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Finite element error analysis for a system coupling surface evolution to diffusion on the surface

Klaus Deckelnick ‡ Vanessa Styles §

Abstract

We consider a numerical scheme for the approximation of a system that couples the evolution of a two–dimensional hypersurface to a reaction–diffusion equation on the surface. The surfaces are assumed to be graphs and evolve according to forced mean curvature flow. The method uses continuous, piecewise linear finite elements in space and a backward Euler scheme in time. Assuming the existence of a smooth solution we prove optimal error bounds both in $L^\infty(L^2)$ and in $L^2(H^1)$. We present several numerical experiments that confirm our theoretical findings and apply the method in order to simulate diffusion induced grain boundary motion.

Key words. surface PDE, forced mean curvature flow, diffusion induced grain boundary motion, finite elements, error analysis

AMS subject classifications. 65M60, 65M15, 35R01

1 Introduction

In this paper we analyse a finite element scheme for approximating a system which couples diffusion on a surface to an equation that determines the evolution of the surface. More precisely, we want to find a family of surfaces $(\Gamma(t))_{t \in [0,T]} \subset \mathbb{R}^3$ and a function $w : \bigcup_{t \in [0,T]} (\Gamma(t) \times \{t\}) \to \mathbb{R}$ such that

\begin{align}
V &= H + f(w) \quad \text{on } \Gamma(t), \quad t \in (0,T], \quad (1.1a) \\
\partial^* \! w &= \Delta_{\Gamma} w + H V w + g(V,w) \quad \text{on } \Gamma(t), \quad t \in (0,T]. \quad (1.1b)
\end{align}

Here, $V$ and $H$ are the normal velocity and the mean curvature of $\Gamma(t)$ corresponding to the choice $\nu$ of a unit normal, while $\Delta_{\Gamma}$ denotes the Laplace–Beltrami operator on $\Gamma(t)$. Furthermore, $\partial^* \! w = w_t + V \frac{\partial w}{\partial \nu}$ is the material derivative of $w$ and $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R}^2 \to \mathbb{R}$ are given functions. We are interested in surfaces $\Gamma(t)$ which can be represented as the graph of a function $u : \bar{\Omega} \times [0,T] \to \mathbb{R}$, i.e.

\[ \Gamma(t) = \{ (x, u(x,t)) \in \mathbb{R}^3 \mid x \in \bar{\Omega} \} \quad (1.2) \]

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where $\Omega \subset \mathbb{R}^2$ is a bounded domain with a smooth boundary. Thus, $(\Gamma(t))_{t \in [0,T]}$ is a family of surfaces with boundary, which evolves according to forced mean curvature flow in the cylindrical set $A = \overline{\Omega} \times \mathbb{R}$. In what follows we consider the following boundary conditions:

$$\nu \cdot \nu_{\partial A} = 0 \quad \text{on } \partial \Gamma(t), \quad t \in (0,T], \quad (1.3a)$$
$$w = 0 \quad \text{on } \partial \Gamma(t), \quad t \in (0,T]. \quad (1.3b)$$

Here, $\nu_{\partial A}$ is the unit outward normal to $\partial A$, so that we assume that the evolving surfaces meet the boundary of the cylinder at a right angle. Finally, we impose the initial conditions

$$\Gamma(0) = \Gamma^0, \quad w(\cdot,0) = w^0 \text{ on } \Gamma^0, \quad (1.4)$$

where $\Gamma^0 = \{(x,u^0(x)) \mid x \in \Omega\}$ and $u^0 : \Omega \to \mathbb{R}$ as well as $w^0 : \Gamma^0 \to \mathbb{R}$ are given functions.

The system $(1.1a)$, $(1.1b)$ occurs e.g. in the modeling of diffusion induced grain boundary motion, see [8], [5] and Section 5.3. Further examples of systems that arise by coupling a geometric evolution equation to a PDE on the evolving surface can be found in [7, Section 10].

A semi-discrete finite element scheme for the approximation of $(1.1a)$, $(1.1b)$ in the case that $\Gamma(t)$ is a closed curve has first been analysed by Pozzi and Stinner in [12]. Using a tangentially modified parametrisation of the evolving curves, [1] obtains error bounds for a corresponding fully discrete scheme. In [13] this idea is applied to the case of open curves $\Gamma(t)$ meeting a given boundary orthogonally. In each of these papers the error bounds are optimal in $H^1$.

A first error analysis involving the evolution of two-dimensional closed (i.e. compact without boundary) surfaces was obtained in [9] for a regularized version of $(1.1a)$. Extending ideas used in the error analysis for pure mean curvature flow in [10], Kovács, Li and Lubich obtain in [11] a convergence proof for the system $(1.1a)$, $(1.1b)$ in the case of closed surfaces. The scheme uses polynomials of degree at least two and is based on a system coupling the variable $w$ in $(1.1b)$ with the velocity, the normal and the mean curvature of $\Gamma(t)$. The error estimates are optimal in $H^1$, while the restriction on the polynomial degree is essentially used to guarantee, via inverse estimates, that the discrete surfaces are non-degenerate.

The purpose of our paper is to derive and analyse a simple, fully discrete finite element scheme for the system $(1.1a)$, $(1.1b)$ when the evolving surfaces are of the form $(1.2)$. In order to translate $(1.1a)$, $(1.1b)$ into problems which are posed on $\overline{\Omega} \times [0,T]$ we introduce

$$Q(u) = \sqrt{1 + |\nabla u|^2}.$$

Then, the upward pointing unit normal $\nu(u)$, the normal velocity $V$ and the mean curvature $H$ of $\Gamma(t)$ are given by

$$\nu(u) = \frac{1}{Q(u)}(-\nabla u,1), \quad V = \frac{u_t}{Q(u)} \quad \text{and} \quad H = \nabla \cdot \left( \frac{\nabla u}{Q(u)} \right) \quad (1.5)$$

respectively. Furthermore, if we denote by $n$ the outward unit normal to $\partial \Omega$, then $\nu_{\partial A} = (n,0)$ and hence $\nu(u) \cdot \nu_{\partial A} = -\frac{\nabla u \cdot n}{Q(u)}$. If we let $\tilde{w} : \Omega \times [0,T] \to \mathbb{R}$, $\tilde{w}(x,t) := w(x,u(x,t),t)$ then we may write $(1.1a)$, $(1.3a)$ as

$$\frac{u_t}{Q(u)} - \nabla \cdot \left( \frac{\nabla u}{Q(u)} \right) + f(\tilde{w}) = 0 \quad \text{in } \Omega \times (0,T]; \quad (1.6)$$
$$\frac{\nabla u \cdot n}{Q(u)} = 0 \quad \text{on } \partial \Omega \times (0,T]. \quad (1.7)$$
Let us next rewrite (1.1b) in terms of \( \tilde{w} \). To do so, we make use of the formulae (2.1) and (2.2) in [7], which yield (temporarily suppressing the dependence on \( t \))

\[
(\nabla_1 w)(\Phi(x)) = \sum_{i,j=1}^{2} g^{ij}(x) \tilde{w}_{x_j}(x) \Phi_{x_i}(x),
\]

(1.8)

\[
(\Delta w)(\Phi(x)) = \frac{1}{\sqrt{q(x)}} \sum_{i,j=1}^{2} \frac{\partial}{\partial x_j} \left( g^{ij}(x) \sqrt{q(x)} \tilde{w}_{x_i}(x) \right).
\]

(1.9)

In the above, \( \Phi(x) = (x, u(x)) \) and \( (g^{ij}) \) is the inverse matrix of \( (g_{ij}) \), where \( g_{ij} = \Phi_{x_i} \cdot \Phi_{x_j} = \delta_{ij} + u_{x_i} u_{x_j}, \ i, j = 1, 2 \). Furthermore, \( q = \det(g_{ij}) = 1 + |\nabla u|^2 = Q(u)^2 \). A simple calculation shows that

\[
(g^{ij}) = I - \frac{\nabla u \otimes \nabla u}{Q(u)^2}.
\]

We can expand the velocity vector \((0, u_t)\) for the evolving family of graphs in terms of \( \Phi_{x_1}, \Phi_{x_2} \) and \( \nu(u) \) as follows

\[
(0, u_t) = V \nu(u) + \sum_{k=1}^{2} \frac{u_t u_{x_k}}{Q(u)^2} \Phi_{x_k}.
\]

Combining this relation with (1.8) we find

\[
\begin{align*}
\tilde{w}_t &= w_t + \nabla w \cdot (0, u_t) = w_t + V \frac{\partial w}{\partial \nu} + \nabla_1 w \cdot \sum_{k=1}^{2} \frac{u_t u_{x_k}}{Q(u)^2} \Phi_{x_k} \\
&= \partial^* w + \sum_{i,j,k=1}^{2} g^{ij} \tilde{w}_{x_j} \frac{u_t u_{x_k}}{Q(u)^2} \Phi_{x_i} \cdot \Phi_{x_k} = \partial^* w + \frac{u_t}{Q(u)^2} \nabla \tilde{w} \cdot \nabla u.
\end{align*}
\]

Recalling (1.5) we deduce that

\[
\partial^* w - H V w = \tilde{w}_t - \frac{u_t}{Q(u)} \frac{\nabla u}{Q(u)} \cdot \nabla \tilde{w} - \frac{u_t}{Q(u)} \nabla \cdot \left( \frac{\nabla u}{Q(u)} \right) \tilde{w} = \tilde{w}_t - \frac{u_t}{Q(u)} \nabla \cdot \left( \tilde{w} \frac{\nabla u}{Q(u)} \right).
\]

Hence, (1.1b), (1.3b) take the form

\[
\begin{align*}
\tilde{w}_t - \frac{1}{Q(u)} \sum_{i,j=1}^{2} \left( g^{ij} Q(u) \tilde{w}_{x_i} \right) &= \frac{u_t}{Q(u)} \nabla \cdot \left( \tilde{w} \frac{\nabla u}{Q(u)} \right) + g(V, \tilde{w}) \quad \text{in } \Omega \times (0, T); \quad (1.10) \\
\tilde{w} &= 0 \quad \text{on } \partial \Omega \times (0, T). \quad (1.11)
\end{align*}
\]

For ease of notation we will from now on write again \( w \) instead of \( \tilde{w} \). Our discretisation will be based on a weak formulation of the system (1.6), (1.10) and uses continuous, piecewise linear finite elements in space and a backward Euler scheme in time, see Section 2. A crucial point in the error analysis is the uniform control of the gradient of the discrete height function. This control is achieved with the help of a superconvergence estimate between the discrete height and a nonlinear projection previously employed in [3] for the numerical analysis of the mean curvature flow of graphs. The properties of this projection and a suitable projection for the function \( w \) are collected in Section 3. As our main results we obtain an \( O(\tau + h) \)–error bound in \( H^1 \) and an \( O(\tau + h^2 |\log h|^2) \)–estimate in \( L^2 \) both for \( u \) and \( w \), provided that the time step \( \tau \) is appropriately related to the mesh size \( h \). To the best of our knowledge, a quasi-optimal
Let us finish the introduction with a few comments on our notation. We shall denote the norm of the Sobolev space $W^{m,p}(\Omega)$ ($m \in \mathbb{N}_0, 1 \leq p \leq \infty$) by $\| \cdot \|_{m,p}$. For $p = 2$, $W^{m,2}(\Omega)$ will be denoted by $H^m(\Omega)$ with norm $\| \cdot \|_m$, where we simply write $\| \cdot \| = \| \cdot \|_0$.

## 2 Weak formulation and finite element approximation

In what follows we make the following assumptions on the data and the solution $(u, w)$:

(A1) $f \in C^0_{\text{loc}}(\mathbb{R})$ and $g : \mathbb{R}^2 \to \mathbb{R}$ has the form

$$g(r, s) = \alpha(r) \beta(s) + \tilde{\beta}(s),$$

where $\beta, \tilde{\beta} \in C^0_{\text{loc}}(\mathbb{R})$ and $\alpha(r) = \begin{cases} \alpha_1 \vert r \vert, & r \geq 0, \\ \alpha_2 \vert r \vert, & r < 0 \end{cases}$ for some $\alpha_1, \alpha_2 \in \mathbb{R}$.

(A2) $(u, w)$ solves (1.6), (1.7), (1.10), (1.11) and satisfies

$$u \in L^\infty((0, T); H^4(\Omega)) \cap L^2((0, T); H^5(\Omega)), \quad u_t \in L^\infty((0, T); H^2(\Omega)) \cap L^2((0, T); H^3(\Omega))$$

$$\nabla u_t \in L^\infty(\Omega \times (0, T)), \quad u_{tt} \in L^\infty((0, T); H^1(\Omega));$$

$$w \in C^0([0, T]; W^{2,\infty}(\Omega)), \quad w_t \in C^0([0, T]; W^{1,\infty}(\Omega) \cap H^2(\Omega)), \quad w_{tt} \in L^\infty((0, T); L^2(\Omega)).$$

Multiplying (1.6) by $\varphi \in H^1(\Omega)$ and integrating by parts yields the weak formulation

$$\int_\Omega \frac{u_t}{Q(u)} \varphi \, dx + \int_\Omega \nabla u \cdot \nabla \varphi \, dx = \int_\Omega f(w) \varphi \, dx \quad \forall \varphi \in H^1(\Omega).$$

In order to derive a weak formulation for (1.10) we proceed as in [7, Section 5] and calculate for a test function $\eta \in H^1_0(\Omega)$

$$\frac{d}{dt} \int_\Omega w \eta Q(u) dx = \int_\Omega w_t \eta Q(u) dx + \int_\Omega w \eta [Q(u)]_t dx = \sum_{i,j=1}^2 \int_\Omega \left( g^{ij} w_{x_i} Q(u) \right)_{x_j} \eta dx$$

$$+ \int_\Omega w_t \nabla \cdot \left( w \frac{\nabla u}{Q(u)} \right) \eta dx + \int_\Omega w \eta \nabla u \cdot \nabla u_t \frac{Q(u)}{Q(u)} dx + \int_\Omega g(V, w) \eta Q(u) dx$$

$$= - \sum_{i,j=1}^2 \int_\Omega g^{ij} w_{x_i} \eta_{x_j} Q(u) dx - \int_\Omega u_t \frac{\nabla u \cdot \nabla \eta}{Q(u)} w dx + \int_\Omega g(V, w) \eta Q(u) dx$$

$$= - \int_\Omega E(\nabla u) \nabla w \cdot \nabla \eta dx - \int_\Omega \nabla u \cdot \nabla \eta V w dx + \int_\Omega g(V, w) \eta Q(u) dx,$$

where $V$ is given by (1.5) and

$$E(p) = \sqrt{1 + |p|^2} \left( I - \frac{p \otimes p}{1 + |p|^2} \right), \quad p \in \mathbb{R}^2.$$
Note that for all \( p, \xi \in \mathbb{R}^2 \)

\[
E(p)\xi \cdot \xi = \sqrt{1 + |p|^2}(|\xi|^2 - \frac{(\xi \cdot p)^2}{1 + |p|^2}) \geq \sqrt{1 + |p|^2} |\xi|^2(1 - \frac{|p|^2}{1 + |p|^2}) = \frac{|\xi|^2}{\sqrt{1 + |p|^2}}.
\] (2.8)

Next, let \((T_h)_{0 < h \leq h_0}\) be a family of triangulations of \(\Omega\), where we allow boundary elements to have one curved face in order to avoid the analysis of domain approximation. We denote by \(h := \max_{S \in T_h} \text{diam}(S)\) the maximum mesh size and assume that the triangulation is quasiuniform in the sense that there exists \(\kappa > 0\) which is independent of \(h\), such that each \(S \in T_h\) is contained in a ball of radius \(\kappa^{-1}h\) and contains a ball of radius \(\kappa h\). Our finite element spaces are given by

\[
X_h = \{ \varphi_h \in C^0(\bar{\Omega}) \mid \varphi_h \text{ is a linear polynomial on each } S \in T_h \}, \quad X_{h0} = X_h \cap H^1_0(\Omega),
\]

where we note that in the curved elements \(\varphi_h\) is a composition of a linear polynomial with a suitably defined nonlinear mapping from \(S\) to the unit triangle. We refer to [14] for a detailed construction of \(X_h\). The following well–known estimates will be useful:

\[
\|\nabla \varphi_h\| \leq c h^{-1} \|\varphi_h\| \quad \forall \varphi_h \in X_h; \quad (2.9)
\]
\[
\|\nabla \varphi_h\|_{0,\infty} \leq c h^{-1} \|\nabla \varphi_h\| \quad \forall \varphi_h \in X_h; \quad (2.10)
\]
\[
\|\varphi_h\|_{0,\infty} \leq c \|\log h\| \|\varphi_h\|_1 \quad \forall \varphi_h \in X_h. \quad (2.11)
\]

Finally, let \(\tau > 0\) be a time step and \(t_m = m\tau, \; m = 0, \ldots, M\), where \(M = \frac{T}{\tau}\). In what follows, an upper index \(m\) will refer to the time level \(m\).

Our discretisation reads: Given \(u^m_h \in X_h, w^m_h \in X_{h0}\), first find \(u^{m+1}_h \in X_h\) such that

\[
\frac{1}{\tau} \int_{\Omega} \left( \frac{(u^{m+1}_h - u^m_h)}{Q(u^m_h)} \varphi_h \right) dx + \int_{\Omega} \frac{\nabla u^{m+1}_h \cdot \nabla \varphi_h}{Q(u^m_h)} dx = \int_{\Omega} f(w^m_h) \varphi_h dx
\] (2.12)

for all \(\varphi_h \in X_h\). Afterwards, find \(w^{m+1}_h \in X_{h0}\) such that

\[
\frac{1}{\tau} \left( \int_{\Omega} w^{m+1}_h \eta_h Q(u^m_h) dx - \int_{\Omega} w^m_h \eta_h Q(u^m_h) dx \right) + \int_{\Omega} E(\nabla u^{m+1}_h) \nabla w^{m+1}_h \eta_h dx = -\int_{\Omega} \nabla u^{m+1}_h \cdot \nabla \eta_h V^{m+1}_h w^m_h dx + \int_{\Omega} g(V^{m+1}_h, w^m_h) \eta_h Q(u^m_h) dx
\] (2.13)

for all \(\eta_h \in X_{h0}\). Here, \(V^{m+1}_h = \frac{u^{m+1}_h - u^m_h}{\tau Q(u^m_h)}\). We note that each time step requires the consecutive solution of two linear systems. In view of (2.8) it is easily seen that \(u^{m+1}_h \in X_h\) and \(w^{m+1}_h \in X_{h0}\) exist and are uniquely determined. The algorithm is initialised by \(u^0_h = \widehat{u}_0^0, \; w^0_h = \widehat{w}_0^0\), given by (3.1) and (3.6) defined in the next section. Our main result reads as follows:

**Theorem 2.1.** There exist \(h_0 > 0, \; \delta_0 > 0\) such that for all \(0 < h \leq h_0\) and all \(\tau > 0\) satisfying \(\tau \leq \delta_0 h \log h^{-\frac{1}{2}}\) the following error bounds hold:

\[
\max_{0 \leq m \leq M} \left[ \|u^m_h - u^m_h\|^2 + \|w^m - w^m_h\|^2 \right] + \sum_{m=0}^{M-1} \tau \|u^m_h - u^{m+1}_h\|^2 \leq c(\tau^2 + h^4 \log h^4),
\]

\[
\max_{0 \leq m \leq M} \|\nabla(u^m_h - u^m_h)\|^2 + \sum_{m=0}^{M} \tau \|\nabla(w^m - w^m_h)\|^2 \leq c(\tau^2 + h^2).
\]
3 Projections

Our error analysis relies on the use of suitable Ritz projections of the solutions $u$ and $w$. Omitting the time dependence for a moment we define for a given function $u \in H^1(\Omega)$ the minimal surface type projection $\hat{u}_h \in X_h$ by

$$
\int_{\Omega} \frac{\nabla \hat{u}_h \cdot \nabla \varphi_h}{Q(\hat{u}_h)} \, dx + \int_{\Omega} \hat{u}_h \varphi_h \, dx = \int_{\Omega} \frac{\nabla u \cdot \nabla \varphi_h}{Q(u)} \, dx + \int_{\Omega} u \varphi_h \, dx \quad \forall \varphi_h \in X_h. \tag{3.1}
$$

Note that we have added the zero order term in order to ensure the $H^1(\Omega)$–coercivity of the problem. For functions that also depend on $t$ we have the following error bounds.

**Lemma.** Assume that $u$ satisfies (2.2) and (2.3). Then

$$
\sup_{0 \leq t \leq T} \| (u - \hat{u}_h)(t) \| + h \sup_{0 \leq t \leq T} \| \nabla (u - \hat{u}_h)(t) \| \leq c h^2, \tag{3.2}
$$

$$
\sup_{0 \leq t \leq T} \| (u - \hat{u}_h)(t) \|_{0, \infty} + h \sup_{0 \leq t \leq T} \| \nabla (u - \hat{u}_h)(t) \|_{0, \infty} \leq c h^2 | \log h|, \tag{3.3}
$$

$$
\sup_{0 \leq t \leq T} \| (u_t - \hat{u}_{h,t})(t) \| \leq c h^2 | \log h|^2, \tag{3.4}
$$

$$
\sup_{0 \leq t \leq T} \| \nabla (u_t - \hat{u}_{h,t})(t) \| \leq c h. \tag{3.5}
$$

**Proof.** The proofs of (3.2) and (3.3) follow from [6] (see p. 160) using that $u(\cdot, t) \in H^4(\Omega) \subset W^{2, \infty}(\Omega)$ for every $t \in [0, T]$. The arguments required to show (3.4) and (3.5) can be found in [2, Section 4] for the case of homogeneous Dirichlet boundary conditions. In order to prove (3.5) for (3.1) one proceeds in the same way as in [2], p. 202 to obtain

$$
\| \nabla (u_t - \hat{u}_{h,t}) \|^2 \leq c h \| \nabla (u_t - \hat{u}_{h,t}) \| (\| \nabla u_t \|_{0, \infty} + \| u_t \|_2) + c h^2 \| \nabla u_t \|_{0, \infty} \| u_t \|_2,
$$

which yields (3.5) taking into account (2.2) and (2.3). The bound (3.4) can be shown for the Neumann case by modifying the dual problem on top of p. 203 in [2] as follows:

$$
- \nabla \cdot (F'(\nabla u) \nabla u) + v = u_t - \hat{u}_{h,t} \quad \text{in } \Omega, \quad F'(\nabla u) \nabla v \cdot n = 0 \quad \text{on } \partial \Omega,
$$

where $F(p) = p/\sqrt{1 + |p|^2}$, $p \in \mathbb{R}^2$. □

Let us next use $\hat{u}_h$ in order to define a projection $\hat{w}_h \in X_{h0}$ of $w$ as follows:

$$
\int_{\Omega} E(\nabla \hat{w}_h) \nabla \hat{w}_h \cdot \nabla \eta_h \, dx = \int_{\Omega} E(\nabla u) \nabla w \cdot \nabla \eta_h \, dx \quad \forall \eta_h \in X_{h0}. \tag{3.6}
$$

**Lemma.** Assume that $w$ satisfies (2.4). Then

$$
\sup_{0 \leq t \leq T} \| \nabla (w - \hat{w}_h)(t) \| \leq c h, \tag{3.7}
$$

$$
\sup_{0 \leq t \leq T} \| (w - \hat{w}_h)(t) \| \leq c h^2 | \log h|, \tag{3.8}
$$

$$
\sup_{0 \leq t \leq T} \| \nabla (w_t - \hat{w}_{h,t})(t) \| \leq c h, \tag{3.9}
$$

$$
\sup_{0 \leq t \leq T} \| (w_t - \hat{w}_{h,t})(t) \| \leq c h^2 | \log h|^2. \tag{3.10}
$$

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Proof. Using (3.2)–(3.5), these bounds have been obtained in [4, Appendix] for a slightly more complicated projection, see (2.22) in that paper. The same arguments can be applied to our case where we note that the matrix valued function $E(p)$ used in [4] differs from (2.7) by a factor of $1 + |p|^2$. However, since $\nabla u$ and $\nabla \hat{u}_h$ vary in a bounded set that is independent of $h$, the analysis in [4] also applies to (3.6). \(\square\)

We set
\[
\rho_u = u - \hat{u}_h
\]
and for later use we record the following estimates, which will be helpful in retaining the optimality of the error bounds:

**Lemma.** 3.3. Suppose that $F : \mathbb{R}^2 \to \mathbb{R}$ is twice continuously differentiable and that $u \in W^{1,\infty}(\Omega)$. Then we have for $f \in W^{1,1}_0(\Omega)$
\[
|\int_{\Omega} (F(\nabla u) - F(\nabla \hat{u}_h)) f \, dx| \leq ch^2 |\log h| \|f\|_{1,1}.
\]

**Proof.** Noting (3.11) we have
\[
\int_{\Omega} (F(\nabla u) - F(\nabla \hat{u}_h)) f \, dx = \int_{\Omega} F'(\nabla u) \cdot \nabla \rho_u f \, dx + R,
\]
where
\[
|R| = \int_{\Omega} \int_0^1 (F'(\nabla u - s\nabla \rho_u) - F'(\nabla u))ds \cdot \nabla \rho_u f \, dx \leq c\|\nabla \rho_u\|_{0,\infty} \|\nabla \rho_u\| \|f\| \leq ch^2 |\log h| \|f\|_{1,1}
\]
in view of (3.2) and (3.3) and the embedding $W^{1,1}(\Omega) \hookrightarrow L^2(\Omega)$. Integration by parts together with (3.3) yields
\[
|\int_{\Omega} F'(\nabla u) \cdot \nabla \rho_u f \, dx| = | - \int_{\Omega} \nabla \cdot (F'(\nabla u) f) \rho_u \, dx| \leq c\|\rho_u\|_{0,\infty} \|f\|_{1,1} \leq ch^2 |\log h| \|f\|_{1,1}
\]
and the result follows. \(\square\)

**Lemma.** 3.4. Suppose that $f \in H^1_0(\Omega) \cap C^0(\overline{\Omega})$ with $f \in H^2(T)$ for all $T \in T_h$. Then
\[
|\int_{\Omega} f\left(\frac{\nabla u}{Q(u)} - \frac{\nabla \hat{u}_h}{Q(\hat{u}_h)}\right) \cdot \nabla \varphi_h \, dx| \leq c h |\log h| \|\varphi_h\| \left(\sum_{T \in T_h} \|f\|_{H^2(T)}^2\right)^{\frac{1}{2}} \quad \forall \varphi_h \in X_h.
\]

If in addition, $f \in H^2(\Omega)$, then
\[
|\int_{\Omega} f\left(\frac{\nabla u}{Q(u)} - \frac{\nabla \hat{u}_h}{Q(\hat{u}_h)}\right) \cdot \nabla \varphi_h \, dx| \leq c h^2 |\log h| \|\varphi_h\|_1 \|f\|_2 \quad \forall \varphi_h \in X_h.
\]

**Proof.** In view of the definition (3.1) of $\hat{u}_h$ we obtain
\[
\int_{\Omega} f\left(\frac{\nabla u}{Q(u)} - \frac{\nabla \hat{u}_h}{Q(\hat{u}_h)}\right) \cdot \nabla \varphi_h \, dx
\]
\[
= \int_{\Omega} \left(\frac{\nabla u}{Q(u)} - \frac{\nabla \hat{u}_h}{Q(\hat{u}_h)}\right) \cdot \nabla (f \varphi_h) \, dx - \int_{\Omega} \varphi_h \left(\frac{\nabla u}{Q(u)} - \frac{\nabla \hat{u}_h}{Q(\hat{u}_h)}\right) \cdot \nabla f \, dx.
\]
Here, \( I_h \) denotes the Lagrange interpolation operator. An interpolation estimate implies
\[
|I| \leq \|\nabla\rho_u\|_{0,\infty}\|\nabla(f\varphi_h - I_h(f\varphi_h))\|_{0,1} \leq ch^2|\log h|\sum_{T \in T_h} \|D^2(f\varphi_h)\|_{L^1(T)}
\]
\[
\leq ch^2|\log h|\|\varphi_h\|_1\left(\sum_{T \in T_h} \|f\|_{H^2(T)}^2\right)^{1/2}.
\]
Next,
\[
|II| \leq \|\rho_u\|_{0,\infty}\|I_h(f\varphi_h)\|_{0,1} \leq ch^2|\log h|(\|f\varphi_h\|_{0,1} + \|f\varphi_h - I_h(f\varphi_h)\|_{0,1})
\]
\[
\leq ch^2|\log h|\|\varphi_h\|_1\left(\sum_{T \in T_h} \|f\|_{H^2(T)}^2\right)^{1/2}.
\]
Finally,
\[
|III| \leq \|\nabla\rho_u\|_{0,\infty}\|\varphi_h\|\|\nabla f\|_1 \leq ch|\log h|\|\varphi_h\|\|f\|_1,
\]
while applying Lemma 3.3 with \( F_i(p) = \frac{p}{\sqrt{1+|p|^2}} \), yields in the case that \( f \in H^2(\Omega) \)
\[
|III| \leq \sum_{i=1}^2 \int_\Omega (F_i(\nabla u) - F_i(\nabla \tilde{u})) \varphi_h f_x, dx \leq ch^2|\log h|\|\varphi_h\nabla f\|_{1,1} \leq ch^2|\log h|\|\varphi_h\|_1\|f\|_2.
\]
The result now follows from the above bounds together with (2.9). \( \square \)

4 Error Analysis

Let us begin with two useful estimates involving the quantities \( Q \) and \( \nu \).

**Lemma 4.1.** Let \( u, v \in W^{1,\infty}(\Omega) \). Then we have a.e. in \( \Omega \):
\[
|\nabla(v - u)| \leq \left(1 + \sup_{\Omega} |\nabla v|\right)Q(u)|\nu(v) - \nu(u)|;
\]
\[
Q(v) - Q(u) = \frac{\nabla u}{Q(u)} \cdot \nabla(v - u) + \frac{|\nabla(v - u)|^2}{2Q(u)} - \frac{(Q(v) - Q(u))^2}{2Q(u)}.
\]

**Proof.** The estimate (4.1) is a consequence of the relation
\[
\nabla v - \nabla u = Q(u)\left(\frac{\nabla v}{Q(v)} - \frac{\nabla u}{Q(u)}\right) + Q(u)\left(\frac{1}{Q(u)} - \frac{1}{Q(v)}\right)\nabla v
\]
and the fact that \( \nu(u) = \left(\frac{-\nabla u}{Q(u)}; \frac{1}{Q(u)}\right) \), while (4.2) follows from a straightforward calculation. \( \square \)

Let us decompose the errors \( e_u^m = u^m - u_h^m, e_v^m = w^m - w_h^m \) as follows:
\[
e_u^m = (u^m - \tilde{u}_h^m) + (\tilde{u}_h^m - u_h^m) =: \rho_u^m + e_{u,h}^m,
\]

\[
e_v^m = (w^m - \tilde{w}_h^m) + (\tilde{w}_h^m - w_h^m) =: \rho_v^m + e_{v,h}^m.
\]
Thus, it follows from (4.1) that

\[ e^m_w = (w^m - \tilde{w}^m_h) + (\tilde{w}^m_h - u^m_h) =: \rho^m_w + e^m_{h,w} \]  

(4.4)

and note that \( e^m_{h,u} \in X_h, e^m_{h,w} \in X_{h,0} \). It will be convenient to introduce the quantities

\[ A^m := \int_\Omega |\nu(u^m_h) - \nu(\tilde{u}^m_h)|^2 Q(u^m_h) \, dx, \]

(4.5)

\[ B^m := \frac{1}{2} A^m - \int_\Omega \mathbf{d}^m : \nabla e^m_{h,u} \rho^m_w \, dx, \]

(4.6)

where

\[ \mathbf{d}^m = - \frac{u^m_i \nabla u^m}{\sqrt{1 + |\nabla u^m|^2}}. \]

(4.7)

We shall use an induction argument and claim that

\[ B^m + \frac{\theta^2}{2} \int_\Omega (e^m_{h,w})^2 Q(u^m_h) \, dx \leq (\tau^2 + h^4 |\log h|^4) e^{\mu T}, \quad m = 0, 1, \ldots, M \]

(4.8)

provided that \( \tau \leq \delta_0 h |\log h|^{-\frac{1}{4}} \). The constants \( \delta_0, 0 < \theta \leq 1 \) and \( \mu > 0 \) are independent of \( h \) and \( \tau \) and will be chosen a posteriori. To begin, choose \( h_0 > 0 \) so small that

\[ h^2 |\log h|^5 e^{\mu T} \leq \frac{1}{2} \text{ and } |\log h| \geq \frac{1}{\theta^2} \quad \text{for all } 0 < h \leq h_0. \]

(4.9)

Clearly, (4.8) holds for \( m = 0 \) since \( e^0_{h,u} = e^0_{h,w} = 0 \) by the choice of our initial data for the scheme. Let us assume that it is true for some \( m \in \{0, \ldots, M-1\} \). Then we have for \( 0 < h \leq h_0 \) that

\[ B^m + \frac{\theta^2}{2} \int_\Omega (e^m_{h,w})^2 Q(u^m_h) \, dx \leq (\delta_0^2 h^2 |\log h|^{-1} + h^4 |\log h|^4) e^{\mu T} \leq h^2 |\log h|^{-1}, \]

(4.10)

provided that \( \delta_0 \) and \( \mu \) satisfy

\[ \delta_0^2 e^{\mu T} \leq \frac{1}{2}. \]

(4.11)

In what follows we shall denote by \( c \) a generic constant that is independent of \( \delta_0, \theta \) and \( \mu \). We infer from an inverse estimate, (4.10), the fact that \( Q(u^m_h) \geq 1 \) and (4.9) that

\[ \|w^m_h\|_{0,\infty} \leq \|\tilde{w}^m_h\|_{0,\infty} + \|e^m_{h,w}\|_{0,\infty} \leq c + ch^{-1}\|e^m_{h,w}\| \leq c + \frac{c}{\theta} |\log h|^{-\frac{1}{4}} \leq c. \]

(4.12)

Next, we deduce with the help of \( \|\nabla \tilde{u}^m_h\|_{0,\infty} \leq c \) and (2.10) that

\[ \sup_{\Omega} Q(u^m_h) \leq 1 + \sup_{\Omega} |\nabla u^m_h| \leq 1 + \|\nabla \tilde{u}^m_h\|_{0,\infty} + \|\nabla e^m_{h,u}\|_{0,\infty} \leq c + ch^{-1}\|\nabla e^m_{h,u}\|. \]

(4.13)

It follows from (4.1) that

\[ |\nabla e^m_{h,u}| = |\nabla (u^m_h - \tilde{u}^m_h)| \leq (1 + \sup_{\Omega} |\nabla \tilde{u}^m_h|)Q(u^m_h)|\nu(u^m_h) - \nu(\tilde{u}^m_h)| \leq c |\nu(u^m_h) - \nu(\tilde{u}^m_h)|Q(u^m_h). \]

Thus,

\[ \|\nabla e^m_{h,u}\|^2 \leq c \int_{\Omega} |\nu(u^m_h) - \nu(\tilde{u}^m_h)|^2 Q(u^m_h)^2 \, dx \leq c \sup_{\Omega} Q(u^m_h) A^m \]
Evaluating (2.5) at $4.1$ The graph equation

If we insert

and hence

Combining (4.18) with (2.12) we obtain the error relation

Here, $\varphi_t \leq h_1$ provided that $0 < h \leq h_1$ for some sufficiently small $0 < h_1 \leq h_0$. Furthermore, we infer from (4.10), (4.14) and (4.15) that

and therefore

provided that $0 < h \leq h_1$ for some sufficiently small $0 < h_1 \leq h_0$. Furthermore, we infer from (4.10), (4.14) and (4.15) that

The graph equation

Evaluating (2.5) at $t = t_m$ and using the definition (3.1) of $u_h$ we derive for $\varphi \in X_h$

and hence

and using the definition (3.1) of $u_h$ we derive for $\varphi \in X_h$

and hence

Here, $R^m = \frac{1}{\Omega} \frac{(u^{m+1} - u^m)}{\tau} + \rho^m_u$, so that in view of (3.2)

Combining (4.18) with (2.12) we obtain the error relation

If we insert $\varphi = \frac{1}{\tau} (e_{h,u}^{m+1} - e_{h,u}^m)$ into (4.20) we derive

\[
\frac{1}{\tau^2} \int \frac{(e_u^{m+1} - e_u^m)^2}{Q(u_h^m)} dx + \frac{1}{\tau} \int \left( \frac{\nabla \hat{u}_h^{m+1}}{Q(u_h^m)} - \frac{\nabla u_h^{m+1}}{Q(u_h^m)} \right) \cdot \nabla \varphi_h dx + \int \frac{\nabla (e_{h,u}^{m+1} - e_{h,u}^m)}{Q(u_h^m)} \cdot \nabla (e_{h,u}^{m+1} - e_{h,u}^m) dx
\]
In order to proceed we make use of the analysis in [3] for the mean curvature flow of graphs subject to Dirichlet boundary conditions. The relation (4.21) corresponds to [3, (3.12)] where we use \( e_u^m, e_{h,u}^m, \rho_u^m \) instead of \( e^m, e_h^m, e^m \) respectively. Furthermore, our remainder term \( R^m \) is defined in a different way and the term \( A_5 \) is not present in [3]. We shall refer to the calculations in [3] whenever possible and focus on the changes due to the differences mentioned above and the use of a Neumann boundary condition. To begin, it follows from Lemma 2 in [3] that

\[
\frac{1}{\tau^2} \int_\Omega \left( \frac{\nabla \tilde{u}_{h}^{m+1}}{Q(u_{h}^m)} - \frac{\nabla u_{h}^{m+1}}{Q(u_{h}^m)} \right) \cdot \nabla (e_{h,u}^{m+1} - e_{h,u}^m) \, dx + \frac{1}{4\tau} \int_\Omega \left| \nabla (e_{h,u}^{m+1} - e_{h,u}^m) \right|^2 \, dx \geq \frac{1}{2\tau} (A^{m+1} - A^m) + c(A^m + A^{m+1}) - c\tau^2. \tag{4.22}
\]

The lemma holds under the condition that \( h^{-2} A^m \leq \gamma \) and \( \gamma > 0 \) is sufficiently small, which can be achieved in view of (4.17) if \( 0 < h \leq h_2 \) and \( h_2 \leq h_1 \) is small enough.

Let us consider the terms on the right hand side of (4.21). The term \( S_1 \) is estimated in (i) at the bottom of page 352 in [3], so that

\[
|A_1| \leq \frac{\delta}{\tau^2} \int_\Omega \frac{(e_{u}^{m+1} - e_{u}^m)^2}{Q(u_{h}^m)} \, dx + \frac{c}{\delta} h^4 |\log h|^4. \tag{4.23}
\]

The integral \( A_2 \) is treated in (ii) on page 353 in [3] and uses integration by parts for the term

\[
\int_\Omega \frac{\nabla (u_{h}^{m+1} - u_{h}^m) \cdot \nabla (e_{h,u}^{m+1} - e_{h,u}^m)}{Q(u_{h}^m)} \, dx.
\]

Since \( \nabla (u_{h}^{m+1} - u_{h}^m) \cdot n = 0 \) on \( \partial\Omega \) in view of (1.7) the boundary integral vanishes and we obtain in the same way as in [3]

\[
|A_2| \leq \frac{\delta}{\tau} \int_\Omega \frac{\left| \nabla (e_{h,u}^{m+1} - e_{h,u}^m) \right|^2}{Q(u_{h}^m)} \, dx + \frac{\delta}{\tau^2} \int_\Omega \frac{(e_{u}^{m+1} - e_{u}^m)^2}{Q(u_{h}^m)} \, dx + \frac{c}{\delta}(\tau^2 + h^4 |\log h|^4). \tag{4.24}
\]

The term \( A_3 \) is handled in (iii) on pages 353 to 356 in [3]. It again involves an integration by parts, namely for the term

\[
-\frac{1}{\tau^2} \int_\Omega (u_{h}^{m+1} - u_{h}^m)(e_{h,u}^{m+1} - e_{h,u}^m)b^m \cdot \nabla \rho_u^m \, dx,
\]

which is \( II \) at the top of page 354. Here \( b^m = B(\nabla u^m) \) with \( B_i(p) = \frac{\partial}{\partial p_i} \left( \frac{1}{\sqrt{1 + |p|^2}} \right) = -\frac{p_i}{\sqrt{1 + |p|^2}}. \)

As a result, the boundary integral reads

\[
-\frac{1}{\tau^2} \int_{\partial\Omega} (u_{h}^{m+1} - u_{h}^m)(e_{h,u}^{m+1} - e_{h,u}^m)b^m \cdot n \rho_u^m \, do = 0,
\]

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since $b^n \cdot n = -\frac{\nabla u^n \cdot n}{\sqrt{1 + |\nabla u^n|^2}} = 0$ on $\partial \Omega$ again by (1.7). Thus we obtain from [3] (see top of page 356) that

$$|A_3| \leq \frac{1}{\tau} \int_{\Omega} d^{m+1} \cdot \nabla e_{h,u}^{m+1} \rho_u^{m+1} dx - \frac{1}{\tau} \int_{\Omega} d^m \cdot \nabla e_{h,u}^m \rho_u^m dx$$

$$+ \frac{6\delta}{\tau^2} \int_{\Omega} \frac{(e_{h,u}^{m+1} - e_{h,u}^m)^2}{Q(\hat{u}_h^m)} dx$$

$$+ \frac{2\delta}{\tau} \int_{\Omega} \frac{\|\nabla (e_{h,u}^{m+1} - e_{h,u}^m)\|^2}{Q(\hat{u}_h^m)} dx + \frac{c}{\delta} h^4 |\log h|^4 + \frac{c}{\delta} (A^m + A^{m+1})$$

(4.25)

with $d^m$ as in (4.7) (see top of page 355). Next, (4.19) implies that

$$|A_4| \leq \frac{1}{\tau} \sup_{t_m \leq t \leq t_{m+1}} \|u(t, \cdot) - \hat{u}_{h,t}(\cdot, t)\| \leq c\tau h^2 |\log h|^2$$

by (3.4). Recalling (4.12) and the assumption that $f \in C^{0,1}_\text{loc} (\mathbb{R})$ we obtain in a similar way

$$|A_5| \leq \frac{c}{\tau} \int_{\Omega} \frac{(e_{w}^m - e_{h,u}^m)^2}{Q(\hat{u}_h^m)} dx \leq \frac{\delta}{\tau^2} \int_{\Omega} \frac{(e_{h,u}^{m+1} - e_{h,u}^m)^2}{Q(\hat{u}_h^m)} dx + \frac{c}{\delta} (\|e_w^m\|^2 + h^4 |\log h|^4).$$

(4.26)

(4.27)

If we insert (4.22)–(4.27) into (4.21) we obtain after multiplying by $\tau$ and choosing $\delta > 0$ sufficiently small

$$\frac{1}{2\tau} \int_{\Omega} \frac{(e_{h,u}^{m+1} - e_{h,u}^m)^2}{Q(\hat{u}_h^m)} dx + \frac{1}{8} \int_{\Omega} \frac{\|\nabla (e_{h,u}^{m+1} - e_{h,u}^m)\|^2}{Q(\hat{u}_h^m)} dx + \left(1 - c\tau\right) A^{m+1} - \int_{\Omega} d^{m+1} \cdot \nabla e_{h,u}^{m+1} \rho_u^{m+1} dx$$

$$\leq (1 + c\tau) B^m - \int_{\Omega} d^m \cdot \nabla e_{h,u}^m \rho_u^m dx + c\tau (\tau^2 + h^4 |\log h|^4) + c\tau \|e_w^m\|^2.$$

Recalling the definition of $B^m$ (4.6), and noting (4.17) and (3.2), we deduce that

$$\frac{1}{2\tau} \int_{\Omega} \frac{(e_{h,u}^{m+1} - e_{h,u}^m)^2}{Q(\hat{u}_h^m)} dx + \frac{1}{8} \int_{\Omega} \frac{\|\nabla (e_{h,u}^{m+1} - e_{h,u}^m)\|^2}{Q(\hat{u}_h^m)} dx + (1 - c\tau) B^{m+1}$$

$$\leq (1 + c\tau) B^m + c\tau h^2 (\|\nabla e_{h,u}^{m+1}\| + \|\nabla e_{h,u}^m\|) + c\tau (\tau^2 + h^4 |\log h|^4) + c\tau \|e_w^m\|^2.$$

(4.28)

The second term on the right hand side of (4.28) is estimated by

$$c\tau h^2 (\|\nabla e_{h,u}^{m+1}\| + \|\nabla e_{h,u}^m\|) \leq \tau h^2 (\|\nabla (e_{h,u}^{m+1} - e_{h,u}^m)\| + 2 \|\nabla e_{h,u}^m\|)$$

$$\leq \frac{1}{16} \int_{\Omega} \frac{\|\nabla (e_{h,u}^{m+1} - e_{h,u}^m)\|^2}{Q(\hat{u}_h^m)} dx + c\tau \|\nabla e_{h,u}^m\|^2 + c\tau h^4$$

$$\leq \frac{1}{16} \int_{\Omega} \frac{\|\nabla (e_{h,u}^{m+1} - e_{h,u}^m)\|^2}{Q(\hat{u}_h^m)} dx + c\tau B^m + c\tau h^4,$$

where we have used (4.16) in the last step. Inserting this estimate into (4.28) we infer that

$$\frac{1}{2\tau} \int_{\Omega} \frac{(e_{h,u}^{m+1} - e_{h,u}^m)^2}{Q(\hat{u}_h^m)} dx + B^{m+1} + \frac{1}{16} \int_{\Omega} \frac{\|\nabla (e_{h,u}^{m+1} - e_{h,u}^m)\|^2}{Q(\hat{u}_h^m)} dx$$

The second term on the right hand side of (4.28) is estimated by
4.2 The surface PDE

We deduce from (4.29) and the induction hypothesis (4.8) together with (4.15)

\[ \frac{1}{2c} \frac{1}{\tau} \| e^{m+1}_u - e^m_u \|^2 + E^{m+1} \leq \left( \tau^2 + h^4 \log |h|^{4} \right) e^{\mu t m} \left( 1 + c \tau \left( 1 + \frac{1}{\theta^2} \right) \right) \]

provided that

\[ \mu \geq c(1 + \frac{1}{\theta^2}). \]  

Note that for the last inequality in (4.30) we have used again (4.9), (4.11) and the fact that \( \tau \leq \delta_0 h |\log h|^{-\frac{1}{2}} \). In particular, we can repeat the arguments leading to (4.15) and (4.16) and obtain

\[ \sup_{\Omega} Q(u^{m+1}_h) \leq c \quad \text{and} \quad \| \nabla e^{m+1}_{h,u} \| \leq c h^2 |\log h|^{-1}. \]

4.2 The surface PDE

As already mentioned in the Introduction the error analysis of the surface equation is laborious. Much of this work is related to the handling of differences of the form \( Q(u^{m+1}_h) - Q(u^{m+1}_{h,0}) \), which are typically split into \( Q(u^{m+1}_h) - Q(u^{m+1}_{h,0}) \) and \( Q(u^{m+1}_{h,0}) - Q(u^{m+1}_h) \). The second term can be bounded in terms of \( \nabla e^{m+1}_{h,u} \), which is naturally controlled within our induction. On the other hand, simply estimating the first term by \( \nabla \rho^{m+1}_u \) will frequently lead to suboptimal error bounds, which are not sufficient to control the gradient of the discrete height function uniformly. Instead, we will try to exploit the structure of \( Q(u) \) and frequently apply integration by parts to take advantage of the quadratic convergence of \( \rho^{m+1}_u \).

Evaluating (2.6) at \( t = t_{m+1} \) and using the definition (3.6) we obtain for \( \eta_h \in X_{h,0} \)

\[ \int_{\Omega} (w Q(u)) (\cdot, t_{m+1}) \eta_h dx + \int_{\Omega} E(\nabla \tilde{w}^{m+1}_h) \nabla \tilde{w}^{m+1}_h \cdot \nabla \eta_h dx \]

\[ = - \int_{\Omega} \nabla u^{m+1} \cdot \nabla \eta_h V^{m+1} w^{m+1} dx + \int_{\Omega} g(V^{m+1}, w^{m+1}) \eta_h Q(u^{m+1}) dx. \]

If we combine this relation with (2.13) we deduce

\[ \int_{\Omega} e^{m+1}_{h,w} \eta_h Q(u^{m+1}_h) dx - \int_{\Omega} e^{m+1}_{h,w} \eta_h Q(u^{m+1}_h) dx + \tau \int_{\Omega} E(\nabla u^{m+1}_h) \nabla e^{m+1}_{h,w} \cdot \nabla \eta_h dx \]

\[ = \int_{\Omega} \left( \tilde{w}^{m+1}_h Q(u^{m+1}_h) - \tilde{w}^m_h Q(u^m_h) - \tau (w Q(u))(\cdot, t_{m+1}) \right) \eta_h dx \]

\[ + \tau \int_{\Omega} E(\nabla u^{m+1}_h) \nabla \tilde{w}^{m+1}_h \cdot \nabla \eta_h dx \]

\[ + \tau \int_{\Omega} (V^{m+1}_h w^{m+1}_h \nabla u^{m+1}_h - V^{m+1}_h w^{m+1}_h \nabla u^{m+1}_h) \cdot \nabla \eta_h dx \]

\[ + \tau \int_{\Omega} g(V^{m+1}_h, w^{m+1}_h) Q(u^{m+1}_h) - g(V^{m+1}_h, w^{m+1}_h) Q(u^{m+1}_h) \eta_h dx. \]  

(4.33)

Inserting \( \eta_h = e^{m+1}_{h,w} \) we derive after some straightforward manipulations

\[ \frac{1}{2} \int_{\Omega} (e^{m+1}_{h,w})^2 Q(u^{m+1}_h) dx + \tau \int_{\Omega} E(\nabla u^{m+1}_h) \nabla e^{m+1}_{h,w} \cdot \nabla e^{m+1}_{h,w} dx + \frac{1}{2} \int_{\Omega} (e^{m+1}_{h,w} - e^m_{h,w})^2 Q(u^m_h) dx \]
\[
\begin{align*}
&= \frac{1}{2} \int_{\Omega} (e^{m}_{h,w})^2 Q(u^{m}_{h}) \, dx + \frac{1}{2} \int_{\Omega} (e^{m+1}_{h,w})^2 (Q(u^{m}_{h}) - Q(u^{m+1}_{h})) \, dx \\
& \quad + \int_{\Omega} (\vec{w}^{m+1}_{h} Q(u^{m+1}_{h}) - \vec{w}^{m}_{h} Q(u^{m}_{h}) - \tau (wQ(u))_t (\cdot, t_{m+1}) e^{m+1}_{h,w} \, dx \\
& \quad + \tau \int_{\Omega} (E(\nabla u^{m+1}_{h}) - E(\nabla \vec{u}^{m+1}_{h})) \nabla \vec{e}^{m+1}_{h,w} \cdot \nabla e^{m+1}_{h,w} \, dx \\
& \quad + \tau \int_{\Omega} (V^{m+1}_{h} w^{m+1}_{h} \nabla u^{m+1}_{h} - V^{m+1}_{h} w^{m+1}_{h} \nabla u^{m+1}_{h}) \cdot \nabla e^{m+1}_{h,w} \, dx \\
& \quad + \tau \int_{\Omega} (g(V^{m+1}_{h} w^{m+1}_{h} Q(u^{m+1}_{h}) - g(V^{m+1}_{h} w^{m}_{h}) Q(u^{m+1}_{h}) e^{m+1}_{h,w} \, dx \\
& =: \frac{1}{2} \int_{\Omega} (e^{m}_{h,w})^2 Q(u^{m}_{h}) \, dx + \sum_{i=1}^{5} B_i, \quad (4.34)
\end{align*}
\]

(i) Rearranging the estimate \(\frac{\nabla u^{m}_{h} \cdot \nabla u^{m+1}_{h} + 1}{Q(u^{m}_{h}) Q(u^{m+1}_{h})} = \nu(u^{m}_{h}) \cdot \nu(u^{m+1}_{h}) \leq 1\) implies that
\[
Q(u^{m}_{h}) - Q(u^{m+1}_{h}) \leq \frac{\nabla u^{m}_{h}}{Q(u^{m}_{h})} \cdot (u^{m}_{h} - u^{m+1}_{h}),
\]
so that
\[
B_1 \leq \frac{1}{2} \int_{\Omega} (e^{m+1}_{h,w})^2 \nabla u^{m}_{h} \cdot \nabla (u^{m}_{h} - u^{m+1}_{h}) \, dx = \frac{1}{2} \int_{\Omega} (e^{m+1}_{h,w})^2 \nabla u^{m}_{h} \cdot \nabla (u^{m}_{h} - u^{m+1}_{h}) \, dx \\
+ \frac{1}{2} \int_{\Omega} (e^{m+1}_{h,w})^2 (\nabla u^{m}_{h} - \nabla u^{m}_{h}) \cdot \nabla (u^{m}_{h} - u^{m+1}_{h}) \, dx =: B_{1,1} + B_{1,2}.
\]

Integration by parts along with an inverse estimate yields
\[
B_{1,1} = \int_{\Omega} (e^{m+1}_{h,w})^2 \nabla e^{m+1}_{h,w} \cdot \nabla (u^{m+1}_{h} - u^{m}_{h}) \, dx + \frac{1}{2} \int_{\Omega} (e^{m+1}_{h,w})^2 \nabla \cdot \left( \frac{\nabla u^{m}_{h}}{Q(u^{m}_{h})} \right) (u^{m+1}_{h} - u^{m}_{h}) \, dx \\
\leq c \int_{\Omega} (|e^{m+1}_{h,w}| + |e^{m+1}_{h,w}|^2) (|e^{m+1}_{h,w} - e^{m}_{h,w}| + |u^{m+1}_{h} - u^{m}_{h}|) \, dx \\
\leq c \cdot \|e^{m+1}_{h,w}\|_{0,\infty} \|e^{m+1}_{h,w}\|_{1} \|e^{m+1}_{h,w} - e^{m}_{h,w}\| + c \tau \sup_{t_{m} \leq t \leq t_{m+1}} \|u_{|t}\|_{0,\infty} \|e^{m+1}_{h,w}\|_{1} \|e^{m+1}_{h,w}\|_{1} \\
\leq ch^{-1} \|e^{m+1}_{h,w}\|_{1} \|e^{m+1}_{h,w}\|_{1} \|e^{m+1}_{h,w} - e^{m}_{h,w}\| + c \tau \|e^{m+1}_{h,w}\|_{1} \|e^{m+1}_{h,w}\|_{1}.
\]

Next, we deduce from (2.10), (4.16) and (3.3) that
\[
\|\nabla e^{m}_{u}\|_{0,\infty} \leq \|\nabla e^{m}_{h,w}\|_{0,\infty} + \|\nabla e^{m}_{h,w}\|_{0,\infty} \leq ch^{-1} \|\nabla e^{m}_{h,w}\| + ch \log h \leq c \|\log h\|^{-\frac{1}{2}}
\]
and therefore by (2.11), (2.9) and (3.4)
\[
B_{1,2} \leq c \|\nabla e^{m}_{u}\|_{0,\infty} \int_{\Omega} (e^{m+1}_{h,w})^2 (|\nabla (e^{m+1}_{h,w} - e^{m}_{h,w})| + |\nabla (\vec{e}^{m+1}_{h,w} - \vec{e}^{m}_{h,w})|) \, dx \\
\leq c \|\log h\|^{-\frac{1}{2}} \|e^{m+1}_{h,w}\|_{0,\infty} \|e^{m+1}_{h,w}\| \|\nabla (e^{m+1}_{h,w} - e^{m}_{h,w})| + \tau \\
\leq ch^{-1} \|e^{m+1}_{h,w}\|_{1} \|e^{m+1}_{h,w}\|_{1} \|e^{m+1}_{h,w} - e^{m}_{h,w}\| + c \tau \|e^{m+1}_{h,w}\|_{1} \|e^{m+1}_{h,w}\|_{1} \\
\leq ch^{-1} \|e^{m+1}_{h,w}\|_{1} \|e^{m+1}_{h,w}\| \|e^{m+1}_{h,w} - e^{m}_{h,w}\| + c \tau \|e^{m+1}_{h,w}\|_{1} \|e^{m+1}_{h,w}\|_{1}.
\]
Combining the above bounds we find that

\[ B_1 \leq \varepsilon \tau \| e_{h,w} \|_1^2 + c\varepsilon \tau \| e_{h,w} \|_1 + c\varepsilon h^{-2} \| e_{h,w} \|_1^2 \leq \frac{1}{\tau} \| e_{u} \|_1^2 - \| e_{u} \|_1^2. \] (4.35)

(ii) Let us write

\[
B_2 = \int_{\Omega} (w^{m+1} Q(u^{m+1}) - w^m Q(u^m) - \tau(wQ(u))_{t}(\cdot, t_{m+1})) e_{h,w}^{m+1} \ dx
\]

\[
\text{and} \quad \int_{\Omega} (\rho_{w}^{m+1} - \rho_{w}^{m}) Q(u_{h}^{m+1}) e_{h,w}^{m+1} \ dx
\]

\[
- \int_{\Omega} Q(u_{h}^{m+1}) e_{h,w}^{m+1} \ dx
\]

\[
+ \int_{\Omega} (w^{m+1} - w^{m}) (Q(u_{h}^{m+1}) - Q(u^{m})) e_{h,w}^{m+1} \ dx
\]

\[
+ \int_{\Omega} w^{m} ((Q(\tilde{u}_{h}^{m+1}) - Q(u^{m+1})) - (Q(\tilde{u}_{h}^{m}) - Q(u^{m})) e_{h,w}^{m+1} \ dx
\]

\[
+ \int_{\Omega} w^{m} ((Q(u_{h}^{m+1}) - Q(\tilde{u}_{h}^{m+1})) - (Q(u_{h}^{m}) - Q(\tilde{u}_{h}^{m})) e_{h,w}^{m+1} \ dx
\]

\[
= \sum_{j=1}^{6} B_{2,j}. \quad (4.36)
\]

Recalling (2.3), (2.4), (4.32) and (3.10) we have

\[ |B_{2,1}| + |B_{2,2}| \leq c(\tau + \tau \sup_{t_{m} \leq \cdot \leq t_{m+1}} \| \rho_{w, t} \|) \| e_{h,w}^{m+1} \| \leq c \tau (\tau + h^{2} |\log h|^{2}) \| e_{h,w}^{m+1} \|.
\]

Next, since |\(Q(u_{h}^{m+1}) - Q(u_{h}^{m})| \leq |\nabla(u_{h}^{m+1} - u_{h}^{m})| we obtain with the help of (3.8), (2.11), (3.2), (2.9) and (3.4)

\[ |B_{2,3}| \leq \| \rho_{w} \| \| \nabla(u_{h}^{m+1} - u_{h}^{m}) \| \| e_{h,w}^{m+1} \|_{0,\infty}
\]

\[ \leq c \tau^{2} \log h(\| \nabla(e_{h,w}^{m+1} - e_{h,w}^{m}) \| + \| \nabla(\tilde{e}_{h,w}^{m+1} - \tilde{e}_{h,w}^{m}) \|) \| \log h \|^{2} \| e_{h,w}^{m+1} \|_{1}
\]

\[ \leq c \tau^{2} \log h^{2} \| e_{h,w}^{m+1} \| + \| e_{h,w}^{m+1} \|_{1} \| e_{h,w}^{m+1} \|_{1} + c \tau^{2} \| \log h \|^{2} \| e_{h,w}^{m+1} \|_{1}
\]

\[ \leq c \| e_{h,w}^{m+1} \|_{1}(\| e_{h,w}^{m+1} \| + \| e_{h,w}^{m+1} \| + \tau h^{2} |\log h|^{2}).
\]

Applying Lemma 3.3 to \(f = (w^{m+1} - w^{m})e_{h,w}^{m+1} \) yields

\[ B_{2,4} = \int_{\Omega} (w^{m+1} - w^{m}) (Q(u_{h}^{m+1}) - Q(\tilde{u}_{h}^{m+1})) e_{h,w}^{m+1} \ dx
\]

\[ + \int_{\Omega} (w^{m+1} - w^{m}) (Q(\tilde{u}_{h}^{m+1}) - Q(u_{h}^{m})) e_{h,w}^{m+1} \ dx
\]

\[ \leq c \| w^{m+1} - w^{m} \|_{0,\infty} \| \nabla e_{h,w}^{m+1} \| \| e_{h,w}^{m+1} \| + c \tau^{2} \| \log h \| (w^{m+1} - w^{m}) e_{h,w}^{m+1} \|_{1,1}
\]

\[ \leq c \tau \| e_{h,w}^{m+1} \|_{1}(\| \nabla e_{h,w}^{m+1} \| + h^{2} |\log h|).
\]

Since \(Q(\tilde{u}_{h}) - Q(u)\) \(t = \frac{\nabla u_{h,t} \cdot \nabla \tilde{u}_{h}}{Q(u)} - \frac{\nabla u_{t} \cdot \nabla u}{Q(u)}\) we obtain

\[ B_{2,5} = \int_{t_{m}}^{t_{m+1}} \int_{\Omega} w^{m} (\frac{\nabla u_{h,t} \cdot \nabla \tilde{u}_{h}}{Q(u_{h})} - \frac{\nabla u_{t} \cdot \nabla u}{Q(u)}) e_{h,w}^{m+1} \ dx \ dt \quad (4.37)
\]
We remark that (2.10), (4.16) and (4.32) imply that

\[ \text{(4.37)} \]

After integration by parts we obtain

\[ B = \int_{t_m}^{t_{m+1}} \nabla \cdot (w^m \nabla u_t) \, dx \, dt \]

Another application of Lemma 3.3 yields

\[ I \leq c h^2 |\log h| \int_{t_m}^{t_{m+1}} |w^m e_{h,w}^{m+1} \nabla u_t|_1,dt \leq c \tau h^2 |\log h| |e_{h,w}^{m+1}|_1. \]

If we insert the above estimates into (4.37) we obtain

\[ B_{2.5} \leq c \tau h^2 |\log h|^2 |e_{h,w}^{m+1}|_1. \]

In order to treat \( B_{2.6} \) we write with the help of (4.2)

\[ (Q(u_h^{m+1}) - Q(\tilde{u}_h^{m+1})) - (Q(u_h^m) - Q(\tilde{u}_h^m)) = -\nabla \tilde{u}_h^{m+1} \cdot e_{h,u}^{m+1} + \nabla \tilde{u}_h^m \cdot e_{h,u}^{m} \]

\[ + \frac{|\nabla e_{h,u}^{m+1}|^2}{2Q(\tilde{u}_h^{m+1})} - \frac{|\nabla e_{h,u}^m|^2}{2Q(\tilde{u}_h^m)} - \frac{(Q(u_h^{m+1}) - Q(\tilde{u}_h^{m+1}))^2}{2Q(\tilde{u}_h^{m+1})} + \frac{(Q(u_h^m) - Q(\tilde{u}_h^m))^2}{2Q(\tilde{u}_h^m)} \]

\[ = -\frac{\nabla \tilde{u}_h^{m+1}}{Q(\tilde{u}_h^{m+1})} \cdot e_{h,u}^{m+1} - \nabla \tilde{u}_h^{m+1} \cdot e_{h,u}^{m} + \frac{1}{2} \left( \frac{1}{Q(\tilde{u}_h^{m+1})} - \frac{1}{Q(\tilde{u}_h^m)} \right) \left| \nabla e_{h,u}^{m+1} \right|^2 - \left( Q(u_h^{m+1}) - Q(\tilde{u}_h^{m+1}) \right)^2 \]

\[ + \frac{\nabla e_{h,u}^{m+1} \cdot e_{h,u}^{m+1} + \nabla e_{h,u}^m \cdot e_{h,u}^{m+1}}{2Q(\tilde{u}_h^{m+1})} - \delta_h \left\{ (Q(u_h^{m+1}) - Q(\tilde{u}_h^{m+1})) - (Q(u_h^m) - Q(\tilde{u}_h^m)) \right\}, \]

where

\[ \delta_h = \frac{(Q(u_h^{m+1}) - Q(\tilde{u}_h^{m+1})) + (Q(u_h^m) - Q(\tilde{u}_h^m))}{2Q(\tilde{u}_h^m)}. \]

We remark that (2.10), (4.16) and (4.32) imply that

\[ |\delta_h| \leq \frac{1}{2} \left( |\nabla e_{h,u}^{m+1}| + |\nabla e_{h,u}^m| \right) \leq c h^{-1} \left( \| \nabla e_{h,u}^{m+1} \| + \| \nabla e_{h,u}^m \| \right) \leq c |\log h|^{-\frac{1}{2}} \leq \frac{1}{2}. \]
provided that $0 < h \leq h_3$ and $h_3 \leq h_2$ is small enough. Thus, if we move the last term on the right hand side of (4.38) to the left hand side and divide by $1 + \delta_h \geq \frac{1}{2}$ we end up with

\[
(Q(u_h^{m+1}) - Q(\widehat{u}_h^{m+1})) - (Q(u_h^m) - Q(\widehat{u}_h^m)) = -\frac{\nabla \widehat{u}_h^{m+1}}{Q(\widehat{u}_h^{m+1})} \cdot \nabla (e_h^{m+1} - e_h^m) + \\
\frac{1}{1 + \delta_h} \nabla (e_h^m - e_h^m) \cdot (\delta_h - \frac{\nabla \widehat{u}_h^{m+1}}{Q(\widehat{u}_h^{m+1})} + \frac{\nabla (e_h^{m+1} + \nabla e_h^m)}{2Q(\widehat{u}_h^{m+1})}) + \\
\frac{1}{1 + \delta_h} Q(\widehat{u}_h^m) - Q(\widehat{u}_h^{m+1}) \left( |\nabla e_h^m|^2 - (Q(u_h^m) - Q(\widehat{u}_h^m))^2 \right) + \\
- \frac{1}{1 + \delta_h} \left( \frac{\nabla \widehat{u}_h^{m+1}}{Q(\widehat{u}_h^{m+1})} - \frac{\nabla \widehat{u}_h^m}{Q(\widehat{u}_h^m)} \right) \cdot \nabla e_h^m =: S_1 + S_2 + S_3 + S_4.
\]

Note that in order to derive the form of $S_1$ and $S_2$ we have split $\frac{1}{1 + \delta_h} = -1 + \frac{\delta_h}{1 + \delta_h}$. The fact that $S_1$ does not contain $\delta_h$ will allow us to apply integration by parts to the integral involving this term. From the above calculations we now have

\[
B_{2,0} = \sum_{i=1}^4 \int_{\Omega} w^m S_i e_{h,w}^{m+1} \, dx.
\]

To begin, integration by parts together with (1.7) yields

\[
\int_{\Omega} w^m S_1 e_{h,w}^{m+1} \, dx = \int_{\Omega} w^m e_{h,w}^{m+1} \left( \frac{\nabla u_h^{m+1}}{Q(u_h^{m+1})} - \frac{\nabla \widehat{u}_h^{m+1}}{Q(\widehat{u}_h^{m+1})} \right) \cdot \nabla (e_h^{m+1} - e_h^m) \, dx
\]

\[
= \int_{\Omega} w^m e_{h,w}^{m+1} \left( \frac{\nabla u_h^{m+1}}{Q(u_h^{m+1})} - \frac{\nabla \widehat{u}_h^{m+1}}{Q(\widehat{u}_h^{m+1})} \right) \cdot \nabla (e_h^{m+1} - e_h^m) \, dx
\]

\[+ \int_{\Omega} \nabla \cdot \left( \frac{w^m e_{h,w}^{m+1} \nabla u_h^{m+1}}{Q(u_h^{m+1})} \right) \cdot (e_h^{m+1} - e_h^m) \, dx.
\]

Using Lemma 3.4 and (3.4) for the first term we obtain

\[
\int_{\Omega} w^m S_1 e_{h,w}^{m+1} \, dx \leq c h \log h \|e_h^{m+1} - e_h^m\|_1 \left( \sum_{t \in T_h} \|w^m e_{h,w}^{m+1}\|_{H^2(T)}^2 \right)^{\frac{1}{2}} + c \|e_h^{m+1} - e_h^m\|_{L^1} \|e_h^{m+1}\|_{L^1}
\]

\[
\leq c h \log h \|e_h^{m+1}\|_1 \|e_h^{m+1} - e_h^m\|_1 \leq c \|e_h^{m+1}\|_{L^1} \|e_h^{m+1} - e_h^m\|_1,
\]

where we also exploited the fact that the second derivatives of $e_h^{m+1}$ vanish. Since $1 + \delta_h \geq \frac{1}{2}$ and $|\delta_h| \leq \frac{1}{2} (|\nabla e_h^{m+1}| + |\nabla e_h^m|)$ we derive with the help of (4.16), (4.32), (2.9) and (2.11)

\[
\int_{\Omega} w^m S_2 e_{h,w}^{m+1} \, dx \leq c \|\nabla (e_h^{m+1} - e_h^m)\|_{L^1} \left( \|\nabla e_h^{m+1}\|_1 + \|\nabla e_h^m\|_1 \right) \|e_h^{m+1}\|_1
\]

\[
\leq c h \log h \|e_h^{m+1} - e_h^m\|_1 \|\log h\|_{L^1} \|\nabla e_h^{m+1}\|_1 \leq c \|e_h^{m+1} - e_h^m\|_{L^1} \|e_h^{m+1}\|_1
\]

Finally, we deduce with the help of (4.39), (2.11) and (4.16)

\[
\int_{\Omega} w^m (S_3 + S_4) e_{h,w}^{m+1} \, dx \leq c \int_{\Omega} \|\nabla (\widehat{u}_h^{m+1} - \widehat{u}_h^m)\|_{L^1} \left( \|\nabla e_h^{m+1}\|_1 + \|\nabla e_h^m\|_1 \right) \|e_h^{m+1}\|_1 \, dx
\]

\[
\leq c \|\nabla (\widehat{u}_h^{m+1} - \widehat{u}_h^m)\|_{L^1} \|\nabla e_h^{m+1}\|_1 \|e_h^{m+1}\|_{L^1} + \|e_h^{m+1}\|_{L^1}
\]

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\[
\leq c\tau \|\nabla e_{h,u}^{m+1}(h\|e_{h,w}^{m+1}||_1 + \|e_{h,u}^{m+1}\|) \leq c\tau \|\nabla e_{h,u}^{m+1}||e_{h,w}^{m+1}||_1.
\]

Collecting the above estimates and recalling (3.4) we obtain

\[
B_{2,6} \leq c\|e_{h,w}^{m+1}\|_1(\|e_{h,u}^{m+1} - e_{h}^{m+1}|| + \tau h^2 |\log h|^2 + \tau \|\nabla e_{h,u}^{m+1}\|).
\]

If we insert the bounds for \(B_{2,j}, j = 1, \ldots, 6\) into (4.36) we obtain

\[
B_2 \leq c\|e_{h,w}^{m+1}\|_1(\tau^2 + \tau h^2 |\log h|^2 + \|e_{h,u}^{m+1} - e_{h}^{m+1}\| + \tau \|\nabla e_{h,u}^{m+1}\| + \tau \|\nabla e_{h,u}^{m+1}\|)
\]

\[
\leq \varepsilon\tau \|e_{h,w}^{m+1}\|_1^2 + c_\varepsilon \frac{1}{\tau} \|e_{h,u}^{m+1} - e_{h}^{m+1}\|^2 + c_\varepsilon \tau (\tau^2 + h^4 |\log h|^4 + \|\nabla e_{h,u}^{m+1}\|^2 + \|\nabla e_{h,u}^{m+1}\|^2).
\]

(iii) Recalling (2.7) it is not difficult to verify that \(|E(p) - E(q)| \leq c|p - q|\) and hence

\[
B_3 \leq C\tau \|\nabla e_{h,w}^{m+1}\||\|e_{h,w}^{m+1}\| \leq \varepsilon\tau \|\nabla e_{h,w}^{m+1}\|^2 + c_\varepsilon \tau \|\nabla e_{h,w}^{m+1}\|^2.
\]

(iv) In view of the definition of \(V^{m+1}\) and \(V_{h,w}^{m+1}\) we have

\[
B_4 = \int_\Omega \left( (u_{h}^{m+1} - u_{h}^{m})w_{h}^{m} \nabla u_{h}^{m+1} \right) \cdot \nabla e_{h,w}^{m+1} \, dx
\]

\[
= - \int_\Omega (e_{u}^{m+1} - e_{u}^{m})w_{h}^{m} \nabla u_{h}^{m+1} \cdot \nabla e_{h,w}^{m+1} \, dx - \int_\Omega (u_{h}^{m+1} - u_{h}^{m})w_{h}^{m} \nabla u_{h}^{m+1} \cdot \nabla e_{h,w}^{m+1} \, dx
\]

\[
+ \int_\Omega (u_{h}^{m+1} - u_{h}^{m})w_{h}^{m} \nabla u_{h}^{m+1} \cdot \nabla e_{h,w}^{m+1} \, dx
\]

\[
+ \int_\Omega ((u_{h}^{m+1} - u_{h}^{m})w_{h}^{m} + \tau u_{h}^{m+1}(w_{h}^{m} - w_{h}^{m+1})) \nabla u_{h}^{m+1} \cdot \nabla e_{h,w}^{m+1} \, dx = \sum_{i=1}^{5} B_{4,i}.
\]

It follows from (4.12) and (4.19) that

\[
B_{4,1} + B_{4,2} + B_{4,3} + B_{4,5} \leq c\|\nabla e_{h,w}^{m+1}\|(\|e_{h,u}^{m+1} - e_{h}^{m+1}\| + \|e_{h,w}^{m+1}\| + \|\nabla e_{h,u}^{m+1}\| + \tau^2),
\]

while Lemma 3.4 implies

\[
B_{4,4} \leq ch^2 |\log h||(e_{h,w}^{m+1}||_1 ||(u_{h}^{m+1} - u_{h}^{m})w_{h}^{m}||_2 \leq c\tau h^2 |\log h||e_{h,w}^{m+1}||_1.
\]

In conclusion

\[
B_4 \leq \varepsilon\tau \|e_{h,w}^{m+1}\|_1^2 + c\varepsilon \frac{1}{\tau} \|e_{h,u}^{m+1} - e_{h}^{m+1}\|^2 + c\varepsilon \tau \|\nabla e_{h,u}^{m+1}\|^2 + \tau^2 + h^4 |\log h|^2.
\]
so that
\[ B_5 = \sum_{i=1}^{4} \tau \int_{\Omega} S_i e_{h,w}^{m+1} \, dx =: \sum_{i=1}^{4} B_{5,i}. \]

Since \( \beta, \tilde{\beta} \in C^{0,1}_{\text{loc}}(\mathbb{R}) \) we obtain from (4.12), (4.32) and (3.8)
\[
|B_{5,1}| + |B_{5,4}| \leq c\tau \|w^{m+1} - u_h^{m}\|\|e^{m+1}_{h,w}\| \leq c\tau (\|w^{m+1} - w^m\| + \|\rho^m\| + \|e^m_{h,w}\|)\|e^{m+1}_{h,w}\|
\leq c\tau (\tau + h^2 |\log h| + \|e^m_{h,w}\|)\|e^{m+1}_{h,w}\|.
\]

Next, we deduce with the help of the global Lipschitz continuity of \( r \mapsto \alpha(r) \) and (4.19) that
\[
|B_{5,2}| \leq c\tau\|u_t^{m+1} - u_h^{m}\|\|e^{m+1}_{h,w}\| \leq c\tau^2\|e^{m+1}_{h,w}\| + c\|e^{m+1}_{u} - e^{m}_{u}\|\|e^{m+1}_{h,w}\|.\]

Applying Lemma 3.3 with \( f = \tilde{\beta}(w^{m+1})e^{m+1}_{h,w} \) we infer that
\[
|B_{5,3}| \leq c\tau h^2 |\log h||\tilde{\beta}(w^{m+1})e^{m+1}_{h,w}|_{1,1} \leq c\tau h^2 |\log h||e^{m+1}_{h,w}|_1.
\]

After collecting the above estimates we obtain
\[
B_5 \leq \varepsilon \tau\|e^{m+1}_{h,w}\|^2 + c\varepsilon \frac{1}{\tau}\|e^{m+1}_{u} - e^{m}_{u}\|^2 + c\varepsilon \tau (\|e^{m}_{h,w}\|^2 + \tau^2 + h^4 |\log h|^2).
\]

(4.43)

If we insert (4.35), (4.40), (4.41), (4.42) and (4.43) into (4.34), use Poincaré’s inequality and observe (2.32) together with (4.32) we derive
\[
\frac{1}{2} \int_{\Omega} (e^{m+1}_{h,w})^2 Q(u^{m+1}_h) \, dx + \tau \varepsilon_0 \|\nabla e^{m+1}_{h,w}\|^2 + \frac{1}{2} \|e^{m+1}_{h,w} - e^m_{h,w}\|^2
\leq \frac{1}{2} \int_{\Omega} (e^{m}_{h,w})^2 Q(u^m_h) \, dx + \varepsilon \tau \|\nabla e^{m+1}_{h,w}\|^2 + c\varepsilon \tau (\tau^2 + h^4 |\log h|^4 + \|\nabla e^{m+1}_{h,w}\|^2 + \|\nabla e^m_{h,w}\|^2)
\leq c \varepsilon \frac{1}{\tau}\|e^{m+1}_{u} - e^{m}_{u}\|^2 + c\varepsilon \frac{1}{\tau}\|e^{m+1}_{h,w} - e^m_{h,w}\|.
\]

(4.44)

In view of (4.10), (4.30) and (4.9) we have
\[
h^{-2}\|e^{m+1}_{h,w}\|\frac{1}{\tau}\|e^{m+1}_{u} - e^{m}_{u}\| \leq 2h^{-2}(\|e^{m}_{h,w}\|^2 + \|e^{m+1}_{h,w} - e^m_{h,w}\|^2) \frac{1}{\tau}\|e^{m+1}_{u} - e^{m}_{u}\|^2
\leq 4 \frac{1}{\theta^2} \frac{1}{\tau}\|e^{m+1}_{u} - e^{m}_{u}\|^2 + c|\log h|^{-\frac{1}{2}}\|e^{m+1}_{h,w} - e^m_{h,w}\|^2
\leq c \frac{1}{\tau}\|e^{m+1}_{u} - e^{m}_{u}\|^2 + c|\log h|^{-\frac{1}{2}}\|e^{m+1}_{h,w} - e^m_{h,w}\|^2.
\]

Using this bound in (4.44) and choosing \( \varepsilon \) and \( h_0 \) sufficiently small we obtain
\[
\frac{1}{2} \int_{\Omega} (e^{m+1}_{h,w})^2 Q(u^{m+1}_h) \, dx + \frac{\varepsilon_0}{2} \|\nabla e^{m+1}_{h,w}\|^2 \leq \frac{1}{2} \int_{\Omega} (e^{m}_{h,w})^2 Q(u^m_h) \, dx + c\|e^{m+1}_{h,w}\|^2
\leq (1 + c \frac{1}{2}) \int_{\Omega} (e^{m+1}_{h,w})^2 Q(u^{m+1}_h) \, dx + c\|e^{m+1}_{h,w}\|^2
+ c \frac{1}{2\tau} \int_{\Omega} \frac{(e^{m+1}_{h,w} - e^m_{h,w})^2}{Q(u^m_h)} \, dx,
\]

(4.45)
where we used (4.16) and the fact that $Q(\tilde{u}_h^m), Q(u_h^m) \leq c$ in order to derive the last estimate. Multiplying (4.45) by $\theta^2$ ($0 < \theta \leq 1$) and adding the result to (4.29) we obtain with the help of our induction hypothesis (4.8)

$$B^{m+1} + \frac{\theta^2}{2} \int_{\Omega} (e_{h,w}^{m+1})^2 Q(u_h^m) \, dx + \tau \frac{\theta^2 c_0}{2} \|\nabla e_{h,w}^{m+1}\|^2$$

$$+ (1 - e\theta^2) \frac{1}{2\tau} \int_{\Omega} (e_{h,w}^{m+1} - e_{u}^{m})^2 \frac{Q(u_h^m)}{Q(\tilde{u}_h^m)} \, dx + \frac{1}{16} - c\tau \int_{\Omega} \frac{|\nabla(e_{h,u}^m - e_{u}^m)|^2}{Q(\tilde{u}_h^m)} \, dx$$

$$\leq (1 + \frac{c}{\theta^2 \tau})(B^m + \frac{\theta^2}{2} \int_{\Omega} (e_{h,w}^m)^2 Q(u_h^m) \, dx + c\tau (\tau^2 + \tau h^4 \log h^4))$$

$$\leq \left(1 + c\tau \left(1 + \frac{1}{\theta^2}\right)\right)(\tau^2 + h^4 \log h^4)e^{\mu t^m} \leq (\tau^2 + h^4 \log h^4)e^{\mu t^m + 1},$$

(4.46)

provided that

$$\mu \geq c(1 + \frac{1}{\theta^2}).$$

(4.47)

We are now in position to specify the choice of the constants $\theta, \mu, \delta_0$ and $h_0$. To begin, choose $0 < \theta \leq 1$ such that $1 - e\theta^2 \geq \frac{1}{4}$ in the second line of (4.46). Next choose $\mu > 0$ to satisfy (4.31) and (4.47) and then $\delta_0 > 0$ to satisfy (4.11). Finally, $h_0 > 0$ is fixed by (4.9) and additional smallness conditions on $h$ that were required in the course of the calculations.

5 Numerical Results

We begin this section by investigating the experimental order of convergence (eoc) of our scheme and then we display some simulations of diffusion induced grain boundary motion. Throughout the computations in this section we choose a uniform time step $\tau = h^2$.

5.1 Experimental order of convergence

We set $\Omega := \{x \in \mathbb{R}^2 \mid |x| < 1\}$, $T = 0.1$ and choose $f(w) = w^2$ as well as $g(V,w) = V w$. We consider $u, w : \Omega \times [0,T] \rightarrow \mathbb{R}$ given by

Example 1 $u(x,t) = 5 \sin(t)(1 - |x|^2)$, $w(x,t) = e^{-t}(1 + |x|^2)$;

Example 2 $u(x,t) = 5 \sin(t)(1 + (1 - |x|^2)^2)$; $w(x,t) = e^{-t}(1 + |x|^2)$

and include additional right hand sides in order for $(u, w)$ to be solutions of the corresponding PDEs, while the boundary conditions are $u(x,t) = 0$, $w(x,t) = 2e^{-t}$ in Example 1 and $\frac{\partial u}{\partial n}(x,t) = 0$, $w(x,t) = 2e^{-t}$ in Example 2. Let us point out that the Dirichlet condition for $u$ in Example 1 is not covered by our theory. We commence our numerical results with Figure 1 in which we display the solution $w_h^m$ plotted on the surface $\Gamma_h^m = \{(x, u_h^m(x)) \mid x \in \Omega\}$, at $t^m = 0$ and $t^m = 0.1$, for Example 2. When investigating the experimental order of convergence we monitor the following errors:

$$\mathcal{E}_1 := \max_{0 \leq m \leq M} \|e_w^m\|^2, \mathcal{E}_2 := \sum_{m=1}^M \tau \|\nabla e_w^m\|^2, \mathcal{E}_3 := \max_{0 \leq m \leq M} \|e_u^m\|^2,$$
Figure 1: Example 2, $w^m_h$ plotted on $\Gamma^m_h$ at $t^m = 0.0$ and $t^m = 0.1$.

$$\mathcal{E}_4 := \max_{0 \leq m \leq M} \| \nabla e^m_u \|_2^2, \quad \mathcal{E}_5 := \sum_{m=0}^{M-1} \tau \| \frac{e^{m+1}_u - e^m_u}{\tau} \|_2^2.$$ 

In Tables 1 and 2 we display the values of $\mathcal{E}_i$, $i = 1, \ldots, 5$, evaluated using a quadrature rule of degree 4, for Example 1 and Example 2 respectively. For both examples we see the expected order of convergence, with eocs close to four for $\mathcal{E}_1, \mathcal{E}_3$ and $\mathcal{E}_5$, and eocs close to two for $\mathcal{E}_2$ and $\mathcal{E}_3$. In particular, the results of Example 2 confirm the bounds obtained in Theorem 2.1.

<table>
<thead>
<tr>
<th>h</th>
<th>$\mathcal{E}_1 \times 10^2$</th>
<th>eoc$_1$</th>
<th>$\mathcal{E}_2 \times 10^2$</th>
<th>eoc$_2$</th>
<th>$\mathcal{E}_4 \times 10^4$</th>
<th>eoc$_4$</th>
<th>$\mathcal{E}_5 \times 10^4$</th>
<th>eoc$_5$</th>
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<td>64.7340</td>
<td>-</td>
<td>81.45108</td>
<td>-</td>
<td>35.3598</td>
<td>-</td>
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<td>9.10</td>
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<td>0.00514</td>
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<tr>
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<td>0.0217</td>
<td>2.21</td>
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<td>2.02</td>
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</table>

Table 1: Example 1, the errors, $\mathcal{E}_i$, together with their associated estimated order of convergence, eoc$_i$, for $i = 1, 2, 3, 4, 5$.

<table>
<thead>
<tr>
<th>h</th>
<th>$\mathcal{E}_1 \times 10^2$</th>
<th>eoc$_1$</th>
<th>$\mathcal{E}_2 \times 10^2$</th>
<th>eoc$_2$</th>
<th>$\mathcal{E}_4 \times 10^4$</th>
<th>eoc$_4$</th>
<th>$\mathcal{E}_5 \times 10^4$</th>
<th>eoc$_5$</th>
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<tbody>
<tr>
<td>0.1961</td>
<td>38.25517</td>
<td>-</td>
<td>152.3575</td>
<td>-</td>
<td>99.40560</td>
<td>-</td>
<td>101.7628</td>
<td>-</td>
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<td>0.27386</td>
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</tr>
</tbody>
</table>

Table 2: Example 2, the errors, $\mathcal{E}_i$, together with their associated estimated order of convergence, eoc$_i$, for $i = 1, 2, 3, 4, 5$. 

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5.2 Non–orthogonal boundary contact

Even though we have restricted our error analysis to the case where the evolving surface meets the boundary of the cylinder at a right angle, it is not difficult to apply our approach to the case where it meets the boundary of the cylinder at a given angle $\alpha$. In order to do this, we replace the boundary condition (1.3a) with

$$\nu \cdot \nu_{\partial A} = \cos(\alpha) \quad \text{on } \partial \Gamma(t), \quad t \in (0, T],$$

leading to the following boundary condition for the height function $u$:

$$\frac{\nabla u \cdot n}{Q(u)} = -\cos(\alpha) \quad \text{on } \partial \Omega \times (0, T].$$

The weak formulation for $u$ then takes the form

$$\int_{\Omega} \frac{u_t \varphi}{Q(u)} \, dx + \int_{\Omega} \frac{\nabla u \cdot \nabla \varphi}{Q(u)} \, dx = \int_{\Omega} f(w) \varphi \, dx - \int_{\partial \Omega} \cos(\alpha) \varphi \, dx \quad \forall \varphi \in H^1(\Omega),$$

from which we derive the corresponding finite element approximation replacing (2.12). We set $\Omega := \{x \in \mathbb{R}^2 \mid |x| < 1\}$, $f(w) = w$ and $g(V, w) = |V| w$ and specify the following boundary conditions for $u$ and $w$:

$$\frac{\nabla u \cdot n}{Q(u)} = -\cos(2\pi t - \pi/2) \quad \text{and} \quad w = 1 \quad \text{on } \partial \Omega \times (0, T].$$

As initial data we choose $u^0(x) = 0$ and $w^0(x) = \frac{1}{2}(1 + |x|^2)$. In Figure 2 we display $w^n_h$ on the surface $\Gamma^m_h = \{(x, u^n_h(x)) \mid x \in \Omega\}$ at $t^m = 0, 0.25, 0.35, 0.5, 0.65, 0.75$. As $|\cos(2\pi t - \pi/2)| = 1$ for $t = 0.25, 0.75$, the gradient of $u$ will blow up on the boundary. However, for the mesh sizes we chose the discrete solution was able to flow through these singularities without problems.

5.3 Simulations of diffusion induced grain boundary motion

We conclude our numerical results with two simulations of diffusion induced grain boundary motion. We consider the physical set-up of a film of metal, containing a single grain boundary. We denote the film by $A = \Omega \times [0, 5] \subset \mathbb{R}^3$, with $\Omega = (-2, 2)^2$, and we model the grain boundary by the surface $\Gamma(t) = \{(x, u(x, t)) \mid x \in \Omega\}$. We impose the boundary condition

$$\frac{\partial u}{\partial n}(x, t) = 0 \quad \forall (x, t) \in \partial \Omega \times (0, T]$$

such that the grain boundary meets the boundaries of the film orthogonally. The film is immersed in a solute that diffuses into the grain boundary at the surfaces $x_1 = \pm 2$. We denote the concentration of the solute on the grain boundary by $w(x, t) \in [0, 1]$, for $x \in \Omega$, and we assume that the solute concentration is set to 1 on the surfaces $x_1 = \pm 2$ and satisfies zero flux boundary conditions at the surfaces $x_2 = \pm 2$, i.e.

$$w(x, t) = 1 \quad \text{for } x_1 = \pm 2, \quad \frac{\partial w}{\partial n}(x, t) = 0 \quad \text{for } x_2 = \pm 2.$$
We consider two initial configurations for the grain boundary, in the first we take the grain boundary to be the planar surface $x_3 = 1$ such that $u^0(x) = 1$, while in the second we take

$$u^0(x_1,x_2) = \begin{cases} 1 + \varepsilon & \text{if } x_1 > \frac{\pi \varepsilon}{2} \\ \varepsilon \sin \left( \frac{x_1}{\varepsilon} \right) & \text{if } |x_1| \leq \frac{\pi \varepsilon}{2} \\ 1 - \varepsilon & \text{if } x_1 < -\frac{\pi \varepsilon}{2} \end{cases}$$

(5.1)

with $\varepsilon = 0.4$. For both configurations we assume that the concentration of solute on the grain boundary is initially zero, such that $w^0(x) = 0$ for $x \in \Omega$. In this set-up physically meaningful choices for $f(w)$ and $g(V,w)$ are $f(w) = w^2$ and $g(V,w) = |V| w$. Figure 3 displays the solute concentration, $w_h^m(x)$, plotted on the grain boundary, $\Gamma_h^m = \{(x,w_h^m(x)) \mid x \in \Omega\}$, at times $t^m = 0, 0.1, 0.2, 0.3$. In addition in each plot we display the initial grain boundary, depicted by the blue surface, and the outline of the metallic film $A = \Omega \times [0,5]$. The symmetry of this set-up makes it equatable to the two-dimensional configurations studied in [5] and [13]. In particular we see a travelling wave solution comparable to the ones displayed in Figures 9 and 10 of [5] and Figure 4.4 of [13]. In Figure 4 the initial surface is defined by (5.1) which gives rise to a fully three-dimensional simulation. We display the solute concentration, $w_h^m(x)$, plotted on the grain boundary, at times $t^m = 0, 0.2, 0.4, 0.6$, together with the initial grain boundary and the outline of the film.

References

Figure 3: Travelling wave solution showing the grain boundary with the solute concentration at $t^m = 0, 0.1, 0.2, 0.3$, with $u^0_h \equiv 1$ and $w^0_h \equiv 0$.

(2017).


Figure 4: Evolving grain boundary with the solute concentration at \( t^m = 0, 0.2, 0.4, 0.6 \), with the initial surface defined by (5.1) and \( w^0_h \equiv 0 \).


