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Superconducting analogs of quantum optical phenomena: Macroscopic quantum superpositions and squeezing in a superconducting quantum-interference device ring

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In this paper we explore the quantum behavior of a superconducting quantum-interference device (SQUID) ring which has a significant Josephson coupling energy. We show that the eigenfunctions of the Hamiltonian for the ring can be used to create macroscopic quantum superposition states of the ring. We also show that the ring potential may be utilized to squeeze coherent states. With the SQUID ring as a strong contender as a device for manipulating quantum information, such properties may be of great utility in the future. However, as with all candidate systems for quantum technologies, decoherence is a fundamental problem. In this paper we apply an open systems approach to model the effect of coupling a quantum-mechanical SQUID ring to a thermal bath. We use this model to demonstrate the manner in which decoherence affects the quantum states of the ring.

INTRODUCTION

In two recent publications [1,2] we reported on the theoretical description of a quantum-mechanical superconducting quantum-interference device (SQUID) ring (here, a thick superconducting ring enclosing a single Josephson weak link device) coupled to quantized electromagnetic field (em) oscillator modes. In this work we emphasized that the SQUID ring could be used to control various quantum phenomena involving each of the circuit components of the coupled system via the static magnetic bias flux \( \Phi_x \) applied to the ring. These included frequency conversion between the em modes and quantum entanglement extending across the system, both with relevance to emerging quantum technologies based on Josephson devices [3–13]. Furthermore, work by Friedman et al. on SQUID rings has highlighted another phenomenon of potentially great significance to these incipient technologies, namely, the creation of externally controlled superpositions of macroscopically distinct states in a SQUID ring, or other, Josephson weak link based, circuit configurations [14]. As will become apparent, the creation and control of such states is a natural application for a SQUID ring.

In this paper we consider the creation and control of macroscopic quantum superposition states in a SQUID ring alone, uncoupled to any em oscillator modes. First, we consider the spectral properties of the ring Hamiltonian. Then we observe that at certain points in the bias flux \( \Phi_x \) applied to the ring the eigenfunctions of this Hamiltonian form macroscopic superposition states. We show that a strong enough level of dissipation may destroy the quantum nature of these states, while leaving the flux in the SQUID ring in a statistical mixture of two macroscopically distinguishable states. Following this we demonstrate that a SQUID ring can be used to create (form) a controllable macroscopic superposition of states. In addition, we show that a SQUID ring with a sufficiently large Josephson coupling term in its potential can be used to squeeze coherent states. In this it is apparent that physical phenomena associated with SQUID rings, and with quantum circuits built around SQUID rings, have analogies with effects well known in the field of quantum optics. Indeed, the SQUID ring can be viewed as a nonlinear medium which, for example, can be utilized to generate entanglements, frequency conversion, superposition states, and squeezing. However, the SQUID ring has significant advantages over the generally weakly polynomial nonlinear media of quantum optics which are usually weakly coupled to external em fields. Thus, it is extremely nonperturbative in nature (and concomitantly capable of inducing extremely nonlinear behavior [1,2,15]) with a coupling to em modes that can be adjusted by means of an external bias flux. This would appear to make the SQUID a prime candidate for future developments in what is, in effect, highly nonperturbative quantum optics, albeit at much lower frequencies. In practice, these frequencies would typically be much less than 1 THz for low critical temperature superconductors.

BACKGROUND

Wigner and Weyl functions

Although the Wigner and Weyl functions are familiar to those working in the field of quantum optics [16–18], their use in the quantum description of Josephson weak link circuits, and in particular SQUID rings, appears to be rather
functions for the first four energy eigenstates of the simple harmonic oscillator together with the absolute values of their associated Weyl functions. We note that the Weyl function is, in general, complex valued. In this paper, therefore, we only ever plot its absolute value since, for our purposes, this provides us with sufficient information about the correlations of the wave function.

The SQUID ring Hamiltonian

Over the last two decades SQUID rings, viewed as single, macroscopic, quantum objects, have been the subject of considerable attention theoretically. In early studies the focus was primarily on time-independent properties and the interaction of SQUID rings with external environments [15,20,21]. Of late there has been much interest in time-dependent behavior, for example, in solving the time-dependent Schrödinger equation for a SQUID ring in the presence of a microwave field [22,23]. Recently, significant efforts have been devoted to the experimental measurement and control of macroscopic quantum superposition states in SQUID rings [14]. In this paper we proceed to develop a theoretical description of macroscopic quantum superpositions in SQUID rings, borrowing techniques that are commonly used in quantum optics. We extend the usefulness of this description by considering quantum-mechanical squeezed states in SQUID rings. For both superposition of states and squeezing in these rings we also discuss the effect of dissipation (decoherence).

In the widely used lumped component model of a SQUID ring [15,20] the Hamiltonian takes the form

\[ H = \frac{Q^2}{2C} + \frac{(\Phi - \Phi_c)^2}{2\Lambda} - \hbar \nu \cos \left( \frac{2\pi \Phi}{\Phi_0} \right), \]

where \( \Phi \) and \( Q \) are, respectively, the magnetic flux threading the ring and the electric displacement flux across the weak link (with \( [\Phi, Q] = i\hbar \)), \( \hbar / 2 \) is the matrix element for Josephson tunneling through the weak link (with critical current \( I_c = 2e\nu \)), \( \Phi_0 = h / 2e \) is the superconducting flux quantum, and \( \Lambda \) and \( C \) are, respectively, the ring inductance and the capacitance of the weak link in the ring.

Introducing a unitary translation operator \( T = \exp(-i\Phi Q / \hbar) \), we can then write down the ring Hamiltonian as

\[ H' = T^\dagger HT = \frac{Q^2}{2C} + \frac{\Phi^2}{2\Lambda} - \hbar \nu \cos \left( \frac{2\pi \Phi}{\Phi_0} \right), \]

where it is clear that as \( \hbar \nu \to 0 \) the system behavior reduces to that of a simple harmonic oscillator. Given the relation between our system and the simple harmonic oscillator, we now define creation and annihilation operators in the usual way, i.e., as

\[ a = \sqrt{\frac{C\omega}{2\hbar}}(\Phi - i\frac{Q}{C\omega}), \quad a^\dagger = \sqrt{\frac{C\omega}{2\hbar}}(\Phi + i\frac{Q}{C\omega}). \]

These raising and lowering operators, as used in quantum optics, then allow us to write the ring Hamiltonian in a more convenient form. We also choose to express it in dimension-

FIG. 1. Illustrative example of the (a) Wigner and (b) absolute value of the Weyl functions of the first four energy eigenstates of the simple harmonic oscillator (increasing in energy from left to right).
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were computed to sufficient accuracy. However, for the potential wells shown in Fig. 2, with the very weak coupling between levels in different wells, the energy splittings at these points may be extremely small indeed.

RESULTS

Quantum superpositions of macroscopically distinct states

Ignoring, for the moment, the special cases where the locally s-harmonic oscillator states are degenerate (zero coupling between wells) we instead consider making an equal superposition of two energy eigenstates. We start by assuming that these two states are locally s harmonic in different wells and, for simplicity, take these states to correspond to the lowest-energy levels in each well. Then, as one would expect, we create a superposition of macroscopically distinct states in flux. As this superposition state is no longer an eigenstate of the Hamiltonian for the system, it must evolve with time. This evolution introduces a phase difference between the two stationary (locally s-harmonic) states in the superposition which manifests itself in a time-dependent evolution of the interference term in the Wigner function.

In computing Wigner functions for the SQUID ring it is convenient to introduce equivalent dimensionless position x and momentum p operators in place of \( \Phi \) and \( Q \). These are defined as

\[
\begin{align*}
x &= \sqrt{\frac{\omega}{\hbar}} \Phi, \\
p &= \sqrt{\frac{1}{\hbar \omega}} Q.
\end{align*}
\]

An example of a macroscopic superposition state in a SQUID ring, as illustrated through its computed Wigner function, is shown in Fig. 4, both in perspective and in projection on the \( x-p \) plane. Here, we have taken our standard values of \( \Lambda, C \) and \( \hbar \nu \) (above) and have selected the state \((1/\sqrt{2})(|\Lambda|+|\rangle)\) at the flux bias \( \Phi_x = 0.49\Phi_0 \) of Fig. 2(b). In this example we can distinguish in the Wigner function two macroscopically distinct flux states of the SQUID ring in the \( x-p \) plane (i.e., the Gaussian-like components), separated from one another by an oscillatory region. The latter, oscillatory, region arises because of the quantum coherence between the two separate components in the superposition and demonstrates that we are indeed dealing with a true quantum superposition of macroscopically distinct states in flux. We note that the Wigner function of Fig. 4 has been calculated at a fixed time \( t=0 \). However, its general form does not vary with time. Nevertheless, there is dynamical evolution of the interference term in the superposition but not in the observable flux states.

Tunable superposition of states in a SQUID ring

We now consider the potential with a static bias flux \( \Phi_x = 0.5\Phi_0 \), as shown in the bottom plot of Fig. 2. By examining the first two energy eigenstates we find that we have wave functions which are a symmetric, \(|s\rangle\), and an antisymmetric, \(|a\rangle\), superposition of the lowest energy, locally s-harmonic, oscillator vacuum states of the two middle wells. These two states, which are squeezed, are extremely close in energy due to the height of the barrier between the wells in the potential. Even so, the interference terms in the Wigner function between the states for each of these macroscopically distinct states are \( \pi/2 \) out of phase. However, from the viewpoint of theory we might also consider an equal superposition of these states of the form

\[
\frac{1}{\sqrt{2}}[|s\rangle + \exp(i\theta)|a\rangle],
\]

where \( \theta \) is the phase. Let \( E_s \) and \( E_a \) be the eigenenergies corresponding to the eigenstates \(|s\rangle\) and \(|a\rangle\) so that \( \Delta E = E_s - E_a \) the energy splitting associated with the lifting of the degeneracy concomitant with tunneling between the adjacent wells. Then \( \theta = \Delta E t \). It is seen that this phase \( \theta \) changes as a function of time and any particular value of \( \exp(i\theta) \) is realized cyclically with period \( 2\pi/\Delta E \). If we wished to fix our state in a given superposition, we can do this simply by changing our external magnetic bias flux by a fraction. If this fractional change in bias flux is sufficient, this will remove the coupling between the wells and set \( \theta \). Alternatively, this could be achieved by tuning the barrier height of the SQUID ring (see Fig. 9 later in this paper) so that the Josephson-tunneling energy becomes vanishingly small [24]. The effect of changing \( \theta \) can be dramatic, as illustrated in Fig. 5, where the Wigner function in the \( x-p \) plane is shown for selected values of this parameter. As is apparent, when the wave function for the ring is strongly (but not completely) localized in two or more
regions in the ring potential, it is the quantum interaction between these regions that is responsible for the creation of superposition of macroscopically distinct states.

We note that given the very slightly different energies between the symmetric and antisymmetric superposition states, due to the height of the barrier between the two wells, the superposition (5) will oscillate slowly back and forth between these wells. Using these energies we calculate this period to be 100 ns. This is much longer than the time constant corresponding to the \( \Lambda C \) frequency of the SQUID ring in our example (i.e., for \( C=5 \times 10^{-15} \) F, \( \Lambda=3 \times 10^{-10} \) H this is 7.6 ps) but well within the decoherence times of modern SQUID ring circuits [4, 25–27].

In the above discussion we have considered the development of macroscopic superposition states in a SQUID ring based on a choice of ring parameters which can be realized by fabrication and which are physically reasonable. However, this choice does not connect directly with published experimental data. We will therefore deviate briefly from our standard parameter values and consider an explicit example of superpositions of macroscopic states in a SQUID ring.

In Fig. 5 we show the Wigner functions for three different superpositions of the form

\[
\frac{1}{\sqrt{2}}[|s\rangle + \exp(i\theta)|a\rangle].
\]

In Fig. 7 we show the Wigner functions for three different superpositions of these eigenstates, again both in perspective and projection onto the \( x-p \) plane. We notice that compared with the Wigner functions of Fig. 5 both the states of the macroscopic quantum superposition corresponding to the left- and right-hand wells in Fig. 7 and the region of interference between them display more complex patterns. Given the choice of more highly excited states in the experiment of Friedman et al., this is to be expected. Nevertheless, Fig. 7 demonstrates that the Wigner function (and, of course, the associated Weyl function) can expose sophisticated quantum coherent behavior in SQUID rings.

**Dissipation**

In considering the effect of dissipation on the calculations presented in this paper we have chosen to use a standard approach, and one familiar in quantum optics. This is to be done by coupling the SQUID ring to a decohering monochromatic thermal bath. The master equation for the evolution of the density operator of the system then takes the form [28]...
\[
\frac{\partial \rho}{\partial t} = -\frac{i}{\hbar} [\mathcal{H}, \rho] + \frac{\gamma}{2\hbar} (M + 1)(2a^\dagger a^\dagger - a^\dagger a^\dagger - a^\dagger a^\dagger a^\dagger a^\dagger) + \frac{\gamma}{2\hbar} M(2a^\dagger \rho a - a^\dagger a^\dagger \rho - \rho a^\dagger a^\dagger),
\]

where \(M\) is related to the temperature \(T\) and the frequency \(\omega_0\) of each decohering bath via \(M_i = \exp\left(\frac{\omega_0 T}{\hbar}\right) - 1\), and \(\gamma_i\) is the coupling (damping rate) between each of the components to its respective thermal bath. For the following examples that we will now calculate we set \(T = 1\) K with \(\omega_0 = \omega\).

To illustrate the effect that dissipation has upon a macroscopic superposition state in a SQUID ring, we now solve the master equation (7) for the system evolution, where the ring is taken to be in its ground state at \(\Phi_e = 0.5\Phi_0\). For this computation we return to our initial circuit values of \(C = 5 \times 10^{-15}\) F, \(\Lambda = 3 \times 10^{-10}\) H, and \(\hbar = 0.047\Phi_0^2 / \Lambda\). Here we have chosen to use a decoherence rate of 0.01\(\omega\). In Fig. 8 we have plotted the Wigner and Weyl functions for the ring at particular times. As the system evolves we notice that the Wigner and Weyl functions display very clearly the disappearance of the quantum coherence between the two states of this superposition. We also note that after sufficient time has elapsed the Wigner function still displays two distinct flux probabilities. However, the SQUID ring is longer in a pure state and this represents a classical coin toss probability and not a quantum one. This is also clearly reflected in the disappearance of the symmetrically positioned correlation peaks in the Weyl function at these later times.

**Squeezed states of a SQUID ring**

Just as in quantum optics, where coherent light can be squeezed in the number-phase plane through its interaction with a nonlinear optical medium, we shall now demonstrate that a SQUID ring, with the cosine Hamiltonian (1), can be used to squeeze an initial coherent state. This starting condition in a SQUID ring can be achieved by changing the weak link structure in the ring. As is well known, the combined Josephson critical current in a parallel, two weak link, loop, connected by superconducting wires (a dc SQUID [29]), can be varied by adjusting the magnetic flux threading the loop. Such a structure can serve as the adjustable weak link within a larger diameter SQUID ring. This has been already discussed in several earlier papers, including that of Friedman et al. [14]. The arrangement is depicted in Fig. 9.

Provided the control magnetic flux \(\Phi_{x2}\) threading the minor loop in Fig. 9 is large enough, and the weak links in the loop can be fabricated (in principle) to possess identical criti-
The current density of around 20 kA cm\(^{-2}\) from our given effectively squeezed this state. We also see that for a judicious choice of decoherence rate and interval of time, dissipation can assist in the squeezing process. We note that in the \(\gamma=0.1\) case it can be seen that the SQUID ring decoheres to its (squeezed) vacuum state over a relatively short period. Thus, it appears that these potential wells may be used with facility to generate squeezing of the magnetic flux variable of the ring. It is therefore reasonable to assume that adiabatic changes of this potential would, unless matching conditions exist between local \(s\)-harmonic energy levels of adjacent wells, allow some adjustment of the expectation value of the flux in the ring.

**CONCLUSIONS**

In this paper we have explored two applications of SQUID rings, with strong analogies to the field of quantum...
optics, which may prove to be of great utility given the current interest in quantum circuit technologies. In the first we describe the manner in which we can create macroscopic superposition states in SQUID rings and the way these may be manipulated through an external control flux. In the second we provide a dynamical mechanism for inducing squeezing of the magnetic flux variable in a SQUID ring starting from an almost harmonic-oscillator state. These take advantage of the highly nonperturbative quantum nature of SQUID rings arising from the cosine coupling energy term in the ring Hamiltonian. As we emphasize, both are of great interest in the light of current research in the fields of quantum computing and quantum-information processing [10–13]. Moreover, the results presented in this paper emphasize that the SQUID ring can act as a versatile quantum device and, as we have shown previously, can be used to create correlations across extended, multicomponent, quantum circuit systems [1,2,30–33]. These correlations can, for example, be made manifest as quantum entanglements and quantum frequency up/down-conversion [1,2], with each controlled by the external bias flux applied to the SQUID ring. In this sense these phenomena, including macroscopic superposition states and squeezing, highlight the role of the SQUID ring as the essential machinery for a range of applications in superconducting quantum circuit technologies. This further enhances our understanding of these nonperturbative quantum objects and their possible usefulness in these incipient technologies.

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