Liouville theorems and elliptic gradient estimates for a nonlinear parabolic equation involving the Witten Laplacian

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LIOUVILLE THEOREMS AND ELLIPTIC GRADIENT ESTIMATES FOR A NONLINEAR PARABOLIC EQUATION INVOLVING THE WITTEN LAPLACIAN

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Abstract. In this paper we establish local and global elliptic type gradient estimates for a nonlinear parabolic equation on a smooth metric measure space whose underlying metric and potential satisfy a (k,m)-super Perelman Ricci flow inequality. We discuss a number of applications and implications including curvature free global estimates and some constancy and Liouville type results.

1. Introduction

In this paper we establish elliptic type gradient estimates for positive solutions to a class of nonlinear parabolic equations involving the Witten Laplacian in the context of smooth metric measure spaces where both the metric and potential are time dependent. We also discuss some applications and consequences of these estimates most notably to Liouville theorems and global estimates.

Gradient estimates have a central place in geometric analysis with a vast scope of applications (for some recent work see [2, 6, 9, 11, 14, 15, 16], [5, 23, 25, 29, 31, 32] as well as [3, 4, 10, 13, 27, 34] and the references therein). The estimates of interest in this paper fall under the category of Souplet-Zhang and Hamilton type estimates that were first formulated and proved for the heat equation on static manifolds in [11, 23]. Such estimates along with the differential Harnack and Li-Yau type inequalities have since been extensively studied and applied to families of parabolic equations on manifolds with both static and evolving metrics (e.g., under the Ricci flow and its relatives). They are well known not only for their effectiveness in proving classical Harnack inequalities, Liouville theorems and sharp bounds on heat kernels to mention a few but also more recently and in conjunction with the Perelman entropy method for establishing sharp geometric and functional inequalities in a variety of contexts and settings (see, e.g., [4, 10, 13, 15, 16, 17, 18, 28, 34]). In this paper we make a contribution to the subject by deriving such estimates in the context of smooth metric measure spaces equipped with time dependent metrics and potentials under suitable bounds on the generalised Bakry-Émery curvature tensor before presenting some applications most notably here to Liouville theorems.

To this end suppose that $(M, g, e^{-f} dv)$ is a smooth metric measure space by which it is meant that (M, g) is a complete Riemannian manifold of dimension n (here $n \geq 2$), f is a smooth potential on M and $d\mu = e^{-f} dv$ is a weighted measure on M conformally equivalent to the Riemannain volume measure $dv = dv_g$. (For the significance and role

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of smooth metric measure spaces in modern geometric analysis see [3, 4, 10], [22, 28], the sizeable recent literature, e.g., [8, 16, 17, 19, 24, 30, 33, 34] as well as the discussion in the next section). A (k, m)-super Perelman-Ricci flow on M refers to the evolution of a one parameter family of smooth metric-potential pairs (g, f) on M subject to the relation

$$\frac{1}{2}\frac{\partial g}{\partial t}(x,t) + Ric_f^m(g)(x,t) \geq -kg(x,t), \qquad t > 0. \tag{1.1}$$

Here k, m are real constants with $m \geq n$ (not necessarily an integer). The second order symmetric tensor field $Ric_f^m(g)$ is the Bakry-Émery m-Ricci curvature tensor given in this evolutionary context by [*cf.* also (2.2) below]

$$Ric_f^m(g)(x,t) = Ric(g)(x,t) + \nabla_{g(t)}\nabla_{g(t)} f(x,t) - \frac{\nabla_{g(t)} f \otimes \nabla_{g(t)} f}{m-n}(x,t), \tag{1.2}$$

with $Ric(g)$ the usual Ricci tensor and $\nabla_g \nabla_g f$ the Hessian of f. Note that the definition in (1.2) extends to $m = n$ only for constant functions f hence giving $Ric_f^n(g) \equiv Ric(g)$, and to $m = \infty$ by $Ric_f(g) = Ric_f^\infty(g) \equiv Ric(g) + \nabla_g \nabla_g f$. In this paper we consider positive smooth solutions to the semilinear parabolic equation on M:

$$\frac{\partial u}{\partial t} - \Delta_f u = F(u), \qquad t > 0, \tag{1.3}$$

where $\mathscr{L} = \Delta_f$ denotes the Witten Laplacian (also known as Nelson's diffusion operator in stochastic mechanics, or else the weighted Laplacian). It acts on smooth functions $v \in \mathscr{C}^\infty(M)$ by [1]

$$\Delta_f v = e^f \operatorname{div}(e^{-f}\nabla v) = \Delta v - \langle \nabla f, \nabla v \rangle, \tag{1.4}$$

and constitutes a natural extension of the Laplace-Beltrami operator to the context of smooth metric measure spaces. The Witten Laplacian has close links with Markov processes and probability theory, quantum field theory, Riemannian geometry and more modern areas in mathematical physics. Our goal here is to establish among other things local and global elliptic type gradient estimates for positive smooth solutions to (1.3) when both the metric and potential evolve under a (k, m)-super Perelman-Ricci flow (see Theorem 3.1) and discuss some implications of these estimates.

Regarding the nonlinearity in (1.3) we take $F = F(u)$ a sufficiently smooth function on the half axis $u > 0$. Of particular interest, prompted by the surge of recent research work in literature, is when F has a power-like growth and/or a logarithimic singularity at infinity, e.g., $F(u) = Au^p \log u + Bu^q$ with real p, q or $F(u) = Au\,|\log u|^p + Bu^q$ with real p, q and $p > 1$ (see Sections 5 and 6 for more). It turns out that a crucial role in the estimates is played here by the non-negative, F and u dependent quantity,

$$\mathsf{R}_F(u) = \left[\frac{2u(1+\sqrt{u})F'(u) - (1+2\sqrt{u})F(u)}{2(1+\sqrt{u})^2}\right]_+. \tag{1.5}$$

We discuss some consequences of these estimates and other closely related issues. In particular and as a byproduct we establish some new relative evolutionary estimates that include a Hamilton-Zhang type global gradient estimate to the nonlinear context with dimension free constants. As the static case is a particular instance of the system we also discuss consequences of the gradient estimate to that context. In particular

[1] Here Δ denotes the Laplace-Beltrami operator with local description $[\partial_j(\sqrt{\det g}\, g^{jk}\partial_k)]/\sqrt{\det g}$.

here we establish new Liouville type results for the stationary elliptic counterpart of (1.3): $\Delta_f u + F(u) = 0$ subject to the non-negativity of the Bakry-Émery curvature tensors $\mathscr{R}ic_f^m \geq 0$ in Section 5 and global curvature free gradient estimates for the full parabolic equation (1.3) in Section 6 (see in particular Theorem 6.1 and the subsequent discussion). The non-negativity and/or monotonicity of related entropy like quantities is a question of great contemporary interest [8, 15, 22, 27, 28, 34].

2. Preliminaries

Let (M, g) be a complete n-dimensional Riemannian manifold. Given $f \in \mathscr{C}^\infty(M)$ put $d\mu = e^{-f} dv_g$ where dv_g is the Riemannian volume measure. The triple $(M, g, d\mu)$ is then called a smooth metric measure space or equivalently a weighted manifold (see [3, 4, 10, 19] for nice and exquisite introductions to the subject). [2]

The f-Laplacian (1.4) is the natural substitute for the Laplace-Beltrami operator in this context (in fact as is easily seen the two coincide when the potential f is constant). It is a symmetric diffusion operator with respect to the invariant measure $d\mu = e^{-f} dv_g$ [borrowing the terminology from the theory of Dirichlet forms and Itô's SDE theory prompted by the second integral in (2.1)] and as one readily verifies for $u, v \in \mathscr{C}_0^\infty(M)$:

$$\int_M e^{-f} v \Delta_f u \, dv_g = \int_M -e^{-f} \langle \nabla u, \nabla v \rangle \, dv_g = \int_M e^{-f} u \Delta_f v \, dv_g. \tag{2.1}$$

Next regarding the curvature properties of $(M, g, d\mu)$ the theory of carré du champ for Markov diffusion operators and the curvature dimension condition $\mathrm{CD}(k, m)$ as developed by Bakry-Émery (*see* [3, 4]) naturally lead to the generalised m-Ricci curvature tensor $Ric_f^m = Ric_f^m(g)$ associated with the f-Laplacian defined by

$$Ric_f^m(g) = Ric(g) + \nabla\nabla f - \frac{\nabla f \otimes \nabla f}{m - n}. \tag{2.2}$$

When $m = n$ to make sense of (2.2) one only allows constant functions f as admissible potentials in which case $\Delta_f = \Delta$ and $Ric_f^n(g) \equiv Ric(g)$. By contrast when $m = \infty$ one simply defines $Ric_f(g) := Ric_f^\infty(g) = Ric(g) + \nabla^2 f$. These different Ricci type tensors carry important geometric information and are of great utility in the theory. Note that here we have at our disposal the weighted version of the Bochner-Weitzenböck formula

$$\frac{1}{2}\Delta_f |\nabla u|^2 = |\nabla\nabla u|^2 + \langle \nabla u, \nabla \Delta_f u \rangle + Ric_f(\nabla u, \nabla u). \tag{2.3}$$

Upon switching the inequality sign in (1.1) to equality we get the (k, m) Perelman-Ricci flow [(2.5) below]. Under the additional assumption that the weighted measure $d\mu = e^{-f} dv$ remains static in time, it can be seen that

$$\frac{\partial d\mu_t}{\partial t} = \frac{\partial}{\partial t}\left(e^{-f(t)} \, dv\right) = \left[-\frac{\partial f}{\partial t} + \frac{1}{2}\mathrm{Tr}\left(\frac{\partial g}{\partial t}\right)\right] d\mu_t = 0. \tag{2.4}$$

[2] We also point out that some authors use the term a manifold with density [10, 20].

Here Tr stands for the metric trace on $(0,2)$-tensors and the combined system resulting from the flow and the stationary volume measure constraint takes the form

$$\frac{1}{2}\frac{\partial g}{\partial t}(x,t) + Ric^m_f(g)(x,t) = -kg(x,t), \tag{2.5}$$

$$\frac{\partial f}{\partial t}(x,t) - \frac{1}{2}\mathrm{Tr}\left(\frac{\partial g}{\partial t}\right)(x,t) = 0. \tag{2.6}$$

In particular it follows from (2.5) and (2.6) that the potential f satisfies the evolution equation

$$\frac{\partial f}{\partial t} + \Delta f = \mathrm{Tr}\left(\frac{1}{2}\frac{\partial g}{\partial t} + \nabla\nabla f\right) = -R + \frac{|\nabla f|^2}{m-n} - nk. \tag{2.7}$$

The case $(k,m) = (0,\infty)$ in the system (2.5)-(2.6) is what was introduced by Perelman in [22] as the L^2-gradient flow of the functional $\mathscr{P}(g,f) = \int_M (R + |\nabla f|^2)e^{-f}dv_g$ subject to the stationary volume measure constraint considered above. Here $R = R(g)$ is the scalar curvature, $Ric^m_f(g) = Ric(g) + \nabla^2 f$ is the modified Ricci tensor, and f is the potential satisfying the so-called conjugate or adjoint heat equation,

$$\Box^\star f = -\frac{\partial f}{\partial t} - \Delta f + R = 0. \tag{2.8}$$

The above discussion illustrates another aspect of how the problems considered here link with the larger and more recent body of research on the subject and the remarkable work on the Poincaré conjecture [22]. For further reading and related work with volume measure constraints see [8, 12, 15, 16, 21, 25, 34] and the references therein.

Notation. For any pair of points x, y on M we designate by $d = d_{g(t)}(x,y)$ or $d(x,y;t)$ the Riemannian distance between x, y with respect to the metric $g = g(t)$. Fixing a reference point x_0 on M we denote by $\varrho = \varrho(x,t)$ the geodesic radial variable measuring the distance between x and x_0 at time t. For $R, T > 0$ we introduce the compact set $\mathcal{B}_{R,T} \equiv \mathcal{B}_{R,T}(x_0) \equiv \{(x,t) : d(x,x_0;t) \le R, 0 \le t \le T\} \subset M \times [0,T]$. Throughout we assume $R \ge 2$ and we make use of the notation $s_+ = \max(s,0)$ and $s_- = \max(-s,0)$.

3. A Hamilton-Souplet-Zhang type gradient estimate

The main result here is a local elliptic gradient estimate on positive smooth solutions to the equation (1.3). Here the metric $g = g(t)$ and the potential $f = f(t)$ evolve under the (k,m)-super Perelman-Ricci flow (1.1) [see the discussion following the theorem] and to make it clear the fact that both the metric and the potential are time dependent means that in explicit terms $\Delta_f = \Delta_{g(t)} - \langle \nabla_{g(t)} f, \nabla_{g(t)} \rangle$ [*cf.* (1.4)]. For the purpose of Theorem 3.1 below $x_0 \in M$ and $R \ge 2$ are chosen and are fixed whilst $T > 0$ and $\mathcal{B}_{R,T}$ is as defined above. The estimate makes use of the constants $\mathsf{h}, \mathsf{k} \ge 0$ describing the lower bound $Ric^m_f(g) \ge -\mathsf{k}_m g$ with $\mathsf{k}_m = (m-1)\mathsf{k}$ or $(n-1)\mathsf{k}$ depending on $m < \infty$ or $m = \infty$ respectively and $\partial g/\partial t \ge -2\mathsf{h} g$ in the compact set $\mathcal{B}_{R,T}$. Note finally that due to the local nature of the estimate here all one needs is for u to be a positive solution in an open set containing $\mathcal{B}_{R,T}$.

Theorem 3.1. *Let u be a positive solution to* (1.3) *and* $\{(M, g(t), f(t)) : t \in [0,T]\}$ *be a complete solution to the super Perelman-Ricci flow* (1.1) *with* $Ric^m_f(g) \ge -\mathsf{k}_m g$ *and*

$\partial g/\partial t \geq -2\mathsf{h}g$ for some $\mathsf{h}, \mathsf{k}_m \geq 0$ in $\mathcal{B}_{R,T}$. Then there exists $C > 0$ depending only on n, m such that, for all $(x,t) \in \mathcal{B}_{R/2,T}$ with $t > 0$,

$$\frac{|\nabla \sqrt{u}|}{\sqrt{u}+1} \leq C \left\{ \left(\frac{1}{R} + \sqrt{\frac{\zeta}{R}} + \sqrt{\mathsf{K}} + \frac{1}{\sqrt{t}} \right) \left(\sup_{\mathcal{B}_{R,T}} \sqrt{u} \right) + \sup_{\mathcal{B}_{R,T}} \sqrt{\mathsf{R}_F(u)} \right\}. \tag{3.1}$$

Here $\mathsf{R}_F(u) = \{[2u(1+\sqrt{u})F'(u) - (1+2\sqrt{u})F(u)]/[2(1+\sqrt{u})^2]\}_+$, $\mathsf{K} = \sqrt{\mathsf{h}^2 + \mathsf{k}^2}$ and $\zeta = [Z_{\Delta_f}]_+$ where

$$Z_{\Delta_f} = \max_{(x,t)} \{\Delta_f \varrho(x,t) : d(x,x_0;t) = 1,\ 0 \leq t \leq T\}. \tag{3.2}$$

Moreover in the case $m < \infty$ one can remove the term $\sqrt{\zeta/R}$ in (3.1).

Remark 3.2. From the bounds $Ric_f^m(g) \geq -\mathsf{k}_m g$ and $\partial g/\partial t \geq -2\mathsf{h}g$, one obtains (1.1) with $k = \mathsf{k}_m + \mathsf{h}$, i.e., $k = (m-1)\mathsf{k} + \mathsf{h}$ for $m < \infty$ and $k = (n-1)\mathsf{k} + \mathsf{h}$ for $m = \infty$. The lower bound $Ric_f^m(g) \geq -\mathsf{k}_m g$ here is required for the Wei-Wylie weighted Laplacian comparison theorem [step (e) in the proof] and the bound $\partial g/\partial t \geq -2\mathsf{h}g$ for controlling the time derivative of the geodesic distance, namely, $d_t = \partial[d_{g(t)}(x,x_0)]/\partial t$ [in step (f)]. As is evident from (2.2) a lower bound on $Ric_f(g)$ is a *weaker* assumption than one on $Ric_f^m(g)$ for $m < \infty$ by virtue of $Ric_f^m \geq kg$ implying $Ric_f \geq kg$ but not *vice versa*.

Remark 3.3. In the static case $\partial g/\partial t \equiv 0$ and $\partial f/\partial t \equiv 0$ we set $\mathsf{h} = 0$ and then $\mathsf{K} = \mathsf{k}$. As such Theorem 3.1 also gives local gradient estimates for positive solutions of (1.3) on the *static* metric measure space $(M, g, d\mu)$ with $Ric_f^m(g) \geq -\mathsf{k}_m g$. See Section 5.

If u is a positive bounded solution to (1.3) and the lower bounds $Ric_f^m(g) \geq -\mathsf{k}_m g$ and $\partial g/\partial t \geq -2\mathsf{h}g$ in Theorem 3.1 are global on $M \times [0,T]$, then by passing to $R \nearrow \infty$, we have the following global counterpart of (3.1).

Corollary 3.4. *Under the assumptions of Theorem 3.1 and the global bounds $Ric_f^m(g) \geq -\mathsf{k}_m g$ and $\partial g/\partial t \geq -2\mathsf{h}g$ on $M \times [0,T]$ with $\mathsf{h}, \mathsf{k}_m \geq 0$, if u is a positive bounded solution to (1.3), there exists $C > 0$ such that for $0 < t \leq T$ and $x \in M$,*

$$\frac{|\nabla \sqrt{u}|}{\sqrt{u}+1} \leq C \left\{ \left(\sqrt{\mathsf{K}} + \frac{1}{\sqrt{t}} \right) \left(\sup_{M\times[0,T]} \sqrt{u} \right) + \sup_{M\times[0,T]} \sqrt{\mathsf{R}_F(u)} \right\}. \tag{3.3}$$

Note that the boundedness of u does not necessarily imply that the right-hand side of (3.3) is always finite, however, this is not an issue for the estimate itself (see Section 5 for more on this). The proof of Theorem 3.1 and all the necessary tools span Section 4. In Sections 5 and 6 we give some interesting consequences of the local gradient estimate (3.1) that includes global elliptic and parabolic estimates for positive solutions, a curvature free estimate for the case when M is closed and a number of important Liouville type results on the elliptic (non-evolutionary) counterpart of (1.3).

4. Proof of the Main Estimate and Theorem 3.1

4.1. **Parabolic lemmas.** Before proceeding onto the proof of Theorem 3.1, we present a chain of lemmas that will be used in the course of the argument. First we establish some parabolic estimates under the flow on auxiliary functions involved in the proof. Note that hereafter for convenience we abbreviate the metric inner product by writing $\nabla h_1 \nabla h_2 = \langle \nabla h_1, \nabla h_2 \rangle$. We also write $h_t = \partial h/\partial t$ and as before $\Delta_f h = \Delta h - \langle \nabla f, \nabla h \rangle$.

Lemma 4.1. *Let u be a positive solution to (1.3) and set $h = \sqrt{u}$. Then h is a positive solution to the equation*

$$h_t - \Delta_f h - \frac{|\nabla h|^2}{h} = \frac{F(h^2)}{2h}. \tag{4.1}$$

Proof. A straightforward calculation with $h = \sqrt{u}$ gives $2h_t = u_t/\sqrt{u}$, $2\nabla h = \nabla u/\sqrt{u}$ and $2\Delta h = [\Delta u - |\nabla u|^2/2u]/\sqrt{u}$. Moreover it is easily seen that

$$\frac{|\nabla u|}{u} = |\nabla \log u| = 2|\nabla \log \sqrt{u}| = 2\frac{|\nabla \sqrt{u}|}{\sqrt{u}}.$$

Hence referring to (1.3) and (1.4) and putting the above pieces together we have

$$\begin{aligned} 2\Delta_f h &= 2\Delta h - 2\nabla f \nabla h = [\Delta u - |\nabla u|^2/2u]/\sqrt{u} - \nabla f \nabla u/\sqrt{u} \\ &= [\Delta_f u - |\nabla u|^2/2u]/\sqrt{u} = [u_t - F(u) - |\nabla u|^2/2u]/\sqrt{u} \\ &= [2hh_t - F(h^2) - 2|\nabla h|^2]/h, \end{aligned}$$

and so the conclusion follows immediately. □

Lemma 4.2. *Suppose $g = g(t)$ and $f = f(t)$ are of class $\mathscr{C}^2$ and let h be a positive solution to (4.1). Put $W = |\nabla h|^2/(1+h)^2$. Then*

$$\begin{aligned} \Delta_f W - W_t = &\frac{g_t(\nabla h, \nabla h)}{(1+h)^2} + \frac{2Ric_f^m(\nabla h, \nabla h)}{(1+h)^2} + 2\left|\frac{\nabla^2 h}{1+h} - \frac{\nabla h \otimes \nabla h}{(1+h)^2}\right|^2 \\ &+ \frac{2[\nabla f \otimes \nabla f](\nabla h, \nabla h)}{(m-n)(1+h)^2} - \frac{2(2h+1)\nabla h \nabla W}{(1+h)h} + \frac{2(1+h)W^2}{h^2} \\ &+ \left[\frac{(1+2h)F(h^2) - 2h^2(1+h)F'(h^2)}{(1+h)h^2}\right] W. \end{aligned} \tag{4.2}$$

Proof. By using the description $W = |\nabla h|^2/(1+h)^2$ and the conclusion of Lemma 4.1 let us proceed by explicitly calculating the left-hand side of (4.2). Towards this end we first note that

$$\begin{aligned} W_t = \left[\frac{|\nabla h|^2}{(1+h)^2}\right]_t &= \frac{(|\nabla h|^2)_t}{(1+h)^2} - \frac{2|\nabla h|^2 h_t}{(1+h)^3} \\ &= -\frac{g_t(\nabla h, \nabla h)}{(1+h)^2} + \frac{2\nabla h \nabla h_t}{(1+h)^2} - \frac{2|\nabla h|^2 h_t}{(1+h)^3}, \end{aligned} \tag{4.3}$$

where we have made use of the identity $(|\nabla h|^2)_t = -g_t(\nabla h, \nabla h) + 2\nabla h \nabla h_t$. Similarly, for the gradient and the Laplacian of W we have,

$$\begin{aligned} \nabla W &= \frac{\nabla |\nabla h|^2}{(1+h)^2} - \frac{2|\nabla h|^2 \nabla h}{(1+h)^3}, \\ \Delta W &= \frac{\Delta |\nabla h|^2}{(1+h)^2} - \frac{2|\nabla h|^2 \Delta h}{(1+h)^3} - \frac{4\nabla h \nabla |\nabla h|^2}{(1+h)^3} + \frac{6|\nabla h|^4}{(1+h)^4}, \end{aligned}$$

which then gives

$$\Delta_f W = \Delta W - \nabla f \nabla W = \frac{\Delta_f |\nabla h|^2}{(1+h)^2} - \frac{2|\nabla h|^2 \Delta_f h}{(1+h)^3} - \frac{4\nabla h \nabla |\nabla h|^2}{(1+h)^3} + \frac{6|\nabla h|^4}{(1+h)^4}. \tag{4.4}$$

Now putting together the individual descriptions of W_t and $\Delta_f W$ from (4.3) and (4.4) respectively and taking into account the relevant cancellations we have

$$\begin{aligned}\Delta_f W - W_t =& \frac{\Delta_f |\nabla h|^2}{(1+h)^2} + \frac{g_t(\nabla h, \nabla h)}{(1+h)^2} - \frac{2\nabla h}{(1+h)^2}\nabla\left(\Delta_f h + \frac{|\nabla h|^2}{h} + \frac{F(h^2)}{h}\right)\\ &+ \frac{2|\nabla h|^4}{(1+h)^3 h} + \frac{2|\nabla h|^2}{(1+h)^3}\frac{F(h^2)}{h} - \frac{4\nabla h\nabla|\nabla h|^2}{(1+h)^3} + \frac{6|\nabla h|^4}{(1+h)^4},\end{aligned} \tag{4.5}$$

or after some algebra

$$\begin{aligned}\Delta_f W - W_t =& \frac{\Delta_f |\nabla h|^2}{(1+h)^2} - \frac{2\nabla h\nabla|\nabla h|^2}{(1+h)^2} + \frac{g_t(\nabla h, \nabla h)}{(1+h)^2}\\ &- \frac{2\nabla h}{(1+h)^2}\left(\frac{\nabla|\nabla h|^2}{h} - \frac{|\nabla h|^2\nabla h}{h^2} + \frac{2hF'(h^2)\nabla h}{2h} - \frac{F(h^2)\nabla h}{2h^2}\right)\\ &+ \frac{2|\nabla h|^4}{(1+h)^3 h} + \frac{2|\nabla h|^2}{(1+h)^3}\frac{F(h^2)}{2h} - \frac{4\nabla h\nabla|\nabla h|^2}{(1+h)^3} + \frac{6|\nabla h|^4}{(1+h)^4}.\end{aligned} \tag{4.6}$$

Hence by using the weighted Bochner-Weitzenböck formula (2.3) we can write

$$\begin{aligned}\Delta_f W - W_t =& \frac{2Ric_f^m(\nabla h, \nabla h)}{(1+h)^2} + \frac{2|\nabla^2 h|^2}{(1+h)^2} + \frac{g_t(\nabla h, \nabla h)}{(1+h)^2} + \frac{2[\nabla f\otimes\nabla f](\nabla h, \nabla h)}{(m-n)(1+h)^2}\\ &- \frac{2\nabla h\nabla|\nabla h|^2}{(1+h)^2 h} + \frac{2|\nabla h|^4}{(1+h)^2h^2} - \frac{2|\nabla h|^2 F'(h^2)}{(1+h)^2} + \frac{|\nabla h|^2 F(h^2)}{(1+h)^2 h^2}\\ &+ \frac{2|\nabla h|^4}{(1+h)^3 h} + \frac{|\nabla h|^2}{(1+h)^3}\frac{F(h^2)}{h} - \frac{4\nabla h\nabla|\nabla h|^2}{(1+h)^3} + \frac{6|\nabla h|^4}{(1+h)^4},\end{aligned} \tag{4.7}$$

or upon rearranging terms

$$\begin{aligned}\Delta_f W - W_t =& 2\left|\frac{\nabla^2 h}{1+h} - \frac{\nabla h\otimes\nabla h}{(1+h)^2}\right|^2\\ &+ \frac{2Ric_f^m(\nabla h, \nabla h)}{(1+h)^2} + \frac{g_t(\nabla h, \nabla h)}{(1+h)^2} + \frac{2[\nabla f\otimes\nabla f](\nabla h, \nabla h)}{(m-n)(1+h)^2}\\ &- \frac{2\nabla h\nabla|\nabla h|^2}{(1+h)^2 h} - \frac{2\nabla h\nabla|\nabla h|^2}{(1+h)^3} + \frac{4|\nabla h|^4}{(1+h)^4} + \frac{2|\nabla h|^4}{(1+h)^2 h^2} + \frac{2|\nabla h|^4}{(1+h)^3 h}\\ &+ \frac{|\nabla h|^2 F(h^2)}{(1+h)^2 h^2} + \frac{|\nabla h|^2 F(h^2)}{(1+h)^3 h} - \frac{2|\nabla h|^2 F'(h^2)}{(1+h)^2}.\end{aligned} \tag{4.8}$$

Now substituting and reverting to W in the terms involving $|\nabla h|^2$ in the last two lines this results in:

$$\begin{aligned}\Delta_f W - W_t =& 2\left|\frac{\nabla^2 h}{1+h} - \frac{\nabla h\otimes\nabla h}{(1+h)^2}\right|^2\\ &+ \frac{2Ric_f^m(\nabla h,\nabla h)}{(1+h)^2} + \frac{g_t(\nabla h,\nabla h)}{(1+h)^2} + \frac{2[\nabla f\otimes\nabla f](\nabla h,\nabla h)}{(m-n)(1+h)^2}\\ &- \frac{2\nabla h\nabla W}{h} - \frac{2\nabla h\nabla W}{1+h} + \frac{2(1+h)^2}{h^2}W^2 - \frac{2(1+h)}{h}W^2\\ &+ \frac{WF(h^2)}{h^2} + \frac{WF(h^2)}{(1+h)h} - 2WF'(h^2)\end{aligned} \tag{4.9}$$

which is the desired conclusion. Note that here we have made use of the relation

$$\frac{2\nabla h\nabla W}{1+h} = \frac{2\nabla h\nabla|\nabla h|^2}{(1+h)^3} - \frac{4|\nabla h|^4}{(1+h)^4}. \tag{4.10}$$

This therefore completes the proof. □

Lemma 4.3. *Under the assumptions of Lemma 4.2, if $g = g(t)$, $f = f(t)$ evolve under the (k,m)-super Perelman-Ricci flow (1.1) then*

$$\begin{aligned}\Delta_f W - W_t \geq& -\frac{2(2h+1)\nabla h\nabla W}{(1+h)h} + \frac{2(1+h)W^2}{h^2}\\ &+ \left[\frac{(1+2h)F(h^2) - 2h^2(1+h)F'(h^2)}{(1+h)h^2}\right]W - 2kW.\end{aligned} \tag{4.11}$$

Proof. This is straightforward consequence of (4.2) by using the flow inequality (1.1) and $2[\nabla f\otimes\nabla f](\nabla h,\nabla h)]/[(m-n)(1+h)^2] = 2\langle\nabla f,\nabla h\rangle^2/[(m-n)(1+h)^2] \geq 0$. □

The proof of Theorem 3.1 uses Lemma 4.3 and a space-time localisation using suitable cut-off functions whose main properties appear in Lemma 4.4 below. (For more on the method, its background and underlying ideas see Li and Yau [14], Souplet and Zhang [23], Băileşteanu et al [2], Brighton [6], Wei and Wylie [30] and the references therein. For more discussion on the use of cut-off techniques and localisation in deriving various estimates in PDEs see also [26, 27].) Now fix $R, T > 0$ and then $\tau \in (0, T]$. Denoting by $\varrho(x,t) = d_{g(t)}(x, x_0)$ the geodesic radial variable at time t with respect to the reference point x_0 and $0 \leq t \leq T$ let us write

$$\psi(x,t) = \bar\psi(\varrho(x,t),t), \tag{4.12}$$

for a smooth cut-off function supported in the compact set $\mathcal{B}_{R,T} \subset M \times [0,T]$. The function $\bar\psi = \bar\psi(\varrho,t)$ appearing on the right-hand side of (4.12) is one granted by and described in the following statement (see [2, 6, 23, 31]).

Lemma 4.4. *Given $\tau \in (0,T]$ there exists a smooth function $\bar\psi : [0,\infty)\times[0,T] \to \mathbb{R}$ such that the following properties hold:*

(*i*) supp $\bar\psi(\varrho,t) \subset [0,R]\times[0,T]$ *and* $0 \leq \bar\psi(\varrho,t) \leq 1$ *in* $[0,R]\times[0,T]$.

(*ii*) $\bar\psi = 1$ *in* $[0,R/2]\times[\tau,T]$ *and* $\partial\bar\psi/\partial\varrho = 0$ *in* $[0,R/2]\times[0,T]$, *respectively.*

(*iii*) $|\partial\bar\psi/\partial t| \leq c\bar\psi^{1/2}/\tau$ *on* $[0,\infty)\times[0,T]$ *for some* $c > 0$ *and* $\bar\psi(\varrho,0) = 0$ $\forall\varrho\in[0,\infty)$.

(iv) $-c_a\bar{\psi}^a/R \le \partial\bar{\psi}/\partial\varrho \le 0$ *and* $|\partial^2\bar{\psi}/\partial\varrho^2| \le c_a\bar{\psi}^a/R^2$ *hold on* $[0,\infty)\times[0,T]$ *for every* $0<a<1$ *and some* $c_a>0$.

4.2. **Proof of Theorem 3.1.** Let ψ be as described in (4.12) with $\bar{\psi}$ as in Lemma 4.4. Note that here we have fixed $R \ge 2$, $T>0$ and $0<\tau\le T$. We will show that (3.1) holds for all (x,τ) with $d(x,x_0;\tau)\le R/2$ and then the arbitrariness of τ will grant the assertion for all (x,t) in $\mathcal{B}_{R/2,T}$ with $t\neq 0$. Now proceeding forward a straightforward calculation gives $\Delta_f(\psi W)=\psi\Delta_f W+2\nabla\psi\nabla W+W\Delta_f\psi$ and so when combined with $(\psi W)_t=\psi W_t+W\psi_t$ we can write

$$\left(\Delta_f-\frac{\partial}{\partial t}\right)(\psi W)=\psi\left(\Delta_f-\frac{\partial}{\partial t}\right)W+2\nabla\psi\nabla W+W\left(\Delta_f-\frac{\partial}{\partial t}\right)\psi. \tag{4.13}$$

At this stage we refer to the conclusion in Lemma 4.3 and by substitution and making note of the non-negativity of ψ write

$$\begin{aligned}\left(\Delta_f-\frac{\partial}{\partial t}\right)(\psi W)\ge&-\frac{2(2h+1)\psi\nabla h\nabla W}{(1+h)h}+\frac{2(1+h)\psi W^2}{h^2}+2\nabla\psi\nabla W \qquad (4.14)\\&+\left[\frac{(1+2h)F(h^2)-2h^2(1+h)F'(h^2)}{(1+h)h^2}\right]\psi W+W\left(\Delta_f-\frac{\partial}{\partial t}-2k\right)\psi.\end{aligned}$$

Next by utilising the two basic vector identities $\psi\nabla h\nabla W=\nabla h\nabla(\psi W)-(\nabla h\nabla\psi)W$ and $\nabla\psi\nabla W=(\nabla\psi/\psi)\nabla(\psi W)-(|\nabla\psi|^2/\psi)W$ and upon substituting in (4.14) we have

$$\begin{aligned}\left(\Delta_f-\frac{\partial}{\partial t}\right)(\psi W)\ge&\,2(2h+1)\left[\frac{(\nabla h\nabla\psi)W-\nabla h\nabla(\psi W)}{(1+h)h}\right]+\frac{2\nabla\psi}{\psi}\nabla(\psi W)-\frac{2|\nabla\psi|^2}{\psi}W\\&+\frac{2(1+h)\psi W^2}{h^2}+\left[\frac{(1+2h)F(h^2)-2h^2(1+h)F'(h^2)}{(1+h)h^2}\right]\psi W\\&+W\left(\Delta_f-\frac{\partial}{\partial t}-2k\right)\psi.\end{aligned} \tag{4.15}$$

Assume now that the localised function ψW attains its maximum value on the compact set $\mathcal{B}_{R,T}$ at (x_1,t_1). We suppose without loss of generality that x_1 is not on the cut-locus of M by Calabi's argument [14]. We also assume that $(\psi W)(x_1,t_1)>0$ as otherwise the desired estimate becomes trivial with $W(x,\tau)\le 0$ whenever $d(x,x_0;\tau)\le R/2$. Thus in particular $t_1>0$ by (iii) and at (x_1,t_1) we have $\Delta_f(\psi W)\le 0$, $(\psi W)_t\ge 0$ and $\nabla(\psi W)=0$. Therefore (4.15) after taking into account all the necessary cancellations implies that

$$\begin{aligned}\frac{2(1+h)\psi W^2}{h^2}\le&-2(2h+1)\frac{(\nabla h\nabla\psi)W}{(1+h)h}+\frac{2|\nabla\psi|^2}{\psi}W \qquad (4.16)\\&-\left[\frac{(1+2h)F(h^2)-2h^2(1+h)F'(h^2)}{(1+h)h^2}\right]\psi W-W\left(\Delta_f-\frac{\partial}{\partial t}-2k\right)\psi,\end{aligned}$$

at the point (x_1, t_1). After multiplying through by $h^2/2(1+h)$ this can be rewritten as

$$\begin{aligned}
\psi W^2 \le &- (2h+1)\frac{(\nabla h \nabla \psi) W h}{(1+h)^2} + \frac{|\nabla \psi|^2 W h^2}{\psi(1+h)} \\
&+ \left[\frac{2h^2(1+h)F'(h^2) - (1+2h)F(h^2)}{2(1+h)^2}\right]\psi W \\
&+ k\psi \frac{Wh^2}{1+h} - \frac{Wh^2 \Delta_f \psi}{2(1+h)} + \frac{Wh^2 \psi_t}{2(1+h)}.
\end{aligned} \tag{4.17}$$

The plan is now to exploit the above inequality and the maximal characterisation of (x_1, t_1) to establish the required estimate at the space-time point (x, τ). Towards this end we proceed by considering two cases depending as to whether $d(x_1, x_0; t_1) \ge 1$ or $d(x_1, x_0; t_1) \le 1$. Let us consider the first case. Here we apply the properties (i)-(iv) of ψ as listed in Lemma 4.4, the Cauchy-Schwarz and Young inequalities respectively to obtain suitable upper bounds for each of the six terms on the right-hand side of (4.17).
(a) For the first term by recalling the definition of W, noting $0 < (2h+1)/(1+h) \le 2$ and using basic inequalities we can write

$$\begin{aligned}
-(2h+1)\frac{(\nabla h \nabla \psi) W h}{(1+h)^2} &\le (2h+1)\frac{|\nabla h||\nabla \psi| W h}{(1+h)^2} \\
&\le \frac{(2h+1)h}{1+h}\frac{|\nabla h|}{1+h}|\nabla \psi| W \le 2|\nabla \psi| W^{3/2}\Big(\sup_{\mathcal{B}_{R,T}} h\Big) \\
&\le \frac{1}{8}\psi W^2 + C\left(\frac{|\nabla \psi|(\sup_{\mathcal{B}_{R,T}} h)}{\psi^{3/4}}\right)^4 \le \frac{1}{8}\psi W^2 + \frac{C}{R^4}\Big(\sup_{\mathcal{B}_{R,T}} h\Big)^4.
\end{aligned} \tag{4.18}$$

(b) For the second term noting $1/(1+h) \le 1$ and proceeding in a similar way we have

$$\begin{aligned}
\frac{|\nabla \psi|^2}{\psi}\frac{Wh^2}{1+h} &\le \sqrt{\psi} W \frac{|\nabla \psi|^2}{\psi^{3/2}}\Big(\sup_{\mathcal{B}_{R,T}} h\Big)^2 \\
&\le \frac{1}{8}\psi W^2 + C\left(\frac{|\nabla \psi|^2}{\psi^{3/2}}\right)^2\Big(\sup_{\mathcal{B}_{R,T}} h\Big)^4 \le \frac{1}{8}\psi W^2 + \frac{C}{R^4}\Big(\sup_{\mathcal{B}_{R,T}} h\Big)^4.
\end{aligned} \tag{4.19}$$

(c) For the third term noting $0 \le \psi \le 1$ and taking positive parts we can write

$$\begin{aligned}
&\left[\frac{2h^2(1+h)F'(h^2) - (1+2h)F(h^2)}{2(1+h)^2}\right]\psi W \\
&\le \sqrt{\psi} W \left[\frac{2h^2(1+h)F'(h^2) - (1+2h)F(h^2)}{2(1+h)^2}\right]_+ \le \frac{1}{8}\psi W^2 + C\sup_{\mathcal{B}_{R,T}} \mathsf{R}_F^2(u),
\end{aligned} \tag{4.20}$$

where we have written $\mathsf{R}_F(u) = \{[2h^2(1+h)F'(h^2) - (1+2h)F(h^2)]/[2(1+h)^2]\}_+$.
(d) For the fourth term we can write

$$k\psi\frac{Wh^2}{1+h} \le k_+\sqrt{\psi} W \frac{\sqrt{\psi}}{1+h}\Big(\sup_{\mathcal{B}_{R,T}} h\Big)^2 \le \frac{1}{8}\psi W^2 + Ck_+^2\Big(\sup_{\mathcal{B}_{R,T}} h\Big)^4.$$

(e) For the fifth term we treat the cases $m = \infty$ and $n \le m < \infty$ separately. Before that however we note that in view of the relation $Ric_f(g) = Ric_f^m(g) + [\nabla f \otimes \nabla f]/(m-n)$ (see (2.2) and Remark 3.2), the estimate obtained in the first case under a lower bound on $Ric_f(g)$ remains true also in the second case. However, as will be seen, one can take advantage of the lower bound on $Ric_f^m(g)$ in the second case to obtain a slightly less involved estimate [compare (4.22) with (4.23) below]. Proceeding now onto the first case let us set $\zeta = \max(Z_{\Delta_f}, 0) \geq 0$ where

$$Z_{\Delta_f} = \max_{(x,t)}\{\Delta_f \varrho(x,t) : d(x, x_0; t) = 1,\ 0 \leq t \leq T\}. \tag{4.21}$$

Using $Ric_f(g) \geq -(n-1)\mathsf{k}g$ and the weighted Laplacian comparison theorem (Theorem 3.1 in [30]) we have $\Delta_f \varrho \leq \zeta + (R-1)(n-1)\mathsf{k}$ for $\varrho \geq 1$. From ψ being radial with $\bar\psi_\varrho \leq 0$ it then follows that $\Delta_f \psi = \bar\psi_{\varrho\varrho}|\nabla\varrho|^2 + \bar\psi_\varrho \Delta_f \varrho \geq \bar\psi_{\varrho\varrho} + \bar\psi_\varrho[\zeta + (R-1)(n-1)\mathsf{k}]$ and so $-\Delta_f\psi \leq |\bar\psi_{\varrho\varrho}| + [\zeta + (R-1)(n-1)\mathsf{k}]|\bar\psi_\varrho|$. Hence we can write

$$\begin{aligned}
\frac{h^2 W}{1+h}(-\Delta_f\psi) &\leq \frac{h^2\sqrt{\bar\psi}W}{1+h}\left(\frac{|\bar\psi_{\varrho\varrho}|}{\sqrt{\bar\psi}} + [\zeta + (R-1)(n-1)\mathsf{k}]\frac{|\bar\psi_\varrho|}{\sqrt{\bar\psi}}\right) \\
&\leq \sqrt{\bar\psi}W\left(\frac{|\bar\psi_{\varrho\varrho}|}{\sqrt{\bar\psi}} + [\zeta + (R-1)(n-1)\mathsf{k}]\frac{|\bar\psi_\varrho|}{\sqrt{\bar\psi}}\right)\left(\sup_{\mathcal{B}_{R,T}} h\right)^2 \\
&\leq \frac{1}{8}\bar\psi W^2 + C\left(\frac{|\bar\psi_{\varrho\varrho}|^2}{\bar\psi} + \zeta^2\frac{|\bar\psi_\varrho|^2}{\bar\psi} + n^2\mathsf{k}^2R^2\frac{|\bar\psi_\varrho|^2}{\bar\psi}\right)\left(\sup_{\mathcal{B}_{R,T}} h\right)^4 \\
&\leq \frac{1}{8}\psi W^2 + C\frac{n^2}{R^4}\left(1 + \zeta^2R^2 + \mathsf{k}^2R^4\right)\left(\sup_{\mathcal{B}_{R,T}} h\right)^4.
\end{aligned} \tag{4.22}$$

In the second case using $Ric_f^m(g) \geq -(m-1)\mathsf{k}g$ we have $\Delta_f\varrho \leq (m-1)\sqrt{\mathsf{k}}\coth(\sqrt{\mathsf{k}}\varrho)$ (see [30]). Therefore again by virtue of ψ being radial and $\bar\psi_\varrho \leq 0$ it follows that,

$$\Delta_f\psi = \bar\psi_{\varrho\varrho}|\nabla\varrho|^2 + \bar\psi_\varrho\Delta_f\varrho \geq \bar\psi_{\varrho\varrho} + (m-1)\bar\psi_\varrho\sqrt{\mathsf{k}}\coth(\sqrt{\mathsf{k}}\varrho).$$

Using the bound $\sqrt{\mathsf{k}}\coth(\sqrt{\mathsf{k}}\varrho) \leq \sqrt{\mathsf{k}}\coth(\sqrt{\mathsf{k}}R/2) \leq (2 + \sqrt{\mathsf{k}}R)/R$ for $R/2 \leq \varrho \leq R$ (here we are making use of $v\coth v \leq 1 + v$ and the monotonicity of $\coth v$ for $v > 0$) and noting that $\bar\psi_\varrho \equiv 0$ for $0 \leq \varrho \leq R/2$ it then follows that

$$-\Delta_f\psi \leq -[\bar\psi_{\varrho\varrho} + (m-1)\bar\psi_\varrho\sqrt{\mathsf{k}}\coth(\sqrt{\mathsf{k}}R/2)] \leq |\bar\psi_{\varrho\varrho}| + (m-1)(2/R + \sqrt{\mathsf{k}})|\bar\psi_\varrho|.$$

Therefore we can write

$$\begin{aligned}
\frac{h^2W}{1+h}(-\Delta_f\psi) &\leq \sqrt{\bar\psi}W\left(\frac{|\bar\psi_{\varrho\varrho}|}{\sqrt{\bar\psi}} + (m-1)(2/R + \sqrt{\mathsf{k}})\frac{|\bar\psi_\varrho|}{\sqrt{\bar\psi}}\right)\left(\sup_{\mathcal{B}_{R,T}} h\right)^2 \\
&\leq \frac{1}{8}\bar\psi W^2 + C\left(\frac{|\bar\psi_{\varrho\varrho}|^2}{\bar\psi} + m^2(1/R^2 + \mathsf{k})\frac{|\bar\psi_\varrho|^2}{\bar\psi}\right)\left(\sup_{\mathcal{B}_{R,T}} h\right)^4 \\
&\leq \frac{1}{8}\psi W^2 + C\frac{m^2}{R^4}\left(1 + \mathsf{k}R^2\right)\left(\sup_{\mathcal{B}_{R,T}} h\right)^4.
\end{aligned} \tag{4.23}$$

(f) For the sixth term we first estimate ψ_t as follows: For $d(x_1, x_0; t) \leq R$ and fixed $t > 0$ let $\gamma = \gamma(s) : [0, d] \to M$ be a minimal geodesic with respect to $g(t)$ connecting $x_0 = \gamma(0)$ to $x_1 = \gamma(d)$. Then a standard calculation utilising $\partial g/\partial t \geq -2\mathsf{h}g$ in $\mathcal{B}_{R,T}$

with $\mathsf{h} \ge 0$ as in the theorem gives

$$\begin{aligned}\frac{\partial}{\partial t}\varrho(x_1,t) &= \frac{\partial}{\partial t}d(x_1,x_0;t) = \frac{\partial}{\partial t}\int_0^d |\gamma'(s)|_{g(t)}\,ds = \int_0^d \frac{(\partial g/\partial t)(\gamma',\gamma')}{2|\gamma'|_{g(t)}}\,ds\\ &\ge \int_0^d -\mathsf{h}|\gamma'|_{g(t)}\,ds \ge -\mathsf{h}\varrho(x_1,t) \ge -\mathsf{h}R.\end{aligned}$$

Thus referring to (4.12) and applying the properties of $\bar\psi$ as listed in Lemma 4.4 we deduce that $\psi_t = \bar\psi_t + \bar\psi_\varrho \varrho_t \le \bar\psi_t - \bar\psi_\varrho \mathsf{h}R \le |\bar\psi_t| + |\bar\psi_\varrho|\mathsf{h}R \le C(1/\tau + \mathsf{h})\sqrt{\psi}$. Now with the aid of this estimate of ψ_t we can proceed onto the last term and write

$$\begin{aligned}\frac{h^2W}{1+h}\psi_t =& \frac{\sqrt{\psi}h^2W}{1+h}\frac{\psi_t}{\sqrt{\psi}} \le C\frac{\sqrt{\psi}h^2W}{1+h}\left(\frac{1}{\tau}+\mathsf{h}\right) \qquad (4.24)\\ &\le C\sqrt{\psi}W\left(\frac{1}{\tau}+\mathsf{h}\right)h^2 \le \frac{1}{8}\psi W^2 + C\left(\frac{1}{\tau^2}+\mathsf{h}^2\right)\left(\sup_{\mathcal{B}_{R,T}} h\right)^4.\end{aligned}$$

Completing the estimate of the individual terms on the right-hand side of (4.17), by inserting these estimates back and after basic considerations and adjusting the constants upon noting $R \ge 2$, we arrive at the following bound at (x_1,t_1),

$$\psi W^2 \le \frac{3}{4}\psi W^2 + C\left\{\frac{1}{R^4}\left(1+\zeta^2R^2+\frac{R^4}{\tau^2}+\mathsf{K}^2R^4\right)\left(\sup_{\mathcal{B}_{R,T}} h\right)^4 + \sup_{\mathcal{B}_{R,T}}\mathsf{R}_F^2(u)\right\} \qquad (4.25)$$

where $\mathsf{K} = \sqrt{\mathsf{h}^2+\mathsf{k}^2}$. Referring to the expression on the right-hand side in (4.25), note that in the case resulting from (4.23) in case two in (e), we can remove the term ζ^2R^2. Now invoking the maximal characterisation of (x_1,t_1) we have for all $(x,t) \in \mathcal{B}_{R,T}$ the inequalities $\psi^2W^2(x,t) \le \psi^2W^2(x_1,t_1) \le \psi W^2(x_1,t_1)$ and

$$\psi^2W^2(x,t) \le C\left\{\frac{1}{R^4}\left(1+\zeta^2R^2+\frac{R^4}{\tau^2}+\mathsf{K}^2R^4\right)\left(\sup_{\mathcal{B}_{R,T}} h\right)^4 + \sup_{\mathcal{B}_{R,T}}\mathsf{R}_F^2(u)\right\}.$$

Since $\psi(x,\tau) = 1$ in $d(x,x_0;\tau) \le R/2$ and by definition $W = |\nabla h|^2/(1+h)^2$ we obtain upon recalling $h = \sqrt{u}$ and $|\nabla h| = |\nabla\sqrt{u}|$ this gives

$$\frac{|\nabla\sqrt{u}|}{1+\sqrt{u}} \le C\left\{\left(\frac{1}{R}+\sqrt{\frac{\zeta}{R}}+\frac{1}{\sqrt{\tau}}+\sqrt{\mathsf{K}}\right)\left(\sup_{\mathcal{B}_{R,T}}\sqrt{u}\right) + \sup_{\mathcal{B}_{R,T}}\sqrt{\mathsf{R}_F(u)}\right\}.$$

We now move to the second case and consider $d(x_1,x_0;t_1) \le 1$. As by (ii) in Lemma 4.4 ψ is a constant function in the space direction (when $d(x,x_0;t) \le R/2$, $t \in [0,T]$ with $R \ge 2$), referring to (4.17), all the terms involving spatial derivatives of ψ vanish or simplify (including $\psi_t = \bar\psi_t$) and so we have at the point (x_1,t_1),

$$\begin{aligned}W &\le \left\{\frac{2h^2(1+h)F'(h^2)-(1+2h)F(h^2)}{2(1+h)h^2} + \left(k+\frac{1}{2}\frac{\psi_t}{\psi}\right)\right\}\frac{h^2}{1+h}\\ &\le \left\{\left[\frac{2h^2(1+h)F'(h^2)-(1+2h)F(h^2)}{2(1+h)h^2}\right]_+ + \left(k_+ + \frac{1}{2}\frac{|\bar\psi_t|}{\bar\psi}\right)\right\}\frac{h^2}{1+h}\\ &\le C\{\sup_{\mathcal{B}_{R,T}}\mathsf{R}_F(u) + [k_+ + 1/\tau]h^2\}. \qquad (4.26)\end{aligned}$$

Noting $W(x,\tau) = \psi W(x,\tau) \le \psi W(x_1,t_1) \le W(x_1,t_1)$ due to $\psi(x_1,t_1) \le 1$, $\psi(x,\tau) = 1$ when $d(x,x_0;\tau) \le R/2$, the above and the arbitrariness of $\tau > 0$ easily lead to a special case of (3.1). We have now established the estimate in both cases and so the proof is complete. □

5. Applications to special nonlinearities F and Liouville theorems

In this section we present some corollaries of Theorem 3.1 to the static case (see Remark 3.3). Here $(M, g, e^{-f}dv)$ is a smooth metric measure space where the metric g and the potential f are time independent whilst $Ric_f^m(g) \ge -\mathsf{k}_m g$ in $\mathcal{B}_R$ with $\mathsf{k}_m \ge 0$, $R \ge 2$. For clarification $\mathcal{B}_R \subset M$ is the geodesic ball with centre x_0 and radius $R > 0$. We consider positive bounded solutions to the elliptic equation $\Delta_f u + F(u) = 0$. Since u is a time independent solution of (1.3) setting $t \nearrow \infty$ in (3.1) and noting that the constants do not depend on t gives the corresponding estimate for $x \in \mathcal{B}_{R/2}$

$$\frac{|\nabla \sqrt{u}|}{1+\sqrt{u}} = \frac{|\nabla u|/2}{u+\sqrt{u}} \le C\left\{\left(\frac{1}{R} + \sqrt{\frac{\zeta}{R}} + \sqrt{\mathsf{k}}\right)\left(\sup_{\mathcal{B}_R}\sqrt{u}\right) + \sup_{\mathcal{B}_R}\sqrt{\mathsf{R}_F(u)}\right\}. \tag{5.1}$$

As $g_t \equiv 0$ we have set $\mathsf{h} = 0$ and so $\mathsf{K} = \mathsf{k}$ whilst $\zeta = [Z_{\Delta_f}]_+ = [\max_{\{x:d(x,x_0)=1\}} \Delta_f r]_+$ and $\mathsf{R}_F(u) = \{[2u(1+\sqrt{u})F'(u) - (1+2\sqrt{u})F(u)]/[2(1+\sqrt{u})^2]\}_+$. Now subject to the global bound $Ric_f^m(g) \ge -\mathsf{k}_m g$ on M setting $R \nearrow \infty$ in (5.1) gives the global gradient estimate for positive bounded solutions u to the equation $\Delta_f u + F(u) = 0$, namely,

$$\frac{|\nabla \sqrt{u}|}{1+\sqrt{u}} = \frac{|\nabla u|/2}{u+\sqrt{u}} \le C\left\{\sqrt{\mathsf{k}}\left(\sup_{M}\sqrt{u}\right) + \sup_{M}\sqrt{\mathsf{R}_F(u)}\right\}. \tag{5.2}$$

Of particular interest here are Liouville type theorems; the underlying idea upon referring to (5.2) being that if $Ric_f^m(g) \ge 0$ (i.e., $\mathsf{k} = 0$) and $\mathsf{R}_F(u) \equiv 0$ then combined with the above estimate these together imply that $\nabla u = 0$ and so u is a constant. Below we examine this more closely in the context of certain power-like and logarithmic type nonlinearities F that arise frequently in the literature and include:

(1) $F(u) = Au\,|\log u|^\alpha$ (with real $\alpha > 1$).
(2) $F(u) = Au(\log u)^d$ (with integer $d \ge 1$).
(3) $F(u) = Au^p + Bu^q$ (p, q arbitrary real exponents).
(4) $F(u) = Au^p \log u + Bu^q$ (p, q arbitrary real exponents).

5.1. **The case $F(u) = Au\,|\log u|^\alpha$ with $\alpha > 1$.** The function F here is continuously differentiable and so referring to the description of $\mathsf{R}_F(u)$ [see (1.5)] a basic calculation gives

$$\frac{2h^2(1+h)F'(h^2) - (1+2h)F(h^2)}{2(1+h)^2} = \frac{Au\,|\log u|^\alpha}{2(1+h)^2} + \frac{A\alpha u\,|\log u|^{\alpha-2}\log u}{1+h}. \tag{5.3}$$

Therefore $\mathsf{R}_F(u) = [2^{-1}Au\,|\log u|^\alpha/(1+h)^2 + A\alpha u\,|\log u|^{\alpha-2}\log u/(1+h)]_+$ and so as readily seen $\mathsf{R}_F(u) \le 2^{-1}A_+ u\,|\log u|^\alpha + \alpha u[A\,|\log u|^{\alpha-2}\log u]_+$.

Theorem 5.1. *Let $(M, g, e^{-f}dv)$ be a smooth metric measure space with $Ric_f(g) \ge 0$ and consider the equation $\Delta_f u + Au\,|\log u|^\alpha = 0$ (with $\alpha > 1$) on M. If $A \le 0$ then any solution with $1 \le u \le D$ is a constant. Additionally if $A < 0$ then necessarily $u \equiv 1$.*

Proof. As here $A_+ = 0$ and $[A\,|\log u|^{\alpha-2}\log u]_+ = 0$ by referring to the above calculation $\mathsf{R}_F(u) = 0$. Since by assumption $\mathsf{k} = 0$ it then follows from (5.2) that $\nabla u \equiv 0$ and this leads immediately to the desired conclusion. □

5.2. **The case $F(u) = Au(\log u)^d$ with integer $d \geq 1$.** Comparing with 5.1 above the modulus sign in F has been removed from the $\log u$ term, however, it is evident that due to the sign changing nature of the logarithm some restriction has to be imposed on the exponent (here integer $d \geq 1$) as speaking of arbitrary powers of $\log u$ for $u > 0$ is in general meaningless. Now a short calculation gives

$$\frac{2h^2(1+h)F'(h^2) - (1+2h)F(h^2)}{2(1+h)} = \frac{Au(\log u)^d}{2(1+h)^2} + \frac{Adu(\log u)^{d-1}}{1+h}, \tag{5.4}$$

and so referring to (1.5) we can write

$$\begin{aligned} \mathsf{R}_F(u) &= [Au(\log u)^d/[2(1+h)^2] + Adu(\log u)^{d-1}/(1+h)]_+ \\ &\leq 2^{-1}u[A(\log u)^d]_+ + du[A(\log u)^{d-1}]_+. \end{aligned} \tag{5.5}$$

Theorem 5.2. *Let $(M, g, e^{-f}dv)$ be a smooth metric measure space with $Ric_f(g) \geq 0$ and consider the equation $\Delta_f u + Au(\log u)^d = 0$ (with integer $d \geq 1$) on M. If $A \leq 0$ then any solution with $1 \leq u \leq D$ is a constant. Additionally if $A < 0$ then necessarily $u \equiv 1$.*

Proof. For $A \leq 0$ and $d \geq 2$ even we have $[A(\log u)^d]_+ = 0$ with $[A(\log u)^{d-1}]_+ = 0$ only when $u \geq 1$. For $A \leq 0$ and $d \geq 1$ odd we have $[A(\log u)^{d-1}]_+ = 0$ with $[A(\log u)^d]_+ = 0$ only when $u \geq 1$. Thus $A \leq 0$ and $u \geq 1$ gives $\mathsf{R}_F(u) = 0$. The rest is an in the proof of Theorem 5.1. □

5.3. **The case $F(u) = Au^p + Bu^q$ with real exponents p, q.** A direct calculation in the spirit of those in the previous sections here gives

$$\frac{2h^2(1+h)F'(h^2) - (1+2h)F(h^2)}{2(1+h)^2} = \frac{Au^p}{1+h}\left[p - \frac{1+2h}{2(1+h)}\right] + \frac{Bu^q}{1+h}\left[q - \frac{1+2h}{2(1+h)}\right],$$

and so from (1.5) and by considering positive parts we have

$$\mathsf{R}_F(u) \leq \left[A\frac{(2p-1)+2h(p-1)}{2(1+h)^2}\right]_+ u^p + \left[B\frac{(2q-1)+2h(q-1)}{2(1+h)^2}\right]_+ u^q. \tag{5.6}$$

Now as $0 < h = \sqrt{u} \leq \sqrt{D}$ we have $(2p-1)+2h(p-1) \geq 0$ if $p \geq [1+2\sqrt{D}]/[2+2\sqrt{D}]$ and $(2q-1)+2h(q-1) \leq 0$ if $q \leq 1/2$. Thus in particular if $A \leq 0$, $B \geq 0$ and $p \geq [1+2\sqrt{D}]/[2+2\sqrt{D}]$ and $q \leq 1/2$ it follows that $\mathsf{R}_F(u) \equiv 0$.

Theorem 5.3. *Let $(M, g, e^{-f}dv)$ be a smooth metric measure space with $Ric_f(g) \geq 0$. Assume $0 < u \leq D$ is a solution to the equation $\Delta_f u + Au^p + Bu^q = 0$ on M. If $A \leq 0$, $B \geq 0$, $p \geq [1+2\sqrt{D}]/[2+2\sqrt{D}]$ and $q \leq 1/2$ then u is a constant.*

Note that regardless of the upper bound on u the condition $p \geq [1+2\sqrt{D}]/[2+2\sqrt{D}]$ is always implied by $p \geq 1$. Likewise $q \leq (1+2h)/(2+2h)$ is implied by $q \leq 1/2$.

Proof. From the discussion preceding the theorem, the assumptions on the exponents p, q the solution u and the coefficients A, B give $\mathsf{R}_F \equiv 0$. Hence from the global estimate (5.2) with $\mathsf{k} = 0$ it follows that $\nabla u \equiv 0$. This immediately gives the conclusion. □

5.4. **The case $F(u) = Au^p \log u + Bu^q$.** Here again by a direct differentiation we have $F'(u) = A(pu^{p-1} \log u + u^{p-1}) + Bqu^{q-1}$ and so referring to (1.5) we can write

$$\begin{aligned} \mathsf{R}_F(u) &= \left[\frac{2h^2(1+h)F'(h^2) - (1+2h)F(h^2)}{2(1+h)^2} \right]_+ \qquad (5.7) \\ &= \left[\frac{Au^p}{1+h} \left\{ \frac{[(2p-1)+2h(p-1)] \log u}{2(1+h)} + 1 \right\} + \frac{B[(2q-1)+2h(q-1)]u^q}{2(1+h)^2} \right]_+ \\ &\le \left[A \left\{ \frac{[(2p-1)+2h(p-1)] \log u}{2(1+h)} + 1 \right\} \right]_+ u^p + \left[\frac{B[(2q-1)+2h(q-1)]}{2(1+h)} \right]_+ u^q. \end{aligned}$$

Let us now set $s(p) = [(2p-1)+2h(p-1)]/[2(1+h)]$. Then from the above we have $\mathsf{R}_F \le [A(s(p) \log u + 1)]_+ + [Bs(q)]_+$. In what follows we look more closely at the sign of the term $\log(eu^{s(p)})$ or equivalently that of $s(p) \log u + 1$. Towards this end consider the function

$$\zeta_p(h) = -\frac{1}{s} = \frac{2(1+h)}{(1-2p)+2h(1-p)}, \qquad h \ge 0.$$

As for the range $1/2 < p < 1$ the function ζ_p has a singularity at $h = (2p-1)/[2(1-p)]$ on the half-axis $h > 0$ and is in particular unbounded, hereafter, we restrict attention to p outside this interval, i.e., $p \le 1/2$ or $p \ge 1$. Now it is easily seen that $\zeta_p' < 0$ and so ζ_p is monotonically decreasing on $h \ge 0$. Moreover by direct evaluation $\zeta_p(0) = 2/(1-2p)$ and $\lim \zeta_p(h) = 1/(1-p)$ as $h \nearrow \infty$. Now when $p \le 1/2$ we have $0 < 1/(1-p) \le \zeta_p(h) \le 2/(1-2p)$ and when $p \ge 1$ we have $1/(1-p) \le \zeta_p(h) \le 2/(1-2p) < 0$. Thus depending on the signs of A and $s \log u + 1$ we have the following:

- If $A \le 0$ and $s \log u \ge -1$ then $[A(s \log u + 1)]_+ = 0$. Thus, in terms of p:
  (1) If $p \le 1/2$ then $s < 0$ and so $\log u \le 1/(1-p)$ gives $\log u \le \zeta_p = -1/s$.
  (2) If $p \ge 1$ then $s > 0$ and so $\log u \ge 2/(1-2p)$ gives $\log u \ge \zeta_p = -1/s$.
- If $A \ge 0$ and $s \log u \le -1$ then $[A(s \log u + 1)]_+ = 0$. Thus, in terms of p:
  (1) If $p \le 1/2$ then $s < 0$ and so $\log u \ge 2/(1-2p)$ gives $\log u \ge \zeta_p = -1/s$.
  (2) If $p \ge 1$ then $s > 0$ and so $\log u \le 1/(1-p)$ gives $\log u \le \zeta_p = -1/s$.

As regarding the B term we can argue exactly as in Section 5.3, by putting the above together we have proved the following statement.

Theorem 5.4. *Let $(M, g, e^{-f}dv)$ be a smooth metric measure space with $Ric_f(g) \ge 0$. Assume u is a positive bounded solution to the equation $\Delta_f u + Au^p \log u + Bu^q = 0$ on M. Consider the set of assumptions on A, B, u and the exponents p, q described below:*

$$\begin{cases} A \ge 0 \text{ and either:} \\ p \ge 1, 0 < u \le \exp(1/(1-p)), \text{or} \\ p \le 1/2, u \ge \exp(2/(1-2p)), \end{cases} \qquad \begin{cases} A \le 0 \text{ and either:} \\ p \ge 1, u \ge \exp[2/(1-2p)], \text{or} \\ p \le 1/2, 0 < u \le \exp[1/(1-p)], \end{cases}$$

together with $B \ge 0$ and $q \le 1/2$ or $B \le 0$ and $q \ge 1$. Then if these conditions hold u must be a constant.

6. A global gradient estimate for solutions to (1.3)

We have already seen some implication of the local and global estimates to Liouville type results on the elliptic counterpart of (1.3) in the previous sections. In this section we continue by establishing estimates on the full parabolic equation that are of a global nature where as before the metric and potential evolve under a (k,m)-super Perelman-Ricci flow.

Theorem 6.1. *Assume the pair $(g(t), f(t))_{t\in[0,T]}$ evolves under a (k,m)-super Perelman-Ricci flow and let u be a positive solution to (1.3). Assume $uF(u) \le 0$ and $F'(u) \le 0$ along u. Then if M is closed and $k \ge 0$ we have the global estimate:*

$$|\nabla u(x,t)|^2 \le \frac{1+2kt}{2t}\left[(\sup_M u|_{t=0})^2 - u^2\right] \tag{6.1}$$

for all $x \in M$ and $0 < t \le T$.

Proof. The idea is to get the estimate (6.1) out of the evolution of a suitably chosen quantity depending on the solution u. Towards this end recalling $\partial u/\partial t = \Delta_f u + F(u)$ and the flow inequality $\partial g/\partial t \ge -2(Ric_f^m(g) + kg)$ we can write

$$\begin{aligned}\partial|\nabla u|^2/\partial t &= -g_t(\nabla u, \nabla u) + 2\nabla u \nabla u_t \\ &\le 2Ric_f^m(\nabla u, \nabla u) + 2k|\nabla u|^2 + 2\nabla u\nabla[\Delta_f u + F(u)] \\ &\le \Delta_f|\nabla u|^2 - 2|\nabla^2 u|^2 + 2k|\nabla u|^2 + 2\nabla u \nabla F(u),\end{aligned} \tag{6.2}$$

where in writing the second inequality we have made use of (2.2), the weighted Bochner-Weitzenböck formula (2.3) and $[\nabla f \otimes \nabla f](\nabla u, \nabla u)]/(m-n) = \langle \nabla f, \nabla u\rangle^2/(m-n) \ge 0$ for when $m < \infty$. Likewise

$$\partial u^2/\partial t = 2u[\Delta_f u + F(u)] = \Delta_f u^2 - 2|\nabla u|^2 + 2uF(u). \tag{6.3}$$

Now for $t \ge 0$ and Z to be specified below put $\mathscr{P}_\gamma[u] = \gamma(t)|\nabla u|^2 + Zu^2$. Here γ is a non-negative, smooth but otherwise arbitrary function. Then differentiating and making use of (6.2) and (6.3) gives

$$\begin{aligned}\partial\mathscr{P}_\gamma[u]/\partial t &= \gamma'|\nabla u|^2 + \gamma\partial|\nabla u|^2/\partial t + Z\partial u^2/\partial t \\ &\le \gamma[\Delta_f|\nabla u|^2 - 2|\nabla^2 u|^2 + 2k|\nabla u|^2 + 2\nabla u\nabla F(u)] \\ &\quad + \gamma'|\nabla u|^2 + Z[\Delta_f u^2 - 2|\nabla u|^2 + 2uF(u)],\end{aligned} \tag{6.4}$$

or upon rearranging terms

$$\begin{aligned}\partial\mathscr{P}_\gamma[u]/\partial t \le{}& \Delta_f[\gamma|\nabla u|^2 + Zu^2] + (\gamma' + 2k\gamma - 2Z)|\nabla u|^2 \\ & - 2\gamma|\nabla^2 u|^2 + 2ZuF(u) + 2\gamma\nabla u\nabla F(u).\end{aligned} \tag{6.5}$$

Therefore substituting for $\mathscr{P}_\gamma[u]$ on the right and other considerations lead to

$$\partial\mathscr{P}_\gamma[u]/\partial t \le \Delta_f\mathscr{P}_\gamma[u] + (\gamma' + 2k\gamma - 2Z)|\nabla u|^2 + 2ZuF(u) + 2\gamma\nabla u\nabla F(u). \tag{6.6}$$

Thus in particular upon setting $Z = 1/2$ and rearranging terms we have

$$\partial\mathscr{P}_\gamma[u]/\partial t - \Delta_f\mathscr{P}_\gamma[u] \le (\gamma' + 2k\gamma - 1)|\nabla u|^2 + uF(u) + 2\gamma|\nabla u|^2F'(u). \tag{6.7}$$

Now the function $\gamma(t) = t/(2kt+1)$ with $k, t \ge 0$ is seen to be non-negative, smooth and to satisfy $\gamma(0) = 0$ and $\gamma' + 2k\gamma \le 1$. The conclusion thus follows by an application of the weak maximum principle by virtue of M being compact. □

Remark 6.2. Note that in this argument we did not use the fact that u is positive but only $uF(u) \le 0$ to hold along u.

Specialising to the nonlinearities F introduced and studied in Section 5 we have the following corollaries of Theorem 6.1 for the positive solutions u to equation (1.3).

(1) For the equation $\partial u/\partial t = \Delta_f u + Au\,|\log u|^\alpha$ with $\alpha > 1$ we have from (6.7) that $\mathscr{P}_t \le \Delta_f \mathscr{P} + Au^2\,|\log u|^\alpha + 2tA|\nabla u|^2[|\log u|^\alpha + \alpha\,|\log u|^{\alpha-2}\log u]$. Therefore if $A \le 0$ and $u \ge 1$ then $\mathscr{P}_t \le \Delta_f \mathscr{P}$ and so the global estimate (6.1) holds.

(2) For the equation $\partial u/\partial t = \Delta_f u + Au(\log u)^d$ with $d \ge 1$ integer we have that that $\mathscr{P}_t \le \Delta_f \mathscr{P} + Au^2(\log u)^d + 2tA|\nabla u|^2[(\log u)^d + d(\log u)^{d-1}]$. Hence if $d \ge 2$ is even then exactly as in (1). If $d \ge 1$ is odd then if $A \ge 0$ and $0 < u \le e^{-d}$ or if $A \le 0$ and $u \ge 1$ then in either case the global estimate (6.1) holds.

(3) For the equation $\partial u/\partial t = \Delta_f u + Au^p + Bu^q$ we have again from (6.7) that $\mathscr{P}_t \le \Delta_f \mathscr{P} + Au^{p+1} + Bu^{q+1} + 2t|\nabla u|^2(Apu^{p-1} + Bqu^{q-1})$. As a result if $A, B \le 0$ and $p, q \ge 0$ or if $A \le 0$, $B \ge 0$ and $p \ge 0$, $q \le 0$ with $u \ge (-B/A)^{1/(p-q)}$ then $\mathscr{P}_t \le \Delta_f \mathscr{P}$ and so the global estimate (6.1) holds.

(4) For the equation $\partial u/\partial t = \Delta_f u + Au^p \log u + Bu^q$ noting (6.7) in Theorem (6.1) gives $\mathscr{P}_t \le \Delta_f \mathscr{P} + Au^{p+1}\log u + Bu^{q+1} + 2t|\nabla u|^2[Au^{p-1}(p\log u + 1) + Bqu^{q-1}]$. Thus if $B \le 0$, $q \ge 0$ and either $A \ge 0$, $p \ge 0$ and $0 < u \le e^{-1/p}$ or $A \le 0$, $p \ge 0$ and $u \ge 1$ then $\mathscr{P}_t \le \Delta_f \mathscr{P}$ and so the global estimate (6.1) holds.

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