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SOLITARY WAVES AND EXCITED STATES FOR BOSON STARS

M. MELGAARD* AND F. D. Y. ZONGO

Abstract. We study the nonlinear, nonlocal, time-dependent partial differential equation

\[ i\partial_t \phi = (\sqrt{-\Delta + m^2} - m) \phi - \left( \frac{1}{|x|} + |\phi|^2 \right) \phi \] on \( \mathbb{R}^3 \),

which is known to describe the dynamics of quasi-relativistic boson stars in the mean-field limit. For positive mass parameter \( m > 0 \) we establish existence of infinitely many (corresponding to distinct energies \( \lambda_k \)) travelling solitary waves, \( \varphi_k(x, t) = e^{i\lambda_k t} \phi_k(x - vt) \), with speed \( |v| < 1 \), where \( c = 1 \) corresponds to the speed of light in our choice of units. These travelling solitary waves cannot be obtained by applying a Lorentz boost to a solitary wave at rest (with \( v = 0 \)) because Lorentz covariance fails. Instead we study a suitable variational problem for which the functions \( \phi_k \in H^{1/2}(\mathbb{R}^3) \) arise as solutions (called boosted excited states) to a Choquard type equation in \( \mathbb{R}^3 \), where the negative Laplacian is replaced by the pseudo-differential operator \( \sqrt{-\Delta + m^2} - m \) and an additional term \( i(v \cdot \nabla) \) enters. Moreover, we give a new proof for existence of boosted ground states. The results are based on perturbation methods in critical point theory.

1. Introduction

The Choquard equation in three dimensions reads

\[ -\Delta u - \left( \int_{\mathbb{R}^3} |u(y)|^2 W(x - y) \, dy \right) u(x) = -\lambda u, \]  

where \( W \) is a positive function. It comes from the functional

\[ E_{NR}(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx - \frac{1}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |u(x)|^2 W(x - y)|u(y)|^2 \, dx \, dy, \]  

which, in turn, arises from an approximation to the Hartree-Fock theory of a one-component plasma when \( W(y) = 1/|y| \) (Coulomb case); as suggested by P. Choquard in 1976. If one defines

\[ E_{NR}(\nu) = \inf \{ E_{NR}(u) : u \in H^1(\mathbb{R}^3), \|u\|_{L^2} \leq \nu \} \]

intuition suggests that:

- the energy \( E_{NR}(\nu) \) is finite;

Date: July 22, 2021
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1991 Mathematics Subject Classification. 35Q40, 35Q75, 35Q51, 81Q99, 83C20, 85A15.
Key words and phrases. quasi-relativistic Choquard/Hartree type equation, Boson star, solitary waves, infinitely many solutions, critical point theory.
there is a minimizing $u$ for $E_{NR}(\nu)$ which satisfies the nonlinear Schrödinger equation
$$(-\Delta + W_{u}(x))u(x) = -\lambda u(x)$$
for some $\lambda > 0$, with $W_{u}(x) = -2 \int |u(y)|^2 |x - y|^{-1} \, dy$;

the minimizing $u$ is unique except for translations (i.e. $u(x) \mapsto u(x + a)$, $a \in \mathbb{R}^3$), and $\|u\|_{L^2} = \lambda$. Moreover, $u \in C^\infty(\mathbb{R}^3)$. Hence
$$E_{NR}(\nu) = \inf \{ E_{NR}(u) : u \in H^1(\mathbb{R}^3), \|u\|_{L^2} = \nu \}.$$  \hspace{1cm} (1.4)

All these facts were established by Lieb in 1977 [9]. The mathematical difficulty of the functional is caused by the minus sign in $E$, which makes it impossible to apply standard arguments for convex functionals. Lieb overcame the lack of convexity by using the theory of symmetric decreasing functions.

In 1980 Lions [12] studied both the unconstrained and constrained problems. For the unconstrained problem (1.1) (with $W$ being a positive function as above) he showed under very mild conditions on $W$ – essentially $W$ is required to be spherically symmetric – that, for $\lambda > 0$, (1.1) possesses infinitely many solutions $(u_j)_{j \geq 1}$ such that

- $u_1$ is positive;
- $u_j$ is spherically symmetric;
- furthermore,
$$0 < S_{NR}(u_j) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u_j|^2 + \frac{\lambda}{2} |u_j(x)|^2 \, dx$$
$$- \frac{1}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |u_j(y)|^2 W(x - y) |u_j(x)|^2 \, dxdy \to +\infty \text{ as } j \to \infty.$$

To show these results for the unconstrained problem, Lions applied the original Mountain Pass Theorem by Ambrosetti and Rabinowitz [3, 2]. For the constrained problem, seeking radially symmetric, normalized functions $\|u\|_{L^2} = +1$, or more generally, seeking solutions belonging to
$$C_N = \{ \phi \in H^1_1(\mathbb{R}^3) : \|\phi\|_{L^2} = N \},$$
the situation is more complicated, see Lions [12], and conditions on $W$ are necessary. In the Coulomb case, Lions proves that there exists a sequence $(\lambda_j, u_j)$, with $\lambda_j > 0$, and $u_j$ satisfies (1.1) (with $\lambda = \lambda_j$) and belongs to $C_1$. Moreover, one has that :

- $u_1$ is positive and $E_{NR}(u_1) = \min_{\{v \in H^1(\mathbb{R}^3), \|v\|_{L^2(\mathbb{R}^3)} = 1\}} E_{NR}(v)$;
- $u_j$ is spherically symmetric;
- $0 > E_{NR}(u_j) \to 0$, $\lambda_j \downarrow 0$ as $j \to \infty$, where
$$E_{NR}(u_j) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u_j|^2 - \frac{1}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |u_j(y)|^2 |x - y|^{-1} |u_j(x)|^2 \, dxdy.$$
We may replace the negative Laplace operator by the so-called quasi-relativistic operator, i.e., the pseudodifferential operator \( \sqrt{-\Delta + m^2} - m \); this is the kinetic energy operator of a relativistic particle of mass \( m \geq 0 \). It is defined via multiplication in the Fourier space with the symbol \( \sqrt{\xi^2 + m^2} - m \), which is frequently used in relativistic quantum physics models as a suitable replacement of the full (matrix valued) Dirac operator.

In the present paper we begin by considering the time-dependent equation

\[
\begin{align*}
    i\partial_t \psi &= \left( \sqrt{-\Delta + m^2} - m \right) \psi - \left( \frac{1}{|x|} \ast |\psi|^2 \right) \psi \quad \text{on } \mathbb{R}^3, \\
\end{align*}
\]

where \( \psi(x,t) \) is a complex-valued wave function, and the symbol \( \ast \) designates the convolution on \( \mathbb{R}^3 \). In suitable physical units, the convolution kernel \( |x|^{-1} \) represents the Newtonian gravitational potential. The equation (1.5) arises as an effective dynamical description for an \( N \)-body quantum system of relativistic bosons with two-body interaction given by Newtonian gravity, as recently shown by Elgart and Schlein [5]. This system models a Boson star [14, 16, 17], characterized by a quasi-relativistic regime, where effects of special relativity (taken into account by the operator \( \sqrt{-\Delta + m^2} - m \)) are important, but effects of the general theory of relativity are negligible.

The nonlinearity in (1.5) is focusing (see, e.g., [18]) and, therefore, there exist solitary wave solutions (or solitary waves)

\[
\psi(x,t) = e^{it\lambda} \phi(x),
\]

where \( \phi \in H^{1/2}(\mathbb{R}^3) \) (see Section 3 for the definition of this Sobolev space) is defined as a minimizer of

\[
\mathcal{E}(\phi) = \frac{1}{2} \int_{\mathbb{R}^3} \psi(\sqrt{-\Delta + m^2} - m) \phi \, dx - \frac{1}{4} \int \left( \frac{1}{|x|} \ast |\phi|^2 \right) |\phi|^2 \, dx
\]

such that

\[
\int_{\mathbb{R}^3} |\psi(x,t)|^2 \, dx = N.
\]

Any such minimizer \( \phi \) is referred to as a ground state [11] and it satisfies the associated Euler-Lagrange equation

\[
(\sqrt{-\Delta + m^2} - m)\phi - \left( \frac{1}{|x|} \ast |\phi|^2 \right) \phi(x) = -\lambda \phi
\]

for some \( \lambda \in \mathbb{R} \).

In this paper we consider solutions of the form

\[
\psi(x,t) = e^{it\lambda} \phi_v(x - vt)
\]

with some \( \lambda \in \mathbb{R} \) and traveling velocity \( v \in \mathbb{R}^3 \), such that \( |v| < 1 \) holds (in our choice of units, this means that \( |v| \) is below the speed of light). Solutions of the kind (1.7) cannot be directly obtained from solitary waves at rest (meaning we set \( v \) equal to zero) and invoking a Lorentz boost (see, e.g., [19]) because (1.5) is not Lorentz covariant. To by-pass this obstacle, we insert the ansatz (1.7) into (1.5), which leads to

\[
(\sqrt{-\Delta + m^2} - m)\phi + i(v \cdot \nabla)\phi - \left( \frac{1}{|x|} \ast |\phi|^2 \right) \phi = -\lambda \phi.
\]
This is the Euler-Lagrange equation for the following minimization problem associated with functional
\[ E_v(\varphi) := E(\varphi) + \frac{i}{2} \int \overline{\varphi} (v \cdot \nabla) \varphi \, dx \quad (1.9) \]
subject to the constraint
\[ N(\varphi) := \int_{\mathbb{R}^3} |\varphi(x,t)|^2 \, dx = N. \quad (1.10) \]

We study the nonlocal and nonlinear problem
\[ \tilde{L}_0 \phi + i(v \cdot \nabla) \phi - \left( \frac{1}{|x|} * |\phi|^2 \right) \phi = -\lambda \phi, \quad (1.11) \]
\[ \|\phi\|_{L^2(\mathbb{R}^3)} = N, \quad (1.12) \]
where \( \tilde{L}_0 = \sqrt{-\Delta + m^2} - m \). We prove existence of multiple solutions, including a minimizer of the corresponding energy functional \( E_v(\cdot) \), viz.
\[ E_v(N) := \inf \{ E_v(\varphi) : \varphi \in H^{1/2}(\mathbb{R}^3), \quad N(\varphi) = N \}, \quad (1.13) \]
where \( H^{1/2}(\mathbb{R}^3) \) is the Sobolev space defined in Section 2. Introduce
\[ \mathcal{M}_N = \{ \psi \in H^{1/2}(\mathbb{R}^3) : N(\psi) = N \}. \quad (1.14) \]

The main theorem is:

**Theorem 1.1.** Assume \( m > 0, \ v \in \mathbb{R}^3 \) with \( |v| < 1 \). Then there exists a positive constant \( N_c(v) \) depending only on \( v \) such that :

1. For \( 0 < N < N_c(v) \), every minimizing sequence of (1.13) is relatively compact in \( \mathcal{M}_N \) up to translations; defined in (1.14). In particular, there exists a minimizer \( \phi \) of \( E_v(\cdot) \) on the admissible set \( \mathcal{M}_N \) such that

\[ \left( \sqrt{-\Delta + m^2} - m + iv \cdot \nabla \right) \phi - \left( \frac{1}{|x|} * |\phi|^2 \right) \phi + \lambda \phi = 0, \quad (1.15) \]
\[ \|\phi\|_{L^2(\mathbb{R}^3)} = N, \quad (1.16) \]

for some \( \lambda > 0 \).

2. For \( 0 < N < N_c(v) \) there exists a sequence \( (\phi_k) \) of distinct solutions of

\[ \left( \sqrt{-\Delta + m^2} - m + iv \cdot \nabla \right) \phi_k - \left( \frac{1}{|x|} * |\phi_k|^2 \right) \phi_k + \lambda_k \phi_k = 0, \quad (1.17) \]
\[ \|\phi_k\|_{L^2(\mathbb{R}^3)} = N, \quad (1.18) \]

for some \( \lambda_k > 0 \) and, moreover,
\[ \lambda_k \to 0, \quad \phi_k \to 0 \quad \text{weakly in} \ H^{1/2}(\mathbb{R}^3). \]

3. For \( N \geq N_c(v) \) no minimizer exists for problem (1.13), even though \( E_v(N) = 0 \) is finite for \( N = N_c(v) \).
We note that radially symmetric solutions to (1.11) do not exist. Indeed if $\phi$ was radial, the left-hand side of
\[ L_0 \phi - \left( \frac{1}{|x|} \ast |\phi|^2 \right) \phi + \lambda \phi = -i(v \cdot \nabla)\phi \]
is radial. However, the right-hand side is
\[ -iv \cdot \nabla \phi = -i \frac{x}{|x|} \phi'(|x|) \]
which is radial if and only if $\phi' \equiv 0$.

For $v \equiv 0$, the first rigorous study of (1.11) was performed by Lieb and Yau [11] in a slightly different context, when the constraint is replaced by $\| \phi \|_{L^2} = N$. They established the existence of a symmetric decreasing minimizer provided $N < N_b$ for some number $N_b$. For the same problem, but allowing $v \neq 0$, Fröhlich et al [7] proved existence of minimizers (or boosted ground states) by using Lions’ concentration-compactness method, and they also proved item 3 above. In the present paper we will prove existence of boosted excited states (item 2 in Theorem 1.1), as well as existence of minimizers (boosted ground states, see item 1 in Theorem 1.1) by completely different methods. The methods are based on smooth perturbed minimization principles (item 1) and perturbed variational principles (item 2), using explicitly Morse-type information on Palais-Smale sequences (see Section 5) to avoid the possibility of “vanishing eigenvalues”.

2. Preliminaries

Throughout the paper we denote by $C$ (with or without indices) various constants whose precise value is of no importance. Let $\mathbb{R}^N$ be the $N$-dimensional Euclidean space. We set
\[ B_R = \{ x \in \mathbb{R}^N : |x| < R \}, \quad B(x, R) = \{ y \in \mathbb{R}^N : |x - y| < R \}. \]
By $S^{N-1}$ we will denote the unit sphere in $\mathbb{R}^N$.

Functions. By $C_0^\infty$, $C^\infty$, and $L^p$ we refer to the standard function spaces. For a measure space $(\mathcal{M}, \mu)$, $\mu$ being a $\sigma$-finite measure, the weak $L^p$ space (or Marcinkiewicz space) is defined as the space $L^p_w$ of measurable functions $\phi$ such that
\[ \sup_{t>0} \mu(\{ x : |\phi(x)| > t \})^{1/p} < \infty. \]

Sobolev spaces. Denoting the Fourier-Plancherel transform of $u \in L^2(\mathbb{R}^3)$ by $\hat{u}$, we define
\[ H^{1/2}(\mathbb{R}^3) = \{ \phi \in L^2(\mathbb{R}^3) : (1 + |\xi|)^{1/2} \hat{\phi} \in L^2(\mathbb{R}^3) \}, \]
which, endowed with the scalar product
\[ \langle \phi, \psi \rangle_{H^{1/2}(\mathbb{R}^3)} = \int_{\mathbb{R}^3} (1 + |\xi|) \hat{\phi}(\xi) \overline{\psi}(\xi) \, d\xi, \]
becomes a Hilbert space; evidently, $H^1(\mathbb{R}^3) \subset H^{1/2}(\mathbb{R}^3)$. We have that $C_0^\infty(\mathbb{R}^3)$ is dense in $H^{1/2}(\mathbb{R}^3)$ and the continuous embedding $H^{1/2}(\mathbb{R}^3) \hookrightarrow L^r(\mathbb{R}^3)$ holds whenever $r \in [2, 3]$ [1, Theorem 7.57]. Moreover, we shall use that any weakly convergent sequence in $H^{1/2}(\mathbb{R}^3)$ has a pointwise convergent subsequence.
Homotopic families. If a group $G$ acts on two topological spaces $\mathcal{X}$ and $\mathcal{Y}$, we say that a function $f : \mathcal{X} \to \mathcal{Y}$ is $G$-equivariant if $f(g \cdot x) = gf(x)$ for all $g \in G$ and $x \in \mathcal{X}$. We denote by $C_G(\mathcal{X}, \mathcal{Y})$ the set of all $G$-equivariant functions.

Let $\Omega$ be a compact subset of $\mathbb{R}^n$, $n \geq 1$ and let $M$ be a complete $C^2$-Riemannian manifold. Assume that $G$ is a compact Lie group acting freely and differentiably on $M$ and $\Omega$. A family $\mathcal{F}$ of sets of the form

$$\{ f(\Omega) : f \in C_G(\Omega, M) \}$$

is called a $G$-homotopic family of dimension $n$. Here $C_G(\Omega, M)$ is the set of all $G$-equivariant continuous $f : \Omega \to M$.

Operators. Let $T$ be a self-adjoint operator on a Hilbert space $H$ with domain $D(T)$. The spectrum and resolvent set are denoted by $\sigma(T)$ and $\rho(T)$, respectively. We use standard terminology for the various parts of the spectrum. The spectral family associated to $T$ is denoted by $E_T(\lambda)$, $\lambda \in \mathbb{R}$.

3. Basic set-up and assumptions

We define the following quadratic form, associated to the kinetic energy,

$$I_0[\phi] := \| \hat{\phi}(\xi) \|^2_{L^2(\mathbb{R}^3, (\sqrt{2\pi |\xi|^2 + m^2} - m)^2 d\xi)}$$

on $H^{1/2}(\mathbb{R}^3)$. It is convenient to introduce

$$I_0[\phi] := \| \hat{\phi}(\xi) \|^2_{L^2(\mathbb{R}^3, (\sqrt{2\pi |\xi|^2} + m^2) d\xi)}$$

and

$$I_v[\phi] := I_0[\phi] - \int_{\mathbb{R}^3} v \cdot \xi |\hat{\phi}(\xi)|^2 d\xi$$

where $v \in \mathbb{R}^3$ and $|v| < 1$. Moreover, we define (arising from the direct Coulomb energy)

$$\mathcal{J}_{1/|x|}(\psi, \phi) := \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \psi(x)\phi(y) |x - y| dxdy,$$  

whenever it makes sense. We consider the following functional $\mathcal{E}_v : H^{1/2}(\mathbb{R}^3) \to \mathbb{R}$ defined by

$$\phi \mapsto \frac{1}{2} I_v[\phi] - \frac{1}{2} m \| \phi \|^2_{L^2} - \frac{1}{4} \mathcal{J}_{1/|x|}(|\phi|^2, |\phi|^2),$$

At this place we do not focus on whether the functionals are well-defined or not, this will be discussed in detail in the sequel. We do, however, point out that $\mathcal{E}_v$ is real-valued. Clearly, it suffices to show that $I_v$ is real-valued and this immediately follows from two applications of Plancherel’s theorem.

Assumption 3.1. Let $W$ be a nonnegative, nonzero measure such that there exist $K \geq 1$, $p_k \in (1, \infty)$, with $k \in [1, K]$, and functions $W_k$ satisfying

$$W = \sum_{k=1}^{K} W_k, \quad W_k \in L^{p_k}_{w}(\mathbb{R}^3).$$
4. Auxiliary facts and results

We summarize some basic facts, starting with [7, Lemma B.1].

**Lemma 4.1.** For any $v \in \mathbb{R}^3$ with $|v| < 1$, there exists (an optimal) a constant $S_v$ such that

$$
\int_{\mathbb{R}^3} \left( \frac{1}{|x|} * |\psi|^2 \right) |\psi|^2 \, dx \leq S_v \langle \psi, \left( \sqrt{-\Delta} + iv \cdot \nabla \right) \psi \rangle \langle \psi, \psi \rangle \tag{4.1}
$$

is valid for all $\psi \in H^{1/2}(\mathbb{R}^3)$. Furthermore,

$$
S_v = \frac{2}{\langle Q_v, Q_v \rangle} \tag{4.2}
$$

where $Q_v \in H^{1/2}(\mathbb{R}^3)$, $Q_v \not\equiv 0$, is an optimiser for (4.1) and it obeys

$$
\sqrt{-\Delta} Q_v + i(v \cdot \nabla) Q_v - \left( \frac{1}{|x|} * |Q_v|^2 \right) |Q_v|^2 = -Q_v. \tag{4.3}
$$

Moreover,

$$
S_{v=0} < \frac{\pi}{2}, \quad S_{v=0} \leq S_v \leq (1 - |v|)^{-1} S_{v=0}.
$$

We note that (see also [7, Appendix C]):

**Lemma 4.2.** For every $v \in \mathbb{R}^3$ with $|v| < 1$, there exist $C_1, C_2 > 0$ such that

$$
C_1(|\xi| + m) \leq \sqrt{\xi^2 + m^2} - (v \cdot \xi) \leq C_2(|\xi| + m). \tag{4.4}
$$

**Proof.** It is easy to show that there exists $0 < \delta < 1$ such that $\sqrt{x^2 + m^2} \geq (1 - \delta)x + \delta m$ for every $x, m > 0$. The latter inequality, together with $v \cdot \xi \geq -|v||\xi|$, implies that

$$
\sqrt{|\xi|^2 + m^2} - v \cdot \xi \geq (1 - \delta)|\xi| + \delta m - |v||\xi| = (1 - \delta - |v|)|\xi| + \delta m \geq C_v(|\xi| + m),
$$

where we have chosen $\delta$ such that $\delta < 1 - |v|$ and $C_v = \min(1 - |v| - \delta, \delta)$. This establishes the first inequality of (4.4). As for the second inequality we use the fact that $|v \cdot \xi| \leq |\xi| + m$ and $\sqrt{|\xi|^2 + m^2} \leq |\xi| + m$, which yield

$$
\sqrt{|\xi|^2 + m^2} - (v \cdot \xi) \leq 2(|\xi| + m).
$$

This completes the proof. \qed

We also need [7, Lemma A.4]:

**Lemma 4.3.** Assume $m > 0$, $v \in \mathbb{R}^3$ with $|v| < 1$. Then the functional $I_v$ is well-defined and weakly lower semi-continuous on $H^{1/2}(\mathbb{R}^3)$. Furthermore, if $\lim_{k \to \infty} I_v(\phi_k) = I_v(\phi)$ holds, then $\phi_k \to \phi$ strongly in $H^{1/2}(\mathbb{R}^3)$ as $k$ tends to $\infty$.

For the sake of completeness, we include its proof.
Proof. An application of Plancherel’s theorem yields

\[ I_v[\phi] = \int_{\mathbb{R}^3} |\hat{\phi}(\xi)|^2 (\sqrt{\xi^2 + m^2} - (v \cdot \xi)) \, d\xi. \]

In view of Lemma 4.2 we have that

\[ C_{v,1}(|\xi| + m) \leq \sqrt{\xi^2 + m^2} - (v \cdot \xi) \leq C_2(|\xi| + m). \tag{4.5} \]

As a consequence, \( C_{1,v} \|\phi\|_{H^{1/2}} \leq I_v[\phi] \leq C_2 \|\phi\|_{H^{1/2}} \) and if \( \phi \in H^{1/2}(\mathbb{R}^3) \) then \( I_v[\cdot] \) is well-defined. Moreover,

\[ \|\phi\|_{C_v} := \sqrt{I_v[\phi]} \tag{4.6} \]

defines a norm, which is equivalent to \( \|\cdot\|_{H^{1/2}} \). In particular, weak and strong convergence for the two norms coincide. By virtue of (3.3) we identify \( \|\phi\|_{C_v} \) with the \( L^2 \)-norm of \( \hat{\phi} \) taken with respect to the integration measure

\[ d\mu = (\sqrt{\xi^2 + m^2} - (v \cdot \xi)) \, d\xi. \tag{4.7} \]

We now deduce the first assertion from the weakly lower semicontinuity of the \( L^2(\mathbb{R}^3; d\mu) \)-norm. As for the second assertion, it is a consequence of the Brezis-Lieb Lemma (see, e.g., [2, Lemma 11.9] or [10]). \( \Box \)

We recall [7, Lemma 2.1]:

Lemma 4.4. Assume \( m \geq 0, v \in \mathbb{R}^3 \) with \( |v| < 1 \). Then

\[ 2\mathcal{E}_v(\psi) \geq \left( 1 - \frac{N}{N_c(v)} \right) \langle \psi, (\sqrt{-\Delta} + iv \cdot \nabla)\psi \rangle - mN \tag{4.8} \]

for all \( \psi \in H^{1/2}(\mathbb{R}^3) \) with \( \mathcal{N}(\psi) = N \). Moreover,

\[ E_v(N) \geq -\frac{1}{2} mN \text{ for } 0 < N < N_c(v) \tag{4.9} \]

and \( E_v(N) = -\infty \) for \( N > N_c(v) \). In particular, any minimizing sequence of the problem (1.13) is bounded from below whenever \( 0 < N < N_c(v) \).

For the class of potentials \( W \) in Assumption 3.1 we have:

Lemma 4.5. Let Assumption 3.1 be satisfied. Let \( r \in [2,3] \) and suppose that the sequence \( (\phi_j)_{j \geq 1} \) is bounded in \( L^r(\mathbb{R}^3) \), and that \( \phi_j \rightharpoonup \phi \) strongly in \( L^r(\mathbb{R}^3) \). Then

\[ (W * \phi_j^2)\phi_j \rightharpoonup (W * \phi^2)\phi, \quad \text{as } j \to \infty, \tag{4.10} \]

and

\[ \mathcal{J}_W(|\phi_j|^2, |\phi_j|^2) \text{ converges to } \mathcal{J}_W(|\phi|^2, |\phi|^2) \text{ as } j \to \infty, \tag{4.11} \]

where \( \mathcal{J}_W \) is defined as a convolution, analogously to \( \mathcal{J}_{1/|x|} \) in (3.4).

Proof. The sequence \( (\phi_j^2)_{j \geq 1} \) is bounded in \( L^r(\mathbb{R}^3) \), \( s \in [1, \frac{3}{2}] \) because \( \phi_j \) is bounded in \( L^r(\mathbb{R}^3) \), \( r \in [2,3] \). The generalized Young inequality and the hypothesis \( W \in L^p_w(\mathbb{R}^3) \) imply that \( W * \phi_j^2 \) is bounded in \( L^q(\mathbb{R}^3) \) with \( 3/2 < q < \infty \). An application of Lebesgue’s
dominated convergence theorem shows that \( W \ast \phi_j^2 \) converges strongly to \( W \ast |\phi|^2 \) in \( L^q(\mathbb{R}^3) \). Let \( \psi_j = W \ast |\phi_j|^2 \), and \( w \in \text{H}^{1/2} \). Then
\[
|\langle \psi_j \phi_j - \psi \phi, w \rangle_{\text{H}^{-1/2, \text{H}^{1/2}}} | = |\langle \psi_j \phi_j - \psi \phi + \psi \phi - \psi \phi, w \rangle_{\text{H}^{-1/2, \text{H}^{1/2}}} |
\leq C \|\psi_j \phi_j - \phi\|_{L^2} + \|\psi \phi - \psi \phi\|_{L^2}
\]
The Hölder inequality yields
\[
\|\psi_j \phi_j - \phi\|_{L^2} \leq \|\psi_j^2\|_{L^l}\|\phi - \phi\|^2_{L^m}
\]
with \((1/l) + (1/m) = 1\); this holds because \( m \in [1, 3/2] \) and \( l \in (3/4, \infty) \). The uniform boundedness of \( \psi_j \in L^q(\mathbb{R}^3) \), \( q \in (3/2, \infty) \), together with the strong convergence of \( \phi_j \) to \( \phi \) in \( L^r \), \( r \in [2, 3] \), and the strong convergence of \( \psi_j \) to \( \psi \) in \( L^g \), allow us to deduce that \( \langle \psi_j \phi_j - \psi \phi, w \rangle_{\text{H}^{-1/2, \text{H}^{1/2}}} \to 0 \) as \( j \to \infty \). Therefore,
\[
\psi_j \phi_j \rightharpoonup_{\text{H}^{-1/2}} \psi \phi,
\]
i.e., \((4.10)\) holds. On the other hand, the boundedness of \( \phi_j \) in \( \text{H}^{1/2}(\mathbb{R}^3) \) and the boundedness of \( W \ast \phi_j^2 \) in \( L^g \), imply that the sequence \((W \ast \phi_j^2)\phi_j^2\) is bounded in \( L^1 \). These facts, in conjunction with the pointwise convergence of \((W \ast \phi_j^2)\phi_j^2\) to \((W \ast \phi^2)\phi^2\) in \( \mathbb{R}^3 \) and the Lebesgue’s dominated convergence theorem yields \((4.11)\). \(\square\)

We need the following form and its associated self-adjoint operator.

**Lemma 4.6.** Assume that \( 0 < N < N_c(v) = 2/S_v \). The sesquilinear form defined on \( \text{H}^{1/2}(\mathbb{R}^3) \) by
\[
f_v[\phi, \psi] = l_v[\phi, \psi] - m\langle \phi, \psi \rangle_{L^2} - \frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\phi(x)^2 |\psi(y)|^2}{|x - y|} \, dx \, dy \tag{4.12}
\]
generates a unique self-adjoint operator \( F_v \) associated with the differential expression
\[
\sqrt{-\Delta + m^2} + i(v \cdot \nabla) - m - \frac{1}{2} \left( \frac{1}{|x|} * |\phi|^2 \right).
\]

**Proof.** We already know that \( l_v[\cdot, \cdot] \) is a closed, nonnegative form. Since, by invoking Lemma 4.1,
\[
|(1/2) J_{1/|x|} (u, u) | \leq \frac{1}{2} S_v l_v[u] \|u\|_{L^2}^2 < l_v[u],
\]
where the last inequality follows from the assumption \( 0 < N < N_c(v) \), the KLMN theorem [15, Theorem 3.7] immediately yields the result. \(\square\)

The following abstract operator result goes back to Lions [13, Lemma II.2].

**Lemma 4.7.** Let \( T \) be a self-adjoint operator on a Hilbert space \( \mathcal{H} \) and let \( \mathcal{H}_1, \mathcal{H}_2 \) two subspaces of \( \mathcal{H} \) such that \( \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \), \( \dim \mathcal{H}_1 = h_1 < \infty \) and \( P_2P_1 \geq 0 \), where \( P_2 \) is the orthogonal projection onto \( \mathcal{H}_2 \). Then \( T \) has at most \( h_1 \) negative eigenvalues.

We will use the following result repeatedly:

**Lemma 4.8.** Suppose \( m > 0, v \in \mathbb{R}^3 \) with \( |v| < 1 \) and \( 0 < N < N_c(v) \). Let \( F_v \) be the quasi-relativistic Choquard type operator in Lemma 4.6. Then, for each integer \( k \), there is a \( \delta > 0 \) such that \( F_v \) admits at least \( k \) eigenvalues strictly below \(-\delta\).
Proof. In view of Glazman’s lemma (see, e.g., [15, Lemma A.3]) we shall find for each integer \( d \) a subspace \( S_d \) of dimension \( d \) such that

\[
\max\{ \langle f, u \rangle_{L^2} : u \in S_d, \|u\|_{L^2} = N \} < 0.
\]

Let \( V_d \) be the \( d \)-dimensional subspace of \( H^{1/2} \) spanned by the first \( d \) eigenfunctions \( u_n \) of \(-\Delta\) on \( \mathbb{R} = \{ |x| \leq 2 \} \) with Dirichlet boundary conditions on \( \partial \mathbb{R} \). Choose a function \( u \in V_d \cap M_N \) and let \( u_\kappa(x) = \kappa^{-3/2}u(x/\kappa) \). Then \( \|u_\kappa\|_{L^2} = N \). We are going to use that for two forms \( t_1, t_2 \) satisfying \( D(t_1) \subset D(t_2) \) and \( t_2 \leq t_1 \), one has \( \text{Coun}(-\epsilon_{\kappa,N}; T_1) \leq \text{Coun}(-\epsilon_{\kappa,N}; T_2) \). In our case we have that \( H^1(\mathbb{R}^3) \subset H^{1/2}(\mathbb{R}^3) \) and, moreover,

\[
I_v[u] \leq C\|u\|^2_{H^1}, \quad \forall u \in H^1(\mathbb{R}^3).
\]

Therefore, it suffices to estimate

\[
f_v[u_\kappa] \leq \frac{C}{\kappa^2} \int_{\mathbb{R}^3} |\nabla u|^2 dx - \frac{1}{2} \int \frac{\phi^2(x)u^2(y)}{|x - \kappa y|} dxdy
\]

\[
\leq \frac{C}{\kappa^2} \int_{\mathbb{R}^3} |\nabla u|^2 dx - \frac{1}{2\kappa} \int \frac{\phi^2(x)u^2(y)}{|x| + |y|} dxdy \quad (\text{for } \kappa \gg 1)
\]

Since

\[
\int \int \frac{\phi^2(x)u^2(y)}{|x| + |y|} dxdy \leq \int \int \frac{\phi^2(x)u^2(y)}{|x - y|} dxdy < +\infty,
\]

It is now enough to choose any \( d \)-dimensional space of functions \( u \) as above and then let \( S_d \) be the space obtained by rescaling them as above \((u \rightarrow u_\kappa)\). Then there exists \( \kappa_0 > 1 \) such that for every \( \kappa \geq \kappa_0 \), we have \( f_v[\phi_\kappa] < 0 \). \( \square \)

5. Summary of critical point theory by Lions-Fang-Ghoussoub

By combining [8, Theorem 11.1] and [8, Remark 11.13] we obtain the following result, which is a simplified version of the original theorem first presented in [6]. Below, as usual, \( d\mathcal{J} \) and \( d^2\mathcal{J} \) denote the first and second variation of a functional \( \mathcal{J} \).

**Theorem 5.1.** Suppose \( G \) is a compact Lie group acting freely and differentiable on a complete \( C^2 \)-Riemannian manifold \( \mathcal{X} \). Let \( \mathcal{J} \) be a \( G \)-invariant \( C^2 \) functional on \( \mathcal{X} \) with \( d\mathcal{J} \) and \( d^2\mathcal{J} \) being Hölder continuous on \( \mathcal{X} \) and suppose \( H \) is a \( G \)-homotopic family of dimension \( k \), i.e., a set of the kind

\[
H = \{ h(D) : h \in C_G(D, \mathcal{X}) \},
\]

where \( D \) is a fixed \( G \)-invariant compact subset of \( \mathbb{R}^k \), and \( C_G(D, \mathcal{X}) \) is the set of all \( G \)-equivariant continuous functions \( h : D \rightarrow \mathcal{X} \). Consider

\[
l = \inf_{A \in H} \max_{x \in \mathcal{X}} \mathcal{J}(x)
\]

Then for any min-max sequence \( \{A_j\}_{j \in \mathbb{N}} \) in \( H \) there exists sequences \( \{x_j\}_j \) in \( \mathcal{X} \) and \( \{\delta_j\}_j \) in \( \mathbb{R}^+ \) with \( \lim_{j \to \infty} \delta_j = 0 \) such that

1. \( x_j \in A_j \) for each \( j \),
2. \( \lim_{j \to \infty} \mathcal{J}(x_j) = l \),
Proposition 6.1. Assume that $\phi$ satisfies (P1) $\lim_{j \to \infty} dJ(x_j) = 0$,
(P2) $\lim_{j \to \infty} d^2J(x_j) = 0$.
(P3) There exists a sequence $\{\delta(j)\}$ of positive real numbers such that $\delta(j) \to 0$ for every $j$, $d^2J(x_j)$ has at most $k$ eigenvalues below $-\delta(j)$.

Definition 5.2. A $C^2$-function on a $C^2$-Riemannian manifold $\mathcal{X}$ is said to have the Palais-Smale condition at level $l$, around the set $\mathcal{H}$ and of order less than $k$ (in short, (PS)$_{\mathcal{H},l,k^{-}}$), if a sequence $\{x_j\}_j$ in $\mathcal{X}$ is relatively compact whenever the sequence satisfies the following conditions:

(P1) $\lim_{j} J(x_j) = l$
(P2) $\lim_{j} dJ(x_j) = 0$
(P3) There exists a sequence $\{\delta(j)\}$ of positive real numbers such that $\delta(j) \to 0$ for every $j$, $d^2J(x_j)$ has at most $k$ eigenvalues below $-\delta(j)$.

6. Convergence of Palais-Smale type sequences

The following result is instrumental in the proof of the main theorem.

Proposition 6.1. Assume that $l \in \mathbb{R}$, that $n \in \mathbb{N}$ and let $0 < N < N_c(v)$. Let $(\phi_j) \subset \mathcal{M}_N$ be a sequence satisfying the Palais-Smale condition at level $l$ and of order less than $n$, i.e., for some sequence $(\delta_j)_j$ of positive reals such that $\delta_j \to 0$ and some real sequence $(\lambda_j)_j$, the following three conditions hold:
(i) $\lim_{j} E_v(\phi_j) = l$;
(ii) $\lim_{j} dE_v(\phi_j) = 0$;
(iii) there exists a sequence $(\delta_j)$ of positive reals with $\delta_j \downarrow 0$ such that for each $j$, $d^2E_v(\phi_j)$ has at most $n$ eigenvalues below $-\delta_j$.

Then any sequence $(\phi_j)$ satisfying (i)-(iii) is relatively compact in $\mathcal{M}_N$ up to translations. Moreover, the limit $\phi$ of $\phi_j$ satisfies the Choquard type equation

$$
(\sqrt{-\Delta + m^2} - m + iv \cdot \nabla) \phi - \left(\frac{1}{|x|} * |\phi_j|^2\right) \phi + \lambda \phi = 0,
$$

$$
\|\phi\|_{L^2(\mathbb{R}^3)} = N,
$$

for some $\lambda > 0$.

Proof. Henceforth let $(\phi_j) \in \mathcal{M}_N$, be a sequence satisfying (i)-(iii). By hypothesis (ii), there exists a sequence $(\lambda_j)$ of reals such that

$$
(\sqrt{-\Delta + m^2} - m + iv \cdot \nabla) \phi_j - \frac{1}{2} \left(\frac{1}{|x|} * |\phi_j|^2\right) \phi_j + \lambda_j \phi_j \to 0 \text{ in } L^2(\mathbb{R}^3)
$$

as $j \to \infty$. First we prove that there exists $0 < \lambda < \infty$ such that $\lambda < \lambda_j$. To prove existence of a lower bound we use the second order information summarized in hypothesis (iii). Indeed, we have

$$
\langle d^2E_v(\phi_j), \varphi, \varphi \rangle = \langle (\sqrt{-\Delta + m^2} - m + iv \cdot \nabla) \varphi, \varphi \rangle - \int_{\mathbb{R}^3} \left(\frac{1}{|x|} * |\phi_j|^2\right) |\varphi(x)|^2 dx
$$

$$
+ (\lambda_j + \delta_j)\|\varphi\|_{L^2}^2 - 2 \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\phi_j(x)\phi_j(y)\varphi(x)\varphi(y)}{|x - y|} dx dy \geq 0.
$$
By the hypothesis (i) and (ii) we have

\[ f_\psi[\psi, \psi] + (\lambda_j + \delta_j)\|\psi\|_{L^2}^2 \geq 0, \tag{6.4} \]

where \( f_\psi \) is the form in (4.12), \( \delta_j \downarrow 0 \) and \( \psi \) belongs to a closed subspace of \( H^{1/2} \) with finite codimension \( 1 + n \). By invoking Lemma 4.7, we deduce that the family of operator

\[ L_j = L_v - mI - \frac{1}{|x|} * |\phi_j|^2 \tag{6.5} \]

has at most \( n + 1 \) eigenvalues strictly less than \(- (\lambda_j + \delta_j)\). Lemma 4.8 ensures that there exists \( \delta > 0 \) (independent of \( j \)) such that \( L_j \) has at least \( 1 + n \) eigenvalues strictly below \(- \delta\). As a consequence, we infer that \( \lambda_j + \delta_j > \delta, \forall j \). Since \( \delta_j \downarrow 0 \) as \( j \to \infty \), we conclude that, for \( j \) large enough, \( \lambda_j \geq \delta > 0, \forall j \). The uniform upper bound follows from the information level and the first order information. Thus, by the Bolzano-Weierstrass theorem there exists a subsequence still denoted \( \lambda_j \) such that \( \lambda_j \to \lambda > 0 \).

By the hypothesis (i) and (ii) we have

\[ \{ \begin{align*}
&\mathcal{E}_v(\phi_j) \text{ is bounded,} \\
&(\sqrt{-\Delta + m^2} - m + iv \cdot \nabla) \phi_j + \lambda_j \phi_j - \left( \frac{1}{|x|} * |\phi_j|^2 \right) \phi_j \xrightarrow{H^{-1/2}} 0
\end{align*} \]

Let \( \epsilon_j = \mathcal{E}_v(\phi_j) \). We begin by proving that \( (\phi_j)_{j \geq 1} \) is a bounded sequence in \( H^{1/2}(\mathbb{R}^3) \). Now, for some \( C_1 > 0 \), we have that

\[ \mathcal{E}_v(\phi_j) = \frac{1}{2} L_v[\phi_j] - \frac{m}{2} \|\phi_j\|_{L^2}^2 - \frac{1}{4} \mathcal{J}_1/|x|(\phi_j^2, \phi_j^2) \leq C_1. \tag{6.6} \]

Moreover,

\[ L_v[\phi_j] - m \|\phi_j\|_{L^2}^2 = 2\mathcal{E}_v(\phi_j) + \frac{1}{2} \mathcal{J}_1/|x|(\phi_j^2, \phi_j^2) \tag{6.7} \]

and

\[ L_v[\phi_j] + (\lambda_j - m) \|\phi_j\|_{L^2}^2 - \mathcal{J}_1/|x|(\phi_j^2, \phi_j^2) = \langle \epsilon_j, \phi_j \rangle_{H^{-1/2}, H^{1/2}}. \tag{6.8} \]

By combining (6.7) and (6.8) we obtain

\[ L_v[\phi_j] - m \|\phi_j\|_{L^2}^2 - \lambda_j \|\phi_j\|_{L^2}^2 = 4\mathcal{E}_v(\phi_j) - \langle \epsilon_j, \phi_j \rangle \tag{6.9} \]

Using the level information and the first order information in conjunction with (\( \lambda_j \)) being a bounded sequence, we deduce that

\[ L_v[\phi_j] - m \|\phi_j\|_{L^2}^2 \leq C_2 \text{ and thus } L_v[\phi_j] \leq C_3. \tag{6.10} \]

Since \( L_v[\cdot] \) defines a norm which is equivalent to the \( H^{1/2}(\mathbb{R}^3) \)-norm, we deduce that \( (\phi_j)_{j \geq 1} \) is a bounded sequence in \( H^{1/2}(\mathbb{R}^3) \) and \( N(\phi_j) = N \) for every \( j \). By the Banach-Alaoglu theorem there exists a subsequence of \( \phi_j \) (still denoted \( \phi_j \)) such that \( \phi_j \to \phi \) in \( H^{1/2}(\mathbb{R}^3) \). We show that \( \phi \) is nonzero by proving that, for all \( R > 0 \),

\[ \lim_{j \to \infty} \sup_{y \in \mathbb{R}^3} \int_{|x-y|<R} |\phi_j|^2 \, dx > 0. \tag{6.11} \]

Firstly, we note that

\[ E_v(N) < -\frac{1}{2} \left( 1 - \sqrt{1 - v^2} \right) mN. \tag{6.12} \]
This is a consequence of [7, Lemma 2.2], which asserts that
\[
E_v(N) \leq -\frac{1}{2} \left( 1 - \sqrt{1 - v^2} \right) mN + E_{vNR}^v(N),
\]
(6.13)
where
\[
E_{vNR}^v(N) = \inf \left\{ E_{vNR}^v(\psi) : \psi \in H^1(\mathbb{R}^3), \quad \mathcal{N}(\psi) = N \right\}
\]
with
\[
E_{vNR}^v(\psi) := \sqrt{1 - v^2} \int_{\mathbb{R}^3} |\nabla \psi|^2 - \frac{1}{4} \int \frac{1}{|x|} |\psi|^2 |\psi|^2 dx.
\]
From [9] we know that \( E_{vNR}^v(N) \leq E_{vNR}^v(\psi) < 0 \) and, by using this in (6.13), we obtain (6.12).

To prove (6.11), we argue by contradiction, so suppose
\[
\lim_{j \to \infty} \sup_{y \in \mathbb{R}^3} \int_{|x-y|<R} |\phi_j|^2 dx = 0 \quad (6.14)
\]
for all \( R > 0 \). Then [7, Lemma A.1] implies that
\[
\lim_{j \to \infty} \int_{\mathbb{R}^3} \left( \frac{1}{|x|} * |\phi_j|^2 \right) |\phi_j|^2 = 0.
\]
Since
\[
\langle \psi, L_0 \psi \rangle + i \langle \psi, (v \cdot \nabla) \psi \rangle \geq - \left( 1 - \sqrt{1 - v^2} \right) mN,
\]
we deduce that
\[
E_v(N) \geq -\frac{1}{2} \left( 1 - \sqrt{1 - v^2} \right) mN.
\]
(6.15)
But (6.15) contradicts (6.12). Hence we have reached a contradiction and, therefore, the weak limit \( \phi \) is nonzero. From (6.8) we get that
\[
\lambda_j \|\phi_j\|^2_{L^2} = \langle \epsilon_j, \phi_j \rangle - 2E_v(\phi_j) + \frac{1}{2} J_{1/|x|}(\phi_j^2, \phi_j^2)
\]
and, therefore,
\[
\lim_{j} \sup \lambda_j \|\phi_j\|^2_{L^2} = \lim_{j} \langle \epsilon_j, \phi_j \rangle - 2 \liminf_j E_v(\phi_j) + \frac{1}{2} \lim_j J_{1/|x|}(\phi_j^2, \phi_j^2)
\]
\[
\leq \lambda \|\phi\|^2_{L^2} \leq \liminf_j \lambda_j \|\phi_j\|^2_{L^2},
\]
(6.16)
whence
\[
\lim_{j} \sup \lambda_j \|\phi_j\|^2_{L^2} \leq \lambda \|\phi\|^2_{L^2} \leq \liminf_j \lambda_j \|\phi_j\|^2_{L^2},
\]
or
\[
\lim_{j} \sup \lambda_j \leq \frac{\lambda}{N} \|\phi\|^2_{L^2} \leq \liminf_j \lambda_j .
\]
We thus conclude that
\[
\|\phi\|^2_{L^2} = N
\]
and we infer that \( \phi_j \) converges to \( \phi \) in \( L^2(\mathbb{R}^3) \).
Finally, we show that $\phi_j$ converges strongly to $\phi$ in $H^{1/2}(\mathbb{R}^3)$. By passing to the limit in (6.8) we get, using Lemma 4.5,
\[
\lim_j t_n[\phi_j] = -\lim_j \left[ (\lambda_j - m)\|\phi_j\|_{L^2}^2 - J_{1/|x|}(\phi^2_j, \phi^2_j) \right] \\
= - \left[ (\lambda - m)\|\phi\|_{L^2}^2 - J_{1/|x|}(\phi^2, \phi^2) \right] \\
= t_n[\phi]
\]
and since $t_n[\cdot]$ as a norm is equivalent to $H^{1/2}$, this shows that $\phi_j$ converges strongly to $\phi$ in $H^{1/2}(\mathbb{R}^3)$.

7. Proof of Theorem 1.1

We are ready to prove Theorem 1.1

Proof of Theorem 1.1.

Assertion 1. We will show that the functional $E_v$ restricted to $M_N$ verifies (PS)$_{l,k}$ for every $l \in \mathbb{R}$ and any $k \in \mathbb{N}$. From Lemma 4.4 we have that $E_v(\cdot)$ is bounded from below on $M_N$ whenever $0 < N < N_c(v)$ and we may therefore conclude existence of a minimizing sequence $(\tilde{\phi}_j)$ to (1.14). To prove relative compactness we will now prove that the hypothesis (ii) and (iii) in Proposition 6.1 are fulfilled.

For a complete metric space $(X, d)$ introduce $Q$ as the set of functions that can be written in the form
\[
q(x) = \frac{1}{2} \sum_{k=1}^{\infty} \alpha_k d(x, v_k)^2
\]
for some convergent sequence $(v_k)$ and $\alpha_k \geq 0$ such that $\sum_{k=1}^{\infty} \alpha_k = 1$. In our case we have $X = M_N$ and $d(\cdot, v) = \|\cdot - v\|_{H^{1/2}}$ for some $v$. An application of the Borwein-Preiss smooth variational principle [4] (see also [8]) provides us with a new minimizing sequence $(\phi_j)$ such that
\[
\|\phi_j - \tilde{\phi}_j\|_{H^{1/2}} \to 0.
\]
We also have that $(\phi_j)$ minimises
\[
E_v(\cdot) + \frac{1}{j} q_j(\cdot)
\]
on $M_N$ with $q_j \in Q$. This new minimizing sequence satisfies the assumptions of Proposition 6.1 with $n = 1$ therein. Existence of a minimum for $0 < N < N_c(v)$ now follows.

Assertion 2. Since $E_v$ is even in $\phi$, we use a min-max method to obtain critical points. We will prove that there exists a critical point at infinitely many distinct levels. We note that $Z_2 := \{-1, 1\}$ equipped with multiplication as binary operation and the discrete topology can be considered to be a compact Lie group. As in [13, 8] we use a min-max method of the form
\[
\min_{f \in C_G(\mathbb{S}^{k-1}, M_N)} \max_{\phi \in f(\mathbb{S}^{k-1})} E_v(\phi),
\]
where $G = Z_2 \sim \{-1, 1\}$ acts on the Euclidean space $\mathbb{S}^{k-1}$ of $\mathbb{R}^k$, i.e.,
\[
(\pm 1, x) \mapsto \pm x, \quad x \in \mathbb{R}^k.
\]
The action of $\mathbb{Z}_2$ on $M_N$ is chosen as $(\pm 1, \phi) \mapsto \pm \phi$, $\phi \in M_N$. Existence of a minimum for $0 < N < N_c(v)$ now follows.
\(H_N\). For each \(k \in \mathbb{N}\), we consider the following homotopic class of order \(k\),

\[
H_k = \{ M : M = f(S^{k-1}) : f \in C_{Z_2}(S^{k-1}, M_N) \}
\]

where \(C_{Z_2}(S^{k-1}, M_N)\) is the set of all \(Z_2\)-equivariant continuous functions. We then define

\[
l_k := \inf_{M \in H_k} \max_{\phi \in M} E_v(\phi).
\]

(7.2)

We shall apply Theorem 5.1 by Fang-Ghoussoub which enable us to obtain Palais-Smale sequences satisfying the assumptions of Proposition 6.1. Evidently, the min-max (7.1) is defined with \(\mathbb{Z}_2\)-homotopic classes of dimension \(k\) for each \(k\) (Choose \(X = M_N\) and \(D = S^{k-1}\)). An application of Theorem 5.1 provides us with a sequence \((\phi_j)_j\) that satisfies the assumptions of Proposition 6.1 with \(n = k\). Therefore, the sequence \((\phi_j)_j\) converges up to a subsequence, to some critical point \(\phi\) of \(E_v\) on \(M_N\). The monotonicity \(l_k \leq l_{k+1}\) of \((l_k)_k\) is a direct consequence of how we have defined \(H_k\) and since \(E_v\) is bounded from below on \(M_N\) we immediately get \(l_k > -\infty\). By arguing as in the proof of Lemma 4.8, one can easily deduce that, for each \(k \geq 1\), there exists a \(k\)-dimensional subspace \(H_k\) of \(H^{1/2}(\mathbb{R}^3)\) and a \(\delta > 0\) such that

\[
I_v[\phi] - m\|\phi\|_{L^2}^2 - \int_{\mathbb{R}^3} \left( \frac{1}{|x|} \ast |\phi|^2 \right) |\phi|^2 \leq -\delta < 0, \quad \phi \in H_k, \quad \|\phi\|_{L^2} = 1.
\]

The set of these \(\phi\) yields a sphere \(\tilde{S}^{k-1}\) homeomorphic to \(S^{k-1}\) and which belongs to \(M_N\). This clearly implies that \(l_k < 0\). Now let \(g_r : H_n \to \mathbb{R}^k\) be a continuous and linear function such that \(g_r(S^{k-1}) = \tilde{S}^{k-1}\). Let \(e\) be an embedding of \(\tilde{S}^{k-1}\) onto \(M_N\). Note that \(e \circ g_r \in C_{Z_2}(S^{k-1}, M_N)\) and also note that since the global minimum is finite \(E_v|_{M_N} \geq C\). Therefore \(\{l_k\}_{k=1}^\infty \subset (-\infty, 0]\) and by, if necessary, going to a subsequences, that it is strictly increasing. Since, for all \(k \geq 1\), \(l_k < 0\) there exists a \(Z_2\)-equivariant function \(f_k\) such that

\[
l_k \leq \max_{f_k(S^{k-1})} E_v < \frac{l_k}{2}.
\]

(7.3)

Let \(\{\psi_m\}_{m=1}^\infty\) be a basis of \(H^{1/2}(\mathbb{R}^3)\). Denote by \(W_k\) as the subspace spanned by \(\{\psi_m\}_{m=1}^k\). Define \(V_k\) as the orthogonal complement of \(W_{k-1}\) and assume that \(M_k \cap V_k = \emptyset\). Let \(\pi_k\) be the orthogonal projection from \(H^{1/2}(\mathbb{R}^3)\) onto \(W_k\). Then, bearing in mind that \(V_{k+1} = \text{Ker}(\pi_k) \subset V_k\), we have that \(\pi_{k-1}(M_k) \subset V_{k-1} \setminus \{0\} \cong \mathbb{R}^k \setminus \{0\}\), where \(M_k = f_k(S^{k-1})\), we have existence of a continuous and odd map from \(S^{k-1}\) to \(\mathbb{R}^{k-1} \setminus \{0\}\). From the Borsuk-Ulam theorem we will now get existence of two antipodal points on \(S^{k-1}\) which maps (due to symmetry) to zero and we have thus arrived at a contradiction. Therefore, \(M_k \cap V_k \neq \emptyset\).

For each \(k \geq 1\), we can thus fix some \(\phi_k \in M_k \cap V_k\). Now, \(E_v(\phi_k) \leq \frac{l_k}{2} < 0 = E_v(0)\) and \(\phi_k \to 0\) weakly in \(H^{1/2}(\mathbb{R}^3)\). In view of Lemma 4.3 and Lemma 4.5, \(E_v\) is weakly lower semicontinuous and, consequently, \(0 = E_v(0) \leq \liminf_{k \to \infty} E_v(\phi_k) \leq 0\) which, together with (7.3) implies that \(\lim_{n \to \infty} l_k = 0\) as claimed. Finally, we note that the construction of the levels implies that \(-\infty < l_{k-1} < l_k = E_v(\phi_k) < l_{k+1} < 0\) with \(\phi_k\) being the critical point on level \(l_k\) and, therefore, \(E_v(\phi_k) \to 0\) as claimed. 3. This is established in [7, Theorem 1(ii)].

\(\square\)

Regularity properties of the solutions can be deduced as in [7, Theorem 3].
Acknowledgement. The first author acknowledges financial support from the Dr Perry James Browne Research Centre on Mathematics and its Applications. The second author is grateful to the IMU-Simons African Fellowship Program for supporting his research stay at the University of Sussex.

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