Regularity, symmetry and asymptotic behaviour of solutions for some Stein-Weiss type integral systems

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REGULARITY, SYMMETRY AND ASYMPTOTIC BEHAVIOR OF SOLUTIONS FOR SOME STEIN-WEISS TYPE INTEGRAL SYSTEMS

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Abstract. We consider the positive solutions of some integral systems related to the static Hartree type equations:

$$\begin{cases} u(x) = \displaystyle\int_{\mathbb{R}^N} \frac{u^{p-1}(y)v(y)}{|x-y|^{N-\tau}}dy, & \text{in} \quad \mathbb{R}^N, \\ v(x) = \displaystyle\int_{\mathbb{R}^N} \frac{u^p(y)}{|x|^\alpha |x-y|^\mu |y|^\beta}dy, & \text{in} \quad \mathbb{R}^N, \end{cases}$$

where $N \geq 3$, $p \geq 1$, $0 < \mu, \tau < N$, $\alpha, \beta \geq 0$ and $0 < \alpha + \beta + \mu \leq N$. Firstly, assuming that the exponent p belongs to some suitable interval depending on the parameters μ, τ, α, β, we are able to prove some nonexistence results for the positive solution. Secondly, we also establish some qualitative results for the integrable solution of the system like regularity, symmetry and asymptotic behaviour. As a corollary, we deduce the corresponding results for the equivalent weighted Hartree type nonlocal equations. The results obtained in this paper generalize and complement the existing results in the literature.

1. Introduction and main results

In this paper, we study some integral systems related to the weighted Hartree type equations:

$$\begin{cases} u(x) = \displaystyle\int_{\mathbb{R}^N} \frac{u^{p-1}(y)v(y)}{|x-y|^{N-\tau}}dy, & x \in \mathbb{R}^N, \\ v(x) = \displaystyle\int_{\mathbb{R}^N} \frac{u^p(y)}{|x|^\alpha |x-y|^\mu |y|^\beta}dy, & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $N \geq 3$, $p \geq 1$, $0 < \mu, \tau < N$, $\alpha, \beta \geq 0$ and $0 < \alpha + \beta + \mu \leq N$. If $\tau = 2$, then the system in (1.1) is an equivalent integral form of the following nonlocal equation with Stein-Weiss type convolution part

$$-\Delta u = \left(\int_{\mathbb{R}^N} \frac{u^p(y)}{|x|^\alpha |x-y|^\mu |y|^\beta}dy \right) u^{p-1}, \quad x \in \mathbb{R}^N. \tag{1.2}$$

We have paid much attention to the classification of the solutions of equation (1.2) in recent years. On the one hand, for $\alpha = \beta = 0$, (1.2) is reduced to

$$-\Delta u = (|x|^{-\mu} * u^p)u^{p-1}, \quad x \in \mathbb{R}^N. \tag{1.3}$$

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The static Hartree type nonlocal equation (1.3), also called the Schrödinger-Newtown equation [23], has been studied in several contexts and plays an essential role in mathematics and physics. This kind of equation is related to the Hartree-Fock theory associated with the purely Coulomb Hamiltonian of electrons interacting with static nuclei [37]. There are many works on the topic of the Liouville theorem and qualitative analysis of the positive solutions for (1.3). If $N \geq 5$, $p = 2$ and $\mu = 4$, Miao, Xu and Zhao [41] established the existence of solutions of the Hartree equation. The appearance of the convolution term in (1.3) creates difficulties and, therefore, it is useful to consider its equivalent integral system involving the Riesz potential. Among them, Liu [35], Lei [33], Du and Yang [17] studied equation (1.3) with critical exponent $\frac{2N-\mu}{N-2}$ by analyzing the corresponding integral system. They discussed the regularity, symmetry and asymptotic behaviour of positive solutions of equation (1.3). Furthermore, Du and Yang [17], Giacomoni, Wei and Yang [26] obtained the nondegeneracy of the unique solutions for the equation when μ close to N or 0 respectively. In addition, Lei [34] was concerned about the existence and nonexistence of solution of (1.3) with $\mu = N - 2$ for three cases: subcritical case, critical case, and supercritical case, by means of the method of moving plane, the shooting method and the Pohozaev identity. On the other hand, For $\alpha = \beta \neq 0$, the authors in [16] proved the existence of positive ground state solutions for (1.2) in the critical case by a nonlocal version of concentration-compactness principle. They also considered the regularity of positive solutions and proved the symmetry of these solutions by the moving plane method in integral forms [6]. The readers may turn to [28, 34, 36, 40, 42] and the references therein for more backgrounds about the general Hartree type equations. The classification results have been successfully used to investigate the critical Choquard equation or the Hartree system arising from the BEC theory, see [1, 2, 16, 20–22] for example, where the authors had studied the existence, multiplicity and concentration of different types of solutions.

The purpose of the present paper is to investigate a general form of the integral system (1.1) and to study the properties of the solutions for equation (1.2) with $\alpha = \beta$, we will see how the interval of the exponent p that depending on the parameters τ, μ, α, β will affect the qualitative properties of the solutions. The main results of this paper are stated in the following theorems. In this paper, We denote $\gamma := \alpha + \beta + \mu$ for simplicity.

Firstly, we obtain a nonexistence result.

Theorem 1.1. *Let $N \geq 3$. If $1 \leq p \leq \frac{3N-\gamma-\tau}{2(N-\tau)}$, then system (1.1) has no positive solutions.*

The results in [8] show that if a solution of (1.6) satisfies

$$\int_{\mathbb{R}^N} u^{\frac{N(p-1)}{\tau}} dx < \infty,$$

then u is bounded, and it has higher regularity which implies $u \in L^{q_0}(\mathbb{R}^N)$ for any $q_0 > \frac{N}{N-\tau}$. It inspires us to call a solution of (1.1) an integrable solution provided $u \in L^{\frac{2N(p-1)}{N-\gamma+\tau}}(\mathbb{R}^N)$. In Section 2, we will investigate the higher regularity of an integrable solution u of (1.1).

Denote by

$$I_1 := \begin{cases} \left(\dfrac{3N-\gamma-\tau}{2(N-\tau)}, +\infty\right), & if \quad \tau+\gamma > N; \\ \left(\dfrac{2N}{N-\tau+\gamma}, \dfrac{2(N-\gamma)}{N-\tau-\gamma}\right) \cap \left(\dfrac{2N-\gamma-\tau}{\gamma-\tau}, +\infty\right), & if \quad \tau+\gamma \le N. \end{cases}$$

Then the following theorem shows the boundedness and integrability.

Theorem 1.2. *Let $p \in I_1$ and let (u, v) be a pair of positive solutions of system (1.1) with $u \in L^{\frac{2N(p-1)}{N-\gamma+\tau}}(\mathbb{R}^N)$. Then*

1. *$u \in L^s(\mathbb{R}^N)$ for $\frac{1}{s} \in (0, \frac{N-\tau}{N})$; $u \notin L^s(\mathbb{R}^N)$ for $\frac{1}{s} \ge \frac{N-\tau}{N}$.*
2. *u is bounded. Moreover, $|x|^\alpha v$ is bounded if $\tau+\gamma \le N + 2(\mu+\beta)$.*

Remark 1.3. *Some special cases of Theorem 1.2 are already known from [33–35]. If $u \in L^{\frac{2N(p-1)}{N-\gamma+\tau}}(\mathbb{R}^N)$ and (u, v) is a pair of positive solutions of (1.1), from the proof of Theorem 1.2 in Section 2, we can conclude that the integral interval of v can also be extended, that is, $v \in L^r(\mathbb{R}^N)$ where r satisfies $\frac{1}{r} \in \left(0, \frac{\gamma}{N}\right)$ if $\tau+\gamma \le N$, and $\frac{1}{r} \in \left(0, \frac{\gamma}{N}\right) \cap \left(0, \frac{p(N-\tau)-N+\gamma}{N}\right)$ if $\tau+\gamma > N$.*

In the following we will apply the method of moving planes in integral forms [6,11] to study the radial symmetry and the asymptotic behaviour of the integrable solutions of system (1.1).

Theorem 1.4. *Let $p \in I_1$ and let (u, v) be a pair of positive solutions of the system (1.1) with $u \in L^{\frac{2N(p-1)}{N-\gamma+\tau}}(\mathbb{R}^N)$. Then u, v are radially symmetric and decreasing about the origin.*

The study of asymptotic behaviour plays an essential role in the existence or the nonexistence of positive solutions. Obviously, Theorem 1.1 implies that if system (1.1) has a pair of positive solutions, then $p > \frac{3N-\gamma-\tau}{2(N-\tau)}$. Naturally, we get $N-\tau > \frac{N-\gamma+\tau}{2(p-1)}$. We say that the radial solution u decays fast with the rate $N-\tau$, if there exists a constant $C > 0$ such that $\frac{1}{C} \le u(x)|x|^{N-\tau} \le C$ as $|x| \to \infty$. Similarly, we say u decays slowly with the rate $\frac{N-\gamma+\tau}{2(p-1)}$ if there exists a constant $C > 0$ such that $\frac{1}{C} \le u(x)|x|^{\frac{N-\gamma+\tau}{2(p-1)}} \le C$ as $|x| \to \infty$.

Theorem 1.5. *Let $p \in I_1$ and (u, v) be a pair of positive solutions of (1.1).*

1. *If $u \in L^{\frac{2N(p-1)}{N-\gamma+\tau}}(\mathbb{R}^N)$, then $u(x) \backsimeq A_0$ as $|x| \to 0$, where $A_0 = \displaystyle\int_{\mathbb{R}^N} \frac{u^{p-1}(y)v(y)}{|y|^{N-\tau}} dy$.*
2. *If $u \in L^{\frac{2N(p-1)}{N-\gamma+\tau}}(\mathbb{R}^N)$, then u decays fast with the rate $N-\tau$. Moreover, $u(x) \backsimeq \frac{B_0}{|x|^{N-\tau}}$ as $|x| \to \infty$, where $B_0 = \displaystyle\int_{\mathbb{R}^N} u^{p-1}(y)v(y)dy$.*
3. *Suppose that u is bounded and decaying. If $u \notin L^{\frac{2N(p-1)}{N-\gamma+\tau}}(\mathbb{R}^N)$, then u decays almost slowly with the rate $\frac{N-\gamma+\tau}{2(p-1)}$.*

Here we say that a solution u decays almost slowly with the rate $\frac{N-\gamma+\tau}{2(p-1)}$, if the decay rate of u is neither slower than $C|x|^{\frac{N-\gamma+\tau}{2(p-1)}}$ nor faster than $C|x|^{\frac{N-\gamma+\tau}{2(p-1)}}$.

In Section 3, we study positive solutions of the equation

$$-\Delta u=\left(\int_{\mathbb{R}^N}\frac{u^p(y)}{|x|^\alpha|x-y|^\mu|y|^\alpha}dy\right)u^{p-1},\quad x\in\mathbb{R}^N. \tag{1.4}$$

As we can see, a solution of equation (1.4) must satisfy (1.1) with $\tau=2$ and $\alpha=\beta$. For this special case, in order to simplify the discussion about the positive solutions of (1.4), we may introduce

$$I_2:=\begin{cases}\left(\frac{3N-\gamma-2}{2(N-2)},+\infty\right), & \text{if } \gamma>N-2;\\ \left(\frac{2N}{N+\gamma-2},\frac{2(N-\gamma)}{N-\gamma-2}\right)\cap\left(\frac{2N-\gamma-2}{N-2},+\infty\right), & \text{if } \gamma\le N-2,\end{cases}$$

where $\gamma=2\alpha+\mu$. All the above results for the corresponding integral system are still true for equation (1.4); indeed, we can easily obtain qualitative results such as the boundedness and integrability results, symmetry and asymptotic behaviour. Furthermore, we can draw the following conclusions.

Corollary 1.6. *Let $N\ge 3$. If $1\le p\le\frac{3N-\gamma-2}{2(N-2)}$, then equation (1.4) has no positive classical solutions.*

A positive solution u belonging to the homogeneous Sobolev space $D^{1,2}(\mathbb{R}^N)$ is called a weak solution of equation (1.4), if

$$\int_{\mathbb{R}^N}\nabla u\nabla\varphi dx=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^p(y)u^p(x)\varphi(x)}{|x|^\alpha|x-y|^\mu|y|^\alpha}dxdy,\quad\forall\varphi\in C_c^\infty(\mathbb{R}^N).$$

Definition 1.7. *A classical solution $u\in C^2(\mathbb{R}^N)$ of equation (1.4) is called*

(1) *an integrable solution if $u\in L^{\frac{2N(p-1)}{N-\gamma+2}}(\mathbb{R}^N)$.*
(2) *a finite energy solution if $\int_{\mathbb{R}^N}|\nabla u|^2dx=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^p(y)u^p(x)}{|x|^\alpha|x-y|^\mu|y|^\alpha}dxdy$.*

Next, we are going to study the relationship between the integrable solutions and the finite energy solutions of equation (1.4).

Theorem 1.8. *Assume that $u\in C^2(\mathbb{R}^N)$ is a classical solution of equation (1.4) with $p\in I_2$. If $u\in L^{\frac{2N(p-1)}{N-\gamma+2}}(\mathbb{R}^N)$, then $\int_{\mathbb{R}^N}\frac{u^p(y)}{|x|^\alpha|x-y|^\mu|y|^\alpha}dyu^p(x)\in L^1(\mathbb{R}^N)$ and $u\in D^{1,2}(\mathbb{R}^N)$. Moreover,*

$$\int_{\mathbb{R}^N}|\nabla u|^2dx=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^p(y)u^p(x)}{|x|^\alpha|x-y|^\mu|y|^\alpha}dxdy.$$

As we all know, $p=\frac{2N-2\alpha-\mu}{N-2}$ is the upper critical exponent of the weighted Hardy-Littlewood-Sobolev inequality. Apparently, if $p=\frac{2N-2\alpha-\mu}{N-2}\in I_2$, then the following theorem indicates that if $p\neq\frac{2N-2\alpha-\mu}{N-2}$, there does not exist solutions of equation (1.4) in $L^{\frac{2N(p-1)}{N-\gamma+2}}(\mathbb{R}^N)$.

Theorem 1.9. *Assume that u is a classical solution of equation (1.4) with $p\in I_2$. If $u\in L^{\frac{2N(p-1)}{N-\gamma+2}}(\mathbb{R}^N)$, then $p=\frac{2N-2\alpha-\mu}{N-2}$. Hence, $L^{\frac{2N(p-1)}{N-\gamma+2}}(\mathbb{R}^N)=L^{2^*}(\mathbb{R}^N)$.*

According to the above theorem, we can get some sufficient and necessary conditions for the classification results.

Theorem 1.10. *If $u \in C^2(\mathbb{R}^N)$ is a classical solution of equation (1.4) with $p \in I_2$, then the following items are equivalent:*

(1) $u \in L^{\frac{2N(p-1)}{N-\gamma+2}}(\mathbb{R}^N)$;
(2) *u belongs to the homogeneous Sobolev space $D^{1,2}(\mathbb{R}^N)$;*
(3) *u is bounded and decaying fast with the rate of $N-2$.*

As far as we know, the nonlocal convolutionary term of the Hartree type equation (1.2) is relevant to the following famous weighted Hardy-Littlewood-Sobolev inequality which was demonstrated by Stein and Weiss [43].

Proposition 1.11 (Weighted HLS inequality [39])**.** *Let $1 < t, r < \infty$, $0 < \mu < N$, $\alpha + \beta \geq 0$ and $0 < \alpha + \beta + \mu \leq N$, $f \in L^t(\mathbb{R}^N)$, and $h \in L^r(\mathbb{R}^N)$. There exists a sharp constant $C_{t,r,\alpha,\beta,\mu,N}$ such that*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{f(x)h(y)}{|x|^\alpha |x-y|^\mu |y|^\beta} dxdy \leq C(t, r, N, \mu, \alpha, \beta)\|f\|_t \|h\|_r,$$

where $\frac{1}{t} + \frac{1}{r} + \frac{\alpha+\beta+\mu}{N} = 2$ and $1 - \frac{1}{t} - \frac{\mu}{N} < \frac{\alpha}{N} < 1 - \frac{1}{t}$, and C is independent of f and h.

Moreover, for any $h \in L^r(\mathbb{R}^N)$, we have

$$\left\| \int_{\mathbb{R}^N} \frac{h(y)}{|x|^\alpha |x-y|^\mu |y|^\beta} dy \right\|_s \leq C(s, N, \mu, \alpha, \beta)\|h\|_r,$$

where s satisfies $1 + \frac{1}{s} = \frac{1}{r} + \frac{\alpha+\beta+\mu}{N}$ and $\frac{\alpha}{N} < \frac{1}{s} < \frac{\alpha+\mu}{N}$.

For the special case $\alpha = \beta = 0$, it is the classical Hardy-Littlewood-Sobolev inequality [39] which has been investigated extensively by several authors over many years. Lieb [38] classified all the maximizers of the HLS functional under constraints and obtained the best constant. Moreover, he posed the classification of the positive solutions of

$$u(x) = \int_{\mathbb{R}^N} \frac{u(y)^{\frac{N+\tau}{N-\tau}}}{|x-y|^{N-\tau}} dy, \quad x \in \mathbb{R}^N \tag{1.5}$$

as an open problem. In [18] and [19], Dou and Zhu classified the extremal functions and computed the best constant for the reversed HLS inequality and the sharp HLS inequality on the upper half space, respectively. As a matter of fact, (1.5) arises as an Euler-Lagrange equation for a functional under a constraint of the classical HLS inequality (See [38, 39]). Later, Chen, Li and Ou [7] developed the method of moving planes in integral forms to prove that any critical points of the functional was radially symmetric and assumed the unique form and gave a positive answer to Lieb's open problem. Furthermore, they also considered a general integral equation

$$u(x) = \int_{\mathbb{R}^N} \frac{u^p(y)}{|x-y|^{N-\tau}} dy, \quad x \in \mathbb{R}^N \tag{1.6}$$

in subcritical cases and supercritical cases, separately. In [8], the authors proved that there does not exist regular solutions of (1.6) when $1 < p < \frac{N+\tau}{N-\tau}$, and equation (1.6) possesses both symmetric solutions and non-symmetric solutions when $p > \frac{N+\tau}{N-\tau}$. Obviously, integral equation (1.5) is equivalent to the fractional equation

$$(-\Delta)^{\frac{\tau}{2}} u = u^{\frac{N+\tau}{N-\tau}}, \quad x \in \mathbb{R}^N. \tag{1.7}$$

If $N \geq 3$, $\tau = 2$, it is worth noting that equation (1.7) turns to

$$-\Delta u = u^{\frac{N+2}{N-2}}, \quad x \in \mathbb{R}^N, \tag{1.8}$$

which is related to the Euler-Lagrange equation of the extremal functions of the Sobolev inequality and is a special case of the Lane-Emden equation

$$-\Delta u = u^p, \quad x \in \mathbb{R}^N. \tag{1.9}$$

It is acknowledged that, for $0 < p < \frac{N+2}{N-2}$, Gidas and Spruck [25] proved that (1.9) has no positive solutions. This result is optimal in the sense that for any $p \geq \frac{N+2}{N-2}$, there are infinitely many positive solutions to (1.9). Gidas, Ni and Nirenberg [24], Caffarelli, Gidas and Spruck [5] proved the symmetry and uniqueness of the positive solutions respectively. Chen and Li [9], Li [31] simplified the results above as an application of the moving plane method, and Li [32] used the moving sphere method. Actually, the classification of the solutions of equation (1.8) plays an important role in the Yamabe problem, the prescribed scalar curvature problem on Riemannian manifolds and the *a priori* estimates in nonlinear equations. Furthermore, for the more general case that τ is any even number between 0 and N, Wei and Xu [44] generalized the classification of the solutions of the fractional equation (1.7).

For the doubly weighted case, Lieb [38] proved the existence of sharp constant for the case that one of r and t equals 2 or $r = t$. Furthermore, Beckner [3,4] gave the sharp constant for $1 < r, t < \infty$ with $\frac{1}{r} + \frac{1}{t} = 1$. It is obvious that the corresponding Euler-Lagrange equations for weighted HLS inequality is the system of integral equations

$$\begin{cases} u(x) = \displaystyle\int_{\mathbb{R}^N} \frac{v^q(y)}{|x|^\alpha |x-y|^\mu |y|^\beta} dy, \\ v(x) = \displaystyle\int_{\mathbb{R}^N} \frac{u^p(y)}{|x|^\beta |x-y|^\mu |y|^\alpha} dy, \end{cases} \tag{1.10}$$

where $0 < p, q < +\infty$, $0 < \mu < N$, $\frac{\alpha}{N} < \frac{1}{p+1} < \frac{\mu+\alpha}{N}$ and $\frac{1}{p+1} + \frac{1}{q+1} = \frac{\mu+\alpha+\beta}{N}$. In [13] and [27], the authors obtained the symmetry, monotonicity and the optimal integrability results for (1.10). Notice that integral system (1.10) is equivalent to the nonlinear singular PDE system

$$\begin{cases} (-\Delta)^{\frac{N-\mu}{2}}(|x|^\alpha u(x)) = \dfrac{v^q(x)}{|x|^\beta}, \\ (-\Delta)^{\frac{N-\mu}{2}}(|x|^\beta v(x)) = \dfrac{u^p(x)}{|x|^\alpha}, \end{cases} \tag{1.11}$$

for the special case $\mu = N - 2$, Chen and Li [14] proved the uniqueness of the solutions for (1.11), and they also classified the solutions and obtained the best constant in the corresponding weighted HLS inequality when $\alpha = \beta$ and $p = q$. Then Lei, Li and Ma [30] studied the asymptotic radial symmetry and growth estimates of positive solutions for (1.10). Later, Liu and Lei [29] obtained the nonexistence results for (1.10), and considered the existence of positive solutions for the following weighted system with double bounded coefficients

$$\begin{cases} u(x) = c_1(x) \displaystyle\int_{\mathbb{R}^N} \frac{v^q(y)}{|x|^\alpha |x-y|^\mu |y|^\beta} dy, \\ v(x) = c_2(x) \displaystyle\int_{\mathbb{R}^N} \frac{u^p(y)}{|x|^\beta |x-y|^\mu |y|^\alpha} dy, \end{cases} \tag{1.12}$$

where $0 < p, q < +\infty$, $0 < \mu + \alpha + \beta < N$, $\frac{\alpha}{N} < \frac{1}{p+1} < \frac{\mu+\alpha}{N}$ and $\frac{\beta}{N} < \frac{1}{q+1} < \frac{\mu+\beta}{N}$. More generally, Chen, Liu and Lu [12] discussed the weighted Hardy-Sobolev type system

$$\begin{cases} u(x) = \displaystyle\int_{\mathbb{R}^N} \frac{f_1(u(y), v(y))}{|x|^\alpha |x-y|^\mu |y|^\beta} dy, \\ v(x) = \displaystyle\int_{\mathbb{R}^N} \frac{f_2(u(y), v(y))}{|x|^\beta |x-y|^\mu |y|^\alpha} dy, \end{cases} \tag{1.13}$$

where

$$f_1(u(y), v(y)) = \lambda_1 u^{p_1}(y) + \mu_1 v^{q_1}(y) + \gamma_1 u^{\alpha_1}(y) v^{\beta_1}(y),$$
$$f_2(u(y), v(y)) = \lambda_2 u^{p_2}(y) + \mu_2 v^{q_2}(y) + \gamma_2 u^{\alpha_2}(y) v^{\beta_2}(y),$$

and when none of λ_i, μ_i and γ_i is zero $(i = 1, 2)$, they established the symmetry, integrability and C^∞ regularity results of the solutions. For the special case of (1.13), if $\alpha = \beta = 0$, there are some work concerning the system of integral equations

$$\begin{cases} u(x) = \displaystyle\int_{\mathbb{R}^N} \frac{u^p(y) v^q(y)}{|x-y|^{N-\gamma}} dy, \\ v(x) = \displaystyle\int_{\mathbb{R}^N} \frac{u^q(y) v^p(y)}{|x-y|^{N-\gamma}} dy, \end{cases} \tag{1.14}$$

where $0 < \gamma < N$, $1 \le p, q \le \frac{N+\gamma}{N-\gamma}$ with $p + q \le \frac{N+\gamma}{N-\gamma}$. For $\gamma = 2$, Li and Ma [28] proved the symmetry and uniqueness of the positive solutions for (1.14) with critical exponents.

The paper is organized as follows. In Section 2, we study the integral system (1.1) and prove the nonexistence, regularity and exponential decay of the solutions. In Section 3, we investigate the classification and asymptotic behaviours of the solutions for a weighted Hartree type equation. In this paper, c, C will be used to denote different constants.

2. An integral system

In this section, we are going to investigate the properties of the solutions of integral system (1.1), such as nonexistence, regularity, symmetry and decay results.

2.1. **Nonexistence.** First we will prove a nonexistence result for system (1.1). We may assume that $N \ge 3$.

Proof of Theorem 1.1. We consider the cases $1 \le p < \frac{3N-\gamma-\tau}{2(N-\tau)}$ and $p = \frac{3N-\gamma-\tau}{2(N-\tau)}$ separately.

Firstly, for the case $1 \le p < \frac{3N-\gamma-\tau}{2(N-\tau)}$, if (u, v) is a pair of positive solutions of (1.1), we can argue by contradiction. In fact, for $|x| > 1$, if $y \in B_1(0)$, then $|x-y| \le |x| + |y| \le 2|x|$, thus we have

$$u(x) \ge \int_{B_1(0)} \frac{u^{p-1}(y) v(y)}{|x-y|^{N-\tau}} dy \ge c \frac{1}{|x|^{N-\tau}} \int_{B_1(0)} u^{p-1}(y) v(y) dy. \tag{2.1}$$

Additionally, we get

$$u(x) \geq \frac{c}{|x|^{N-\tau}} = \frac{c}{|x|^{a_0}}, \quad a_0 = N - \tau. \tag{2.2}$$

Notice that $\frac{|x|}{2} \leq |y| \leq \frac{3|x|}{2}$ for $|x| > 2$ and $y \in B_{\frac{|x|}{2}}(x)$, we may find that

$$\begin{aligned} v(x) &\geq \int_{B_{\frac{|x|}{2}}(x)} \frac{u^p(y)}{|x|^{\alpha}|x-y|^{\mu}|y|^{\beta}} dy \geq c \int_{B_{\frac{|x|}{2}}(x)} \frac{1}{|x-y|^{\mu}|x|^{pa_0+\alpha+\beta}} dy \\ &\geq \frac{c}{|x|^{pa_0+\alpha+\beta}} \int_{B_{\frac{|x|}{2}}(0)} \frac{1}{|y|^{\mu}} dy \\ &\geq \frac{c}{|x|^{pa_0+\alpha+\beta+\mu-N}} \\ &= \frac{c}{|x|^{b_0}}, \end{aligned}$$

where $b_0 = pa_0 + \gamma - N$. By this estimate, for $|x| > 4$ and $y \in B_{\frac{|x|}{2}}(x)$, we can get

$$\begin{aligned} u(x) &\geq \int_{B_{\frac{|x|}{2}}(x)} \frac{u^{p-1}(y)v(y)}{|x-y|^{N-\tau}} dy \geq c \int_{B_{\frac{|x|}{2}}(x)} \frac{1}{|x-y|^{N-\tau}|x|^{(p-1)a_0+b_0}} dy \\ &\geq \frac{c}{|x|^{(p-1)a_0+b_0}} \int_{B_{\frac{|x|}{2}}(0)} \frac{1}{|y|^{N-\tau}} dy \\ &\geq \frac{c}{|x|^{(p-1)a_0+b_0-\tau}} \\ &= \frac{c}{|x|^{a_1}}, \end{aligned}$$

where $a_1 = (p-1)a_0 + b_0 - \tau$. By induction, there exists $R > 0$ such that for $|x| > R$, $u(x)$ and $v(x)$ satisfy

$$u(x) \geq \frac{c}{|x|^{a_k}}, \quad v(x) \geq \frac{c}{|x|^{b_k}}.$$

Here

$$a_k = (p-1)a_{k-1} + b_{k-1} - \tau, \quad b_k = pa_k + \gamma - N, \quad k = 1, 2, \ldots$$

Therefore,

$$a_k = (2p-1)^k a_0 + \left[1 + (2p-1) + \cdots + (2p-1)^{k-1}\right](\gamma - N - \tau).$$

We have the following two cases:

Case I. If $p = 1$, then we obtain

$$a_k = a_0 + k(\gamma - N - \tau),$$

which implies $a_{k_0} < 0$ for large k_0.
Case II. If $1 < p < \frac{3N-\gamma-\tau}{2(N-\tau)}$, we get

$$\begin{aligned} a_k =&(2p-1)^k a_0 + \frac{1-(2p-1)^k}{1-(2p-1)}(\gamma - N - \tau) \\ =&\left(a_0 - \frac{\mu - N - \tau}{2-2p}\right)(2p-1)^k + \frac{\gamma - N - \tau}{2-2p}. \end{aligned}$$

Noting that $a_0 = N - \tau$ and $1 < p < \frac{3N-\gamma-\tau}{2(N-\tau)}$, we have

$$a_0 < \frac{\gamma - N - \tau}{2-2p}.$$

Consequently, letting $k \to +\infty$, we know

$$a_k \to -\infty,$$

which implies $a_{k_0} < 0$ for sufficiently large k_0. Thus, for fix $x \in B_{\frac{R}{2}}(0)$, we can obtain

$$v(x) \geq \int_{\mathbb{R}^N - B_R(0)} \frac{u^p(y)}{|x|^\alpha |x-y|^\mu |y|^\beta} dy \geq c \int_{\mathbb{R}^N - B_R(0)} \frac{1}{|y|^{\beta+\mu+pa_{k_0}}} dy \geq c \int_R^\infty r^{N-1-\mu-\beta-pa_{k_0}} dr = \infty,$$

which is impossible.

Secondly, for the case $p = \frac{3N-\gamma-\tau}{2(N-\tau)}$, we will prove the result by showing

$$\| u^{p-1} v \|_{L^1(\mathbb{R}^N)} = 0.$$

By repeating the same arguments for (2.1), for $R > 0$, we have

$$u(x) \geq \int_{B_R(0)} \frac{u^{p-1}(y)v(y)}{|x-y|^{N-\tau}} dy \geq c \frac{1}{(|x|+R)^{N-\tau}} \int_{B_R(0)} u^{p-1}(y)v(y)dy \geq \frac{c}{(|x|+R)^{N-\tau}}. \quad (2.3)$$

Similarly, we also obtain

$$v(x) \geq \int_{B_R(0)} \frac{u^p(y)}{|x|^\alpha |x-y|^\mu |y|^\beta} dy \geq c \frac{1}{(|x|+R)^{\alpha+\beta+\mu}} \int_{B_R(0)} u^p(y) dy.$$

From

$$\begin{aligned} \int_{B_R(0)} u^p(x)dx \geq& c \int_{B_R(0)} \frac{1}{(|x|+R)^{p(N-\tau)}} dx \left(\int_{B_R(0)} u^{p-1}(y)v(y)dy\right)^p \\ \geq& cR^{N-p(N-\tau)} \left(\int_{B_R(0)} u^{p-1}(y)v(y)dy\right)^p, \end{aligned}$$

we obtain

$$v(x) \geq c \frac{1}{(|x|+R)^{\alpha+\beta+\mu}} R^{N-p(N-\tau)} \left(\int_{B_R(0)} u^{p-1}(y)v(y)dy\right)^p. \quad (2.4)$$

Combining (2.3) with (2.4), we get

$$\begin{aligned}\int_{B_R(0)} u^{p-1}(x)v(x)dx \geq &\int_{B_R(0)} \frac{c}{(|x|+R)^{(p-1)(N-\tau)+\gamma}} R^{N-p(N-\tau)}dx \left(\int_{B_R(0)} u^{p-1}(y)v(y)dy\right)^p \\ \geq & cR^{2N-p(N-\tau)-(p-1)(N-\tau)-\gamma} \left(\int_{B_R(0)} u^{p-1}(y)v(y)dy\right)^p \\ \geq & cR^{2N-(2p-1)(N-\tau)-\gamma} \left(\int_{B_R(0)} u^{p-1}(y)v(y)dy\right)^p .\end{aligned} \tag{2.5}$$

Since $p = \frac{3N-\gamma-\tau}{2(N-\tau)}$, let $R \to +\infty$, we have

$$\int_{\mathbb{R}^N} u^{p-1}(y)v(y)dy \leq C.$$

Therefore $u^{p-1}v \in L^1(\mathbb{R}^N)$.

Analogous to the estimate of (2.5), integrating on $B_{2R}(0) - B_R(0)$, we get

$$\begin{aligned}\int_{B_{2R}(0)-B_R(0)} u^{p-1}(x)v(x)dx \geq &\int_{B_{2R}(0)-B_R(0)} \frac{c}{(|x|+R)^{(p-1)(N-\tau)+\gamma}} R^{N-p(N-\tau)}dx \left(\int_{B_R(0)} u^{p-1}(y)v(y)dy\right)^p \\ \geq & cR^{2N-(2p-1)(N-\tau)-\gamma} \left(\int_{B_R(0)} u^{p-1}(y)v(y)dy\right)^p \\ \geq & c\left(\int_{B_R(0)} u^{p-1}(y)v(y)dy\right)^p .\end{aligned}$$

By letting $R \to +\infty$, we get

$$\|u^{p-1}v\|_{L^1(\mathbb{R}^N)} = 0.$$

Hence, the proof is completed. □

The above proof indicates that the assumption

$$p > \frac{3N-\gamma-\tau}{2(N-\tau)}$$

is necessary for the existence of positive solutions of system (1.1).

2.2. **Regularity.** We proceed to discussing the integrability of the solutions of system (1.1) by using the following regularity lifting theorem (See Theorem 3.3.1 in [10]).

Lemma 2.1. *Let X and Y be Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. The subspace of X and Y, $Z = X \cap Y$ is endowed with the norm defined by*

$$\|\cdot\|_Z := \sqrt[p]{\|\cdot\|_X^p + \|\cdot\|_Y^p}, \quad p \in [1, \infty) .$$

Suppose that $\mathscr{T}$ is a contraction map from Banach space X into itself and from Banach space Y into itself. If $f \in X$ and there exists a function $g \in Z = X \cap Y$ such that $z{=}\mathscr{T}f + g$, then f also belongs to Z.

First, we may assume that $\tau + \gamma > N$. Then we have

Lemma 2.2. *Let* $N \geq 3, \tau+\gamma > N$. *If* (u,v) *is a pair of positive solutions of* (1.1) *and* $u \in L^{\frac{2N(p-1)}{N-\gamma+\tau}}(\mathbb{R}^N)$ *with* $p \in \left(\frac{3N-\gamma-\tau}{2(N-\tau)}, +\infty\right)$, *then* $(u,v) \in L^s(\mathbb{R}^N) \times L^r(\mathbb{R}^N)$ *with* $\frac{1}{s} \in \left(0, \frac{N-\tau}{N}\right)$ *and* $\frac{1}{r} \in \left(\frac{\tau+\gamma-N}{2N}, \frac{N-\tau+\gamma}{2N}\right)$.

Proof. For $A > 0$, define

$$u_A(x) = \begin{cases} u(x), & u(x) > A \text{ or } |x| > A; \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, we can define $v_A(x)$. Take $s > \frac{N}{N-\tau}$, and

$$\frac{1}{r} = \frac{1}{s} - \frac{N-\gamma-\tau}{2N}.$$

Suppose $g \in L^s(\mathbb{R}^N)$ and $f \in L^r(\mathbb{R}^N)$, we can define

$$\begin{aligned} T_1 f(x) &= \int_{\mathbb{R}^N} \frac{u_A^{p-1}(y) f(y)}{|x-y|^{N-\tau}} dy, \\ T_2 g(x) &= \int_{\mathbb{R}^N} \frac{u_A^{p-1}(y) g(y)}{|x|^\alpha |x-y|^\mu |y|^\beta} dy, \\ F(x) &= \int_{\mathbb{R}^N} \frac{(u(y)-u_A(y))^{p-1} v(y)}{|x-y|^{N-\tau}} dy, \\ G(x) &= \int_{\mathbb{R}^N} \frac{(u-u_A)^p(y)}{|x|^\alpha |x-y|^\mu |y|^\beta} dy, \end{aligned}$$

and the operator

$$\begin{aligned} &T : L^s(\mathbb{R}^N) \times L^r(\mathbb{R}^N) \to L^s(\mathbb{R}^N) \times L^r(\mathbb{R}^N), \\ &T(g,f) = (T_1 f, T_2 g) \end{aligned}$$

with the norm $\|T(g,f)\|_{s\times r} = \|T_1 f\|_s + \|T_2 g\|_r$. Evidently, (u,v) solves the operator equation

$$(g,f) = T(g,f) + (F,G).$$

By the weighted Hardy-Littlewood-Sobolev inequality and the Hölder inequality, we have

$$\|T_1 f\|_s \leq C \|u_A^{p-1} f\|_{\frac{Ns}{N+\tau s}} \leq C \|u_A\|^{p-1}_{\frac{2N(p-1)}{N-\gamma+\tau}} \|f\|_r \tag{2.6}$$

and

$$\|T_2 g\|_r \leq C \|u_A^{p-1} g\|_{\frac{Nr}{N+(N-\gamma)r}} \leq C \|u_A\|^{p-1}_{\frac{2N(p-1)}{N-\gamma+\tau}} \|g\|_s. \tag{2.7}$$

By virtue of $u \in L^{\frac{2N(p-1)}{N-\gamma+\tau}}(\mathbb{R}^N)$, we can choose A large enough such that $C\|u_A\|^{p-1}_{\frac{2N(p-1)}{N-\gamma+\tau}} < 1$. Thus, T is a contraction map from $L^s(\mathbb{R}^N) \times L^r(\mathbb{R}^N)$ into itself; here

$$\frac{1}{s} \in \left(0, \frac{N-\tau}{N}\right), \quad \frac{1}{r} \in \left(\frac{\gamma+\tau-N}{2N}, \frac{N+\gamma-\tau}{2N}\right). \tag{2.8}$$

Set $s_0 = \frac{2N(p-1)}{N-\gamma+\tau}$, there exists a r_0 such that $u \in L^{s_0}(\mathbb{R}^N)$ and $v \in L^{r_0}(\mathbb{R}^N)$ where r_0, s_0 satisfies (2.8). In fact, applying the weighted Hardy-Littlewood-Sobolev inequality to the second integral equation of the system (1.1), we get

$$\|v\|_{r_0} \le C\|u^p\|_{\frac{Nr_0}{N+(N-\gamma)r_0}}.$$

From

$$\frac{pNr_0}{N+(N-\gamma)r_0} = s_0,$$

it follows that

$$r_0 = \frac{2N(p-1)}{p\tau-(p-2)(N-\gamma)}.$$

Note that r_0, s_0 satisfies (2.8) since $p \in \left(\frac{3N-\gamma-\tau}{2(N-\tau)}, +\infty\right)$, the operator T is also a contraction map from $L^{s_0}(\mathbb{R}^N) \times L^{r_0}(\mathbb{R}^N)$ into itself. According to the derivation of (2.6) and (2.7), we find

$$\|F\|_s \le C\|(u-u_A)^{p-1}v\|_{\frac{Ns}{N+\tau s}} \le C\|u-u_A\|^{p-1}_{\frac{2N(p-1)}{N-\gamma+\tau}}\|v\|_r,$$

and

$$\|G\|_r \le C\|u-u_A\|^p_{\frac{Npr}{N+(N-\gamma)r}}.$$

If we write $X = L^{s_0}(\mathbb{R}^N) \times L^{r_0}(\mathbb{R}^N)$ and $Y = L^s(\mathbb{R}^N) \times L^r(\mathbb{R}^N)$, we deduce $(u,v) \in L^s(\mathbb{R}^N) \times L^r(\mathbb{R}^N)$ by the regularity lifting theorem. □

Lemma 2.3. *Under the assumptions of Lemma 2.2,* $(u,v) \in L^s(\mathbb{R}^N) \times L^r(\mathbb{R}^N)$ *for* $\frac{1}{s} \in \left(0, \frac{N-\tau}{N}\right), \frac{1}{r} \in \left(0, \frac{p(N-\tau)-N+\gamma}{N}\right) \cap \left(0, \frac{\gamma}{N}\right)$.

Proof. Applying the weighted Hardy-Littlewood-Sobolev inequality to the second formula in the system (1.1) yields

$$\|v\|_r \le C\|u\|_s^p, \quad \frac{1}{r} = \frac{p}{s} - \frac{N-\gamma}{N}.$$

From Lemma 2.2, we can choose suitable $\frac{1}{s} \in \left(0, \frac{N-\tau}{N}\right)$ such that $v \in L^r(\mathbb{R}^N)$ for $\frac{1}{r} \in \left(0, \frac{p(N-\tau)-N+\gamma}{N}\right)$. In fact, for any p,

$$\frac{\gamma+\tau-N}{2N} < \frac{n+\gamma-\tau}{2N} < \frac{p(N-\tau)-N+\gamma}{N},$$

together with the continuity of the function $s \longmapsto \frac{p}{s} - \frac{N-\gamma}{N}$, we immediately obtain that $v \in L^r(\mathbb{R}^N)$ for $\frac{1}{r} \in \left(0, \frac{p(N-\tau)-N+\gamma}{N}\right) \cap \left(0, \frac{\gamma}{N}\right)$. □

Next we will focus on the integrability of solutions of (1.1) for the case $\tau+\gamma \le N$.

Lemma 2.4. *Let* $N \ge 3, \tau+\gamma \le N$. *If* (u,v) *is a pair of positive solutions of* (1.1) *and* $u \in L^{\frac{2N(p-1)}{N-\gamma+\tau}}(\mathbb{R}^N)$ *for* $p \in \left(\frac{2N}{N-\tau+\gamma}, \frac{2(N-\gamma)}{N-\tau-\gamma}\right)$, *then* $(u,v) \in L^s(\mathbb{R}^N) \times L^r(\mathbb{R}^N)$, *for* $\frac{1}{s} \in \left(\frac{N-\gamma-\tau}{2N}, \frac{N+\gamma-\tau}{2N}\right), \frac{1}{r} \in \left(0, \frac{\gamma}{N}\right)$.

Proof. Take $r > \frac{N}{\gamma}$, and set

$$\frac{1}{s} = \frac{1}{r} + \frac{N-\gamma-\tau}{2N}.$$

Similar to the proof of Lemma 2.2, the operator T remains a contraction map from $L^s(\mathbb{R}^N)\times L^r(\mathbb{R}^N)$ into itself, where s, r satisfying

$$\frac{1}{r}\in\left(0,\frac{\gamma}{N}\right),\quad \frac{1}{s}\in\left(\frac{N-\gamma-\tau}{2N},\frac{N+\gamma-\tau}{2N}\right). \tag{2.9}$$

Set $s_0{=}\frac{2N(p-1)}{N-\gamma+\tau}$ and $r_0=\frac{2N(p-1)}{p\tau-(p-2)(N-\gamma)}$. Since $p\in\left(\frac{2N}{N-\tau+\gamma},\frac{2(N-\gamma)}{N-\tau-\gamma)}\right)$, we know s_0 and r_0 satisfy (2.9). Thus, T still is a contraction map from $L^{s_0}(\mathbb{R}^N)\times L^{r_0}(\mathbb{R}^N)$ into itself. Therefore, by the regularity lifting theorem as well, we get $(u,v)\in L^s(\mathbb{R}^N)\times L^r(\mathbb{R}^N)$, for $\frac{1}{s}\in\left(\frac{N-\gamma-\tau}{2N},\frac{N+\gamma-\tau}{2N}\right), \frac{1}{r}\in\left(0,\frac{\gamma}{N}\right)$. □

For $\tau+\gamma>N$, we proved that $u\in L^s(\mathbb{R}^N)$ for any $\left(0,\frac{N-\tau}{N}\right)$, which inspires us to infer the integrability interval of u can also be extended to $\left(0,\frac{N-\tau}{N}\right)$ when $\tau+\gamma\le N$.

Lemma 2.5. *Let $N\ge 3, \tau+\gamma\le N$. If (u,v) is a pair of positive solutions of (1.1) and $u\in L^{\frac{2N(p-1)}{N-\gamma+\tau}}(\mathbb{R}^N)$ for $p\in\left(\frac{2N}{N-\tau+\gamma},\frac{2(N-\gamma)}{N-\tau-\gamma}\right)\cap\left(\frac{2N-\tau-\gamma}{N-\tau},+\infty\right)$, then $(u,v)\in L^s(\mathbb{R}^N)\times L^r(\mathbb{R}^N)$ for $\frac{1}{s}\in\left(0,\frac{N-\tau}{N}\right)$ and $\frac{1}{r}\in\left(0,\frac{\gamma}{N}\right)$.*

Proof. By the estimate in (2.6), we have

$$\|u\|_{\sigma_j}\le C\|u^{p-1}v\|_{\frac{N\sigma_j}{N+\sigma_j\tau}}\le C\|u\|^{p-1}_{\sigma_{j-1}}\|v\|_r, \tag{2.10}$$

for $j=1,2\cdots$ and $\frac{1}{r}\in\left(0,\frac{\gamma}{N}\right)$, where σ_j satisfying

$$\frac{1}{\sigma_j}<\frac{N-\tau}{N}\quad\text{and}\quad\frac{1}{\sigma_j}=\frac{p-1}{\sigma_{j-1}}+\frac{1}{r}-\frac{\tau}{N}. \tag{2.11}$$

Choose a suitable $\frac{1}{\sigma_0}\in\left(\frac{N-\gamma-\tau}{2N},\frac{N+\gamma-\tau}{2N}\right)$, we know $u\in L^{\sigma_1}(\mathbb{R}^N)$ satisfying (2.11), which means

$$\frac{1}{\sigma_1}\in\left(\max\left\{0,\frac{N-\gamma-\tau}{2N}(p-1)-\frac{\tau}{N}\right\},\quad\min\left\{\frac{N-\tau}{N},\frac{N+\gamma-\tau}{2N}(p-1)+\frac{\gamma-\tau}{N}\right\}\right).$$

In view of $p\in\left(\frac{2N}{N-\tau+\gamma},\frac{2(N-\gamma)}{N-\tau-\gamma}\right)$, we can easily see that the new interval covers $\left(\frac{N-\gamma-\tau}{2N},\frac{N+\gamma-\tau}{2N}\right)$. From (2.11) it is not difficult to get

$$\frac{1}{\sigma_j}-\frac{1}{\sigma_{j-1}}=\frac{p-2}{\sigma_{j-1}}+\frac{1}{r}-\frac{\tau}{N}. \tag{2.12}$$

Take $\frac{1}{\sigma_0}=\frac{N-\gamma-\tau+\varepsilon}{2N}, \frac{1}{r}=\frac{\varepsilon}{N}$ and $\frac{1}{\sigma_0}=\frac{N+\gamma-\tau-\varepsilon}{2N}, \frac{1}{r}=\frac{\gamma-\varepsilon}{N}$, where $\varepsilon>0$ is sufficiently small. We claim $\frac{1}{\sigma_j}$ is decreasing and increasing as $j\to\infty$ respectively.

Therefore, by finite steps, we get $u\in L^s(\mathbb{R}^N)$ for any $\frac{1}{s}\in\left(0,\frac{N-\tau}{N}\right)$. Otherwise, notice that $\left\{\frac{1}{\sigma_j}\right\}$ is bounded, we assume $\frac{1}{\sigma_j}$ converges to L as $j\to\infty$. In addition, $L\in\left(0,\frac{N-\tau}{N}\right)$. Next we derive contradictions.

Case I. Take $\frac{1}{\sigma_0}=\frac{N-\gamma-\tau+\varepsilon}{2N}$ and $\frac{1}{r}=\frac{\varepsilon}{N}$.

Indeed, since

$$\frac{1}{\sigma_1}-\frac{1}{\sigma_0}=\frac{p-2}{\sigma_0}+\frac{\varepsilon-\tau}{N},$$

by letting $\varepsilon \to 0$, it follows that

$$\frac{1}{\sigma_1} \leq \frac{1}{\sigma_0}.$$

Form (2.11) we obtain

$$\lim_{\varepsilon\to 0}\frac{1}{\sigma_1} \leq \frac{N-\gamma-\tau}{2N},$$

which implies $\frac{1}{\sigma_1} < \frac{1}{\sigma_0}$ as ε enough small. Namely, $\frac{1}{\sigma_j}$ is decreasing as $j \to \infty$. Letting $j \to \infty$ in (2.11), we lead

$$L = \frac{\tau-\varepsilon}{N(p-2)},$$

which implies $p > 2$. By virtue of $p < 2 + \frac{2\tau}{N-\gamma-\tau}$, we obtain $L \geq \frac{1}{\sigma_0}$ which is impossible.

Case II. Take $\frac{1}{\sigma_0} = \frac{N+\gamma-\tau-\varepsilon}{2N}$ and $\frac{1}{r} = \frac{\gamma-\varepsilon}{N}$.

In fact, since $p \in \left(\frac{2N}{N-\tau+\gamma}, \frac{2(N-\gamma)}{N-\tau-\gamma}\right) \cap \left(\frac{2N-\tau-\gamma}{N-\tau}, +\infty\right)$ and

$$\frac{1}{\sigma_1} - \frac{1}{\sigma_0} = \frac{p-2}{\sigma_0} + \frac{\gamma-\varepsilon-\tau}{N},$$

letting $\varepsilon \to 0$, it follows

$$\frac{1}{\sigma_1} \geq \frac{1}{\sigma_0}.$$

Form (2.11) we get

$$\lim_{\varepsilon\to 0}\frac{1}{\sigma_1} \geq \frac{N+\gamma-\tau}{2N},$$

which implies $\frac{1}{\sigma_1} > \frac{1}{\sigma_0}$ as ε enough small. Namely, $\frac{1}{\sigma_j}$ is increasing as $j \to \infty$. Letting $j \to \infty$ in (2.11), we have

$$L = \frac{\tau-\gamma+\varepsilon}{N(p-2)}.$$

(i) If $\tau < \gamma$, then $p < 2$. Notice $p > 2 + \frac{\tau-\gamma}{N-\tau}$, we obtain $L \geq \frac{N-\tau}{N}$ which is a contradiction.

(ii) If $\tau = \gamma$, then $L \to 0$ as $\varepsilon \to 0$; this contradicts $L > \frac{1}{\sigma_0}$.

(iii) If $\tau > \gamma$, then $p < 2$. Notice that when $p > 2+\frac{2(\tau-\gamma)}{N+\gamma-\tau}$, it follows that $\lim\limits_{\varepsilon\to 0}(L-\frac{1}{\sigma_0}) \leq 0$, which implies $L \leq \frac{1}{\sigma_0}$ for sufficiently small ε. This is impossible. □

Lemma 2.6. *Under the assumptions of Theorem 1.2, $u \notin L^s(\mathbb{R}^N)$ for $\frac{1}{s} \geq \frac{N-\tau}{N}$ and $v \notin L^r(\mathbb{R}^N)$ for $\frac{1}{r} \geq \frac{\gamma}{N}$.*

Proof. In view of (2.2), for $|x| > 2$, we get

$$u(x) \geq \frac{c}{|x|^{N-\tau}}.$$

Since $\frac{1}{s} \geq \frac{N-\tau}{N}$, we have that

$$\int_{\mathbb{R}^N} u^s(x)dx \geq \int_{\mathbb{R}^N-B_2(0)} \frac{c}{|x|^{s(N-\tau)}}dx = \int_2^\infty cr^{N-1-s(N-\tau)}dr = \infty.$$

Similarly, for $|x|>2$, we obtain

$$v(x)\ge\int_{B_1(0)}\frac{u^p(y)}{|x|^{\alpha}|x-y|^{\mu}|y|^{\beta}}dy\ge\frac{c}{|x|^{\alpha+\beta+\mu}}\int_{B_1(0)}u^p(y)dy=\frac{c}{|x|^{\gamma}}.$$

Thus, we have $v\notin L^r(\mathbb{R}^N)$ for $\frac{1}{r}\ge\frac{\gamma}{N}$. This completes the proof. □

In the following, we prove the boundedness of the integrable solutions by using the integrability results.

Proof of Theorem 1.2.
(1). Lemma 2.2 - Lemma 2.6 shows that the integrability results is true.
(2). **Step 1.** For $R>0$, we have that

$$u(x)=\int_{B_R(x)}\frac{u^{p-1}(y)v(y)}{|x-y|^{N-\tau}}dy+\int_{\mathbb{R}^N-B_R(x)}\frac{u^{p-1}(y)v(y)}{|x-y|^{N-\tau}}dy:=L_1+L_2.$$

Take $\frac{1}{s}=\frac{\varepsilon}{(p-1)N}$ and $\frac{1}{r}=\frac{\varepsilon}{N}$ such that $\frac{p-1}{s}+\frac{1}{r}<\frac{\tau}{N}$, with $\varepsilon>0$ suitably small. From Lemma 2.2 - Lemma 2.5, we get $(u,v)\in L^s(\mathbb{R}^N)\times L^r(\mathbb{R}^N)$. By the Hölder inequality, we have

$$\begin{aligned}L_1\le&\|u\|_s^{p-1}\|v\|_r\left(\int_{B_R(x)}|x-y|^{\frac{\tau-N}{1-\frac{p-1}{s}-\frac{1}{r}}}dy\right)^{1-\frac{p-1}{s}-\frac{1}{r}}\\ \le&C\left(\int_{B_R(0)}|y|^{\frac{\tau-N}{1-\frac{p-1}{s}-\frac{1}{r}}}dy\right)^{1-\frac{p-1}{s}-\frac{1}{r}}\\ \le&C\left(\int_0^R r^{N+\frac{\tau-N}{1-\frac{p-1}{s}-\frac{1}{r}}}\frac{dr}{r}\right)^{1-\frac{p-1}{s}-\frac{1}{r}}<\infty.\end{aligned}\tag{2.13}$$

In order to estimate L_2, we can choose suitable $\frac{1}{s}\in\left(0,\frac{N-\tau}{N}\right)$ and $\frac{1}{r}\in\left(0,\frac{\gamma}{N}\right)$ such that $\frac{p-1}{s}+\frac{1}{r}=1$. In fact, if $\tau+\gamma>N$, we take $\frac{1}{s}=\frac{N-\gamma+\tau+\varepsilon}{2N(p-1)}$ and $\frac{1}{r}=\frac{N+\gamma-\tau-\varepsilon}{2N}$ for sufficiently small $\varepsilon>0$. When $\tau+\gamma\le N$, in view of $p>\frac{2N-\gamma-\tau}{N-\tau}$, we observe that $(p-1)\frac{N-\tau}{N}+\frac{\gamma}{N}>1$. From the continuity of the function $\left(\frac{1}{s},\frac{1}{r}\right)\longmapsto\frac{p-1}{s}+\frac{1}{r}$, we infer that there exists suitable $\frac{1}{s}\in\left(0,\frac{N-\tau}{N}\right)$ and $\frac{1}{r}\in\left(0,\frac{\gamma}{N}\right)$ such that $\frac{p-1}{s}+\frac{1}{r}=1$.

Thus, by the Hölder inequality and the integrability results, it follows that

$$L_2\le R^{\tau-N}\int_{\mathbb{R}^N-B_R(x)}u^{p-1}(y)v(y)dy\le R^{\tau-N}\|u\|_s^{p-1}\|v\|_r<\infty.$$

Combining the estimates of L_1 and L_2, we infer that u is bounded.
Step 2. Next we prove that $|x|^{\alpha}v(x)\in L^{\infty}(\mathbb{R}^N)$. Obviously,

$$|x|^{\alpha}v(x)=\int_{\mathbb{R}^N}\frac{u^p(y)}{|x-y|^{\mu}|y|^{\beta}}dy,$$

whence, for any $r>0$, we have

$$||x|^{\alpha}v(x)|\le\int_{B_r(0)}\frac{u^p(y)}{|x-y|^{\mu}|y|^{\beta}}dy+\int_{\mathbb{R}^N-B_r(0)}\frac{u^p(y)}{|x-y|^{\mu}|y|^{\beta}}dy:=K_1+K_2.\tag{2.14}$$

On the one hand, for $x \in \mathbb{R}^N - B_{2r}(0)$, we have $|x-y| > |y|$, then

$$K_1 = \int_{B_r(0)} \frac{u^p(y)}{|x-y|^\mu |y|^\beta} dy < \int_{B_r(0)} \frac{u^p(y)}{|y|^{\mu+\beta}} dy \leq \|u\|_\infty^p \int_{B_r(0)} \frac{1}{|y|^{\mu+\beta}} dy < \infty,$$

For $x \in B_{2r}(0)$,

$$K_1 = \int_{B_r(0)} \frac{u^p(y)}{|x-y|^\mu |y|^\beta} dy \leq \int_{B_r(0)} \frac{u^p(y)}{|y|^{\mu+\beta}} dy + \int_{B_{3r}(x)} \frac{u^p(y)}{|x-y|^{\mu+\beta}} dy \leq \|u\|_\infty^p \int_{B_r(0)} \frac{1}{|y|^{\mu+\beta}} dy + \|u\|_\infty^p \int_{B_{3r}(x)} \frac{1}{|x-y|^{\mu+\beta}} dy < \infty.$$

Thus, we obtain $K_1 < \infty$.

On the other hand,

$$K_2 = \int_{\mathbb{R}^N - B_r(0)} \frac{u^p(y)}{|x-y|^\mu |y|^\beta} dy = \int_{(\mathbb{R}^N - B_r(0)) \cap B_r(x)} \frac{u^p(y)}{|x-y|^\mu |y|^\beta} dy + \int_{(\mathbb{R}^N - B_r(0)) \cap (\mathbb{R}^N - B_r(x))} \frac{u^p(y)}{|x-y|^\mu |y|^\beta} dy := K_{21} + K_{22}.$$

As for the preceding estimates, we have

$$K_{21} = \int_{(\mathbb{R}^N - B_r(0)) \cap B_r(x)} \frac{u^p(y)}{|x-y|^\mu |y|^\beta} dy \leq \frac{1}{r^\beta} \int_{(\mathbb{R}^N - B_r(0)) \cap B_r(x)} \frac{u^p(y)}{|x-y|^\mu} dy \leq \frac{1}{r^\beta} \int_{B_r(x)} \frac{u^p(y)}{|x-y|^\mu} dy \leq \frac{1}{r^\beta} \|u\|_\infty^p \int_{B_r(x)} \frac{1}{|x-y|^\mu} dy < \infty,$$

And we also have

$$K_{22} = \int_{(\mathbb{R}^N - B_r(0)) \cap (\mathbb{R}^N - B_r(x))} \frac{u^p(y)}{|x-y|^\mu |y|^\beta} dy \leq \int_{\mathbb{R}^N - B_r(0)} \frac{u^p(y)}{|y|^{\mu+\beta}} dy + \int_{\mathbb{R}^N - B_r(x)} \frac{u^p(y)}{|x-y|^{\mu+\beta}} dy.$$

Since

$$\int_{\mathbb{R}^N - B_r(0)} \frac{u^p(y)}{|y|^{\mu+\beta}} dy \leq \|u\|^p_{L^{\frac{pk}{k-1}}(\mathbb{R}^N - B_r(0))} \left\| \frac{1}{|y|^{\mu+\beta}} \right\|_{L^k(\mathbb{R}^N - B_r(0))} < \infty,$$

where $\frac{N}{\mu+\beta} < k < \frac{N}{N-p(N-\tau)}$ if $N - p(N-\tau) > 0$ while $k > \frac{N}{\mu+\beta}$ if $N - p(N-\tau) \leq 0$, from which we conclude that $K_{22} < \infty$. Hence, we easily see $K_2 < \infty$.

Therefore, combining with (2.14), we can conclude that

$$|x|^\alpha v(x) \in L^\infty(\mathbb{R}^N). \tag{2.15}$$

The proof is complete. □

2.3. **Symmetry.** By using the method of moving planes [7,8], we are able to prove the symmetry of the integrable solutions. For $\lambda \in \mathbb{R}$, define

$$\Sigma_\lambda = \{x = (x_1, \dots, x_n) \,:\, x_1 < \lambda\}, \quad x^\lambda = (2\lambda - x_1, \dots, x_n),$$
$$u_\lambda(x) = u(x^\lambda), \quad v_\lambda(x) = v(x^\lambda),$$

and

$$\Sigma_\lambda^u = \{x \in \Sigma_\lambda \,:\, u(x) > u_\lambda(x)\}, \quad \Sigma_\lambda^v = \{x \in \Sigma_\lambda \,:\, v(x) > v_\lambda(x)\}.$$

Lemma 2.7. *If (u, v) is a pair of positive solutions of the system (1.1), then we have*

$$u(x) - u_\lambda(x) = \int_{\Sigma_\lambda} \left(\frac{1}{|x-y|^{N-\tau}} - \frac{1}{|x^\lambda - y|^{N-\tau}} \right) \left[u^{p-1}(y)v(y) - u_\lambda^{p-1}(y)v_\lambda(y) \right] dy \quad (2.16)$$

and

$$v(x) - v_\lambda(x) = \int_{\Sigma_\lambda} \frac{1}{|x-y|^\mu} \left(\frac{u^p(y)}{|x|^\alpha |y|^\beta} - \frac{u_\lambda^p(y)}{|x^\lambda|^\alpha |y^\lambda|^\beta} \right) dy + \int_{\Sigma_\lambda} \frac{1}{|x^\lambda - y|^\mu} \left(\frac{u_\lambda^p(y)}{|x|^\alpha |y|^\beta} - \frac{u^p(y)}{|x^\lambda|^\alpha |y|^\beta} \right) dy. \quad (2.17)$$

Proof. By direct calculation, it is not difficult to get

$$\begin{aligned} u(x) &= \int_{\Sigma_\lambda} \frac{u^{p-1}(y)v(y)}{|x-y|^{N-\tau}} dy + \int_{\mathbb{R}^N - \Sigma_\lambda} \frac{u^{p-1}(y)v(y)}{|x-y|^{N-\tau}} dy \\ &= \int_{\Sigma_\lambda} \frac{u^{p-1}(y)v(y)}{|x-y|^{N-\tau}} dy + \int_{\Sigma_\lambda} \frac{u_\lambda^{p-1}(y)v_\lambda(y)}{|x-y^\lambda|^{N-\tau}} dy \end{aligned}$$

and

$$u_\lambda(x) = \int_{\Sigma_\lambda} \frac{u^{p-1}(y)v(y)}{|x-y^\lambda|^{N-\tau}} dy + \int_{\Sigma_\lambda} \frac{u_\lambda^{p-1}(y)v_\lambda(y)}{|x-y|^{N-\tau}} dy.$$

Since $|x^\lambda - y^\lambda| = |x - y|$ and $|x - y^\lambda| = |x^\lambda - y|$, we have

$$u(x) - u_\lambda(x) = \int_{\Sigma_\lambda} \left(\frac{1}{|x-y|^{N-\tau}} - \frac{1}{|x^\lambda - y|^{N-\tau}} \right) \left[u^{p-1}(y)v(y) - u_\lambda^{p-1}(y)v_\lambda(y) \right] dy.$$

Similarly, we obtain

$$v(x) = \int_{\Sigma_\lambda} \frac{u^p(y)}{|x|^\alpha |x-y|^\mu |y|^\beta} dy + \int_{\Sigma_\lambda} \frac{u_\lambda^p(y)}{|x|^\alpha |x-y^\lambda|^\mu |y^\lambda|^\beta} dy$$

and

$$v_\lambda(x) = \int_{\Sigma_\lambda} \frac{u^p(y)}{|x^\lambda|^\alpha |x^\lambda - y|^\mu |y|^\beta} dy + \int_{\Sigma_\lambda} \frac{u_\lambda^p(y)}{|x^\lambda|^\alpha |x^\lambda - y^\lambda|^\mu |y^\lambda|^\beta} dy,$$

from which we can deduce that

$$\begin{aligned} v(x) - v_\lambda(x) &= \int_{\Sigma_\lambda} \left(\frac{1}{|x|^\alpha |x-y|^\mu |y|^\beta} - \frac{1}{|x^\lambda|^\alpha |x^\lambda - y|^\mu |y|^\beta} \right) u^p(y) dy \\ &\quad + \int_{\Sigma_\lambda} \left(\frac{1}{|x|^\alpha |x^\lambda - y|^\mu |y^\lambda|^\beta} - \frac{1}{|x^\lambda|^\alpha |x^\lambda - y^\lambda|^\mu |y^\lambda|^\beta} \right) u_\lambda^p(y) dy \\ &= \int_{\Sigma_\lambda} \frac{1}{|x-y|^\mu} \left(\frac{u^p(y)}{|x|^\alpha |y|^\beta} - \frac{u_\lambda^p(y)}{|x^\lambda|^\alpha |y^\lambda|^\beta} \right) dy + \int_{\Sigma_\lambda} \frac{1}{|x^\lambda - y|^\mu} \left(\frac{u_\lambda^p(y)}{|x|^\alpha |y^\lambda|^\beta} - \frac{u^p(y)}{|x^\lambda|^\alpha |y|^\beta} \right) dy. \end{aligned}$$

□

For $p \geq 2$, we have

Lemma 2.8. *Under the assumption of Theorem 1.4, for any $\lambda < 0$, there exists a positive constant C such that*

$$\|u-u_\lambda\|_{L^{s_0}(\Sigma_\lambda^u)} \leq C\left[\|u\|^q_{L^{s_0}(\mathbb{R}^N)}\|u\|^{p-1}_{L^{s_0}(\Sigma_\lambda^u)} + \|u\|^p_{L^{s_0}(\Sigma_\lambda^v)}\|u\|^{q-1}_{L^{s_0}(\Sigma_\lambda^u)}\right]\|u-u_\lambda\|_{L^{s_0}(\Sigma_\lambda^u)}, \quad (2.18)$$

where $s_0 = \frac{2N(p-1)}{N-\gamma+\tau}$.

Proof. We firstly divide the domain of integration into four disjoint parts as

$$\Sigma_\lambda = \{\Sigma_\lambda^u \cap \Sigma_\lambda^v\} \cup \{\Sigma_\lambda^u - \Sigma_\lambda^v\} \cup \{\Sigma_\lambda^v - \Sigma_\lambda^u\} \cup \{\Sigma_\lambda - \Sigma_\lambda^u - \Sigma_\lambda^v\}.$$

If $x \in \Sigma_\lambda^u \cap \Sigma_\lambda^v$, using the Mean Value Theorem, we have

$$u^{p-1}v - u_\lambda^{p-1}v_\lambda \leq (p-1)u^{p-2}v(u-u_\lambda) + u^{p-1}(v-v_\lambda).$$

If $x \in \Sigma_\lambda^u - \Sigma_\lambda^v$, by the Mean Value Theorem, we get

$$u^{p-1}v - u_\lambda^{p-1}v_\lambda \leq (p-1)u^{p-2}v(u-u_\lambda).$$

If $x \in \Sigma_\lambda^v - \Sigma_\lambda^u$, we obtain

$$u^{p-1}v - u_\lambda^{p-1}v_\lambda \leq u^{p-1}(v-v_\lambda).$$

If $x \in \Sigma_\lambda - \Sigma_\lambda^u - \Sigma_\lambda^v$, then

$$u^{p-1}v - u_\lambda^{p-1}v_\lambda \leq 0.$$

Therefore, for any $x \in \Sigma_\lambda$, since $|x^\lambda - y| \geq |x-y|$, from (2.16) we have

$$u(x) - u_\lambda(x) \leq (p-1)\int_{\Sigma_\lambda^u} \frac{u^{p-2}v(u-u_\lambda)}{|x-y|^{N-\tau}}dy + \int_{\Sigma_\lambda^v}\frac{u^{p-1}(v-v_\lambda)}{|x-y|^{N-\tau}}dy. \quad (2.19)$$

For $x \in \Sigma_\lambda$, then $|x| > |x^\lambda|$, from (2.17), it follows

$$\begin{aligned} v(x) - v_\lambda(x) &\leq \frac{1}{|x|^\alpha}\int_{\Sigma_\lambda}\left(\frac{1}{|x-y|^\mu} - \frac{1}{|x^\lambda - y|^\mu}\right)\left(\frac{u^p(y)}{|y|^\beta} - \frac{u_\lambda^p(y)}{|y^\lambda|^\beta}\right)dy \\ &\leq \int_{\Sigma_\lambda^u}\frac{u^p(y) - u_\lambda^p(y)}{|x|^\alpha|x-y|^\mu|y|^\beta}dy. \end{aligned} \quad (2.20)$$

By using the Mean Value Theorem, we also obtain that

$$v(x) - v_\lambda(x) \leq p\int_{\Sigma_\lambda^u}\frac{u^{p-1}(y)\left[u(y) - u_\lambda(y)\right]}{|x|^\alpha|x-y|^\mu|y|^\beta}dy.$$

Clearly, we know $(u,v) \in L^{s_0}(\mathbb{R}^N) \times L^{r_0}(\mathbb{R}^N)$, where $s_0 = \frac{2N(p-1)}{N-\gamma+\tau}$ and $r_0 = \frac{2N(p-1)}{p\tau-(p-2)(N-\gamma)}$. Applying the Hardy-Littlewood-Sobolev inequality and the Hölder inequality, we deduce that

$$\begin{aligned} \|u-u_\lambda\|_{L^{s_0}(\Sigma_\lambda^u)} \leq & C\|u^{p-2}v(u-u_\lambda)\|_{L^{\frac{Ns_0}{N+\tau s_0}}(\Sigma_\lambda^u)} + C\|u^{p-1}(v-v_\lambda)\|_{L^{\frac{Ns_0}{N+\tau s_0}}(\Sigma_\lambda^v)} \\ \leq & C\|u\|^{p-2}_{L^{s_0}(\Sigma_\lambda^u)}\|v\|_{L^{r_0}(\Sigma_\lambda^u)}\|u-u_\lambda\|_{L^{s_0}(\Sigma_\lambda^u)} + C\|u\|^{p-1}_{L^{s_0}(\Sigma_\lambda^v)}\|v-v_\lambda\|_{L^{r_0}(\Sigma_\lambda^v)}. \end{aligned} \quad (2.21)$$

Similarly, from the Hölder inequality and the weighted HLS inequality, we have

$$\|v-v_\lambda\|_{L^{r_0}(\Sigma_\lambda^v)} \le C\|u^{p-1}(u-u_\lambda)\|_{L^{\frac{Nr_0}{N+(N-\alpha-\beta-\mu)r_0}}(\Sigma_\lambda^u\cap(\Sigma_\lambda^v))} \le C\|u\|^{p-1}_{L^{s_0}(\Sigma_\lambda^u)}\|u-u_\lambda\|_{L^{s_0}(\Sigma_\lambda^u)}. \tag{2.22}$$

In addition, we can also easily get

$$\|v\|_{L^{r_0}(\Sigma_\lambda^u)} \le C\|u^p\|_{L^{\frac{Nr_0}{N+(N-\alpha-\beta-\mu)r_0}}(\Sigma_\lambda^u)} \le C\|u\|^p_{L^{s_0}(\Sigma_\lambda^u)}. \tag{2.23}$$

Inserting (2.22) and (2.23) into (2.21), we obtain

$$\|u-u_\lambda\|_{L^{s_0}(\Sigma_\lambda^u)} \le C|u\|^p_{L^{s_0}(\Sigma_\lambda^u)}\|u\|^{p-2}_{L^{s_0}(\Sigma_\lambda^u)}\|u-u_\lambda\|_{L^{s_0}(\Sigma_\lambda^u)}+C\|u\|^{p-1}_{L^{s_0}(\Sigma_\lambda^v)}\|u\|^{p-1}_{L^{s_0}(\Sigma_\lambda^u)}\|u-u_\lambda\|_{L^{s_0}(\Sigma_\lambda^u)}.$$

Therefore, there exists a positive constant C such that

$$\|u-u_\lambda\|_{L^{s_0}(\Sigma_\lambda^u)} \le C\left[\|u\|^p_{L^{s_0}(\Sigma_\lambda^u)}\|u\|^{p-2}_{L^{s_0}(\Sigma_\lambda^u)}+\|u\|^{p-1}_{L^{s_0}(\Sigma_\lambda^v)}\|u\|^{p-1}_{L^{s_0}(\Sigma_\lambda^u)}\right]\|u-u_\lambda\|_{L^{s_0}(\Sigma_\lambda^u)}.$$

This completes the proof. □

In the following we will prove that for sufficiently negative λ, we have that

$$u(x) \le u(x^\lambda), \quad v(x) \le v(x^\lambda), \quad \forall x \in \Sigma_\lambda.$$

Thus, we are able to provide a beginning of the procedure of moving plane methods.

Lemma 2.9. *Under the assumption of Theorem 1.4, there exists $M > 0$ such that for $\lambda < -M$, we have*

$$u(x) \le u_\lambda(x), v(x) \le v_\lambda(x) \quad \forall x \in \Sigma_\lambda. \tag{2.24}$$

Proof. Since $u(x)$ and $v(x)$ are integrable, letting $\lambda \to -\infty$, we get

$$C\left[\|u\|^p_{L^{s_0}(\Sigma_\lambda^u)}\|u\|^{p-2}_{L^{s_0}(\Sigma_\lambda^u)}+\|u\|^{p-1}_{L^{s_0}(\Sigma_\lambda^v)}\|u\|^{p-1}_{L^{s_0}(\Sigma_\lambda^u)}\right] < 1. \tag{2.25}$$

Hence, as $\lambda \to -\infty$, from Lemma 2.8, we have

$$\|u(x)-u_\lambda(x)\|_{L^{s_0}(\Sigma_\lambda^u)} = 0, \quad \|v(x)-v_\lambda(x)\|_{L^{r_0}(\Sigma_\lambda^v)} = 0,$$

which implies $\Sigma_\lambda^u = \Sigma_\lambda^v = \emptyset$. Therefore, there exists $M > 0$ such that for $\lambda < -M$, we have $u(x) \le u(x^\lambda)$ and $v(x) \le v(x^\lambda)$ in Σ_λ. □

From Lemma 2.9, we can begin to move the plane $T_\lambda = \{x \in \mathbb{R}^N \mid x_1 = \lambda\}$ from $-\infty$ to the right as long as (2.24) holds. Naturally, we denote

$$\lambda_0 = \sup\{\lambda : u(x) \le u_\rho(x), v(x) \le v_\rho(x), x \in \Sigma_\rho, \rho \le \lambda\}.$$

By virtue of a similar argument as in Lemma 2.9 from λ near $+\infty$, we have $\lambda_0 < +\infty$. Next we conclude that

Lemma 2.10. *Under the assumption of Theorem 1.4, for any $\lambda_0 < 0$, we have*

$$u(x) \equiv u_{\lambda_0}(x), \ v(x) \equiv v_{\lambda_0}(x) \quad \forall x \in \Sigma_{\lambda_0}. \tag{2.26}$$

Proof. Suppose that at $\lambda_0 < 0$, we have $u(x) \leq u_{\lambda_0}(x)$ and $v(x) \leq v_{\lambda_0}(x)$, but $u(x) \not\equiv u_{\lambda_0}(x)$ or $v(x) \not\equiv v_{\lambda_0}(x)$ on Σ_{λ_0}. We claim that there exists an $\varepsilon > 0$ such that $u(x) \leq u_\lambda(x)$ and $v(x) \leq v_\lambda(x)$ on Σ_λ for any $\lambda \in [\lambda_0, \lambda_0 + \varepsilon)$.

In fact, for any $\eta > 0$, we can choose suitable $R > 0$ large enough such that

$$C\left[\|u\|^p_{L^{s_0}(\mathbb{R}^N - B_R(0))}\|u\|^{p-2}_{L^{s_0}(\mathbb{R}^N - B_R(0))} + \|u\|^{p-1}_{L^{s_0}(\mathbb{R}^N - B_R(0))}\|u\|^{p-1}_{L^{s_0}(\mathbb{R}^N - B_R(0))}\right] < \eta. \tag{2.27}$$

For such $R > 0$ and $\lambda > \lambda_0$, we can prove that the measure of $\Sigma_\lambda \cap B_R(0)$ goes to 0 as $\lambda \to \lambda_0$. In the case

$$v(x) \not\equiv v_{\lambda_0}(x) \quad \forall x \in \Sigma_{\lambda_0}.$$

By (2.16), we have

$$u(x) < u_{\lambda_0}(x), \ \forall x \in \Sigma_{\lambda_0}.$$

Naturally, for any $\delta > 0$, we define

$$D_\delta = \{x \in \Sigma_{\lambda_0} \cap B_R(0) : u_{\lambda_0}(x) - u(x) > \delta\},$$

$$E_\delta = \{x \in \Sigma_{\lambda_0} \cap B_R(0) : u_{\lambda_0}(x) - u(x) \leq \delta\},$$

and

$$G_\lambda = (\Sigma_\lambda - \Sigma_{\lambda_0}) \cap B_R(0).$$

Clearly, we get

$$\lim_{\delta \to 0} \mathcal{L}(E_\delta) = 0, \tag{2.28}$$

and

$$\lim_{\lambda \to \lambda_0} \mathcal{L}(G_\lambda) = 0, \tag{2.29}$$

where $\mathcal{L}$ is the Lebesgue measure. For any $x \in \Sigma^u_\lambda \cap D_\delta$, since

$$u(x) - u_\lambda(x) = u(x) - u_{\lambda_0}(x) + u_{\lambda_0}(x) - u_\lambda(x) > 0,$$

then

$$u_{\lambda_0}(x) - u_\lambda(x) > u_{\lambda_0}(x) - u(x) > \delta.$$

Thus, by the Chebyshev inequality, for fixed $\delta > 0$, we obtain that

$$\mathcal{L}(\Sigma^u_\lambda \cap D_\delta) \leq \frac{1}{\delta^{s_0}}\int_{\Sigma^u_\lambda \cap D_\delta} |u_{\lambda_0}(x) - u_\lambda(x)|^{s_0} dx \leq \frac{1}{\delta^{s_0}}\int_{B_R(0)} |u_{\lambda_0}(x) - u_\lambda(x)|^{s_0} dx \to 0 \tag{2.30}$$

if $\lambda \to \lambda_0$. Notice

$$\Sigma^u_\lambda \cap B_R(0) \subset (\Sigma^u_\lambda \cap D_\delta) \cup E_\delta \cup G_\lambda,$$

From (2.28)-(2.30), as $\lambda \to \lambda_0$ and $\delta \to 0$, we can easily get

$$\mathcal{L}(\Sigma^u_\lambda \cap B_R(0)) \to 0. \tag{2.31}$$

Combining (2.31) with (2.27), there exists an $\varepsilon > 0$ such that for any $\lambda \in [\lambda_0, \lambda_0 + \varepsilon)$,

$$C\left[\|u\|^p_{L^{s_0}(\Sigma^u_\lambda)}\|u\|^{p-2}_{L^{s_0}(\Sigma^u_\lambda)} + \|u\|^{p-1}_{L^{s_0}(\Sigma^v_\lambda)}\|u\|^{p-1}_{L^{s_0}(\Sigma^u_\lambda)}\right] \leq \frac{1}{2}.$$

By the same arguments as above, we can conclude $\Sigma^u_\lambda = \Sigma^v_\lambda = \emptyset$. Therefore, there exists an $\varepsilon > 0$ such that $u(x) \leq u_\lambda(x)$ and $v(x) \leq v_\lambda(x)$ on Σ_λ for any $\lambda \in [\lambda_0, \lambda_0 + \varepsilon)$. □

Proof of Theorem 1.4. Analogously, we can move the plane from $+\infty$ to left, and define

$$\lambda_1 = \inf\left\{\lambda \mid u(x) \le u_\rho(x),\ v(x) \le v_\rho(x),\ x \in \Sigma'_\rho,\ \rho \ge \lambda\right\},$$

where $\Sigma'_\rho = \left\{x \in \mathbb{R}^N \mid x_1 > \rho\right\}$. If $\lambda_0 = \lambda_1 \neq 0$, then both u and v are radially symmetric and decreasing about the plane $x_1 = \lambda_0$, which implies $u(x) \equiv u_{\lambda_0}(x)$ and $v(x) \equiv v_{\lambda_0}(x)$ on Σ_{λ_0}. Since $|x-y| < |x^{\lambda_0} - y|$ and $|y| > |y^{\lambda_0}|$, we easily deduce from (2.20) that

$$v(x) - v_{\lambda_0}(x) \le \int_{\Sigma_{\lambda_0}} \frac{1}{|x|^\alpha}\left(\frac{1}{|x-y|^\mu} - \frac{1}{|x^{\lambda_0}-y|^\mu}\right)\left(\frac{1}{|y|^\beta} - \frac{1}{|y^{\lambda_0}|^\beta}\right) u^p_{\lambda_0} dy < 0.$$

Therefore, from (2.16), we have

$$0 = u(x) - u_{\lambda_0}(x) = \int_{\Sigma_{\lambda_0}} \left(\frac{1}{|x-y|^{N-\tau}} - \frac{1}{|x^{\lambda_0}-y|^{N-\tau}}\right)(v - v_{\lambda_0}) u^{p-1}_{\lambda_0} dy < 0,$$

which is impossible. Thus, we get $\lambda_0 = \lambda_1 = 0$. Since the direction of x_1 is arbitrary, we conclude that u, v are radially symmetric and decreasing about origin. □

2.4. **Asymptotic behaviour.** Next we will obtain the main results of Theorem 1.5 by proving the following two Lemmas.

Lemma 2.11. *Let $p \in I_2$ and let (u, v) be a pair of positive solutions of (1.1) with $u \in L^{\frac{2N(p-1)}{N-\gamma+\tau}}(\mathbb{R}^N)$. Then we have*

1. *For small $|x|$, $u(x) \backsimeq A_0$, where $A_0 = \displaystyle\int_{\mathbb{R}^N} \frac{u^{p-1}(y)v(y)}{|y|^{N-\tau}} dy$.*
2. *For large $|x|$, $u(x) \backsimeq \dfrac{B}{|x|^{N-\tau}}$, where $B = \displaystyle\int_{\mathbb{R}^N} u^{p-1}(y)v(y)dy$.*

Proof. (1) We first claim that A_0 is finite.

Step 1. In fact, for small $|x|$, fix $\delta > 0$,

$$A_0 = \int_{B_\delta(0)} \frac{u^{p-1}(y)v(y)}{|y|^{N-\tau}} dy + \int_{\mathbb{R}^N - B_\delta(0)} \frac{u^{p-1}(y)v(y)}{|y|^{N-\tau}} dy := J_1 + J_2.$$

By an estimate similar to the one in (2.13), we have

$$\begin{aligned} J_1 \le & \|u\|_s^{p-1} \|v\|_r \left(\int_{B_\delta(0)} |y|^{\frac{\tau - N}{1 - \frac{p-1}{s} - \frac{1}{r}}} dy\right)^{1 - \frac{p-1}{s} - \frac{1}{r}} \\ \le & C\left(\int_0^\delta r^{N + \frac{\tau-N}{1-\frac{p-1}{s}-\frac{1}{r}}} \frac{dr}{r}\right)^{1-\frac{p-1}{s}-\frac{1}{r}} < \infty, \end{aligned} \tag{2.32}$$

where $\frac{1}{s}=\frac{\varepsilon}{(p-1)N}$ and $\frac{1}{r}=\frac{\varepsilon}{N}$ such that $\frac{p-1}{s}+\frac{1}{r}<\frac{\tau}{N}$, with $\varepsilon>0$ suitable small. Let k_1 be a number and let k_2 be its dual number such that $\frac{1}{k_1}+\frac{1}{k_2}=1$. Then

$$\begin{aligned}
J_2 \leq & \left(\int_{\mathbb{R}^N-B_\delta(0)}\frac{1}{|y|^{k_1(N-\tau)}}dy\right)^{\frac{1}{k_1}}\left(\int_{\mathbb{R}^N-B_\delta(0)}(u^{p-1}(y)v(y))^{k_2}dy\right)^{\frac{1}{k_2}}\\
\leq & C\|u\|_s^{p-1}\|v\|_r\left(\int_{\mathbb{R}^N-B_\delta(0)}\frac{1}{|y|^{k_1(N-\tau)}}dy\right)^{\frac{1}{k_1}}\\
\leq & C\|u\|_s^{p-1}\|v\|_r\left(\int_\delta^\infty r^{N-k_1(N-\tau)}\frac{dr}{r}\right)^{\frac{1}{k_1}}<\infty.
\end{aligned}$$

Here, we require $N-k_1(N-\tau)<0$ and $\frac{(p-1)k_2}{s}+\frac{k_2}{r}=1$, that is,

$$\frac{1}{k_1}<\frac{N-\tau}{N},\quad \frac{1}{k_2}=\frac{p-1}{s}+\frac{1}{r}>\frac{\tau}{N}. \tag{2.33}$$

In fact, take $\frac{1}{s}=\frac{N-\gamma+\tau+\varepsilon}{2N(p-1)}$, and $\frac{1}{r}=\max(\frac{\tau+\gamma-N+\varepsilon}{2N},\frac{\varepsilon}{2N})$ satisfying (2.33), with $\varepsilon>0$ sufficiently small, the integrability results shows that $(u,v)\in L^s(\mathbb{R}^N)\times L^r(\mathbb{R}^N)$. From the estimates of J_1 and J_2, we easily deduce that A_0 is finite.

Step 2. Now we are able to get the estimates when $|x|$ small. As $|x|\to 0$, we claim that

$$\left|\int_{\mathbb{R}^N}\frac{u^{p-1}(y)v(y)}{|x-y|^{N-\tau}}dy-\int_{\mathbb{R}^N}\frac{u^{p-1}(y)v(y)}{|y|^{N-\tau}}dy\right|\to 0. \tag{2.34}$$

For fixed $\delta>0$ in Step 1,

$$\begin{aligned}
&|\int_{\mathbb{R}^N}\frac{u^{p-1}(y)v(y)}{|x-y|^{N-\tau}}dy-\int_{\mathbb{R}^N}\frac{u^{p-1}(y)v(y)}{|y|^{N-\tau}}dy|\\
&\quad\leq\int_{B_{\delta(0)}}\frac{u^{p-1}(y)v(y)}{|x-y|^{N-\tau}}dy+\int_{B_{\delta(0)}}\frac{u^{p-1}(y)v(y)}{|y|^{N-\tau}}dy+\int_{\mathbb{R}^N-B_{\delta(0)}}\left|\frac{u^{p-1}(y)v(y)}{|x-y|^{N-\tau}}-\frac{u^{p-1}(y)v(y)}{|y|^{N-\tau}}\right|dy\\
&\quad:=N_1+N_2+N_3.
\end{aligned}\tag{2.35}$$

Here we assume that $\delta>2|x|$. Take $\frac{1}{s}=\frac{\varepsilon}{(p-1)N}$ and $\frac{1}{r}=\frac{\varepsilon}{N}$ such that $\frac{p-1}{s}+\frac{1}{r}<\frac{\tau}{N}$, with $\varepsilon>0$ suitable small, then $(u,v)\in L^s(\mathbb{R}^N)\times L^r(\mathbb{R}^N)$. Dividing N_1 into three parts, similar to the arguments as (2.13), we have

$$
\begin{aligned}
N_{11} := \int_{B_{\frac{|x|}{2}}(x)} \frac{u^{p-1}(y)v(y)}{|x-y|^{N-\tau}}dy &\le \|u\|_s^{p-1}\|v\|_r \left(\int_{B_{\frac{|x|}{2}}(x)} |x-y|^{\frac{\tau-N}{1-\frac{p-1}{s}-\frac{1}{r}}}dy\right)^{1-\frac{p-1}{s}-\frac{1}{r}}\\
&\le C\left(\int_{B_{\frac{|x|}{2}}(0)} |y|^{\frac{\tau-N}{1-\frac{p-1}{s}-\frac{1}{r}}}dy\right)^{1-\frac{p-1}{s}-\frac{1}{r}}\\
&\le C\left(\int_0^{\frac{|x|}{2}} r^{N+\frac{\tau-N}{1-\frac{p-1}{s}-\frac{1}{r}}}\frac{dr}{r}\right)^{1-\frac{p-1}{s}-\frac{1}{r}}\\
&\le C|x|^{d_1} \le C\delta^{d_1},
\end{aligned}
$$

where $d_1 = N(1-\frac{p-1}{s}-\frac{1}{r}) + (\tau - N) > 0$. Moreover, from the Hölder inequality, we obtain

$$
\begin{aligned}
N_{12} := \int_{B_{2|x|}(0)-B_{\frac{|x|}{2}}(x)} \frac{u^{p-1}(y)v(y)}{|x-y|^{N-\tau}}dy \le& C\int_{B_{2|x|}(0)-B_{\frac{|x|}{2}}(x)} \frac{u^{p-1}(y)v(y)}{|x|^{N-\tau}}dy\\
\le& C\frac{1}{|x|^{N-\tau}}\int_{B_{2|x|}(0)} u^{p-1}(y)v(y)dy\\
\le& C\|u\|_s^{p-1}\|v\|_r|x|^{d_1} \le C\delta^{d_1}.
\end{aligned}
$$

In addition, $|x-y| > \frac{|y|}{2}$ when $y \in B_\delta(0) - B_{2|x|}(0)$, then

$$
\begin{aligned}
N_{13} := \int_{B_\delta(0)-B_{2|x|}(0)} \frac{u^{p-1}(y)v(y)}{|x-y|^{N-\tau}}dy \le& C\int_{B_\delta(0)-B_{2|x|}(0)} \frac{u^{p-1}(y)v(y)}{|y|^{N-\tau}}dy\\
\le& C\int_{B_\delta(0)} \frac{u^{p-1}(y)v(y)}{|y|^{N-\tau}}dy = CN_2.
\end{aligned}
$$

Combining the estimates of N_{11}, N_{12} and N_{13}, together with (2.32), it follows that

$$
N_1 + N_2 = N_{11} + N_{12} + N_{13} + N_2 \le C\delta^{d_1}.
$$

For $|y| \ge \delta$ and $|x|$ small, we have

$$
\frac{u^{p-1}(y)v(y)}{|x-y|^{N-\tau}} \le C\frac{u^{p-1}(y)v(y)}{|y|^{N-\tau}}.
$$

By virtue of $\int_{\mathbb{R}^N} \frac{u^{p-1}(y)v(y)}{|y|^{N-\tau}}dy < \infty$, using the Lebesgue dominated convergence theorem, we get $N_3 \to 0$ as $|x| \to 0$. Letting $\delta \to 0$ in (2.35), we see (2.34) holds.

(2). We take suitable s and r the same as in the proof of Theorem 1.2, satisfying $\frac{p-1}{s} + \frac{1}{r} = 1$. Applying the Hölder inequality, we have

$$
B \le \|u\|_s^{p-1}\|v\|_r < \infty.
$$

For fixed $R > 0$,

$$
\begin{aligned}
|x|^{N-\tau}u(x) - B =& \int_{\mathbb{R}^N} \left(\frac{|x|^{N-\tau}}{|x-y|^{N-\tau}} - 1 \right) u^{p-1}(y)v(y)dy \\
=& \int_{B_R(0)} \left(\frac{|x|^{N-\tau}}{|x-y|^{N-\tau}} - 1 \right) u^{p-1}(y)v(y)dy + \int_{\mathbb{R}^N - B_R(0)} \left(\frac{|x|^{N-\tau}}{|x-y|^{N-\tau}} - 1 \right) u^{p-1}(y)v(y)dy \\
:=& G_1 + G_2.
\end{aligned}
$$

For large $|x|$, using the Lebesgue dominated convergence theorem, from

$$
|G_1| \leq \int_{B_R(0)} \left| \frac{|x|^{N-\tau}}{|x-y|^{N-\tau}} - 1 \right| u^{p-1}(y)v(y)dy \leq C \int_{B_R(0)} u^{p-1}(y)v(y)dy < \infty,
$$

we get $\lim\limits_{|x|\to\infty} |G_1| = 0$. Decompose it into two parts by

$$
G_{21} = \int_{(\mathbb{R}^N - B_R(0)) - B_{\frac{|x|}{2}}(x)} \frac{|x|^{N-\tau}}{|x-y|^{N-\tau}} u^{p-1}(y)v(y)dy
$$

and

$$
G_{22} = \int_{B_{\frac{|x|}{2}}(x)} \frac{|x|^{N-\tau}}{|x-y|^{N-\tau}} u^{p-1}(y)v(y)dy.
$$

Notice $|x-y| \geq \frac{|x|}{2}$ when $y \in (\mathbb{R}^N - B_R(0)) - B_{\frac{|x|}{2}}(x)$, we have

$$
G_{21} \leq C \int_{\mathbb{R}^N - B_R(0)} u^{p-1}(y)v(y)dy
$$

which implies $G_{21} \to 0$ as $R \to +\infty$.

Next, we estimate G_{22} when $|x| \to +\infty$. Clearly, from Theorem 1.4, we know u, v are radially symmetric and decreasing about $x_0 = 0$. Hence we can write

$$
U(r) = U(|x|) = u(x), \quad V(r) = V(|x|) = v(x).
$$

Thus, we deduce that for $y \in B_{\frac{|x|}{2}}(x)$,

$$
u(y) \leq u(\frac{x}{2}) = U(\frac{|x|}{2}), \quad v(y) \leq v(\frac{x}{2}) = V(\frac{|x|}{2}).
$$

Therefore,

$$
\begin{aligned}
G_{22} \leq& |x|^{N-\tau} U^{p-1}(\frac{|x|}{2}) V(\frac{|x|}{2}) \int_{B_{\frac{|x|}{2}}(x)} \frac{dy}{|x-y|^{N-\tau}} \\
\leq& C|x|^{N-\tau} U^{p-1}(\frac{|x|}{2}) V(\frac{|x|}{2}) \int_0^{\frac{|x|}{2}} r^{\tau} \frac{dr}{r} \\
\leq& C|x|^{N} U^{p-1}(\frac{|x|}{2}) V(\frac{|x|}{2}).
\end{aligned} \tag{2.36}
$$

By choosing suitable s, r such that $\frac{p-1}{s} + \frac{1}{r} > 1$, where s, r to be determined later, together with the integrability results, we get $(u, v) \in L^s(\mathbb{R}^N) \times L^r(\mathbb{R}^N)$.

In fact, when $\tau+\gamma>N$, we take $\frac{1}{s}=\frac{N-\gamma+\tau+2\varepsilon}{2N(p-1)}$ and $\frac{1}{r}=\frac{N+\gamma-\tau-\varepsilon}{2N}$ for sufficiently small $\varepsilon>0$. When $\tau+\gamma\leq N$, in view of $p>\frac{2N-\gamma-\tau}{N-\tau}$, we observe that $(p-1)\frac{N-\tau}{N}+\frac{\gamma}{N}>1$. From the continuity of the function $(\frac{1}{s},\frac{1}{r})\longmapsto\frac{p-1}{s}+\frac{1}{r}$, we infer that there exists suitable $\frac{1}{s}\in(0,\frac{N-\tau}{N})$ and $\frac{1}{r}\in(0,\frac{\gamma}{N})$ such that $\frac{p-1}{s}+\frac{1}{r}>1$.

Since u,v are decreasing about $x_0=0$, we have

$$U^s(\frac{|x|}{2})|x|^N\leq C\int_{B_{\frac{|x|}{2}}(0)-B_\rho(0)}u^s(y)dy\leq C$$

and

$$V^r(\frac{|x|}{2})|x|^N\leq C,$$

which implies

$$U(\frac{|x|}{2})\leq C|x|^{-\frac{N}{s}},\quad V(\frac{|x|}{2})\leq C|x|^{-\frac{N}{r}}. \tag{2.37}$$

Inserting (2.37) into (2.36), as $|x|\to+\infty$, we have

$$G_{22}\leq C|x|^{N(1-\frac{p-1}{s}-\frac{1}{r})}\to 0.$$

Therefore,

$$\lim_{|x|\to\infty}|x|^{N-\tau}u(x)-B=\lim_{|x|\to\infty}(G_1+G_{21}+G_{22})=0.$$

The proof is complete. □

We can see from Lemma 2.11 that the integrable solution of (1.1) decays fast with the rate of $N-\tau$.

Lemma 2.12. *Assume that the positive solution u is bounded and decay at infinity, we have the following conclusion:*

(1) *If $\theta<\frac{N-\gamma+\tau}{2(p-1)}$, then there does not exist a constant $C>0$ such that $|x|^\theta u(x)\geq C$ for large $|x|$.*

(2) *Suppose $u\notin L^{\frac{2N(p-1)}{N-\gamma+\tau}}(\mathbb{R}^N)$. If $\theta>\frac{N-\gamma+\tau}{2(p-1)}$, then there does not exist a constant $C>0$ such that $|x|^\theta u(x)\leq C$ for large $|x|$.*

Proof. (1). The decay rate of u is not slower than $\frac{N-\gamma+\tau}{2(p-1)}$. In fact, if there exists a constant $C>0$ such that $|x|^\theta u(x)\geq C$ for large $|x|$, where $\theta<\frac{N-\gamma+\tau}{2(p-1)}$. Similar to the proof in the proof of Theorem 1.1, we shall define $a_0=\theta$. Then it is true that $a_0-\frac{N-\gamma+\tau}{2(p-1)}<0$, so that $v(x)=\infty$ which is impossible.

(2). Assume $\theta>\frac{N-\gamma+\tau}{2(p-1)}$, there exists a constant $C>0$ such that $|x|^\theta u(x)\leq C$ for large $|x|$, we can deduce a contradiction that $u\in L^{\frac{2N(p-1)}{N-\gamma+\tau}}(\mathbb{R}^N)$.

Indeed, we have

$$\int_{\mathbb{R}^n}u^{\frac{2N(p-1)}{N-\gamma+\tau}}dx\leq\int_{B_R(0)}u^{\frac{2N(p-1)}{N-\gamma+\tau}}dx+\int_{\mathbb{R}^N-B_R(0)}u^{\frac{2N(p-1)}{N-\gamma+\tau}}dx\leq C_1+C_2\int_R^\infty r^{N-\theta\frac{2N(p-1)}{N-\gamma+\tau}}\frac{dr}{r}<\infty,$$

which is a contradiction. □

3. Conclusions for weighted Hartree type equation

In this section, we will study the qualitative results about the solution of (1.4). First, similar to Theorem 1.4 in [33], we have the following facts.

Lemma 3.1. *If $u \in C^2(\mathbb{R}^N)$ is a classical solution of (1.4), then u satisfies the integral equation*

$$\begin{cases} u(x) = \displaystyle\int_{\mathbb{R}^N} \frac{u^{p-1}(y)v(y)}{|x-y|^{N-2}}dy, & x \in \mathbb{R}^N, \\ v(x) = \displaystyle\int_{\mathbb{R}^N} \frac{u^p(y)}{|x|^\alpha |x-y|^\mu |y|^\alpha}dy, & x \in \mathbb{R}^N. \end{cases} \tag{3.1}$$

Next, we prove that an integrable solution of (1.4) is a finite energy solution, by utilizing the Pohozaev identity.

Lemma 3.2 (See [16])**.** *If $u \in W^{2,2}_{loc}(\mathbb{R}^N)$ is a positive solution of (1.4) with $\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u^p(x)u^p(y)}{|x|^\alpha|x-y|^\mu|y|^\alpha}dxdy < \infty$ and $\int_{\mathbb{R}^N} |\nabla u|^2 dx < \infty$, then the following identity holds*

$$\frac{2N-2\alpha-\mu}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u^p(x)u^p(y)}{|x|^\alpha|x-y|^\mu|y|^\alpha}dxdy = \frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla u|^2dx.$$

Proof of Theorem 1.8. We divide the proof into three steps.
Step 1. We will prove that $u^p v \in L^1(\mathbb{R}^N)$. According to Lemma 3.1, u solves (3.1), then $v > 0$. Assume $(u, v) \in L^s(\mathbb{R}^N) \times L^r(\mathbb{R}^N)$ is a pair of positive solutions of (3.1). Clearly, (u, v) satisfies the regularity results of the integral system. Take $\frac{1}{r} = \frac{\gamma}{2N}$. Then, by the Hölder inequality, we have

$$\int_{\mathbb{R}^N} u^p v dx \le \|u^p\|_t \|v\|_r$$

where $\frac{1}{t} = 1 - \frac{1}{r} = \frac{2N-\gamma}{2N}$. If $u \in L^{pt}(\mathbb{R}^N)$, then $u^p v \in L^1(\mathbb{R}^N)$. Indeed, since

$$p > \frac{3N-\gamma-\tau}{2(N-\tau)} > \frac{2N-\gamma}{2(N-\tau)},$$

it implies $pt > \frac{N}{N-\tau}$. Therefore, $u \in L^{pt}(\mathbb{R}^N)$, by Theorem 1.2.
Step 2. We claim $\nabla u \in L^2(\mathbb{R}^N)$.
Write $B_R = B_R(0)$. In view of $2^* > \frac{N}{N-2}$, from Theorem 1.2 (1), we can see that $u \in L^{2^*}(\mathbb{R}^N)$. Define a cut-off function

$$\xi_R(x) = \begin{cases} 1, |x| \le R \\ 0, |x| \ge 2R \end{cases}$$

which satisfies $|\xi_R(x)| \le 1$. Multiplying (1.4) by $u\xi_R^2(x)$ and integrating on B_{3R}, we have

$$\int_{B_{3R}} -\Delta u u \xi_R^2 dx = \int_{B_{3R}} u^p v \xi_R^2 dx.$$

Integrating by part, it follows that

$$\int_{B_{3R}} |\nabla u|^2 dx + 2\int_{B_{3R}} \nabla u \xi_R u \nabla \xi_R dx = \int_{B_{3R}} u^p v \xi_R^2 dx. \tag{3.2}$$

By the Cauchy inequality, for ε small enough, we get

$$\left|\int_{B_{3R}} \nabla u \xi_R u \nabla \xi_R dx\right| \leq \varepsilon \int_{B_{3R}} |\nabla u \xi_R|^2 dx + C_\varepsilon \int_{B_{3R}} |u \nabla \xi_R|^2 dx. \tag{3.3}$$

Since $u \in L^{2^*}(\mathbb{R}^N)$, applying the Hölder inequality to obtain

$$\int_{B_{3R}} |u \nabla \xi_R|^2 dx \leq \|u\|_{2^*}^2 \|(\nabla \xi_R)^2\|_{\frac{N}{2}} \leq C. \tag{3.4}$$

Combining (3.2)-(3.4) and $u^p v \in L^1(\mathbb{R}^N)$, by letting $R \to \infty$, we can deduce that $\nabla u \in L^2(\mathbb{R}^N)$.

Step 3. From $\nabla u \in L^2(\mathbb{R}^N)$ and $u \in L^{2^*}(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} \left(|u|^{2^*} + |\nabla u|^2\right) dx < \infty,$$

which implies there exists R_j such that

$$\lim_{R_j} \int_{\partial B_{R_j}} \left(|u|^{2^*} + |\nabla u|^2\right) ds = 0.$$

Multiplying (1.4) by u and integrating on B_R, we have

$$\int_{B_R} |\nabla u|^2 dx - \int_{\partial B_R} u \frac{\partial u}{\partial n} ds = \int_{B_R} u^p v dx,$$

where n is the outward unit normal vector on B_R. By the Hölder inequality, we see that

$$\left|\int_{\partial B_R} u \frac{\partial u}{\partial n} ds\right| \leq C R^{\frac{N-1}{N} - \frac{1}{2} - \frac{1}{2^*}} \left(R \int_{\partial B_R} |\nabla u|^2 ds\right)^{\frac{1}{2}} \left(R \int_{\partial B_R} |u|^{2^*} ds\right)^{\frac{1}{2^*}} \to 0$$

if $R = R_j \to \infty$. Therefore, as $R = R_j \to \infty$, we obtain

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx = \int_{\mathbb{R}^N} u^p v dx.$$

This ends the proof. $\square$

Theorem 1.8 shows that if $u \in L^{2^*}(\mathbb{R}^N)$ is a positive solution of (1.4), then $\int_{\mathbb{R}^N} \frac{u^p(y)}{|x|^\alpha |x-y|^\mu |y|^\alpha} dy u^p(x) \in L^1(\mathbb{R}^N)$ if and only if $\nabla u \in L^2(\mathbb{R}^N)$.

Theorem 1.9 implies that $p = \frac{2N-2\alpha-\mu}{N-2}$ is necessary for the existence of integrable solutions of (1.4) if $p \in I_2$.

Proof of Theorem 1.9. According to Theorem 1.8, we get

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^p(x) u^p(y)}{|x|^\alpha |x-y|^\mu |y|^\alpha} dx dy < \infty$$

and

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx < \infty.$$

Moreover, we have

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u^p(x)u^p(y)}{|x|^\alpha |x-y|^\mu |y|^\alpha} dxdy. \tag{3.5}$$

Combining (3.5) and Lemma 3.2, we obtain that

$$\frac{2N-2\alpha-\mu}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u^p(x)u^p(y)}{|x|^\alpha |x-y|^\mu |y|^\alpha} dxdy = \frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u^p(x)u^p(y)}{|x|^\alpha |x-y|^\mu |y|^\alpha} dxdy.$$

Namely,

$$\left(\frac{2N-2\alpha-\mu}{2p} - \frac{N-2}{2}\right)\int_{\mathbb{R}^n}\int_{\mathbb{R}^N} \frac{u^p(x)u^p(y)}{|x|^\alpha |x-y|^\mu |y|^\alpha} dxdy = 0,$$

which implies $p = \frac{2N-2\alpha-\mu}{N-2}$. □

Proof of Theorem 1.10.
(1) ⇒ (2): Theorem 1.8 shows (2).
(2) ⇒ (3): Evidently, (2) shows that (3) is true.
(3) ⇒ (1): Let (3) hold. By virtue of $p > \frac{3N-\gamma-2}{2(N-2)}$ which shows $N - \frac{2N(p-1)}{N-\gamma+2}(N-2) < 0$, for fixed $R > 0$, we have

$$\int_{\mathbb{R}^N} u^{\frac{2N(p-1)}{N-\gamma+2}}(x)dx \le \int_{B_R(0)} u^{\frac{2N(p-1)}{N-\gamma+2}}(x)dx + \int_{\mathbb{R}^N - B_R(0)} u^{\frac{2N(p-1)}{N-\gamma+2}}(x)dx \le C_1 + C_2\int_R^\infty r^{N-\frac{2N(p-1)}{N-\gamma+2}(N-2)}\frac{dr}{r} < \infty.$$

This completes the proof. □

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