)$  with  $y = sk^2$ , achieved as  $f_{\text{PT}}(y) = \lim_{m \rightarrow \infty} \Gamma(m, my)/\Gamma(m)$  [16] (see also [45]). For large  $\Gamma_k^{(2)} \gg k^2$ , the flow then becomes exponentially suppressed  $\propto \exp(-\Gamma_k^{(2)}/k^2)$ , rather than algebraically. Therefore we expect that the case  $1/m = 0$  gives a fast convergence of the derivative expansion, as also suggested by previous numerical estimates of the critical indices to lower orders in the expansion [22–24]. We come back to the optimization w.r.t. the parameter  $m$  in Sec. V. After taking  $1/m = 0$ , the remaining  $s$ -integration in (9) is performed analytically. We introduce

$$\begin{aligned} t &= \ln(k/\Lambda), & x &= k^{1-D/2-\eta/2} \phi, & \hat{p} &= p/k, \\ a_0 &= A_0 k^{-2+\eta} & v(x) &= k^{-D} V(\phi), & z(x) &= k^\eta Z(\phi), \\ & & w(x) &= k^{2+\eta} W(\phi), & & \end{aligned} \quad (10)$$

where we have rescaled dimensionful variables in units of  $k$ . In particular the relation between  $Z(\phi)$ , defined in (6), and its dimensionless counterpart  $z(x)$  defines the field and the anomalous dimension  $\eta$ . It is understood that  $v, z$  and  $w$  are functions of  $t$  and  $x$ . Below, we denote derivatives w.r.t.  $x$  as e.g.  $\partial_x v \equiv v'$ . In the parametrization (12), the explicit  $k$ -dependence of the differential equations (9) is factored into the variables. We finally obtain

$$\begin{aligned} \partial_t Y + D_Y Y - D_x x Y' &= \int \frac{d^D \hat{p}}{(2\pi)^D} e^{-(a_0/z)} K_Y \\ \text{with } Y &= \{v', z, w\} \end{aligned} \quad (11)$$

The terms on the left-hand sides display the canonical and anomalous scaling of the fields  $D_x = [\phi]$  and the variables  $D_Y$ , with

$$\begin{aligned} D_x &= \frac{1}{2}(D - 2 + \eta), & D_{v'} &= \frac{1}{2}(D + 2 - \eta), \\ D_z &= -\eta, & D_w &= -(2 + \eta). \end{aligned} \quad (12)$$

We note that the scaling dimensions  $D_x$  and  $D_{v'}$  ( $D_z$  and  $D_w$ ) are positive (negative) for  $\eta \geq 0$  and  $D \geq 2$ . The terms on the right-hand sides parametrize the nontrivial interactions induced by (6) under the renormalization group. The integral kernels  $K_Y$  are related to the kernels  $K_X|_{s=1}$  in (9) via the relations  $K_{v'} = -\partial_x(a_0/z)$ ,  $K_z = K_z k^{D+\eta}$ , and  $K_w = K_w k^{D+2+\eta}$ .

#### IV. RESULTS

In this section, we analyze the physics of (11) at the Wilson-Fisher fixed point, which, in  $D = 3$  dimensions, corresponds to the unique nontrivial solution  $Y_*(x) \neq 0$  of

$$\partial_t Y_* = 0. \quad (13)$$

In the limit of large  $x \gg 1$ , the rhs of (11) is exponentially suppressed and the fixed point solution is dominated by the scaling of the fields and variables,

$$Y_*(x) \propto x^{D_Y/D_x}. \quad (14)$$

Consequently, the solutions  $z_*(x)$  and  $w_*(x)$  vanish asymptotically because  $D_z$  and  $D_w$  are  $< 0$ . The algebraic suppression is the more pronounced, the larger  $-D_Y$ . For  $v'$ , we find a rising behavior for large  $x$  because  $D_{v'} > 0$ . For small  $x \lesssim 1$ , the interaction terms become relevant, and the complexity of the equations makes it necessary to use numerical methods. Here, we solve (11) with (13) for  $v'_*(x)$ ,  $z_*(x)$  and  $w_*(x)$  without making any further expansions such as e.g. polynomial expansions.

The results are displayed in Fig. 1. Including the wave function renormalization  $z_*(x)$ , the first derivative of the potential  $v'_*(x)$  changes only mildly from the local potential approximation result. The inclusion of  $w_*(x)$  leaves  $v'_*(x)$  practically unchanged, whereas the wave function renormalization  $z_*(x)$  increases mildly, though only for larger  $x$ . We note that  $w_*(x)$  is very small and negative for small  $x$ , enhancing its impact for smaller  $x$ . Some characteristic values of the scaling solution are given in Table I. Including second (fourth) order operators, the vacuum expectation value changes approximately by 1% (0.1%), the curvature  $v''$  by 10% (1.5%), and the wave function renormalization by 5% (0.5%).

We point out that the quantitative relevancy of operators in the effective action correlates with their scaling dimension. This is already visible from the results to lower orders in the derivative expansion. We have  $D_{v'} > 0 > D_z > D_w$ , which materializes at the fixed point (13) as variations in the scaling solutions  $v'_*$ ,  $z_*$  and  $w_*$  of order 1,  $10^{-1}$ , and

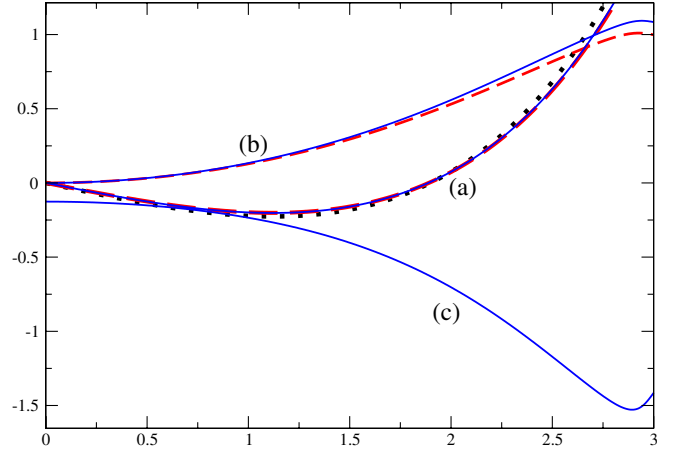


FIG. 1 (color online). Wilson-Fisher fixed point in  $D = 3$  for (a) the first derivative of the potential  $v'_*(x)$ , (b) the deviation of the wave function renormalization from its classical value  $10(z_*(x) - 1)$ , and (c) the four-derivative operator  $10^3 w_*(x)$ . Coding: local potential approximation (black, dotted curve), 2nd order derivative expansion (red, dashed curves), 4th order derivative expansion (blue, continuous curves).

$10^{-3}$ , respectively, (see Fig. 1). Quantitatively, this can be understood as follows. The exponential suppression of terms on the right-hand side of (11) for the large-field variable  $x$  implies that the large-field behavior of operators is solely determined by their mass dimension, see (14). The transition from small-field behavior to large-field asymptotics is exponentially strong, thereby restricting the impact of higher derivative operators to the small-field regime. The same observation applies to the variation of the fixed point solution and for the eigenperturbations at criticality under successive extensions from  $V(\phi) \rightarrow V(\phi)$ ,  $Z(\phi) \rightarrow V(\phi)$ ,  $Z(\phi)$ ,  $W(\phi)$ .

Next, we comment on the approximation (6). A full  $\mathcal{O}(\partial^4)$  order calculation in the derivative expansion of the effective action requires the inclusion of the terms  $H$  and  $J$ , see (7). Close to the free field theory limit, the three terms in (7) scale identically, but this degeneracy is lifted at a nontrivial fixed point solution. The mass dimensions of  $W$ ,  $H$  and  $J$  in (7) are different, and increasingly negative, e.g.  $D_w \equiv [W] = -(2 + \eta)$ ,  $D_h \equiv [H] = -\frac{1}{2}(5 + 3\eta)$  and  $D_j \equiv [J] = -(3 + 2\eta)$  in  $D = 3$  dimensions. Hence  $0 > D_w > D_h > D_j$  and, consequently, the scaling solutions  $h_*(x)$  and  $j_*(x)$  will be suppressed compared to  $w_*(x)$ , see (14). Therefore we expect that the impact of

TABLE I. Potential minimum  $x_0$ , curvature, and other reference values of the scaling solution.

approximation	$x_0$	$v''(0)$	$v''(x_0)$	$z(x_0)$	$w(0)$
$\mathcal{O}(\partial^0)$	1.899	-0.297	0.672	1	0
$\mathcal{O}(\partial^2)$	1.889	-0.266	0.601	1.047	0
$\mathcal{O}(\partial^4)$	1.888	-0.267	0.609	1.050	$-1.2610^{-4}$

$H$  and  $J$  on scaling exponents is subleading, analogous to the pattern observed to lower orders in the derivative expansion. We also note that the suppression, in general, will depend quantitatively on the regularization. The suppression is exponential for the background field flow used here, and hence stronger than the power-law suppression observed for standard flows in a derivative expansion, see [46–48].

Small deviations from the fixed point  $\Phi_{Y,\lambda}(x) = Y(x) - Y_*(x)$  are classified according to their universal scaling exponents  $\lambda$ . In the vicinity of the fixed point the eigenperturbations obey the eigenvalue equation

$$\partial_t \Phi_{Y,\lambda} = \lambda \Phi_{Y,\lambda}. \quad (15)$$

We solve (15) using (11) and the fixed point solution (13) to find the leading and subleading eigenvalues as well as the eigensolutions. The leading eigenperturbations  $\Phi(x)$  are symmetric under  $x \leftrightarrow -x$ . The eigenvalues obey  $\lambda_0 < 0 < \lambda_1 < \lambda_2 < \dots$  with  $\lambda_0 \equiv -1/\nu$  and  $\lambda_1 \equiv \omega$  in the statistical physics literature. The eigenperturbations  $\Phi(x)$  which are antisymmetric under  $x \leftrightarrow -x$  have eigenvalues  $0 < \bar{\lambda}_1 < \bar{\lambda}_2 < \dots$ , and the smallest eigenvalue is denoted as  $\bar{\lambda}_1 \equiv \omega_5$  in the literature; see [49] for a determination of  $\omega_5$  in the local potential approximation. Our results for the eigenperturbations are given in Figs. 2 and 3. The inclusion of  $z_*(x)$  and  $w_*(x)$  changes the eigenperturbations with eigenvalue  $\nu$  ( $\omega$ ) only mildly from the local potential approximation. For the scaling exponents and the anomalous dimension, we find

$$\nu = 0.6247, \quad \eta = 0.0313, \quad \omega = 0.865, \quad \omega_5 = 2.563. \quad (16)$$

The numerical precision for  $\omega$  ( $\omega_5$ ) is of the order 0.1% (1%). Comparing with lower orders in the derivative expansion,

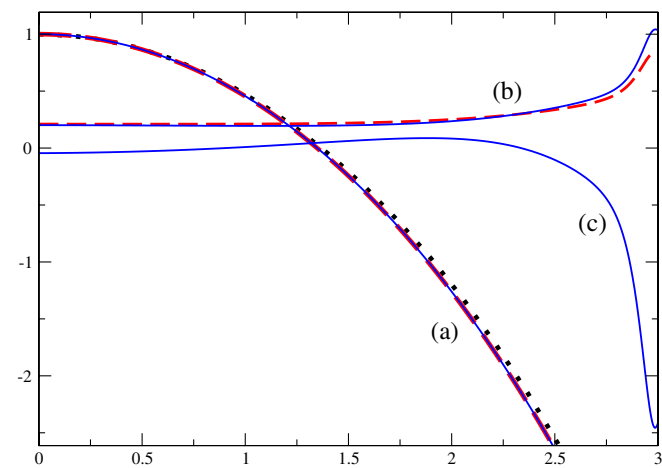


FIG. 2 (color online). Eigenperturbations  $\Phi(x)$  with eigenvalue  $\nu$  at the Wilson-Fisher fixed point for (a) the potential  $v_*(x)$ , (b)  $z_*(x)$  (rescaled by a factor 10), and for (c)  $w_*(x)$  (rescaled by a factor  $10^3$ ). Coding as in Fig. 1.

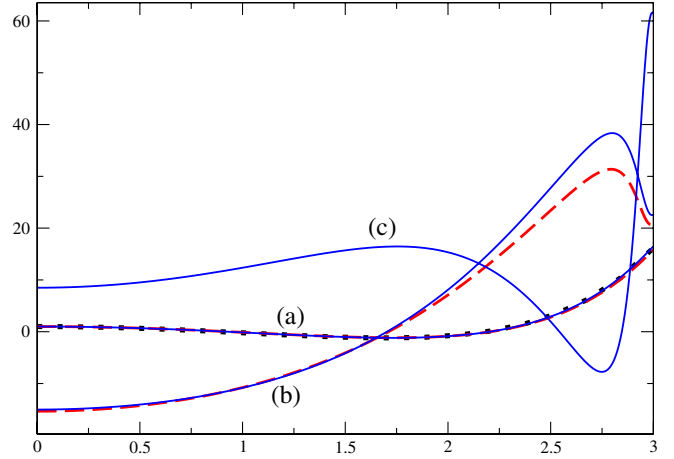


FIG. 3 (color online). Eigenperturbations  $\Phi(x)$  with eigenvalue  $\omega$  at the Wilson-Fisher fixed point for (a) the potential  $v_*(x)$ , (b)  $z_*(x)$  (rescaled by a factor 10), and for (c)  $w_*(x)$  (rescaled by a factor  $10^3$ ). Coding as in Fig. 1.

Table II, we conclude that, up to the order considered, the derivative expansion of the background field flow (11) displays a very good numerical convergence. According to our previous discussion, the inclusion of the terms  $J$  and  $H$ , defined in (7) should produce smaller changes than those induced by  $W$  and, in agreement with the supposed convergence of the derivative expansion, one expects smaller corrections from higher orders of the expansion.

In Table III a) and b), we compare the “world average” of scaling exponents in three dimensions as compiled in [2] with our findings (16). Most exponents agree on the percent level and below, in particular, the indices  $\nu$ ,  $\gamma$ ,  $\delta$  and  $\beta$  which are predominantly sensitive to the field dependence of vertices at vanishing momentum. The anomalous dimension  $\eta$ , and the exponents  $\alpha$  and  $\omega$  are subleading and more sensitive to the momentum structure of propagators and vertices. Consequently, their precision is lower. Interestingly, the exponent  $\omega$  already agrees with the world average within 3%. The exponent  $\alpha$  and the anomalous dimension  $\eta$  only agree within 15% with the best values quoted in the literature. The same pattern persists the comparison with recent high-accuracy results from Monte Carlo simulations  $\nu_{\text{MC}} = 0.63002(10)$ ,  $\omega_{\text{MC}} = 0.832(6)$ ,  $\eta_{\text{MC}} = 0.3627(10)$  [50]. Here, the indices  $\nu$ ,  $\Delta = \nu\omega$ ,  $\omega$  and  $\eta$  agree to within 0.8%, 3%, 4% and 13%, respectively.

TABLE II. Anomalous dimension, leading and subleading scaling exponents, and the Wegner corrections  $\Delta = \nu\omega$  and  $\Delta_5 = \nu\omega_5$  for different orders in the derivative expansion (see text).

approximation	$\eta$	$\nu$	$\omega$	$\omega_5$	$\Delta$	$\Delta_5$
$\mathcal{O}(\partial^0)$	0	0.6260	0.762	2.163	0.477	1.354
$\mathcal{O}(\partial^2)$	0.0330	0.6244	0.852	2.459	0.532	1.535
$\mathcal{O}(\partial^4)$	0.0313	0.6247	0.865	2.563	0.540	1.601

TABLE III. Comparison of a) the world average of theoretical predictions [2], with our results b) (16) and c) (19), also using scaling and hyper-scaling relations.

info	$\gamma$	$\nu$	$\eta$	$\alpha$	$\beta$	$\delta$	$\omega$
a) world average	1.237 2(5)	0.630 1(4)	0.036 4(5)	0.110(1)	0.326 5(3)	4.789(2)	0.84(4)
b) this work	1.2298	0.6247	0.0313	0.1259	0.3221	4.818	0.865
c) impl. opt.	1.233	0.627	0.034	0.119	0.324	4.803	0.839

## V. OPTIMIZATION

In this section, we discuss the optimization of results based on a background field flow. It is well-known that physical observables within an approximation of the functional flow can depend on the shape of the momentum cutoff function  $R$  and its parameters, e.g.  $m$  for the regulator  $f_{\text{PT}}$ . The reason for this is that the cutoff function, a nontrivial function of momenta, couples to all operators in the effective action. Neglecting some operators means that some cutoff-dependent back coupling in (2) is missing. Formally, within some systematic expansion of the flow equation, one obtains the exponents as a series

$$\nu_{\text{phys}} = \nu_{(0)}(R) + \nu_{(1)}(R) + \nu_{(2)}(R) + \dots, \quad (17)$$

where the contribution from every single order  $\nu_{(n)}(R)$  may depend on the cutoff function  $R$ , and only the full physical result will be independent thereof. Optimization is based on the observation that the convergence of (17), and similarly for other observables, is improved through optimized choices for  $R$ , i.e., in our case, the parameter  $m$ .

To evaluate the  $m$ -dependence of our results—and also as a consistency check for our numerical codes—we have recalculated the exponents to leading and second order in the derivative expansion using (5) for other values of the cutoff parameter  $m$  [22–24]. The output is displayed in Table IV and, as expected, the results fully agree with earlier findings. We add the following observations. The exponent  $\nu(m)$  is monotonous, covering the physical value  $\nu_{\text{av}} \approx 0.63$ . Therefore, a minimum sensitivity condition  $\partial \nu(m)/\partial m = 0$  has no solution for all  $m$  considered. Rather the values at the boundary of the parameter interval (in our case  $1/m = 0$ ) should serve as best choices [8,51,52]. This is different from the pattern observed using the standard flow in the same approximation [9]. The index

TABLE IV. Variation of scaling exponents with cutoff shape parameter  $m$ ; background field flow (4) to leading and second order in the derivative expansion.

$m$	2	2.5	3	4	$\infty$
$\eta _{\mathcal{O}(\partial^2)}$	0.065	0.056	0.051	0.045	0.033
$\nu _{\mathcal{O}(\partial^0)}$	0.660	0.650	0.644	0.638	0.626
$\nu _{\mathcal{O}(\partial^2)}$	0.632	0.631	0.630	0.629	0.624
$\omega _{\mathcal{O}(\partial^0)}$	0.628	0.656	0.674	0.698	0.762
$\omega _{\mathcal{O}(\partial^2)}$	0.677	0.702	0.725	0.756	0.852

$\omega(m)$  is equally monotonous, approaching the world average value  $\omega_{\text{av}} \approx 0.84$  from below,  $\omega(m) < \omega_{\text{av}}$ . We note that the physical value for  $\nu$  is matched at  $1/m \approx 0.11$ .

To second order in the derivative expansion, all three observables  $\nu$ ,  $\omega$  and  $\eta$  are monotonous functions of  $m$ . The ranges covered include the physical value in all three cases. For  $\nu$  ( $\omega$ ), the relative change from leading to second order is minimal at  $1/m = 0$  ( $1/m \approx 0.4$ ). We can use our results to second order in the derivative expansion to identify the value of the cutoff parameter  $m = m_{\text{av}}$  for which  $\eta$ —the least well-determined index—matches best with the prediction from the “world average” or Monte Carlo studies. We find  $1/m_{\text{av}} \approx 0.08$ , with

$$\nu = 0.626, \quad \eta = 0.036, \quad \omega = 0.823. \quad (18)$$

Because of the saturating behavior of the indices with increasing  $m$ , the values in (18) are very close to those corresponding to  $1/m \approx 0.11$ , found by matching  $\nu$  at lowest order. The difference between (18) and Monte Carlo results for  $\nu$ ,  $\omega$  and  $\Delta$  reduces to 0.7%, 1.6% and 2.3%, respectively. The improved agreement shows that the scaling exponents display the correct cross dependences, also supporting small values for the inverse cutoff parameter  $1/m$ . The fit  $1/m_{\text{av}} \approx 0.08$  comes out slightly larger than  $1/m = 0$ , a consequence of the anomalous dimension being underestimated in the latter case. The cross correlation amongst  $\nu$ ,  $\omega$  and  $\eta$  as functions of the cutoff parameter  $m$  is similar to the strong cross correlation observed earlier in the local potential approximation [26].

This suggests that, at fourth order, the estimate of the exponents for instance at  $1/m = 0.08$  would sensibly improve the agreement with the Monte Carlo results. Also note that at large  $m$ , the difference  $m \frac{dX}{dm} |_{\mathcal{O}(\partial^2)} - m \frac{dX}{dm} |_{\mathcal{O}(\partial^4)}$  in the  $m$ -dependence of exponents with  $X = \nu$ ,  $\omega$  or  $\eta$  between the 2nd order and 4th order results are small. Then, at  $1/m \approx 0.08$ , we arrive at

$$\nu = 0.627, \quad \eta = 0.034, \quad \omega = 0.839. \quad (19)$$

The difference between (16) and (19) serves as a measure for the variation of indices within the stable domain of RG flows to this order in the approximation. In Table III c), we compile our findings (19) and compare with the world average. The agreement with Table III a) becomes significantly enhanced. For the indices  $\gamma$ ,  $\nu$ ,  $\eta$ ,  $\alpha$ ,  $\beta$ ,  $\delta$ ,  $\Delta = \omega\nu$  and  $\omega$  we find an accuracy of 3%, 0.5%, 7%, 8%, 0.3%, 0.2% and 0.6%, respectively. The same quality persists the

comparison with recent high-accuracy results from Monte Carlo simulations [50]. Here, the indices  $\nu$ ,  $\Delta = \nu\omega$ ,  $\omega$  and  $\eta$  agree to within 0.5%, 0.4%, 0.9% and 6%, respectively, which is a clear improvement over the results at  $1/m = 0$  (see Sec. IV).

In summary, the comparison of background field flows with Monte Carlo results consistently favors small values for  $1/m$ . The excellent agreement of indices around  $1/m \approx 0.08$  indicates that the operators retained in our approximation display the physically expected cross correlations. This nontrivial result lends additional support for the present setup and the internal consistency of the underlying approximations.

## VI. VARIATION WITH DIMENSIONALITY

In this section, we consider the Wilson-Fisher fixed point away from three dimensions. It is a useful consistency check to understand the global  $D$ -dependence of our findings, and their interpolation between the known results in two and four dimensions. Furthermore, probing the local  $D$ -dependence by perturbing the system with  $\frac{d}{dD}|_{D=3}$  provides insights into the structural stability of our setup. In fact, as can be seen from (11) and (12), small variations of  $D$  probe how sensitive the system is to quantum corrections via the anomalous dimension. Suppose we are interested in a physical observable  $X$ , then the relative variation of  $X$  with  $D$  around the dimensionality of interest serves as an indicator for the stability in the observable  $X$ . A strong sensitivity indicates that the observable  $X$  will depend more strongly on the accuracy in the anomalous dimension.

With increasing dimensionality  $D > 3$ , the scaling exponents approach mean field values at  $D = 4$  with  $\eta|_{D=4} = 0$ ,  $\nu|_{D=4} = \frac{1}{2}$  and  $\omega|_{D=4} = 0$ . For decreasing  $D < 3$ , further higher-order critical points become accessible whenever  $n \approx D/(D-2)$  becomes an integer, with  $n = 3$  corresponding to the Wilson-Fisher fixed point. In  $D = 2$ , scaling exponents and the anomalous dimension take the known values  $\nu|_{D=2} = 1$ ,  $\omega|_{D=2} = 2$  and  $\eta|_{D=2} = \frac{1}{4}$ .

We have computed  $\nu$ ,  $\omega$  and  $\eta$  in the vicinity of three dimensions, see Table V. Below  $D \leq 2.7$ , the identification of the scaling solution becomes numerically more demanding. This should be related to the appearance of a competing scaling solution, which becomes available when  $D \approx 8/3$ . From the data, we find

$$\frac{d\nu}{dD} = -0.18, \quad \frac{d\omega}{dD} = -0.65, \quad \frac{d\eta}{dD} = -0.08. \quad (20)$$

for the first derivatives at  $D = 3$ . Note that the derivatives would read  $-0.25$ ,  $-1$  and  $-0.125$  for a simple linear interpolation between the analytically known results at  $D = 2$  and 4. Fitting the data points for  $\nu$  and  $\eta$  ( $\omega$ ) with a cubic (quadratic) polynomial in  $D$ , and extrapolating we find  $\nu|_{D=4} \approx 0.49$ ,  $\omega|_{D=4} \approx 0.02$  and  $\eta|_{D=4} \approx -0.02$ . In the opposite limit, extrapolation leads to  $\nu|_{D=2} \approx 0.92$ ,

TABLE V. Variation of  $\nu$ ,  $\omega$  and  $\eta$  with dimensionality  $D$  to leading and second order in the derivative expansion (see text).

$D$	$\nu _{\mathcal{O}(\partial^0)}$	$\omega _{\mathcal{O}(\partial^0)}$	$\eta _{\mathcal{O}(\partial^2)}$	$\eta _{\mathcal{O}(\partial^4)}$
3.3	0.577 915	0.559 475		
3.2	0.592 702	0.628 553		
3.1	0.608 674	0.696 103		
3.0	0.625 979	0.762 204	0.0330	0.0313
2.9	0.644 808	0.826 85	0.0418	0.0400
2.8	0.665 407	0.8899	0.0519	0.0502
2.7	0.688	0.949	0.0637	0.0621

$\omega|_{D=2} \approx 1.3$ , and  $\eta|_{D=2} \approx 0.2$ . These estimates are fully consistent with the expected behavior, and the slight deviations at the endpoints serve as (rough) indicators for the underlying error. The extrapolated result for  $\eta$  is smaller than the exact one  $\eta = \frac{1}{4}$ , and the fourth-order result is slightly smaller than the second-order result. This suggests that our result slightly underestimates the value for  $\eta$  at  $D = 3$ , though a definite conclusion would require more data points for  $\eta$  in  $2 < D < 3$ . For a study of  $\eta$  and  $\nu$  in  $1 < D < 4$  dimensions using an optimized standard flow to second order in the derivative expansion, we refer to [53].

Next, we estimate the relative variation of scaling exponents and we write  $dX/dD = -A_X X$  for  $X = \nu, \omega, \eta$ . Using (20), we find

$$A_\nu = 0.28(1), \quad A_\omega = 0.85(1), \quad A_\eta = 2.5(1) \quad (21)$$

in three dimensions. The smallness of  $A_\nu$  explains the high accuracy achieved for  $\nu$  already to low orders in a derivative expansion. In turn, the large value of  $A_\eta$  explains the stronger sensitivity of  $\eta$  on the approximation. Furthermore, the pattern  $A_\nu < A_\omega < A_\eta$  suggests that the expected accuracy in  $\omega$  should be better than the one in  $\eta$ , and worse than the one in  $\nu$ . This is in accord with the pattern observed in our results, see Sec. IV, and with the earlier functional RG results discussed below (see Sec. VIII and IX).

## VII. CROSS CORRELATIONS

Cross correlations amongst scaling exponents provide insights into the finer structure of the theory, and into the inner working of the approximation in place. Within the local potential approximation, cross correlations are strong [26], and only weakly dependent on the cutoff  $R_k$ , in particular, for optimized flows [6]. A similar cross correlation has been observed based on hierarchical RG transformations, thereby providing a link between the cutoff ( $R_k$ ) dependence of the continuum RG and finite step size effects in discrete versions thereof [26].

Here, we are interested in the correlations to higher order in the derivative expansion. To set the stage, we perform a linear interpolation for the derivatives based on the known results at  $D = 2$  and  $D = 4$ . We find



$$\frac{d\omega}{d\nu} = 4, \quad \frac{d\eta}{d\nu} = 0.5, \quad \frac{d\eta}{d\omega} = 0.125. \quad (22)$$

Within our functional RG setup, we access the cross correlation of exponents by keeping the regulator fixed, and by exploiting that (20) represent full variations with  $D$ . Since  $\eta(D)$  is monotonous in  $D$ , at least in the region of interest (see Table V), we invert  $\eta(D)$  into  $D(\eta)$  to obtain the functions  $\nu(\eta) \equiv \nu(D(\eta))$  and  $\omega(\eta) \equiv \omega(D(\eta))$  which encode the cross correlation of scaling exponents. In three dimensions, their first derivatives read

$$\frac{d\omega}{d\nu} = 3.63, \quad \frac{d\eta}{d\nu} = 0.45, \quad \frac{d\eta}{d\omega} = 0.124. \quad (23)$$

Note that  $\frac{d\omega}{d\nu} \frac{d\nu}{d\eta} \frac{d\eta}{d\omega} = 1$  to within 0.03%, which is smaller than the error in (23). We conclude that the derivatives  $\frac{d\eta}{d\nu}$ ,  $\frac{d\omega}{d\nu}$  and  $\frac{d\eta}{d\omega}$  at  $D = 3$  agree to within 10%, 10%, and 1% with the linear approximation (22), respectively. Our result (23) compares well with the estimate  $\frac{d\eta}{d\nu}|_{\epsilon\text{-exp}} = 0.59$  obtained from a modified epsilon expansion by Guida and Zinn-Justin [54]. We note that  $\frac{d\eta}{d\nu}|_{\text{fRG}} < \frac{d\eta}{d\nu}|_{\text{lin}} < \frac{d\eta}{d\nu}|_{\epsilon\text{-exp}}$ . The double-logarithmic derivatives follow from (21) in an obvious manner, e.g.  $\frac{d \ln \omega}{d \ln \nu} \equiv A_\omega/A_\nu$ , leading to the estimates  $\frac{d \ln \omega}{d \ln \nu} \approx 3$ ,  $\frac{d \ln \eta}{d \ln \nu} \approx 9$  and  $\frac{d \ln \eta}{d \ln \omega} \approx 3$ , consistent with the sensitivity observed in our results.

### VIII. CONVERGENCE

Despite the small anomalous dimension, the Wilson-Fisher fixed point shows nonperturbative features. While

little is known about the absolute convergence of systematic approximations to (2) in the nonperturbative regime, the numerical convergence of expansions can be accessed order-by-order [28]. In this section, we discuss the convergence of the derivative expansion (see Table VI) by comparing results for  $\nu$ ,  $\eta$  and  $\omega$  from different realizations of the functional renormalization group including the standard flow (2), background field flows (4), and the Wilson-Polchinski flow (see [28] for an earlier overview). We have omitted data points which are not based on an (at least partly) optimized choice for the momentum cutoff, e.g. sharp cutoff results [60].

To leading order in the derivative expansion, the full cutoff dependence of  $\nu(R)$  is known within the standard flow [9,26], within the Wilson-Polchinski flow [15] where the result is unique, and, partly, within background field flows [21–24]. For the standard flow, the best result is given in b) [9], achieved for suitably optimized regulators. High-accuracy expressions for the exponents are given in [55] and are in full agreement with findings from the Wilson-Polchinski flow [15] in c). The background field flow covers a larger range of values for  $\nu(m)$ , the smallest one given in a). Comparing a) with b), we note that the leading index  $\nu$  (subleading index  $\omega$ ) is slightly (significantly) closer to the physical result in the setup a).

For the  $\mathcal{O}(\partial^2)$  approximation, we report the exponents from the standard flow based on an optimized algebraic (power-law) cutoffs [27] in d), a standard exponential cutoff [56,57] in e), an optimized exponential cutoff [47] in f), and a flat optimized cutoff [47] in g). Note that algebraic (power-law) cutoffs of [27] leads to slowly converging

TABLE VI. Comparison of results from the functional renormalization group within various approximations (see text). Local potential approximation ( $\partial^0$ ): a) background field flow (bf); b) standard flow (st); c) Wilson-Polchinski flow (WP). Derivative expansion to second order ( $\partial^2$ ): d) - g) standard flow (various cutoffs); h) background field flow. Derivative expansion to second order with matching of the anomalous dimension: i) Wilson-Polchinski flow; j) background field flow. Derivative expansion to fourth order ( $\partial^4$ ): k) standard flow; l) background field flow; m) background field flow with implicit optimization. Mixed approximation retaining momentum- and field-dependences (mixed): n) standard flow.

	info	$\eta$	$\nu$	$\omega$	refs.
a)	$\partial^0$ , bf	0	0.6260	0.7622	[23,24], this work
b)	$\partial^0$ , st	0	0.649 561 $\cdots$	0.655 746 $\cdots$	[9,28,55]
c)	$\partial^0$ , WP	0	0.649 561 $\cdots$	0.655 746 $\cdots$	[15,55]
d)	$\partial^2$ , st, alg	0.053 93	0.6181	0.8975	[27]
e)	$\partial^2$ , st, exp	0.0467	0.6307	$\cdots$	[56,57]
f)	$\partial^2$ , st, exp, opt	0.0443	0.6281	$\cdots$	[47]
g)	$\partial^2$ , st, opt	0.0470	0.6260	$\cdots$	[47]
h)	$\partial^2$ , bf	0.0330	0.624	0.852	[23], this work
i)	$\partial^2$ , WP, $\eta$ -matching	0.038	0.625	0.77	[58]
j)	$\partial^2$ , bf, $\eta$ -matching	0.036	0.626	0.823	this work
k)	$\partial^4$ , st, exp, opt	0.033	0.632	$\cdots$	[48]
l)	$\partial^4$ , bf	0.0313	0.6247	0.865	this work
m)	$\partial^4$ , bf, implicit	0.034	0.627	0.839	this work
n)	mixed, st, exp, opt	0.039	0.632	0.78	[59]

flows within the derivative expansion [28], which is already visible within the local potential approximation [9]. The comparatively large estimate for  $\eta$  in e) is a consequence thereof. Below, we will retain e) for a conservative error estimate. Comparing d)-g) with h), we note that the indices  $\nu$  and  $\omega$  differ only slightly amongst the different implementations. In contrast, the anomalous dimension  $\eta$  varies more strongly, about  $\pm 25\%$ . In the standard flow, the anomalous dimension stays above 4%, whereas the background field flow leads to a result below 4%, closer to the physical value.

Results to second order in the derivative expansion are also available within Polchinski's formulation of the renormalization group [3,61]. The Wilson-Polchinski flow is linked to (2) by a Legendre transform, implying that derivative expansions are inequivalent beyond the trivial order. A significant cutoff dependence, in particular, for  $\eta$ , is observed [58,61,62], which calls for a stability-based optimization of the cutoff [6,10,15]. A prediction for Ising exponents is achieved at  $O(\partial^2)$  by tuning the cutoff to the desired value for  $\eta$ , say  $\eta \approx 0.038$  [58], and using a minimum sensitivity condition to identify the remaining exponents (see also [61,62]). This leads to  $\nu \approx 0.625$  and  $\omega \approx 0.77$  [58], summarized in Table VI i). The predictions for  $\nu$  and  $\omega$  are in the expected range of values, showing that the Wilson-Polchinski flow displays the correct cross correlation of scaling exponents. It will be interesting to see whether a fourth-order computation stabilises the result. For comparison, we have added in Table VI j) our result (18) from the background field flow to second order in the derivative expansion, where  $\eta$  has been matched to the world average and Monte Carlo result. The Wilson-Polchinski and background field estimates agree very well for  $\nu$ , and differ by less than 8% for the exponent  $\omega$ . The background field value is much closer to the expected value. Note that this procedure is not applicable for the standard flow to second order, because the anomalous dimension stays above 4% for all cutoffs and cannot be matched to the physical value.

Beyond  $O(\partial^2)$  in the derivative expansion, we cite the fourth-order computation by [48] in k), which is compared with our result (16) in l), the optimized background field result (19) in m) and the ‘‘mixed’’ analysis of [59] in n), which is optimized using the principle of minimum sensitivity [8,51,52]. The approach n) retains momentum- and field-dependences in the ansatz for the effective action, amended by approximations on the level of the flow; see [29] for technical details. The results for  $\nu$  in all approaches are very close to the world average  $\nu_{\text{av}} = 0.6301(4)$ . The value for  $\omega$  from background field flows l) and m) are closest to the world average  $\omega_{\text{av}} = 0.84(4)$ . All values for  $\eta$  are now below 4%. Still, a slight variation of  $\eta$  remains visible which makes the anomalous dimension the least well-determined observable in Table VI. We note that the prediction for  $\eta$  based on k) and n) are equally

close to the world average  $\eta_{\text{av}} = 0.0362(4)$ , approaching it from opposite sides. This is interesting because n) should have a better access to the momentum dependence of propagators. We suspect that the approximations on the level of the flow exercised in [29] are responsible for this pattern. The  $\eta$ -values from background field flows approach the physical value from below, with m) being closest to the expected value.

The mean values based on all data points in Table VI are  $\langle \nu \rangle_{\text{IRG}} = 0.630$  and  $\langle \omega \rangle_{\text{IRG}} = 0.790$ . For the anomalous dimension, we find  $\langle \eta \rangle_{\text{IRG}} = 0.0312(0.0397)$ , depending on whether we retain (suppress) the data points to leading order in the derivative expansion with  $\eta = 0$ . (We come back to a detailed discussion of mean values and systematic errors in Sec. IX.)

We use the numerical convergence of the derivative expansion for a crude error estimate. For the standard flow with order-by-order optimized exponential cutoff function  $R_k(q^2) \propto \alpha q^2 / (\exp q^2/k^2 - 1)$  we compare the leading order result  $\eta = 0$  and  $\nu = 0.6506$  [6] with higher orders in the derivative expansion Table VI f) and k). This leads to  $\nu \approx 0.637 \pm 2\%$  and  $\eta \approx 0.0387 \pm 15\%$ . Retaining only the two best values for  $\nu$  improves the error estimate,  $\nu \approx 0.630 \pm 0.3\%$ . The relative change  $\Delta \nu / \nu$  reads  $3.5 \times 10^{-2}$  ( $6.3 \times 10^{-3}$ ) at second (fourth) order in the derivative expansion. For the background field flow with cutoff  $m \rightarrow \infty$ , we compare Table VI a), h) and l), leading to  $\nu \approx 0.625 \pm 0.4\%$  and  $\eta = 0.0322 \pm 2\%$ . Hence, in the approximation (4), the numerical convergence of background field flows is slightly faster.

We conclude that the derivative expansion of the functional renormalization group, together with suitably optimized regulators, shows a very good numerical convergence up to high order for both standard and background field flows.

## IX. SYSTEMATIC ERRORS

Estimating systematic errors is common practice in e.g. lattice approaches and resummations of perturbation theory. Here we discuss how analogous estimates can be achieved for the functional renormalization group, where physical observables are obtained by projecting the full flow in ‘‘theory space’’—the infinite dimensional space of operators parametrizing the effective action  $\Gamma_k$ —onto a subset thereof. This step implies an approximation and is a potential source for systematic errors. The flexibility of the formalism, however, allows for many different projections. Then the quantitative comparison of different projections gives access to the systematic uncertainty.

We recall that, in general, approximations to the flow equation (2) enter via operators neglected in the effective action  $\Gamma_k$ , approximations on the level of the flow  $\partial_t \Gamma_k$ , and the choice for the momentum cutoff  $R_k$ . These aspects are partly intertwined, to the least because a momentum cutoff introduces a nontrivial momentum structure into the flow.

In general, the operator content is central. A similar importance should be given to approximations on the level of the flow  $\partial_t \Gamma_k$ , which feed back into the determination of scaling exponents. The regulator is crucial for the stability and convergence of the RG flow [6]. Within given approximations for  $\Gamma_k$  and  $\partial_t \Gamma_k$ , the regulator can be optimized to maximize the physics content in the flow, and to minimize cutoff artifacts. Uncertainties due to the boundary condition for the effective action are irrelevant for fixed point solutions. We conclude that systematic errors should only be derived from ‘cutoff-optimized’ results to eliminate cutoff artifacts [6,9].

Next we employ this reasoning to the data collected in Table VI. A first estimate for the systematic error is obtained by taking a weighted average over representative entries for each projection method (standard flow, Wilson-Polchinski flow, background field flow), disregarding further details of the approximations. Common to the data points is that the underlying regulators are, at least partially, optimized [6]. We first consider the data points Table VI a), b), f), h), k), l), n) to obtain

$$\nu = 0.631^{+0.018}_{-0.006}, \quad \eta = 0.036^{+0.008}_{-0.005}, \quad \omega = 0.783^{+0.082}_{-0.127}. \quad (24)$$

For  $\eta$ , we only took the data with  $\eta \neq 0$  into account. The mean values (24) change by less than 0.1% (1.5%) for  $\nu$ ,  $\eta$  ( $\omega$ ) had we included the data points i), j) and m) based on some additional input. Hence (24) represents an average with equal weight for the different implementations of the functional flow. Note that the width of the error bars, roughly a standard deviation, are set by the least advanced approximations.

An improved estimate is obtained by retaining only the most advanced results in Table VI, ie. k), l) and n), all of which are based on a similar operator content, supported by a partial cutoff optimization [6], but differ in the projection technique. We recall that in k) a standard full fourth-order derivative expansion is used, together with a polynomial expansion in the fields [48]; in n) a mixed approximation is employed retaining momentum- and field-dependences, but neglecting loop momenta of certain vertex functions [59]; in l) a background field flow is used within a fourth-order derivative expansion and without polynomial expansion in the fields, but neglecting higher-

order flow terms and subleading fourth-order derivative operators in the action (this work). The qualitative differences in the approximation make sure that the computations project in different manners onto the Wilson-Fisher fixed point, thereby probing the systematic error. We find

$$\nu = 0.630^{+0.002}_{-0.005}, \quad \eta = 0.034^{+0.005}_{-0.003}, \quad \omega = 0.823^{+0.043}_{-0.043}. \quad (25)$$

Note that we omit the data set m) from this estimate to achieve a conservative error bar and an equal weight between projection methods. From (24) to (25) the error bars are reduced by at least a factor of 2. The mean values for  $\nu$ ,  $\omega$  and  $\eta$  are shifted by 0.2%, 5% and 6%, respectively. The shift in the mean values from (24) to (25) is of a similar size as the estimated error in (25).

In Table VII, the combined functional RG results (25) are compared with the  $\epsilon$ -expansion, resummed perturbation theory, Monte Carlo simulations, and a world average of theoretical predictions. It shows that the functional RG results agree very well with results from other methods within systematic errors and on the level of the mean values. The results are also compatible with recent experimental results, e.g.  $\eta = 0.041 \pm 0.005$  and  $\nu = 0.632 \pm 0.002$  [63], with experimental errors slightly larger than those from theory (see [64,65] for overviews). Expected errors from the functional RG are presently about an order of magnitude larger than those from e.g. numerical simulations, and more data and extended approximations are required to further reduce the systematic uncertainty. In particular, the value for  $\omega$  in (25) is presently only based on two data points. Here, it would be useful to know the value from the standard flow at fourth order in the derivative expansion to improve the error estimate in Table VII. Natural candidates for further data points are approximations of the Wilson-Polchinski equation beyond second order in the derivative expansion, or approximations with an improved access to the full momentum structure of propagator and vertices.

## X. DISCUSSION

The computation of universal scaling exponents is an important testing ground for methods in quantum field theory and statistical physics. We have obtained new results for the indices  $\nu$ ,  $\omega$ ,  $\omega_5$  and  $\eta$  of the 3d Ising

TABLE VII. Comparison of results from the functional renormalization group with resummed perturbation theory, Monte Carlo simulations,  $\epsilon$ -expansion, and a world average.

	$\eta$	$\nu$	$\omega$	ref./year
resummed PT	0.033 5(25)	0.630 4(13)	0.799 (11)	[54] (1998)
$\epsilon$ -expansion	0.036 0(50)	0.629 0(25)	0.814 (18)	[54] (1998)
world average	0.036 4(5)	0.630 1(4)	0.84(4)	[2] (2000)
Monte Carlo	0.036 27(10)	0.630 02(10)	0.832(6)	[50] (2010)
functional RGs	0.034(5)	0.630(5)	0.82(4)	this work

universality class using functional renormalization group methods within a background field formulation. Our analysis complements earlier studies without background fields. Our findings to second (18) and fourth order (16) and (19) in the derivative expansion agree very well with other theoretical studies. The indices also display the correct cross dependences. This nontrivial result lends further support to the underlying approximations.

We have also studied the cross correlations of exponents, and their sensitivity to tiny variations of the dimensionality. The latter correlates with the expected error of exponents within the derivative expansion. As a result (21), the index  $\nu$  shows a weak, the subleading index  $\omega$  a moderate, and the anomalous dimension a strong dependence on dimensional variations. We conclude that the achievable precision in these observables follows the same pattern. This is confirmed by the data (25).

The flexibility of the functional renormalization group allows for different projections onto the Wilson-Fisher fixed point. We have exploited this freedom to estimate the systematic uncertainty of scaling exponents using all available data. The resulting mean values and error estimates (25) agree very well with results from resummed perturbation theory and lattice simulations. More work and further data points are required to reduce the error bars,

which are similar to those from experiments, but larger than those from recent numerical simulations. Natural candidates for further data points are e.g. Wilson-Polchinski flows to fourth order in the derivative expansion, and approximation schemes with an improved access to the momentum structure of propagators and vertices.

In addition, we have analyzed the convergence of the derivative expansion, comparing data from standard flows, background field flows, and the Wilson-Polchinski flow. Background field flows lead systematically to smaller values for  $\eta$ , and the derivative expansion converges very fast. Standard flows provide narrower bounds on exponents, while the derivative expansion shows a slightly slower rate of convergence. For the Wilson-Polchinski flow, structural arguments suggest that approximations beyond the leading order are more sensitive to the cutoff. Still, good results are available to second order, provided  $\eta$  is matched. It will thus be interesting to extend these studies beyond the Ising universality class.

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- [1] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (Clarendon, Oxford, 1989).
  - [2] A. Pelissetto and E. Vicari, *Phys. Rep.* **368**, 549 (2002).
  - [3] J. Polchinski, *Nucl. Phys.* **B231**, 269 (1984).
  - [4] C. Wetterich, *Phys. Lett. B* **301**, 90 (1993).
  - [5] K. G. Wilson and J. B. Kogut, *Phys. Rep.* **12**, 75 (1974).
  - [6] D. F. Litim, *Phys. Lett. B* **486**, 92 (2000).
  - [7] D. F. Litim, *Phys. Rev. D* **64**, 105007 (2001).
  - [8] D. F. Litim, *Int. J. Mod. Phys. A* **16**, 2081 (2001).
  - [9] D. F. Litim, *Nucl. Phys.* **B631**, 128 (2002).
  - [10] J. M. Pawłowski, *Ann. Phys. (N.Y.)* **322**, 2831 (2007).
  - [11] C. Bagnuls and C. Bervillier, *Phys. Rep.* **348**, 91 (2001).
  - [12] J. Berges, N. Tetradis, and C. Wetterich, *Phys. Rep.* **363**, 223 (2002).
  - [13] J. Polonyi, *Central Eur. J. Phys.* **1**, 1 (2003).
  - [14] D. F. Litim and J. M. Pawłowski, in *The Exact Renormalization Group*, edited by Krasnitz *et al.* (World Scientific, Singapore, 1999), p. 168.
  - [15] D. F. Litim, *J. High Energy Phys.* **07** (2005) 005.
  - [16] D. F. Litim and J. M. Pawłowski, *Phys. Rev. D* **65**, 081701 (2002).
  - [17] D. F. Litim and J. M. Pawłowski, *Phys. Rev. D* **66**, 025030 (2002).
  - [18] D. F. Litim and J. M. Pawłowski, *Phys. Lett. B* **546**, 279 (2002).
  - [19] D. F. Litim, J. M. Pawłowski, and L. Vergara, *arXiv:hep-th/0602140*.
  - [20] S. B. Liao, *Phys. Rev. D* **53**, 2020 (1996).
  - [21] O. Bohr, B. J. Schaefer, and J. Wambach, *Int. J. Mod. Phys. A* **16**, 3823 (2001).
  - [22] A. Bonanno and D. Zappalà, *Phys. Lett. B* **504**, 181 (2001).
  - [23] M. Mazza and D. Zappalà, *Phys. Rev. D* **64**, 105013 (2001).
  - [24] D. F. Litim and J. M. Pawłowski, *Phys. Lett. B* **516**, 197 (2001).
  - [25] Y. Meurice, *J. Phys. A* **40**, R39 (2007).
  - [26] D. F. Litim, *Phys. Rev. D* **76**, 105001 (2007).
  - [27] T. R. Morris, *Phys. Lett. B* **329**, 241 (1994).
  - [28] D. F. Litim, *J. High Energy Phys.* **11** (2001) 059.
  - [29] J. -P. Blaizot, R. Mendez Galain, and N. Wschebor, *Phys. Lett. B* **632**, 571 (2006).
  - [30] D. F. Litim, *arXiv:hep-ph/9811272*.
  - [31] B. Bergerhoff, F. Freire, D. Litim, S. Lola, and C. Wetterich, *Phys. Rev. B* **53**, 5734 (1996).
  - [32] H. Gies, *Phys. Rev. D* **66**, 025006 (2002).
  - [33] J. M. Pawłowski, D. F. Litim, S. Nedelko, and L. v. Smekal, *Phys. Rev. Lett.* **93**, 152002 (2004).
  - [34] D. F. Litim, *arXiv:0810.3675*.
  - [35] M. Reuter and C. Wetterich, *Nucl. Phys.* **B417**, 181 (1994).
  - [36] F. Freire, D. F. Litim, and J. M. Pawłowski, *Phys. Lett. B* **495**, 256 (2000).
  - [37] D. F. Litim and J. M. Pawłowski, *J. High Energy Phys.* **09** (2002) 049.

- [38] D. F. Litim, *Phys. Lett. B* **393**, 103 (1997).
- [39] F. Freire and D. F. Litim, *Phys. Rev. D* **64**, 045014 (2001).
- [40] D. Zappalà, *Phys. Lett. A* **290**, 35 (2001).
- [41] A. Bonanno and G. Lacagnina, *Nucl. Phys.* **B693**, 36 (2004).
- [42] M. Consoli and D. Zappalà, *Phys. Lett. B* **641**, 368 (2006).
- [43] P. Castorina, M. Mazza, and D. Zappalà, *Phys. Lett. B* **567**, 31 (2003).
- [44] A. Bonanno and M. Reuter, *J. High Energy Phys.* 02 (2005) 035.
- [45] D. Zappalà, *Phys. Rev. D* **66**, 105020 (2002).
- [46] H. Ballhausen, arXiv:hep-th/0303070.
- [47] L. Canet, B. Delamotte, D. Mouhanna, and J. Vidal, *Phys. Rev. D* **67**, 065004 (2003).
- [48] L. Canet, B. Delamotte, D. Mouhanna, and J. Vidal, *Phys. Rev. B* **68**, 064421 (2003).
- [49] D. F. Litim and L. Vergara, *Phys. Lett. B* **581**, 263 (2004).
- [50] M. Hasenbusch, *Phys. Rev. B* **82**, 174433 (2010).
- [51] P. M. Stevenson, *Phys. Rev. D* **23**, 2916 (1981).
- [52] S. -B. Liao, J. Polonyi, and M. Strickland, *Nucl. Phys.* **B567**, 493 (2000).
- [53] H. Ballhausen, J. Berges, and C. Wetterich, *Phys. Lett. B* **582**, 144 (2004).
- [54] R. Guida and J. Zinn-Justin, *J. Phys. A* **31**, 8103 (1998).
- [55] C. Bervillier, A. Juttner, and D. F. Litim, *Nucl. Phys.* **B783**, 213 (2007).
- [56] S. Seide and C. Wetterich, *Nucl. Phys.* **B562**, 524 (1999).
- [57] G. Von Gersdorff and C. Wetterich, *Phys. Rev. B* **64**, 054513 (2001).
- [58] R. D. Ball, P. E. Haagensen, J. I. Latorre, and E. Moreno, *Phys. Lett. B* **347**, 80 (1995).
- [59] F. Benitez, J. P. Blaizot, H. Chate, B. Delamotte, R. Mendez-Galain, and N. Wschebor, *Phys. Rev. E* **80**, 030103 (2009).
- [60] A. Hasenfratz and P. Hasenfratz, *Nucl. Phys.* **B270**, 687 (1986); *Helv. Phys. Acta* **59**, 833 (1986).
- [61] C. Bervillier, *J. Phys. Condens. Matter* **17**, S1929 (2005).
- [62] J. Comellas, *Nucl. Phys.* **B509**, 662 (1998).
- [63] A. Lytle and D. T. Jacobs, *J. Chem. Phys.* **120**, 5709 (2004).
- [64] M. Barmatz, I. Hahn, J. A. Lipa, and R. V. Duncan, *Rev. Mod. Phys.* **79**, 1 (2007).
- [65] J. V. Sengers and J. G. Shanks, *J. Stat. Phys.* **137**, 857 (2009).