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Article (Published Version)

Antunes, Nuno D and Bettencourt, Luís M A (1998) The length distribution of vortex strings in U(1) equilibrium scalar field theory. *Physical Review Letters*, 81 (15). pp. 3083-3086. ISSN 0031-9007

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The Length Distribution of Vortex Strings in $U(1)$ Equilibrium Scalar Field Theory

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(Received 7 July 1998)

We characterize the statistical length distribution of vortex strings by studying the nonperturbative thermodynamics of scalar $U(1)$ field theory in 3D. All parameters characterizing the length distributions exhibit critical behavior at T_c and we measure their associated critical exponents. Moreover, we show that vortex strings are generally self-seeking at finite temperature and that there exists a natural separation between long strings and small loops based on their spatial correlation behavior. [S0031-9007(98)07128-2]

PACS numbers: 11.27.+d, 05.70.Fh, 98.80.Cq

The accurate knowledge of the thermodynamic density and length distribution of a population of topological defects is a fundamental ingredient for scenarios of defect formation and evolution in the early Universe and in the laboratory. Although a phenomenological model exists, due to Kibble [1] and Zurek [2], for predicting defect densities it does not provide us with a microscopic understanding of the dynamics of defect populations, their length distribution, and correlations. These aspects of the defect network at the time of formation will have a crucial impact on the subsequent network evolution.

In the past, several attempts have been made at characterizing the statistics of defect networks. Analytical studies of cosmic string distributions at finite temperature were based on the assumption that strings form a gas of noninteracting random walks [3]. The generalization to realistic interactions is problematic and yields only qualitative predictions. Numerical support for the free random walk picture was found from Vachaspati-Vilenkin type algorithms [4]. There, phases of an underlying field are thrown down at random on sites of a lattice and vortex lines are formed by joining points around which the phases wind about the circle following the shortest path. This yields a (lattice-type dependent) fraction of long string of 67%–80% and a length distribution that is Brownian for long enough strings. However, these studies correspond to the *infinite* temperature limit of the underlying field theory [5], where it becomes trivially free, and therefore have no direct predictive power at criticality, when defects are formed.

Recently, developments in computer power made possible the first studies of defect network properties directly from the nonperturbative thermodynamics of field theories. These focused their attention on defect densities. In this Letter we proceed to characterize in detail the equilibrium statistics of a network of vortex strings in scalar $U(1)$ field theory in 3D. We measure several universal critical exponents associated with the string length distribution and construct a detailed picture of the critical behavior of the theory in terms of vortex strings.

The adoption of a dual perspective, *i.e.*, of the description of the mechanisms underlying criticality in terms of

nonlinear solutions of the theory was found to be the key to understanding the Kosterlitz-Thouless [6] transition in 2D, which proceeds by vortex-antivortex pair unbinding. Generally, given a dual model, critical behavior can be described equivalently in either the fundamental or the dual picture, the choice between either point of view being made based on computational advantage. However, the construction of exact dual theories proves very difficult in $D \geq 3$, except when supersymmetry is unbroken. The measurements of the string network universal critical exponents presented below provide us with direct constraints on dual field theories to $U(1) \lambda \phi^4$ in 3D [7].

Although $U(1) \lambda \phi^4$ is the simplest field theory containing vortex lines the model is of direct relevance to the thermodynamic description of liquid ^4He and of strong type II superconductors (such as most new high- T_c materials) in the absence of an external field. It also embodies applications to cosmology, *e.g.*, the phase transition associated with the spontaneous breakdown of Peccei-Quinn symmetry and the formation of axionic strings [8].

The methods used to drive the system to canonical equilibrium are familiar from stochastic field quantization. A more detailed description can be found elsewhere [5]. The equations of motion for the fields ϕ_i are

$$(\partial_t^2 - \nabla^2)\phi_i - m^2\phi_i + \lambda\phi_i(\phi_i^2 + \phi_j^2) + \eta\dot{\phi}_i = \Gamma_i, \quad (1)$$

where $i, j \in \{1, 2\}$ and $i \neq j$ in Eq (1). $\Gamma_i(x)$ is the Gaussian noise characterized by

$$\langle \Gamma_i(x) \rangle = 0, \quad \langle \Gamma_i(x)\Gamma_j(x') \rangle = \frac{2\eta}{\beta} \delta_{ij}\delta(x - x'), \quad (2)$$

where x, x' denote spacetime coordinates. Equations (1) and (2) ensure that the fluctuation-dissipation theorem applies in equilibrium, which will result for large times at temperature $T = 1/\beta$. We use a computational domain of size $N^3 = 100^3$, lattice spacing $\Delta x = 0.5$ and impose periodic boundary conditions in space. At criticality all universal exponents are independent of Δx . Once in equilibrium, we construct the string network from the phase of the complex field $\Phi = \phi_1 + i\phi_2$, at regular time

intervals. In all measurements presented below a number of samples between 400 and 1200 was used. The inverse critical temperature β_c , measured from field correlators, is $\beta_c = 1.91 \pm 0.01$ [5].

We separate strings as usual in cosmology into two distinct populations, one of loops, vortex lines shorter than $\sim N^2$ and another of long strings longer than any loop. Note that there is *a priori* no reason for this separation or for the definition of the boundary between the two classes. First we deal with the statistics of loops. Characteristic loop distributions above and below the critical temperature are shown in Fig. 1.

In order to characterize the loop length distribution $n(l)$ quantitatively we fit it for each temperature to an expression of the form

$$\langle n(l) \rangle = A l^{-\gamma} \exp(-\beta \sigma_{\text{eff}} l), \quad (3)$$

where the average is over the canonical ensemble. σ_{eff} is the T dependent effective string tension and has dimensions of mass per unit length. The temperature at which it vanishes defines the so-called Hagedorn temperature T_H , the point at which long string appears in the system. We present our estimate of errors, mostly resulting from performing multiparameter fits, in the final measurements of universal quantities only. Statistical errors are generally very small. γ is related to the string interactions. For Brownian (i.e., noninteracting) strings $\gamma = 5/2$, whereas $\gamma > 5/2$ or $\gamma < 5/2$ for self-avoiding or self-seeking walks, respectively. The loop distributions measured in equilibrium are in fact very well fit by Eq. (3), with $A(\beta)$, $\gamma(\beta)$ and $\sigma_{\text{eff}}(\beta)$ determined by χ^2 minimization. These three quantities undergo radical behavior changes near the T_c lending support to a recent claim [5] that the criti-

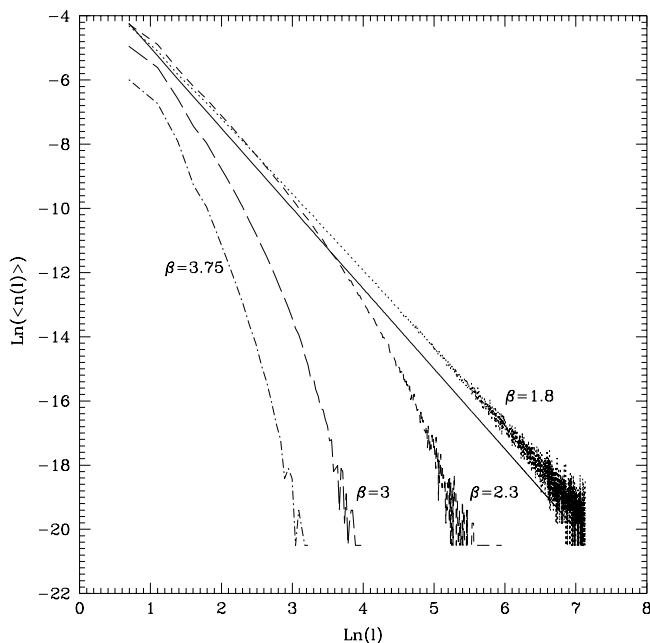


FIG. 1. The loop length distribution for several β . An arbitrary $l^{-5/2}$ distribution (solid) is plotted for comparison.

cal temperature for the fields, T_c and the Hagedorn temperature for the strings, T_H coincide in the infinite volume limit.

Figure 2 shows the temperature dependence of γ and σ_{eff} . We observe that as the critical point is approached from below σ_{eff} vanishes. A fit to σ_{eff} in this regime gives

$$\sigma_{\text{eff}}(\beta) \propto (\beta - \beta_H)^{\nu_{\text{st}}}$$

with $\beta_H = 1.96 \pm 0.01$, $\nu_{\text{st}} = 1.50 \pm 0.01$, and where $\beta_H = 1/T_H$. Note that β_H does not coincide exactly with β_c . This discrepancy was observed previously in [5], where it was shown that as the computational domain is increased the temperature at which long strings first appear β_H migrates to β_c . In addition, the relative precision of the measurement of σ_{eff} close to β_c is poor since it requires a large number of long loops, lying on the boundary of the long string/loop separation, e.g., for $\beta = 1.975$, we have that $l \sim (\beta \sigma_{\text{eff}})^{-1} = 0.44 \times 10^4 \sim N^2$. Below a more accurate measurement of β_H will be obtained from the behavior of the length distribution of long strings.

Also very interesting is the value of ν_{st} . This universal exponent for σ_{eff} in the global $U(1)$ theory, corresponds to the exponent $\nu_{\text{dual}} = \nu_{\text{st}}/2$, associated with the correlation length of its dual field theory [7]. We believe this measurement provides a first direct determination leading to the classification of the universality class to which the dual formulation belongs. Note that $\nu_{\text{dual}} = 0.75$ differs clearly from the mean-field prediction $\nu_{\text{MF}} = 0.5$, for any $O(N)$, $N \geq 1$, scalar field theory. Remarkably ν_{dual} coincides with the *exact* value for the critical exponent of an ensemble of interacting polymers in 2D [9].

As for γ , it is especially interesting to note that $\gamma < 5/2$ generally at any finite temperature; see Fig. 2. The

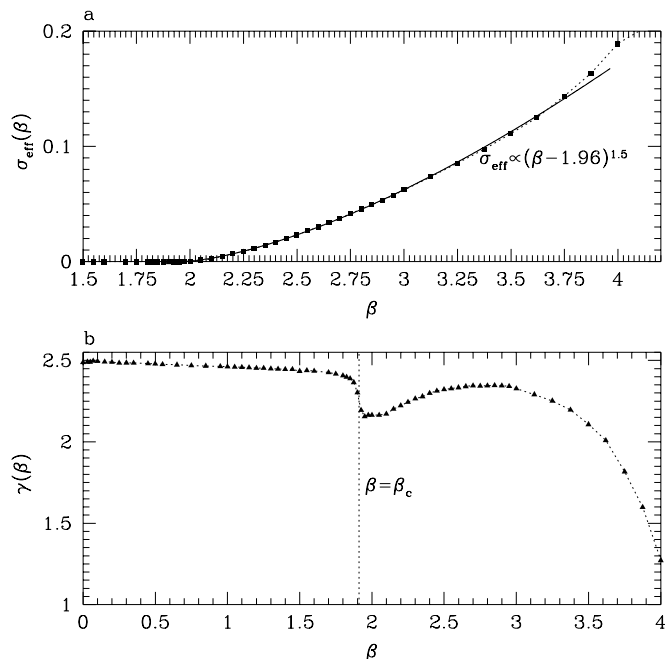


FIG. 2. (a) $\sigma_{\text{eff}}(\beta)$ and (b) $\gamma(\beta)$.

discrepancy between our observations and the Brownian result cannot be attributed to small computational domain effects. The computation of the Brownian distribution in a finite volume generates exponential corrections [10] to Eq. (3), which fail to describe the changes in the distributions correctly and are, moreover, β independent. Above T_c , $\gamma(\beta) \approx 5/2 - 0.0385\beta$ and $\gamma(\beta_c) = 2.23 \pm 0.04$, [11].

Figure 3 shows the β dependence of A and of the average *walk step size* $a(\beta)$, which we define in terms of $A(\beta)$ and $\gamma(\beta)$ by

$$A = \frac{1}{3} \left(\frac{3}{2\pi} \right)^{3/2} a^{\gamma-3}, \quad (4)$$

where the arbitrary numerical prefactor was taken from the computation of $\langle n(l) \rangle$ for random walks [10]. We observe that the amplitude of the distribution $A(\beta)$ varies drastically with temperature in the vicinity of T_c .

$A(\beta)$ alone determines the β -dependent probability of finding the smallest loops. In the vicinity of the critical point A suffers a dramatic drop until β_c is hit, followed by a sudden increase. At $\beta = \beta_c$ $dA/d\beta$ is discontinuous. The sharp drop in A signifies that $a(\beta)$ has grown and with it the possibility for producing longer strings.

From general considerations at a second order phase transition we can expect that all characteristic lengths will present divergences as $T \rightarrow T_c$. $a(\beta)$ is no exception. Note, however, that while $\xi(\beta)$, the correlation length associated with the connected field 2-point function, measures isotropic correlations of field excitations, a refers to correlations along the strings. It is therefore interesting to measure the critical exponents associated with a .

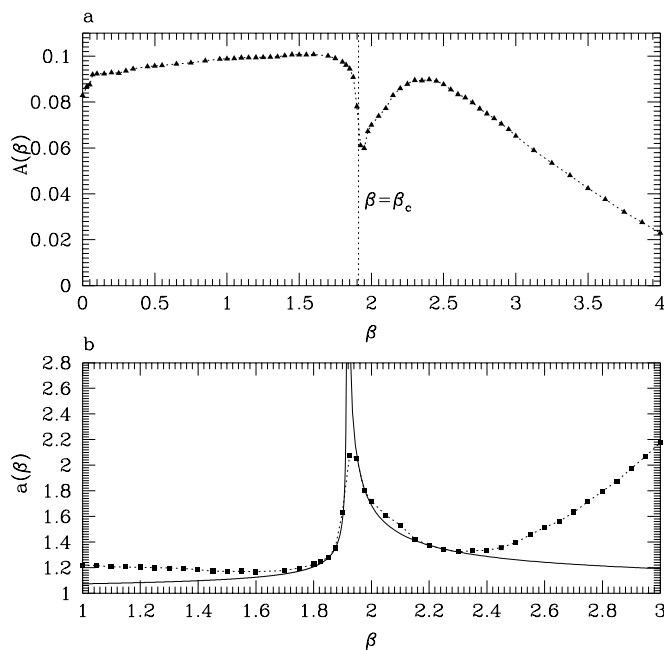


FIG. 3. (a) The amplitude of the loop distribution $A(\beta)$ and (b) the average step length $a(\beta)$ and the corresponding best fits Eqs. (5) at criticality.

Figure 3b shows the best fits to $a(\beta)$ at criticality:

$$\begin{aligned} a(\beta) - 1 &\propto 1/(\beta - \beta_c)^{\nu_a^+}, & \beta > \beta_c, \\ a(\beta) - 1 &\propto 1/(\beta_c - \beta)^{\nu_a^-}, & \beta < \beta_c, \end{aligned} \quad (5)$$

with $\nu_a^+ = \nu_a^- = 0.5 \pm 0.1$. These values are typical of mean-field predictions and differ from $\nu = 0.671$ [9], associated with ξ .

Above the critical point the behavior of A is given by $A(\beta) \approx 0.092 + 0.006\beta$. Note that in addition, given γ and above T_c ($\sigma_{\text{eff}} = 0$), the total density of loops

$$\rho_{\text{loop}} = \sum_{l=4}^{\infty} l \langle n(l) \rangle = \frac{A}{4^{(\gamma-2)} (\gamma-2)} \quad (\gamma > 2) \quad (6)$$

is proportional to A in the infinite volume. For $\beta = 0$, when the distribution becomes Brownian the value of A does indeed exactly coincide with our measurement of $\rho_{\text{loop}}(0) = A(0) = 0.083$ in [5].

In Fig. 4, we characterize the length distribution of long strings and show the mean distance between two string segments $R(n)$ as a function of the number of steps (in lattice units) along the string. The distribution of long string, is notoriously difficult to measure because the total number of long strings in any computational domain is typically only a few. Here we could probe 600–1200 independent string networks, a large number compared to any study in a computational domain of this size. However, in such a set of data the number of long strings of a given length is never more than one in the total ensemble of measurements and a direct fit to the data becomes impossible. To obviate this problem we adopted a maximum likelihood test, see, e.g., [12], to fit the data from all samples to a distribution of the form $Al^{-\gamma_l}$. In the free random walk approximation $\gamma_l = 1$, as the result of the periodicity of the boundary conditions [10].

Figure 4a shows the temperature dependence of γ_l . For $T > T_c$, γ_l is close to the Brownian value $\gamma_l = 1$ converging to it as $\beta \rightarrow 0$, as $\gamma_l \approx 1 + 0.1147\beta$. In the small band of temperatures below T_c for which we identified long string, γ_l assumes much larger values.

This sharp increase in γ_l is an indication that the distribution no longer has the simple form of a power law. In analogy to the form of the loop distribution it marks the onset of exponential suppression. Note that it was necessary to look at strings classified as long to notice this suppression, allowing us a more precise insight into the temperature variation of σ_{eff} , just below criticality. This behavior reinforces our previous conjecture that true long string is present in the system only above T_c which would, in the infinite volume limit, coincide with T_H .

A complementary test on the interacting nature of the strings is provided by $R(n)$, shown in Fig. 4b. Generally, one expects $R(n) \sim n^\alpha$, where n is the number of steps in lattice units. For Brownian walks it is well known that $\alpha = 0.5$. On the other hand $\alpha > 0.5$ if the interactions are repulsive ($\alpha \approx 0.59$ for self-avoiding walks) and $\alpha \approx 1/3$ if they are attractive at low T [13].

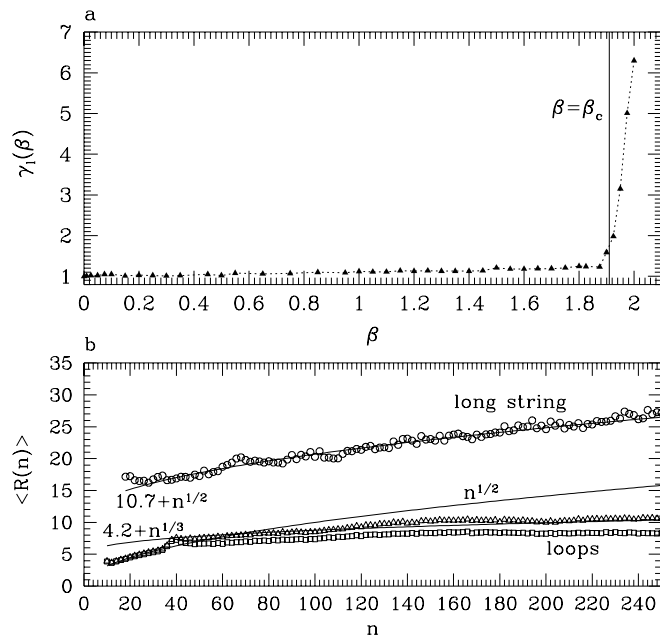


FIG. 4. (a) $\gamma_l(\beta)$ for the long string length distribution and (b) $R(n)$ at $\beta = 1.8$ for loops (Δ) and long strings (\circ) and for loops at $\beta = 2.05$ (\square). Solid curves show the behavior of free random ($\alpha = 0.5$) and self-seeking ($\alpha = 1/3$) walks.

In $U(1)$ scalar field theory two string segments, at distance d interact with potential $V(d) \approx \vec{n}_1 \cdot \vec{n}_2 \ln(r/m)$, for $d \geq m^{-1}$, the width of the string. Here \vec{n}_i is the unit vector normal to the i th string segment times the topological charge of the vortex. Thus vortices interact with a potential which grows with distance. At short distances $d < m^{-1}$ the interactions vanish since the superposition of two vortices (or of a vortex-antivortex) is also a solution of the static field equations.

Small loops of string are the direct equivalent in 3D of vortex-antivortex pairs in 2D, since each segment will tend on the average to see, normal to the string, another string segment with opposite vorticity. This fact permits the nucleation of small loops of string below T_c .

As T increases longer loops can be nucleated. The nature of the interactions makes it advantageous to create loops where the correlations between segments of opposite vorticity are maximal. Consequently, loops behave on the average as random walks for a short distance, while the interactions are irrelevant, and beyond become self-seeking; see Fig. 4b. This distance increases with T .

As the critical point is approached the total string density mounts to a universal critical value $\rho_{\text{tot}}(\beta_c) \approx 0.2$. At this point enough string exists in the system that the effects of the interactions shield each others out efficiently. Precisely then a population of long strings can be nucleated, interacting with itself through the background loop sea with a potential that is almost vanishing. In this sense long strings are born quasi-Brownian. This fact is demonstrated in Fig. 4b. After a short transient for small distances, where they are self-avoiding (otherwise they would have

become a loop), long strings assume an extraordinarily clear random walk behavior, corresponding to Hausdorff dimension $D_H \approx 2$. This establishes the difference between the two string populations, since loops remain self-seeking ($D_H \approx 3$) even for $T > T_c$; see Fig. 4.

In conclusion, we have found quantitative evidence that the phase transition in 3D scalar $U(1)$ field theory proceeds by vortex string nucleation and in particular is triggered by long string proliferation, as often suggested [14]. We established that a population of quasi-Brownian long strings exists above T_c only, and that it is different from small loops which remain self-seeking. We characterized the length distribution of both loops and long strings, and measured associated universal critical exponents. The latter provide the first direct constraints on field theories dual to the $U(1)$ scalar model in 3D. The critical behavior of the length distributions confirms our previous claim that the Hagedorn phase transition in the strings coincides with T_c . The Ginzburg temperature ($\beta_G \approx 2.34$) seems to play no particular role in the thermodynamics.

We thank M. Hindmarsh, J. Kottmann, and A. Schakel for useful discussions, the Department of Computer Science at the T. U. of Berlin for usage of their Cray T3E and A. Yates for his string tracing routine. This work was partially supported by the European Science Foundation. N.D.A. is supported by JNICT, Contract No. PRAXIS XXI/BD/13765/97.

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