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Semi-classical transport theory for non-Abelian plasmas

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Abstract

We review a semi-classical transport theory for non-Abelian plasmas based on a classical picture of coloured point particles. Within this formalism, kinetic equations for the mean particle distribution, the mean fields and their fluctuations are obtained using an ensemble-average in phase space. The framework permits the integrating-out of fluctuations in a systematic manner. This leads to the derivation of collision integrals, noise sources and fluctuation-induced currents for the effective transport equations of QCD. Consistency with the non-Abelian gauge symmetry is established, and systematic approximation schemes are worked out. In particular, the approach is applicable to both in- and out-of-equilibrium plasmas. The formalism is applied explicitly to a hot and weakly coupled QCD plasma slightly out of equilibrium. The physics related to Debye screening, Landau damping or colour conductivity is deduced in a very simple manner. Effective transport equations are computed to first and second order in moments of the fluctuations. To first order, they reproduce the seminal hard-thermal-loop effective theory. To second order, the fluctuations induce collisions amongst the quasi-particles, leading to a Langevin-type transport equation. A complementary Langevin approach is discussed as well. Finally, we show how the approach can be applied to dense quark matter systems. In the normal phase, the corresponding kinetic equations lead to the hard-dense-loop effective theory. At high density and low temperature diquark condensates are formed, changing the ground state of QCD. In the superconducting phase with two massless quark flavours, a transport equation for coloured excitations is given as well. Possible future applications are outlined.

Contents

I. INTRODUCTION

A. Motivation

In recent years, there has been an increasing interest in the dynamics of non-Abelian plasmas at both very high temperature and density. One of the most spectacular predictions of quantum chromodynamics is asymptotic freedom, which implies that quarks and gluons behave as free particles in such extreme conditions because their coupling becomes very weak at short distances. It is expected that a specific state of matter with quarks and gluons unconfined – the so-called quark-gluon plasma – can exist. Many efforts for its experimental detection in the core region of heavy-ion collisions will be made at RHIC and LHC within the next years.

Other possible applications concern relativistic non-Abelian plasmas in extreme cosmological and astrophysical conditions, like the electro-weak plasma in the early universe, the physics of dense neutron stars or the physics of supernovae explosions. If baryogenesis can be understood within an electroweak scenario, an understanding of the physics of the electroweak plasma in the unbroken phase is essential for a computation of the rate of baryon number violation. In different astrophysical settings, the density reached can be such that the hadrons melt into their fundamental constituents, which gives rise to a very rich phenomenology.

It is therefore mandatory to devise reliable and maniable theoretical tools for a quantitative description of non-Abelian plasmas both in and out-of equilibrium. While some progress has been achieved in the recent years, we are still far away from having a satisfactory understanding of the dynamics of non-Abelian plasmas. There are different approaches in the literature in studying non-Abelian plasmas, ranging from thermal quantum field theory to transport equations or lattice studies. Every approach has its advantages and drawbacks, and each one appears to be suited to answer a specific subset of questions. A quantum field theoretical description should be able to describe all possible aspects of the plasma. Most applications have concerned to the weakly coupled plasma close to equilibrium. But even there, the situation is complicated due to the non-perturbative character of long-wavelength excitations. Lattice simulations, which in principle can handle large gauge couplings, have proven particularly successful for non-perturbative studies of static quantities like equal-time correlation functions. However, it seems very difficult to employ them for the dynamical case. Furthermore, the standard Monte Carlo simulations of QCD with a finite quark chemical potential fail. A kinetic or transport theory approach has proven most efficient for the computation of macroscopic properties of the plasma, like transport coefficients such as viscosities or conductivities. In turn, a direct evaluation of transport coefficients in the framework of quantum field theory is quite involved. Already the leading order result at weak coupling requires the resummation of infinitely many perturbative loop

diagrams, an analysis which at present has only been performed for a scalar theory.

A new semi-classical transport theory, based on a classical point particle picture, has been introduced recently [105–108]. There are several motivations for such a formalism. First of all, it is expected that the main characteristics of non-Abelian plasmas can be understood within simple semi-classical terms. Secondly, this approach allows for the study of transport phenomena without the difficulties inherent to a full quantum field theoretical analysis. Finally, the formalism is even applicable to hot out-of-equilibrium plasmas, which are of particular relevance for the case of heavy ion collisions, or to dense quark matter in the superconducting phase.

This article reviews in detail the conceptual framework for the semi-classical approach. In addition, we discuss applications to the case of a hot and weakly coupled non-Abelian plasma, and to dense quark matter systems. In the remainder of the introduction, we review the ideas behind the semi-classical approximation which are at the basis of the present formalism. We also review the present understanding for constructing effective theories of hot and weakly coupled non-Abelian plasmas, where the formalism is finally put to work. For an application to the physics of dense quark matter, we briefly summarise the present understanding about superconducting phases of matter at large baryonic density. A detailed outline of the review is given at the end of the introduction.

B. Semi-classical approach for hot plasmas

There are several reasons for considering a semi-classical approximation. A heuristic argument is that the occupation number per mode vector $|\mathbf{p}|$ of 'soft' ($\hbar|\mathbf{p}| \ll T$) gauge fields in a hot plasma at temperature T is very high, due to the Bose-Einstein enhancement. This suggests that the long wave-length limit corresponds to the classical limit where the fundamental constant $\hbar$ vanishes. Therefore, one has reasons to believe that the soft *quantum* fields are, to leading order, well approximated by soft *classical* ones. Such a reasoning has been substantiated by various workers in the field (see [38] for a recent review). On the other hand, the 'hard' modes of the plasma cannot be approximated this way as their occupation number is of order unity. However, it has been established that weakly coupled hard modes behave like quasi-particles [33]. They can be described, to leading order in a gradient expansion, by an ensemble of coloured *classical point particles* moving on world lines. Therefore it is conceivable that the main characteristics of such plasmas can be understood within a purely semi-classical language.

For QED plasmas, semi-classical methods have been known and applied for a long while [97,94]. They consist in describing the charged constituents of the plasma as classical point particles moving on world lines. Their interactions are determined self-consistently through the Maxwell equations induced by the current of the particles. On the mean

field level, the resulting Boltzmann equation for the one-particle distribution function is known as the Vlasov equation. Beyond the mean field approximation, several approaches for the construction of a full kinetic theory for such plasmas are known in the literature, the most famous one being the BBGKY hierarchy for correlator functions within non-relativistic statistical mechanics [20]. Alternative approaches have been put forward as well. Of particular interest is the approach by Klimontovitch, who constructed a kinetic theory on the basis of the one-particle distribution function and the correlators of fluctuations about them [94]. This leads naturally to a description in terms of mean fields and fluctuations. Conceptually, the new ingredient in his approach is that the plasma is *not* considered as a continuous medium. Instead, the stochastic fluctuations of the particles are taken into account, and the dissipative character of effective long-range interactions enters naturally in this framework. The procedure leads to a Boltzmann-Langevin type of effective transport equations, which, on phenomenological grounds, have been already proposed for non-charged particles by Bixon and Zwanzig [26]. This approach has been extended for the study of nuclear collisions [127,14,54] (see [2] for a recent review of applications to nuclear dynamics). It also appeared that systematic approximations within the Klimontovitch approach are better behaved than those based on the BBGKY hierarchy [94]. Finally, this approach permits the derivation of collision integrals, like the Balescu-Lenard one for Abelian plasmas [99,94].

For non-Abelian plasmas, semi-classical transport equations can be obtained in essentially two distinct manners. The first one starts from a quantum field theoretical framework, which is used to construct a (quantum) transport theory, for example in terms of Wigner functions, or for hierarchies of Schwinger-Dyson equations [89,20,61]. The semi-classical approximation is performed in a second step on the level of the transport theory, that is, on the 'macroscopic' level (see, for example [61,117,27]). Alternatively, one may perform the semi-classical approximation already on the 'microscopic' level, in analogy to the Abelian case outlined above. In this case the concept of an ensemble of classical *coloured* point particles moving on world lines has to be invoked. The new ingredients are the $SU(N)$ colour charges of the particles. Their classical equations of motion for high dimensional representations were first given by Wong [146] and can be understood as equations of motion for expectation values of quantum wave packets, as shown by Brown and Weisberger [49].

A transport equation based on the classical point particle approximation has been given by Heinz [69–72]. It consists in a Boltzmann equation for a one-particle distribution function for gluons, quarks or anti-quarks with a – yet unspecified – collision term. On the mean field level, neglecting collision terms, these equations are known as the non-Abelian Vlasov equations. These are intimately linked to a gradient expansion of the Wigner transform, as pointed out by Winter [145], and to the quantum Boltzmann equation, as discussed by Elze and Heinz [61]. An important step forward in the understanding of the semi-classical transport theory has been achieved by Kelly et al. [91,92]. They noticed that

a gauge-consistent solution of non-Abelian Vlasov equations to leading order in the gauge coupling reproduces precisely the hard thermal loop effective kinetic theory. In addition, the authors clarified the role of the non-Abelian colour charges as non-canonical phase space variables and their explicit link to canonical Darboux variables. This formalism has also been applied to magnetic screening [112], and to cold dense plasmas [111] which are characterised by a large chemical potential.

The effects of non-Abelian fluctuations have to be considered to go beyond the Vlasov approximation. In the context of QCD transport theory this was pointed out by Selikhov, who, motivated by the earlier work of Klimontovitch, derived a collision term of the Balescu-Lenard type for non-Abelian Boltzmann equations [133]. This method, applied by Selikhov and Gyulassy to the problem of colour conductivity, uncovered a logarithmic sensitivity of the colour relaxation time scale [134,135]. However, in their considerations only the local part of the corresponding collision term has been identified, which implies that the corresponding colour current is not covariantly conserved. Along similar lines, Markov and Markova applied the procedure of Klimontovitch to a classical non-Abelian plasma and formally derived a Balescu-Lenard collision integral [113]. The strategy is similar to Selikhov's approach, except that it embarks from a purely classical starting point. This approach overlooked the important point that the colour charges are non-standard phase space variables, which is crucial for a definition of an ensemble average. Also, neither the non-linear higher-order effects due to the non-Abelian interactions have been considered, nor the requirements implied due to gauge symmetry.

A fully self-contained approach, aimed at filling this gap in the literature of classical non-Abelian plasmas was presented recently and is the subject of the review [105–108,103,104,109]. It is based on a classical point particle picture and uses the Klimontovitch procedure, extended to the non-Abelian case, to describe non-Abelian fluctuations. The essential contribution is considering the non-Abelian colour charges as dynamical variables and introducing the concept of ensemble average to the non-Abelian kinetic equations. Equally important is the consistent treatment of the intrinsic non-linearities of non-Abelian gauge interactions. The fundamental role of fluctuations in the quasi-particle distribution function has been worked out, and results in a recipe as to how effective semi-classical transport equations can be *derived* in a systematic manner. This set of coupled dynamical equations for mean fields and correlators of fluctuations should be enough to consider all transport phenomena in the plasma, at least in the domain where the underlying point particle picture is applicable. This procedure could even be applicable for out-of-equilibrium situations, since the derivation of the transport equation does not depend on the system being in equilibrium or not. Although we are not applying the formalism to plasmas out-of-equilibrium, this observation could open a door for interesting further applications in a domain relevant for future experiments. It would be very interesting to investigate out-of-equilibrium situations and plasma instabilities within the present transport theory.

C. Hot QCD plasmas

The case of a hot and weakly coupled non-Abelian plasma close to thermal equilibrium has already proven quite rich and involved in structure due to the non-perturbative character of the long-wavelength fluctuations [102,68]. We shall apply the aforementioned formalism in detail to the weakly coupled plasma to show how the physics related to Debye screening, Landau damping and colour relaxation can be understood very efficiently within this simple semi-classical framework.

Let us briefly summarise the present status for constructing effective (transport) theories for hot and weakly coupled non-Abelian plasmas close to thermal equilibrium at temperature T (see Fig. 1). The physics for the whole range of momentum modes of the non-Abelian fields requires the framework of thermal QCD [90,100,138]. *Effective* theories for the long-range modes are obtained when high-momentum modes are integrated-out. There is one conceptual limitation for the use of perturbative methods due to the non-perturbative magnetic sector of QCD, which corresponds to momentum scales about $\sim g^2 T$, the magnetic mass scale. The interactions of modes with smaller momenta are strictly non-perturbative [68], hence, any perturbative scheme for integrating-out modes with momenta $\sim |\mathbf{p}|$ relies on $\sim g^2 T/|\mathbf{p}|$ as the effective expansion parameter [34].

The first step towards obtaining an effective theory for the long wave-length excitations has been made by Pisarski [122] and Braaten and Pisarski [43,45]. Standard thermal perturbation theory is plagued by severe infrared divergences due to massless modes. Braaten and Pisarski proposed the resummation of all 1-loop diagrams with hard internal momenta and soft external ones, the seminal Hard Thermal Loops (HTL). Here, 'hard' refers to momenta of the order of the temperature $|\mathbf{p}| \sim T$. We denote momenta with $|\mathbf{p}| \sim gT$ as 'soft' (sometimes also referred to as 'semi-hard' in the literature). The HTL-resummed gluon propagator has its poles not on the light cone and the dispersion relation yields, apart from a complicated momentum dependence, the Debye (screening) mass $\sim gT$ for the chromo-electric fields. The HTL polarisation tensor also has an imaginary part, which describes the emission and absorption of soft gluons by the hard modes, known as Landau damping. The resulting effective theory for the soft modes contains highly non-local interactions in space and time, induced as HTL corrections to the propagator and to the vertices. It leads to gauge-invariant results for physical observables like the soft gluon damping rate [42,44]. An effective action for the HTLs was given by Taylor and Wong [141]. Further aspects of the HTL effective theory, for example their link to Chern-Simons theory [58,59,82], to Wess-Zumino-Novikov-Witten actions [118] and their Hamiltonian structure [119] have been considered subsequently.

A local formulation of the HTL effective theory was given by Blaizot and Iancu [28–30] and by Nair [118,119]. Blaizot and Iancu managed to reformulate the HTL effective theory within the language of kinetic theory. Such equations are similar to those for the HTL effec-

tive theory of QED plasmas as considered by Silin [136]. The derivation has been achieved invoking a truncation to a Schwinger-Dyson hierarchy. This has lead to a transport equation for the distribution function describing the hard or particle-like degrees of freedom. The advantage of a kinetic description is that the *non-local* interactions in the HTL effective theory are replaced by a *local* transport theory. The crucial step is to consider the quasi-particle distribution function as independent degrees of freedom, describing the hard excitations of the plasma. This also has lead to a local expression for the HTL energy in terms of the soft gauge fields and the colour current density [118,30]. It is worthwhile pointing out that the HTL effective theory can be derived within a semi-classical transport theory based on a point particle picture [91,92].
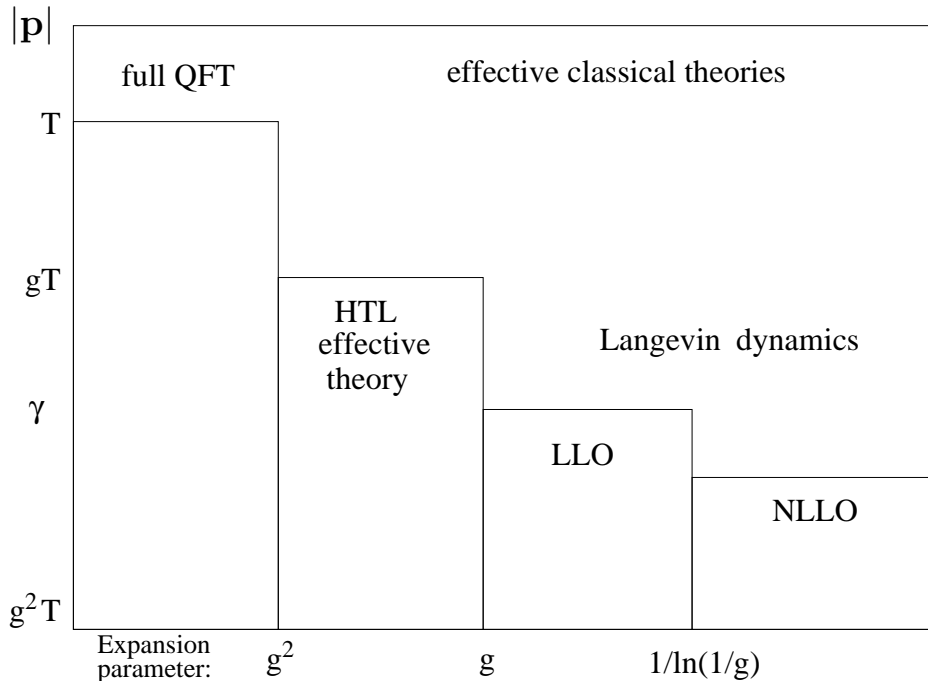
Figure 1: Schematic diagram for the series of effective theories for hot and weakly coupled non-Abelian plasmas close to thermal equilibrium. The physics of the hard modes with momenta $|\mathbf{p}| \sim T$ or larger needs the full thermal QCD. Effective *classical* theories are found for modes $|\mathbf{p}| \ll T$. The hard-thermal-loop (HTL) effective theory integrates-out the hard modes, and is effective for modes at about the Debye mass, $|\mathbf{p}| \sim gT$. It can be written as a collisionless Boltzmann equation. The effective expansion parameter is $\sim g^2$. A collisional Boltzmann equation is found after integrating-out the modes $|\mathbf{p}| \sim gT$ to leading logarithmic order (LLO), which is an expansion in g. The effective theory for the ultra-soft gauge fields with spatial momenta $|\mathbf{p}| \ll gT$ is a Langevin-type dynamical equation. The next step integrates-out the modes with $|\mathbf{p}| \sim \gamma \sim g^2 T \ln(1/g)$, which is an expansion in $1/\ln(1/g)$ and yields next-to-leading-logarithmic order (NLLO) corrections without changing the qualitative form of the effective theory [37].

Some attempts have been made for obtaining effective theories beyond the HTL approximation [80,11]. Eventually, Bödeker showed the way how the Debye scale can be integrated-out within a quantum field-theoretical framework [34–36]. His derivation relied on a semi-classical approximation which treats the soft modes as classical fields, and made use of the local expression for the HTL effective energy which allowed him to define a weight function and to perform classical thermal averages over initial conditions. This procedure has also been understood as an appropriate resummation of certain classes of Feynman diagrams [35,32]. To leading logarithmic order (LLO), the resulting effective theory corresponds to a Langevin-type Boltzmann equation, including a collision term and a source for stochastic noise. The physics behind it describes the damping of colour excitations due to their scattering with the hard particles in the plasma. A dynamical scale γ appeared, which corresponds to the damping rate of hard gluons in the plasma. It is of the order of $\sim g^2 T \ln(1/g)$.

These findings initiated further developments in the field. Arnold, Son and Yaffe [9,10] interpreted the kinetic equation in terms of Lenz Law and gave an alternative derivation of Bödeker's collision term and the related noise source. The very same effective kinetic theory has also been obtained within the semi-classical approach which will be discussed in the present article [105,106]. Subsequently, and making use of an additional fluctuation-dissipation relation, Valle-Basagoiti presented an equivalent set of transport equations [143]. Finally, Blaizot and Iancu extended their earlier work to higher order and derived the collision term from a truncated Schwinger-Dyson hierarchy [31].

The Boltzmann-Langevin equation, when solved to leading order in the overdamped limit $p_0 \ll |\mathbf{p}| \ll \gamma$ results in a very simple Langevin equation for the ultra-soft gauge fields only [36]. This effective theory is, quite remarkably, ultra-violet finite [9] and has been used for numerical simulations to determine the hot sphaleron transition rate [116]. Some consideration beyond LLO have been made in [35,32]. A non-local Langevin equation, valid to leading order in g and to all orders in $\ln(1/g)$ has been given by Arnold [7]. It is valid for frequencies $p_0 \sim g^4 T$, and has been used by Arnold and Yaffe [12,13] to push the computation of the colour conductivity to the next-to-leading logarithmic order (NLLO). They made use of the stochastic quantisation method [148] to convert the stochastic dynamical equations into path integrals [7], which can be treated with standard techniques. It is interesting that the effective theory for the gauge fields remains of the Langevin-type even at NLLO. A local Langevin equation, valid for frequencies $p_0 \lesssim g^2 T$, has been given by Bödeker [39]. It is expected that the UV divergences are local as well, in contrast to those associated to the original Boltzmann-Langevin equation [34].

Thus, it is fair to say that the physics related to Debye screening, Landau damping, and colour relaxation in the close-to-equilibrium plasma is by now well understood. A variety of different and complementary approaches have lead to identical effective transport equations. All approaches made use of some semi-classical approximation in the course of

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their considerations. Interestingly, these characteristics of a hot plasma can be understood within a simple semi-classical language straight away.

D. Dense quark matter

So far we have discussed effective theories for the long distance physics of QCD plasmas at very high temperature. It is also interesting to consider situations when the baryonic density is very high, while the temperature is low. Usually such a QCD plasma is called dense quark matter, or simply quark matter. This state of matter could be realized in various astrophysical settings, such as in the core of neutron stars, collapsing stars and supernova explosions. In the presence of strange quarks, and at zero pressure, quark matter might even be stable, in which case quark stars could exist in nature.

It has been known for a long time that cold dense quark matter should exhibit the phenomenon of colour superconductivity [22,23,15]. The present microscopic understanding of colour superconductivity relies on techniques of BCS theory [132] adapted to dense quark matter (see Refs. [126,4] for recent reviews and related literature). At asymptotically large baryonic densities, and because of asymptotic freedom, the strong gauge coupling constant becomes small. Then, the relevant degrees of freedom of the system are those of quarks, filling up their corresponding Fermi seas up to the value of the Fermi energy $E_F = \mu$, where μ is the quark chemical potential. These highly degenerate fermionic systems are very unstable to attractive interactions. In dense quark matter the attractive interaction among quarks is provided by one-gluon exchange in an antitriplet colour channel. This leads to the formation of diquark condensates, in analogy to the Cooper pairs of electromagnetic superconductors.

In QCD a diquark condensate cannot be colour neutral, and thus the colour symmetry is spontaneously broken and gluons acquire a mass through the Anderson-Higgs mechanism. Because the gauge symmetry is $SU_c(3)$ and there are some flavour symmetries for massless or light quarks, the possible patterns of symmetry breaking are richer than in the Abelian case. One finds different phases of quark matter, according to the number of quark flavours that participate in the condensation. The form of the diquark condensate is dictated by Pauli's principle and by the fact that it should minimise the free energy of the system. The different colour superconducting phases of quark matter are characterised by the total or partial breaking of the gauge group, and by the possible existence of Nambu-Goldstone modes associated to the breaking of the global symmetries. Using standard techniques of BCS theory [132] it is possible to study the microscopic properties of the colour superconductor in the weak coupling regime.

While our knowledge of the microscopic behaviour of quark matter is increasing, little is known about its macroscopic behaviour. Is quark matter a dissipative or dissipativeless

system? Is it such a good heat and electricity conductor as an electromagnetic superconductor? To answer those questions, it is mandatory to compute the transport coefficients of quark matter in their different possible phases. Kinetic theory provides the perfect framework for such a computation.

A classical transport equation for the gapped quasiparticles of a two-flavour colour superconductor has been proposed recently [109]. When the temperature is increased, thereby melting the diquark condensates, the transport equation reduces to the classical transport equation valid for a non-Abelian plasma in the unbroken phase. In the close-to-equilibrium case and in the Vlasov approximation, the leading-order solution to the transport equation reproduces the one-loop gluon polarisation tensor for small gluon energy and momenta, as found within a quantum field theoretical computation. It is also worth mentioning that the microscopic dynamics of the gapped quasiparticles is not governed by the Wong equations, as in the normal phase.

E. Further applications

The starting point of this approach are the classical equations of motion obeyed by coloured point particles, which, in the unbroken phase of a non-Abelian gauge theory, are the Wong equations [146]. In the past, these equations have also been studied for other purposes, and we briefly summarise the main applications here. We also comment on other uses of classical methods for studying the quark-gluon plasma.

After Wong proposed the equations of motion for classical Yang-Mills particles, a number of publications have been concerned with a more fundamental understanding of them, either by providing corresponding point particle Lagrangians or by establishing a link between point particle Lagrangians and one-loop effective actions in quantum field theories. Balachandran et al. [17,18] and Barducci et al. [21] proposed different Lagrangians which lead to the Wong equations. Upon quantisation, the several different choices describe particles which belong to reducible or irreducible representations of the Lie group. A unified description of the different choices was given in [18]. Balachandran et al. showed that upon quantisation, some of the parameters which appear in the Lagrangian are restricted to a certain set of values. This reflects the fact that the spectrum of the Casimir invariants of the Lie group is discrete.

A different line of research concerned the link between the point particle Lagrangian, on one side, and quantum field theory on the other. Brown and Weisberger [49] argued that the Wong equations can be interpreted as classical equations of motion for expectation values of quantum fields. Strassler [140] showed that the one-loop effective action in quantum field theory can be expressed in terms of a quantum mechanical path integral over a point particle Lagrangian. In the case of non-Abelian gauge theories, D'Hoker and Gagne [55,56]

gave the world-line Lagrangian for a non-Abelian gauge field theory. Pisarski [124] noted that this Lagrangian is identical to the one for Wong particles. It has been suggested that this intimate link may provide a deeper explanation for the applicability of the semi-classical approximation. Jalilian-Marian et al. [84] followed this line of research. They derived, within a real time many-body formalism for the world line action, a set of effective transport equations which closely resemble those studied here.

Gibbons et al. [63], and Holm and Kupershmidt [77] employed the Wong equations to derive a set of chromohydrodynamic transport equations. These equations are the non-Abelian analogues of the magnetohydrodynamic equations for charged fluids. These authors did not attempt, however, to link their approach with a quantum field theoretical analysis. At present, it is not clear whether consistent chromohydrodynamic equations can be derived from QCD in the first place, due to the (yet) unknown dynamics of the chromomagnetic fields in the plasma. In this respect, the Abelian and non-Abelian cases are qualitatively different, because magnetic fields are never screened in a Coulomb plasma, while chromomagnetic fields are supposed to be screened by a non-perturbative magnetic mass.

The transport equations associated to the Wong particles have been used to study non-Abelian dynamics, both analytically and numerically, in combination with lattice simulations. Based on assumptions linked to the colour flux model, some aspects of the quark-gluon plasma during the very early stages of an ultrarelativistic heavy-ion collision have been studied in [120,121,57]. This concerns the production and evolution of a quark [120] or a quark-gluon plasma [121], created in a constant colour-electric field. The production of gluons from a space-time dependent chromofield has been discussed in [57]. The use of lattice simulations in combination with the Wong point particle degrees of freedom have first been pointed out by Hu and Müller [79], and Hu, Müller and Moore [115]. Here, the classical Yang-Mills equation is formulated on a spatial lattice, following the standard Kogut-Susskind implementation. Then one adds the classical point particle degrees of freedom. This technique has been used to construct a lattice implementation of the HTL effective theory [115], which allowed for the computation of the Chern-Simons diffusion rate, a quantity which is essential for the evaluation of baryogenesis in an electroweak scenario.

A few further applications and extensions have been considered in the literature. In a quantum mechanical framework, the non-Abelian charges have been used to describe the non-Abelian analogue of the Aharanov-Bohm effect [16]. The dual of the Wong equations have been studied in [53]. These are the equations obeyed by particles which are the non-Abelian analogues of the Dirac magnetic monopole of electromagnetism. Explicit analytical solutions to the Wong equations for several coloured point particles, have been found in [96]. The extension of Wong's equations for QCD to curved space time has been studied by Brandt, Frenkel and Taylor [48]. They constructed the corresponding effective action and obtained an exact, but implicit, solution of the classical Boltzmann equation. Semi-classical methods have been applied in the context of small x physics. Here, the Wong equations

have been used to construct a small x effective action [83], which opens an interesting door to future applications.

An important domain of research concerns the computation of transport coefficients of hot and dense non-Abelian plasmas, such as the shear and bulk viscosities, heat and electrical conductivities, and baryon, lepton and flavour diffusion. In the past, a derivation of shear and bulk viscosity from quantum field theory has only been performed for a scalar theory [85,86]. Based on classical transport theory, a number of leading order computations have been done, though not necessarily within the Wong particle picture. The first computations made use of a relaxation time approximation, which allowed a correct determination of the functional dependence of transport coefficients on the gauge coupling [78]. The considerations in [25,75,76,88,114,24] improved on the relaxation time approximation in that the relevant collision terms were deduced from the scattering amplitudes resulting from the particle interactions. All earlier computations have been recently reviewed in [8], where some numerical errors in the existing literature were detected and corrected. At present, the results of these transport coefficients are only known to leading logarithmic order in the non-Abelian gauge coupling. While computations of transport coefficients are typically based on linear response, the case of non-linear response has recently been emphasised in [52], for the example of the quadratic shear viscosity of a weakly coupled scalar field. It has been argued that an intimate link between classical transport theory and response theory ensures that the non-linear response is correctly described by classical transport theory.

F. Outline

This review presents an approach to semi-classical transport theory for non-Abelian plasmas based on a classical point particle picture. Both conceptual and computational issues are considered. The first part, Sections II – V, addresses the various conceptual aspects of the approach, while the second part, Sections VI – X, presents an application to hot plasmas close to thermal equilibrium, and to dense quark matter. We summarise, whenever appropriate, the main results at the end of the sections.

In Section II, the microscopic formalism is reviewed. This starts with the classical equations of motions for coloured point particles carrying a non-Abelian colour charge (Section II A) and the derivation as equations of motions of expectation values for quantum wave packets (Section II B). The basic definitions of microscopic kinetic functions are given (Section II C) and the phase space variables associated to the colour charges introduced (Section II D). The dynamical equations for the distribution function (Section II E) and the microscopic gauge symmetry (Section II F) are discussed.

In Section III, the step from a microscopic to a macroscopic formulation is performed. The general assumptions made when switching to an effective description are explained

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(Section III A), followed by a brief description of the Gibbs ensemble average which is the starting point for the subsequent applications (Section III B). Finally, the basic equal-time correlation functions for the quasi-particle distribution function are given explicitly (Section III C)

In Section IV, the ensemble average is performed for the transport equations itself. The split of the distribution function and the gauge fields into mean and fluctuating parts is considered next (Section IV A), followed by a derivation of the effective transport equations in their most general form for mean fields and correlators (Section IV B). The new terms in the effective kinetic equations are interpreted as collision integrals, sources for stochastic noise, and fluctuation-induced currents (Section IV C). Systematic approximation schemes, able to truncate the infinite hierarchy of coupled differential equations, are detailed (Section IV D). This is followed by a discussion of the basic macroscopic conservation laws (Section IV E) and the kinetic entropy (Section IV F). The section is finished by a brief discussion (Section IV G).

All aspects connected to the requirements of gauge symmetry in the effective transport theory are discussed in Section V. The intimate relationship to the background field method and the invariance under both the background and the fluctuation field gauge transformations are discussed (Section V A). Current conservation for the mean and the fluctuation field implies non-trivial cross-dependences amongst different correlation functions. Their consistency is shown for the general case (Section V B), and for approximations to it (Section V C).

The remaining part is dedicated to applications of the method to hot non-Abelian plasmas close to thermal equilibrium. In Section VI, the relevant physical scales and parameters for classical (Section VI A) and quantum plasmas (Section VI B) are discussed.

In Section VII we discuss how the HTL effective theory is recovered within the present formalism. To leading order in the gauge coupling one obtains the non-Abelian Vlasov equation (Section VII A). Their solution (Section VII B) allows us to identify all HTL amplitudes, including the HTL polarisation tensor (Section VII C). As an application, it is shown how a local expression for the Hamiltonian and the Poynting vector is obtained (Section VII D).

Non-Abelian fluctuations have to be taken into account beyond the HTL approximation. This is done in Section VIII. All approximations are controlled by a small gauge coupling, and the leading order dynamical equations are given (Section VIII A). The dynamics of fluctuations is solved explicitly in terms of initial fluctuations of the quasi-particle distribution function (Section VIII B). The basic equal-time correlators are obtained, and the example of Landau damping is discussed (Section VIII C). The domain of validity is derived from the two-particle correlators and the associated correlation length (Section VIII D). The correlators in the effective transport equation are evaluated, and the relevant collision

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integral (Section VIII E) and the corresponding noise source (Section VIII F) are identified to leading logarithmic accuracy. The resulting transport equation is discussed. Iterative solutions allow the computation of the ultra-soft amplitudes (Section VIII G). In the over-damped limit, Bödeker's Langevin-type dynamical equation for the ultra-soft gauge fields is recovered (Section VIII H).

In Section IX, a phenomenological approach to non-Abelian fluctuations is discussed. It is based on the idea of coarse-graining the microscopic transport equations (Section IX A). We describe the line of reasoning for the example of classical dissipative systems (Section IX B). This is extended to the case of non-Abelian plasmas, where the basic spectral functions and equal-time correlators for stochastic fluctuations are derived from the kinetic entropy (Section IX C). As an application, Bödeker's effective kinetic theory is re-considered and shown to be compatible with the fluctuation-dissipation theorem (Section IX D). We close with a discussion of the results and further applications (Section IX E).

In Section X we consider dense quark matter. When the effects of quark pairing can be neglected, the transport equations are the same as for the hot non-Abelian plasma (Section X A). In the superconducting phase, the ground state is given by a diquark condensate (Section X B). For two massless quark flavours, the thermal colour excitations of the condensate are described by quasiparticles (Section X C). The corresponding transport equation is given and solved to leading order (Section X D). We close with a brief discussion of the results (Section X E).

Two Appendices contain technical details.

II. MICROSCOPIC APPROACH

The microscopic approach to semi-classical transport theory considers an ensemble of classical point particles. In the Abelian case, these are simply electrons or ions, interacting self-consistently through the Maxwell equations. Based on this picture, a complete effective theory for classical Coulomb or Abelian plasmas has been worked out in the literature (see for example [93–95,99,137]).

For the non-Abelian case, the concept of an electro-magnetically charged classical point particle is replaced by a *coloured* point particle, where 'colour' stands for a non-Abelian colour charge. The classical equations of motion for such particles have been given by Wong [146], and a transport theory based on it has been discussed in [69–74,61]. In this section, we review the microscopic approach to non-Abelian plasmas based on classical equations of motions for such 'particles'. We also introduce the basic notation to be used in the following sections for the construction of a kinetic theory [67].

A. Wong equations

Let us consider a system of particles carrying a non-Abelian colour charge Q^a, where the colour index runs from $a = 1$ to $N^2 - 1$ for a $SU(N)$ gauge group. Within a microscopic description, the trajectories in phase space are known exactly. The trajectories $\hat{x}(\tau), \hat{p}(\tau)$ and $\hat{Q}(\tau)$ for every particle are solutions of their classical equations of motions, known as the Wong equations [146]

$$m\frac{d\hat{x}^\mu}{d\tau} = \hat{p}^\mu , \tag{2.1a}$$

$$m\frac{d\hat{p}^\mu}{d\tau} = g\hat{Q}^a\, F_a^{\mu\nu}(\hat{x})\, \hat{p}_\nu , \tag{2.1b}$$

$$m\frac{d\hat{Q}^a}{d\tau} = -gf^{abc}\hat{p}^\mu\, A_\mu^b(\hat{x})\, \hat{Q}^c . \tag{2.1c}$$

Here, A_μ denotes the microscopic gauge field. A dependence on spin degrees of freedom, which can be incorporated as well [71], is not considered in the present case. The microscopic field strength $F_{\mu\nu}^a$ and the energy momentum tensor of the gauge fields $\Theta^{\mu\nu}$ are given by

$$F_{\mu\nu}^a[A] = \partial_\mu A_\mu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c , \tag{2.2}$$

$$\Theta^{\mu\nu}[A] = \tfrac{1}{4}g^{\mu\nu}F_{\rho\sigma}^a F_a^{\rho\sigma} + F_a^{\mu\rho}F_\rho^{a\ \nu} \tag{2.3}$$

and f^{abc} are the structure constants of $SU(N)$. We set $c = k_B = \hbar = 1$ and work in natural units, unless otherwise indicated. Note that the non-Abelian charges are also subject to

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dynamical evolution. Eq. (2.1c) can be rewritten as $D_\tau Q = 0$, where $D_\tau = \frac{d\hat{x}^\mu}{d\tau} D_\mu$ is the covariant derivative along the world line, and $D_\mu^{ac}[A] = \partial_\mu \delta^{ac} + g f^{abc} A_\mu^b$ the covariant derivative in the adjoint representation.

The colour current can be constructed once the solutions of the Wong equations are known. For every particle it reads

$$j_a^\mu(x) = g \int d\tau \frac{d\hat{x}^\mu}{d\tau} \hat{Q}_a(\tau) \delta^{(4)}[x - \hat{x}(\tau)] . \tag{2.4}$$

Employing the Wong equations (2.1) we find that the current is covariantly conserved, $D_\mu j^\mu = 0$ [146]. Similarly, the energy momentum tensor of the particles is given by [146]

$$t^{\mu\nu}(x) = \int d\tau \frac{d\hat{x}^\mu}{d\tau} \hat{p}^\nu(\tau) \delta^{(4)}[x - \hat{x}(\tau)] . \tag{2.5}$$

The Wong equations couple to classical non-Abelian gauge fields. The Yang-Mills equations are

$$D_\mu F^{\mu\nu}(x) = J^\nu(x) . \tag{2.6}$$

The source for the Yang-Mills fields

$$J^\nu(x) = \sum_{\text{particles}} j^\nu(x) \tag{2.7}$$

is given by the sum of the currents of all particles.

B. Classical limit in a non-Abelian quantum field theory

One may wonder under which conditions the point particle picture and Wong equations are a good approximation to the full quantum field theory. Originally, the Wong equations have been derived as the non-Abelian generalisation of equations of motion for electrically charged point particles. Here, we shall outline how the Wong equations are obtained as equations of motions for wave packets, in a gauge field theory coupled to matter. The derivation is valid for a system with matter interacting with non-Abelian gauge fields, and it does not apply for the gluons themselves. This line of reasoning is due to Brown and Weisberger [49]. It is argued that these equations derive from the conservation laws

$$\partial_\nu t^{\mu\nu} = F_a^{\mu\nu} J_\nu^a , \tag{2.8a}$$

$$D_\mu J^\mu = 0 , \tag{2.8b}$$

as classical equations of motion for sufficiently localised quantum states (or 'wave packets'), provided that the gauge fields are 'soft', *i.e.* vary only slowly over typical scales associated to the particles. We begin with the following definitions,

$$X^\mu(t) = \frac{\int d^3x\, x^\mu\, \langle t^{00}(\mathbf{x}, t)\rangle}{\int d^3x\, \langle t^{00}(\mathbf{x}, t)\rangle} \equiv \frac{1}{P^0} \int d^3x\, x^\mu \langle t^{00}(\mathbf{x}, t)\rangle \,, \tag{2.9a}$$

$$P^\mu(t) = \int d^3x\, \langle t^{0\mu}(\mathbf{x}, t)\rangle \,, \tag{2.9b}$$

$$g\, Q_a(t) = \int d^3x\, \langle J_a^0(\mathbf{x}, t)\rangle \,. \tag{2.9c}$$

The variable $P^\mu(t)$ describes the particle's mechanical four-momentum at the time $t \equiv x_0 \equiv X_0$ as an expectation value of the quantum fields, $X^i(t)$ the center-of-energy expectation value of the quantum fields, and $Q_a(t)$ the expectation value for the associated colour charge. Making use only of the conservation laws Eqs. (2.8), and partial integrations, one derives the following equation of motion for the particles,

$$\frac{dX^i}{dt} = \frac{P^i}{P^0} + \frac{1}{(P^0)^2} \int d^3x\, d^3y\, \langle F^{0k}(\mathbf{x}, t)\, J^k(\mathbf{x}, t)\rangle \langle t^{00}(\mathbf{y}, t)\rangle (x^i - y^i) \,, \tag{2.10a}$$

$$\frac{dP^\mu}{dt} = \int d^3x\, \langle F_a^{\mu\nu}(\mathbf{x}, t) J_\nu^a(\mathbf{x}, t)\rangle \,, \tag{2.10b}$$

$$g\, \frac{dQ_a}{dt} = -g\, f_{abc} \int d^3x\, \langle\, A_\mu^b(\mathbf{x}, t)\, J^{\mu,c}(\mathbf{x}, t)\rangle \,. \tag{2.10c}$$

These equations of motion still involve integrals over all space on the right-hand side. To obtain a classical limit, these equations can be simplified in the case where the characteristic length-scales of the particles are much smaller than those associated to the gauge fields. In this case, the particle current $J(\mathbf{x}, t)$ is localised close to the location of the particle, and equally $t^{00}(\mathbf{x}, t)$. If the gauge fields do vary very slowly over these short distance scales, we can perform the following approximation,

$$\int d^3x\, \langle F^{\mu\nu}(\mathbf{x}, t) J_\rho(\mathbf{x}, t)\rangle \approx F^{\mu\nu}(\mathbf{X}, t) \int d^3x\, \langle J_\rho(\mathbf{x}, t)\rangle \,, \tag{2.11a}$$

by replacing the gauge fields in the integrand through their values at the location of the particle. Equally, we approximate

$$\int d^3x\, \langle A^\mu(\mathbf{x}, t) J_\rho(\mathbf{x}, t)\rangle \approx A^\mu(\mathbf{X}, t) \int d^3x\, \langle J_\rho(\mathbf{x}, t)\rangle \,. \tag{2.11b}$$

This implies in addition that

$$\int d^3x\, d^3y\, \langle F^{0k}(\mathbf{x}, t)\, J^k(\mathbf{x}, t)\rangle \langle t^{00}(\mathbf{y}, t)\rangle (x^i - y^i) \approx 0 \tag{2.11c}$$

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to leading order. The approximation Eq. (2.11) corresponds to the leading order in a gradient expansion. Employing Eq. (2.11), the equations of motion become

$$\frac{dX^i}{dt} = \frac{P^i}{P^0} \,, \tag{2.12a}$$

$$\frac{dP^\mu}{dt} = F_a^{\mu\nu}(\mathbf{X}, t) \int d^3x \, \langle J_\nu^a(\mathbf{x}, t) \rangle \,, \tag{2.12b}$$

$$g \frac{dQ_a}{dt} = -g \, f_{abc} \, A_\mu^b(\mathbf{X}, t) \int d^3x \, \langle J^{\mu,c}(\mathbf{x}, t) \rangle \,. \tag{2.12c}$$

We shall now exploit the fact that the particle's mass m with

$$m^2 = P^\mu P_\mu \tag{2.13}$$

is a constant of motion, $dm^2/dt = 0$. With Eq. (2.12b), this yields the constraint

$$0 = F_a^{\mu\nu}(\mathbf{X}, t) \, P_\mu \, P_0 \int d^3x \, \langle J_\nu^a(\mathbf{x}, t) \rangle \tag{2.14}$$

which has to hold for any field configuration. The field strength is an arbitrary antisymmetric tensor, therefore, the constraint implies

$$P_\mu \int d^3x \, \langle J_\nu^a(\mathbf{x}, t) \rangle = P_\nu \int d^3x \, \langle J_\mu^a(\mathbf{x}, t) \rangle \,. \tag{2.15}$$

This is automatically the case if the current density is proportional to the momentum density. Evaluating (2.15) for $\nu = 0$ and using the definition (2.9c) we indeed find

$$P_0 \int d^3x \, \langle J_\mu^a(\mathbf{x}, t) \rangle = g \, P_\mu \, Q^a \,, \tag{2.16}$$

and the approximate equations of motion read

$$\frac{dX^i}{dt} = \frac{P^i}{P^0} \,, \tag{2.17a}$$

$$\frac{dP^\mu}{dt} = g \, F_a^{\mu\nu} \frac{P^\nu}{P^0} \, Q^a \,, \tag{2.17b}$$

$$\frac{dQ_a}{dt} = -g \, f_{abc} \, A_\mu^b \frac{P^\mu}{P^0} \, Q^c \,. \tag{2.17c}$$

Let us finally introduce the proper time τ for the particles, which serves as a normalisation condition for Eq. (2.17a). The proper time relates the mass of the particles to the 00-component of the energy-momentum tensor through the requirement $m \, dX/d\tau = P$, hence

$$m \frac{d}{d\tau} = P^0 \frac{d}{dt} \,. \tag{2.18}$$

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Using Eqs. (2.17) and (2.18), the equations of motion become

$$m\frac{dX^\mu}{d\tau} = P^\mu \ , \tag{2.19a}$$

$$m\frac{dP^\mu}{d\tau} = gQ^a \, F_a^{\mu\nu} \, P_\nu \ , \tag{2.19b}$$

$$m\frac{dQ^a}{d\tau} = -gf^{abc} \, P^\mu A_\mu^b \, Q^c \ , \tag{2.19c}$$

and agree with the equations found by Wong, Eqs. (2.1), if the replacements $X \to \hat{x}$, $P \to \hat{p}$ and $Q \to \hat{Q}$ are made. We conclude that the Wong equations are the leading order approximate equations of motions for point particles, if the induced gauge fields are soft. This scale separation between hard particles and soft fields is at the root of the present approach.

C. Microscopic distribution functions

Instead of describing every particle individually, it is convenient to introduce a phase space density for the ensemble of particles, that is a distribution function which depends on the whole set of coordinates x^μ, p^μ and Q_a. To that end, we introduce two functions $n(x, p, Q)$ and $f(x, p, Q)$, which only differ by appropriately chosen normalisation factors. We begin with the function

$$n(x, p, Q) = \sum_i \int d\tau \, \delta^{(4)}[x - \hat{x}_i(\tau)] \, \delta^{(4)}[p - \hat{p}_i(\tau)] \, \delta^{(N^2-1)}[Q - \hat{Q}_i(\tau)] \ , \tag{2.20}$$

where the index i labels the particles. This distribution function is constructed in such a way that the colour current

$$J_a^\mu(x) = g \int d^4p \, d^{(N^2-1)}Q \, \frac{p^\mu}{m} \, Q_a \, n(x, p, Q) \tag{2.21}$$

coincides with the sum over all currents associated to the individual particles $J_a^\mu = \sum_i j_a^\mu$, Eq. (2.7). The current is covariantly conserved, $D_\mu J^\mu = 0$. It is convenient to make the following changes in the choice of the distribution function. For convenience, we introduce a momentum and a group measure such that the physical constraints like the on-mass shell condition, positive energy and conservation of the group Casimirs are factored out into the phase space measure. Consider the momentum measure

$$dP = d^4p \, 2\theta(p_0) \, \delta(p^2 - m^2) \ , \tag{2.22}$$

which accounts for the on-mass-shell constraint. The measure for the colour charges is

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$$dQ = d^3Q \, c_R \, \delta(Q_a Q_a - q_2) \,, \tag{2.23}$$

in the case of $SU(2)$. For $SU(3)$ the measure is

$$dQ = d^8Q \, c_R \, \delta(Q_a Q_a - q_2) \, \delta(d_{abc} Q^a Q^b Q^c - q_3) \,. \tag{2.24}$$

For $SU(N)$, $N - 1$ δ-functions ensuring the conservation of the set of $N - 1$ Casimirs have to be introduced into the measure for the colour charges. We have also introduced the representation-dependent normalisation constant c_R into the measure, which is fixed by the normalisation condition $\int dQ = 1$. Furthermore, we have $\int dQ Q_a = 0$. The quadratic Casimir C_2 is defined as

$$\int dQ Q_a Q_b = C_2 \delta_{ab} \,, \tag{2.25}$$

and depends on the group representation of the particles. For particles in the adjoint representation of $SU(N)$ we have $C_2 = N$ (gluons). For particles in the fundamental representation, $C_2 = \frac{1}{2}$ (quarks). Notice that the colour charges have to be quantised within a quantum field theoretical approach.

We will define a second distribution function $f(x, p, Q)$ such that the physical constraints within $n(x, p, Q)$ have been factored out,

$$dP \, dQ \, f(x, p, Q) = d^3p \, \frac{dp_0}{m} \, d^{(N^2-1)}Q \, n(x, p, Q) \,. \tag{2.26}$$

With this convention the colour current of the particles Eq. (2.21) now reads

$$J_a^\mu(x) = g \int dP dQ \, p^\mu \, Q_a \, f(x, p, Q) \,. \tag{2.27}$$

The energy-momentum tensor associated to the particles is

$$t^{\mu\nu}(x) = \int dP dQ \, p^\mu p^\nu \, f(x, p, Q) \tag{2.28}$$

when expressed in terms of the distribution function f.

D. Phase space

We have introduced the distribution function $f(x, p, Q)$ to further a description in phase space. While $\mathbf{x}$ and $\mathbf{p}$ are standard phase space variables with a canonical Poisson bracket, the colour charges Q_a are *not*. However, it is always possible to define the set of canonical (Darboux) variables associated to the Q_a charges. For $SU(N)$, there are $N(N -$

1)/2 pairs of canonical variables which we denote as $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_{N(N-1)/2})$ and $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_{N(N-1)/2})$. The canonical variables define the canonical Poisson bracket

$$\{A, B\}_{PB} = \frac{\partial A}{\partial x_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial x_i} + \frac{\partial A}{\partial \phi_a}\frac{\partial B}{\partial \pi_a} - \frac{\partial B}{\partial \phi_a}\frac{\partial A}{\partial \pi_a} \ , \tag{2.29}$$

and obey trivially

$$\{x_i, p_k\}_{PB} = \delta_{ik} \ , \qquad \{\phi_a, \pi_b\}_{PB} = \delta_{ab} \ . \tag{2.30}$$

The colour charges Q_a are a representation of $SU(N)$. When expressed as functions of the canonical variables, their Poisson bracket reads

$$\{Q_a, Q_b\}_{PB} = f_{abc}Q_c \ , \tag{2.31}$$

where f_{abc} are the structure constants of $SU(N)$.

The explicit construction of Darboux variables for $SU(2)$ and $SU(3)$ has been performed in [91]. Let us first consider the $SU(2)$-case. We define the set of variables ϕ_1, π_1 and J by the implicit transformation

$$Q_1 = \cos \phi_1 \sqrt{J^2 - \pi_1^2} \ , \qquad Q_2 = \sin \phi_1 \sqrt{J^2 - \pi_1^2} \ , \qquad Q_3 = \pi_1 \ , \tag{2.32}$$

where the variable π_1 is bounded by $-J \le \pi_1 \le J$. The variables ϕ_1, π_1 form a canonically conjugate pair and obey Eq. (2.30), while J is fixed by the value of the quadratic Casimir, which is constant under the dynamical evolution. One confirms that Eq. (2.32) obey Eq. (2.31) with $f_{abc} = \epsilon_{abc}$. The phase space volume element Eq. (2.23) becomes

$$dQ = d\pi_1 \, d\phi_1 \, dJ \, J \, c_R \, \delta(J^2 - q_2) \tag{2.33}$$

in terms of the Darboux variables. With the above change of variables, one can fix the value of the representation-dependent normalisation constant c_R introduced in (2.23). From the condition $\int dQ = 1$ one finds $c_R = 1/2\pi\sqrt{q_2}$. From the condition $\int dQ Q_a Q_b = C_2 \delta_{ab}$ one gets $q_2 = 3C_2$. This entirely fixes the value of c_R as a function of C_2.

The group $SU(3)$ has eight charges, $(Q_1, \ldots, Q_8)$ and two conserved quantities, the quadratic and the cubic Casimirs, $Q^a Q_a$ and $d_{abc} Q^a Q^b Q^c$, respectively. The phase-space colour measure is quoted above in (2.24). As in the $SU(2)$ case, new coordinates $(\phi_1, \phi_2, \phi_3, \pi_1, \pi_2, \pi_3, J_1, J_2)$ may be introduced by means of the following transformations [87]:

$$\begin{aligned}
Q_1 &= \cos \phi_1 \, \pi_+ \, \pi_- \ , & Q_2 &= \sin \phi_1 \, \pi_+ \, \pi_- \ , \\
Q_3 &= \pi_1 \ , \\
Q_4 &= C_{++} \, \pi_+ \, A + C_{+-} \, \pi_- \, B \ , & Q_5 &= S_{++} \, \pi_+ \, A + S_{+-} \, \pi_- \, B \ , \\
Q_6 &= C_{-+} \, \pi_- \, A - C_{--} \, \pi_+ \, B \ , & Q_7 &= S_{-+} \, \pi_- \, A - S_{--} \, \pi_+ \, B \ , \\
Q_8 &= \pi_2 \ ,
\end{aligned} \tag{2.34}$$

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in which we have used the definitions:

$$\pi_+ = \sqrt{\pi_3 + \pi_1} \;, \qquad\qquad \pi_- = \sqrt{\pi_3 - \pi_1} \;,$$
$$C_{\pm\pm} = \cos\left[\tfrac{1}{2}(\pm\phi_1 + \sqrt{3}\phi_2 \pm \phi_3)\right] \;, \qquad S_{\pm\pm} = \sin\left[\tfrac{1}{2}(\pm\phi_1 + \sqrt{3}\phi_2 \pm \phi_3)\right] \;, \tag{2.35}$$

and A, B are given by

$$A = \frac{1}{2\pi_3}\sqrt{\left(\frac{J_1 - J_2}{3} + \pi_3 + \frac{\pi_2}{\sqrt{3}}\right)\left(\frac{J_1 + 2J_2}{3} + \pi_3 + \frac{\pi_2}{\sqrt{3}}\right)\left(\frac{2J_1 + J_2}{3} - \pi_3 - \frac{\pi_2}{\sqrt{3}}\right)} \;,$$

$$B = \frac{1}{2\pi_3}\sqrt{\left(\frac{J_2 - J_1}{3} + \pi_3 - \frac{\pi_2}{\sqrt{3}}\right)\left(\frac{J_1 + 2J_2}{3} - \pi_3 + \frac{\pi_2}{\sqrt{3}}\right)\left(\frac{2J_1 + J_2}{3} + \pi_3 - \frac{\pi_2}{\sqrt{3}}\right)} \;.$$

Note that in this representation, the set (Q_1, Q_2, Q_3) forms an $SU(2)$ subgroup with quadratic Casimir $Q_1^2 + Q_2^2 + Q_3^2 = \pi_3^2$. It can be verified, using the values of the structure constants given in Tab. 1, that the expressions above for $Q_1, \ldots, Q_8$ form a representation of the group $SU(3)$.

f_{abc}	f_{123}	f_{147}	f_{156}	f_{246}	f_{257}	f_{345}	f_{367}	f_{458}	f_{678}
	1	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$

Table 1: The non-zero constants f_{abc} for $SU(3)$.

d_{abc}	d_{118}	d_{146}	d_{157}	d_{228}	d_{247}	d_{256}	d_{338}	d_{344}	d_{355}	d_{366}	d_{377}	d_{448}	d_{558}	d_{668}	d_{778}	d_{888}
	$\frac{1}{\sqrt{3}}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{3}}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{3}}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$

Table 2: The non-zero constants d_{abc} for $SU(3)$.

As is implicit in the above, the two Casimirs depend only on J_1 and J_2. They can be computed, using the values given in the Tab. 2, as:

$$Q^a Q_a = \frac{1}{3}(J_1^2 + J_1 J_2 + J_2^2) \;, \tag{2.36a}$$

$$d_{abc}Q^a Q^b Q^c = \frac{1}{18}(J_1 - J_2)(J_1 + 2J_2)(2J_1 + J_2) \;. \tag{2.36b}$$

The phase-space colour measure for $SU(3)$, given in (2.24), may be transformed to the new coordinates through use of (2.34) and evaluation of the Jacobian

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$$\left| \frac{\partial(Q_1, Q_2, \ldots, Q_8)}{\partial(\phi_1, \phi_2, \phi_3, \pi_1, \pi_2, \pi_3, J_1, J_2)} \right| = \frac{\sqrt{3}}{48} \, J_1 \, J_2 \, (J_1 + J_2) \,. \tag{2.37}$$

The measure reads:

$$dQ = c_R \, d\phi_1 \, d\phi_2 \, d\phi_3 \, d\pi_1 \, d\pi_2 \, d\pi_3 \, dJ_1 \, dJ_2 \, \frac{\sqrt{3}}{48} \, J_1 \, J_2 \, (J_1 + J_2) \, \delta\Big(\frac{1}{3}(J_1^2 + J_1 J_2 + J_2^2) - q_2\Big) \, \times$$
$$\delta\Big(\frac{1}{18}(J_1 - J_2)(J_1 + 2J_2)(2J_1 + J_2) - q_3\Big) \,. \tag{2.38}$$

Since the two Casimirs are linearly independent, the delta-functions uniquely fix both J_1 and J_2 to be representation-dependent constants. Upon integrating over J_1 and J_2, (2.38) reduces to a constant times the proper canonical volume element $\prod_{i=1}^{3} d\phi_i \, d\pi_i$. The value of the normalisation constant c_R will now depend both on q_2 and q_3.

For $SU(N)$, the canonical variables can be constructed along similar lines [3]. For the quadratic and cubic Casimir, one finds $q_2 = (N^2 - 1)C_2$, and $C_2 = \frac{1}{2}$ for particles in the fundamental (quarks), and $C_2 = N$ for particles in the adjoint (gluons). The constant q_3 reads $q_3 = (N^2 - 4)(N^2 - 1)/4N$ for particles in the fundamental, and $q_3 = 0$ for particles in the adjoint. We also comment in passing that in the pure classical framework, the quadratic Casimir C_2 carries the dimensions of $\hbar c$. After quantisation, the quadratic Casimirs should take quantised values proportional to $\hbar$. The Poisson brackets then have to be replaced by commutators.

The microscopic phase space density, expressed in terms of the real phase space variables, is given by

$$\hat{n}(\mathbf{x}, \mathbf{p}, \boldsymbol{\phi}, \boldsymbol{\pi}) = \sum_i \delta^{(3)}[\mathbf{x} - \hat{\mathbf{x}}_i(t)] \, \delta^{(3)}[\mathbf{p} - \hat{\mathbf{p}}_i(t)] \, \delta[\boldsymbol{\phi} - \hat{\boldsymbol{\phi}}_i(t)] \, \delta[\boldsymbol{\pi} - \hat{\boldsymbol{\pi}}_i(t)] \,, \tag{2.39}$$

where the sum runs over all particles of the system, and $(\hat{\mathbf{x}}_i, \hat{\mathbf{p}}_i, \hat{\boldsymbol{\phi}}_i, \hat{\boldsymbol{\pi}}_i)$ refers to the trajectory of the i-th particle in phase space. Then $\hat{n} \, d\mathbf{x} \, d\mathbf{p} \, d\boldsymbol{\phi} d\boldsymbol{\pi}$ gives the number of particles at time t in an infinitesimal volume element of phase space around the point $z = (\mathbf{x}, \mathbf{p}, \boldsymbol{\phi}, \boldsymbol{\pi})$. The function $\hat{n}(\mathbf{x}, \mathbf{p}, \boldsymbol{\phi}, \boldsymbol{\pi})$ agrees with the microscopic function $f(x, p, Q)$ introduced above, except for a representation-dependent normalisation constant.

E. Dynamical equations and conservation laws

Now we come to the dynamical equation of the microscopic distribution functions $n(x, p, Q)$, $\hat{n}(\mathbf{x}, \mathbf{p}, \boldsymbol{\phi}, \boldsymbol{\pi})$ and $f(x, p, Q)$, which will serve as the starting point for the subsequent formalism. Although the independent degrees of freedom are given by the phase space variables $(\mathbf{x}, \mathbf{p}, \boldsymbol{\phi}, \boldsymbol{\pi})$, it is more convenient to derive the dynamical equations in terms

of the variables (x, p, Q). The Darboux variables will become important when an ensemble average is defined in the following section. Secondly, we note that the dynamical equation for $n(x, p, Q)$ is the same as for $f(x, p, Q)$. This is so because the physical constraints which we have factored out to obtain $f(x, p, Q)$ are not affected by the Wong equations. Employing Eqs. (2.1), we find

$$p^\mu \left(\frac{\partial}{\partial x^\mu} - g f^{abc} A_\mu^b Q^c \frac{\partial}{\partial Q^a} - g Q_a F_{\mu\nu}^a \frac{\partial}{\partial p_\nu} \right) f(x, p, Q) = 0 \ , \tag{2.40a}$$

which can be checked explicitly by direct inspection of Eq. (2.20) into Eq. (2.40a) [92]. Equivalently, one could have made use of Liouville's theorem $df/d\tau = 0$, which states that the phase space volume is conserved. In combination with Eqs. (2.1), one obtains Eq. (2.40a). In a self-consistent picture this equation is completed with the Yang-Mills equation,

$$(D_\mu F^{\mu\nu})_a(x) = J_a^\nu(x) \ , \tag{2.40b}$$

and the current being given by Eq. (2.27). It is worth noticing that Eqs. (2.40) are exact in the sense that no further approximations apart from the quasiparticle picture have been made. This Boltzmann equation looks formally as *collisionless*. However, it effectively contains collisions inasmuch as the Wong equations account for them, that is, due to the long range interactions between the particles.

For the microscopic energy-momentum tensor of the gauge fields Eq. (2.3) we find

$$\partial_\mu \Theta^{\mu\nu}(x) = -F_a^{\nu\mu}(x) J_\mu^a(x) \ . \tag{2.41}$$

On the other hand, using Eq. (2.40a) and the definition Eq. (2.28) we find

$$\partial_\mu t^{\mu\nu}(x) = F_a^{\nu\mu}(x) J_\mu^a(x) \tag{2.42}$$

for the energy-momentum tensor of the particles, hence

$$\partial_\mu T^{\mu\nu}(x) = 0, \quad T^{\mu\nu}(x) = \Theta^{\mu\nu}(x) + t^{\mu\nu}(x) \tag{2.43}$$

which establishes that the combined energy-momentum tensor of the particles and the fields is conserved.

F. Gauge symmetry

To finish the discussion of the microscopic description of the system, let us recall the gauge symmetry properties of the Wong Eqs. (2.1) and the set of microscopic dynamical

equations (2.40) (a detailed discussion is given in Section V). With Q_a and $F^a_{\mu\nu}$ transforming in the adjoint representation, the Wong equations are invariant under gauge transformations. The equation (2.1c) ensures that the set of $N-1$ Casimir of the $SU(N)$ group is conserved under the dynamical evolution. For $SU(2)$, it is easy to verify explicitly the conservation of the quadratic Casimir $Q_a Q_a$. For $SU(3)$, both the quadratic and cubic Casimir $d_{abc}Q_a Q_b Q_c$, where d_{abc} are the symmetric structure constants of the group, are conserved under the dynamical evolution. The last conservation can be checked using (2.1c) and a Jacobi-like identity which involves the symmetric d_{abc} and antisymmetric f_{abc} constants [92].

From the definition of the distribution function $f(x, p, Q)$ we conclude that it transforms as a scalar under (finite) gauge transformations, $f'(x, p, Q') = f(x, p, Q)$. This implies the gauge covariance of Eq. (2.40b) because the current Eq. (2.27) transforms like the vector Q_a in the adjoint representation. The non-trivial dependence of $f(x, p, Q)$ on the non-Abelian colour charges implies that the partial derivative $\partial_\mu f(x, p, Q)$ does not transform as a scalar. Instead, its covariant derivative $D_\mu f(x, p, Q)$, which is given by

$$D_\mu[A]f(x, p, Q) \equiv [\partial_\mu - g f^{abc} Q_c A_{\mu,b} \partial^Q_a] f(x, p, Q) \,, \tag{2.44}$$

does. Notice that Eq. (2.44) combines the first two terms of Eq. (2.40a). Here and in the sequel we use the shorthand notation $\partial_\mu \equiv \partial/\partial x^\mu$, $\partial^p_\mu \equiv \partial/\partial p^\mu$ and $\partial^Q_a \equiv \partial/\partial Q^a$. The invariance of the third term in Eq. (2.40a) follows from the trivial observation that $Q_a F^a_{\mu\nu}$ is invariant under gauge transformations, which establishes the gauge invariance of Eq. (2.40a). This terminates the review of the basic microscopic quantities.

III. MACROSCOPIC APPROACH

A. General considerations

Within the semi-classical approach introduced in Section II, all information about properties of the non-Abelian plasma is given by the microscopic dynamical equations as written down in the previous section. However, for most situations not all the microscopic information is of relevance. Of main physical interest are the characteristics of the system at large length scales. This includes quantities like damping rates, colour conductivities or screening lengths within the kinetic regime, or transport coefficients like shear or bulk viscosities within the hydrodynamic regime. The microscopic length scales, like typical inter-particle distances, are much smaller than such macroscopic scales.

There are two closely related aspects worth noticing when performing the transition from a microscopic to an effective, or macroscopic, description. We first observe that the classical problem as described in the previous section is well-posed only if all initial conditions for the particles are given. If the system under study contains a large number of particles it is impossible to follow their individual trajectories. A natural step to perform is to switch to a *statistical* description of the system. In this way, the stochastic character of the initial conditions are taken into account. It follows that the microscopic distribution function can no longer be considered a deterministic, but rather a *stochastic* quantity. This program is worked out in detail in the following two sections.

Given the statistical ensemble which represents the state of the system, the macroscopic properties should be given as functions of the fundamental parameters and the interactions between the particles. This requires an appropriate definition of macroscopic quantities as ensemble averages. Within kinetic theory, the basic 'macroscopic' quantity is the one-particle distribution function, from which all further macroscopic observables can be derived. The aim of a kinetic theory is to construct, with as little restrictions or assumptions as possible, a closed set of transport equations for this distribution function [67]. Such an approach assumes implicitly that the 'medium', described by the distribution function, is continuous. If the medium is *not* continuous, stochastic fluctuations due to the particles can be taken into account as well, and their consistent inclusion leads to *effective* transport equations for correlators of fluctuations and the one-particle distribution function [93–95]. The random fluctuations of the distribution function are at the root of the dissipative character of the effective transport theory.

An alternative reading of the above invokes the notion of *coarse-graining*. This amounts to an averaging of both the microscopic distribution function and of the non-Abelian fields over characteristic physical volumes. The resulting effective kinetic equations dissipative due to the coarse-graining over microscopic quantities, and require the consistent inclusion of a corresponding noise term. This is very similar to the phenomenological

Langevin approach to dissipative systems [97]. In the regime where fluctuations can be taken as linear these two approaches are equivalent [94]. We come back to this point of view in Section IX, where its application to the theory of non-Abelian fluctuations in plasmas is discussed [108].

In this section, we work out the first line of thought. The basics related to the Gibbs ensemble average in phase space are discussed, and the basic correlators reflecting the stochastic fluctuations are derived. In the following section, this procedure is applied to the microscopic transport equations, ultimately resulting in a closed set of macroscopic transport equations.

B. Ensemble average

As we are studying classical point particles in phase space, the appropriate statistical average corresponds to the Gibbs ensemble average for classical systems [94,97]. We will review the main features of this procedure as defined in phase space. Let us remark that this derivation is completely general, valid for any classical system, and does not require equilibrium situations.

We introduce two basic functions. The first one is the phase space density function $n(z)$ which gives, after integration over a phase space volume element, the number of particles contained in that volume. Microscopically the phase space density function reads

$$n(z) = \sum_{i=1}^{L} \delta[z - z_i(\tau)] \ , \tag{3.1}$$

where z are the phase space coordinates, and z_i the trajectory of the particle i in phase space. Let us also define the distribution function ρ of the microstates of a system of L identical classical particles. Due to Liouville's theorem, $d\rho/dt = 0$. Thus, it can be normalised as

$$\int dz_1 dz_2 \ldots dz_L \ \rho(z_1, z_2, \ldots, z_L, t) = 1 \ . \tag{3.2}$$

For simplicity we have considered only one species of particles. The generalisation to several species of particles is straightforward.

The statistical average of any function G defined in phase space is given by

$$\langle G \rangle = \int dz_1 dz_2 \ldots dz_L \ G(z_1, z_2, \ldots, z_L) \ \rho(z_1, z_2, \ldots, z_L, t) \ . \tag{3.3}$$

A particularly important example is the one-particle distribution function, which is obtained from ρ as

$$f_1(z_1, t) = V \int dz_2 \ldots dz_L \; \rho(z_1, z_2, \ldots, z_L, t) \; . \tag{3.4}$$

Here V denotes the phase space volume. Correspondingly, the two-particle distribution function is

$$f_2(z_1, z_2, t) = V^2 \int dz_3 \ldots dz_L \; \rho(z_1, z_2, \ldots, z_L, t) \; , \tag{3.5}$$

and similarly for the k-particle distribution functions. A complete knowledge of ρ would allow us to obtain all the set of $(f_1, f_2, \ldots, f_L)$ functions; this is, however, not necessary for our present purposes.

Notice that we have allowed for an explicit dependence on the time t of the function ρ, as this would typically be the case in out-of-equilibrium situations. We will drop this t dependence from now on to simplify the formulas.

Using the above definition one can obtain the first moment (mean value) of the microscopic phase space density. The statistical average of this function is

$$\langle n(z) \rangle = \int dz_1 dz_2 \ldots dz_L \; \rho(z_1, z_2, \ldots, z_L) \sum_{i=1}^{L} \delta(z - z_i) = \tfrac{L}{V} f_1(z) \; . \tag{3.6}$$

The second moment $\langle n(z)\, n(z') \rangle$ can similarly be computed, and it is not difficult to see that it gives

$$\langle n(z)\, n(z') \rangle = \tfrac{L}{V} \delta(z - z') f_1(z) + \tfrac{L(L-1)}{V^2} f_2(z, z') \; . \tag{3.7}$$

Let us now define a deviation of the phase space density from its mean value

$$\delta n(z) \equiv n(z) - \langle n(z) \rangle \; . \tag{3.8}$$

By definition $\langle \delta n(z) \rangle = 0$, although the second moment of this statistical fluctuation does not vanish in general, since

$$\langle \delta n(z)\, \delta n(z') \rangle = \langle n(z) n(z') \rangle - \langle n(z) \rangle \langle n(z') \rangle \; . \tag{3.9}$$

If the number of particles is large, $L \gg 1$, we have

$$\langle \delta n(z)\, \delta n(z') \rangle = \left(\tfrac{L}{V} \right) \delta(z - z') f_1(z) + \left(\tfrac{L}{V} \right)^2 g_2(z, z') \; , \tag{3.10}$$

where the function

$$g_2(z, z') = f_2(z, z') - f_1(z) f_1(z') \tag{3.11}$$

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measures the two-particle correlations in the system. Notice that the above statistical averages are well defined in the thermodynamic limit, $L, V \to \infty$ but L/V remaining constant.

Similarly, one can define the k-point correlator of fluctuations, and the k-point correlation function $g_k(z, z', \ldots)$. In an ideal (non-interacting) system, the k-particle distribution function factorizes $f_k = \prod_{i=1}^{k} f_1$, and hence $g_k \equiv 0$, simply because all particles in the system are statistically independent from each other. Interactions induce correlations among particles. Typically, higher order correlations depend on the distance between particles and have a characteristic (finite) range or correlation length. Exceptions are met close to critical points of phase transitions, where correlation lengths tend to diverge.

Starting from the Liouville equation, obeyed by the distribution function ρ, it is possible to deduce a set of chained equations for the k-point distribution functions f_k. For non-relativistic systems this is the BBGKY hierarchy. These equations exhibit a hierarchical structure: the determination of the k-particle distribution function requires the knowledge of the $(k+1)$ particle function. Alternatively, one can describe the set of equations obeyed by the correlation functions g_k. These equations are non-linear due to the non-linear relationship between f_k and g_k. In the following section we will describe a different approach. It is based on deducing the equations for the statistical fluctuations, and their correlators. When using some approximate methods, this approach is more effective in a number of cases, as we will explicitly illustrate in the following sections.

C. Basic equal-time correlators

We now return to the case of our concern. The statistical averages have to be performed in phase space. The phase space density function Eq. (2.39) is a function of the time t, the vectors $\mathbf{x}$ and $\mathbf{p}$, and the set of canonical variables $\boldsymbol{\phi}$ and $\boldsymbol{\pi}$. We scale for later convenience the density factors L/V into the mean functions $\langle f \rangle$. Those small changes in the normalisation simplify slightly the notations of the equations. Also, to adopt a unified description of both the classical and quantum plasmas, from now on we will use dimensionless distribution functions, replacing the measure $d^3x d^3p$ by $d^3x d^3p/(2\pi\hbar)^3$ (although working in natural units $\hbar = 1$). This change in the measure also affect the normalisation of the basic correlators, as we will show below.

We now turn to the basic correlators which will be of relevance for later applications. For a classical plasma, the basic equal-time correlator obtained from averaging over initial conditions follows from Eq. (3.10) after the redefinitions as indicated above as

$$\langle \delta f_{\mathbf{x},p,Q} \, \delta f_{\mathbf{x}',p',Q'} \rangle_{t=0} = (2\pi)^3 \delta^{(3)}(\mathbf{x} - \mathbf{x}') \delta^{(3)}(\mathbf{p} - \mathbf{p}') \delta(Q - Q') \, \bar{f}$$
$$+ \tilde{g}_2(\mathbf{x}, p, Q; \mathbf{x}', p', Q') \tag{3.12}$$

for each species of particles and each internal degree of freedom. The function $\tilde{g}_2$ comes

from the two-particle correlator, and

$$\delta(Q - Q') = \frac{1}{c_R}\delta(\boldsymbol{\phi} - \boldsymbol{\phi}') \, \delta(\boldsymbol{\pi} - \boldsymbol{\pi}) \, , \tag{3.13}$$

and $\boldsymbol{\phi}$, $\boldsymbol{\pi}$ are the Darboux variables associated to the colour charges Q_a. The appearance of the factor $1/c_R$ in the above expression is due to the change of normalisation factors associated to the functions n and f.

Within the semi-classical approach, the quantum statistical properties of the particles are taken into account as well. For bosons, and for every internal degree of freedom, this amounts to replacing Eq. (3.12) by

$$\langle \delta f_{\mathbf{x},p,Q} \, \delta f_{\mathbf{x}',p',Q'} \rangle_{t=0} = (2\pi)^3 \delta^{(3)}(\mathbf{x} - \mathbf{x}') \delta^{(3)}(\mathbf{p} - \mathbf{p}') \delta(Q - Q') \bar{f}_{\mathrm{B}}(1 + \bar{f}_{\mathrm{B}})$$
$$+ \tilde{g}_2^{\mathrm{B}}(\mathbf{x}, p, Q; \mathbf{x}', p', Q') \, , \tag{3.14}$$

for the quadratic correlator of fluctuations. For fermions, the corresponding equal-time correlator is

$$\langle \delta f_{\mathbf{x},p,Q} \, \delta f_{\mathbf{x}',p',Q'} \rangle_{t=0} = (2\pi)^3 \delta^{(3)}(\mathbf{x} - \mathbf{x}') \delta^{(3)}(\mathbf{p} - \mathbf{p}') \delta(Q - Q') \bar{f}_{\mathrm{F}}(1 - \bar{f}_{\mathrm{F}})$$
$$+ \tilde{g}_2^{\mathrm{F}}(\mathbf{x}, p, Q; \mathbf{x}', p', Q') \, . \tag{3.15}$$

The functions $\tilde{g}_2^{\mathrm{B}}$ or $\tilde{g}_2^{\mathrm{F}}$ are the bosonic or fermionic two-particle correlation function, up to a normalisation factor. The above relations could be derived from first principles in a similar way as Eq. (3.12). In the limit $\bar{f}_{\mathrm{B/F}} \ll 1$ they reduce to the correct classical value. We present a justification of the use of the above correlators in Appendix B. It also has to be pointed out that the correlators (3.14) and (3.15) have been derived for the cases of both an ideal gas of bosons and ideal gas of fermions close to equilibrium. Hence, the above correlators can be taken as the correct answer in the case where the non-Abelian interactions are perturbative.

IV. EFFECTIVE TRANSPORT THEORY

The goal of transport theory is to derive a closed system of dynamical equations based on the one-particle distribution function from which all macroscopic characteristics can be derived [67]. Given the prescription as to how statistical averages over the particles in phase space have to be performed, we apply this formalism to the quasi-particle distribution function, the non-Abelian gauge fields and the dynamical equations themselves. This first step results in a set of transport equations for the distribution function coupled to correlators of statistical fluctuations [106]. The dynamical equations are worked out in their most general form. Integrating-out the fluctuations, in a second step, yields the seeked-for effective kinetic theory for the mean fields only. Such a procedure amounts to a derivation of collision terms, noise sources and fluctuation-induced currents. As usual, applications are tied to certain systematic approximations, which are discussed as well. We mainly follow the lines of reasoning as first outlined in [105,106]. For a brief summary, see [108].

A. Mean fields vs. fluctuations

To perform the step from the microscopic to the macroscopic formulation of the problem, we take the ensemble average of the microscopic equations (2.40). As argued above, this implies that the distribution function $f(x, p, Q)$, which in the microscopic picture is a deterministic quantity, now has a probabilistic nature and can be considered as a random function, given by its mean value and statistical (random) fluctuation about it. Let us define the quantities

$$f(x, p, Q) = \bar{f}(x, p, Q) + \delta f(x, p, Q) , \tag{4.1a}$$

$$J_a^\mu(x) = \bar{J}_a^\mu(x) + \delta J_a^\mu(x) , \tag{4.1b}$$

where the quantities carrying a bar denote the mean values, e.g. $\bar{f} = \langle f \rangle$, $\bar{J} = \langle J \rangle$, while the mean value of the statistical fluctuations vanish by definition, $\langle \delta f \rangle = 0$ and $\langle \delta J \rangle = 0$. This separation into the mean distribution function and the mean current on the one side, and their fluctuations on the other, takes into account in particular the stochastic (or source) fluctuations of the one-particle distribution function. These fluctuations in the quasi-particle distribution function and in the induced current are responsible for fluctuations in the gauge fields as well, and we therefore split the gauge fields accordingly as

$$A_\mu^a(x) = \bar{A}_\mu^a(x) + a_\mu^a(x) , \tag{4.2a}$$

$$\langle A \rangle = \bar{A} , \quad \langle a \rangle = 0 . \tag{4.2b}$$

Notice that the split of the gauge fields Eq. (4.2a) has to be seen on a different footing as the split for the one-particle distribution function. These gauge field degrees of freedom

are not defined in phase space. Their fluctuations are induced by those of the particles. We postpone a detailed discussion on further implications due to gauge symmetry until Section V.

Effectively, such a split corresponds to a separation of the low frequency or long wavelength modes associated to the mean quantities from the high frequency or short wavelength modes associated to the fluctuations. As we shall see below, the relevant momentum scales depend on the approximations employed. They are identified explicitly for a plasma close to thermal equilibrium (see Sections VI-VIII).

The induced fluctuations in the gauge fields Eq. (4.2a) require additionally the split of the field strength tensor as

$$F_{\mu\nu}^a = \bar{F}_{\mu\nu}^a + f_{\mu\nu}^a \ , \tag{4.3a}$$

$$\bar{F}_{\mu\nu}^a = F_{\mu\nu}^a[\bar{A}] \ , \tag{4.3b}$$

$$f_{\mu\nu}^a = (\bar{D}_\mu a_\nu - \bar{D}_\nu a_\mu)^a + g f^{abc} a_\mu^b a_\nu^c \ . \tag{4.3c}$$

We used $\bar{D}_\mu \equiv D_\mu[\bar{A}]$. The term $f_{\mu\nu}^a$ contains terms linear and quadratic in the fluctuations. Note that the statistical average of the field strength $\langle F_{\mu\nu}^a \rangle$ is not only given by $\bar{F}_{\mu\nu}^a$, but rather by

$$\langle F_{\mu\nu}^a \rangle = \bar{F}_{\mu\nu}^a + g f^{abc} \langle a_\mu^b a_\nu^c \rangle \ , \tag{4.4}$$

due to quadratic terms contained in $f_{\mu\nu}^a$.

B. Effective transport equations

We now perform the step from the microscopic to the macroscopic Boltzmann equation by taking the statistical average of Eqs. (2.40). This yields the dynamical equation for the mean values,

$$p^\mu \left(\bar{D}_\mu - g Q_a \bar{F}_{\mu\nu}^a \partial_p^\nu \right) \bar{f} = \langle \eta \rangle + \langle \xi \rangle \ . \tag{4.5a}$$

We have made use of the covariant derivative of f as introduced in Eq. (2.44). The macroscopic Yang-Mills equations are

$$\bar{D}_\mu \bar{F}^{\mu\nu} + \langle J_{\text{fluc}}^\nu \rangle = \bar{J}^\nu \ . \tag{4.5b}$$

In Eqs. (4.5), we collected all terms quadratic or cubic in the fluctuations into the functions $\eta(x, p, Q), \xi(x, p, Q)$ and $J_{\text{fluc}}(x)$. These terms are qualitatively new as they are not present

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in the original set of microscopic transport equations. Their physical relevance is discussed in Section IV C below. Written out explicitly, they read

$$
\begin{aligned}
\eta(x, p, Q) \equiv & \ gQ_a\, p^\mu \left(\bar{D}_\mu a_\nu - \bar{D}_\nu a_\mu \right)^a \partial_p^\nu \delta f(x, p, Q) \\
& + g^2\, Q_a\, p^\mu\, f^{abc}\, a_\mu^b a_\nu^c\, \partial_p^\nu \delta f(x, p, Q) \ ,
\end{aligned}
\tag{4.6a}
$$

$$
\begin{aligned}
\xi(x, p, Q) \equiv & \ g\, p^\mu f^{abc} Q^c\, a_\mu^b\, \partial_Q^a \delta f(x, p, Q) \\
& + g^2 p^\mu f^{abc} Q^c\, a_\mu^a a_\nu^b\, \partial_p^\nu \bar{f}(x, p, Q),
\end{aligned}
\tag{4.6b}
$$

$$
\begin{aligned}
J_{\text{fluc}}^{a\nu}(x) \equiv & \ g \left[f^{dbc} \bar{D}_{ad}^\mu a_{b\mu} a_c^\nu + f^{abc} a_{b\mu} (\bar{D}^\mu a^\nu - \bar{D}^\nu a^\mu)_c \right] \\
& + g^2 f^{abc} f^{cde}\, a_{b\mu}\, a_d^\mu\, a_e^\nu \ .
\end{aligned}
\tag{4.6c}
$$

We remark in passing that the split of the fluctuation-induced terms into η and ξ is, to some extend, arbitrary. The term η stems entirely from the fluctuations of the field strength tensor Eq. (4.3c) and the fluctuation of the distribution function. In turn, ξ contains two contributions of different origin: the fluctuation fields from the covariant derivative term, and the fluctuation gauge fields to quadratic order of the field strength tensor multiplied with the mean value of the distribution function. The first term is due to fluctuations of the 'drift term' in the Boltzmann equation. The second term can be seen as a fluctuation-induced force. Both vanish identically in the Abelian case (see Section IV C below).

The effective transport equations (4.5) are not yet a closed system of differential equations involving only the mean fields. They still contain correlators of fluctuations, for which the appropriate transport equations have to be studied separately. They are obtained by subtracting Eqs. (4.5) from Eqs. (2.40). The result is

$$
\begin{aligned}
p^\mu \left(\bar{D}_\mu - gQ_a \bar{F}_{\mu\nu}^a \partial_p^\nu \right) \delta f = & \ gQ_a(\bar{D}_\mu a_\nu - \bar{D}_\nu a_\mu)^a p^\mu \partial_\nu^p \bar{f} \\
& + g p^\mu a_{b\mu} f^{abc} Q_c \partial_a^Q \bar{f} \\
& + \eta + \xi - \langle \eta + \xi \rangle
\end{aligned}
\tag{4.7a}
$$

$$
\left[\bar{D}^2 a^\mu - \bar{D}^\mu(\bar{D}_\nu a^\nu) \right]^a + 2g f^{abc} \bar{F}_b^{\mu\nu} a_{c\nu} = \delta J^{a\mu} - J_{\text{fluc}}^{a\mu} + \langle J_{\text{fluc}}^{a\mu} \rangle \ .
\tag{4.7b}
$$

The above set of dynamical equations – in addition to the initial conditions as derived in the previous section from the Gibbs ensemble average – is at the basis for a description of all transport phenomena in the plasma.

While the dynamics of the mean fields Eqs. (4.5) depends on correlators quadratic and cubic in the fluctuations, the dynamical equations for the fluctuations Eqs. (4.7) also depend on higher order terms (up to cubic order) in the fluctuations themselves. The dynamical equations for the higher order correlation functions are contained in Eqs. (4.7). To see this, consider for example the dynamical equation for the correlators $\langle \delta f\, \delta f \rangle$. After multiplying Eq. (4.7a) with $\delta f'$ and taking the statistical average, we obtain

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$$p^\mu \left(\bar{D}_\mu - g Q_a \bar{F}^a_{\mu\nu} \partial^\nu_p \right) \langle \delta f \, \delta f' \rangle = g Q_a p^\mu \partial^p_\nu \bar{f} \, \left\langle (\bar{D}_\mu a_\nu - \bar{D}_\nu a_\mu)^a \delta f' \right\rangle$$

$$+ g p^\mu f^{abc} Q_c \partial^Q_a \bar{f} \, \langle a_{b,\mu} \, \delta f' \rangle$$

$$+ \langle (\eta + \xi - \langle \eta + \xi \rangle) \, \delta f' \rangle \; . \tag{4.8}$$

(To simplify the notation, we have not given the arguments of all fields explicitly. In particular, $\langle \delta f \, \delta f' \rangle$ means $\langle \delta f(x, p, Q) \, \delta f(x', p', Q') \rangle$, and the derivatives act only on the (x, p, Q) dependences.) In the same way, we find for $\langle \delta f \, \delta f' \, \delta f'' \rangle$ the dynamical equation

$$p^\mu \left(\bar{D}_\mu - g Q_a \bar{F}^a_{\mu\nu} \partial^\nu_p \right) \langle \delta f \, \delta f' \, \delta f'' \rangle = g Q_a p^\mu \partial^p_\nu \bar{f} \, \left\langle (\bar{D}_\mu a_\nu - \bar{D}_\nu a_\mu)^a \, \delta f' \, \delta f'' \right\rangle$$

$$+ g p^\mu f^{abc} Q_c \partial^Q_a \bar{f} \, \langle a_{b,\mu} \, \delta f' \, \delta f'' \rangle$$

$$+ \langle (\eta + \xi - \langle \eta + \xi \rangle) \, \delta f' \, \delta f'' \rangle \; , \tag{4.9}$$

and similarly for higher order correlators. Typically, the dynamical equations for correlators of n fluctuations will couple to correlators ranging from the order $(n-1)$ up to order $(n+2)$ in the fluctuations. From cubic order onwards, the back-coupling contains terms non-linear in the correlation functions.

The correlators of gauge field fluctuations, like $\langle a \, \delta f' \rangle$ and $\langle a \, a' \rangle$ or higher order ones, are related to those of the one-particle distribution function through the Yang-Mills equations. For example, from Eq. (4.7b), we deduce for the quadratic correlators

$$\left[\left(\bar{D}^2 \delta^{\mu\nu} - \bar{D}^\mu \bar{D}^\nu \right)^{ab} + 2 g f^{acb} \bar{F}^{\mu\nu}_c \right] \langle a_{\nu b} \, \delta f' \rangle = \int dP dQ \, p^\mu Q^a \, \langle \delta f \, \delta f' \rangle$$

$$- \langle (J^{\mu a}_{\text{fluc}} - \langle J^{\mu a}_{\text{fluc}} \rangle) \, \delta f' \rangle \tag{4.10}$$

and

$$\left[\left(\bar{D}^2 \delta^{\mu\nu} - \bar{D}^\mu \bar{D}^\nu \right)^{ab} + 2 g f^{acb} \bar{F}^{\mu\nu}_c \right] \left\langle a_{\nu b} \, a'_{\rho d} \right\rangle = \int dP dQ \, p^\mu Q^a \, \left\langle \delta f \, a'_{\rho d} \right\rangle$$

$$- \left\langle (J^{\mu a}_{\text{fluc}} - \langle J^{\mu a}_{\text{fluc}} \rangle) \, a'_{\rho d} \right\rangle \; . \tag{4.11}$$

In this manner, the full hierarchy of coupled dynamical equations for all n-point correlation functions are obtained. The initial conditions are the equal-time correlation functions as derived from the ensemble average.

The resulting hierarchy of dynamical equations for the correlators is very similar to the BBGKY hierarchy within non-relativistic statistical mechanics [20]. A decisive difference stems from the fact that the present set of dynamical equations is dissipative even in their un-approximated form, while the complete BBGKY hierarchy remains time-inversion invariant [20,95].

C. Collision integrals, noise and induced currents

Let us comment on the qualitatively new terms η, ξ and J_{fluc} as defined in Eqs. (4.6). In the effective Boltzmann equation, the functions $\langle\eta\rangle$ and $\langle\xi\rangle$ appear only after the splitting Eqs. (4.1) has been performed. These terms are qualitatively different from those already present in the initial transport equation. The correlators $\langle\eta\rangle$ and $\langle\xi\rangle$ are interpreted as effective collision integrals of the macroscopic Boltzmann equation. The fluctuations in the distribution function of the particles induce fluctuations in the gauge fields, while the gauge field fluctuations, in turn, induce fluctuations in the motion of the quasi-particles. In the present formalism, the correlators of statistical fluctuations have the same effect as collisions. This yields a precise recipe for obtaining collision integrals within semi-classical transport theory.

The term $\langle\eta\rangle$ contains the correlator $\langle f^a_{\mu\nu}\delta f\rangle$ between the fluctuations of the field strength and the fluctuations of the distribution function. In the Abelian limit, only the collision integral $\langle\eta\rangle$ survives and Eqs. (4.5) and (4.7) reduce to the known set of kinetic equations for Abelian plasmas [94]. Then, $\langle\eta\rangle$ can be explicitly expressed as the Balescu-Lenard collision integral [19,101] after solving the dynamical equations for the fluctuations and computing the correlators involved [94,99]. This proves in a rigorous way the correspondence between fluctuations and collisions in an Abelian plasma.

The term $\langle\xi\rangle$ contains two contributions. The term proportional to $\langle\delta D\delta f\rangle$ leads to a collision integral due to the fluctuations of the 'drift covariant derivative'. The term proportional to $\langle\delta f^a_{\mu\nu}\rangle\bar{f}$ is interpreted as a fluctuation-induced force term, because $\langle\delta f^a_{\mu\nu}\rangle$ is to be seen as a fluctuation-induced field strength in the effective transport equation. Both terms describe a purely non-Abelian effect, they vanish identically in the Abelian limit.

At the same time we observe the presence of stochastic noise in the effective equations. The noise originates in the source fluctuations of the particle distributions and induces *field-independent* fluctuations to the gauge fields. The corresponding terms in the effective Boltzmann equations are therefore η, ξ and J_{fluc} at vanishing mean field or mean current. An explicit example is given in Section VIII F.

Finally, we observe the presence of a fluctuation-induced current $\langle J_{\text{fluc}}\rangle$ in the effective Yang-Mills equation for the mean fields. This current, due to its very nature, stems from the induced correlations of gauge field fluctuations. It vanishes identically in the Abelian case. While the collision integrals are linear in the quasi-particle fluctuations, the induced current only contains the gauge field fluctuations. As the fluctuations of the one-particle distribution function are the basic source for fluctuations, we expect that a non-vanishing induced current will appear as a subleading effect.

In order to find explicitly the collision integrals, noise sources or the fluctuation-induced currents for non-Abelian plasmas, one has to solve first the dynamical equations for

the fluctuations in the background of the mean fields. This step amounts to incorporating the fluctuations within the mean particle distribution function ('integrating-out' the fluctuations). In general, this is a difficult task, in particular due to the non-linear terms present in Eqs. (4.7). As argued above, this will only be possible when some approximations have been performed.

D. Systematic approximations

The coupled set of dynamical equations, as derived and presented here within a semiclassical point particle picture, are exact. No further approximations apart from the original assumption have been made. In order to solve the fluctuation dynamics, it is necessary to apply some systematic approximations, or to find a reasonable truncation for the hierarchy of dynamical equations for correlator functions. With 'solving' we have in mind finding explicit solutions to the dynamics of fluctuations. When reinserted into the mean field equations it should be possible to obtain explicit expressions for them. Such a procedure amounts to incorporating the physics at larger scales, as described by the fluctuations, into the mean quasi-particle distribution function.

Here, two systematic approximation schemes are outlined: an expansion in moments of the fluctuations and an expansion in a small gauge coupling. Although they have distinct origins in the first place, we will see below (Section V) that they are intimately linked due to the requirements of gauge invariance.

1. Expansion in moments of the distribution function

An expansion in moments of the fluctuations has its origin in the framework of kinetic equations. Effectively, the kinetic equations describe the coherent behaviour of the particles within some physically relevant volume. This coherent behaviour is described by the *plasma parameter* ϵ, the inverse of which measures the number of particles within a physically relevant volume element as described by the one-particle distribution function. In a close-to-equilibrium plasma, the plasma parameter is given by the ratio between the cube of the mean particle distance and the Debye radius (see Section VI). The fluctuations in the number of particles become arbitrarily small if the physical volume – or the number of particles contained in it – can be made arbitrarily large. For realistic situations, both the physical volume and the particle number are finite. Still, the fluctuations remain at least parametrically small and suppressed by the plasma parameter [94,97]. Hence, the underlying expansion parameter for an expansion in moments of fluctuations is a small plasma parameter

$$\epsilon \ll 1 \ . \tag{4.12}$$

The leading order approximation in an expansion in moments is the *first moment approximation*. It consists in imposing

$$f = \bar{f} \ , \quad \text{or} \quad \delta f \equiv 0 \ , \tag{4.13}$$

and corresponds to neglecting all fluctuations throughout. Sometimes it is referred to as the mean field or *Vlasov approximation*. It leads to a closed system of equations for the mean one-particle distribution function and the gauge fields. In particular, the corresponding Boltzmann equation is dissipationless. It remains time-inversion invariant[*] as does the original microscopic transport equation.

Beyond leading order, the *second moment approximation* takes into account the corrections due to correlators up to quadratic order in the fluctuations $\langle \delta f \delta f \rangle$. All higher order correlators like

$$\langle \delta f_1 \delta f_2 \ldots \delta f_n \rangle = 0 \tag{4.14}$$

for $n > 2$ are neglected within the dynamical equations for the mean fields and the quadratic correlators. This approximation is viable if the fluctuations remain sufficiently small (see also Section VI). We remark that the second moment approximation, the way it is introduced here, and unlike the first moment approximation, no longer yields a closed system of dynamical equations for the one-particle distribution function and quadratic correlators. The reason for this is that the initial conditions for the evolution of correlators, which are given by the equal-time correlation functions as derived from the Gibbs ensemble average, still do involve the two-particle correlation functions. Hence, in addition to Eq. (4.14), we have to require that two-particle correlators remain small as compared to products of one-particle distribution functions,

$$g_2 \ll f_1 \, f_1 \ , \tag{4.15}$$

so that $f_2 \approx f_1 f_1$ (see Eq. (3.11)). This is the case if the effective dynamical equations are valid at scales larger than typical scales of two-particle correlations. Once the method is put to work, it is possible to check explicitly whether Eq. (4.15) holds true or not. The combined approximations Eqs. (4.14) and (4.15) are known as the *approximation of second correlation functions*, sometimes also referred to as the *polarisation approximation* [94].

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  [*]This holds as long as the initial conditions do not violate explicitly time-inversion invariance (cf. Landau damping) [94].

For the dynamical equations of the fluctuations Eqs. (4.7) these approximations imply that the terms non-linear in the fluctuations should be neglected to leading order. This corresponds to setting

$$\eta - \langle \eta \rangle = 0 \ , \tag{4.16a}$$

$$\xi - \langle \xi \rangle = 0 \ , \tag{4.16b}$$

$$J_{\mathrm{fluc}} - \langle J_{\mathrm{fluc}} \rangle = 0 \ . \tag{4.16c}$$

The essence of this step is that the dynamical equations for the correlators become homogeneous. It is easy to see that Eq. (4.8) or Eq. (4.9) depend only on quadratic or cubic correlators, respectively, once Eqs. (4.16) are imposed. This approximation permits truncating the infinite hierarchy of equations for the mean fields and the correlators of fluctuations down to a closed system of differential equations for both mean quantities and quadratic correlators. The mean fields then couple only to quadratic correlators, and all higher order correlators couple amongst themselves. This turns the dynamical equation for the fluctuations Eqs. (4.7) into a differential equation linear in the fluctuations. Notice, however, that the approximation Eq. (4.16) can be improved even within the second moment approximation, if these differences are found to be again linear in the fluctuations.

The polarisation approximation is the minimal choice necessary to genuinely describe dissipative processes, because it takes into account the feed-back of stochastic fluctuations within the particle distribution function.

In the light of the discussion in Section IV C, the second moment approximation Eq. (4.14) can be interpreted as neglecting three-particle collisions in favour of two-particle collisions within the collision integrals for the mean fields. Notice that no explicit correlators higher than cubic order appear in the collision integral Eq. (4.5a). Effective four- or more-particle interactions will only appear due to the back-coupling of the quadratic and cubic correlators to higher order ones. The approximation Eq. (4.16) for the fluctuations can be interpreted as neglecting the back-coupling of collisions to the dynamics of the fluctuations. Beyond leading order, this approximation is modified and the right-hand side of Eqs. (4.16) will be replaced by the effective collision terms as obtained to leading order. If such terms turn out to be linear in the fluctuations we can perform a 'resummed' polarisation approximation, taking the higher order effects iteratively into account.

2. Expansion in the gauge coupling

A qualitatively different approximation scheme concerns the non-Abelian sector of the theory, characterised by a small gauge coupling g. It is possible to perform a systematic perturbative expansion in powers of the gauge coupling g, keeping only the leading order

terms. This can be done because the differential operator appearing in the effective Boltzmann equation Eq. (4.5a) admits such an expansion. In a small coupling expansion, the force term $g\, p^{\mu} Q_a \bar{F}^a_{\mu\nu} \partial^{\nu}_p$ is suppressed by a power of g as compared to the leading order term $p^{\mu} \bar{D}_{\mu}$. Notice that expanding the covariant derivative term $p^{\mu} \bar{D}_{\mu}$ of Eq. (2.44) into powers of g is not allowed as it will break gauge invariance. In this spirit, we expand

$$\bar{f} = \bar{f}^{(0)} + g\, \bar{f}^{(1)} + g^2\, \bar{f}^{(2)} + \dots \tag{4.17}$$

and similarly for δf. This is at the basis for a systematic organisation of the dynamical equations in powers of g.

To leading order, this concerns in particular the cubic correlators appearing in $\langle \eta \rangle$ and $\langle J_{\text{fluc}} \rangle$. They are suppressed by a power of g as compared to the quadratic ones. Hence, the second moment approximation and an expansion in a small gauge coupling are mutually compatible. At the same time, the quadratic correlator $\sim f_{abc} \langle a^b\, a^c \rangle$ within $\langle \xi \rangle$ is also suppressed by an additional power of g and should be suppressed to leading order. We shall show in the following section that such approximations are consistent with the mean field gauge symmetry.

A word of caution is due at this point. While a small gauge coupling appears to be at the basis for perturbative expansions, it cannot be excluded that *another* dimensionless expansion parameter becomes relevant due to particular dynamical properties of the system. Indeed, as we shall see below in the close-to-equilibrium plasma, at higher order the natural expansion parameter happens to be $[\ln(1/g)]^{-1}$ instead of g. This implies that expansions like Eq. (4.17) might be feasible only for the first few terms.

In principle, after these approximations are done, it should be possible to express the correlators of fluctuations appearing in Eqs. (4.5) through known functions. This requires finding a solution of the fluctuation dynamics first.

E. Conservation laws

After this discussion of the basic set of dynamical equations we return to the conservation laws for the energy-momentum tensor and current conservation. In the same spirit as the splitting of the fundamental variables into fluctuations and mean values we split the energy-momentum tensor of the gauge fields into the part from the mean fields and the fluctuations, according to

$$\Theta^{\mu\nu} = \bar{\Theta}^{\mu\nu} + \theta^{\mu\nu} \,, \tag{4.18a}$$

$$\bar{\Theta}^{\mu\nu} = \tfrac{1}{4} g^{\mu\nu} \bar{F}^a_{\rho\sigma} \bar{F}^{\rho\sigma}_a + \bar{F}^{\mu\rho}_a \bar{F}^{a\ \nu}_{\rho} \,, \tag{4.18b}$$

$$\theta^{\mu\nu} = \tfrac{1}{2} g^{\mu\nu} \bar{F}^a_{\rho\sigma} f^{\rho\sigma}_a + \bar{F}^{\mu\rho}_a f^a_{\rho\nu} + \bar{F}^{\nu\rho}_a f^a_{\rho\mu} + \tfrac{1}{4} g^{\mu\nu} f_{\rho\sigma,a} f^{\rho\sigma,a} + f^{\mu\rho}_a f^{a\ \nu}_{\rho} \,. \tag{4.18c}$$

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The term $\theta^{\mu\nu}$ contains the fluctuations up to quartic order. Due to the non-linear character of the theory, we find that the ensemble average of the energy momentum tensor is not only given by $\bar{\Theta}^{\mu\nu}$, but

$$\langle \Theta^{\mu\nu} \rangle = \bar{\Theta}^{\mu\nu} + \langle \theta^{\mu\nu} \rangle \ . \tag{4.19}$$

The dynamical equation for the energy momentum tensor of the gauge fields comes from the average of Eq. (2.41). The corresponding one for the particles is found after integrating Eq. (2.40a) over $dPdQ\,p^{\mu}$. The two of them read

$$\partial_{\nu}\bar{\Theta}^{\mu\nu} + \partial_{\nu}\langle\theta^{\mu\nu}\rangle = -\bar{F}_a^{\mu\nu}\bar{J}_{\nu a} - \langle f_a^{\mu\nu}\delta J_{\nu a}\rangle - \langle f_a^{\mu\nu}\rangle\bar{J}_{\nu}^a \ , \tag{4.20a}$$

$$\partial_{\nu}\bar{t}^{\mu\nu} = \bar{F}_a^{\mu\nu}\bar{J}_{\nu a} + \langle f_a^{\mu\nu}\delta J_{\nu a}\rangle + \langle f_a^{\mu\nu}\rangle\bar{J}_{\nu}^a \ . \tag{4.20b}$$

Hence, we confirm that the total mean energy-momentum tensor $\langle T^{\mu\nu} \rangle$ is conserved,

$$\partial_{\nu}\langle T^{\mu\nu} \rangle = 0 \ . \tag{4.21}$$

The condition for the microscopic current conservation translates, after averaging, into two equations, one for the mean fields, and another one for the fluctuation fields. From $\langle D_{\mu}J^{\mu} \rangle = 0$ we obtain

$$(\bar{D}_{\mu}\bar{J}^{\mu})_a + gf_{abc}\langle a_{\mu}^b\delta J^{c\mu}\rangle = 0 \ . \tag{4.22}$$

For the fluctuation current, we learn from $D_{\mu}J^{\mu} - \langle D_{\mu}J^{\mu} \rangle = 0$ that

$$(\bar{D}_{\mu}\delta J^{\mu})_a + gf_{abc}\left(a_{\mu}^b\bar{J}_c^{\mu} + a_{\mu}^b\delta J_c^{\mu} - \langle a_{\mu}^b\delta J_c^{\mu}\rangle\right) = 0 \ . \tag{4.23}$$

Similar equations are obtained from the Yang-Mills equations themselves. Here, we only remark that these two set of equations are mutually consistent, which is shown in Section V B.

F. Entropy

Finally, we shall also introduce the kinetic entropy S associated to these particles. The entropy density $S^{\mu}(x)$, as a function of the one-particle distribution function, is defined as

$$S^{\mu}(x) = \int dPdQ\,p^{\mu}\,\Sigma[f](x,p,Q) \ , \tag{4.24}$$

from which the entropy obtains as

$$S(t) = \int d^3x\,S_0(x) \ . \tag{4.25}$$

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The function $\Sigma[f](x, p, Q)$ depends on the statistics of the particles. For classical (Maxwell-Boltzmann) statistics, we have [67]

$$\Sigma_{\text{cl}}[f] = -f\,(\ln f - 1) \tag{4.26}$$

while for quantum (Bose-Einstein or Fermi-Dirac) statistics, we use

$$\Sigma_{\text{qm}}[f] = -f \ln f \pm (1 \pm f) \ln (1 \pm f) \tag{4.27}$$

instead. The '−' sign stands for bosonic degrees of freedom, and the '+' sign for fermionic ones. Microscopically, the entropy Eq. (4.24) is conserved, $dS/dt = 0$. This follows from the vanishing of

$$\partial_\mu S^\mu(x) = 0 \; , \tag{4.28}$$

which, for both classical or quantum plasmas, can be deduced from inserting the microscopic Boltzmann equation Eq. (2.40a) into Eq. (4.24). Ultimately, this is linked to the fact that the Boltzmann equation contains no explicit collision term.

On the macroscopic level, and after separation into mean field contributions and fluctuations, the entropy four-flow reads

$$S^\mu(x) = \bar{S}^\mu(x) + \Delta S^\mu(x) \tag{4.29a}$$

$$\bar{S}^\mu(x) = \int dP dQ \, p^\mu \, \Sigma[\bar{f}] \tag{4.29b}$$

$$\Delta S^\mu(x) = \int dP dQ \, p^\mu \left[\Delta\Sigma^{(1)} + \Delta\Sigma^{(2)} \right] \; . \tag{4.29c}$$

We have separated the terms of linear order in δf into $\Delta\Sigma^{(1)}$, and all the higher order terms into $\Delta\Sigma^{(2)}$. They read explicitly

$$\Delta\Sigma_{\text{cl}}^{(1)} = -\delta f \, \ln \bar{f} \tag{4.30a}$$

$$\Delta\Sigma_{\text{cl}}^{(2)} = -(\bar{f} + \delta f) \ln \left(1 + \delta f/\bar{f} \right) + \delta f \tag{4.30b}$$

for the classical plasma, and

$$\Delta\Sigma_{\text{qm}}^{(1)} = -\delta f \left[\ln \bar{f} - \ln(1 \pm \bar{f}) \right] \tag{4.31a}$$

$$\Delta\Sigma_{\text{qm}}^{(2)} = -(\bar{f} + \delta f) \ln \left(1 + \delta f/\bar{f} \right) \pm (1 \pm \bar{f} \pm \delta f) \ln \left[1 \pm \delta f/(1 \pm \bar{f}) \right] \tag{4.31b}$$

for quantum plasmas. The presence of fluctuations is closely linked to dissipative processes. In particular, the mean entropy density is no longer given by the entropy density of the mean particle distribution, but rather by

$$\langle S^\mu(x) \rangle = \bar{S}^\mu(x) + \int dP dQ \, p^\mu \, \langle \Delta\Sigma^{(2)} \rangle \; , \tag{4.32}$$

which involves arbitrarily high order correlation functions of the fluctuations.

G. Discussion

A self-contained semi-classical transport theory has been derived from a microscopic point particle picture. The ensemble average transformed the microscopic kinetic equation into a coupled set of dynamical equations for mean distribution functions, mean fields, and correlator functions of fluctuations. Usually, a kinetic description considers the plasma as a continuous medium. Here, the stochastic fluctuations are taken into account as well. The source of stochastic noise is given by the fluctuations of the one-particle distribution function. These enter the initial conditions for the dynamics of the correlation functions. Fluctuations in the gauge fields are induced by the latter.

The split into mean quantities and fluctuations is convenient for several reasons. First of all, it separates the short-scale characteristics of the plasma, associated to the fluctuations, from the large-scale ones, associated to the mean quantities. Second, the set of coupled dynamical equations can be reduced to an effective transport equation for the mean fields only, at least within some approximations. This amounts to the 'integrating-out' of fluctuations. Only then are the modes associated to the fluctuations incorporated in the quasi-particle distribution function.

Two systematic approximation schemes have been discussed, an expansion in the plasma parameter and an expansion in a small gauge coupling. These schemes are mutually compatible and linked further by the requirement of gauge invariance (Section V). On a technical level, this procedure corresponds to a recipe for *deriving* collision integrals and the corresponding noise sources for the effective transport equation, and a fluctuation-induced current for the Yang-Mills equation.

We stress that the present formalism is applicable for both in- and out-of-equilibrium situations. This is due to the fact that the statistical properties of the system are all encoded in the Gibbs ensemble average, which in turn does not rely on a close-to-equilibrium situation. A detailed discussion of the gauge symmetry, and in particular the consistency of the split Eq. (4.2a), is given in Section V.

V. GAUGE SYMMETRY

The formalism developed in the two preceding sections is based on a split of non-Abelian gauge fields into a mean field and a fluctuation field. Accordingly, their original dynamical equation, the Yang-Mills equation, splits into two separate ones. Ultimately, one aims at integrating-out the fluctuation fields such that the remaining effective theory only involves mean fields. It remains to be shown that such a procedure is consistent with the requirements of gauge symmetry.

The idea of splitting gauge fields into two parts in order to integrate-out the fluctuation part is not new. The *background field method* is precisely one such formalism based on a path integral approach.[†] Within the background field method, the gauge fields in the path integral are formally separated into a mean field piece and a quantum piece. The original gauge symmetry splits accordingly into a *background gauge symmetry* under which the quantum field transforms homogeneously, and a *quantum gauge symmetry*, under which the mean field transforms trivially. The background field formalism allows the derivation of an effective theory for the mean fields only, which corresponds to the integrating-out of the quantum fluctuations. It is to be noticed that the background field is an auxiliary field, which is identified with the mean field only after the quantum field has been integrated out. The quantum gauge symmetry is the physical gauge symmetry, which, after the quantum field is integrated out, is inherited by the mean gauge field symmetry. The converse is not true [1]. An application of the background field method within the QCD transport equation for Wigner functions has been considered by Elze [60].

The present formalism is very similar to such a procedure. Here, we aim at integrating-out induced stochastic fluctuations as opposed to quantum ones. Furthermore, after having integrated-out these fluctuations within a given approximation, the resulting effective Boltzmann equation can be seen as the generating functional for the mean gauge field interactions entering the effective Yang-Mills equations.

In this section the requirements of gauge symmetry are exploited. It is shown that the present approach is consistent within the background field approach. This discussion will concern the consistency of the general set of equations. The question of consistent approximations will be raised as well. In this section, we shall for convenience switch to a matrix notation, using the conventions $A \equiv A^a t_a$, $Q \equiv Q^a t_a$ etc., as well as $[t_a, t_b] = f_{abc} t^c$ and $\text{Tr } t_a t_b = -\frac{1}{2}\delta_{ab}$.

[†]For a discussion of the background field method applied to QCD, see Abbott [1].

A. Background gauge symmetry vs. fluctuation gauge symmetry

To begin with, let us consider finite gauge transformations, parametrised as

$$gA'_\mu = U(x)(\partial_\mu + gA_\mu)U^{-1}(x) , \qquad (5.1a)$$

$$U(x) = \exp\left[-g\epsilon^a(x)t_a\right] , \qquad (5.1b)$$

with parameter $\epsilon^a(x)$. Under these transformations, we have the transformation laws

$$Q' = U(x)\, Q\, U^{-1}(x) , \qquad (5.2a)$$

$$\partial'_Q = U^{-1}(x)\, \partial_Q\, U(x) , \qquad (5.2b)$$

$$F'_{\mu\nu} = U(x)\, F_{\mu\nu}\, U^{-1}(x) . \qquad (5.2c)$$

From the definition of the microscopic distribution function $f(x, p, Q)$ we conclude that it transforms as a scalar under (finite) gauge transformations,

$$f'(x, p, Q') = f(x, p, Q) , \qquad (5.3)$$

which has been shown in [92] and establishes that the microscopic set of equations (2.40) transform covariantly under the gauge transformations Eq. (5.1).

When switching to a macroscopic description a statistical average has to be performed. The averaging procedure $\langle\dots\rangle$ as defined in Section III is naturally invariant under gauge transformations. It remains to be shown that the subsequent split of the gauge field into a mean (or background) field and a fluctuation field respects the gauge symmetry. We split the gauge field as

$$A_\mu = \bar{A}_\mu + a_\mu \qquad (5.4a)$$

$$\langle A \rangle = \bar{A} + \langle a \rangle . \qquad (5.4b)$$

For the time being, $\bar{A}$ is an arbitrary constant, and in particular, we shall *not* yet require $\langle a \rangle = 0$. The field $\bar{A}$ is identified as the mean field only when the additional constraint $\langle a \rangle = 0$ is employed. Only then does the dynamical equation reduce to those discussed in the preceding section.

The separation Eq. (5.4) is very similar to what is done in the background field method [1,60]. Two symmetries are left after the splitting is performed, the *background gauge symmetry*,

$$g\bar{A}'_\mu = U(x)(\partial_\mu + g\bar{A}_\mu)U^{-1}(x) , \qquad (5.5a)$$

$$a'_\mu = U(x)\, a_\mu\, U^{-1}(x) , \qquad (5.5b)$$

and the *fluctuation gauge symmetry*,

$$g\bar{A}'_\mu = 0 \;, \tag{5.6a}$$

$$ga'_\mu = U(x)\left(\partial_\mu + g(\bar{A}_\mu + a_\mu)\right)U^{-1}(x) \;. \tag{5.6b}$$

Under the background gauge symmetry, the fluctuation field transforms covariantly (as a vector in the adjoint).

In the first step, we split the microscopic Boltzmann equation (2.40a) according to Eq. (5.4a). It follows trivially that the resulting equation is invariant under both the mean field symmetry Eq. (5.5) and the fluctuation field symmetry Eq. (5.6), if both $\bar{f}$ and δf transform as f, that is, as scalars.

Turning to the macroscopic equations, we perform the statistical average and split the transport equation into those for mean fields and fluctuations. We employ the fundamental requirement $\langle \delta f \rangle = 0$, but leave $\langle a \rangle$ unrestricted. It is useful to rewrite the effective transport equations in matrix convention. We have

$$p^\mu\left(\bar{D}_\mu + 2g\,\mathrm{Tr}\,(Q\bar{F}_{\mu\nu})\,\partial_p^\nu\right)\bar{f} = \langle \eta \rangle + \langle \xi \rangle + \langle \zeta \rangle \;, \tag{5.7a}$$

$$\left[\bar{D}_\mu, \bar{F}^{\mu\nu}\right] + \langle J_{\mathrm{fluc}}^\nu \rangle + \langle J_{\mathrm{lin}}^\nu \rangle = \bar{J}^\nu \;. \tag{5.7b}$$

Notice that Eqs. (5.7) appears to be of the same form as Eqs. (4.5), except for the new terms $\langle \zeta \rangle$ and $\langle J_{\mathrm{lin}}^\nu \rangle$, which contain the pieces linear in $\langle a \rangle$. The functions η, ξ and J_{fluc} read

$$\eta(x,p,Q) = -2g\,\mathrm{Tr}\,\left(Q\left[\bar{D}_\mu, a_\nu\right] - Q\left[\bar{D}_\nu, a_\mu\right]\right)p^\mu\partial_p^\nu\delta f(x,p,Q)$$
$$-2g^2\,\mathrm{Tr}\,(Q\left[a_\mu, a_\nu\right])p^\mu\partial_p^\nu\delta f(x,p,Q) \;, \tag{5.8a}$$

$$\xi(x,p,Q) = -2g\,p^\mu\,\mathrm{Tr}\,([Q,\partial^Q]\,a_\mu)\,\delta f(x,p,Q)$$
$$-2g^2 p^\mu\,\mathrm{Tr}\,([a_\mu, a_\nu]Q)\,\partial_p^\nu\bar{f}(x,p,Q) \;, \tag{5.8b}$$

$$J_{\mathrm{fluc}}^\nu(x) = g\left[\bar{D}^\mu, [a_\mu, a^\nu]\right] + g\left[a_\mu, [\bar{D}^\mu, a^\nu]\right] - g\left[a_\mu, [\bar{D}^\nu, a^\mu]\right]$$
$$+g^2\left[a_\mu, [a^\mu, a^\nu]\right] \;. \tag{5.8c}$$

while the linear terms ζ and J_{lin} are given by

$$\zeta(x,p,Q) = -2g\,\mathrm{Tr}\,\left(Q[\bar{D}_\mu, a_\nu] - Q[\bar{D}_\nu, a_\mu]\right)p^\mu\partial_p^\nu\bar{f}(x,p,Q)$$
$$-2g\,p^\mu\,\mathrm{Tr}\,\left([Q,\partial^Q]a_\mu\right)\bar{f}(x,p,Q) \;, \tag{5.9a}$$

$$J_{\mathrm{lin}}^\nu(x) = g\left[a_\mu, \bar{F}^{\mu\nu}\right] + \left[\bar{D}_\mu, [\bar{D}^\mu, a^\nu]\right] - \left[\bar{D}_\mu, [\bar{D}^\nu, a^\mu]\right] \;. \tag{5.9b}$$

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For $\langle a \rangle = 0$, the terms $\langle \zeta \rangle$ and $\langle J_{\text{lin}} \rangle$ vanish, and Eqs. (5.7) reduce to Eqs. (4.5). Along the same lines, the fluctuation dynamics becomes

$$p^\mu \left(\bar{D}_\mu - g \, \text{Tr} \left(Q \bar{F}_{\mu\nu} \right) \partial_p^\nu \right) \delta f = -2g \text{Tr} \left(Q[\bar{D}_\mu, a_\nu] - Q[\bar{D}_\nu, a_\mu] \right) p^\mu \partial_p^\nu \bar{f}$$

$$-2g \, p^\mu \, \text{Tr} \left([Q, \partial^Q] a_\mu \right) \bar{f} - \langle \zeta \rangle$$

$$+\eta + \xi - \langle \eta + \xi \rangle \, , \tag{5.10a}$$

$$\left[\bar{D}_\nu, [\bar{D}^\nu, a^\mu] \right] - \left[\bar{D}^\mu, [\bar{D}_\nu, a^\nu] \right] + 2g[\bar{F}^{\mu\nu}, a_\nu] = \delta J^\mu - J^\mu_{\text{fluc}} + \langle J^\mu_{\text{fluc}} + J^\mu_{\text{lin}} \rangle \, . \tag{5.10b}$$

Again, the vanishing of $\langle a \rangle$, and hence of $\langle \zeta \rangle$ and $\langle J_{\text{lin}} \rangle$, reduces Eqs. (5.10) to Eqs. (4.7).

It is straightforward, if tedious, to confirm that this coupled set of differential equations (5.7) to (5.10) is invariant under both the fluctuation gauge symmetry Eqs. (5.6) and under the background field symmetry Eqs. (5.5). It suffices to employ the cyclicity of the trace, and to note that a_μ and background covariant derivatives of it transform covariantly. This establishes that the full gauge symmetry of the underlying microscopic set of equations is respected at the effective mean field level.

The next step, to finally obtain the set of equations given in the preceding section, involves the requirement that the statistical average of the gauge field fluctuation vanishes, $\langle a \rangle = 0$. This additional constraint is fully compatible with the background gauge symmetry, as $\langle a \rangle = 0$ is invariant under Eqs. (5.5). Any inhomogeneous transformation law for a, and in particular Eq. (5.6), can no longer be a symmetry of the macroscopic equations as the constraint $\langle a \rangle = 0$ is not invariant under the fluctuation gauge symmetry. This is similar to what happens in the background field method, where the fluctuation gauge symmetry can no longer be seen once the expectation value of the fluctuation field is set to zero. As we have just verified, the symmetry Eqs. (5.6) is observed in both Eqs. (4.5a) and (4.7a), as long as the terms linear in $\langle a \rangle$ are retained.

The value for a is obtained when the dynamical equations for the fluctuations are solved explicitly. This requires that some gauge for the fluctuation field has to be fixed. For any (approximate) explicit solution which expresses the fluctuation field a as a functional of the source fluctuations of the particle distribution function one has to check for consistency that the intial constraint $\langle a \rangle = 0$ is satisfied. If the solution a turns out to be a *linear* functional of $\delta f|_{t=0}$, this is automatically satisified. An example for this is encountered in Section VIII. This justifies the dynamical equations as given in Section IV.

B. Current conservation

In Eqs. (4.22) and (4.23), we have given the equations which imply the covariant current conservation of the mean and the fluctuation current. However, this information is

contained both in the transport and in the Yang-Mills equation. It remains to be shown that these equations are self-consistent. For the reasons detailed above it is sufficient to consider from now on only the case $\langle a \rangle = 0$.

We start with the mean current $\bar{J}$. Performing $g \int dPdQ\, Q$ of the transport equation Eq. (5.7a), we find

$$0 = [\bar{D}_\mu, \bar{J}^\mu] + g\langle [a_\mu, \delta J^\mu] \rangle \; . \tag{5.11}$$

This is Eq. (4.22). In order to obtain Eq. (5.11), we made use of the following moments of the effective transport equation,

$$\int dP\; \eta(x, p, Q) = 0 \; , \tag{5.12a}$$

$$\int dP\; p^\mu F_{\mu\nu} \partial_p^\nu f(x, p, Q) = 0 \; , \tag{5.12b}$$

$$\int dPdQ\, Q\; \xi(x, p, Q) = -[a_\mu, \delta J^\mu] \; , \tag{5.12c}$$

$$g \int dPdQ\, Q\; p^\mu \bar{D}_\mu \bar{f}(x, p, Q) = [\bar{D}_\mu, \bar{J}^\mu] \; . \tag{5.12d}$$

On the other hand we can simply take the background-covariant derivative of the mean field Yang-Mills equation, Eq. (5.7b), to find

$$0 = [\bar{D}_\mu, \bar{J}^\mu] - [\bar{D}_\mu, \langle J_{\text{fluc}}^\mu \rangle] \; . \tag{5.13}$$

This equation has to be consistent with Eq. (5.11). Combining them, we end up with the *consistency condition*

$$0 = [\bar{D}_\mu, \langle J_{\text{fluc}}^\mu \rangle] + g\langle [a_\mu, \delta J^\mu] \rangle \; . \tag{5.14}$$

This consistency condition links the background covariant derivative of some correlator of induced gauge field fluctuations with the correlator between the current fluctuations and the gauge field fluctuations. Such a condition can hold because the gauge field fluctuations are induced by those of the current.

In order to prove the consistent current conservation for the mean fields Eq. (5.14), and the corresponding equation for the fluctuation current, it is useful to establish explicitly the following identity

$$0 = [\bar{D}_\mu, J_{\text{fluc}}^\mu] + g[a_\mu, \delta J^\mu] + g[a_\mu, \langle J_{\text{fluc}}^\mu \rangle] \; . \tag{5.15}$$

The check of Eq. (5.15) is algebraic, and it will make use of symmetry arguments like the antisymmetry of the commutator and the tensors $\bar{F}_{\mu\nu}, f_{\mu\nu}$, and of the Jacobi identity

$[t_a, [t_b, t_c]] + [t_b, [t_c, t_a]] + [t_c, [t_a, t_b]] = 0$. The identity $[\bar{D}_\mu, \bar{D}_\nu] = g\bar{F}_{\mu\nu}$ is employed as well. To simplify the computation, we will separate the fluctuation part of the field strength Eq. (4.3c) into the terms linear and quadratic in a, according to

$$f_{\mu\nu} = f_{1,\mu\nu} + f_{2,\mu\nu} \ , \tag{5.16a}$$

$$f_{1,\mu\nu} = [\bar{D}_\mu, a_\nu] - [\bar{D}_\nu, a_\mu] \ , \tag{5.16b}$$

$$f_{2,\mu\nu} = g[a_\mu, a_\nu] \ . \tag{5.16c}$$

Recall furthermore, using Eq. (5.8c) and Eq. (5.10b), that

$$J^\mu_{\mathrm{fluc}} = [\bar{D}_\nu, f_2^{\nu\mu}] + g[a_\nu, f_1^{\nu\mu} + f_2^{\nu\mu}] \ , \tag{5.17}$$

$$\delta J^\mu = [\bar{D}_\nu, f_1^{\nu\mu}] + g[a_\nu, \bar{F}^{\nu\mu}] + J^\mu_{\mathrm{fluc}} - \langle J^\mu_{\mathrm{fluc}} \rangle \tag{5.18}$$

are functions of the fluctuation field a. The first term of Eq. (5.15) reads, after inserting J_{fluc} from Eq. (5.17),

$$[\bar{D}_\mu, J^\mu_{\mathrm{fluc}}] = [\bar{D}_\nu, [\bar{D}_\mu, f_2^{\mu\nu}]] + g[\bar{D}_\nu, [a_\mu, f_1^{\mu\nu}]] + g[\bar{D}_\nu, [a_\mu, f_2^{\mu\nu}]] \ . \tag{5.19}$$

Using δJ from Eq. (5.18), it follows for the second term of Eq. (5.15)

$$g[a_\mu, \delta J^\mu] = g^2[a_\nu, [a_\mu, \bar{F}^{\mu\nu}]] + g[a_\nu, [\bar{D}_\mu, f_1^{\mu\nu}]] + g\,[a_\nu, J^\nu_{\mathrm{fluc}}] - g\,[a_\nu, \langle J^\nu_{\mathrm{fluc}} \rangle] \ . \tag{5.20}$$

The last term of Eq. (5.20) will be cancelled by the last term in Eq. (5.15). We show now that the first three terms of Eq. (5.19) and Eq. (5.20) do cancel one by one. The first term in Eq. (5.19) can be rewritten as

$$[\bar{D}_\nu, [\bar{D}_\mu, f_2^{\mu\nu}]] = [[\bar{D}_\nu, \bar{D}_\mu], f_2^{\mu\nu}] - [\bar{D}_\nu, [\bar{D}_\mu, f_2^{\mu\nu}]] = \tfrac{1}{2}g[\bar{F}_{\nu\mu}, f_2^{\mu\nu}] \ . \tag{5.21}$$

Similarly, the first term of Eq. (5.20) yields

$$g^2[a_\nu, [a_\mu, \bar{F}^{\mu\nu}]] = -g^2[\bar{F}^{\mu\nu}, [a_\nu, a_\mu]] - g^2[a_\nu, [a_\mu, \bar{F}^{\mu\nu}]] = -\tfrac{1}{2}g[\bar{F}_{\mu\nu}, f_2^{\nu\mu}] \ . \tag{5.22}$$

For the second term in Eq. (5.19) we have

$$g[\bar{D}_\nu, [a_\mu, f_1^{\mu\nu}]] = g[a_\mu, [\bar{D}_\nu, f_1^{\mu\nu}]] + g[[\bar{D}_\nu, a_\mu], f_1^{\mu\nu}] = -g[a_\mu, [\bar{D}_\nu, f_1^{\nu\mu}]] \ , \tag{5.23}$$

which equals (minus) the second term of Eq. (5.20). Finally, consider the third term of Eq. (5.20),

$$\begin{aligned}
g\,[a_\nu, J^\nu_{\mathrm{fluc}}] &= g^2[a_\nu, [\bar{D}_\mu, [a^\mu, a^\nu]]] + g^2[a_\nu, [a_\mu, f^{\mu\nu}]] \\
&= \tfrac{1}{2}g[f_2^{\mu\nu}, f_{1,\mu\nu}] - g[\bar{D}_\mu, [f_2^{\mu\nu}, a_\nu]] - \tfrac{1}{2}g[f_2^{\mu\nu}, f_{1,\mu\nu}] \\
&= -g[\bar{D}_\mu, [a_\nu, f_2^{\nu\mu}]] \ , \tag{5.24}
\end{aligned}$$

which equals (minus) the third term of Eq. (5.19). This establishes Eq. (5.15).

Returning to our main line of reasoning we take the average of Eq. (5.15) which reduces it to Eq. (5.14) and establishes the self-consistent conservation of the mean current. The analogous consistency equation for the fluctuation current follows from Eq. (5.10a) after performing $g \int dP dQ\, Q$, and reads

$$0 = [\bar{D}_\mu, \delta J^\mu] + g[a_\mu, \delta J^\mu] + g[a_\mu, \bar{J}^\mu] - g\langle[a_\mu, \delta J^\mu]\rangle \ . \tag{5.25}$$

This is Eq. (4.23). Here, in addition to Eq. (5.12), we made use of

$$2\, g \int dP dQ\, Q \ \mathrm{Tr}\left([Q, \partial^Q] a_\mu\right) \bar{f}(x, p, Q) = g[a_\mu, \bar{J}^\mu] \ . \tag{5.26}$$

The background covariant derivative of Eq. (5.10b) is given as

$$0 = [\bar{D}_\mu, \delta J^\mu] + g\left[a_\nu, [\bar{D}_\mu, \bar{F}^{\mu\nu}]\right] - [\bar{D}_\mu, J^\mu_{\mathrm{fluc}}] + [\bar{D}_\mu, \langle J^\mu_{\mathrm{fluc}}\rangle] \ . \tag{5.27}$$

Subtracting these equations yields the consistency condition

$$0 = [\bar{D}_\mu, J^\mu_{\mathrm{fluc}}] + g[a_\mu, \delta J^\mu] - [\bar{D}_\mu, \langle J^\mu_{\mathrm{fluc}}\rangle]$$
$$- g\langle[a_\mu, \delta J^\mu]\rangle + g[a_\mu, \bar{J}^\mu] - g\left[a_\nu, [\bar{D}_\mu, \bar{F}^{\mu\nu}]\right] \ . \tag{5.28}$$

Eq. (5.28) is a consistency condition which links different orders of the fluctuations of gauge fields with those of the current. Using Eqs. (5.7b), (5.14) and (5.15) we confirm Eq. (5.28) explicitly. This establishes the self-consistent conservation of the fluctuation current.

C. Approximations

We close this section with a comment on the consistency of *approximate* solutions. The consistent current conservation can no longer be taken for granted when it comes to finding approximate solutions of the equations. On the other hand, finding an explicit solution will require some type of approximations to be performed. The relevant question in this context is to know which approximations will be consistent with gauge invariance.

Consistency with gauge invariance requires that approximations have to be consistent with the background gauge symmetry. From the general discussion above we can already conclude that dropping any of the explicitly written terms in Eqs. (4.5) to (4.7) is consistent with the background gauge symmetry Eq. (5.5). This holds in particular for the first and second moment approximations Eq. (4.14) as well as for the polarisation approximation Eq. (4.16).

The first moment approximation Eq. (4.13) is automatically consistent with the covariant current conservation, simply because the approximate equations are structurally the same as the microscopic ones.

Consistency of the polarisation approximation Eq. (4.16) with covariant current conservation turns out to be more restrictive. Employing $J_{\text{fluc}} = \langle J_{\text{fluc}} \rangle$ implies that Eq. (5.14) is only satisfied if in addition

$$0 = \left[\bar{D}_\nu, \langle [a_\mu, [a^\mu, a^\nu]] \rangle \right] \tag{5.29}$$

holds true. This is in accordance with neglecting cubic correlators for the collision integrals.

Similarly, the consistent conservation of the fluctuation current implies the consistency condition Eq. (5.28), and holds if

$$0 = [a_\mu, \langle J^\mu_{\text{fluc}} \rangle] \ . \tag{5.30}$$

It is interesting to note that the consistent current conservation relates the second moment approximation with the neglection of correlators of gauge field fluctuations. We conclude, that Eqs. (4.16) with (5.29) and (5.30) form a gauge-consistent set of approximations.

This terminates the general discussion of a semi-classical transport theory built upon a microscopical point particle picture. The following sections discuss the weakly coupled plasmas close to thermal equilibrium, and the techniques are put to work.

VI. PLASMAS CLOSE TO EQUILIBRIUM

In the remaining part of the article, we employ the present formalism to the classical and the quantum non-Abelian plasma close to equilibrium. Prior to this, we discuss briefly the relevant physical scales for relativistic classical and quantum plasmas close to equilibrium. Here, we restore the fundamental constants $\hbar, c$ and k_B in the formulas.

A. Classical plasmas

To discuss the relevant physical scales in the classical non-Abelian plasma, it is convenient to discuss first the simpler Abelian case, which has been considered in detail in the literature [97,94]. At equilibrium the classical distribution function is given by the relativistic Maxwell distribution,

$$\bar{f}^{\mathrm{eq}}(p_0) = \exp\left(\frac{\mu - p_0}{k_B T}\right) \ , \tag{6.1}$$

where μ is the chemical potential. The mean density of particles $\bar{N}$ is then deduced from the above distribution function. If we neglect the masses of the particles $(m \ll T)$, then

$$\bar{N} = 8\pi \left(\frac{k_B T}{2\pi \hbar c}\right)^3 e^{\mu/k_B T} \ . \tag{6.2}$$

The value of the fugacity of the system $z = e^{\mu/k_B T}$ is then fixed by knowing the mean density of the plasma [‡]. The interparticle distance is then $\bar{r} \sim \bar{N}^{-1/3}$. As we are considering a classical plasma, we are assuming $\bar{r} \gg \lambda_{\mathrm{dB}}$, where λ_{dB} is the de Broglie wave length, $\lambda_{\mathrm{dB}} \sim \hbar/p$, with p some typical momenta associated to the particles, so that $p \sim k_B T/c$. The previous inequality is satisfied if $z \ll 1$, which is the condition under which quantum statistical effects can be neglected.

Another typical scale in a plasma close to equilibrium is the Debye length r_D. The Debye length is the distance over which the screening effects of the electric fields in the plasma are felt. For an electromagnetic plasma, the Debye length squared is given by [94]

$$r_D^2 = \frac{k_B T}{4\pi \bar{N} e^2} \ . \tag{6.3}$$

[‡]Note that the dependence on $\hbar$ of the mean density arises only because our momentum measure is $d^3 p/(2\pi \hbar)^3$; it is just a normalisation constant.

Notice that the electric charge contained in the above formula is a dimensionful parameter: it is just the electric charge of the point particles of the system.

In the classical case, and in the absence of the fundamental constant $\hbar$, the only dimensionless quantities that can be constructed from the basic scales of the problem are dimensionless ratios of the basic scales of the problem. The most important one is the *plasma parameter* ϵ. The plasma parameter is defined as the ratio [94]

$$\epsilon = \frac{\bar{r}^3}{r_D^3} \; . \tag{6.4}$$

The quantity $1/\epsilon$ gives the number of particles contained in a sphere of radius r_D. If $\epsilon \ll 1$ this implies that a large number of particles are in that sphere, and thus a large number of particles are interacting in this volume, and the collective character of their interactions in the plasma cannot be neglected. For the kinetic description to make sense, ϵ has to be small [94]. This does not require, in general, that the interactions have to be weak and treated perturbatively.

Let us now consider the non-Abelian plasma. The interparticle distance is defined as in the previous case. The main difference with respect to the Abelian case concerns the Debye length, defined as the distance over which the screening effects of the non-Abelian electric fields in the plasma are noticed. It reads

$$r_D^2 = \frac{k_B T}{4\pi \bar{N} g^2 C_2} \; , \tag{6.5}$$

where C_2, defined in Eq. (2.25), is a dimensionful quantity, carrying the same dimensions as the electric charge squared in Eq. (6.3). The coupling constant g is a dimensionless parameter. In the non-Abelian plasma one can also construct the plasma parameter, defined as in Eq. (6.4).

It is interesting to note that there are two natural dimensionless parameters in the non-Abelian plasma: ϵ and g. The condition for the plasma parameter being small translates into

$$\left(\frac{4\pi C_2}{k_B T}\right)^{3/2} \bar{N}^{1/2} g^3 \ll 1 \; , \tag{6.6}$$

which is certainly satisfied for small gauge coupling constant $g \ll 1$. But it can also be fulfilled for a rarefied plasma. Thus, one may have a small plasma parameter *without* having a small gauge coupling constant. This is an interesting observation, since the inequalities $\epsilon \ll 1$ and $g \ll 1$ have different physical meanings. A small gauge coupling constant allows us to treat the non-Abelian interaction perturbatively, while $\epsilon \ll 1$ just means having a collective field description of the physics occurring in the plasma. In principle, these two

situations are different. If we knew how to treat the non-Abelian interactions *exactly*, we could also have a kinetic description of the classical non-Abelian plasmas without requiring $g \ll 1$.

B. Quantum plasmas

Now we consider the quantum non-Abelian plasma, and consider the quantum counterparts of all the above quantities, as derived from quantum field theory. For a quantum plasma at equilibrium the one particle distribution function is

$$\bar{f}_{\mathrm{B}}^{\mathrm{eq}}(p_0) = \frac{1}{\exp\left(\frac{p_0 - \mu}{k_B T}\right) - 1} \ , \tag{6.7a}$$

$$\bar{f}_{\mathrm{F}}^{\mathrm{eq}}(p_0) = \frac{1}{\exp\left(\frac{p_0 \mp \mu}{k_B T}\right) + 1} \ , \tag{6.7b}$$

where the subscripts 'B' and 'F' refer to the Bose-Einstein and Fermi-Dirac statistics, respectively. In the fermionic distribution function the $\mp$ sign refers to partices/antiparticles, respectively.

Note that in the limit of low occupation numbers, one can recover the classical distribution function from the quantum one. This happens for large values of the fugacity. Also we should point out that a chemical potential associated to a specific species of particles can only be introduced if there is a conserved charge associated to them. Since it is impossible to associate a global $U(1)$ symmetry to the gluons, one cannot introduce a gluonic chemical potential.

In the remaining part of this article, we will mainly study the physics of non-Abelian plasmas close to thermal equilibrium, and put $\mu = 0$. If we further neglect the masses of the particles, then the mean density is $\bar{N} \sim (k_B T / \hbar c)^3$. The interparticle distance $\bar{r} \sim \bar{N}^{-1/3}$ becomes of the same order as the de Broglie wavelength, which is why quantum statistical effects cannot be neglected in this case.

The value of the Debye mass is obtained from quantum field theory. It depends on the specific quantum statistics of the particles and their representation of $SU(N)$. From the quantum Debye mass one can deduce the value of the Debye length, which is of order

$$r_D^2 \sim \frac{1}{g^2} \left(\frac{\hbar c}{k_B T}\right)^2 \ . \tag{6.8}$$

It is not difficult to check that the plasma parameter, defined as in Eq. (6.4), becomes proportional to g^3. Thus ϵ is small *if and only if $g \ll 1$*. This is so, because in a quantum field theoretical formulation one does not have the freedom to fix the mean density $\bar{N}$ in

an arbitrary way, as in the classical case (*i.e.*, one cannot introduce a fugacity for gluons). This explains why the kinetic description of a quantum non-Abelian plasma is deeply linked to the small gauge coupling regime of the theory.

VII. HTL EFFECTIVE THEORY

In this section we consider the example of the HTL effective theory. In physical terms, it describes the leading-order chromoelectrical screening effects in a non-Abelian plasmas close to thermal equilibrium [100]. The HTL-resummed gluon propagator no longer has poles on the light cone, and the dispersion relation, which is a complicated function of momenta, yields a screening mass – the Debye mass m_D – for the chromo-electric fields. Also, the HTL polarisation tensor has an imaginary part which is responsible for the absorption and emission of soft gluons by the particles, known as Landau damping. The corresponding effective action for QCD is highly non-local, but can be brought into a local form when written as a transport equation.

Within the conceptual framework laid-out in the previous sections the HTL effective theory can be obtained in a very simple manner [91,92]. A prerequisite for a kinetic description to be viable is a small plasma parameter $\epsilon \ll 1$. We shall ensure this by assuming that the temperature is sufficiently high such that the gauge coupling as a function of temperature obeys

$$g \ll 1 \ . \tag{7.1}$$

We also assume that all particle masses m are small compared to both the temperature and the Debye mass which allows considering them as massless. It is then shown that the HTL effective theory emerges within the simplest non-trivial approximation to the set of transport equations derived in Section IV, which is the *first moment approximation*. Hence, fluctuations will play no part in the effective mean field dynamical equations. Solving the approximate transport equations within the first moment approximation and to leading order in the gauge coupling reproduces the HTL effective theory. In the present section we drop the bar on the mean fields for notational simplicity.

A. Non-Abelian Vlasov equations

We begin with the set of mean field equations (4.5) and neglect the effect of statistical fluctuations entirely, $\delta f \equiv 0$. In that case, Eqs. (4.5) become the non-Abelian Vlasov equations [69]

$$p^\mu D_\mu f = g \, p^\mu \, Q_a F^a_{\mu\nu} \, \partial^\nu_p f \ , \tag{7.2a}$$

$$D_\mu F^{\mu\nu} = J^\nu \ , \tag{7.2b}$$

where the colour current is given by

$$J_a^\mu(x) = g \sum_{\substack{\text{helicities} \\ \text{species}}} \int dP dQ \, Q_a p^\mu f(x, p, Q) \ . \tag{7.2c}$$

We will omit the species and helicity indices on the distribution functions, and in the sequel, we will also omit the above sum, in order to keep the notation as simple as possible. Equation (7.2a) is then solved perturbatively, as it admits a consistent expansion in powers of g. Close to equilibrium, we expand the distribution function as in Eq. (4.17) up to leading order in the coupling constant

$$f(x, p, Q) = f^{\text{eq}}(p_0) + g f^{(1)}(x, p, Q) \ . \tag{7.3}$$

In the strictly classical approach, the relativistic Maxwell distribution Eq. (6.1) at equilibrium is used, which is semi-classically replaced by the corresponding quantum distributions Eqs. (6.7). Here, we consider massless particles with two helicities as internal degrees of freedom.

It is convenient to rewrite the equations in terms of current densities. The momentum measure we use is

$$dP = \frac{d^4 p}{(2\pi)^3} \, 2\Theta(p_0) \, \delta(p^2) \tag{7.4}$$

for massless particles. Consider the current densities

$$J_{a_1 \cdots a_n}^\rho(x, p) = g \, p^\rho \int dQ \, Q_{a_1} \cdots Q_{a_n} f(x, p, Q), \tag{7.5a}$$

$$\mathcal{J}_{a_1 \cdots a_n}^\rho(x, \mathbf{v}) = \int d\tilde{P} \, J_{a_1 \cdots a_n}^\rho(x, p) \ . \tag{7.5b}$$

Here we introduced the vector $v^\mu = (1, \mathbf{v})$, where $\mathbf{v}$ describes the velocity of the particle with $\mathbf{v}^2 = 1$. The measure $d\tilde{P}$ integrates over the radial components. It is related to Eq. (2.22) by $dP = d\tilde{P} d\Omega / 4\pi$, and reads

$$d\tilde{P} = \frac{1}{2\pi^2} \, dp_0 \, d|\mathbf{p}| \, |\mathbf{p}|^2 \, 2\Theta(p_0) \, \delta(p^2) \ . \tag{7.6}$$

The colour current is obtained by performing the remaining angle integration

$$J_a^\mu(x) = \int \tfrac{d\Omega}{4\pi} \mathcal{J}_a^\mu(x, v). \tag{7.7}$$

For massless particles, a simple consequence of these definitions is $v^\mu \mathcal{J}_a^0(x, v) = \mathcal{J}_a^\mu(x, v)$, a relation we will use continously.

We now insert Eq. (7.3) into Eqs. (7.2) and expand in powers of g. The leading order term $p \cdot D f^{\text{eq.}}(p_0)$ vanishes. After multiplying Eq. (7.2a) by $g Q_a p^\rho / p_0$, summing over two helicities, and integrating over $d\tilde{P} dQ$, we obtain for the mean current density at order g

$$v^\mu D_\mu \mathcal{J}^\rho(x, \mathbf{v}) = m_D^2 v^\rho v^\mu F_{0\mu}(x) , \tag{7.8a}$$

$$D_\mu F^{\mu\nu}(x) = J^\nu(x) , \tag{7.8b}$$

with the Debye mass

$$m_D^2 = -\frac{g^2 C_2}{\pi^2} \int_0^\infty dp\, p^2 \frac{df^{\mathrm{eq}}(p)}{dp} . \tag{7.9}$$

In the classical case the Debye mass is given by the inverse of the Debye length Eq. (6.5), as a function of the mean density. In the quark-gluon plasma, with gluons in the adjoint representation, $C_2 = N$, and N_F quarks and N_F antiquarks in the fundamental representation, $C_2 = 1/2$, and all the particles carrying two helicities, the Debye mass reads

$$m_D^2 = \frac{g^2 T^2}{3} \left(N + \frac{N_F}{2} \right) . \tag{7.10}$$

From Eq. (7.8a) we can estimate the typical momentum scale of the mean fields. If the effects of statistical fluctuations are neglected, the typical momentum scales associated to the mean current and the mean field strength are of the order of the Debye mass m_D. Here and in the sequel, we will refer to those scales as soft scales. The momentum scales with momenta $\ll m_D$ will be referred to as ultra-soft from now on.

The Boltzmann equation (7.8a) is consistent with current conservation. This follows easily from the $\rho = 0$ component of Eq. (7.8a), which yields

$$D_\mu J^\mu = \int \frac{d\Omega}{4\pi} D_\mu \mathcal{J}^\mu(x, \mathbf{v}) = m_D^2 F_{0\mu} \int \frac{d\Omega}{4\pi} v^\mu = 0 . \tag{7.11}$$

The last equation vanishes because both the angle average of $\mathbf{v}$ and the component F^{00} vanish.

B. Solution to the transport equation

In a first step we have found the Boltzmann equation to leading order in the Vlasov approximation. In a second step, we are interested in integrating-out the quasi-particle degrees of freedom. This amounts to solving the Boltzmann equation and to express the induced current explicitly as a functional of the soft gauge fields [28]. The solution to Eq. (7.8a) is constructed with the knowledge of the retarded Green's function

$$i v^\mu D_\mu G_{\mathrm{ret}}(x, y; \mathbf{v}) = \delta^{(4)}(x - y) . \tag{7.12}$$

It reads

$$G_{\text{ret}}(x, y; \mathbf{v})_{ab} = -i\theta(x_0 - y_0)\delta^{(3)}\left(\mathbf{x} - \mathbf{y} - \mathbf{v}(x_0 - y_0)\right) U_{ab}(x, y) \ . \tag{7.13}$$

Here, we introduced the parallel transporter $U_{ab}[A]$ which is defined via a path-ordered exponential as

$$U(x, y) = P \exp\left(ig \int_y^x dz^\mu\, A_\mu(z)\right) \ , \tag{7.14a}$$

$$U_{ab}(x, x) = \delta_{ab} \ , \tag{7.14b}$$

and obeys

$$v^\mu D_\mu^x\, U_{ab}(x, y)\big|_{y = x - vt} = 0 \ . \tag{7.14c}$$

Using Eq. (7.13), one finds for the current density

$$\mathcal{J}_a^\mu(x, \mathbf{v}) = -m_D^2\, v^\mu v^\nu \int_0^\infty d\tau\, U_{ab}(x, x - v\tau)F_{\nu 0, b}(x - v\tau) \tag{7.15}$$

and for the HTL effective current

$$J_a^\mu(x) = -m_D^2 \int \frac{d\Omega_\mathbf{v}}{4\pi}\, v^\mu v^\nu \int_0^\infty d\tau\, U_{ab}(x, x - v\tau)F_{\nu 0, b}(x - v\tau) \ . \tag{7.16}$$

The above colour current agrees with the HTL colour current, if one uses the value of the Debye mass for the quark-gluon plasma, Eq. (7.10).

Inserting Eq. (7.16) into Eq. (7.8b) yields an effective theory for the soft gauge fields only. This final step can be seen as integrating-out the particles from the dynamical equations for the soft non-Abelian fields. The soft current $J[A]$ of Eq. (7.16) is related as $J(x) = -\delta\Gamma_{\text{HTL}}[A]/\delta A(x)$ to the generating functional $\Gamma_{\text{HTL}}[A]$ for soft amplitudes. Integrating the HTL current is known to give the HTL effective action $\Gamma_{\text{HTL}}[A]$ explicitly [43–46,62,141,58,59,92]. A simple and elegant expression for the HTL effective action was given in [46], and it reads

$$\Gamma_{\text{HTL}}[A] = \frac{m_D^2}{2} \int \frac{d\Omega_\mathbf{v}}{4\pi} \int d^4x\, d^4y\, \text{Tr}\left(F_{\mu\nu}(x)\langle x|\frac{v^\nu v^\rho}{-(v \cdot D)^2}|y\rangle F_\rho^\mu(y)\right) \ . \tag{7.17}$$

The leading-order effective action for soft gauge fields is then given by adding the HTL effective action to the Yang-Mills one.

C. Soft amplitudes

As a first application, we consider the polarisation tensor for soft gauge fields. The HTL colour current can be expanded in powers of the gauge fields as

$$J_\mu^a[A] = \Pi_{\mu\nu}^{ab} \, A_b^\nu + \frac{1}{2}\Pi_{\mu\nu\rho}^{abc} \, A_b^\nu \, A_c^\rho + \dots \; , \tag{7.18}$$

where the expansion coefficients (or 'soft amplitudes') correspond to 1PI-irreducible amplitudes in thermal equilibrium.

We solve the transport equation in momentum space [147,111] in order to find the explicit expressions of the HTLs. Using the Fourier transform

$$\mathcal{J}_a^\mu(k,v) = \int d^4x \, e^{ik\cdot x} \, \mathcal{J}_a^\mu(x,v) \tag{7.19}$$

we can write the transport equation in momentum space

$$v \cdot k \, \mathcal{J}_a^\mu(k,v) + igf_{abc} \int \frac{d^4q}{(2\pi)^4} \, v \cdot A^b(k-q) \, \mathcal{J}^{\mu c}(q,v)$$
$$= -m_D^2 v^\mu \left[v \cdot k \, A_0^a(k) - k_0 \, v \cdot A^a(k) + igf_{abc} \int \frac{d^4q}{(2\pi)^4} \, v \cdot A^b(k-q) A_0^c(q) \right] \; . \tag{7.20}$$

Now, after assuming that $\mathcal{J}_a^\mu(k,v)$ can be expressed as an infinite power series in the gauge field $A_\mu^a(k)$, Eq. (7.20) can be solved iteratively for each order in the power series. We impose retarded boundary conditions by the prescription $p_0 \to p_0 + i\epsilon$, with $\epsilon \to 0^+$. The first order solution is

$$\mathcal{J}_a^{\mu\,(1)}(k,v) = m_D^2 \, v^\mu \left(k_0 \frac{v \cdot A_a(k)}{v \cdot k} - A_a^0(k) \right) \; . \tag{7.21}$$

Inserting Eq. (7.21) in Eq. (7.20) allows solving for the second order term in the series, which reads

$$\mathcal{J}_a^{\mu\,(2)}(k,v) = -igm_D^2 f_{abc} \int \frac{d^4q}{(2\pi)^4} \, v^\mu q_0 \frac{v \cdot A^b(k-q) \, v \cdot A^c(q)}{(v \cdot k)(v \cdot q)} \; . \tag{7.22}$$

The n-th order term $(n > 2)$ can be expressed as a function of the $(n-1)$-th one as

$$\mathcal{J}_a^{\mu\,(n)}(k,v) = -igf_{abc} \int \frac{d^4q}{(2\pi)^4} \frac{v \cdot A^b(k-q)}{v \cdot k} \mathcal{J}_c^{\mu\,(n-1)}(q,v) \; . \tag{7.23}$$

The complete expression of the induced colour current is thus given by

$$J_a^\mu(x) = \int \frac{d^4k}{(2\pi)^4} \, e^{-ik\cdot x} \sum_{n=1}^\infty J_a^{\mu\,(n)}(k)$$
$$= \int \frac{d\Omega_{\mathbf{v}}}{4\pi} \int \frac{d^4k}{(2\pi)^4} \, e^{-ik\cdot x} \sum_{n=1}^\infty \mathcal{J}_a^{\mu\,(n)}(x,v) \; . \tag{7.24}$$

To obtain the corresponding n-point HTL amplitude, one only needs to perform $n - 1$ functional derivatives of the current with respect to the vector gauge fields. The leading order coefficient is given by the HTL polarisation tensor. This is given by

$$\Pi^{\mu\nu}_{ab}(k) = \delta_{ab} m_D^2 \left(-g^{\mu 0} g^{\nu 0} + k_0 \int \frac{d\Omega_{\mathbf{v}}}{4\pi} \frac{v^\mu v^\nu}{k \cdot v} \right) . \tag{7.25}$$

It obeys $k_\mu \Pi^{\mu\nu}_{ab}(k) = 0$ due to gauge invariance, and agrees with the HTL polarisation tensor of QCD [100,92], if one uses the quantum Debye mass Eq. (7.10).

The polarisation tensor has an imaginary part, due to

$$\frac{1}{k \cdot v + i0^+} = \mathcal{P} \frac{1}{k \cdot v} - i\pi \delta(k \cdot v) . \tag{7.26}$$

The imaginary part corresponds to Landau damping and describes the emission and absorption of soft gluons by the hard particles. It can be expressed as

$$\mathrm{Im}\, \Pi^{\mu\nu}_{ab}(k) = -\delta_{ab} m_D^2 \pi k_0 \int \frac{d\Omega_{\mathbf{v}}}{4\pi} v^\mu v^\nu \, \delta(k \cdot v) . \tag{7.27}$$

Notice the appearance of the δ-function under the angle average. Because of $\mathbf{v}^2 = 1$ it implies that Eq. (7.27) is only non-vanishing for space-like momenta with $|\mathbf{k}| \geq k_0$. It is closely related to fluctuations within the plasma due to the fluctuation-dissipation theorem. We shall come back to this point in Section VIII C.

The polarisation tensor can be projected into their longitudinal (L) and transverse (T) components as

$$\Pi^{00}_{ab}(k_0, \mathbf{k}) = \delta_{ab} \, \Pi_L(k_0, \mathbf{k}) , \tag{7.28a}$$

$$\Pi^{0i}_{ab}(k_0, \mathbf{k}) = \delta_{ab} \, k_0 \frac{k^i}{|\mathbf{k}|^2} \, \Pi_L(k_0, \mathbf{k}) , \tag{7.28b}$$

$$\Pi^{ij}_{ab}(k_0, \mathbf{k}) = \delta_{ab} \left[\left(\delta^{ij} - \frac{k^i k^j}{|\mathbf{k}|^2} \right) \Pi_T(k_0, \mathbf{k}) + \frac{k^i k^j}{|\mathbf{k}|^2} \frac{k_0^2}{|\mathbf{k}|^2} \Pi_L(k_0, \mathbf{k}) \right] , \tag{7.28c}$$

where

$$\Pi_L(k_0, \mathbf{k}) = m_D^2 \left(\frac{k_0}{2|\mathbf{k}|} \left(\ln \left| \frac{k_0 + |\mathbf{k}|}{k_0 - |\mathbf{k}|} \right| - i\pi \, \Theta(|\mathbf{k}|^2 - k_0^2) \right) - 1 \right) , \tag{7.29a}$$

$$\Pi_T(k_0, \mathbf{k}) = -m_D^2 \frac{k_0^2}{2|\mathbf{k}|^2} \left[1 + \frac{1}{2} \left(\frac{|\mathbf{k}|}{k_0} - \frac{k_0}{|\mathbf{k}|} \right) \left(\ln \left| \frac{k_0 + |\mathbf{k}|}{k_0 - |\mathbf{k}|} \right| - i\pi \, \Theta(|\mathbf{k}|^2 - k_0^2) \right) \right] . \tag{7.29b}$$

Similarly, all HTLs, such as $\Pi^{abc}_{\mu\nu\rho}$, are obtained.

The poles of the longitudinal and transverse parts of the gluon propagator give the dispersion laws for the collective excitations in the non-Abelian plasma

$$|\mathbf{k}|^2 - \mathrm{Re}\,\Pi_L(k_0, \mathbf{k})|_{k_0 = \omega_L(\mathbf{k})} = 0 \ , \tag{7.30a}$$

$$k_0^2 - |\mathbf{k}|^2 + \mathrm{Re}\,\Pi_T(k_0, \mathbf{k})|_{k_0 = \omega_T(\mathbf{k})} = 0 \ . \tag{7.30b}$$

The plasma frequency ω_{pl} follows from Eq. (7.30) as

$$\omega_{\mathrm{pl}}^2 = \frac{1}{3} m_D^2 \ . \tag{7.31}$$

For generic external momenta the dispersion relations can only be solved numerically. In turn, if the spatial momenta are much smaller than the plasma frequency $|\mathbf{k}| \ll \omega_{\mathrm{pl}}$, solutions to Eq. (7.30) can be expanded in powers of $|\mathbf{k}|^2/\omega_{\mathrm{pl}}^2$ as

$$\omega_L^2(\mathbf{k}) = \omega_{\mathrm{pl}}^2 \left[1 + \frac{3}{5} \frac{|\mathbf{k}|^2}{\omega_{\mathrm{pl}}^2} + \mathcal{O}(\frac{|\mathbf{k}|^4}{\omega_{\mathrm{pl}}^4}) \right] \ , \tag{7.32a}$$

$$\omega_T^2(\mathbf{k}) = \omega_{\mathrm{pl}}^2 \left[1 + \left(1 + \frac{1}{5} \right) \frac{|\mathbf{k}|^2}{\omega_{\mathrm{pl}}^2} + \mathcal{O}(\frac{|\mathbf{k}|^4}{\omega_{\mathrm{pl}}^4}) \right] \ . \tag{7.32b}$$

D. Energy-momentum tensor

For a second application, we come back to the conservation laws for the energy momentum tensor in the Vlasov approximation. They have been given in Eq. (4.20), and simplify in the present approximation to

$$\partial_\nu \Theta^{\mu\nu}(x) = -F_a^{\mu\nu}(x) J_{\nu a}(x) \ , \tag{7.33a}$$

$$\partial_\nu t^{\mu\nu}(x) = F_a^{\mu\nu}(x) J_{\nu a}(x) = \int \frac{d\Omega}{4\pi} F_a^{\mu\nu}(x)\, v_\nu\, \mathcal{J}_{0,a}(x, \mathbf{v}) \ . \tag{7.33b}$$

In the second line we have inserted the current density $\mathcal{J}(x, \mathbf{v})$ on the right-hand side. Following an observation due to Blaizot and Iancu [30], it is possible to derive an explicit expression for the $t^{\mu 0}$ components of the particle's energy-momentum tensor. Indeed, making use of the Boltzmann equation (7.8a), we can substitute the term $F^{0\nu} v_\nu$ in Eq. (7.33b) to obtain

$$\partial_\mu t^{\mu 0}(x) = m_D^2 \int \frac{d\Omega}{4\pi} [v^\mu D_\mu W(x, \mathbf{v})]^a W_a(x, \mathbf{v}) \ . \tag{7.34}$$

Here, we found it convenient to introduce $W_a(x, \mathbf{v}) \equiv m_D^{-2} \mathcal{J}_{0,a}(x, \mathbf{v})$. Using the identity $\partial_\mu(A^a B_a) = (D_\mu A)^a B_a + A^a (D_\mu B)_a$, we can trivially integrate this equation to find, apart from an integration constant

$$t^{\mu 0}(x) = \frac{1}{2} m_D^2 \int \frac{d\Omega}{4\pi} v^\mu W_a(x, \mathbf{v}) W_a(x, \mathbf{v}) \ . \tag{7.35}$$

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Combining it with the energy-momentum tensor of the mean fields $T^{\mu\nu} = \Theta^{\mu\nu} + t^{\mu\nu}$, we obtain for the total energy

$$T^{00}(x) = \frac{1}{2}\left[\mathbf{E}_a(x)\cdot\mathbf{E}_a(x) + \mathbf{B}_a(x)\cdot\mathbf{B}_a(x) + m_D^2\int\frac{d\Omega}{4\pi}\,W_a(x,\mathbf{v})\,W_a(x,\mathbf{v})\right] \qquad (7.36)$$

where we have introduced the colour electric field $E^i \equiv F^{i0}$ and the colour magnetic field $B^i \equiv \frac{1}{2}\epsilon^{ijk}F^{jk}$. For the energy flux (or Poynting vector), the result reads

$$T^{i0}(x) = [\mathbf{E}_a(x)\times\mathbf{B}_a(x)]^i + \frac{1}{2}m_D^2\int\frac{d\Omega}{4\pi}\,v^i\,W_a(x,\mathbf{v})\,W_a(x,\mathbf{v})\ . \qquad (7.37)$$

The local expression Eq. (7.36) for the energy has proven quite useful for the integrating-out of modes at the Debye scale [34], and for certain lattice implementations [41].

VIII. BEYOND THE VLASOV APPROXIMATION

In this section, the physics related to colour relaxation in a non-Abelian plasma is studied. The HTL transport equation was found to be collisionless, which implies that colour relaxation effects cannot be described in this approximation. The effects of collisions have to be taken into account. The corresponding effective transport theory was first derived in [34–36] (see also the discussion in Section I C).

Within the present approach the physics at length scales larger than the inverse Debye mass can be probed once the gauge field modes with momenta about m_D have been integrated out, that is, incorporated within the quasi-particle distribution function [105,106,108]. The simplest approximation which includes the genuine effects due to source fluctuations of the quasi-particle distribution function is the *polarisation approximation* as discussed in Section IV D. The polarisation approximation requires that the two-particle correlation functions remain small within a Debye volume. This is the case if the plasma parameter is sufficiently small, which is assumed anyhow.

Here, a detailed derivation is given for both the mean field equations at leading order beyond the HTL effective theory and an effective theory for the ultra-soft gauge fields to leading logarithmic order. We proceed in two steps. The first step consists in solving the dynamical equations for the fluctuations as functions of initial fluctuations of the particle distribution function. From this, a collision term and a noise source for the effective mean field Boltzmann equation are obtained. Here, it will be necessary to perform a leading logarithmic approximation, assuming that the gauge coupling is sufficiently small to give

$$g \ll 1 \quad \text{and} \quad \frac{1}{\ln 1/g} \ll 1 \ . \tag{8.1}$$

In a second step the quasi-particle degrees of freedom are integrated out as well. This implies that the mean field transport equation has to be solved. This expresses the induced current explicitly as a functional of the ultra-soft gauge fields only. Ultimately, we shall see that a simple Langevin-type dynamical equation emerges [34]. Here, we extend the reasoning as presented in [105,106].

A. Leading order dynamics

We now allow for small statistical fluctuations $\delta f(x, p, Q)$ around Eq. (7.3), writing

$$f(x, p, Q) = \bar{f}^{\text{eq.}}(p_0) + g\bar{f}^{(1)}(x, p, Q) + \delta f(x, p, Q) \ , \tag{8.2}$$

and rewrite the approximations to (4.5) and (4.7) in terms of current densities and their fluctuations. Note that the fluctuations $\delta f(x, p, Q)$ in the close-to-equilibrium case are

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already of the order of g. This observation is important for the consistent approximation in powers of the gauge coupling. As a consequence, the term $g\bar{f}^{(1)}$ in Eq. (8.2) will now account for the ultra-soft modes for momenta $\ll m_D$. Integrating-out the fluctuations results in an effective theory for the latter.

As before, we obtain the dynamical equation for the mean current density at leading order in g, after multiplying (4.5a) by $g Q_a p^\rho/p_0$, summing over two helicities, and integrating over $d\tilde{P}dQ$. The result is

$$v^\mu \bar{D}_\mu \bar{\mathcal{J}}^\rho + m_D^2 v^\rho v^\mu \bar{F}_{\mu 0} = \langle \eta^\rho \rangle + \langle \xi^\rho \rangle \;, \tag{8.3a}$$

$$\bar{D}_\mu \bar{F}^{\mu\nu} + \langle J_{\text{fluc}}^\nu \rangle = \bar{J}^\nu \;. \tag{8.3b}$$

In a systematic expansion in g, we have to neglect cubic correlator terms as compared to quadratic ones, as they are suppressed explicitly by an additional power in g. Therefore, we find to leading order

$$\eta_a^\rho = g \int d\tilde{P} \, \frac{p^\rho}{p_0} (\bar{D}_\mu a_\nu - \bar{D}_\nu a_\mu)^b \, \partial_p^\nu \, \delta J_{ab}^\mu(x,p) \;, \tag{8.4a}$$

$$\xi_a^\rho = -g f_{abc} v^\mu \, a_\mu^b \, \delta \mathcal{J}^{c,\rho} \;, \tag{8.4b}$$

$$J_{\text{fluc}}^{\rho,a} = g f^{dbc} \Big(\bar{D}_\mu^{ad} a_b^\mu a_c^\rho + \delta^{ad} a_\mu^b \big(\bar{D}^\mu a^\rho - \bar{D}^\rho a^\mu \big)^c \Big) \;. \tag{8.4c}$$

The same philosophy is applied to the dynamical equations for the fluctuations. To leading order in g, the result reads

$$\Big(v^\mu \bar{D}_\mu \, \delta \mathcal{J}^\rho \Big)_a = -m_D^2 v^\rho v^\mu \Big(\bar{D}_\mu a_0 - \bar{D}_0 a_\mu \Big)_a - g f_{abc} v^\mu a_\mu^b \bar{\mathcal{J}}^{c,\rho} \;, \tag{8.5a}$$

$$v^\mu \Big(\partial_\mu \delta_{ac} \delta_{bd} + g \bar{A}_\mu^m \big(f_{amc} \delta_{bd} + f_{bmd} \delta_{ac} \big) \Big) \delta \mathcal{J}_{cd}^\rho = g v^\mu a_\mu^m \big(f_{mac} \delta_{bd} + f_{mbd} \delta_{ac} \big) \bar{\mathcal{J}}_{cd}^\rho \;, \tag{8.5b}$$

$$\Big(\bar{D}^2 a^\mu - \bar{D}^\mu (\bar{D}a) \Big)_a + 2g f_{abc} \bar{F}_b^{\mu\nu} a_{c,\nu} = \delta J_a^\mu \;. \tag{8.5c}$$

Notice that the dynamical equations (8.5a) and (8.5c) can also be obtained from the HTL effective equations Eqs. (7.8) within the present approximation. It suffices to expand Eqs. (7.8) to linear order in $\bar{A}_\mu \to \bar{A}_\mu + a_\mu$ and $\bar{\mathcal{J}} \to \bar{\mathcal{J}} + \delta \mathcal{J}$, which gives Eq. (8.5a) for the dynamics of $\delta \mathcal{J}$ and Eq. (8.5c) for the dynamics of a_μ.

The typical momentum scale associated to the fluctuations can be estimated from Eq. (8.5). We find that it is of the order of the Debye mass $\sim m_D$, that is, of the same order as the mean fields in Eqs. (7.8). This confirms explicitly the discussion made above. The typical momentum scales associated to the mean fields in Eq. (8.3) are therefore $\ll m_D$.

B. Integrating-out the fluctuations

We solve the equations for the fluctuations Eqs. (8.5) with an initial boundary condition for δf, and $a_\mu(t = 0) = 0$. Exact solutions to Eqs. (8.5a) and (8.5b) can be obtained. It is convenient to proceed as follows [94]. We separate the colour current fluctuations into a source part and an induced part,

$$\delta\mathcal{J}^\mu = \delta\mathcal{J}^\mu_{\text{source}} + \delta\mathcal{J}^\mu_{\text{induced}} . \tag{8.6}$$

The induced piece $\delta\mathcal{J}^\mu_{\text{induced}}$ is the part of the current which contains the dependence on a_μ, and thus takes the polarisation effects of the plasma into account. The source piece $\delta\mathcal{J}^\mu_{\text{source}}$ is the part of the current which depends only on the initial condition, given by the solution of the homogeneous equation Eq. (8.7). This splitting will be useful later on since ultimately all the relevant correlators can be expressed in terms of correlators of $\delta\mathcal{J}^\mu_{\text{source}}$.

We start by solving the homogeneous differential equation

$$v^\mu \bar{D}_\mu \, \delta\mathcal{J}^\rho(x, \mathbf{v}) = 0 , \tag{8.7}$$

with the initial condition $\delta\mathcal{J}^\mu_a(t = 0, \mathbf{x}, \mathbf{v})$. It is not difficult to check, by direct inspection, that the solution to the homogeneous problem is

$$\delta\mathcal{J}^\rho_{a,\text{source}}(x, \mathbf{v}) = \bar{U}_{ab}(x, x - vt) \, \delta\mathcal{J}^\rho_b(t = 0, \mathbf{x} - \mathbf{v}t, \mathbf{v}) . \tag{8.8a}$$

The full solution of Eq. (8.5a) is now constructed using the retarded Green's function Eq. (7.13). For $x_0 \equiv t \geq 0$ the induced piece can be expressed as

$$\delta\mathcal{J}^\rho_{a,\text{induced}}(x, \mathbf{v}) = -\int_0^\infty d\tau \, \bar{U}_{ab}(x, x_\tau) \Big[m_D^2 v^\rho v^\mu \left(\bar{D}_\mu a_0 - \bar{D}_0 a_\mu \right)^b (x_\tau)$$
$$+ g f_{bdc} v^\mu a^d_\mu(x_\tau) \bar{\mathcal{J}}^\rho_c(x_\tau, v) \Big] . \tag{8.8b}$$

We have introduced

$$x_\tau \equiv x - v\tau , \quad \text{thus} \quad x_t = (0, \mathbf{x} - \mathbf{v}t) . \tag{8.9}$$

Since $a_\mu(t = 0) = 0$, one can check that the above current obeys the correct initial condition.

Let us remark that the induced solution Eq. (8.8b) can be obtained directly from the explicit solution of the HTL current density Eq. (7.15) by expanding it to linear order in small deviations about the mean gauge fields. The right-hand side of Eq. (7.15) depends only on the gauge fields $\bar{A}(x_\tau)$. Linearising Eq. (7.15) to leading order about the mean gauge field $\bar{A}(x_\tau) + \delta A(x_\tau)$ yields

$$\delta \mathcal{J}^{\mu}_{a,\text{induced}}(x, \mathbf{v}) = -m_D^2 \, v^{\mu} v^{\nu} \int_0^{\infty} d\tau \, \Big[\bar{U}_{ab}(x, x_{\tau}) \, \delta \bar{F}_{\nu 0, b}(x_{\tau})$$
$$+ \delta \bar{U}_{ab}(x, x_{\tau}) \, \bar{F}_{\nu 0, b}(x_{\tau}) \, . \Big] \tag{8.10}$$

In order to evaluate Eq. (8.10) explicitly, we need to know that

$$\delta F^b_{\mu\nu}(x_{\tau}) = \left(\bar{D}_{\mu} \, \delta A_{\nu} - \bar{D}_{\nu} \, \delta A_{\mu} \right)^b (x_{\tau}) \ + O[(\delta A)^2] \ . \tag{8.11}$$

For the second term in Eq. (8.10), we make use of an identity for the variation of the parallel transporter Eq. (7.14a) with respect to the gauge fields $\delta \bar{U}(x, x_{\tau})/\delta A(x_{\sigma})$, namely

$$\delta \bar{U}_{ab}(x, x_{\tau}) = \int_0^{\infty} d\sigma \bar{U}_{ac}(x, x_{\sigma}) \, [g \, f_{cde} \, v^{\mu} \, \delta A_{\mu}(x_{\sigma})] \, \bar{U}_{eb}(x_{\sigma}, x_{\tau}) \ . \tag{8.12}$$

Using Eq. (8.12) and the explicit expression Eq. (7.15), we confirm that Eq. (8.10) coincides with the explicit result for the induced current Eq. (8.8b), if we replace δA by a.

The equation (8.5b) can be solved in a similar way. The solution is

$$\delta \mathcal{J}^{\rho}_{ab}(x, v) = \bar{U}_{am}(x, x_t) \bar{U}_{bn}(x, x_t) \, \delta \mathcal{J}^{\rho}_{mn}(x_t, v)$$
$$- g \int_0^{\infty} d\tau \, \bar{U}_{am}(x, x_{\tau}) \bar{U}_{bn}(x, x_{\tau}) \, (f_{mpc}\delta_{nd} + f_{npd}\delta_{mc}) \, v^{\mu} a^p_{\mu}(x_{\tau}) \bar{\mathcal{J}}^{\rho}_{cd}(x_{\tau}, v) \ . \tag{8.13}$$

Now we seek solutions to Eq. (8.5c) with the colour current of the fluctuation as found above. However, notice that this equation is non-local in a_{μ}, which makes it difficult to find exact solutions. Nevertheless, one can solve the equation in an iterative way, by making a double expansion in both $g\bar{A}$ and $g\bar{\mathcal{J}}$. This is possible since the parallel transporter $\bar{U}$ admits an expansion in $g\bar{A}$, so that the current $\delta \mathcal{J}^{\rho}$ can be expressed as a power series in $g\bar{A}$

$$\delta \mathcal{J}^{\rho} = \delta \mathcal{J}^{\rho(0)} + \delta \mathcal{J}^{\rho(1)} + \delta \mathcal{J}^{\rho(2)} + \cdots \ , \tag{8.14}$$

and thus Eq. (8.5c) can be solved for every order in $g\bar{A}$. To lowest order in $g\bar{A}$, using $\bar{U}_{ab} = \delta_{ab} + \mathcal{O}(g\bar{A})$, Eq. (8.5c) becomes

$$\partial^{\mu} \left[\partial_{\mu} a^{(0)}_{\nu, a} - \partial_{\nu} a^{(0)}_{\mu, a} \right] = \delta J^{(0)}_{\nu, a} \ . \tag{8.15}$$

Using the one-sided Fourier transform [§] [94], and Eq. (8.8), we find

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[§]The one-sided Fourier transform with respect to the time variable is defined as $F(\omega) = \int_0^{\infty} dt \, e^{i\omega t} F(t)$

$$\delta J_{a+}^{\mu\,(0)}(k) = \Pi_{ab}^{\mu\nu}(k) a_{\nu,b}^{(0)}(k)$$

$$-g f_{abc} \int \frac{d\Omega_{\mathbf{v}}}{4\pi} \frac{1}{-i\,k\cdot v} \int \frac{d^4q}{(2\pi)^4} v^\rho a_\rho^{b(0)}(q)\, \bar{\mathcal{J}}^{\mu,c}(k-q,v)$$

$$+ \int \frac{d\Omega_{\mathbf{v}}}{4\pi} \frac{\delta \mathcal{J}_a^\mu(t=0,\mathbf{k},v)}{-i\,k\cdot v}\ , \tag{8.16}$$

where $\Pi_{ab}^{\mu\nu}(k)$ is the polarisation tensor Eq. (7.25). Retarded boundary conditions are assumed above, with the prescription $k_0 \to k_0 + i0^+$.

We solve Eq. (8.15) iteratively in momentum space for a_μ as an infinite power series in $g\bar{\mathcal{J}}$,

$$a_\mu^{(0)} = a_\mu^{(0,0)} + a_\mu^{(0,1)} + a_\mu^{(0,2)} + \dots \tag{8.17}$$

where the second index counts the powers of the background current $g\bar{\mathcal{J}}$.

Notice that in this type of Abelianised approximation, the equation (8.15) has a (perturbative) Abelian gauge symmetry associated to the fluctuation a_μ. This symmetry is only broken by the term proportional to $\bar{\mathcal{J}}$ in the current. It is an exact symmetry for the term $a_\mu^{(0,0)}$ in the above expansion. We will use this perturbative gauge symmetry in order to simplify the computations, and finally check that the results of the approximate collision integrals do not depend on the choice of the fluctuation gauge.

Using the one-sided Fourier transform, we find the following results for the longitudinal fields, in the gauge $\mathbf{k} \cdot \mathbf{a}^{(0,0)} = 0$,

$$a_{0,a+}^{(0,0)}(k) = \frac{1}{\mathbf{k}^2 - \Pi_L} \int \frac{d\Omega_{\mathbf{v}}}{4\pi} \frac{\delta \mathcal{J}_{0,a}(t=0,\mathbf{k},v)}{-i\,k\cdot v}\ , \tag{8.18a}$$

$$a_{0,a+}^{(0,1)}(k) = \frac{-g f_{abc}}{\mathbf{k}^2 - \Pi_L} \int \frac{d\Omega_{\mathbf{v}}}{4\pi} \frac{1}{-i\,k\cdot v} \int \frac{d^4q}{(2\pi)^4} v^\mu a_\mu^{b(0,0)}(q)\, \bar{\mathcal{J}}_0^c(k-q,v)\ , \tag{8.18b}$$

while we find

$$a_{i,a+}^{T(0,0)}(k) = \frac{1}{-k^2 + \Pi_T} \int \frac{d\Omega_{\mathbf{v}}}{4\pi} \frac{\delta \mathcal{J}_{i,a}^T(t=0,\mathbf{k},v)}{-i\,k\cdot v}\ , \tag{8.19a}$$

$$a_{i,a+}^{T(0,1)}(k) = \frac{-g f_{abc}}{-k^2 + \Pi_T} P_{ij}^T(\mathbf{k}) \int \frac{d\Omega_{\mathbf{v}}}{4\pi} \frac{1}{-i\,k\cdot v} \int \frac{d^4q}{(2\pi)^4} v^\mu a_\mu^{b(0,0)}(q)\, \bar{\mathcal{J}}_j^c(k-q,v)\ , \tag{8.19b}$$

for the transverse fields. The functions $\Pi_{L/T}(k)$ are the longitudinal/transverse polarisation tensors of the plasma, $P_{ij}^T(\mathbf{k}) = \delta_{ij} - k_i k_j / \mathbf{k}^2$ the transverse projector, and $a_i^T \equiv P_{ij}^T a_j$.

In the approximation $g \ll 1$, it will be enough to consider the solution of leading (zeroth) order in $g\bar{A}$, and the zeroth and first order in $g\bar{\mathcal{J}}$. The remaining terms are subleading in the leading logarithmic approximation. However, in principle all tools are available to compute the complete perturbative series. If we could solve Eq. (8.5c) exactly, it would not be necessary to use this perturbative expansion.

C. Correlators and Landau damping

With the explicit expressions obtained in Eqs. (8.16), (8.18) and (8.19), we can express all fluctuations in terms of initial conditions $\delta \mathcal{J}_a^\mu(t = 0, \mathbf{x}, v)$ and the mean fields. From Eq. (3.12) one deduces the statistical average over colour current densities $\delta \mathcal{J}$. We expand the momentum δ-function in polar coordinates

$$\delta^{(3)}(\mathbf{p} - \mathbf{p}') = \frac{1}{p^2}\, \delta(p - p')\, \delta^{(2)}(\Omega_\mathbf{v} - \Omega_{\mathbf{v}'})\,, \tag{8.20}$$

where $\Omega_\mathbf{v}$ represents the angular variables associated to the vector $\mathbf{v} = \mathbf{p}/|\mathbf{p}|$. After some simple integrations we arrive at

$$\left\langle \delta \mathcal{J}_\mu^a(t = 0, \mathbf{x}, v)\, \delta \mathcal{J}_\nu^b(t = 0, \mathbf{x}', v') \right\rangle = 2g^2 B_C\, C_2\, \delta^{ab}\, v_\mu v_\nu'\, \delta^{(3)}(\mathbf{x} - \mathbf{x}')\, \delta^{(2)}(\Omega_\mathbf{v} - \Omega_{\mathbf{v}'})$$
$$+\, \tilde{g}_{2,\mu\nu}^{ab}(\mathbf{x}, v; \mathbf{x}', v')\,, \tag{8.21}$$

where $v^\mu = (1, \mathbf{v})$, and

$$B_C = \frac{2}{\pi} \int_0^\infty dp\, p^2\, \bar{f}^{\text{eq}}(p)\,, \tag{8.22}$$

for classical statistics. For a quantum plasma, the value of the constant B_C is obtained from the quantum correlators Eq. (3.14) and Eq. (3.15). For bosonic statistics it reads

$$B_C = \frac{2}{\pi} \int_0^\infty dp\, p^2\, \bar{f}_B^{\text{eq}}(p) \left(1 + \bar{f}_B^{\text{eq}}(p)\right)\,, \tag{8.23}$$

while for fermionic statistics it is

$$B_C = \frac{2}{\pi} \int_0^\infty dp\, p^2\, \bar{f}_F^{\text{eq}}(p) \left(1 - \bar{f}_F^{\text{eq}}(p)\right)\,. \tag{8.24}$$

The function $\tilde{g}_{2,\mu\nu}^{ab}$ is obtained from the two-particle correlation function $\tilde{g}_2$. Notice that we have neglected the piece $g\bar{f}^{(1)}$ above, as this is subleading in an expansion in g. Since we know the dynamical evolution of all fluctuations we can also deduce the dynamical evolution of the correlators of fluctuations, with the initial condition Eq. (3.12). This corresponds to solving Eq. (4.8) in the present approximation.

From the explicit solution Eq. (8.8) and the average Eq. (8.21) we then find, at leading order in g and neglecting the non-local term in Eq. (8.21),

$$\left\langle \delta \mathcal{J}_{\text{source}}^{a,\mu}(x, v)\, \delta \mathcal{J}_{\text{source}}^{b,\nu}(x', v') \right\rangle = 2g^2 B_C\, C_2\, \delta^{(3)}[\mathbf{x} - \mathbf{x}' - \mathbf{v}(t - t')]\, \delta^{(2)}(\Omega_\mathbf{v} - \Omega_{\mathbf{v}'})$$
$$\times\, v^\mu v'^\nu\, \bar{U}^{ac}(x, x - vt)\, \bar{U}^{bc}(x', x' - v't')\,. \tag{8.25}$$

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Expanding the parallel transporter $\bar{U}$, and switching to momentum space we find the spectral density to zeroth order in $g\bar{A}$

$$\left\langle \delta\mathcal{J}_\mu^a \, \delta\mathcal{J}_\nu^b \right\rangle_{k,v,v'}^{\text{source}\,(0)} = 2g^2 B_C \, C_2 \, \delta^{ab} \, v_\mu v'_\nu \, \delta^{(2)}(\Omega_{\mathbf{v}} - \Omega_{\mathbf{v}'}) \, (2\pi)\, \delta(k\cdot v) \; . \tag{8.26}$$

This is the basic correlator reflecting the microscopic fluctuations of the quasi-particle distribution function within the plasma, and it is all that is required to derive the collision integral relevant for colour conductivity.

As a simple and illustrative example let us reconsider the case of Landau damping. As discussed in Section VII C, the HTL polarisation tensor Eq. (7.25) has an imaginary part which describes the absorption of soft gluonic fields by the hard particles. This imaginary part is closely linked to fluctuations in the gauge fields, which can be seen as follows. We compute the correlator of two transverse fields a, and in particular consider only the part which corresponds to the source fluctuations of the particle distribution function. They yield the field-independent part of the correlator. Hence, we compute the self-correlator of Eq. (8.19a), using Eq. (8.26), and arrive at

$$\left\langle a_{i,a}^{T(0,0)}(k)\, a_{j,b}^{T(0,0)}(q) \right\rangle = g^2 B_C \, C_2 \delta_{ab} (2\pi)^4 \delta^{(4)}(k+q)$$
$$\times \frac{P_{ik}^T(\mathbf{k}) P_{jl}^T(\mathbf{k})}{|-k^2 + \Pi_T|^2} \int \frac{d\Omega_{\mathbf{v}}}{4\pi} v_k v_l \, \delta(k\cdot v) \; . \tag{8.27}$$

Comparing Eq. (8.27) with Eq. (7.27), the above correlator can be written as

$$\left\langle a_{i,a}^{T(0,0)}(k)\, a_{j,b}^{T(0,0)}(q) \right\rangle = \frac{4\pi T}{k_0} \frac{\text{Im}\,\Pi_{ij,T}^{ab}(k)}{|-k^2 + \Pi_T|^2} (2\pi)^3 \delta^{(4)}(k+q) \; . \tag{8.28}$$

Here, we have used the relation

$$2g^2 C_2 B_C = 4\pi T m_D^2 \; . \tag{8.29}$$

For a quark-gluon plasma, with gluons in the adjoint representation, $C_2 = N$, and N_F quarks and antiquarks in the fundamental representation, $C_2 = 1/2$, a similar relation can be written, after summing over species of particles. Thus

$$\sum_{\text{species}} C_2 B_C = \frac{2N}{\pi} \int_0^\infty dp p^2 \bar{f}_{\text{B}}^{\text{eq}}(1 + \bar{f}_{\text{B}}^{\text{eq}}) + \frac{2N_F}{\pi} \int_0^\infty dp p^2 \bar{f}_{\text{F}}^{\text{eq}}(1 - \bar{f}_{\text{F}}^{\text{eq}}) \; . \tag{8.30}$$

And the relation Eq. (8.29) now reads

$$2g^2 \sum_{\text{species}} C_2 B_C = 4\pi T m_D^2 \tag{8.31}$$

Equation (8.28) is a form of the fluctuation-dissipation theorem, which links the dissipative process, Landau damping, with (induced) statistical fluctuations of the gauge fields.

D. Domain of validity

The present considerations are based on the polarisation approximation, introduced in Section IV. It amounts to a truncation of the hierarchy of dynamical equations for correlation functions, neglecting two-particle correlations g_2. This has lead to a closed set equations for the mean fields and their fluctuations. It remains to be shown that two-particle correlations g_2 indeed remain small (see Eq. (3.10) and Eq. (3.11)).

Here, we compute two-particle correlators in the static limit, based on the solution for the fluctuation dynamics as found in Section VIII B. We shall find that these correlations are only negligible when the distance between particles is sufficiently large. This correlation length defines the domain of validity of our equations.

For convenience, we consider the colour current density fluctuations $\delta\mathcal{J}$. An analogous reasoning applies for the one-particle distribution function. We split the colour current density fluctuations as in Eq. (8.6) into a source (or free) part, and an induced part. The statistical correlator of the source densities has already been displayed in Eq. (8.25) and Eq. (8.26). We just note that the equal time correlator Eq. (8.26) was deduced from the local term of the initial time correlator of Eq. (3.12) which is the free term. To obtain Eq. (8.26), one has to multiply the local term in Eq. (3.12) by colour charges, and integrate over the charges and the modulus of the momentum. The time evolution of δJ^{sou} is given by the free dynamical equations. The effect of interactions in the plasma are responsible for two-particle correlations $\tilde{g}_2$ in Eq. (3.12). In our approach, this corresponds to the part of the correlators which can be computed from δJ^{ind}. More specifically, we have

$$\left\langle \delta\mathcal{J}_0^a \, \delta\mathcal{J}_0^b \right\rangle = \left\langle \delta\mathcal{J}_0^a \, \delta\mathcal{J}_0^b \right\rangle^{\text{sou}} + \left\langle \delta\mathcal{J}_0^{a,\text{ind}} \, \delta\mathcal{J}_0^{b,\text{sou}} \right\rangle + \left\langle \delta\mathcal{J}_0^{a,\text{sou}} \, \delta\mathcal{J}_0^{b,\text{ind}} \right\rangle + \left\langle \delta\mathcal{J}_0^{a,\text{ind}} \, \delta\mathcal{J}_0^{b,,\text{ind}} \right\rangle .$$

$$(8.32)$$

We identify the two-particle correlation function with the correlators arising from δJ^{ind}. This part of the current takes into account the interactions. Since, ultimately, δJ^{ind} is expressed in terms of δJ^{sou} and the mean fields, we have a general recipe to compute these correlation functions. Here, we compute the above correlators using the results given in Section VIII B. From Eq. (8.26), we have

$$\left\langle \delta J_0^a \, \delta J_0^b \right\rangle_k^{\text{sou},(0,0)} = \int \frac{d\Omega_{\mathbf{v}}}{4\pi} \int \frac{d\Omega_{\mathbf{v}}'}{4\pi} \left\langle \delta\mathcal{J}_0^a \, \delta\mathcal{J}_0^b \right\rangle_{k,v,v'}^{\text{source}\,(0,0)}$$

$$= g^2 B_C \, C_2 \, \delta^{ab} \int \frac{d\Omega_{\mathbf{v}}}{4\pi} \delta(k \cdot v) . \qquad (8.33)$$

To lowest order, we find

$$\left\langle \delta J_0^{a,\text{sou}} \, \delta J_0^{b,\text{ind}} \right\rangle_k^{(0,0)} = 2\pi T m_D^2 \, \delta^{ab} \frac{\Pi_L(k)}{\mathbf{k}^2 - \Pi_L(k)} \int \frac{d\Omega_{\mathbf{v}}}{4\pi} \delta(k \cdot v) , \qquad (8.34a)$$

$$\left\langle \delta J_0^{a,\text{ind}} \, \delta J_0^{b,\text{ind}} \right\rangle_k^{(0,0)} = 2\pi T m_D^2 \, \delta^{ab} \left(\frac{\Pi_L(k)}{\mathbf{k}^2 - \Pi_L(k)} \right)^2 \int \frac{d\Omega_{\mathbf{v}}}{4\pi} \, \delta(k \cdot v) \ . \tag{8.34b}$$

Here, we have used Eq. (8.29). Collecting terms and taking the static limit $k_0 = 0$, we obtain

$$\left\langle \delta J_0^a \, \delta J_0^b \right\rangle_{\mathbf{k}}^{(0,0)} = 2\pi T m_D^2 \, \delta^{ab} \left(1 - 2\frac{m_D^2}{\mathbf{k}^2 + m_D^2} + \frac{m_D^4}{(\mathbf{k}^2 + m_D^2)^2} \right) \ . \tag{8.35}$$

The non-local pieces represent the effect of correlations in the system. Taking the inverse Fourier transform, we find

$$\left\langle \delta J_0^a \, \delta J_0^b \right\rangle_{\mathbf{r}}^{(0,0)} = 2\pi T m_D^2 \, \delta^{ab} \left(\delta^{(3)}(\mathbf{r}) - \frac{m_D^2}{2\pi r} e^{-r m_D} + \frac{m_D^3}{8\pi} e^{-r m_D} \right) \ . \tag{8.36}$$

In the static limit, the two-particle correlations are exponentially suppressed at distances $r \gg 1/m_D$. This is the domain of validity of the polarisation approximation. Using the expansion introduced to compute the correlators, one could equally compute the terms $\langle \delta J_0^a \, \delta J_0^b \rangle^{(n,m)}$. However, those terms are suppressed in the weak coupling expansion. Notice also that infrared (IR) problems in the computation of the correlation functions do show up, due to the unscreened magnetic modes (see below). These IR problems provide an additional limitation for the domain of validity of the transport equations. The IR problems are, supposedly, cured by the non-perturbative appearance of a magnetic mass at order $g m_D$.

E. Collision integrals

We are now ready to compute at leading order in g the collision integrals appearing on the right-hand side of Eq. (8.3a). We shall combine the expansions introduced earlier to expand the collision integrals in powers of $\bar{\mathcal{J}}$, while retaining only the zeroth order in $g\bar{A}$,

$$\langle \xi \rangle = \langle \xi^{(0)} \rangle + \langle \xi^{(1)} \rangle + \langle \xi^{(2)} \rangle + \dots \ , \tag{8.37}$$

and similarly for $\langle \eta \rangle$ and $\langle J_{\text{fluc}} \rangle$. We find that the induced current $\langle J_{\text{fluc}}^{(0)} \rangle$ vanishes, as do the fluctuation integrals $\langle \eta^{(0)} \rangle$ and $\langle \xi^{(0)} \rangle$. The vanishing of $\langle J_{\text{fluc}}^{(0)} \rangle$ is deduced trivially from the fact that $\langle a_a^{(0,0)} a_b^{(0,0)} \rangle \sim \delta_{ab}$, while this correlator always appears contracted with the antisymmetric constants f_{abc} in J_{fluc}. To check that $\langle \eta^{(0)} \rangle = 0$, one needs the statistical correlator $\langle \delta J_a^\mu \delta J_{ab}^\rho \rangle$, which is proportional to $\sum_a d_{aab} = 0$ for $SU(N)$. The vanishing of $\langle \eta^{(0)} \rangle$ is consistent with the fact that in the Abelian limit the counterpart of $\langle \eta \rangle$ vanishes at equilibrium [94]. Finally, $\langle \xi^{(0)} \rangle = 0$ due to a contraction of f_{abc} with a correlator symmetric in the colour indices.

In the same spirit we evaluate the terms in the collision integrals containing one $\bar{\mathcal{J}}$ field and no background gauge $\bar{A}$ fields. Consider

$$
\begin{aligned}
\left\langle \xi_{\rho,a}^{(1)}(x,\mathbf{v}) \right\rangle &\equiv \langle \xi_{\rho,a}(x,\mathbf{v}) \rangle |_{\bar{A}=0,\text{ linear in } \bar{J}} \\
&= g f_{abc} v^{\mu} \Big\{ - \left\langle a_{\mu,b}^{(0,1)}(x)\, \delta\mathcal{J}_{\rho,c}^{(0)}(x,\mathbf{v}) \right\rangle \\
&\qquad\quad + g f_{cde} v^{\nu} \int_0^{\infty} d\tau\, \bar{\mathcal{J}}_{\rho,e}(x_{\tau},\mathbf{v}) \left\langle a_{\mu,b}^{(0,0)}(x)\, a_{\nu,d}^{(0,0)}(x_{\tau}) \right\rangle \Big\}.
\end{aligned}
\tag{8.38}
$$

Using the values for a_{μ} and $\delta\mathcal{J}^{(0)}$ as found earlier, we obtain in momentum space** (see Appendix B for a detailed computation of the correlators)

$$
\left\langle \xi_{\rho,a}^{(1)}(k,\mathbf{v}) \right\rangle \Big|_{\text{LLO}} = -\frac{g^4 C_2 N B_C}{4\pi} v_{\rho} \int \frac{d\Omega_{\mathbf{v}'}}{4\pi} C_{\text{LLO}}(\mathbf{v},\mathbf{v}') \left[\bar{\mathcal{J}}_a^0(k,\mathbf{v}) - \bar{\mathcal{J}}_a^0(k,\mathbf{v}') \right],
\tag{8.39}
$$

where

$$
C(\mathbf{v},\mathbf{v}') = \int \frac{d^4 q}{(2\pi)^4} \left| \frac{v_i P_{ij}^T(q) v_j'}{-q^2 + \Pi_T} \right|^2 (2\pi)\delta(q\cdot v)(2\pi)\delta(q\cdot v')
\tag{8.40}
$$

still has to be evaluated within LLO.

To arrive at the above expression we have used the $SU(N)$ relation $f_{abc}f_{abd} = N\delta_{cd}$. Within the momentum integral, we have neglected in the momenta of the mean fields, k, in front of the momenta of the fluctuations, q. As we discussed above, the momenta associated to the background fields are much smaller than those associated to the fluctuations which justifies this approximation to leading order. This is precisely what makes the collision integral, which in principle contains a convolution over momenta, local in k-space (resp. x-space). The only remaining non-locality stems from the angle convolution of Eq. (8.40). Notice that we have only given the part arising from the transverse fields a, as the one associated to the longitudinal modes is subleading. This is easy to see once one realises that the above integral is logarithmically divergent in the infrared region, while the longitudinal contribution is finite. At this point, we can also note that the collision integral computed this way is independent of the perturbative Abelian gauge used to solve equation (8.15). This is so because the collision integral computed this way can always be expressed in terms of the imaginary parts of the polarisation tensors (7.27) in the plasma, which are known to be gauge-independent.

In any case, the transverse polarisation tensor Π_T vanishes at $q_0 = 0$, and the dynamical screening is not enough to make Eq. (8.40) finite. An IR cutoff must be introduced by

---

**Note the typo in Eq. (7.39) of [106] where a factor $1/4\pi$ is missing.

hand in order to evaluate the integral. With a cutoff of order $\Lambda \approx g m_D$ we thus find at logarithmic accuracy

$$C_{\mathrm{LLO}}(\mathbf{v}, \mathbf{v}') = \frac{2}{\pi^2 m_D^2} \ln\left(1/g\right) \frac{(\mathbf{v} \cdot \mathbf{v}')^2}{\sqrt{1 - (\mathbf{v} \cdot \mathbf{v}')^2}} \ . \tag{8.41}$$

The logarithmic dependence on the gauge coupling comes from the integral

$$\int_{\Lambda}^{m_D} \frac{dq}{q} = \ln \frac{m_D}{\Lambda} \ . \tag{8.42}$$

The natural cut-off, as we shall see below, is rather given by the hard gluon damping rate. At logarithmic accuracy, their difference has no effect.

Using also the relation Eq. (8.29) we finally arrive at the collision integral to leading logarithmic accuracy,

$$\left\langle \xi_{\rho,a}^{(1)}(x, v) \right\rangle \Big|_{\mathrm{LLO}} = -\frac{g^2}{4\pi} NT \ln\left(1/g\right) v_\rho \int \frac{d\Omega_{\mathbf{v}'}}{4\pi} I(\mathbf{v}, \mathbf{v}') \bar{\mathcal{J}}_a^0(x, \mathbf{v}'), \tag{8.43a}$$

$$I(\mathbf{v}, \mathbf{v}') = \delta^{(2)}(\mathbf{v} - \mathbf{v}') - K(\mathbf{v}, \mathbf{v}') \tag{8.43b}$$

$$K(\mathbf{v}, \mathbf{v}') = \frac{4}{\pi} \frac{(\mathbf{v} \cdot \mathbf{v}')^2}{\sqrt{1 - (\mathbf{v} \cdot \mathbf{v}')^2}} \ , \tag{8.43c}$$

where we have introduced $\delta^{(2)}(\mathbf{v} - \mathbf{v}') \equiv 4\pi \delta^{(2)}(\Omega_{\mathbf{v}} - \Omega_{\mathbf{v}}')$, $\int \frac{d\Omega_{\mathbf{v}}}{4\pi} \delta^{(2)}(\mathbf{v} - \mathbf{v}') = 1$. The collision integral has first been obtained in [34], and subsequently in [9,105,106,143,31].

We can verify explicitly that the collision integral to leading logarithmic accuracy is consistent with gauge invariance. This should be so, as the approximations employed have been shown in Section V C on general grounds to be consistent with gauge invariance. Evaluating the correlator in Eq. (4.22) to leading logarithmic accuracy yields

$$g f_{abc} \left\langle a_\mu^b(x) \delta J_c^\mu(x) \right\rangle \Big|_{\mathrm{LLO}} = -\frac{g^2}{4\pi} NT \ln\left(1/g\right) \int \frac{d\Omega_{\mathbf{v}}}{4\pi} \frac{d\Omega_{\mathbf{v}'}}{4\pi} I(\mathbf{v}, \mathbf{v}') \bar{\mathcal{J}}_a^0(x, \mathbf{v}') \ , \tag{8.44}$$

which vanishes, because

$$\int \frac{d\Omega_{\mathbf{v}}}{4\pi} I(\mathbf{v}, \mathbf{v}') = 0 \ . \tag{8.45}$$

We thus establish that $\bar{D}_\mu \bar{J}^\mu = 0$, in accordance with Eq. (8.3b) in the present approximation.

F. Stochastic noise

The collision integral obtained above describes a dissipative process in the plasma. In principle it could trigger the system to abandon equilibrium [95]. Whenever dissipative processes are encountered, it is important to also identify the stochastic source related to it. This is the essence of the fluctuation-dissipation theorem [97,95]. Phenomenologically, this is well known, and sometimes used the other way around: imposing the fluctuation-dissipation theorem allows one to identify a source for stochastic noise with the strength of its self-correlator fixed by the dissipative processes.

In the present formalism, it is possible to identify directly the source for stochastic noise which prevents the system from abandoning equilibrium as discussed in section IV C. The relevant noise term is given by the field-independent, that is, the *source* fluctuations in $\xi^{(0)}$,

$$\xi_a^{\rho(0)}(x, \mathbf{v}) \equiv -g f_{abc} v^\mu a_\mu^b(x) \delta \mathcal{J}^{\rho,c}(x, \mathbf{v}) \Big|_{\bar{A}=0, \bar{J}=0} . \tag{8.46}$$

While its average vanishes, $\langle \xi^{(0)}(x, \mathbf{v}) \rangle = 0$, its correlator

$$\left\langle \xi_a^{\rho(0)}(x, \mathbf{v}) \, \xi_b^{\sigma(0)}(y, \mathbf{v}') \right\rangle = g^2 f_{apc} f_{bde} v^\mu v'^\nu \left\langle a_\mu^p(x) \, \delta \mathcal{J}_{\text{source}}^{\rho,c}(x, \mathbf{v}) \, a_\nu^d(y) \, \delta \mathcal{J}_{\text{source}}^{\sigma,e}(y, \mathbf{v}') \right\rangle^{(0)} \tag{8.47}$$

does not. In order to evaluate this correlator we switch to Fourier space. Within the second moment approximation we expand the correlator $\langle \delta f \delta f \delta f \delta f \rangle$ into products of second order correlators $\langle \delta f \delta f \rangle \langle \delta f \delta f \rangle$ and find

$$\left\langle \xi_a^{\rho(0)}(k, \mathbf{v}) \, \xi_b^{\sigma(0)}(p, \mathbf{v}') \right\rangle = g^2 f_{apc} f_{bde} v^\mu v'^\nu \int \frac{d^4 q}{(2\pi)^4} \int \frac{d^4 r}{(2\pi)^4}$$

$$\times \Big\{ \left\langle a_{\mu p}^{(0,0)}(q) \, a_{\nu d}^{(0,0)}(r) \right\rangle \left\langle \delta \mathcal{J}_{\text{source}}^{(0)\rho,c}(k - q, \mathbf{v}) \, \delta \mathcal{J}_{\text{source}}^{(0)\sigma,e}(p - r, \mathbf{v}') \right\rangle$$

$$+ \left\langle a_{\mu p}^{(0,0)}(q) \, \delta \mathcal{J}_{\text{source}}^{(0)\sigma,e}(p - r, \mathbf{v}') \right\rangle \left\langle \delta \mathcal{J}_{\text{source}}^{(0)\rho,c}(k - q, \mathbf{v}) \, a_{\nu d}^{(0,0)}(r) \right\rangle \Big\} \tag{8.48}$$

In the leading logarithmic approximation we retain only the contributions from the transverse modes. Evaluating the correlators leads to

$$\left\langle \xi_{(0)}^{\mu,a}(x, \mathbf{v}) \, \xi_{(0)}^{\nu,b}(y, \mathbf{v}') \right\rangle = \frac{g^6 N C_2^2 B_C^2}{(2\pi)^3 m_D^2} \ln (1/g) \, v^\mu v'^\nu \, I(\mathbf{v}, \mathbf{v}') \, \delta^{ab} \, \delta^{(4)}(x - y) . \tag{8.49}$$

After averaging over the angles of $\mathbf{v}$ and $\mathbf{v}'$, and using the relation Eq. (8.29), the correlator becomes

$$\left\langle \xi_{(0)}^{i,a}(x) \, \xi_{(0)}^{j,b}(y) \right\rangle = 2T \frac{m_D^2}{3} \frac{g^2}{4\pi} N T \ln (1/g) \, \delta^{ab} \, \delta^{ij} \, \delta^{(4)}(x - y) . \tag{8.50}$$

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In particular, all correlators $\langle \xi_{(0)}^0(x)\xi_{(0)}^\mu(y) \rangle$ vanish. Eq. (8.50) identifies $\xi_{(0)}^i(x)$ as a source of white noise. The noise term has been derived in [34], and subsequently in [9,105,106].

The presence of this noise term does not interfere with the covariant current conservation confirmed at the end of the previous section. This can be seen as follows. The noise term enters Eq. (8.44) as the angle average over the 0-component of $\xi_{(0)}^\mu(x, \mathbf{v})$. As we have established above, the logarithmically enhanced contribution from the noise source stems from its correlator Eq. (8.49). Averaging the temporal component of Eq. (8.49) over the angles of $\mathbf{v}$, and using Eq. (8.45), it follows that

$$\left\langle \xi_{(0)}^{0,a}(x, \mathbf{v}') \int \frac{d\Omega_\mathbf{v}}{4\pi} \, \xi_{(0)}^{0,b}(y, \mathbf{v}) \right\rangle = 0 \ . \tag{8.51}$$

We thus conclude, that the temporal component of the noise, $\xi_{(0)}^0(x, \mathbf{v})$, has no preferred $\mathbf{v}$-direction, which implies that $\int \frac{d\Omega}{4\pi}\xi_{(0)}^0(x, \mathbf{v}) = 0$ in the leading logarithmic approximation. Thus, the mean current conservation is not affected by the noise term.

G. Ultrasoft amplitudes

After integrating-out the statistical fluctuations to leading logarithmic order, we end up with the following set of mean field equations [34] (from now on, we drop the bar to denote the mean fields),

$$v^\mu D_\mu \mathcal{J}^\rho(x, \mathbf{v}) = -m_D^2 v^\rho v^\mu F_{\mu 0}(x) - \gamma \, v^\rho \int \frac{d\Omega_{\mathbf{v}'}}{4\pi} \, I(\mathbf{v}, \mathbf{v}') \mathcal{J}^0(x, \mathbf{v}') + \zeta^\rho(x, \mathbf{v}) \,, \tag{8.52a}$$

$$D_\mu F^{\mu\nu} = J^\nu \equiv \int \frac{d\Omega_\mathbf{v}}{4\pi} \, \mathcal{J}^\nu(x, \mathbf{v}) \,. \tag{8.52b}$$

Here, we denote by $\zeta^\rho(x, \mathbf{v})$ the stochastic noise term identified in the preceding section, its correlator given by Eq. (8.49). We also introduced

$$\gamma = \frac{g^2}{4\pi}NT \ln\left(\frac{1}{g}\right) \,, \tag{8.53}$$

which is identified as (twice) the damping rate for the ultra-soft currents [123]. We refer to Eq. (8.52a) as a Boltzmann-Langevin equation as it accounts for quasi-particle interactions via a collision integral as well as for the stochastic character of the underlying fluctuations in the distribution function.

In Eq. (8.52), both the quasi-particle degrees of freedom and the ultra-soft gauge modes are present. In order to integrate-out the quasi-particle degrees of freedom it is necessary to solve Eq. (8.52a) explicitly to obtain the current as a functional of the ultra-soft gauge

fields $J[A]$, from which a generating functional $J(x) = -\delta\Gamma[A]/\delta A(x)$ could in principle be deduced in full analogy to the HTL case. The ultra-soft amplitudes of the plasma can be deduced from $J[A]$ itself [35,32], as done in Eq. (7.18) for the soft amplitudes,

$$J_\mu^a[A] = \Pi_{\mu\nu}^{ab} A_b^\nu + \frac{1}{2}\Pi_{\mu\nu\rho}^{abc} A_b^\nu A_c^\rho + \dots \tag{8.54}$$

With this in mind, we consider again the Boltzmann-Langevin equation (8.52a) for the quasi-particle distribution function. It has three distinct scale parameters: the temperature T, the Debye mass m_D, and the damping term γ. In the leading logarithmic approximation, these scales are well separated,

$$g^2 T \ll \gamma \ll m_D \ll T \;, \tag{8.55}$$

and at least logarithmically larger than the non-perturbative scale of the magnetic mass. This is why Eq. (8.52a) is dominated by different terms, depending on the momentum range considered. For hard momenta, Eq. (8.52a) is only dominated by the left-hand side, reducing it to the (trivial) current of hard particles moving on world lines. For momenta about the Debye mass, the term proportional to m_D^2 becomes equally important, while the noise term and the collision integral remain suppressed by γ/m_D. The resulting current is then given by Eq. (7.16), the HTL current. Momentum modes below the Debye mass are affected by the damping term of the collision integral. Close to the scale of the Debye mass, the higher order corrections to Eq. (7.16) are obtained as an expansion in $\gamma/v \cdot D$. We write

$$\mathcal{J}^\mu(x,\mathbf{v}) = \sum_{n=0}^\infty \mathcal{J}_{(n)}^\mu(x,\mathbf{v}) \;, \tag{8.56}$$

where the current densities $\mathcal{J}_{(n)}^\mu(x,\mathbf{v})$ obey the differential equations

$$v^\mu D_\mu \mathcal{J}_{(0)}^\nu(x,\mathbf{v}) = -m_D^2 v^\nu v^\mu F_{\mu 0}(x) + \zeta^\nu(x,\mathbf{v}) \;, \tag{8.57a}$$

$$v^\mu D_\mu \mathcal{J}_{(n)}^\nu(x,\mathbf{v}) = -\gamma v^\nu \int \frac{d\Omega_{\mathbf{v}'}}{4\pi} I(\mathbf{v},\mathbf{v}')\mathcal{J}_{(n-1)}^0(x,\mathbf{v}') \;. \tag{8.57b}$$

Apart from the noise term, the leading order term in this expansion, $\mathcal{J}_{(0)}^\nu(x,\mathbf{v})$, coincides with the HTL current Eq. (7.16). All higher order terms $\mathcal{J}_{(n)}^\mu(x,\mathbf{v})$ are smaller by powers of $\sim (\gamma/v \cdot D)^n$, and recursively given by

$$\mathcal{J}_{(0)}^\nu(x,\mathbf{v}) = \int_0^\infty d\tau U(x, x-v\tau) \left\{-m_D^2 v^\nu v^\mu F_{\mu 0}(x-v\tau) + \zeta^\nu(x-v\tau,\mathbf{v})\right\} \;, \tag{8.58a}$$

$$\mathcal{J}_{(n)}^\nu(x,\mathbf{v}) = -\gamma \int_0^\infty d\tau U(x, x-v\tau) v^\nu \int \frac{d\Omega_{\mathbf{v}'}}{4\pi} \mathcal{I}(\mathbf{v},\mathbf{v}') \mathcal{J}_{(n-1)}^0(x,\mathbf{v}') \;. \tag{8.58b}$$

This recursive expansion is consistent with covariant current conservation. For every partial sum up to order n, we have

$$D_\mu \left(J^\mu_{(0)} + J^\mu_{(1)} + \ldots + J^\mu_{(n)} \right) = 0 \; . \tag{8.59}$$

This expansion has been considered in [106]. It describes correctly how the presence of the collision integral modifies the ultra-soft current. From Eq. (8.58), all ultrasoft amplitudes of the gauge fields at equilibrium can be deduced. The noise source does not contribute when amplitudes like the coefficients in the expansion Eq. (8.54) are evaluated at equilibrium. Some explicit results for this case have been given recently [35,32].

The expansion Eq. (8.58) has a limited domain of validity because the effective expansion parameter grows large for both small frequencies k_0 and small momenta $\mathbf{k}$. This implies that the overdamped regime where $v \cdot D \ll \gamma$ cannot be reached. Alternatively, one can separate the local from the non-local part of the collision integral to perform an expansion in the latter only. The effective expansion parameter is then $\gamma/(v \cdot D + \gamma)$, which has a better infrared behaviour. It is expected that the expansion is much better for the spatial than for the temporal component of $\mathcal{J}^\rho(x, \mathbf{v})$. This is so, because the term proportional to the non-local part $K(\mathbf{v}, \mathbf{v}')$ of the collision integral in Eq. (8.52a) gives no contribution to the dynamical equations of the spatial component J^i after angle averaging Eq. (8.52a) over the directions of $\mathbf{v}$. However, for the dynamical equation of the temporal component, this term precisely cancels the local damping term, which is of course a direct consequence of current conservation.

In this light, we decompose the current as in Eq. (8.56), but expanding effectively in $\gamma/(v \cdot D + \gamma)$. We find the differential equations

$$(v^\mu D_\mu + \gamma)\mathcal{J}^\rho_{(0)}(x, \mathbf{v}) = -m^2_D v^\rho v^\mu F_{\mu 0}(x) + \zeta^\rho(x, \mathbf{v}) \; , \tag{8.60a}$$

$$(v^\mu D_\mu + \gamma)\mathcal{J}^\rho_{(n)}(x, \mathbf{v}) = \gamma \, v^\rho \int \frac{d\Omega_{\mathbf{v}'}}{4\pi} \, K(\mathbf{v}, \mathbf{v}')\mathcal{J}^0_{(n-1)}(x, \mathbf{v}'). \tag{8.60b}$$

The retarded Green's function G_{ret} obeys

$$i \left(v^\mu D_\mu + \gamma \right) G_{\text{ret}}(x, y; \mathbf{v}) = \delta^{(4)}(x - y) \; , \tag{8.61}$$

and reads, for $t = x_0 - y_0$,

$$G_{\text{ret}}(x, y; \mathbf{v})_{ab} = -i\theta(t)\delta^{(3)} \left(\mathbf{x} - \mathbf{y} - \mathbf{v}t \right) \exp(-\gamma t) \, U_{ab}(x, y) \; . \tag{8.62}$$

The iterative solution to the Boltzmann-Langevin equation is

$$\mathcal{J}^\rho_{(0)}(x, \mathbf{v}) = \int_0^\infty d\tau \exp(-\gamma\tau) \, U(x, x_\tau) \left\{ -m^2_D \, v^\rho v^j \, F_{j0}(x_\tau) + \zeta^\rho(x_\tau, \mathbf{v}) \right\} \; , \tag{8.63a}$$

$$\mathcal{J}^\rho_{(n)}(x, \mathbf{v}) = \gamma \int_0^\infty d\tau \exp(-\gamma\tau) \, U(x, x_\tau) \, v^\rho \int \frac{d\Omega_{\mathbf{v}'}}{4\pi} \, K(\mathbf{v}, \mathbf{v}') \, \mathcal{J}^0_{(n-1)}(x_\tau, \mathbf{v}') \; . \tag{8.63b}$$

This expansion in consistent with current conservation, if the angle average of $\mathcal{J}_{(n)}^0$ vanishes for some n. This follows from taking the temporal component of Eqs. (8.60) and averaging the equation over $\mathbf{v}$, to find

$$D_0 J_{(0)}^0 + D_i J_{(0)}^i = -\gamma J_{(0)}^0 \; , \tag{8.64a}$$

$$D_0 J_{(n)}^0 + D_i J_{(n)}^i = \gamma J_{(n-1)}^0 - \gamma J_{(n)}^0 \tag{8.64b}$$

for the individual contributions, and

$$D_\mu \left(J_{(0)}^\mu + J_{(1)}^\mu + \ldots + J_{(n)}^\mu \right) + \gamma J_{(n)}^0 = 0 \tag{8.64c}$$

for their sum, which is consistent if $\gamma J_{(n)}^0$ is vanishing for some n.

To leading order, the ultra-soft colour current $\mathcal{J}_{(0)}^i(x, \mathbf{v})$ in Eq. (8.63a) has the same functional dependence on the field strength and on the parallel transporter as the soft colour current Eq. (7.16). There is, however, an additional damping factor $\exp(-\gamma\tau)$ in the integrand.

H. Langevin dynamics

Finally, we consider the overdamped regime (or quasi-local limit) of the above equations. This is the regime where $k_0 \ll |\mathbf{k}| \ll \gamma$. Consider the mean field currents Eq. (8.63). The terms contributing to these currents are exponentially suppressed for times τ much larger than the characteristic time scale $1/\gamma$. On the other hand, the fields occurring in the integrand typically vary very slowly, that is on time scales $\ll 1/m_{\rm D}$. Thus, in the quasi-local limit we can perform the approximations

$$U_{ab}(x, x - v\tau) \approx U_{ab}(x, x) = \delta_{ab} \; , \tag{8.65a}$$

$$F_{j0}(x - v\tau) \approx F_{j0}(x) \; . \tag{8.65b}$$

In this case the remaining integration can be performed. The solution for the spatial current $J^i(x)$ stems entirely from the leading order term Eq. (8.63a). All higher order corrections vanish, because they are proportional to

$$\int \frac{d\Omega_{\mathbf{v}}}{4\pi} \, \mathbf{v} \, K(\mathbf{v}, \mathbf{v}') \tag{8.66}$$

which vanishes. For the complete set of gauge field equations in the quasi-local limit we also need to know $J^0(x)$. The iterative solution Eq. (8.63) gives $J_{(n)}^0(x) = 0$ to any finite order. Therefore, we use instead the unapproximated dynamical equation for $\mathcal{J}^0(x, \mathbf{v})$, which yields, averaged over the directions of $\mathbf{v}$, current conservation. In combination with

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the solution Eq. (8.68a) for the spatial current, the Boltzmann-Langevin equation (8.52) becomes

$$D_\mu F^{\mu i} = \sigma E^i + \nu^i \ , \tag{8.67a}$$

$$D_i E^i = J^0 \ , \tag{8.67b}$$

$$D_0 J^0 = -\sigma J^0 - D_i \nu^i \ . \tag{8.67c}$$

In the limit, where the temporal derivative term $D_0 F^{0i}$ can be neglected, one finally obtains for the spatial current from Eq. (8.63)

$$J_a^i = \sigma E_a^i + \nu_a^i \ , \tag{8.68a}$$

$$\sigma = \frac{4\pi m_D^2}{3 N g^2 T \ln(1/g)} \ , \tag{8.68b}$$

where σ denotes the colour conductivity of the plasma. The noise term reads

$$\nu(x) = \frac{1}{\gamma} \int \frac{d\Omega_{\mathbf{v}}}{4\pi} \zeta(x, v) \ , \tag{8.68c}$$

$$\left\langle \nu_a^i(x) \, \nu_b^j(y) \right\rangle = 2 \, T \, \sigma \, \delta^{ij} \, \delta_{ab} \, \delta^{(4)}(x - y) \ . \tag{8.68d}$$

The noise term appearing in the Yang-Mills equation becomes white noise within this last approximation. The fluctuation-dissipation theorem is fulfilled because the strength of the noise-noise correlator Eq. (8.68c) is precisely given by the dissipative term of Eq. (8.68a). This is the simplest form of the FDT. The colour conductivity in the quasi-local limit has been obtained originally by Bödeker [34].

It is worth pointing out that already in the leading logarithmic approximation the noise term appearing in the Yang Mills equation is not white, except in the local limit Eq. (8.68c). The noise in the Boltzmann-Langevin equation, on the other hand, is white (see Eq. (8.50)), when averaged over the directions of $\mathbf{v}$.

IX. LANGEVIN APPROACH

A. Coarse-graining

In this section, we consider a more phenomenologically inspired Langevin-type approach to fluctuations in non-Abelian plasmas [107]. Let us step back to the original microscopic transport equation (2.40). The microscopic distribution function is a strongly fluctuating quantity at scales associated to mean particle distances, namely $\sim 1/T$ for a plasma close to thermal equilibrium. In order to obtain an effective transport theory at length scales much larger than the typical inter-particles distances, one may wish to coarse-grain the distribution function and the non-Abelian fields over scales characteristic for the problem under investigation. With coarse-graining, we have in mind a volume average over a characteristic (physical) volume and/or over characteristic time scales. Such a coarse-graining results in a coarse-grained distribution function which is considerably smoother than the microscopic one.

The appropriate physical volume depends on the particular physical problem under investigation. As a guideline, one would like to have a *scale separation* such that the coarse-graining scale is large as compared to typical length scale at which two-particle correlations grow large. This way, it is ensured that two- and higher-particle correlators remain small within a coarse-graining volume. In addition, the coarse-graining volume should be sufficiently large, such that the remaining particle number fluctuations of the one-particle distribution function within the coarse-graining volume remain parametrically small. Finally, the coarse-graining scale should be smaller than typical relaxation or damping scales of the problem under investigation. It cannot be guaranteed on general grounds that these requirements can be met. For a hot plasma close to equilibrium, however, the appropriate coarse-graining scale is given by the Debye radius.

Performing such a procedure with the microscopic transport equation (2.40a), we expect to obtain a Boltzmann-Langevin-type of equation for the coarse-grained one-particle distribution function $f(x, p, Q)$, namely

$$p^{\mu} \left(\frac{\partial}{\partial x^{\mu}} - g f^{abc} A_{\mu}^{b} Q^{c} \frac{\partial}{\partial Q^{a}} - g Q_{a} F_{\mu\nu}^{a} \frac{\partial}{\partial p_{\nu}} \right) f(x, p, Q) = C[f](x, p, Q) + \zeta(x, p, Q) . \quad (9.1)$$

Here, the transport equation contains an effective – but not yet specified – collision term $C[f]$, and an associated source for stochastic noise ζ. In the collisionless limit $C = \zeta = 0$, the above set of transport equation reduces to those introduced by Heinz [69]. In the general case, however, the right-hand side of Eq. (9.1) does not vanish due to effective interactions (collisions) in the plasma, resulting in the term $C[f]$. In writing Eq. (9.1), we have already made the assumption that the one-particle distribution function f is a fluctuating quantity. This is quite natural having in mind that f describes a coarse-grained

microscopic distribution function for coloured point particles, and justifies the presence of the stochastic source ζ in the transport equation.

For non-charged particles, a similar (phenomenological) kinetic equation has been considered by Bixon and Zwanzig [26]. This philosophy and its extension for out-of-equilibrium situations have also been applied to nuclear dynamics [127,14,54]. For a recent review of different approaches, we refer to [2]. Some recent approaches which include stochastic noise sources within a Schwinger-Dyson approach have been reported as well [51]. For a stochastic interpretation of the Kadanoff-Baym equations, see [64].

The coarse-grained transport equation is accompanied by a corresponding Yang-Mills equation,

$$
(D_\mu F^{\mu\nu})_a = J_a^\nu(x) \equiv g \sum_{\substack{\text{helicities} \\ \text{species}}} \int dP\, dQ\, Q_a\, p^\nu\, f(x,p,Q)\;, \tag{9.2}
$$

which contains, to leading order, the current due to the quasi-particles, but no fluctuation-induced current.

Given the stochastic dynamical equation Eq. (9.1), the question arises as to what can be said on general grounds about the spectral functions of $f(x,p,Q)$ and $\zeta(x,p,Q)$. For simplicity, we shall assume that the dissipative processes are known close to equilibrium, but no further information is given regarding the underlying fluctuations. This way of proceeding is *complementary* to the procedure of [105,106] as worked out in Sections III and IV, where the right-hand side of Eq. (9.1) has been obtained from correlators of the microscopic statistical fluctuations.

Here, we show that the spectral function of the fluctuations close to equilibrium can be obtained from the knowledge of the kinetic entropy of the plasma. The spectral function of the noise source is shown to be linked to the dissipative term in the effective transport equation. This gives a well-defined prescription as to how the correct source for noise can be identified *without* the detailed knowledge of the underlying microscopic dynamics responsible for the dissipation. The basic idea behind this approach relies on the essence of the fluctuation-dissipation theorem. While this theorem is more general, here we will only discuss the close to equilibrium situation. According to the fluctuation-dissipation theorem, if a fluctuating system remains close to equilibrium, then the dissipative process occurring in it are known. Vice versa, if one knows the dissipative process in the system, one can describe the fluctuations without an explicit knowledge of the microscopic structure or processes in the system. The cornerstone of our approach is the kinetic entropy of the fluctuating system, which serves to identify the associated thermodynamical forces. This leads to the spectral function for the deviations from the non-interacting equilibrium.

B. Classical dissipative systems

Before entering into the discussion of plasmas, we will illustrate this way of proceeding by reviewing the simplest setting of classical linear dissipative systems [97]. A generalisation to the more complex case of non-Abelian plasmas will then become a natural step to perform. We consider a classical homogeneous system described by a set of variables x_i, where i is a discrete index running from 1 to n. These variables are normalised in such a way that their mean values at equilibrium vanish. The entropy of the system is a function of the quantities x_i, $S(x_i)$. If the system is at equilibrium, the entropy reaches its maximum, and thus $(\partial S / \partial x_i)_{\rm eq} = 0$, $\forall i$. If the system is taken slightly away from equilibrium, then one can expand the difference $\Delta S = S - S_{\rm eq}$, where $S_{\rm eq}$ is the entropy at equilibrium, in powers of x_i. If we expand up to quadratic order, then

$$\Delta S = \frac{1}{2} \left(\frac{\partial^2 S}{\partial x_i \partial x_j} \right)_{\rm eq} x^i x^j \equiv -\frac{1}{2} \beta_{ij} x^i x^j \ . \tag{9.3}$$

The matrix β_{ij} is symmetric and positive-definite, since the entropy reaches a maximum at equilibrium. The thermodynamic forces F_i are defined as the gradients of ΔS

$$F_i = -\frac{\partial \Delta S}{\partial x_i} \ . \tag{9.4}$$

For a system close to equilibrium the thermodynamic forces are linear functions of x_i, $F_i = \beta_{ij} x^j$. If the system is at equilibrium, the thermodynamic forces vanish. In more general situations the variables x_i will evolve in time. The time evolution of these variables is given as functions of the thermodynamical forces. In a close to equilibrium case one can expect that the evolution is linear in the forces

$$\frac{dx^i}{dt} = -\gamma^{ij} F_j \ , \tag{9.5}$$

which, in turn, can be expressed as

$$\frac{dx^i}{dt} = -\lambda^{ij} x_j \ . \tag{9.6}$$

Within a phenomenological Langevin approach a white noise source is added to account for the underlying fluctuations. Otherwise, the system would abandon equilibrium. Hence, we write instead

$$\frac{dx^i}{dt} = -\lambda^{ij} x_j + \zeta^i \ . \tag{9.7}$$

The first term on the right-hand side describes the mean regression of the system towards equilibrium, while the second term is the source for stochastic noise. The quantities γ^{ij} are

known as the kinetic coefficients, and it is not difficult to check that $\gamma_{ij} = \lambda_{ik}\beta_{kj}^{-1}$. From the value of the coefficients β_{ij} one can deduce the equal time correlator

$$\langle x_i(t)x_j(t)\rangle = \beta_{ij}^{-1} \ , \tag{9.8}$$

which is used to obtain Einstein's law

$$\left\langle x^i(t)F_j(t)\right\rangle = \delta_j^i \ . \tag{9.9}$$

After taking the time derivative of Eq. (9.9), assuming that the noise is white and Gaussian

$$\left\langle \zeta^i(t)\zeta^j(t')\right\rangle = \nu^{ij}\delta(t-t') \ , \tag{9.10}$$

we find that the strength of the noise self-correlator ν is determined by the dissipative process

$$\nu^{ij} = \gamma^{ij} + \gamma^{ji} \ , \tag{9.11}$$

which is the fluctuation-dissipation relation we have been aiming at.

C. Non-Abelian plasmas as a classical dissipative system

We now come back to the case of a non-Abelian plasma and generalise the above discussion to the case of our concern. We will consider the non-Abelian plasma as a classical linear dissipative system, assuming that we know the collision term in the transport equation. In order to adopt the previous reasoning, we have to identify the dissipative term in the transport equation, and to express it as a function of the thermodynamical force obtained from the entropy. The deviation from the equilibrium distribution is given here by

$$\Delta f(x, p, Q) = f(x, p, Q) - f_{\text{eq}}(p_0) \ , \tag{9.12}$$

and replaces the variables x_i discussed above.

1. Classical plasmas

The entropy flux density for classical plasmas has been given in Eq. (4.24) with Eq. (4.26) for the classical plasma. It reads explicitly

$$S_\mu(x) = -\int dP dQ \, p_\mu \, f(x, p, Q) \left(\ln\left(f(x, p, Q)\right) - 1\right) \ . \tag{9.13}$$

The $\mu = 0$ component of Eq. (9.13) gives the entropy density of the system. The entropy itself is then obtained as $S = \int d^3x\, S_0(x)$.

We shall now assume that the deviation of the mean particle number from the equilibrium one is small within a coarse-graining volume. This can always be arranged for at sufficiently small gauge coupling $g \ll 1$, which ensures that fluctuations are parametrically suppressed by the plasma parameter. We then obtain ΔS just by expanding the expression of the entropy density in powers of Δf up to quadratic order. It is important to take into account that we will consider situations where the small deviations from equilibrium are such that both the particle number and the energy flux remain constant, thus

$$\int dPdQ\, p_0 \ \ \Delta f(x, p, Q) = 0 \;, \tag{9.14a}$$

$$\int dPdQ\, p_0 p_\mu\, \Delta f(x, p, Q) = 0 \,. \tag{9.14b}$$

With these assumptions, one reaches

$$\begin{aligned}
\Delta S_0(x) &= -\int dPdQ\, p_0\, \frac{(\Delta f(x, p, Q))^2}{2 f_{\text{eq}}(p_0)} \\
&= -\int \frac{d^3p}{(2\pi)^3}\, dQ \frac{(\Delta f(x, \mathbf{p}, Q))^2}{2 f_{\text{eq}}(\omega_p)} \;,
\end{aligned} \tag{9.15}$$

where we have taken into account the mass-shell condition the second line, with $p_0 = \omega_p = \sqrt{\mathbf{p}^2 + m^2}$. Without loss of generality, we will consider from now on the case of massless particles, such that $\omega_p = p = |\mathbf{p}|$.

The *thermodynamic force* associated to Δf is defined from the entropy as

$$F(x, \mathbf{p}, Q) = -\frac{\delta \Delta S}{\delta \Delta f(x, \mathbf{p}, Q)} = \frac{1}{(2\pi)^3} \frac{\Delta f(x, \mathbf{p}, Q)}{f_{\text{eq}}(p)} \;. \tag{9.16}$$

We linearise the transport equation (9.1) and express the collision integral close to equilibrium in terms of the thermodynamical force. Dividing Eq. (9.1) by p_0 and imposing the mass-shell constraint, we find

$$v^\mu D_\mu \Delta f - g v^\mu Q_a F^a_{\mu 0} \frac{df_{\text{eq}}}{dp} = C[\Delta f](x, \mathbf{p}, Q) + \zeta(x, \mathbf{p}, Q) \;, \tag{9.17}$$

where $v^\mu = p^\mu/p_0 = (1, \mathbf{v})$, with $\mathbf{v}^2 = 1$. We also introduced the shorthand $D_\mu \Delta f \equiv (\partial_\mu - g f^{abc} A_{\mu,b} Q_c \partial^Q_a) \Delta f$ as in Eq. (2.44). It is understood that the collision integral has been linearised, and we write it as

$$C_{\text{lin}}[\Delta f](t, \mathbf{x}, \mathbf{p}, Q) = \int d^3x' d^3p'\, dQ'\, K(\mathbf{x}, \mathbf{p}, Q; \mathbf{x}', \mathbf{p}', Q') \Delta f(t, \mathbf{x}', \mathbf{p}', Q') \,, \tag{9.18}$$

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with $t \equiv x_0$. For simplicity, we take the collision integral local in time, but unrestricted otherwise.[††] The thermodynamical force is linear in Δf. The linearised collision integral can easily be expressed in terms of F as

$$C_{\text{lin}}[F](t, \mathbf{x}, \mathbf{p}, Q) = \int d^3x' d^3p' \, dQ' \, K(\mathbf{x}, \mathbf{p}, Q; \mathbf{x}', \mathbf{p}', Q') \, (2\pi)^3 \, f_{\text{eq}}(p_0) F(t, \mathbf{x}', \mathbf{p}', Q') . \quad (9.19)$$

According to the fluctuation-dissipation relation, the source of stochastic noise has to obey

$$\langle \zeta(x, \mathbf{p}, Q) \zeta(x', \mathbf{p}', Q') \rangle = - \left(\frac{\delta C[F](x, \mathbf{p}, Q)}{\delta F(x', \mathbf{p}', Q')} + \frac{\delta C[F](x', \mathbf{p}', Q')}{\delta F(x, \mathbf{p}, Q)} \right) \quad (9.20)$$

in full analogy to Eq. (9.11). With the knowledge of the thermodynamical force Eq. (9.16) and Eq. (9.18), or simply using the explicit expression Eq. (9.19) for the linearised collision term, we arrive at

$$\langle \zeta(x, \mathbf{p}, Q) \zeta(x', \mathbf{p}', Q') \rangle = -(2\pi)^3 \left(f_{\text{eq}}(p) K(\mathbf{x}, \mathbf{p}, Q; \mathbf{x}', \mathbf{p}', Q') + \text{sym.} \right) \delta(t - t') . \quad (9.21)$$

Here, symmetrisation in $(\mathbf{x}, \mathbf{p}, Q) \leftrightarrow (\mathbf{x}', \mathbf{p}', Q')$ is understood.

Notice that we can derive the equal-time correlator for the deviations from the equilibrium distribution simply from the knowledge of the entropy and the thermodynamical force, exploiting Einstein's law in full analogy to the corresponding relation Eq. (9.8). Using Eq. (9.16) we find

$$\langle \Delta f(x, \mathbf{p}, Q) \Delta f(x', \mathbf{p}', Q') \rangle_{t=t'} = (2\pi)^3 f_{\text{eq}}(p) \delta^{(3)}(\mathbf{x} - \mathbf{x}') \delta^{(3)}(\mathbf{p} - \mathbf{p}') \delta(Q - Q') . \quad (9.22)$$

If the fluctuations Δf have vanishing mean value, then Eq. (9.22) reproduces the well-known result that the correlator of fluctuations at equilibrium is given by the equilibrium distribution function. In order to make contact with the results of [105,106] as discussed in Section III, we go a step further and consider the case where Δf has a non-vanishing mean value to leading order in the gauge coupling. Splitting

$$\Delta f = g \bar{f}^{(1)} + \delta f \quad (9.23)$$

into a deviation of the mean part $\langle \Delta f \rangle = g \bar{f}^{(1)}$ and a fluctuating part $\langle \delta f \rangle = 0$, and using Eq. (9.22), we obtain the equal time correlator for the fluctuations δf as

$$\langle \delta f(x, \mathbf{p}, Q) \delta f(x', \mathbf{p}', Q') \rangle_{t=t'} = (2\pi)^3 f_{\text{eq}}(p) \delta^{(3)}(\mathbf{x} - \mathbf{x}') \delta^{(3)}(\mathbf{p} - \mathbf{p}') \delta(Q - Q')$$
$$- g^2 \bar{f}^{(1)}(x, \mathbf{p}, Q) \bar{f}^{(1)}(x', \mathbf{p}', Q') \big|_{t=t'} . \quad (9.24)$$

This result agrees with the correlator obtained in Eq. (3.12) from the Gibbs ensemble average as defined in phase space in the limit where two-particle correlations are small and given by products of one-particle correlators.

[††]Of course, gauge invariance imposes further conditions on both the collision term and the noise. However, these constraints are of no relevance for the present discussion.

Up to now we have dealt with purely classical plasmas. On the same footing, we can consider the soft and ultra-soft modes in a hot quantum plasma. These can be treated classically as their occupation numbers are large. The sole effect from their quantum nature reduces to the different statistics, Bose-Einstein or Fermi-Dirac as opposed to Maxwell-Boltzmann. The corresponding quantum fluctuation-dissipation theorem reduces to an effective classical one [97,94].

A few changes are necessary to study hot quantum plasmas. As in Section III, we change the normalisation of f by a factor of $(2\pi\hbar)^3$ to obtain the standard normalisation for the (dimensionless) quantum distribution function. Thus, the momentum measure is also modified by the same factor, $dP = d^4p\,2\Theta(p_0)\delta(p^2)/(2\pi\hbar)^3$ for massless particles, and $\hbar = 1$. To check the fluctuation-dissipation relation in this case one needs to start with the correct expression for the entropy for a quantum plasma. The entropy flux density, as a function of $f(x,p,Q)$, is given by Eq. (4.24) and Eq. (4.27), to wit

$$S_\mu(x) = -\int dP dQ\, p_\mu \left(f \ln f \mp (1 \pm f) \ln (1 \pm f) \right) , \qquad (9.25)$$

where the upper or lower sign applies for bosons or fermions. From the above expression of the entropy one can compute ΔS, and proceed exactly as in the classical case, expanding the entropy up to quadratic order in the deviations from equilibrium. Thus, we obtain the noise correlator

$$\langle \zeta(x, \mathbf{p}, Q)\zeta(x', \mathbf{p}', Q') \rangle = -(2\pi)^3 \delta(t - t')$$
$$\times (f_{\text{eq}}(p)[1 \pm f_{\text{eq}}(p)]K(\mathbf{x}, \mathbf{p}, Q; \mathbf{x}', \mathbf{p}', Q') + \text{sym.}) . \qquad (9.26)$$

Again, the spectral functions of the deviations from equilibrium are directly deduced from the entropy. As a result, we find

$$\langle \Delta f(x, \mathbf{p}, Q)\Delta f(x', \mathbf{p}', Q') \rangle_{t=t'} = (2\pi)^3 f_{\text{eq}}(p)[1 \pm f_{\text{eq}}(p)]$$
$$\times \delta^{(3)}(\mathbf{x} - \mathbf{x}')\delta^{(3)}(\mathbf{p} - \mathbf{p}')\delta(Q - Q') . \qquad (9.27)$$

Expanding $\Delta f = g\bar{f}^{(1)} + \delta f$ as above, we obtain the equal time correlator for δf, which agrees with the findings of Eqs. (3.14) and (3.15) in the case where two-particle distribution functions can be expressed as products of one-particle distributions.

With the knowledge of the above spectral functions for the fluctuations in a classical or quantum plasma one can derive further spectral distributions for different physical quantities. In particular, we can find the correlations of the self-consistent gauge field fluctuations once the basic correlators as given above are known. This is how those spectral functions were deduced in [105].

D. Application

As a particular example of the above we consider Bödeker's effective kinetic equations which couple to the ultra-soft gauge field modes. The linearised collision integral has been obtained to leading logarithmic accuracy in Section VIII E. We will first consider the classical plasma for particles carrying two helicities. It is most efficient to write the transport equation not in terms of the full one-particle distribution function, but in terms of the current density

$$\mathcal{J}_a^\rho(x, \mathbf{v}) = \frac{g}{\pi^2} \, v^\rho \int dp \, dQ \, p^2 \, Q_a \, \Delta f(x, \mathbf{p}, Q) \,. \tag{9.28}$$

(Notice that f_{eq} gives no contribution to the current.) The current of Eq. (9.2) follows after integrating over the angles of $\mathbf{v}$, $J_a^\mu(x) = \int \frac{d\Omega}{4\pi} \mathcal{J}_a^\mu(x, \mathbf{v})$. Expressed in terms of Eq. (9.28), the linearised Boltzmann-Langevin equation Eq. (9.17) becomes

$$[v^\mu D_\mu, \mathcal{J}^\rho](x, \mathbf{v}) = -m_D^2 v^\rho v_\mu F^{\mu 0}(x) + v^\rho C[\mathcal{J}^0](x, \mathbf{v}) + \zeta^\rho(x, \mathbf{v}) \,, \tag{9.29}$$

where m_D is the Debye mass

$$m_D^2 = -\frac{g^2 C_2}{\pi^2} \int dp \, p^2 \, \frac{df_{\text{eq}}}{dp} \,, \tag{9.30}$$

and the quadratic Casimir C_2 has been defined in Eq. (2.25). The linearised collision integral is related to Eq. (9.18) by

$$C[\mathcal{J}_a^0](x, \mathbf{v}) = \frac{g^2}{\pi^2} \int d^3x' \, d\Omega_{\mathbf{v}'} \, dp \, dp' \, dQ \, dQ'$$

$$\times p^2 p'^2 \, Q_a \, K(\mathbf{x}, \mathbf{p}, Q; \mathbf{x}', \mathbf{p}', Q') \, \Delta f(t, \mathbf{x}', \mathbf{p}', Q') \tag{9.31}$$

and corresponds precisely to the correlator Eq. (8.43a), to wit

$$C[\mathcal{J}_a^0](x, \mathbf{v}) = -\gamma \int \frac{d\Omega_{\mathbf{v}'}}{4\pi} I(\mathbf{v}, \mathbf{v}') \mathcal{J}_a^0(x, \mathbf{v}') \,, \tag{9.32}$$

where the kernel has been given in Eq. (8.43b),

$$I(\mathbf{v}, \mathbf{v}') = \delta^{(2)}(\mathbf{v} - \mathbf{v}') - \frac{4}{\pi} \frac{(\mathbf{v} \cdot \mathbf{v}')^2}{\sqrt{1 - (\mathbf{v} \cdot \mathbf{v}')^2}} \tag{9.33}$$

and $\gamma = g^2 N T \ln (1/g)/4\pi$. Comparing Eq. (9.31) with Eq. (9.32) we learn that only the part of the kernel K which is symmetric under $(\mathbf{x}, \mathbf{p}, Q) \leftrightarrow (\mathbf{x}', \mathbf{p}', Q')$ contributes in the present case. This part can be expressed as

$$K(\mathbf{x}, \mathbf{p}, Q; \mathbf{x}', \mathbf{p}', Q') = -\gamma \frac{I(\mathbf{v}, \mathbf{v}')}{4\pi p^2} \delta(p - p')\delta(Q - Q')\delta^{(3)}(\mathbf{x} - \mathbf{x}') \,. \tag{9.34}$$

According to our findings above, the self-correlator of the stochastic source for the classical plasma obeys

$$\langle \zeta_a^\mu(x, \mathbf{v})\, \zeta_b^\nu(y, \mathbf{v}') \rangle = \frac{1}{(2\pi^2)^2}\, g^2 \int dp\, dp'\, dQ\, dQ'\, p^2 p'^2\, Q_a Q_b'\, v^\mu v'^\nu \langle \zeta(x, \mathbf{p}, Q)\, \zeta(y, \mathbf{p}', Q') \rangle$$

$$= 2\,\gamma\, T\, m_D^2\, v^\mu v'^\nu\, I(\mathbf{v}, \mathbf{v}')\, \delta_{ab}\, \delta^{(4)}(x - y) \,. \tag{9.35}$$

The helicities of the particles have been taken into account as well. In order to obtain Eq. (9.35), we have made use of Eqs. (9.21), (9.30) to (9.32), and of the relation $f_{\text{eq}} = -T\, df_{\text{eq}}/dp$ for the Maxwell-Boltzmann distribution.

The quantum plasma can be treated in exactly the same way. To confirm Eq. (9.35), we only need to take into account the change of normalisation as commented above, and the relation $f_{\text{eq}}(1 \pm f_{\text{eq}}) = -T\, df_{\text{eq}}/dp$ for the Bose-Einstein and Fermi-Dirac distributions, respectively.

Using the explicit expression for the collision integral and the stochastic noise it is possible to confirm the covariant conservation of the current, $DJ = 0$.

E. Discussion

We thus found that the correlator Eq. (9.35) is in full agreement with the result of Section VIII F for both the classical or the quantum plasma. While this correlator has been obtained in Section VIII F from the corresponding microscopic theory, here, it follows solely from the fluctuation-dissipation theorem. This way, it is established that the effective Boltzmann-Langevin equation found in [34] is indeed fully consistent with the fluctuation-dissipation theorem. More generally, the important observation is that the spectral functions as derived here from the entropy and the fluctuation-dissipation relation do agree with those obtained in Section III from a microscopic phase space average. This guarantees, on the other hand, that the formalism developed in Sections III – V is consistent with the fluctuation-dissipation theorem.

In the above discussion we have considered the stochastic noise as Gaussian and Markovian. This is due to the fact that the small-scale fluctuations (those within a coarse-graining volume) are to leading order well separated from the typical relaxation scales in the plasma. Within the microscpoic approach, these characteristics can be understood ultimately as a consequence of an expansion in a small plasma parameter (or a small gauge coupling). More precisely, the noise follows to be Gaussian due to the polarisation approximation, where higher order correlators beyond quadratic ones can be neglected. The Markovian

character of the noise follows because the ultra-soft modes are well separated from the soft ones, and suppressed in the collision integral to leading order. This way, the collision term and the correlator of stochastic noise are both local in x-space. Going beyond the logarithmic approximation, we expect from the explicit computation in Section VIII E that the coupling of the soft and the ultra-soft modes makes the collision term *non-local* in coordinate space. This non-trivial memory kernel should also result in a non-Markovian, but still Gaussian, source for stochastic noise.

The present line of reasoning can in principle be extended to other approaches. Using the phenomenological derivation of Eq. (9.32) from [9,10], the same arguments as above justify the presence of a noise source with Eq. (9.35) in the corresponding Boltzmann equation [9,10,31,143]. It might also be fruitful to follow a similar line based on the entropy within a quantum field theoretical language. An interesting proposal to self-consistently include the noise within a Schwinger-Dyson approach has been made recently in [51]. Along these lines, it might be feasible to derive the source for stochastic noise directly from the quantum field theory [31].

While we have concentrated the discussion on plasmas close to thermal equilibrium, it is known that a fluctuation-dissipation theorem can be formulated as well for stationary and stable systems out of equilibrium [94]. More generally, we have exploited the fact that the entropy production vanishes. All the information which links the dissipative characteristics with correlators of fluctuations can be deduced from Eq. (4.32) in the case where the entropy production vanishes.

This approach can also be extended to take non-linear effects into account [137]. Both the out-of-equilibrium situations and the non-linear effects can be treated, in principle, with the general formalism as discussed in Sections III – IV.

X. DENSE QUARK MATTER

Until now, the applications of the formalism have been focussed on different aspects of hot non-Abelian plasmas close to thermal equilibrium. In this section we consider the regime of high baryonic density and low temperature. In this part of the QCD phase diagram, quarks form Cooper pairs due to the existence of attractive interactions among them. A colour superconducting phase then arises, typically characterised by the Anderson-Higgs mechanism and the existence of an energy gap associated to the fermionic quasiparticles.

For gluon momenta much larger than the fermionic gap, the effects of diquark condensation are negligible. For scales larger than the gap, but still smaller than the chemical potential, one finds to leading order an effective theory for the gauge fields totally analogous to the HTL theory, the so called hard dense loop (HDL) theory. This effective theory can be derived from classical transport theory [111].

For gluon momenta of the order of, or smaller than, the fermionic gap, the effects of Cooper pairing cannot be neglected. The kinetic equations have to take into account the modification of the quasiparticle dispersion relations, which ultimately reflect the fact that the ground state of QCD in the superconducting phase is not the same as in a normal phase. We present a kinetic equation for the gapped quasiparticles in a two flavour colour superconductor [109].

A. Normal phase

We consider first the regime where the effects of Cooper pairing of quarks can be neglected, and discuss the kinetic equations associated to the normal phase of dense quark matter. At very high baryonic density, the non-Abelian plasma is ultradegenerate. The fermionic equilibrium distribution function, neglecting the effects of Cooper pairing, is given by

$$f^{\text{eq}}(p_0) = \Theta(\mu - p_0) , \tag{10.1}$$

where Θ is the step function, and μ is the quark chemical potential. This distribution function describes a system where all the fermionic energy levels are occupied, according to Pauli's principle, up to the value of the Fermi energy $p_0 = \mu$. At zero temperature there are no real gluons in the system.

The Vlasov approximation presented in Section VII can be applied to this non-thermal situation. The main input is the above equilibrium distribution function, which affects both the value of the Debye mass and the relevant scales of the system. The Debye mass reads

$$m_D^2 = N_F \frac{g^2 \mu^2}{2\pi^2} , \tag{10.2}$$

for N_F different quark flavours. This is to be compared with $m_D^2 \sim g^2 T^2$ as found for a quantum plasma at high temperature.

Formally, the Vlasov approximation to a non-Abelian plasma at either high tempera-ture, or high baryonic density but vanishing temperature in the normal phase, look almost identical. The colour currents obtained in the two cases are the same, only the explicit value of the Debye mass differs. Roughly speaking, one could say that the role played by T in the HTL effective theory is now played by μ. Now μ is the hard scale, while $g\mu$ is the soft one. As in the thermal case, in the ultradegenerate limit $\delta^{(n)} J / \delta A_1 \ldots \delta A_n |_{A=0}$ generates a $n+1$-point amplitude, which looks formally the same as the $n+1$-point HTL. Due to this similarity, these amplitudes were called hard dense loops (HDL) in [111] (see also [144]).

The above considerations neglect the fact that quarks form Cooper pairs, which mod-ify both the shape of the quasiparticle distribution function and also the underlying kinetic equations. However, the Vlasov approximation remains a valid description for specific mo-mentum scales of the plasma. The HDL effective theory can be derived as well from quantum field theory. An explicit computation of the gluon self-energy in the superconducting phase of QCD, for $N_F = 2$ and $N_F = 3$ [129,130], shows that to leading order it reduces to the HDL value in the limit when the gluon momentum p obeys $\mu \gg p \gg \Delta$, where Δ is the value of the gap.

B. Superconducting phases

Let us now consider the case when diquark condensates are formed, modifying the ground state of QCD. The possible phases of QCD depend strongly on the number of quark flavours participating in the condensation. We will briefly review the two mostly studied phases. These are the idealizations of considering pairing of either two or three massless quark flavours. More realistic situations should consider effects due to non-vanishing quark masses , which may lead to an even richer phase diagram. We also restrict the discussion to quark condensation in the lowest angular momentum channel, the spin zero condensates, as this channel is energetically favored.

For two light quark flavours the diquark condensate is such that the $SU_c(3)$ group is broken down to $SU_c(2)$ [5,128]. Thus, five gluons get a mass, while there are three gluons which remain massless, and exhibit confinement. Furthermore, not all the quarks participate in the condensation. More speficially, if we consider up and down quarks of colours red, green and blue, one of the colours, say the blue one, does not participate in the condesation process. Then the blue up and down quarks are gapless. The condensate is such that if one could neglect the effects of the quantum anomaly, which can be done at asymptotically large densities, it would also break a global axial $U_A(1)$. Thus a (pseudo) Nambu-Goldstone mode, similar to the η' meson, is also present. This meson becomes heavy as soon as one

reduces the density of the system, and its mass can be computed using instanton techniques.

For three light quark flavours the pattern of symmetry breaking induced by the condensates is much more involved, as the condensates lock the colour and flavour symmetry transformations (colour-flavour locking or CFL phase) [6]. They break spontaneously both colour, chiral and baryon number symmetry $SU_c(3) \times SU_L(3) \times SU_R(3) \times U_B(1) \rightarrow SU_{c+L+R}(3) \times \mathcal{Z}_2$. As a result, all the gluons become massive, while there are nine Nambu-Goldstone bosons, eight associated to the breaking of chiral symmetry, and one associated to the breaking of baryon number symmetry. At asymptotically large densities, when the effects of the quantum anomaly can be neglected, there is an extra (pseudo) Nambu-Goldstone boson associated to the breaking of $U(1)_A$. In the CFL phase all the quarks of all flavours and colours participate in the condensation. The light modes are then the Nambu-Goldstone bosons, which dominate the long distance physics of the superconductor.

If electromagnetic interactions are taken into account, then both the 2SC and CFL diquark condensates break spontaneously the standard electromagnetic symmetry. However, a linear combination of the original photon and a gluon remains massless in both cases. This new field plays the role of the "in-medium" photon in the supercondutor.

Using standard techniques in BCS theory, it is possible to compute, in the weak coupling limit $g \ll 1$, the microscopic properties of the 2SC and CFL superconductors. This concerns in particular the fermionic gap Δ associated to the quarks, as well as the gluon Meissner masses m_M. In weak coupling there is a hierarchy of scales $\Delta \ll m_M \ll \mu$. Furthermore, it is also possible to compute the relevant properties of the (pseudo) Nambu-Goldstone modes, which acquire masses due to explicit chiral symmetry breaking effects of QCD. The propagation properties of the "in-medium" photon has been obtained as well [110]. While all these computations rely on a weak gauge coupling expansion, which might be unrealistic for the astrophysical settings of interest, they provide both a qualitative and semi-quantitative insight of the main microscopic porperties of quark matter. These studies may be complemented with others based on QCD-inspired models, which might be pushed to the regime of more moderate densities, and thus large couplings. It would be desirable that the microscopic properties of quark matter could also be computed numerically. At present, no reliable numerical algorithms are available for such a study.

C. Quasiparticles in the 2SC phase

To be specific, we restrict the remaining considerations to the case of two massless quark flavours, the 2SC phase. It is shown that coloured quasiparticle excitations of the 2SC condensate can be formulated in terms of a simple transport equation.

The low energy physics of a two-flavour colour superconductor is dominated by its light degrees of freedom. At vanishing temperature, these are the massless gauge bosons,

the gapless quarks and a (pseudo-) Goldstone boson, similar to the η' meson. However, the gapless quarks and the η' meson are neutral with respect to the unbroken $SU(2)$ subgroup. In turn, the condensate, although neutral with respect to the unbroken $SU(2)$, polarises the medium since their constituents carry $SU(2)$ charges. Hence, the dynamics of the light $SU(2)$ gauge fields differs from the vacuum theory. This picture has recently been introduced by Rischke, Son and Stephanov [131]. Their infrared effective theory for momenta $k \ll \Delta$ is

$$S_{\text{eff}}^{T=0} = \int d^4x \left(\frac{\epsilon}{2} \mathbf{E}_a \cdot \mathbf{E}_a - \frac{1}{2\lambda} \mathbf{B}_a \cdot \mathbf{B}_a \right) , \tag{10.3}$$

where $E_i^a \equiv F_{0i}^a$ and $B_i^a \equiv \frac{1}{2}\epsilon_{ijk}F_{jk}^a$ are the $SU(2)$ electric and magnetic fields. The constants ϵ and λ are the dielectric susceptibility and magnetic permeability of the medium. To leading order, $\lambda = 1$ and $\epsilon = 1 + g^2\mu^2/(18\pi^2\Delta^2)$ [131]. As a consequence, the velocity of the $SU(2)$ gluons is smaller than in vacuum. This theory is confining, but the scale of confinement is highly reduced with respect to the one in vacuum with $\Lambda'_{\text{QCD}} \sim \Delta \exp\left(-\frac{2\sqrt{2}\pi}{11}\frac{\mu}{g\Delta}\right)$ [131]. Due to asymptotic freedom, it is expected that perturbative computations are reliable for energy scales larger than Λ'_{QCD}.

At non-vanishing temperature, thermal excitations modify the low energy physics. The condensate melts at the critical temperature $T_c \approx 0.567\Delta_0$ [125] (Δ_0 is the gap at vanishing temperature). We restrict the discussion to temperatures within $\Lambda'_{\text{QCD}} \ll T < T_c$, which provides the basis for the perturbative computations below. In this regime, the main contribution to the long distance properties of the $SU(2)$ fields stems from the thermal excitations of the constituents of the diquark condensate. The thermal excitations of the massless gauge fields contribute only at the order g^2T^2 and are subleading for sufficiently large μ. Those of the gapless quarks and of the η' meson do not couple to the $SU(2)$ gauge fields.

The thermal excitations due to the constituents of the diquark condensate display a quasiparticle structure. This implies that they can be cast into a transport equation. To that end, and working in natural units $k_B = \hbar = c = 1$, we introduce the on-shell one-particle phase space density $f(x, \mathbf{p}, Q)$, $x^\mu = (t, \mathbf{x})$, describing the quasiparticles. The distribution function depends on time, the phase space variables position $\mathbf{x}$, momentum $\mathbf{p}$, and on $SU(2)$ colour charges Q_a, with the colour index $a = 1, 2$ and 3. The quasiparticles carry $SU(2)$ colour charges simply because the constituents of the condensate do. The on-shell condition for massless quarks $m_q = 0$ relates the energy of the quasiparticle excitation to the chemical potential and the gap as

$$p_0 \equiv \epsilon_p = \sqrt{(p-\mu)^2 + \Delta^2(T)} . \tag{10.4}$$

The gap is both temperature and momentum-dependent. From now on, we can neglect its momentum dependence which is a subleading effect. The velocity of the quasiparticles is given by

$$\mathbf{v}_p \equiv \frac{\partial \epsilon_p}{\partial \mathbf{p}} = \frac{|p - \mu|}{\sqrt{(p - \mu)^2 + \Delta^2(T)}} \, \hat{\mathbf{p}} \,, \tag{10.5}$$

and depends on both the chemical potential and the gap. For $\Delta = 0$, the quasiparticles would travel at the speed of light. However, in the presence of the gap $\Delta \neq 0$, their propagation is suppressed, $v_p \equiv |\mathbf{v}_p| \leq 1$.

The one-particle distribution function $f(x, \mathbf{p}, Q)$ obeys a very simple transport equation, given by

$$\left[D_t + \mathbf{v}_p \cdot \mathbf{D} - gQ_a \left(\mathbf{E}^a + \mathbf{v}_p \times \mathbf{B}^a \right) \frac{\partial}{\partial \mathbf{p}} \right] f = C[f] \,. \tag{10.6}$$

Here, we have introduced the short-hand notation of Eq. (2.44) for the covariant derivative acting on f. The first two terms on the left-hand side of Eq. (10.6) combine to a covariant drift term $v_p^\mu D_\mu$, where $v_p^\mu = (1, \mathbf{v}_p)$ and $D_\mu = (D_t, \mathbf{D})$. The terms proportional to the colour electric and magnetic fields provide a force term. The right-hand side of Eq. (10.6) contains a (yet unspecified) collision term $C[f]$.

Notice that Eq. (10.6) has the same structure as the (on-shell) transport equation valid for the unbroken phase of a non-Abelian plasma. All what changes here are the energy and velocity of the quasiparticles. Once the temperature of the system is increased, and the diquark condensates melt and the gap vanishes, $\Delta(T) \to 0$, we recover the transport equation for the unbroken phase.

The thermal quasiparticles carry an $SU(2)$ charge, and hence provide an $SU(2)$ colour current. It is given by

$$J_a^\mu(x) = g \sum_{\substack{\text{helicities} \\ \text{species}}} \int \frac{d^3p}{(2\pi)^3} dQ \, v_p^\mu \, Q_a \, f(x, \mathbf{p}, Q) \,. \tag{10.7}$$

Below, we simply omit a species or helicity index on f, as well as the explicit sum over them. We use the same definition for the colour measure as in Eq. (2.23). The colour current Eq. (10.7) is covariantly conserved for $C[f] = 0$. For $C[f] \neq 0$ a covariantly conserved current implies certain restrictions in the form of the collision term.

D. Vlasov approximation

We will now study the Vlasov approximation, or collisionless dynamics $C[f] = 0$ of the colour superconductor close to thermal equilibrium and to leading order in the gauge coupling. Consider the distribution function

$$f(x, \mathbf{p}, Q) = f^{\text{eq}}(p_0) + g f^{(1)}(x, \mathbf{p}, Q) \,. \tag{10.8}$$

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Here

$$f^{\text{eq.}}(p_0) = \frac{1}{\exp(\epsilon_p/T) + 1} \tag{10.9}$$

is the fermionic equilibrium distribution function and $gf^{(1)}(x, \mathbf{p}, Q)$ describes a slight deviation from equilibrium. For convenience, we also introduce the colour density

$$J_a(x, \mathbf{p}) = g \int dQ\, Q_a f(x, \mathbf{p}, Q) , \tag{10.10}$$

from which the induced colour current of the medium Eq. (10.7) follows as $J_a^\rho(x) = \int \frac{d^3p}{(2\pi)^3} v_p^\rho J_a(x, \mathbf{p})$. Expanding the transport equation (10.6) to leading order in g, and taking the two helicities per quasiparticle into account, we find the transport equation for the colour density as

$$[D_t + \mathbf{v}_p \cdot \mathbf{D}]\, J(x, \mathbf{p}) = 2g^2\, \mathbf{v}_p \cdot \mathbf{E}(x) \frac{df^{\text{eq}}}{d\epsilon_p} . \tag{10.11}$$

The solution of the transport equation reads

$$J_a^\mu(x) = 2g^2 \int \frac{d^3p\, d^4y}{(2\pi)^3}\, v_p^\mu\, \langle x|\frac{1}{(v_p \cdot D)}|y\rangle_{ab}\, \mathbf{v}_p \cdot \mathbf{E}_b(y) \frac{df^{\text{eq}}}{d\epsilon_p} . \tag{10.12}$$

After having solved the transport equation, the relevant information concerning the low energy effective theory is contained in the functional $J[A]$. Notice that the above derivation is analogous to the derivation of the HTL and HDL effective theories from kinetic theory. Owing to this resemblance, we call the diagrams which are derived from Eq. (10.12) as hard superconducting loops (HSL). The HSL effective action follows from Eq. (10.12) by solving $J[A] = -\delta\Gamma_{\text{HSL}}[A]/\delta A$ for $\Gamma_{\text{HSL}}[A]$, and all HSL diagrams can be derived by performing functional derivatives to the effective action (or the induced current). We thus reach to the conclusion that the low energy effective theory for a two-flavour colour superconductor at finite temperature reads $S_{\text{eff}}^T = S_{\text{eff}}^{T=0} + \Gamma_{\text{HSL}}$ to leading order in g. This theory is effective for modes with $k \ll \Delta$.

Let us have a closer look into the induced current, which we formally expand as $J_\mu^a[A] = \Pi_{\mu\nu}^{ab} A_b^\nu + \frac{1}{2}\Gamma_{\mu\nu\rho}^{abc} A_b^\nu A_c^\rho + \ldots$ in powers of the gauge fields. The most relevant information on the thermal effects is contained in the thermal polarisation tensor $\Pi_{\mu\nu}^{ab}$. Using Eq. (10.12), we find

$$\Pi_{ab}^{\mu\nu}(k) = 2g^2\delta_{ab} \int \frac{d^3p}{(2\pi)^3} \frac{df^{\text{eq}}}{d\epsilon_p} \left(g^{\mu 0}g^{\nu 0} - k_0 \frac{v_p^\mu v_p^\nu}{k \cdot v_p} \right) . \tag{10.13}$$

It obeys the Ward identity $k_\mu \Pi_{ab}^{\mu\nu}(k) = 0$. With retarded boundary conditions $k_0 \to k_0 + i0^+$, the polarisation tensor has an imaginary part,

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$$\text{Im}\,\Pi_{ab}^{\mu\nu}(k) = \delta_{ab}\,2\pi g^2 k_0 \int \frac{d^3p}{(2\pi)^3}\frac{df^{\text{eq}}}{d\epsilon_p}v_p^\mu v_p^\nu\,\delta(k\cdot v_p)\,, \tag{10.14}$$

which corresponds to Landau damping. Performing the angular integration, we obtain for the longitudinal and transverse projections of the polarisation tensor

$$\Pi_L(k_0,\mathbf{k}) = \frac{g^2}{\pi^2}\int_0^\infty dp\,p^2\frac{df^{\text{eq}}}{d\epsilon_p}\left[1-\frac{1}{2}\frac{k_0}{|\mathbf{k}|v_p}\left(\ln\left|\frac{k_0+|\mathbf{k}|v_p}{k_0-|\mathbf{k}|v_p}\right|-i\pi\,\Theta(|\mathbf{k}|^2v_p^2-k_0^2)\right)\right]\,, \tag{10.15a}$$

$$\Pi_T(k_0,\mathbf{k}) = \frac{g^2}{2\pi^2}\frac{k_0^2}{|\mathbf{k}|^2}\int_0^\infty dp\,p^2\frac{df^{\text{eq.}}}{d\epsilon_p}\left[1+\frac{1}{2}\left(\frac{|\mathbf{k}|v_p}{k_0}-\frac{k_0}{|\mathbf{k}|v_p}\right)\times\right.$$
$$\left.\left(\ln\left|\frac{k_0+|\mathbf{k}|v_p}{k_0-|\mathbf{k}|v_p}\right|-i\pi\,\Theta(|\mathbf{k}|^2v_p^2-k_0^2)\right)\right]\,, \tag{10.15b}$$

where Θ is the step function. We first consider the real part of the polarisation tensor. From Eq. (10.15), and in the limit $k_0 \to 0$, we infer that the longitudinal gauge bosons acquire a thermal mass, the Debye mass, while the transverse ones remain massless. The (square of the) Debye mass is given by

$$m_D^2 = -\frac{g^2}{\pi^2}\int_0^\infty dp\,p^2\frac{df^{\text{eq.}}}{d\epsilon_p} \equiv M^2\,I_0\left(\frac{\Delta}{T},\frac{T}{\mu}\right)\,. \tag{10.16}$$

For convenience, we have factored-out the Debye mass M of the ultradegenerate plasma in the normal phase, $M^2 \equiv g^2\mu^2/\pi^2$. The dimensionless functions

$$I_n\left(\frac{\Delta}{T},\frac{T}{\mu}\right) = -\frac{1}{\mu^2}\int_0^\infty dp\,p^2\frac{df^{\text{eq.}}}{d\epsilon_p}v_p^n \tag{10.17}$$

obey $I_n \geq I_{n+1} > 0$ for all n due to $v_p \leq 1$. Equality holds for vanishing gap. For the physically relevant range of parameters $T < \Delta \ll \mu$, the functions I_n are $\ll 1$. In particular, it is easy to see that $I_n(\infty,0) = 0$: there is no Debye screening for the $SU(2)$ gluons at $T = 0$ in the superconducting phase. In the limit where $\Delta/T \gg 1$, and to leading order in $T/\mu \ll 1$, the Debye mass reduces to

$$m_D^2 = M^2\sqrt{2\pi\frac{\Delta}{T}}\exp(-\Delta/T)\,. \tag{10.18}$$

The dispersion relations for the longitudinal and transverse gluons follow from the poles of the corresponding propagators,

$$\epsilon|\mathbf{k}|^2 - \text{Re}\,\Pi_L(k_0,\mathbf{k})|_{k_0=\omega_L(\mathbf{k})} = 0\,, \tag{10.19a}$$

$$\epsilon k_0^2 - \frac{1}{\lambda}|\mathbf{k}|^2 + \text{Re}\,\Pi_T(k_0,\mathbf{k})|_{k_0=\omega_T(\mathbf{k})} = 0\,. \tag{10.19b}$$

Here, the terms containing $\Pi_{L,T}$ are due to the fermionic quasi-particles, while the terms containing ϵ and λ are the leading order contributions from the effective theory at $T = 0$, introduced in Eq. (10.3). At vanishing temperature, $\Pi_{T,L} = 0$, and only the transverse gluon propagates, but with velocity $v = 1/\sqrt{\epsilon\lambda} \ll 1$. At non-vanishing temperature, a plasmon or longitudinal mode also propagates. Neglecting higher order corrections in k_0 to the polarisation tensor at $T = 0$, the plasma frequency ω_{pl} follows from Eq. (10.19) as

$$\omega_{\mathrm{pl}}^2 = \frac{1}{3\epsilon} M^2 I_2 \left(\frac{\Delta}{T}, \frac{T}{\mu} \right) . \tag{10.20}$$

For generic external momenta the dispersion relations can only be solved numerically. In turn, if the spatial momenta are much smaller than the plasma frequency $|\mathbf{k}| \ll \omega_{\mathrm{pl}}$, solutions to Eq. (10.19) can be expanded in powers of $|\mathbf{k}|^2/\omega_{\mathrm{pl}}^2$ as

$$\omega_L^2(\mathbf{k}) = \omega_{\mathrm{pl}}^2 \left[1 + \frac{3}{5} \frac{I_4}{I_2} \frac{|\mathbf{k}|^2}{\omega_{\mathrm{pl}}^2} + \mathcal{O}(\frac{|\mathbf{k}|^4}{\omega_{\mathrm{pl}}^4}) \right] , \tag{10.21a}$$

$$\omega_T^2(\mathbf{k}) = \omega_{\mathrm{pl}}^2 \left[1 + \left(\frac{1}{\epsilon\lambda} + \frac{1}{5} \frac{I_4}{I_2} \right) \frac{|\mathbf{k}|^2}{\omega_{\mathrm{pl}}^2} + \mathcal{O}(\frac{|\mathbf{k}|^4}{\omega_{\mathrm{pl}}^4}) \right] . \tag{10.21b}$$

Apart from the fact that the $T = 0$ transverse mode does not propagate at the speed of light in vacuum, the ratios of the functions I_n measure the departure of the dispersion relations of the gluons in the 2SC phase with respect to the unbroken phase (see Eq. (7.32)). The quantity $v_*^2 = I_4/I_2$ has the intuitive interpretation of a mean velocity squared of the quasiparticles of the system. An approximate form of the HSL polarisation tensor Eq. (10.15) could be given in terms of this mean velocity (see [47] for the use of a similar approximation).

Let us now consider the imaginary part of Eq. (10.15), which describes Landau damping. Since $v_p \leq 1$, we conclude that Landau damping only occurs for $k_0^2 \leq |\mathbf{k}|^2$. Hence, plasmon and transverse gluon excitations are stable as long as $\omega_{L,T}(\mathbf{k}) > |\mathbf{k}|$. Furthermore, we notice that the imaginary part of Eq. (10.15) is logarithmically divergent: the quasiparticle velocity vanishes for momenta close to the Fermi surface, which is an immediate consequence of the presence of a gap, cf. Eq. (10.5). This divergence does not appear in the real part, because the logarithm acts as a regulator for the $1/v_p$ factor. To leading order in T/μ, and in the region of small frequencies $k_0^2 \ll |\mathbf{k}|^2$, we find at logarithmic accuracy, and for all values of Δ/T,

$$\mathrm{Im}\,\Pi_L(k_0, \mathbf{k}) = -2\pi M^2 \frac{k_0}{|\mathbf{k}|} \frac{\Delta}{T} \frac{\ln\left(|\mathbf{k}|/k_0\right)}{(e^{\Delta/T} + 1)(e^{-\Delta/T} + 1)} , \tag{10.22a}$$

$$\mathrm{Im}\,\Pi_T(k_0, \mathbf{k}) = \pi M^2 \frac{k_0}{|\mathbf{k}|} \left[\frac{1}{e^{\Delta/T} + 1} - 2 \frac{k_0^2}{|\mathbf{k}|^2} \frac{\Delta}{T} \frac{\ln |\mathbf{k}|/k_0}{(e^{\Delta/T} + 1)(e^{-\Delta/T} + 1)} \right] . \tag{10.22b}$$

For small frequencies, Landau damping is dominated by the logarithmic terms, which are proportional to the gap. Once the gap vanishes, subleading terms in $\frac{\Delta}{T}$, not displayed in Eq. (10.22), take over and reduce ImΠ to known expressions for the normal phase.

Finally, we explain how the polarisation tensor, as obtained within the present transport theory, matches the computation of $\Pi^{\mu\nu}$ for external momenta $k_0, |\mathbf{k}| \ll \Delta$ to one-loop order from quantum field theory. The one-loop gluon self-energy for a two-flavour colour superconductor has been computed by Rischke, and the polarisation tensor for the unbroken $SU(2)$ subgroup is given in Eq. (99) of [129]. It contains contributions from particle-particle, particle-antiparticle and antiparticle-antiparticle excitations. The particle-antiparticle contribution to Π^{00} and Π^{0i} at low external momenta, and the antiparticle-antiparticle excitations are subleading. The particle-particle contributions divide into two types. The first ones have poles for gluonic frequencies $k_0 = \pm (\epsilon_p + \epsilon_{p-k})$ and an imaginary part once k_0 exceeds the Cooper pair binding energy 2Δ. These terms are related to the formation or breaking of a Cooper pair, and suppressed for low external gluon momenta. The second type of terms, only non-vanishing for $T \neq 0$, have poles at $k_0 = \pm (\epsilon_p - \epsilon_{p-k})$. For $|\mathbf{k}| \ll \Delta$ we approximate it by $k_0 \approx \pm \frac{\partial \epsilon_p}{\partial \mathbf{p}} \cdot \mathbf{k}$. The prefactor, a difference of thermal distribution functions, is approximated by $f^{\mathrm{eq}}(\epsilon_p) - f^{\mathrm{eq}}(\epsilon_{p-k}) \approx \frac{\partial \epsilon_p}{\partial \mathbf{p}} \cdot \mathbf{k} \frac{df^{\mathrm{eq}}}{d\epsilon_p}$. After simple algebraic manipulations we finally end up with the result given above. We conclude that this part of the one-loop polarisation tensor describes the collisionless dynamics of thermal quasiparticles for a two-flavour colour superconductor. The same type of approximations can be carried out for Π^{ij} to one-loop order. There, apart from the HSL contributions, additional terms arise due to particle-particle and particle-antiparticle excitations, cf. Eq. (112) of [129]. We have not evaluated these terms explicitly. However, we expect them to be subleading or vanishing, as otherwise the Ward identity $k_\mu \Pi^{\mu\nu}_{ab}(k) = 0$ is violated. For $T = 0$, this has been confirmed in [129].

E. Discussion

We have introduced a transport equation for the gapped quarks of two-flavour colour superconductors. Its simple structure is based on the quasiparticle behaviour of the thermal excitations of the condensate, in consistency with the underlying quantum field theory. We have constructed a low temperature infrared effective theory of the superconductor. To leading order, we found Landau damping, and Debye screening of the chromo-electric fields. Beyond leading order, chromo-magnetic fields are damped because they scatter with the quasiparticles. The damping rate is related to the colour conductivity. It should be possible to compute the rate from the transport equation (10.6), amended by the relevant collision term. The latter can be derived, for example, using the methods discussed in the preceeding sections.

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We have neither discussed the transport equations for gapless quarks nor for the η' meson, because they do not carry $SU(2)$ charges. However, their excitations are light compared to the gapped quasiparticles, and dominant for other transport properties such as thermal and electrical conductivities or shear viscosity. The corresponding set of transport equations will be discussed elsewhere.

It would be very interesting to study the transport equations in a three-flavour colour superconductor [6]. For $N_f = 3$ the quark-quark condensate breaks the $SU(3)$ gauge group completely, as well as some global flavour symmetries. Transport phenomena should then be dominated by the Goldstone modes associated to the breaking of the global symmetries. The corresponding transport equations will be substantially different for the two and three flavour case.

XI. SUMMARY AND OUTLOOK

We have reviewed a new approach to the transport theory of non-Abelian plasmas. The formalism relies on a semi-classical approximation and considers, on the microscopic level, a system of classical *coloured* point particles interacting through classical non-Abelian fields. It is assumed that the typical length scales of the particle-like degrees of freedom are much smaller than those associated to the classical non-Abelian fields. This scale separation is at the root of the present formalism. The inclusion of stochastic fluctuations due to the particles is also of crucial importance, as well as the ensemble average in phase space, which takes the colour charges as dynamical variables into account. On the macroscopic level, the formalism results in a set of *effective* transport equations for the quasi-particle distribution function, the mean gauge fields, and their fluctuations. The formalism is consistent with the non-Abelian gauge symmetry.

Approximations have to be employed in order to obtain, or to solve, the effective transport equations. For the integrating-out of fluctuations, systematic expansions schemes, consistent with the non-Abelian gauge symmetry, have been worked out. Ultimately, the procedure corresponds to the derivation of collision integrals, noise sources and fluctuation-induced currents for effective transport equations. The compatibility of the approach with the fluctuation-dissipation theorem was established as well. Of course, reliable physical predictions based on the formalism are only as good as the approximations inherent to the approach. This concerns most notably the quasiparticle picture and the separation of scales. However, for a weakly coupled plasma close to equilibrium, these assumptions are satisfied. Although the Wong particle picture relies on a high dimensional representation for the colour charges, observables involving the quadratic or cubic Casimir are reproduced correctly to leading order.

Interesting applications of the formalism concern hot and weakly coupled plasmas close to thermal equilibrium. We have reviewed how the seminal hard thermal loop effective theory is deduced, based on the simplest approximation compatible with gauge invariance and neglecting fluctuations. This step corresponds to the integrating-out of hard modes with $p \sim T$ to leading order in the gauge coupling. Further, the simplest approximation which includes the genuine effects due to fluctuations was shown to reproduce Bödeker's effective theory at leading logarithmic order. This corresponds to integrating-out the soft modes with $p \sim gT$ to leading logarithmic order. These applications exemplify the efficiency of the formalism. As an aside, we note that the effective theories for both classical and quantum plasmas are identical, except for the value of the Debye mass. It is intriguing that a simple semi-classical transport theory is able to correctly reproduce not only the dynamics of soft non-Abelian fields with momenta about the Debye mass, but as well the dynamics of the ultra-soft gluons at leading logarithmic order. These findings imply a link beyond one-loop between the present formalism and a full quantum field theoretical treatment.

A number of possible applications of the formalism to weakly coupled thermal plasmas are worth being mentioned. We have reviewed the computation of the colour conductivity to leading logarithmic order. In principle, it should be possible to extend the analysis to higher order by solving the dynamical equations for the fluctuations iteratively. A further important application concerns colourless excitations of the plasma. These are responsible for most of the bulk or hydrodynamical properties of the medium, described by transport coefficients such as viscosities, electrical or thermal conductivities. The main contributions to these transport coefficients arise from hard and soft degrees of freedom. A computation of transport coefficients within the present formalism is a feasable task, bearing in mind the efficiency of the formalism. Despite the fact that transport coefficients have already been obtained in the literature to leading logarithmic order, it is worthwhile to derive them from the present formalism, and to even extent the existing results to higher order.

More generally, the formalism leads to an equally good description of other physical systems, where the relevant thermal excitations can be described by quasiparticles, and typical length scales associated to the gauge fields are much larger than those of the quasiparticles. As an example, we have reviewed an application to the physics of dense quark matter in a colour superconducting phase with two massless quark flavours. Based on a semi-classical transport equation for fermionic quasiparticle excitations, we obtained the hard-superconducting loop effective action for the $SU(2)$ gauge fields. It describes the physics of Debye screening and Landau damping for the unbroken non-Abelian gauge fields in the presence of a condensate. It will be interesting to use this formalism for the study of transport coefficients in colour superconducting matter.

All applications of the formalism have been done for weakly coupled systems close to, or slightly out of, thermal equilibrium. It would be interesting to understand if a semi-classical description is viable for strongly coupled plasmas, or for plasmas fully out of equilibrium. A kinetic description of a plasma requires a small plasma parameter. For quantum plasmas, the gauge coupling and the plasma parameter are deeply linked, since a small gauge coupling implies a small plasma parameter, and vice versa. For classical plasmas, the plasma parameter remains an independent parameter and can be made small even for large gauge couplings. This observation may lead to a kinetic description of strongly coupled classical plasmas. In principle, the formalism also applies to plasmas out of equilibrium, simply because the ensemble average does not rely on whether the system is in equilibrium or not. Hence, the formalism provides an interesting starting point for applications to out-of-equilibrium plasmas or to the physics of heavy ion collisions.

ACKNOWLEDGMENTS

A. SAMPLE COMPUTATION FOR CORRELATORS

In this Appendix we explain in more detail how to compute the collision integral $\langle \xi \rangle$ which appears in the transport equation Eq. (8.3a). We write the collision integral in momentum space

$$\langle \xi_a^\rho(k, \mathbf{v}) \rangle = -g \, f_{abc} \int \frac{d^4 p}{(2\pi)^4} \, \langle v^\mu a_{\mu,b}(p) \, \delta \mathcal{J}_c^\rho(k-p, \mathbf{v}) \rangle \, . \tag{A1}$$

One first has to solve the dynamical equations for a and $\delta \mathcal{J}$, in order to express them in terms of the initial conditions. This program has been carried out in Section VIII B, where a and $\delta \mathcal{J}$ have been solved in a series in $g\bar{A}$ and $g\bar{J}$. The correlators are computed analogously.

We begin with the zero order contribution. It is easy to show that the first term in the series vanishes. This is because $\langle a_a^{(0,0)} a_b^{(0,0)} \rangle \propto \delta_{ab}$, and this correlator is contracted with the antisymmetric tensor f_{abc}.

For the first correction in $g\bar{J}$, one needs to evaluate the correlators $\langle a_b^{(0,0)} \delta \mathcal{J}_c^{\rho,(0,1)} \rangle$, and $\langle a_b^{(0,1)} \delta \mathcal{J}_c^{\rho,(0,0)} \rangle$. We will illustrate how to compute the contribution of the first term. The second one is computed in a similar way. We need to evaluate

$$- g \, f_{abc} \int \frac{d^4 p}{(2\pi)^4} \, \langle v^\mu a_{\mu,b}^{(0,0)}(p) \, \delta \mathcal{J}_c^{\rho,(0,1)}(k-p, \mathbf{v}) \rangle \, , \tag{A2}$$

where

$$\delta \mathcal{J}_c^{\rho,(0,1)}(k-p, \mathbf{v}) = -g f_{cde} \frac{1}{-i(k-p) \cdot v} \int \frac{d^4 q}{(2\pi)^4} \, v^\nu a_{\nu,d}^{(0,0)}(q) \bar{\mathcal{J}}_e^\rho(k-p-q, v) \, . \tag{A3}$$

Therefore, we have to evaluate the correlator $\left\langle a_{\mu,b}^{(0,0)}(q) \, a_{\nu,d}^{(0,0)}(p) \right\rangle$, which has been computed in Eq. (8.27) for the transverse components of the gauge fields. These are the ones which give the leading order contribution to Eq. (A2). Using the values of Eq. (8.27), and the $SU(N)$ relation $f_{abc} f_{cbe} = -N \delta_{ac}$, we find

$$- g^4 N B_c C_2 \int \frac{d^4 p}{(2\pi)^4} \int \frac{d\Omega_{\mathbf{v}'}}{4\pi} \left| \frac{\mathbf{v}_i P_{ik}^T(p) \mathbf{v}_k'}{p^2 + \Pi_T} \right|^2 \delta(p \cdot v') \frac{\bar{\mathcal{J}}_a^\rho(k, \mathbf{v})}{-i(k-p) \cdot v} \, . \tag{A4}$$

Using retarded boundary conditions, we split

$$\frac{1}{-i(k-p) \cdot v} = i\mathcal{P} \left(\frac{1}{-i(k-p) \cdot v} \right) + \pi \delta \left((k-p) \cdot v \right). \tag{A5}$$

The term which goes with the principal value will be neglected, because it gives a contribution which is damped at asymptotically large times [99]. In the argument of the δ-function, we neglect ultrasoft momenta in front of the soft ones ($k \ll p$). We thus end up with

$$- \frac{g^4 N B_c C_2}{4\pi} v^\rho \int \frac{d^4 p}{(2\pi)^4} \int \frac{d\Omega_{\mathbf{v}'}}{4\pi} \left| \frac{\mathbf{v}_i P_{ik}^T(p) \mathbf{v}'_k}{p^2 + \Pi_T} \right|^2 (2\pi)\, \delta(p \cdot v')(2\pi)\, \delta(p \cdot v) \bar{\mathcal{J}}_a^0(k, \mathbf{v}) \,. \qquad \text{(A6)}$$

This is the first term in Eq. (8.39). The second term is computed in a similar way, after evaluating the $\langle a_b^{(0,1)} \delta \mathcal{J}_c^{\rho,(0,0)} \rangle$ correlator. Notice that in order to find a local collision integral, the separation of scales soft and ultrasoft is a key ingredient.

B. CORRELATORS OF WIGNER FUNCTIONS IN THE CLASSICAL LIMIT

We have restricted our study to the use of classical and semi-classical methods applied to non-Abelian plasmas. In this Appendix we present a formal justification of the use of the quantum correlators given in Eqs. (3.14) – (3.15). We follow here the arguments and reasoning of [142].

To simplify the analysis we will only consider the Abelian case. The central quantity for a classical transport theory is the one-particle distribution function f. This function is split into its mean value and the fluctuations around it as

$$f(x, p) = \bar{f}(x, p) + \delta f(x, p) \,. \qquad \text{(B1)}$$

We consider the case where the system is homogeneous, thus $\bar{f}$ does not depend on x^μ. We also neglect the effect of interactions. Then the fluctuations obey the equation

$$v^\mu \partial_\mu \, \delta f(x, p) = 0 \,, \qquad \text{(B2)}$$

where $v^\mu = (1, \mathbf{v})$ is the particle four velocity. The correlation function of fluctuations was deduced in Section III, when in the Abelian case, the phase space variables are just $z = (\mathbf{x}, \mathbf{p})$.

Now we turn to the quantum generalisation of the previous formalism. The second quantisation representation of $f(p)$ is the particle occupation number averaged over an statistical ensemble. In a quantum formulation, the occupation number is given by the operator $\hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}}$ averaged over the vacuum state, where $\hat{a}_{\mathbf{p}}^\dagger$ is the creation operator of a particle with momentum $\mathbf{p}$, and $\hat{a}_{\mathbf{p}}$ is the annihilation operator of a particle with momentum $\mathbf{p}$. Therefore, we can identify

$$\phi_{\mathbf{p}} = \int d^3 p' \, \langle \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}'} \rangle \,, \qquad \langle \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}'} \rangle = \phi_{\mathbf{p}} \, \delta^{(3)}(\mathbf{p} - \mathbf{p}') \,. \qquad \text{(B3)}$$

It is useful to introduce the operator

$$\hat{\psi}_{\mathbf{p}}^{(0)}(t) = \hat{a}_{\mathbf{p}} \, e^{-i E_p t} \,, \qquad \text{(B4)}$$

where E_p is the particle energy. We thus have

$$\phi_{\mathbf{p}} = \int d^3 p' \langle \hat{\psi}_{\mathbf{p}}^{(0)\dagger} \hat{\psi}_{\mathbf{p'}}^{(0)} \rangle \ . \tag{B5}$$

The quantum analogue of the classical distribution function is the Wigner operator, defined as

$$\langle \hat{f}(x, p) \rangle = \int d^4 v \, e^{-ip \cdot v} \langle \psi^\dagger(x + \frac{1}{2}v)\psi(x - \frac{1}{2}v) \rangle \tag{B6}$$

We will work with the Fourier transform of the Wigner operator,

$$\hat{f}_{\mathbf{p},\mathbf{k}}^{(0)}(t) = \hat{\psi}_{\mathbf{p}-\mathbf{k}/2}^{(0)\dagger}(t)\hat{\psi}_{\mathbf{p}+\mathbf{k}/2}^{(0)}(t) \ . \tag{B7}$$

One can define the fluctuation operator

$$\delta\hat{f}_{\mathbf{p},\mathbf{k}}^{(0)}(t) = \hat{f}_{\mathbf{p},\mathbf{k}}^{(0)}(t) - \langle \hat{f}_{\mathbf{p},\mathbf{k}}^{(0)}(t) \rangle = \hat{\psi}_{\mathbf{p}-\mathbf{k}/2}^{(0)\dagger}(t)\hat{\psi}_{\mathbf{p}+\mathbf{k}/2}^{(0)}(t) - \langle \hat{\psi}_{\mathbf{p}-\mathbf{k}/2}^{(0)\dagger}(t)\hat{\psi}_{\mathbf{p}+\mathbf{k}/2}^{(0)}(t) \rangle \ . \tag{B8}$$

According to the definition (B4), this operator obeys

$$\frac{\partial}{\partial t}\delta\hat{f}_{\mathbf{p},\mathbf{k}}^{(0)}(t) + i\left(E_{p+k/2} - E_{p-k/2}\right)\delta\hat{f}_{\mathbf{p},\mathbf{k}}^{(0)}(t) = 0 \ . \tag{B9}$$

For $\mathbf{k} \ll \mathbf{p}$ we have

$$E_{p+k/2} - E_{p-k/2} \simeq \mathbf{k} \cdot \frac{\partial E_p}{\partial \mathbf{p}} = \mathbf{k} \cdot \mathbf{v} \ , \tag{B10}$$

and then Eq. (B9) agrees with Eq. (B2). The solution of Eq. (B9) is

$$\delta\hat{f}_{\mathbf{p},\mathbf{k}}^{(0)}(t) = \delta\hat{f}_{\mathbf{p},\mathbf{k}}^{(0)}(0)\exp\left\{-i\left(E_{p+k/2} - E_{p-k/2}\right)t\right\} \ , \tag{B11}$$

$$\delta\hat{f}_{\mathbf{p},\mathbf{k}}^{(0)}(0) = \hat{a}_{\mathbf{p}-\mathbf{k}/2}^\dagger\hat{a}_{\mathbf{p'}+\mathbf{k}/2} - \langle \hat{a}_{\mathbf{p}-\mathbf{k}/2}^\dagger\hat{a}_{\mathbf{p'}+\mathbf{k}/2} \rangle \ , \tag{B12}$$

$$\delta\hat{f}_{\mathbf{p},\mathbf{k},\omega}^{(0)} = \delta\hat{f}_{\mathbf{p},\mathbf{k}}^{(0)}(0)\delta\left(\omega - E_{p+k/2} + E_{p-k/2}\right) \ . \tag{B13}$$

One can now evaluate the correlator of fluctuation operators. One finds

$$\langle \delta\hat{f}_{\mathbf{p},\mathbf{k},\omega}^{(0)}\delta\hat{f}_{\mathbf{p'},\mathbf{k'},\omega'}^{(0)} \rangle = \delta\left(\omega - E_{p+k/2} + E_{p-k/2}\right)\delta\left(\omega' - E_{p'+k'/2} + E_{p'-k'/2}\right) \tag{B14}$$

$$\times \left(\langle \hat{a}_{\mathbf{p}-\mathbf{k}/2}^\dagger\hat{a}_{\mathbf{p}+\mathbf{k}/2}\hat{a}_{\mathbf{p'}-\mathbf{k'}/2}^\dagger\hat{a}_{\mathbf{p'}+\mathbf{k'}/2} \rangle - \langle \hat{a}_{\mathbf{p}-\mathbf{k}/2}^\dagger\hat{a}_{\mathbf{p}+\mathbf{k}/2} \rangle\langle \hat{a}_{\mathbf{p'}-\mathbf{k'}/2}^\dagger\hat{a}_{\mathbf{p'}+\mathbf{k'}/2} \rangle\right) \ .$$

If one decomposes the average of four operators into products of the possible averaged values of pairs of operators, and furthermore one uses the commutation/anticommutation relations of the creation and annihilation operators, one then arrives to

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$$\langle \delta \hat{f}^{(0)}_{\mathbf{p},\mathbf{k},\omega} \delta \hat{f}^{(0)}_{\mathbf{p}',\mathbf{k}',\omega'} \rangle = \phi_{\mathbf{p}-\mathbf{k}/2} \left(1 - \phi_{\mathbf{p}+\mathbf{k}/2} \right) \delta^{(3)}(\mathbf{p} - \mathbf{p}') \delta^{(4)}(k + k') \delta \left(\omega - E_{p+k/2} + E_{p-k/2} \right) \;, \quad \text{(B15)}$$

$$\langle \delta \hat{f}^{(0)}_{\mathbf{p},\mathbf{k},\omega} \delta \hat{f}^{(0)}_{\mathbf{p}',\mathbf{k}',\omega'} \rangle = \phi_{\mathbf{p}-\mathbf{k}/2} \left(1 + \phi_{\mathbf{p}+\mathbf{k}/2} \right) \delta^{(3)}(\mathbf{p} - \mathbf{p}') \delta^{(4)}(k + k') \delta \left(\omega - E_{p+k/2} + E_{p-k/2} \right) \;, \quad \text{(B16)}$$

where $k = (\omega, \mathbf{k})$, and Eq. (B15) refers to the correlators for particles obeying bosonic statistics, while Eq. (B16) refers to particles obeying fermionic statistics. Note that in the limit $\mathbf{k} \ll \mathbf{p}$, the above correlators reduce to

$$\langle \delta \hat{f}^{(0)}_{\mathbf{p},\mathbf{k},\omega} \delta \hat{f}^{(0)}_{\mathbf{p}',\mathbf{k}',\omega'} \rangle_{B/F} = \phi_{\mathbf{p}}^{B/F} \left(1 \mp \phi_{\mathbf{p}}^{B/F} \right) \delta^{(3)}(\mathbf{p} - \mathbf{p}') \delta^{(4)}(k + k') \delta \left(\omega - \mathbf{k} \cdot \mathbf{v} \right) \;. \quad \text{(B17)}$$

This expression corresponds to the Fourier transform of $\langle \delta \hat{f}^{(0)}(x, p) \delta \hat{f}^{(0)}(x', p) \rangle_{t=t' \neq 0}$. It can be deduced from the initial time correlators given in Eqs. (3.15) and (3.14), in the Abelian limit, if the dynamical evolution of the fluctuations is given by Eq. (B2). Furthermore, for low occupation numbers, $\phi_{\mathbf{p}} \ll 1$, one recovers the corresponding classical limit. Note that the factors of (2π) of difference between Eq. (B17) and Eqs. (3.14) and (3.15) can be fixed by choosing the proper normalisation of the momentum measure.

The considerations given above are only valid for free particles. Modifications are necessary in order to include the effects of interactions, in which case the time dependence of the operator $\hat{\psi}_{\mathbf{p}}(t)$ will be a more complicated than in Eq. (B5). Furthermore, it should be kept in mind that to describe the system of relativistic particles in a covariant way, one should introduce positive and negative energy states. Internal degrees of freedom, such as spin, are to be introduced as internal indices as well. The above discussion has also been restricted to the Abelian case. It provides a formal justification of why semi-classical methods can be used to study the physics of specific momentum scales in the plasma. With the same tools, we could study the non-Abelian case as well, only by enlarging the phase-space of the particles. Additional technical difficulties are then encountered.

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