Braneworld flow equations

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We generalize the flow equations approach to inflationary model building to the Randall–Sundrum Type II braneworld scenario. As the flow equations are quite insensitive to the expansion dynamics, we find results similar to, though not identical to, those found in the standard cosmology.

I.   INTRODUCTION

The flow equations approach, pioneered by Hoffman and Turner [1] and by Kinney [2] (see also Refs. [3, 4]), is a means of generating large numbers of inflation models via a random process, permitting for example that ensemble to be compared with observational results as done by the Wilkinson Microwave Anisotropy Probe team [4]. It has however been pointed out by Liddle [5] that the flow equations algorithm is actually rather insensitive to the equations governing inflationary dynamics. In particular, the Friedmann equation is not needed to determine the flow equation trajectories in the space of slow-roll parameters; its only role is to measure the amount of expansion taking place along those trajectories.

In this *Brief Report*, we make an explicit study of the effect of modifying the Friedmann equation by considering the flow equations in the simplest braneworld inflation scenario, based on the Randall–Sundrum Type II model [6].

II.   STANDARD COSMOLOGY FLOW EQUATIONS

The flow equations are a set of differential equations linking a set of slow-roll parameters defined from the Hubble parameter H. In the standard cosmology, following the notation of Kinney [2], the parameters can be defined as

$$\epsilon(\phi) \equiv \frac{m_{\mathrm{Pl}}^2}{4\pi} \left(\frac{H'(\phi)}{H(\phi)} \right)^2 ; \tag{1}$$

$$^{\ell}\lambda_{\mathrm{H}} \equiv \left(\frac{m_{\mathrm{Pl}}^2}{4\pi} \right)^{\ell} \frac{(H')^{\ell-1}}{H^{\ell}} \frac{d^{(\ell+1)}H}{d\phi^{(\ell+1)}} \quad ; \quad \ell \geq 1, \tag{2}$$

where primes are derivatives with respect to the scalar field ϕ. Using the relation

$$\frac{d}{dN} = \frac{m_{\mathrm{Pl}}^2}{4\pi} \frac{H'}{H} \frac{d}{d\phi}, \tag{3}$$

where we define the number of *e*-foldings N as *decreasing* with increasing time, yields the flow equations

$$\frac{d\epsilon}{dN} = \epsilon(\sigma + 2\epsilon);$$

$$\frac{d\sigma}{dN} = -5\epsilon\sigma - 12\epsilon^2 + 2(^2\lambda_{\mathrm{H}}); \tag{4}$$

$$\frac{d(^{\ell}\lambda_{\mathrm{H}})}{dN} = \left[\frac{\ell-1}{2}\sigma + (\ell-2)\epsilon \right] (^{\ell}\lambda_{\mathrm{H}}) + {}^{\ell+1}\lambda_{\mathrm{H}} ; \ \ell \geq 2,$$

where $\sigma \equiv 2(^1\lambda_{\mathrm{H}}) - 4\epsilon$ is a convenient definition.

As pointed out in Ref. [5], these equations actually have limited dynamical input from inflation, since the above derivation has been made without reference to the Friedmann equation. Indeed, if written in the form $d/d\phi$ they are a set of identities true for any function $H(\phi)$, and the reparametrization to d/dN modifies only the measure along the trajectories, not the trajectories themselves. In that light it seems surprising that they can say much about inflation at all, but it turns out that the flow equations can be viewed as a (rather complicated) algorithm for generating functions $\epsilon(\phi)$ which have a suitable form to be interpreted as inflationary models [5].

Ref. [5] implied that the flow equation predictions ought to be little changed by moving to the braneworld, although this modifies the Friedmann equation. However this statement needs explicit justification, because although the trajectories are unaffected, there are several changes which affect the predictions: the measure of *e*-foldings along the trajectories changes, the endpoints of the trajectories change, and the equations relating the slow-roll parameters to the observables change. In this short article we investigate these effects, restricting ourselves to the high-energy regime of the Randall–Sundrum Type II model [6].

III.   BRANEWORLD FLOW EQUATIONS

We follow the notation of Ref. [7]. In the Randall–Sundrum Type II braneworld model [6], the Friedmann equation in the high-energy regime can be written as

$$H = \frac{4\pi}{3M_5^3}\rho, \tag{5}$$

where M_5^3 is the five-dimensional Planck mass, related to the brane tension λ by $M_5^3 \equiv (4\pi\lambda/3)^{1/2}m_{\mathrm{Pl}}$. The scalar wave equation is unchanged, and a useful expression is

$$\dot{\phi} = -\frac{M_5^3}{4\pi}\frac{H'}{H}. \tag{6}$$

Although they could be used, the standard cosmology definitions of the slow-roll parameters, Eqs. (1) and (2), are actually not very convenient in the braneworld scenario, in particular because $\epsilon = 1$ is no longer the condition to end inflation. Instead, following Ref. [7], we define new slow-roll parameters relevant to the high-energy regime as follows:

$$\epsilon_{\rm H} = \frac{M_5^3}{4\pi} \frac{H'^2}{H^3} \,; \tag{7}$$

$$\eta_{\rm H} + \epsilon_{\rm H} = \frac{M_5^3}{4\pi} \frac{H''}{H^2} \,. \tag{8}$$

By analogy to the standard cosmology case, we define the higher-order slow-roll parameters as

$$^{\ell}\lambda_{\rm H} \equiv \left(\frac{M_5^3}{4\pi}\right)^{\ell} \frac{(H')^{\ell-1}}{H^{2\ell}} \frac{d^{(\ell+1)}H}{d\phi^{(\ell+1)}} \quad ; \quad \ell \geq 1\,, \tag{9}$$

where one can identify $^1\lambda_1 \equiv \eta_{\rm H} + \epsilon_{\rm H}$.

From Eq. (6), the relation between ϕ and N, measuring the length along the trajectories, is changed to

$$\frac{d}{dN} = \frac{M_5^3}{4\pi} \frac{H'}{H^2} \frac{d}{d\phi}\,. \tag{10}$$

From these definitions, and taking the convention $\dot{\phi} > 0$, we find a set of flow equations

$$\frac{d\epsilon}{d\phi} = \frac{H'}{H}(2\eta_{\rm H} - \epsilon_{\rm H})\,; \tag{11}$$

$$\frac{d\eta_{\rm H}}{d\phi} = \frac{H'}{H}\left[\frac{^2\lambda_{\rm H}}{\epsilon_{\rm H}} - 4\eta_{\rm H} - \epsilon_{\rm H}\right]\,; \tag{12}$$

$$\frac{d(^{\ell}\lambda_{\rm H})}{d\phi} = \frac{1}{\epsilon_{\rm H}} \frac{H'}{H} \left\{ [(\ell-1)\eta_{\rm H} - (\ell+1)\epsilon_{\rm H}]\,(^{\ell}\lambda_{\rm H}) \right. \tag{13}$$
$$\left. +^{(\ell+1)}\lambda_{\rm H} \right\} \,; \quad \ell \geq 2\,,$$

which leads to the following set of flow equations for the braneworld

$$\frac{d\epsilon_{\rm H}}{dN} = \epsilon_{\rm H}(\sigma_{\rm H} + 3\epsilon_{\rm H})\,; \tag{14}$$

$$\frac{d\sigma_{\rm H}}{dN} = -8\epsilon_{\rm H}\sigma_{\rm H} - 30\epsilon_{\rm H}^2 + 2(^2\lambda_{\rm H})\,; \tag{15}$$

$$\frac{d(^{\ell}\lambda_{\rm H})}{dN} = \left[\frac{\ell-1}{2}\sigma_{\rm H} + (\ell-3)\epsilon_{\rm H}\right](^{\ell}\lambda_{\rm H}) \tag{16}$$
$$+^{(\ell+1)}\lambda_{\rm H}; \; \ell \geq 2\,,$$

where $\sigma_{\rm H} \equiv 2\eta_{\rm H} - 4\epsilon_{\rm H}$.

Note that the braneworld slow-roll parameters were purposefully defined [7] so that the end of inflation is at $\epsilon_{\rm H} = 1$, and so that standard expression for the spectral index n still applies; however the expression for the tensor-to-scalar ratio r is modified from the usual $r \simeq \epsilon$.[1]

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[1] The recent literature has two different definitions of r in common use, the other one being 16 times this one. We use this convention to follow Kinney [2].
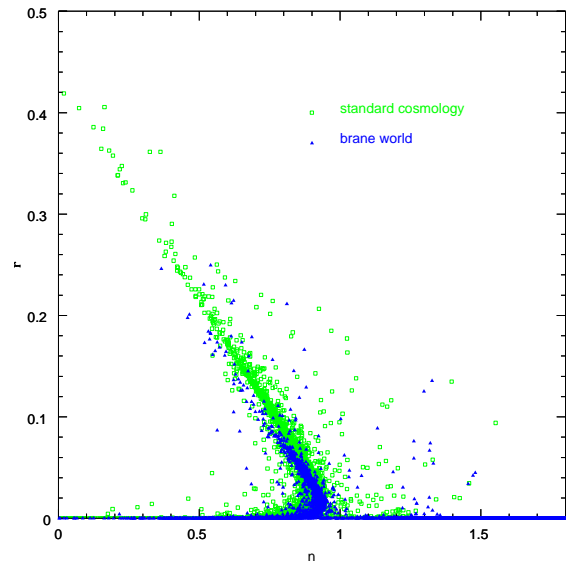

FIG. 1: Distribution of observables for the standard and braneworld cosmologies.

The observable quantities at first-order are given by [8]

$$r \simeq \frac{3}{2}\epsilon_{\rm H}\,; \tag{17}$$

$$n \simeq 1 + \sigma_{\rm H}\,; \tag{18}$$

$$\frac{dn}{d\ln k} \simeq 30\epsilon_{\rm H}^2 + 8\epsilon_{\rm H}\sigma_{\rm H} - 2(^2\lambda_{\rm H})\,. \tag{19}$$

Following Ref. [2], we can analyze the fixed points where all the derivatives vanish, for which the conditions are

$$\epsilon_{\rm H} =^{\ell}\lambda_{\rm H} = 0, \quad \sigma_{\rm H} = {\rm const}\,. \tag{20}$$

As in the standard cosmology, these correspond to $r = 0$ and are stable for $n > 1$. The $n < 1$ branch is stable for integration backwards in time.

We solve the flow equations following the method of Kinney [2], both for the standard cosmology where we verify his results and for the system of braneworld flow equations written above. We consider 60,000 initial conditions drawn from the ranges Kinney uses (taking the same ranges also for the braneworld slow-roll parameters). The results are shown in Fig. 1, with observables computed to first-order in slow-roll.

For both models, the majority of the points lie effectively on the $r = 0$ axis, and in addition we see in each case the now-familiar swathe of points following a tight diagonal locus to large values of r and $1 - n$ [1, 2], plus some other scattered points. The two distributions are extremely similar, though the swathe for the braneworld is slightly below that of the standard cosmology. We conclude therefore that the change in the dynamical equations does not significantly affect the distribution of points in the space of observables.

IV. CONCLUSIONS

We have modified the flow equations approach to implement it in the high-energy regime of the Randall–Sundrum Type II braneworld cosmology. Although the flow equation trajectories are independent of the dynamical equation driving inflation, changes do occur because the measure of length (i.e. the number of e-foldings) along the trajectories changes, because the point corresponding to the end of inflation changes, and because the formulae giving the observables change. Nevertheless, we have shown that those effects are small and that the distribution of observables predicted by the braneworld flow equations is very similar to that of the standard cosmology.

Acknowledgments

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