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Topological Methods for Strong Local Minimizers and Extremals of Multiple Integrals in the Calculus of Variations

by

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Abstract

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and consider the energy functional

$$F[u, \Omega] := \int_{\Omega} F(\nabla u(x)) \, dx,$$

over the space $\mathcal{A}_p(\Omega) := \{ u \in W^{1,p}(\Omega, \mathbb{R}^n) : u|_{\partial \Omega} = x, \det \nabla u > 0 \text{ a.e. in } \Omega \}$, where the integrand $F : M_{n \times n} \to \mathbb{R}$ is quasiconvex, sufficiently regular and satisfies a $p$-coercivity and $p$-growth for some exponent $p \in [1, \infty]$. A motivation for the study of above energy functional comes from nonlinear elasticity where $F$ represents the elastic energy of a homogeneous hyperelastic material and $\mathcal{A}_p(\Omega)$ represents the space of orientation preserving deformations of $\Omega$ fixing the boundary pointwise. The aim of this thesis is to discuss the question of multiplicity versus uniqueness for extremals and strong local minimizers of $F$ and the relation it bares to the domain topology. Our work, building upon previous works of others, explicitly and quantitatively confirms the significant role of domain topology, and provides explicit and new examples as well as methods for constructing such maps.

Our approach for constructing strong local minimizers is topological in nature and is based on defining suitable homotopy classes in $\mathcal{A}_p(\Omega)$ (for $p \geq n$), whereby minimizing $F$ on each class results in, modulo technicalities, a strong local minimizer. Here we work on a prototypical example of a topologically non-trivial domain, namely, a generalised annulus, $\Omega = \{ x \in \mathbb{R}^n : a < |x| < b \}$, with $0 < a < b < \infty$. Then the associated homotopy classes of $\mathcal{A}_p(\Omega)$ are infinitely many when $n = 2$ and two when $n \geq 3$. In contrast, for constructing explicitly and directly solutions to the system of Euler-Lagrange equations associated to $F$ we introduce a topological class of maps referred to as generalised twists and relate the problem to extremising an associated energy on the compact Lie group $\text{SO}(n)$. The main result is a surprising discrepancy between even and odd dimensions. In even dimensions the latter system of equations admits infinitely many smooth solutions, modulo isometries, amongst such maps whereas in odd dimensions this number reduces to one. Even more surprising is the fact that in odd dimensions the functional $F$ admits strong local minimizers yet no solution of the Euler-Lagrange equations can be in the form of a generalised twist. Thus the strong local minimizers here do not have the symmetry one intuitively expects!
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To my family & friends
# Contents

Declaration i  
Abstract ii  
Acknowledgements iii  
Notation vii  

1 Introduction 1  
1.1 Background .................................................. 1  
1.2 Outline of the Thesis ...................................... 4  

2 Continuous self-maps of annuli 11  
2.1 Degree of continuous self-maps on annuli ............... 11  
2.2 Homotopy characterisation and strong local minimizers 13  

3 Measure-preserving maps and generalised twists 16  
3.1 Generalised twists ............................................ 16  
3.2 The $p$-energy restricted to the loop space ............... 21  
3.3 Minimizing $p$-stationary loops ............................. 25  
3.4 Generalised twists as classical solutions ................. 27  
3.5 A characterisation of all twist solutions .................. 41  
3.6 Limiting behaviour of the generalised twists as the inner hole shrinks to a point 44  

4 Quasiconvexity and uniqueness of stationary points 49  
4.1 Quasiconvexity and uniqueness in starshaped domain .... 49  
4.2 Uniqueness theorems on starshaped domain ................ 56  

5 Polyconvexity and generalised twists 58  
5.1 Generalised twists and the space of orientation preserving maps .... 58  
5.2 The energy restricted to the space of twists ............... 62  
5.3 Energy minimizing loops in homotopy classes ............. 65  
5.4 Alternative construction of multiple stationary loops .... 68  
5.5 The restricted versus the full Euler-Lagrange equations .... 73  
5.6 A characterisation of all twist solutions .................. 80  
5.7 The limiting behaviour of twists when the inner boundary converges to a point 83  

Appendices 88
Notation

\( I_n \quad \text{Identity matrix of size } n \)
\( O(n) \quad \text{Orthogonal group} \)
\( \text{SL}(n) \quad \text{Special linear group} \)
\( \text{SO}(n) \quad \text{Special orthogonal group} \)
\( \mathbb{Z}_2 \quad \text{Cyclic group of order } 2 \)
\( \mathbb{Z} \quad \text{Group of integers} \)
\( \mathbb{C} \quad \text{Field of complex numbers} \)
\( \mathbb{R}^n \quad \text{Euclidean real } n\text{-space} \)
\( \mathbb{C}^n \quad \text{Euclidean complex } n\text{-space} \)
\( M_{n \times n} \quad \text{Algebra of } n\text{-by-}n \text{ matrices} \)
\( \theta \quad \text{Unit normal vector} \)
\( B \quad \text{Unit open ball in } \mathbb{R}^n \)
\( S^{n-1} \quad \text{Unit sphere in } \mathbb{R}^n \)
\( \mathcal{L}^n \quad \text{Lebesgue measure} \)
\( \omega_n \quad \mathcal{L}^n(B) := \text{The Lebesgue measure of } B \)
\( \mathcal{H}^{n-1} \quad (n - 1)\text{-dimensional Hausdorff measure} \)
\( L^p(X,Y) \quad L^p \text{ space} \)
\( W^{1,p}(X,Y) \quad \text{Sobolev space} \)
\( \| \cdot \|_{L^p} \quad L^p \text{ norm} \)
\( \| \cdot \|_{W^{1,p}} \quad W^{1,p} \text{ norm} \)
\( \mathcal{C}(X,Y) \quad \text{Space of continuous functions from } X \text{ to } Y \)
\( \mathcal{C}_c(X,Y) \quad \text{Space of functions in } \mathcal{C}(X,Y) \text{ with compact support} \)
\( 1_E \quad \text{Characteristic function of } E \)
\( \nabla u \quad \text{Gradient of } u \)
\( \text{div } u \quad \text{Divergence of } u \)
\( \Delta u \quad \text{Laplacian of } u \)
\[ \Delta_p u \quad \text{p-Laplacian of } u \]
\[ A^t \quad \text{Transpose of } A \]
\[ tr \ A \quad \text{Trace of } A \]
\[ A : B \quad \text{Inner product of matrices } A \text{ and } B \]
\[ \det A \quad \text{Determinant of } A \]
\[ \text{cof } A \quad \text{Cofactor of } A \]
\[ e^A \quad \text{Exponential of } A \]
\[ \langle x, y \rangle \quad \text{Inner product of vectors } x \text{ and } y \]
\[ x \otimes y \quad \text{Tensor product of vectors } x \text{ and } y \]
\[ \{ e_1, \ldots, e_n \} \quad \text{Standard basis of } \mathbb{R}^n \]
\[ \partial \Omega \quad \text{Frontier or boundary of } \Omega \]
\[ \overline{\Omega} \quad \text{Closure of } \Omega \]
\[ \pi_l[X] \quad \text{l-th homotopy group of } X \]
\[ [q] \quad \text{Floor of } q \]
Chapter 1

Introduction

A large number of problems in mathematics, physics and engineering sciences naturally lead to minimizing an energy functional $F : \mathcal{A} \to \mathbb{R}$ over a set $\mathcal{A}$. Problems of this type appear in a variety of areas ranging from analysis and geometry, e.g., harmonic maps, minimal surfaces and their higher dimensional counterparts to more applied branches in economy, optimization, materials science, e.g., nonlinear elasticity, optimal-shape design, modeling of solid-solid phase transitions and liquid crystals.

A general strategy for proving existence of a minimizer is the direct methods of the calculus of variations. It is based on the observation that if the set $\mathcal{A}$ admits a topology $\tau$ with respect to which the following two properties hold:

1. $F$ is $\tau$-coercive,  
2. $F$ is $\tau$-lower semicontinuous,

then there exists an $a \in \mathcal{A}$ such that $F(a) = \inf_{\mathcal{A}} F$. 

1.1 Background

In continuum theories of solid mechanics, specifically elasticity theory, the response of a hyperelastic material subject to external excitations, in the form of applied forces: body and surface forces, as well as boundary displacement, is described by minimization of the total elastic energy

$$F[u, \Omega] := \int_{\Omega} F(\nabla u(x)) \, dx.$$  (1.1)
Chapter 1. Introduction

Here $\Omega \subset \mathbb{R}^n$ is the region occupied by the body, $u : \Omega \rightarrow \mathbb{R}^n$ represents the deformation which is described on parts or whole of the boundary $\partial \Omega$, $\nabla u : \Omega \rightarrow M_{n \times n}$ is the deformation gradient and $F : M_{n \times n} \rightarrow \mathbb{R}$ is the stored energy density. As matter cannot interpenetrate itself the deformation is taken orientation preserving and thus locally invertible, that is, $\det \nabla u > 0$ (almost) everywhere in the domain. Moreover to comply with physics and to avoid unrealistic hypotheses the stored energy density $F$ is taken quasiconvex or often polyconvex but strictly non-convex. (See Ball [6], [7] or Dacorogna [22].)

The aim of this thesis is to investigate the effect of domain topology and geometry on multiplicity versus uniqueness of minimizers (local or global) as well as extremals of $F$. The earliest example of non-uniqueness of extremals for energies of the type described and over spaces of deformations keeping the boundary pointwise fixed, i.e., agreeing with the linear map identity, is a heuristic example of John [41] and [42]. Indeed John considers a two dimensional annulus as the underlying domain and argues that by considering deformations, where a typical representative of each is one keeping the inner boundary fixed while rotating the outer boundary by an integer multiple of $2\pi$, one can define infinitely many distinct classes of non-homotopic self-maps of the annulus and thus arrive at multiple equilibria, see the work by John [41] and [42].

![Figure 1.1: The image of a line segment under a self map of the annulus that keeps the inner boundary fixed and rotates the outer boundary anti-clockwise by $2\pi$.](image)

A more rigorous treatment of this example is due to Post & Sivalogonathan [52] where the authors use the notion of winding number of planar curves to define suitable homotopy classes in the corresponding Sobolev spaces and then proceed by minimizing the energy on each such homotopy class. It is important to note that the use of winding numbers although works well in this example cannot be immediately extended to more complicated plane geometries as well as to higher dimensions as faced and expressed by the authors. The difficulty stems from the fact that in higher dimensions a simply connected domain (i.e., one in which every closed curve is homotopic

---

3 Realistically $n = 2$ or $3$, however, in this exposition and throughout this thesis, unless otherwise specified, we do not restrict the dimension to these two integers.

4 The example of the two dimensional annulus and its infinite homotopy classes of self-maps was known to topologists much earlier in the century (cf. Dehn [23]). These are nowadays known as Dehn-twists and are instrumental in the study of mapping class groups of surfaces.
Chapter 1. Introduction

Figure 1.2: An example of a smooth simply connected domain in $\mathbb{R}^3$ whose spaces of self-maps admits multiple homotopy classes.

In contrast to the non-uniqueness results expressed above there are also examples where by imposing stringent conditions on the domain one can arrive at uniqueness of minimizers and extremals. The first result in this direction is the work of Knops & Stuart [44] where for similar type of energies subject to the domain being starshaped any linear map is the unique minimizer as well as the unique extremal of the energy subject to its own boundary condition. That is any other sufficiently regular extremal must coincide with the latter map. (See Taheri [71] for a different proof and for an analogous result for strong local minimizers. Also Ball [2] and Spadaro [61].)

In this thesis we aim to analyze this distinction more closely and examine a particular geometry, with no restriction on the dimension, where the energy functional $F$ admits infinitely many smooth extremals as well as multiple local minimizers. Indeed, the thesis can be divided into roughly two parts: the first half focuses on domains $\Omega$ with a non-trivial topology, and as a prototype example of such domains, we restrict to generalised annuli, that is, domains in the form $\Omega = \{ x \in \mathbb{R}^n : a < |x| < b \}$ with $0 < a < b < \infty$. We proceed by introducing a class of maps, referred to as generalised twists (see Definition 3.1.1) and examine them as possible solutions to the system of Euler-Lagrange equations associated with $F$ (both in the so-called compressible and incompressible cases); the second half focuses on the other extreme, that is, domains $\Omega$ with a trivial topology, where, here, the prototype example are starshaped domains. We address the question of uniqueness of extremals and strong local minimizers using a method reminiscent of that in [71] by Taheri.
1.2 Outline of the Thesis

To outline in more detail the plan of the thesis and a discussion of the results. In the second chapter bring together some basic properties of the space of self-maps of generalised annuli that are required for the development of the thesis. In the third chapter, we take $\Omega \subset \mathbb{R}^n$ such annulus and consider the energy functional (1.1) over the space of admissible maps

$$\mathcal{A}_p(\Omega) := \left\{ u \in W^{1,p}_\varphi(\Omega, \mathbb{R}^n) : \text{det} \nabla u = 1 \text{ a.e. in } \Omega \right\},$$

where

$$W^{1,p}_\varphi(\Omega, \mathbb{R}^n) = \left\{ u \in W^{1,p}(\Omega, \mathbb{R}^n) : u|_{\partial \Omega} = \varphi \right\}.$$

Here $W^{1,p}(\Omega, \mathbb{R}^n)$ is the standard Sobolev space of vector valued $L^p$ integrable functions defined on $\Omega$, having $L^p$ integrable distributional derivatives, and equipped with the norm

$$\|u\|_{W^{1,p}}^p := \|u\|^p_{L^p} + \|\nabla u\|^p_{L^p}.$$

Our terminology throughout is in agreement with that used by Adams [1] and Ziemer [79]. The boundary map $\varphi$ is taken linear; indeed the case $\varphi = x$ (identity) is of particular interest to us.

With regards to the integrand $F$ we assume $F : \mathbb{M}_{n \times n} \to \mathbb{R}$ to be continuous and to satisfy the following set of hypotheses:

[H1] (Growth condition) There exists $c_1 > 0$ such that for all $\xi \in \mathbb{M}_{n \times n}$ we have that

$$|F(\xi)| \leq c_1 (1 + |\xi|^p).$$

[H2] (Coercivity condition) There exists $c_2 > 0$ such that for all $\xi \in \mathbb{M}_{n \times n}$ we have that

$$c_2 |\xi|^p - c_1 \leq F(\xi).$$

[H3] (Quasiconvexity at $\xi$) For fixed $\xi \in \mathbb{M}_{n \times n}$ and all $\phi \in \mathcal{C}_c^\infty(\Omega, \mathbb{R}^n)$ we have that

$$\int_{\Omega} \left( F(\xi + \nabla \phi(x)) - F(\xi) \right) dx \geq 0.$$

If, additionally, the inequality is strict for $\phi \neq 0$ then $F$ is referred to as being strictly quasiconvex at $\xi$. (If the subscript $\xi$ is omitted $F$ is taken quasiconvex everywhere.)

In Chapter 3 we are primarily concerned with the problem of extremising the energy functional
(1.1) over the space (1.2) and examining a class of maps of topological significance as solutions to the associated system of Euler-Lagrange equations

\[
\begin{align*}
\text{div } \mathcal{G}[x, \nabla u(x)] &= 0 \quad x \in \Omega, \\
\det \nabla u(x) &= 1 \quad x \in \Omega, \\
u(x) &= \varphi(x) \quad x \in \partial \Omega,
\end{align*}
\]

where, we have set

\[
\mathcal{G}[x, \xi] = F_\xi(\xi) - p(x)\xi^{-t} =: \mathcal{T}[x, \xi] \xi^{-t},
\]

for \( x \in \Omega, \xi \in M_{n \times n} \) satisfying \( \det \xi = 1 \) and \( p \) a suitable Lagrange multiplier while

\[
\mathcal{T}[x, \xi] = F_\xi(\xi)\xi^{-t} - p(x)I_n.
\]

In the language of elasticity, the tensor fields (1.4) and (1.5) are referred to as the Piola-Kirchhoff and the Cauchy stress tensors respectively and the Lagrange multiplier \( p \) is better known as the hydrostatic pressure i.e., see Ciarlet [20].

While the linear map \( u = \varphi \) serves as the unique minimizer of \( F \) over \( \mathcal{A}_p(\Omega) \) little is known about the structure and features of the solution set to this system of Euler-Lagrange equations e.g., multiplicity versus uniqueness, existence of strong local minimizers, partial regularity, the nature and form of singularities, symmetries, etc. (see, e.g., [4], [8], [9], [24], [44], [52], [55], [68]).

We contribute towards understanding aspects of these questions by way of presenting multiple solutions to the above system of equations. For most of Chapter three we specialise to \( F(\xi) = p^{-1} |\xi|^p \) \((p > 1)\), that is, the so-called \( p \)-Dirichlet energy and proceed by introducing a class of maps, referred to as generalised twists, characterised and defined by

\[
u(x) = Q(r)x,
\]

where \( Q \in C([a,b], \text{SO}(n)) \) and \( r = |x| \). To ensure admissibility, i.e., \( u \in \mathcal{A}_p(\Omega) \) it suffices to impose a further \( p \)-summability on \( \dot{Q} := dQ/dr \) along with \( Q(a) = Q(b) = I_n \). Restricting the \( p \)-energy to the space of such twists we can write

\[
\mathcal{E}_p(\mathcal{Q}) := p \mathcal{E}_p[Q(r)x, \Omega].
\]
\[ \int_{a}^{b} E(r, \dot{Q}) r^{n-1} \, dr, \]

where the integrand itself is given through an integral over the unit sphere, i.e.,

\[ E(r, \xi) := \int_{S^{n-1}} (n + r^2|\xi\theta|^2)^{\frac{p}{2}} dH^{n-1}(\theta). \]

Here, the Euler-Lagrange equation can be shown to be the second order ordinary differential equation

\[ \frac{d}{dr} \left\{ r^{n-1} \left[ E_{\xi}(r, \dot{Q}) \dot{Q}' - QE'_{\xi}(r, \dot{Q}) \right] \right\} = 0. \]

Now in order to characterise among solutions to the above equation, all those which grant a solution to the Euler-Lagrange equations associated with \( F_p \) over \( A_p(\Omega) \) we are confronted with the task of obtaining necessary and sufficient conditions on the vector field

\[ [\nabla u]^{t} \Delta_{p} u = \nabla s + \left\{ r s A^2 - r^2 s (A\theta, \dot{A}\theta) I_n + \frac{1}{r^n} \frac{d}{dr} (r^{n+1} s A) + \frac{1}{r^{n-1}} \frac{d}{dr} (r^{n+1} s |A\theta|^2) I_n \right\} \theta, \]

with \( A = Q'\dot{Q} \) and \( s = (n + r^2|\dot{Q}\theta|)^{\frac{p-2}{2}} \) for it to be a gradient, specifically, to coincide with \( \nabla p \). This analysis occupies a major part of this chapter and is fully settled in Theorem 3.4.2 and Theorem 3.4.3.

**Figure 1.3:** A schematic of how a generalised twist deforms a vertical plane in a three dimensional annulus.

The conclusion that the above analysis bares on to the original Euler-Lagrange equations turns to be a surprising discrepancy between even and odd dimensions. Indeed it follows that in even dimensions the latter system of equations admits infinitely many smooth solutions, modulo isometries, in the form of generalised twists whilst in odd dimensions this number severely reduces to one.
In Chapter 4, we consider $\Omega \subset \mathbb{R}^n$ to be a bounded starshaped domain and the energy functional (1.1) over the space of admissible maps (1.2) when $p \in [1, \infty]$ and while $\varphi = \bar{\xi}x$ and $\bar{\xi} \in M_{n \times n}$ with $\det \bar{\xi} = 1$. Here the integrand $F : M_{n \times n} \to \mathbb{R}$ is of class $C^1$ and for future reference we associate with it the set of hypotheses $[H_1]$, $[H_2]$, $[H_3]$, and $[H_4]_\xi$ (Rank-one convexity at $\xi$) For fixed $\xi \in M_{n \times n}$ and all rank-one $\zeta \in M_{n \times n}$ the function

$$\mathbb{R} \ni t \mapsto F(\xi + t\zeta) \in \mathbb{R},$$

is convex at $t = 0$. If the subscript $\xi$ is omitted $F$ is taken rank-one convex everywhere. \(^5\)

Here we are primarily concerned with the question of uniqueness for solutions to the system of Euler-Lagrange equations (1.3), associated with the energy functional (1.1) over the space of (1.2), as well as that for its strong local minimizers (see Definition 2.2.1).

Indeed, the former, under the stated $C^1$ regularity assumption on $F$, the question of uniqueness of solutions to the associated system of Euler-Lagrange equations [subject to linear boundary conditions] was established in a seminal paper of Knops & Stuart (see [44]). There it is shown that subject to $F$ being of class $C^2$, rank-one convex everywhere and strictly quasiconvex at $\bar{\xi}$ any smooth solution $u$ in a starshaped domain satisfying $\det \nabla u = 1$ in $\Omega$ and $u = \bar{\xi}x$ on $\partial \Omega$ satisfies $u = \bar{\xi}x$ on $\bar{\Omega}$.

In this short chapter we give a new proof of the aforementioned uniqueness result of Knops & Stuart [44]. This is based on firstly removing the measure preserving condition $\det \nabla u = 1$ and considering instead a suitable unconstrained functional [with the aid of the Lagrange multiplier $p$] and secondly utilising the so-called stationarity condition followed by comparison with homogeneous degree-one extensions as introduced in [71] by Taheri. This approach has the advantage of extending the uniqueness result to all weak solutions $u$ of class $C^1$ satisfying the weak form of the stationarity condition (see (4.4) below).

Finally we prove a new uniqueness result for strong local minimizers of $F$ over $A_p(\Omega)$ to the effect that subject to $[H_1]$, $[H_3]_\xi$ alone any such $u \in A_p(\Omega)$ satisfies $F[u, \Omega] = F[\bar{\xi}x, \Omega]$ and therefore subject to the additional strictly quasiconvexity of $F$ at $\xi$ it must be that $u = \bar{\xi}x$ on $\Omega$! We note that in this chapter for technical reasons one needs to restrict to $p \in [n, \infty]$ for the multiplicity result relating to strong local minimizers and to $p \in ]1, \infty[$ for the one relating to smooth solutions.

\(^5\)For a comprehensive treatment of the convexity notions $[H_3]$, $[H_4]$ and their significance in the Calculus of Variations we refer the interested reader to the books [11], [17], [22] and [30]-[32].
In the final Chapter, we return to the domain $\Omega$ in Chapter 3, i.e., generalised annulus, and consider the integral functionals $F$ given by

$$F[u, \Omega] := \int_{\Omega} \left[ \frac{1}{2} |\nabla u|^2 + \phi(\det \nabla u) \right] dx,$$

(1.6)

over the space of admissible maps

$$\mathcal{A}(\Omega) = \left\{ u \in W^{1,2}_\phi(\Omega; \mathbb{R}^n) : \det \nabla u > 0 \ a.e. \ in \ \Omega \right\},$$

(1.7)

where

$$W^{1,2}_\phi(\Omega) := \left\{ u \in W^{1,2}(\Omega; \mathbb{R}^n) : u|_{\partial \Omega} = \varphi \right\},$$

and where $\varphi$ is the identity map.

Regarding the function $\phi$ appearing in the energy functional $F$ we make the following set of hypotheses.

- [h1] $\phi : [0, \infty] \rightarrow [0, \infty]$,
- [h2] $\phi$ is convex,
- [h3] $\phi \in C^2([0, \infty])$,
- [h4] $\phi$ has the two limiting behaviours

$$\lim_{s \downarrow 0} \phi(s) = \lim_{s \uparrow \infty} \frac{\phi(s)}{s} = \infty,$$

(1.8)

- [h5] there exists $\beta > 0$ and $\delta > 0$ such that for all $s \in [0, \infty]$ and $\alpha > 0$ satisfying $|\alpha - 1| < \delta$ we have that

$$|s\phi'(\alpha s)| \leq \beta[\phi(s) + 1].$$

We are primarily concerned with the task of extremising the energy functional $F$ over the space $\mathcal{A}(\Omega)$ and examining a special class of maps as solutions to the corresponding system of Euler-Lagrange equations which can formally be written as

$$\begin{cases}
\Delta u + \nabla \left[ \phi'(\det \nabla u) \ \text{cof} \ \nabla u \right] = 0 & \text{in } \Omega, \\
\det \nabla u > 0 & \text{in } \Omega, \\
u = \varphi & \text{on } \partial \Omega.
\end{cases}$$
Again we proceed by introducing a class of maps, referred to as generalised twists, characterised and defined by

\[ u(x) = G(r)\theta, \]

with

\[ G(r) = f(r)Q(r), \]

where \( r = |x|, \theta = x/|x|, \) \( Q \in C([a,b], SO(n)) \) and \( f \in C[a,b]. \) In addition, to ensure admissibility, i.e., \( u \in A(\Omega) \) it suffices to impose a further \( L^2 \)-summability on \( f := df/dr \) and \( \dot{Q} := dQ/dr \) along with \( \dot{f} > 0 \) \( L^1 \)-a.e. on \( [a,b] \) while \( f(a) = a, f(b) = b \) and \( Q(a) = Q(b) = I_n. \)

Next by restricting the energy functional \( F \) to the space of such twists we can write

\[ \mathcal{E}[Q,f] := \frac{2}{\omega_n^2} F[u,\Omega] = \int_a^b \left\{ \frac{1}{2} \left[ f^2 \left( \frac{n}{2} \right) + |\dot{Q}|^2 \right] + n\dot{f}^2 + 2n\phi'(\dot{f}) \left( \frac{f}{r} \right)^{n-1} \right\} r^{n-1} \, dr, \]

and the corresponding Euler-Lagrange equations can again be formally shown to be the second order system of ordinary differential equations

\[
\begin{align*}
\frac{d}{dr} \left[ r^{n-1} f^2 \frac{d}{dr} \dot{Q} \right] &= 0, \\
\frac{d}{dr} \left[ r^{n-1} \dot{f} + f^{n-1} \phi' \right] &= (n-1)[r^{n-3} f + \dot{f} f^{n-2} \phi'] + \frac{1}{n} r^{n-1} f |\dot{Q}|^2,
\end{align*}
\]

on \( [a,b] \) where \( \phi' = \phi'(\dot{f} \left( \frac{f}{r} \right)^{n-1}). \)

Now in order to characterise among solutions to the above system all those which grant a solution to the Euler-Lagrange equations associated with \( F \) over \( A(\Omega) \) we are confronted with the task of verifying the necessary and sufficient condition \( \dot{Q}(r) \in \mathbb{R}SO(n) \) on \( [a,b] \). This analysis occupies a major part of the chapter and is fully settled in Theorem 5.5.3 and Theorem 5.6.3.

Again the conclusion that this analysis bares on to the original Euler-Lagrange equations turns to be a similar type of discrepancy between even and odd dimensions as that arose from the model in Chapter 3. Indeed it follows that in even dimensions the latter system of equations admit infinitely many smooth solutions, modulo isometries, in the form of generalised twists whilst in odd dimensions this number severely reduces to one.
We end by noting the following in dealing with the polyconvexity in the last chapter.

[1] It is convenient to extend $\phi$ to the entire real line by setting $\phi(s) = \infty$ for $s \in ]-\infty, 0]$. Evidently with this convention for any $u \in W^{1,2}(\Omega, \mathbb{R}^n)$ we have that

$$F[u, \Omega] < \infty \implies \det \nabla u > 0 \ \mathcal{L}^n\text{-a.e. in } \Omega.$$ 

However notice that the reverse implication, in general, need not be true.

[2] The system of Euler-Lagrange equations associated with $F$ for any solution of class $C^2$ can alternatively be expressed as (see Definition 5.6.2)

$$
\begin{cases}
\quad [\nabla u]^t \Delta u + \det \nabla u \nabla \left[ \phi'(\det \nabla u) \right] = 0 \quad \text{in } \Omega, \\
\quad \det \nabla u > 0 \\ 
\quad u = \varphi \\
\end{cases}
$$

in $\Omega$, and on $\partial \Omega$.

This being a consequence of the so-called Piola’s identity (see, e.g., Morrey [48] pp. 122) and the pointwise invertibility of the gradient matrix.
Chapter 2

Continuous self-maps of annuli

The aim of this chapter is to describe the topology of the space of orientation preserving Sobolev maps on $n$-dimensional annuli that are required for the development of the thesis. At the heart of this investigation lies the profound problem of enumerating the path-connected components of its associated space of self-map. The material in this chapter is taken from Shahrokhi-Dehkordi & Taheri [60] and Taheri [68].

Assume to begin that $\Omega = \{ x \in \mathbb{R}^n : a < |x| < b \}$ with $0 < a < b < \infty$ and that the boundary data $\varphi$ in (1.2) is taken $\varphi = x$, the identity map. Then it can be shown that the space of Sobolev maps

$$\mathcal{A}_p(\Omega) := \left\{ u \in W^{1,p}(\Omega, \mathbb{R}^n) : \det \nabla u > 0 \text{ a.e. in } \Omega, u|_{\partial\Omega} = x \right\},$$

with $p \geq n$ embeds continuously and compactly into the space of self-maps of $\Omega$, that is

$$\mathfrak{A}(\Omega) := \left\{ \phi \in \mathcal{C}(\bar{\Omega}, \bar{\Omega}) : \phi(x) = x \text{ for } x \in \partial\Omega \right\}. \quad (2.1)$$

Here $\mathfrak{A} := \mathfrak{A}(\Omega)$ is equipped with the topology of uniform convergence. The reader is referred to Morrey [48] for this last statement or Taheri [67], [70] and [73] for further details and proofs.

2.1 Degree of continuous self-maps on annuli

Definition 2.1.1. (Homotopy)

A pair of maps $\phi_0, \phi_1 \in \mathfrak{A}$ are referred to as homotopic if and only if there exists a continuous map $h : [0, 1] \times \bar{\Omega} \to \bar{\Omega}$ such that

11
Chapter 2. Continuous self-maps of annuli

1. $h(0, x) = \phi_0(x)$ for all $x \in \Omega$,
2. $h(1, x) = \phi_1(x)$ for all $x \in \Omega$,
3. $h(t, x) = x$ for all $t \in [0, 1]$ and $x \in \partial \Omega$.

The collection of all maps homotopic to $\phi \in \mathfrak{A}$ is referred to as the homotopy class of $\phi$ and denoted by $[\phi]$. In order to give a characterisation of the homotopy classes $\{[\phi] : \phi \in \mathfrak{A}\}$, below, we consider the cases $n = 2$ and $n \geq 3$ separately.

**The case ($n = 2$).** Fix $\phi \in \mathfrak{A}$. Then, using polar coordinates, for $\theta \in [0, 2\pi]$ (fixed) the $S^1$-valued curve
\[
\gamma^\theta(r) = \frac{\phi}{|\phi|}(r, \theta) : [a, b] \to S^1,
\]
has a well-defined index or winding number about the origin. Furthermore, in view of continuity of $\phi$, this is independent of the particular choice of $\theta \in [0, 2\pi]$. The latter correspondence will be denoted by
\[
\phi \mapsto \deg\left(\frac{\phi}{|\phi|}\right).
\]
Note that this integer also agrees with the Brouwer degree of the map resulting from identifying $S^1 \cong [a, b]/\{a, b\}$, justified as a result of $\gamma_0(a) = \gamma_0(b)$. On the other hand for a differentiable curve (taking advantage of $S^1 \subset \mathbb{C}$) we specifically have the formula
\[
\deg\left(\frac{\phi}{|\phi|}\right) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z}.
\]

**Theorem 2.1.2.** Let $n = 2$. Then, the map
\[
\deg() : \left\{[\phi] : \phi \in \mathfrak{A}\right\} \to \mathbb{Z},
\]
is a bijection. Moreover, for a pair of maps $\phi_0, \phi_1 \in \mathfrak{A}$, we have that
\[
[\phi_0] = [\phi_1] \iff \deg\left(\frac{\phi_0}{|\phi_0|}\right) = \deg\left(\frac{\phi_1}{|\phi_1|}\right).
\]

**The case ($n \geq 3$).** Fix $\phi \in \mathfrak{A}$. Then, using the identification $\tilde{\Omega} \cong [a, b] \times S^m$, with $m = n - 1$, it is plain that the map
\[
\omega[r](\cdot) = \frac{\phi}{|\phi|}(r, \cdot) : [a, b] \to \mathcal{C}_\varphi(S^m, S^m),
\]
uniquely defines an element of the group $\pi_1(\mathcal{C}_\varphi(S^m, S^m))$. Where $\varphi$ denoted as identity map of the $m$-sphere and $\mathcal{C}_\varphi(S^m, S^m)$ is the path-connected component of $\mathcal{C}(S^m, S^m)$ containing $\varphi$. By considering the action of $\text{SO}(n)$ on $S^m$ (viewed as its group of orientation preserving isometries, i.e.,
Chapter 2. Continuous self-maps of annuli

through

$$\Phi[\cdot] : \text{SO}(n) \mapsto C(S^n, S^m),$$

where $$\Phi[\xi](x) = \xi x$$ for $$x \in S^m$$ it can be shown that the latter induces an isomorphism between

$$\pi_1[\text{SO}(n)] \cong \mathbb{Z}_2 \quad \text{and} \quad \pi_1[C_{\varphi}(S^n, S^m)].$$

Thus, we are naturally lead to the correspondence

$$\phi \mapsto \deg_2(\frac{\phi}{|\phi|}) \in \mathbb{Z}_2.$$

**Theorem 2.1.3.** Let $$n = 3$$. Then, the map

$$\deg_2(\cdot) : \left\{ \left[ \phi \right] : \phi \in \mathfrak{A} \right\} \to \mathbb{Z}_2,$$

is a bijection. Moreover, for a pair of maps $$\phi_0, \phi_1 \in \mathfrak{A}$$, we have that

$$\left[ \phi_0 \right] = \left[ \phi_1 \right] \iff \deg_2(\frac{\phi_0}{|\phi_0|}) = \deg_2(\frac{\phi_1}{|\phi_1|}).$$

**Remark 2.1.4.** In the case of a punctured ball, say, $$\Omega = \mathbb{B} \setminus \{0\}$$ for any pair of maps $$\phi_0, \phi_1 \in \mathfrak{A} := \{ \phi \in C(\overline{\Omega}, \overline{\Omega}) : \phi = \varphi \text{ on } \partial \Omega = \{0\} \cup \partial \mathbb{B} \}$$ the continuous path $$[0, 1] \ni t \mapsto \phi_t := (1 - t)\phi_0 + t\phi_1$$ lies within $$\mathfrak{A}$$ and joins $$\phi_0$$ to $$\phi_1$$. Therefore, here, $$\mathfrak{A}$$ consists of a single component only!

### 2.2 Homotopy characterisation and strong local minimizers

Let $$\Omega$$ as before be an $$n$$-dimensional annulus and $$\mathcal{A}_p(\Omega)$$ the space defined in (1.2). When $$p \in [n, \infty[$$ by taking advantage of the embedding $$\mathcal{A}_p(\Omega) \subset \mathfrak{A}(\Omega)$$ it follows that every $$u \in \mathcal{A}_p := \mathcal{A}_p(\Omega)$$ has a representative (again, denoted $$u$$) in $$\mathfrak{A}$$. Hence, we can set,

[1] ($$n = 2$$) for each $$m \in \mathbb{Z},$$

$$c_m[\mathcal{A}_p] := \left\{ u \in \mathcal{A}_p : \deg\left(\frac{u}{|u|}\right) = m \right\}. \quad (2.2)$$

As a result the latter are pairwise disjoint and that

$$\mathcal{A}_p = \bigcup_{m \in \mathbb{Z}} c_m[\mathcal{A}_p].$$

[2] ($$n \geq 3$$) for each $$\alpha \in \mathbb{Z}_2 = \{0, 1\},$$

$$c_\alpha[\mathcal{A}_p] := \left\{ u \in \mathcal{A}_p : \deg_2\left(\frac{u}{|u|}\right) = \alpha \right\}. \quad (2.3)$$
As a result, again, the latter are pairwise disjoint and that

\[ \mathcal{A}_p = \bigcup_{\alpha \in \mathbb{Z}_2} c_\alpha[A_p]. \]

**Definition 2.2.1. (Strong local minimizer)**

A map \( \bar{u} \in A_p(\Omega) \) is a strong local minimizer of the functional \( F \), given by (1.1), if and only if there exists \( \delta = \delta(\bar{u}) > 0 \) such that \( F[\bar{u}, \Omega] \leq F[u, \Omega] \) for all \( u \in A_p(\Omega) \) satisfying \( \|\bar{u} - u\|_{L^1} < \delta \).

Over the next two propositions, we will show that the homotopy classes of \( c_\ast[A_p] \) are sequentially weakly closed, hence by minimizing \( F \) on each homotopy class we arrive at a strong local minimizer.

Note that when \( p \in [1, n] \) this argument encounters two serious obstacles, firstly, there is no embedding of \( A_p(\Omega) \) into \( A(\Omega) \), and secondly, the determinant function fails to be sequentially weakly continuous.

**Proposition 2.2.2.** Let \( \Omega = \{ x \in \mathbb{R}^n : a < |x| < b \} \) be a generalised annulus and \( F \) an integrand satisfying [H2]. Fix \( p \in [n, \infty[ \), and consider the classes \( c_\ast[A_p] \) as defined either by (2.2) or (2.3).

Then,

1. \( c_\ast[A_p] \) is \( W^{1,p} \)-sequentially weakly closed,
2. for \( u \in c_\ast[A_p] \) and \( s > 0 \) there exists \( \delta = \delta(u, s) > 0 \) such that

\[
\begin{align*}
\{ v \in A_p, \\
\|v - u\|_{L^1} < \delta, \\
F[v, \Omega] < s, \\
\}
\Rightarrow
v \in c_\ast[A_p],
\end{align*}
\]

**Proof.**

1. Let \( (u_j)_{j \in \mathbb{N}} \subset c_\ast[A_p] \) and \( u_j \rightharpoonup u \) in \( W^{1,p}(\Omega, \mathbb{R}^n) \). Then, as a result of \( p \geq n \), by passing to a subsequence (not re-labeled) we have

\[
\det \nabla u_j \rightharpoonup \det \nabla u,
\]

in \( \mathcal{M}(\Omega) \) and so \( u \in A_p \). Moreover, in view of \( u_j \to u \) uniformly on \( \Omega \), an application of Theorem 2.1.2 or Theorem 2.1.3 gives \( u \in c_\ast[A_p] \). This justifies [1].

2. Assume the contrary. Then, there exists \( u \in c_\ast[A_p] \), \( s > 0 \) and \( (v_j)_{j \in \mathbb{N}} \) such that

\[
\begin{align*}
\{ v_j \in A_p, \\
\|v_j - u\|_{L^1} \to 0, \\
F[v_j, \Omega] < s, \\
\}
\]
while $v_j \notin \mathfrak{c}_*[\mathcal{A}_p]$. However the above imply that by passing to a subsequence (not re-labeled) $v_j \to u$ in $W^{1,p}(\Omega,\mathbb{R}^n)$ and as in [1], $v_j \to u$ uniformly on $\bar{\Omega}$. Hence, again either by Theorem 2.1.2 or Theorem 2.1.3,

$$\deg_*(\frac{v_j}{|v_j|}) \to \deg_*(\frac{u}{|u|}).$$

As the above quantities are integers (and the one on the right being a constant) it follows that for $j$ large enough $v_j \in \mathfrak{c}_*[\mathcal{A}_p]$ which is a contradiction. This completes the proof. □

In view of the sequential weak lower semicontinuity of the energy functional $F$, an application of the direct methods of the calculus of variations leads us to the following conclusion.

**Proposition 2.2.3.** Let $\Omega = \{x \in \mathbb{R}^n : a < |x| < b\}$ be a generalised annulus and $F$ an integrand satisfying $[H1]$, $[H2]$ and $[H3]$. Fix $p \in [n,\infty[$ and consider the classes $\mathfrak{c}_*[\mathcal{A}_p]$ as defined either by (2.2) or (2.3). Then, there exists

$$\bar{u} = \bar{u}(x;a,b,\ast) \in \mathfrak{c}_*[\mathcal{A}_p],$$

such that

$$F[\bar{u},\Omega] = \inf_{\mathfrak{c}_*[\mathcal{A}_p]} F[\cdot,\Omega].$$

(2.4)

In addition, for each such $\bar{u}$ there exists $\delta = \delta(\bar{u}) > 0$ such that

$$F[\bar{u},\Omega] \leq F[v,\Omega],$$

(2.5)

for all $v \in \mathcal{A}$ satisfying $\|\bar{u} - v\|_{L^1} < \delta$.

**Proof.** Let $(v_j)_{j \in \mathbb{N}} \subset \mathfrak{c}_*[\mathcal{A}_p]$ be an infimizing sequence, $F[v_j,\Omega] \downarrow \alpha := \inf_{\mathfrak{c}_*[\mathcal{A}_p]} F[\cdot,\Omega]$. Then as $\alpha < \infty$ it follows that by passing to a subsequence (not re-labeled) $v_j \to \bar{u}$ in $W^{1,p}(\Omega,\mathbb{R}^n)$ where by [1] in Proposition 2.2.2, $\bar{u} \in \mathfrak{c}_*[\mathcal{A}_p]$. As a result we can write

$$\alpha \leq F[\bar{u},\Omega] \leq \liminf_{j \to \infty} F[v_j,\Omega] \leq \alpha,$$

and so $\bar{u}$ is a minimizer as required.

To establish the final assertion, fix $\ast$ and $\bar{u}$ as above and with $s = 1 + F[\bar{u},\Omega]$ pick $\delta > 0$ as [2] in Proposition 2.2.2. Then, any $v \in \mathcal{A}_p$ satisfying $\|\bar{u} - v\|_{L^1} < \delta$ also satisfies (2.5). Otherwise $F[v,\Omega] < F[\bar{u},\Omega] < s$ implying that $v \in \mathfrak{c}_*[\mathcal{A}_p]$ and hence in view of $\bar{u}$ being a minimizer, $F[v,\Omega] \geq F[\bar{u},\Omega]$ which is a contradiction. □
Chapter 3

Measure-preserving maps and generalised twists

In this chapter we introduce a class of maps referred to as generalised twists and examine them in connection with the Euler-Lagrange equations associated with the $p$-Dirichlet energy

$$F_p[u, \Omega] := p^{-1} \int_{\Omega} |\nabla u(x)|^p \, dx,$$

with $p \in ]1, \infty[\), over the space of measure preserving maps (1.2). The main result is an interesting discrepancy between even and odd dimensions. Here we show that in even dimensions the latter system of equations admits infinitely many smooth solutions, modulo isometries, amongst such maps. In odd dimensions this number reduces to one. The result relies on a careful analysis of the full versus the restricted Euler-Lagrange equations where a key ingredient is a necessary and sufficient condition for an associated vector field to be a gradient. The material in this chapter is taken from Shahrokhi-Dehkordi & Taheri [59], [60] and partly [69] by Taheri.

3.1 Generalised twists

We begin this section by introducing a class of maps, referred to as generalised twists and then proceed to study some properties of these maps.

Definition 3.1.1. (Generalised twists)

Let $\Omega = \{ x \in \mathbb{R}^n : a < |x| < b \}$. A map $u \in \mathcal{C}(\bar{\Omega}, \bar{\Omega})$ is referred to as a generalised twists if and
only if it can be expressed as
\[ u(x) = G(r)\theta, \tag{3.2} \]
with
\[ G(r) = f(r)Q(r), \]
where \( r = |x|, \theta = x/|x|, Q \in C([a, b], SO(n)) \) and \( f \in C[a, b] \).

When \( n = 2 \) and \( f \equiv r \) a generalised twist can be shown to take, in polar coordinates, the alternative form
\[ (r, \theta) \mapsto (r, \theta + g(r)), \tag{3.3} \]
for a suitable \( g \in C[a, b] \). Maps of the type (3.3) frequently arise in the study of mapping class groups of surfaces and are better known as Dehn-twists, e.g., see Dehn [23]. In higher dimensions, by contrast, no such simple representation of (3.2) is feasible in generalised spherical coordinates, however, the terminology here is suggested by analogy with (3.3) when \( n = 2 \). The continuous function \( G \) in the above definition will be referred to as the twist path. When additionally \( G(a) = G(b) \) we refer to \( G \) as the twists loop.

Notice that as a result of the basic requirement \( \det \nabla u = 1 \) a.e. in \( \Omega \) built into the definition of a generalised twists it follows in particular we assume \( f \equiv r \) in \([a, b], \) see equation (5.2) in Proposition 5.1.1. Therefore along this chapter we assume always \( f(r) = r \) on \([a, b] \).

**Proposition 3.1.2.** Let \( \Omega = \{ x \in \mathbb{R}^n : a < |x| < b \} \). A generalised twist \( u \) lies in \( A_p = A_p(\Omega) \) with \( p \in [1, \infty[ \) provided that the following hold.

1. \( Q \in W^{1,p}([a, b], SO(n)) \),
2. \( Q(a) = Q(b) = I_n \).

Thus, in particular, when a generalised twist \( u \) lies in \( A_p \) its corresponding twist path forms a loop in the pointed space \((SO(n), I_n)\).

**Proof.** Assume that \( u \) is a generalised twist. Then \( u \in A_p(\Omega) \) if and only if the following hold.

(i) \( u = x \) on \( \partial \Omega \),
(ii) \( \det \nabla u = 1 \) in \( \Omega \), and,
(iii) \( \|u\|_{W^{1,1}(\Omega)} < \infty \).

Evidently [2] gives (i). Moreover, a straight-forward calculation gives
\[ \nabla u = Q + r\dot{Q}\theta \otimes \theta \]
where \( r = \| x \|, \theta = x / \| x \| \) and \( \dot{Q} := dQ / dr \). Hence in view of \( \det Q = 1 \) we can write

\[
\det \nabla u = \det (Q + r \dot{Q} \theta \otimes \theta) = \det (I_n + r Q' \dot{Q} \theta \otimes \theta) = 1 + (Q' \dot{Q} \theta, \theta) = 1 + (\dot{Q} \theta, Q \theta) = 1,
\]

where in the last identity we have used the fact that \( \langle Q \theta, Q \theta \rangle = \| \theta \|^2 = 1 \) for all \( \theta \in S^{n-1} \) and so as a result

\[
\frac{d}{dr} \langle Q \theta, Q \theta \rangle = \langle Q \theta, \dot{Q} \theta \rangle + \langle \dot{Q} \theta, Q \theta \rangle = 0.
\]

This therefore gives (ii). Finally, to justify (iii) we first note that

\[
|\nabla u|^2 = tr \left\{ |\nabla u|^2 |\nabla u| \right\} = tr \left\{ (Q + r \dot{Q} \theta \otimes \theta)(Q' + r \theta \otimes \dot{Q} \theta) \right\} = n + 2r \langle Q \theta, \dot{Q} \theta \rangle + r^2 \langle \dot{Q} \theta, \dot{Q} \theta \rangle.
\]

Therefore as a result of \( \langle Q \theta, \dot{Q} \theta \rangle = 0 \) for any \( p \in [1, \infty) \) we have that

\[
|\nabla u|^p = (n + r^2 |\dot{Q} \theta|^2)^{\frac{p}{2}}.
\]

Hence in view of \( |u| = r \sqrt{\langle Q \theta, Q \theta \rangle} = r \) we can write

\[
\int_{\Omega} |u|^p + |\nabla u|^p = \int_a^b \int_{S^{n-1}} \left\{ r^p + (n + r^2 |\dot{Q} \theta|^2)^{\frac{p}{2}} \right\} r^{n-1} d\mathcal{H}^{n-1}(\theta) dr,
\]

and so referring to [1] the conclusion follows. \( \square \)

**Proposition 3.1.3.** Suppose that \( u \) is a generalised twist with the associated twist path \( Q \in C^2([a, b], SO(n)) \). Then for \( p \in [1, \infty] \) we have that

\[
\Delta_p u := \text{div}(|\nabla u|^{p-2}) \nabla u = Q \left[ \nabla s \otimes \theta + \frac{1}{r^n} \frac{d}{dr} \left( r^{n+1} s A \right) + rs A^2 \right] \theta,
\]

(3.4)
where $A = Q^t \dot{Q}$ and $s = s(r, \theta) := (n + r^2|A\theta|^2)^{\frac{p-2}{2}}$.

Proof. [1] ($p = 2$) Referring to Definition 3.1.1 and using the notation $u = (u_1, u_2, \ldots, u_n)$ we can write with the aid of (3.4) in Proposition 3.1.2 that

$$
\Delta u_i = \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left\{ Q_{ij} + r \sum_{k=1}^{n} \dot{Q}_{ik} \theta_k \theta_j \right\} = \sum_{j=1}^{n} \left\{ \dot{Q}_{ij} \theta_j + \theta_j \sum_{k=1}^{n} \dot{Q}_{ik} \theta_k + r \sum_{k=1}^{n} \dot{Q}_{ik} \theta_k \theta_j + \right.
$$

$$
\left. \sum_{k=1}^{n} \dot{Q}_{ik} (\delta_{kj} - \theta_j \theta_k) \theta_j + \sum_{k=1}^{n} \dot{Q}_{ik} \theta_k (1 - \theta_j \theta_j) \right\} = 2 \sum_{j=1}^{n} \dot{Q}_{ij} \theta_j + r \sum_{j=1}^{n} \ddot{Q}_{ij} \theta_j + (n - 1) \sum_{j=1}^{n} \dot{Q}_{ij} \theta_j = (n + 1) \sum_{j=1}^{n} \dot{Q}_{ik} \theta_k + r \sum_{j=1}^{n} \ddot{Q}_{ij} \theta_j.
$$

As this is true for $1 \leq i \leq n$ going back to the original vector notation and using the substitutions $\dot{Q} = QA$ and $\ddot{Q} = Q[\dot{A} + A^2]$ we have that,

$$
\Delta u = [(n + 1) \dot{Q} + r \ddot{Q}] \theta = Q[(n + 1) \dot{A} + r \dot{A} + r A^2] \theta.
$$

which is the required result for $p = 2$. [Note that in this case $s = s(r, \theta) \equiv 1$.]

[2] ($p \in [1, \infty]$) According to definition we have that

$$
\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u) = \text{div}(s \nabla u) = \nabla u \nabla s + s \Delta u.
$$

Now a straight-forward differentiation gives

$$
\nabla s = \nabla (n + r^2|\dot{Q}\theta|^2)^{\frac{p-2}{2}} = \nabla (n + r^2|A\theta|^2)^{\frac{p-2}{2}} = \beta \left[ rA^t A + r^2(A\theta, \dot{A}\theta) I_n \right] \theta.
$$
where $\beta = \beta(r, \theta, p) := (p - 2)(n + r^2|A\theta|^2)^\frac{p-4}{2}$. Thus we can write

$$
\Delta_p u = \nabla u \nabla s + s \Delta u
= Q\left[ I_n + rA\theta \otimes \theta \right] \nabla s + sQ\left[ (n + 1)A + r\dot{A} + rA^2 \right] \theta
= Q\nabla s + r\beta Q\left[ A\theta \otimes \theta \right] \left[ rA^tA + r^2(A\theta, \dot{A}\theta)I_n \right] \theta +
\quad sQ\left[ (n + 1)A + r\dot{A} + rA^2 \right] \theta.
$$

In order to further simplify the second term on the right in the last identity we first notice that

$$
s_r := \frac{\partial s}{\partial r} = \frac{\partial}{\partial r} (n + r^2|A\theta|^2)^\frac{p-2}{2}
= \beta \left[ |A\theta|^2 + r^2(A\theta, \dot{A}\theta) \right],
$$

and consequently

$$
rs_r Q A\theta = r\beta Q \left[ |A\theta|^2 + r^2(A\theta, \dot{A}\theta) \right] A\theta
= r\beta Q \left[ A\theta \otimes \theta \right] \times \left[ rA^tA + r^2(A\theta, \dot{A}\theta)I_n \right] \theta.
$$

Therefore substituting back gives

$$
\Delta_p u = Q\left[ \nabla s \otimes \theta + rs_r A + (n + 1)sA + rs\dot{A} + rsA^2 \right] \theta
= Q\left[ \nabla s \otimes \theta + \frac{1}{r^n} \frac{d}{dr} (r^{n+1}sA) + rsA^2 \right] \theta,
$$

which is the required conclusion. \qed

**Proposition 3.1.4.** Suppose that $u$ is a generalised twist with the associated twist path $Q \in C^2([a, b], \text{SO}(n))$. Then for $p \in [1, \infty]$ we have that

$$
\left[ \nabla u \right]^t \Delta_p u = \nabla s + \left\{ rsA^2 - r^2 s(A\theta, \dot{A}\theta)I_n +
\frac{1}{r^n} \frac{d}{dr} (r^{n+1}sA) + \frac{1}{r^{n-1}} \frac{d}{dr} (r^{n+1}|A\theta|^2)I_n \right\} \theta,
$$

where $A = Q^t\dot{Q}$ and $s = s(r, \theta) = (n + r^2|A\theta|^2)^\frac{p-2}{2}$.\[3.7\]
Proof. In view of (3.4) we have that

\[ \nabla u_t = [Q + r\dot{Q}\theta \otimes \theta]^t = [Q^t + r\theta \otimes \dot{Q}\theta] = [I_n + r\theta \otimes A\theta]Q^t. \]

Therefore by substituting for \[\nabla u_t\] and \[\Delta p\] (from the previous proposition) we arrive at

\[
[\nabla u_t] \Delta p u = \left[ I_n + r\theta \otimes A\theta \right] \times \left[ \nabla s \otimes \theta + \frac{1}{r^n} \frac{d}{dr}(r^{n+1} sA) + rsA^2 \right] \theta \\
= \left[ \nabla s \otimes \theta + \frac{1}{r^n} \frac{d}{dr}(r^{n+1} sA) + rsA^2 \right] \theta + \left[ r(\nabla s, A\theta) + \frac{1}{r^{n-1}} \left( \frac{d}{dr}(r^{n+1} sA), A\theta \right) + r^2 s(A^2, A\theta) \right] \theta.
\]

However, in view of \[A\] being skew-symmetric it can be easily verified that \[\langle A^2, A\theta \rangle = 0\] and in a similar way referring to (3.6)

\[
\langle \nabla s, A\theta \rangle = \langle \beta \left[ rA^t A + r^2 (A\theta, \dot{A}\theta) I_n \right] \theta, A\theta \rangle \\
= \beta r \left\{ \langle A^3\theta, \theta \rangle + r(A\theta, \dot{A}\theta) \langle A\theta, \theta \rangle \right\} = 0.
\]

Thus summarising, we have that

\[
[\nabla u_t] \Delta p u = \nabla s + \left\{ rsA^2 + \frac{1}{r^n} \frac{d}{dr}(r^{n+1} sA) + \frac{1}{r^{n-1}} \left( \frac{d}{dr}(r^{n+1} sA), A\theta \right) I_n \right\} \theta \\
= \nabla s + \left\{ rsA^2 - r^2 s(A\theta, \dot{A}\theta) I_n + \frac{1}{r^n} \frac{d}{dr}(r^{n+1} sA) + \frac{1}{r^{n-1}} \frac{d}{dr}(r^{n+1} s[A\theta]^2) I_n \right\} \theta.
\]

The proof is thus complete. \[\square\]

3.2 The \(p\)-energy restricted to the loop space

For a generalised twist \(u\) referring to (3.5) we have for any \(p \in [1, \infty]\) that

\[
\int_{\Omega} |\nabla u|^p = \int_a^b \int_{\mathbb{R}^{n-1}} (n + r^2(|\dot{Q}\theta|^2)^\frac{p}{2} + r^{n-1} dH^{n-1}(\theta)) \, dr.
\]
Motivated by the above representation in this section we introduce the energy functional

\[ E_p[Q] := \int_a^b E(r, \dot{Q}) r^{n-1} \, dr \]

where the integrand itself is given through the integral

\[ E(r, \xi) = \int_{S^{n-1}} (n + r^2|\xi\theta|^2)^{\frac{p}{2}} \, dH^{n-1}(\theta). \]

Associated with the energy functional \( E_p \) and in line with Proposition 3.1.2 we introduce the space of admissible loops

\[ E_p := \left\{ Q = Q(r) : Q \in W^{1,p}([a, b], SO(n)), \quad Q(a) = Q(b) = I_n. \right\} \]

Our primary objective here is to obtain the Euler-Lagrange equation associated with the energy functional \( E_p \) over the space of loops \( E_p \). In doing so the following observation will prove useful.

**Proposition 3.2.1.** Let \( Q \in SO(n) \) and \( R \in M_{n \times n} \). Then the following are equivalent:

1. \( RQ' + QR' = 0 \),
2. \( R = (F - F^t)Q \) for some \( F \in M_{n \times n} \).

Moreover, \( F \) in [2] is unique if it is assumed skew-symmetric, i.e., \( F^t = -F \).

**Proof.** The implication [2] \( \implies \) [1] follows from a direct verification. For the reverse implication it suffices to assume \( F^t + F = 0 \) and then take \( 2F = RQ' \). \( \square \)

**Proposition 3.2.2.** Let \( p \in [1, \infty] \). Then the Euler-Lagrange equation associated with \( E_p \) over \( E_p \) takes the form

\[ \frac{d}{dr} \left\{ r^{n-1} \left[ E_\xi(r, \dot{Q})Q' - QE_\xi'(r, \dot{Q}) \right] \right\} = 0. \]  \( (3.8) \)

**Proof.** Fix \( Q \in W^{1,p}([a, b], SO(n)) \) and pick a variation \( H \in C_0^\infty([a, b], M_{n \times n}) \). For \( \varepsilon \in \mathbb{R} \) put \( Q_\varepsilon = Q + \varepsilon H \). Then,

\[ Q_\varepsilon Q_\varepsilon' = [Q + \varepsilon H][Q + \varepsilon H]^t \]
\[ = I_n + \varepsilon[HQ' + QH'] + \varepsilon^2 HH'. \]

Hence for \( Q_\varepsilon \) to take values on \( SO(n) \) to the first order it suffices to have

\[ HQ' + QH' = 0, \]
on \([a, b]\). In view of Proposition 3.2.1 this is equivalent to assuming that for some \(F \in C^\infty_0([a, b], M_{n \times n})\) the variation \(H\) has the form

\[
H = (F - F^t)Q.
\]

With this assumption in place we examine the vanishing of the first derivative of the energy, i.e., that indeed

\[
0 = \frac{d}{d\epsilon} \mathcal{E}_p[Q_\epsilon] \bigg|_{\epsilon = 0} = \frac{d}{d\epsilon} \int_a^b E(r, Q_\epsilon) r^{n-1} dr \bigg|_{\epsilon = 0} = \int_a^b \left\{ \frac{\partial E}{\partial \xi}(r, \dot{Q}_r) + \frac{d}{d\epsilon} \dot{Q}_r \right\} r^{n-1} dr \bigg|_{\epsilon = 0} = \int_a^b \left\{ \frac{\partial E}{\partial \xi}(r, \dot{Q}_r) : \left[ (\dot{F} - \dot{F}^t)Q + (F - F^t)\dot{Q} \right] \right\} r^{n-1} dr =: I + II.
\]

We now proceed by evaluating each term separately. Indeed, with regards to the first term we have that

\[
I = \int_a^b \left\{ \frac{\partial E}{\partial \xi}(r, \dot{Q}_r) : \left[ (\dot{F} - \dot{F}^t)Q + (F - F^t)\dot{Q} \right] \right\} r^{n-1} dr = \int_a^b \int_{S^{n-1}} p(n + r^2 |\dot{Q}_r|) \frac{d}{d\epsilon} \left\langle \dot{Q}_r, (F - F^t)\dot{Q} \right\rangle r^{n+1} dr = 0.
\]

On the other hand for the second term a direct verification reveals that

\[
II = \int_a^b \left\{ \frac{\partial E}{\partial \xi}(r, \dot{Q}_r) : (F - F^t)\dot{Q} \right\} r^{n-1} dr = \int_a^b \int_{S^{n-1}} p(n + r^2 |\dot{Q}_r|) \frac{d}{d\epsilon} \left\langle \dot{Q}_r, (F - F^t)\dot{Q} \right\rangle r^{n+1} dr = 0,
\]
as a result of the pointwise identity
\[ \langle \dot{Q} \theta, (F - F^t) \dot{Q} \theta \rangle = 0. \]
Thus, summarising, we have that
\[ \frac{d}{d\varepsilon} \mathcal{E}_\rho \bigg|_{\varepsilon=0} = \int_a^b \left\{ - \frac{d}{dr} \left[ r^{n-1} \frac{\partial \mathcal{E}}{\partial \xi} (r, \dot{Q}) \right] : (F - F^t) \right\} dr = 0. \]

As this is true for every \( F \in \mathcal{C}_0^\infty([a, b], M_{n \times n}) \) it follows that the skew-symmetric part of the tensor field in the brackets in the equation above is zero. This gives the required conclusion. \( \square \)

**Proposition 3.2.3.** The Euler-Lagrange equation associated with \( \mathcal{E}_\rho \) over \( \mathcal{E}_\rho \) can be alternatively expressed as
\[ \int_a^b \int_{S^{n-1}} \left\{ \frac{d}{dr} (r^{n+1} s A) \right\} \theta, (F - F^t) \theta \right\} d\mathcal{H}^{n-1} (\theta) dr = 0, \]
for all \( F \in \mathcal{C}_0^\infty([a, b], M_{n \times n}) \) where \( A = Q^t \dot{Q} \) and \( s = (n + r^2 |A\theta|)^{\frac{n-2}{2}} \).

**Proof.** Referring to the proof of Proposition 3.2.2 and making the substitutions described above for \( A \) and \( s \) we can write
\[ \begin{align*}
0 &= \frac{d}{d\varepsilon} \mathcal{E}_\rho \bigg|_{\varepsilon=0} =: \text{I} \\
&= \int_a^b \left\{ \frac{\partial \mathcal{E}}{\partial \xi} (r, \dot{Q}) : (\dot{F} - \dot{F}^t) \dot{Q} \right\} r^{n-1} dr \\
&= \int_a^b \int_{S^{n-1}} p (r^{n+1} s A) \theta, (\dot{F} - \dot{F}^t) \theta \right\} d\mathcal{H}^{n-1} (\theta) dr \\
&= \int_a^b \int_{S^{n-1}} -p \left\{ \frac{d}{dr} (r^{n+1} s A) \right\} \theta, (F - F^t) \theta \right\} d\mathcal{H}^{n-1} (\theta) dr,
\end{align*} \]
which is the required conclusion. \( \square \)

Any twist loop forming a solution to the Euler-Lagrange equation associated with \( \mathcal{E}_\rho \) over \( \mathcal{E}_\rho \) (as described in the above proposition) will be referred to as a \( p \)-stationary loop.

**Remark 3.2.4.** In view of Proposition 3.2.3 a sufficient condition for an admissible loop \( Q \in \mathcal{E}_\rho \) to be \( p \)-stationary is the stronger condition
\[ \frac{d}{dr} (r^{n+1} s A) = 0. \] (3.9)
Interestingly for \( p = 2 \) the latter is equivalent to the Euler-Lagrange equation described in Proposition 3.2.3 (see [60]). However, in general, i.e., for \( p \neq 2 \), this need not be the case as in the original Euler-Lagrange equation the function \( s \) depends on both \( r \) and \( \theta \). In fact, if, \( s \) were to be independent of \( \theta \) then the Euler-Lagrange equation described in Proposition 3.2.3 could be easily shown to be equivalent to (3.9).
3.3 Minimizing $p$-stationary loops

Consider as in the previous section for $p \in [1, \infty]$ the energy functional

$$E_p(q) = \int_a^b E(r, \dot{q}) r^{n-1} \, dr,$$

with the integrand

$$E(r, \xi) = \int_{\mathbb{S}^{n-1}} (n + r^2 |\xi\theta|^2) \frac{2}{p} d\mathcal{H}^{n-1}(\theta),$$

over the space of admissible loops

$$\mathcal{E}_p := \left\{ q = q(r) : q \in W^{1,p}([a, b], \SO(n)), q(a) = q(b) = I_n \right\}.$$

According to an elementary version of Sobolev embedding theorem any $q \in \mathcal{E}_p$ has a continuous representative (again denoted $q$). Thus each such $q$ represents an element of the fundamental group $\pi_1[\SO(n)]$ which is denoted by $[q]$. As is well-known (see, e.g., Bredon [16])

$$\pi_1[\SO(n)] \cong \begin{cases} \mathbb{Z} & \text{when } n = 2, \\ \mathbb{Z}_2 & \text{when } n \geq 3, \end{cases}$$

and so these facts combined enables one to introduce the following partitioning of the loop space $\mathcal{E}_p$.

[1] $(n = 2)$ for each $m \in \mathbb{Z}$ put

$$\mathcal{c}_m[\mathcal{E}_p] := \left\{ q \in \mathcal{E}_p : [q] = m \right\}.$$

As a result the latter are pairwise disjoint and that

$$\mathcal{E}_p = \bigcup_{m \in \mathbb{Z}} \mathcal{c}_m[\mathcal{E}_p].$$

[2] $(n \geq 3)$ for each $\alpha \in \mathbb{Z}_2 = \{0, 1\}$ put

$$\mathcal{c}_\alpha[\mathcal{E}_p] := \left\{ q \in \mathcal{E}_p : [q] = \alpha \right\}.$$

As a result, again, the latter are pairwise disjoint and that

$$\mathcal{E}_p = \bigcup_{\alpha \in \mathbb{Z}_2} \mathcal{c}_\alpha[\mathcal{E}_p].$$
When $p > 1$ an application of the direct methods of the calculus of variations to the energy functional $E_p$ together with the observation that the homotopy classes $c_\ast [E_p] \subset \mathcal{E}_p$ are sequentially weakly closed gives the existence of multiple minimizing $p$-stationary loops. Note that the sequential weak closedness of the homotopy classes $c_\ast [E_p]$ is a result of $SO(n)$ having a tubular neighbourhood that projects back onto itself and this in turn follows from $SO(n)$ being a smooth compact manifold.

The only missing ingredient in this regard is the following statement implying the coercivity of $E_p$ over $E_p$.

**Proposition 3.3.1.** Let $p \in [1, \infty]$. Then there exists $c = c(n, p) > 0$ such that

$$
\int_{S^{n-1}} |F\theta|^p \, d\mathcal{H}^{n-1}(\theta) \geq c |F|^p,
$$

for every $F \in M_{n \times n}$.

**Proof.** Fix $F \in M_{n \times n}$. Then the non-negative symmetric matrix $F^t F$ is orthogonally diagonalisable, that is, $F^t F = P^t DP$ where $D = \text{diag}(\lambda_1[F^t F], \ldots, \lambda_n[F^t F])$ and $P \in O(n)$. As a result for $\theta \in S^{n-1}$ we can write

$$
|F\theta| = |(F\theta, F\theta)|^{\frac{1}{2}} = |(F^t F\theta, \theta)|^{\frac{1}{2}} = |(P^t DP\theta, \theta)|^{\frac{1}{2}} = |(DP\theta, P\theta)|^{\frac{1}{2}}.
$$

Setting $w := P\theta$ and noting that $O(n)$ acts as the group of isometries on $S^{n-1}$, an application of Jensen’s inequality followed by Hölder’s inequality on finite sequences (see, e.g., [53] or [25]) gives

$$
\left\{ \int_{S^{n-1}} |F\theta|^p \, d\mathcal{H}^{n-1}(\theta) \right\}^{\frac{1}{p}} \geq \int_{S^{n-1}} |F\theta| \, d\mathcal{H}^{n-1}(\theta)
$$

$$
\geq \int_{S^{n-1}} \left\{ \sum_{j=1}^{n} \lambda_j[F^t F]w_j^2(\theta) \right\}^{\frac{1}{2}} \, d\mathcal{H}^{n-1}(\theta)
$$

$$
\geq \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \lambda_j[F^t F] \int_{S^{n-1}} |w_j(\theta)| \, d\mathcal{H}^{n-1}(\theta)
$$

$$
\geq \frac{\alpha_n}{\sqrt{n}} \left\{ \sum_{j=1}^{n} \lambda_j[F^t F] \right\}^{\frac{1}{2}} = \frac{\alpha_n}{\sqrt{n}} |F|.
$$

Hence the conclusion follows with the choice of

$$
c = \alpha_n n^{1 - \frac{2}{p}} \omega_n = \min_{\frac{1}{2} \leq \delta < 1} \left\{ \int_{S^{n-1}} \left| \theta_{j} \right| \, d\mathcal{H}^{n-1}(\theta) \right\}^{p} n^{1 - \frac{2}{p}} \omega_n > 0.
$$

□
Proposition 3.3.2. Let $p \in [1, \infty[$. Then there exists $d = d(n, p, \Omega) > 0$ such that

$$\mathcal{E}_p[Q] \geq d\|Q\|_{W^{1,p}},$$

for all $Q \in \mathcal{E}_p$.

Proof. In view of Proposition 3.3.1 it is enough to note that for $Q \in \mathcal{E}_p$ we can write

$$\mathcal{E}_p[Q] = \int_a^b \int_{\mathbb{S}^{n-1}} (n + r^2|\dot{Q}\theta|^2)^\frac{p}{2} r^{n-1} dH^{n-1}(\theta) dr$$

$$\geq \int_a^b \int_{\mathbb{S}^{n-1}} r^{p+n-1}|\dot{Q}\theta|^p dH^{n-1}(\theta) dr$$

$$\geq c \int_a^b r^{p+n-1}|\dot{Q}|^p dr,$$

and so the conclusion follows by an application of Poincaré inequality. □

Theorem 3.3.3. Let $p \in ]1, \infty[$. Then the following hold.

1. $(n = 2)$ for each $m \in \mathbb{Z}$ there exists $Q_m \in c_m[\mathcal{E}_p]$ such that

$$\mathcal{E}_p[Q_m] = \inf_{c_m[\mathcal{E}_p]} \mathcal{E}_p,$$

2. $(n \geq 3)$ for each $\alpha \in \mathbb{Z}_2$ there exists $Q_\alpha \in c_\alpha[\mathcal{E}_p]$ such that

$$\mathcal{E}_p[Q_\alpha] = \inf_{c_\alpha[\mathcal{E}_p]} \mathcal{E}_p.$$

In either case the resulting minimizers satisfy the corresponding Euler-Lagrange equations (3.8).

We return to the question of existence of multiple $p$-stationary loops having specific relevance to the original energy functional $\mathcal{F}_p$ over the space $\mathcal{A}_p$ towards the end of the paper. Before this, however, we pause to discuss in detail the implications that the original Euler-Lagrange equations [see Definition 3.4.1 below] will exert upon the twist loop associated with a generalised twist.

3.4 Generalised twists as classical solutions

The aim of this section is to give a complete characterisation of all those $p$-stationary loops $Q \in \mathcal{E}_p$ whose resulting generalised twist

$$u = Q(r)x,$$
furnishes a solution to the Euler-Lagrange equations associated with the energy functional \( F_{p} \) over the space \( A_{p} \). To this end we begin by clarifying the notion of a \([\text{classical}]\) solution.

**Definition 3.4.1.** (Classical solution)

A pair \((u, p)\) is said to be a \([\text{classical}]\) solution to the Euler-Lagrange equations associated with the energy functional \((3.1)\) and subject to the constraint \((1.2)\) if and only if

1. \( u \in C^{2}(\Omega, \mathbb{R}^{n}) \cap C(\bar{\Omega}, \mathbb{R}^{n}) \),
2. \( p \in C^{1}(\Omega) \cap C(\bar{\Omega}) \), and
3. \((u, p)\) satisfy the system of equations

\[
\begin{cases}
\text{cof} \nabla u(x)^{-1} \Delta_{p} u(x) = \nabla p(x) & x \in \Omega, \\
\det \nabla u(x) = 1 & x \in \Omega, \\
u(x) = x & x \in \partial \Omega.
\end{cases}
\]

In view of Proposition 3.1.4 the task outlined at the start of this section amounts to verifying that under what additional conditions would the vector field described by the expression on the right in \((3.7)\) be a gradient. The answer to this question is given by the following two theorems.

**Theorem 3.4.2.** Let \( \Omega = \{x \in \mathbb{R}^{n} : a < |x| < b\} \) and consider the vector field \( v \in C^{1}(\Omega, \mathbb{R}^{n}) \) defined in spherical coordinates through

\[
v = \left\{ rsA^{2} - r^{2}s(A\theta, \dot{A}\theta)I_{n} + \frac{1}{r^{n-1}} \frac{d}{dr}(r^{n+1}sA) + \frac{1}{r^{n-1}} \frac{1}{r^{n+1}} d^{2}(r^{n+1}|A\theta|^{2})I_{n} \right\} \theta,
\]

where \( r \in ]a, b[ , \theta \in S^{n-1} , A = A(r) \in C^{1}([a, b[ , M_{n \times n}) \) is skew-symmetric and

\[
s = s(r, \theta) = (n + r^{2}|A\theta|^{2})^{\frac{p-2}{2}}, \tag{3.10}
\]

with \( p \in [1, \infty[ \). Then the following are equivalent.

1. \( v \) is a gradient,
2. \( A^{2} = -\sigma I_{n} \) for some \( \sigma \in C^{1}[a, b[ \) with \( \sigma \geq 0 \) and

\[
\frac{d}{dr}(r^{n+1}sA) = 0. \tag{3.11}
\]

\(^{1}\)Note that \( \Delta_{p} u := \text{div}(|\nabla u|^{p-2} \nabla u) \).
Proof. \([2] \implies [1]\]

Assuming \(\mathbf{A}\) to be skew-symmetric and \(\mathbf{A}^2 = -\sigma \mathbf{I}_n\) it follows that

\[
s = (n + r^2 |\mathbf{A}\theta|^2)^{\frac{p-2}{2}} = (n - r^2 (\mathbf{A}^2\theta, \theta))^{\frac{p-2}{2}} = (n + \sigma r^2)^{\frac{p-2}{2}},
\]

and so in particular \(s = s(r)\). Now referring to (3.11) we can write

\[
0 = \frac{1}{r^n} \left( \frac{d}{dr} (r^{n+1} s \mathbf{A}) \right) \theta, \quad \text{where} \quad t = -p^{-1} (n + r^2 |\mathbf{A}\theta|^2)^{\frac{p-2}{2}}.
\]

As a result the vector field \(\mathbf{v}\) can be simplified and hence re-written in the form

\[
\mathbf{v} = rs \mathbf{A}^2 \theta = -(n + \sigma r^2)^{\frac{p-2}{2}} \sigma \theta.
\]

Denoting now by \(F\) a suitable primitive of \(f(r) := -(n + \sigma r^2)^{\frac{p-2}{2}} \sigma\) it is evident that

\[
\mathbf{v} = \nabla F,
\]

and so \(\mathbf{v}\) is a gradient. This gives [1].

[1] \(\implies [2]\)

For the sake of clarity and convenience we break this part into two steps. In the first step we establish (3.11) and in the second one the particular diagonal form of \(\mathbf{A}^2\). Thus it is important to note that in the first two steps the function \(s\) depends on both \(r\) and \(\theta\)!

**Step 1.** [Justification of (3.11)] We begin by extracting a gradient out of \(\mathbf{v}\) and hence rewriting it in the form

\[
\mathbf{v} = \nabla t + \left\{ \frac{1}{r^n} \frac{d}{dr} (r^{n+1} s \mathbf{A}) + \frac{1}{r^{n-1}} \frac{d}{dr} (r^{n+1} s |\mathbf{A}\theta|^2) \mathbf{I}_n \right\} \theta,
\]

where \(t = -p^{-1} (n + r^2 |\mathbf{A}\theta|^2)^{\frac{p-2}{2}}\).
Chapter 3. Measure-preserving maps and generalised twists

To the vector field \( \mathbf{v} = (v_1, \ldots, v_n) \) we now assign the differential 1-form \( \omega = v_1 dx_1 + \cdots + v_n dx_n \). Then in view of \( \mathbf{v} \) being a gradient, for any closed path \( \gamma \in C^1([0, 2\pi], \mathbb{S}^{n-1}) \) it must be that

\[
0 = \int_{\gamma} \omega = \int_0^{2\pi} \langle \mathbf{v}(r\gamma(t)), r\gamma'(t) \rangle \, dt = \frac{1}{r^n} \int_0^{2\pi} \left( \frac{d}{dr} \left[ r^{n+1} s(r, \gamma(t)) A \right] \gamma(t), r\gamma'(t) \right) \, dt + \frac{1}{r^{n-1}} \int_0^{2\pi} \left( \frac{d}{dr} \left[ r^{n+1} s(r, \gamma(t)) |A\gamma(t)|^2 \right] \gamma(t), r\gamma'(t) \right) \, dt = \frac{1}{r^n} \int_0^{2\pi} \left( \frac{d}{dr} \left[ r^{n+1} s(r, \gamma(t)) A \right] \gamma(t), r\gamma'(t) \right) \, dt,
\]

(3.13)

where in concluding the last line we have used the pointwise identity \( \langle \gamma, \gamma' \rangle = 0 \) which holds as a result of \( \gamma \) taking values on \( \mathbb{S}^{n-1} \) and consequently implying that

\[
0 = \int_0^{2\pi} \left( \frac{d}{dr} \left[ r^{n+1} s(r, \gamma(t)) |A\gamma(t)|^2 \right] \gamma(t), r\gamma'(t) \right) \, dt = \int_0^{2\pi} \frac{d}{dr} \left[ r^{n+1} s(r, \gamma(t)) |A\gamma(t)|^2 \right] r\gamma(t), \gamma'(t) \right) \, dt.
\]

Anticipating on (3.11) we first note that in view of \( A \) being skew-symmetric it can be orthogonally diagonalised, i.e., \(^2\)

\[
A = PDP^T,
\]

(3.14)

where \( P = P(r) \in SO(n) \) and \( D = D(r) \in M_{n \times n} \) is in special block diagonal form, i.e.,

[1] \((n = 2k)\)

\[
D = \text{diag}(d_1 J_2, d_2 J_2, \ldots, d_k J_2),
\]

[2] \((n = 2k + 1)\)

\[
D = \text{diag}(d_1 J_2, d_2 J_2, \ldots, d_k J_2, 0),
\]

with \( \{ \pm d_1 i, \pm d_2 i, \ldots, \pm d_k i \} \) or \( \{ \pm d_1 i, \pm d_2 i, \ldots, \pm d_k i, 0 \} \) denoting the eigen-values of the skew-symmetric matrix \( A \) as well as \( D \) respectively. We emphasize that nowhere in this proof have we assumed continuity or differentiability of \( P = P(r) \) or \( D = D(r) \) with respect to \( r \). These in general need not even be true! \([\text{see, e.g., [43], Chapter five.}]\)

\(^2\)At this stage the reader is encouraged to consult the Appendix at the end of the thesis where some notation as well as basic properties related to the matrix exponential as a mapping between the space of skew-symmetric matrices and the special orthogonal group is discussed.
With the aid of (3.14) and for the sake of convenience we now introduce the skew-symmetric matrix

\[ F = F(r, \theta) := P^t \frac{d}{dr} (r^{n+1}sA)P. \]  

(3.15)

Then a straightforward differentiation shows that

\[ F = P^t \frac{d}{dr} (r^{n+1}sA)P \]

\[ = P^t \left\{ r^n[(n+1)s + rs_\gamma]A + r^{n+1}s\dot{A} \right\} P \]

\[ = P^t \left\{ r^n[(n+1)s + rs_\gamma]PDP^t + r^{n+1}s\dot{A} \right\} P \]

\[ = r^n[(n+1)s + rs_\gamma]D + r^{n+1}sP^t\dot{A}P. \]  

(3.16)

Evidently establishing (3.11) is equivalent to showing that

\[ F(r, \theta) = 0, \]  

(3.17)

for all \( r \in [a, b] \) and all \( \theta \in S^{n-1} \).

On the other hand for each fixed \( r \in [a, b] \) setting \( \omega := P^t\gamma \) [also a closed path in \( C^1([0, 2\pi], S^{n-1}) \)] in (3.13) we have that expressed as

\[ 0 = \int_0^{2\pi} \left\langle \frac{d}{dr} (r^{n+1}sA), \omega, \omega' \right\rangle dt \]

\[ = \int_0^{2\pi} \left\langle P^t \frac{d}{dr} (r^{n+1}sA), \omega, \omega' \right\rangle dt \]

\[ = \int_0^{2\pi} \left\langle F, \omega, \omega' \right\rangle dt, \]

where in the above \( s = s(r, P\omega) \) and \( F = F(r, P\omega) \). Thus the necessary condition (3.13) can be equivalently expressed as

\[ \int_0^{2\pi} \left\langle F, \omega, \omega' \right\rangle dt = 0, \]  

(3.18)

for every closed path \( \omega \in C^1([0, 2\pi], S^{n-1}) \).

With this introduction the conclusion in step 1 now amounts to proving the implication (3.18) \( \implies \) (3.17). This will be established below in a componentwise fashion. Note that in view of the skew-symmetry of \( F \) it suffices to justify the latter in the form \( F_{pq}(r, \theta) = 0 \) only when \( 1 \leq p < q \leq n \).
Indeed consider a parameterised family of closed paths \( \rho \in C^\infty([0,2\pi], S^{n-1}) \) given by

\[
\rho : [0, 2\pi] \ni t \mapsto \rho(t) \in S^{n-1} \subset \mathbb{R}^n,
\]

with

\[
\begin{align*}
\rho_1 & = \sin t \sin \phi_2 \sin \phi_3 \ldots \sin \phi_{n-1}, \\
\rho_2 & = \cos t \sin \phi_2 \sin \phi_3 \ldots \sin \phi_{n-1}, \\
\rho_3 & = \cos \phi_2 \sin \phi_3 \ldots \sin \phi_{n-1}, \\
\vdots & \\
\rho_{n-1} & = \cos \phi_{n-2} \sin \phi_{n-1}, \\
\rho_n & = \cos \phi_{n-1},
\end{align*}
\]

where \( \phi_j \in [0, \pi] \) for all \( 2 \leq j \leq n-1 \). For fixed \( 1 \leq p < q \leq n \) we introduce the matrix \( \Gamma^{pq} \) as that obtained by simultaneously interchanging the first and \( p \)-th and the second and \( q \)-th rows of \( I_n \), i.e.,

\[
\Gamma^{pq} e_j = \begin{cases} 
  e_p & \text{if } j = 1, \\
  e_1 & \text{if } j = p, \\
  e_q & \text{if } j = 2, \\
  e_2 & \text{if } j = q, \\
  e_j & \text{otherwise},
\end{cases}
\]

where \( \{e_1, e_2, \ldots, e_n\} \) denotes the standard basis of \( \mathbb{R}^n \). In view of \( \Gamma^{pq} \in O(n) \) setting \( \omega = \Gamma^{pq} \rho \) it is clear that \( \omega \) is a closed path in \( C^\infty([0,2\pi], S^{n-1}) \).

**Claim 1.** For any skew-symmetric matrix \( F \in M_{n \times n} \) and \( \omega = \Gamma^{pq} \rho \) as above we have that

\[
\int_0^{2\pi} (F\omega(t), \omega'(t)) \, dt = 2\pi (\rho_1^2 + \rho_2^2) F_{pq}.
\]

The proof of this claim follows by direct verification noting that here

\[
\omega'(t) = \Gamma^{pq} \rho'(t) = \Gamma^{pq} (\rho_2, -\rho_1, 0, \ldots, 0).
\]

We now proceed by substituting \( \omega \) as described above into (3.18) and then considering the following two distinct cases.

\[1\] \( p = 2j - 1, q = 2j \) for some \( 1 \leq j \leq k = \lfloor n/2 \rfloor \) In this case by utilising the special block
diagonal form of $D$ a straight-forward calculation shows that

$$s = s(r, P\omega(t)) = (n - r^2(D^2\omega(t), \omega(t))]_{r=2}^{n-2} = (n - r^2(D^2\Gamma_{pq}\rho(t), \Gamma_{pq}\rho(t))]_{r=2}^{n-2} = (n + r^2[d_1^2\rho_p^2 + d_1^2\rho_q^2 + \cdots + d_2^2(\rho_1^2 + \rho_2^2) + \cdots)]_{r=2}^{n-2},$$

is indeed independent of the $t$ variable [as $\rho_1^2 + \rho_2^2$ does not depend on $t$]. Hence the same is true of $F(r, P\omega)$ and so referring to (3.18) and utilising claim 1 we can write

$$0 = \int_0^{2\pi} (F(r, P\omega))_{\omega, \omega'} dt = \int_0^{2\pi} (F(r, P\Gamma_{pq}\rho(t))\Gamma_{pq}\rho(t), \Gamma_{pq}\rho(t)) dt = 2\pi(\rho_1^2 + \rho_2^2)F_{pq}(r, P\omega),$$

which in turn for $\rho_1^2 + \rho_2^2 \neq 0$ gives \footnote{Note that $(\rho_1^2 + \rho_2^2) = \prod_{2 \leq j \leq n-1} \sin^2 \phi_j$ and so $\rho_1^2 + \rho_2^2 = 0 \iff \sum_{2 \leq j \leq n} \rho_j^2 = 1 \iff \phi_j \in \{0, \pi\}$ for some $2 \leq j \leq n-1$. This set is a copy of $S^{n-3}$ lying in $S^{n-1}$.}

$$F_{pq}(r, P\omega) = 0. \quad (3.20)$$

Now to get (3.17) for the latter choice of $p, q$ pick $\theta \in S^{n-1}$ and set $\alpha = [\Gamma_{pq}]'P\theta$. Then $\alpha \in S^{n-1}$ and thus can be written in generalised spherical coordinates as

$$\begin{align*}
\alpha_1 &= \sin \phi_1 \sin \phi_2 \sin \phi_3 \ldots \sin \phi_{n-1}, \\
\alpha_2 &= \cos \phi_1 \sin \phi_2 \sin \phi_3 \ldots \sin \phi_{n-1}, \\
\alpha_3 &= \cos \phi_2 \sin \phi_3 \ldots \sin \phi_{n-1}, \\
&\vdots \\
\alpha_{n-1} &= \cos \phi_{n-2} \sin \phi_{n-1}, \\
\alpha_n &= \cos \phi_{n-1},
\end{align*}$$

where $\phi_1 \in [0, 2\pi]$ and $\phi_j \in [0, \pi]$ for all $2 \leq j \leq n-1$. Considering now the closed path $\rho$ in (3.19) for the latter choice of parameters $\phi_2, \phi_3, \ldots, \phi_{n-1}$ a straight-forward calculation gives

$$s(r, \theta) = (n + r^2|A\theta|^2)^{\frac{n-2}{n-2}} = (n + r^2|D\Gamma_{pq}\phi|^2)^{\frac{n-2}{n-2}} = (n + r^2|D\Gamma_{pq}\rho|^2)^{\frac{n-2}{n-2}}$$
\[= (n + r^2 |\mathbf{A} \mathbf{P} \omega|^2)^{\frac{p-2}{2}}\]

\[= s(r, \mathbf{P} \omega),\]

and so referring to (3.20) for \(\rho_1^2 + \rho_2^2 \neq 0\) we obtain

\[\mathbf{F}_{pq}(r, \theta) = \mathbf{F}_{pq}(r, \mathbf{P} \omega) = 0,\]

as required.

[2] (p, q not as in [1]) Unlike the case with [1] here \(s\) depends explicitly on the \(t\) variable [yet in a specific manner (see below)] whilst \(D_{pq} = 0\) as can be verified by inspecting its block diagonal representation.

Now referring, again, to (3.18) and noting that the \(p\)-th and \(q\)-th components of \(\omega'\) are given by \(\omega'_p = \rho'_1 = \rho_2\) and \(\omega'_q = \rho'_2 = -\rho_1\) [with all the remaining derivatives vanishing] we can write using \(\mathbf{F} = \mathbf{F}(r, \mathbf{P} \omega)\)

\[
0 = \int_0^{2\pi} (\mathbf{F} \omega, \omega') \, dt \\
= \int_0^{2\pi} \left\{ \sum_{j=1}^n \mathbf{F}_{pj} \omega_j \omega'_p + \sum_{j=1}^n \mathbf{F}_{qj} \omega_j \omega'_q \right\} \, dt \\
= \int_0^{2\pi} \left\{ (\mathbf{F}_{pq} \rho_2^2 - \mathbf{F}_{qp} \rho_1^2) + \rho_2 \sum_{j=1}^n \mathbf{F}_{pj} \omega_j - \rho_1 \sum_{j=1}^n \mathbf{F}_{qj} \omega_j \right\} \, dt \\
= \mathbf{I} + \mathbf{II} - \mathbf{III}.\]

In order to evaluate the above terms we first observe that here \(s\) takes the form

\[s = s(r, \mathbf{P} \omega(t)) = (n - r^2 (\mathbf{D}^2 \omega(t), \omega(t)))^{\frac{p-2}{2}} = (n - r^2 (\mathbf{D}^2 \Gamma^{pq} \rho(t), \Gamma^{pq} \rho(t)))^{\frac{p-2}{2}} = (n + r^2 [d_1^2 \rho_1^2 + d_2^2 \rho_2^2 + \cdots + d_2^2 \rho_1^2 + \cdots + d_2^2 \rho_2^2 + \cdots])^{\frac{p-2}{2}} = s(\sin^2 t, \cos^2 t).\]

Returning to (3.21) we have that

\[\mathbf{II} = \int_0^{2\pi} \mathbf{F}_{pj} \omega_j \, dt\]
\[ I = \int_0^{2\pi} \rho_2 \sum_{j \neq q}^{n} \left[ \mathbf{P}^t \frac{d}{dr} (r^{n+1} \mathbf{sA}) \mathbf{P} \right]_{pj} \omega_j \ dt \]
\[ = \sum_{j \neq q}^{n} \left[ \mathbf{P}^t \frac{d}{dr} (r^{n+1} \left\{ \int_0^{2\pi} \rho_2 \mathbf{sA} \ dt \right\} \mathbf{P} \right]_{pj} \omega_j, \]

and in a similar way

\[ III = \int_0^{2\pi} \rho_1 \sum_{j \neq p}^{n} \mathbf{F}_{qj} \omega_j \ dt \]
\[ = \int_0^{2\pi} \rho_1 \sum_{j \neq p}^{n} \left[ \mathbf{P}^t \frac{d}{dr} (r^{n+1} \mathbf{sA}) \mathbf{P} \right]_{qj} \omega_j \ dt \]
\[ = \sum_{j \neq p}^{n} \left[ \mathbf{P}^t \frac{d}{dr} (r^{n+1} \left\{ \int_0^{2\pi} \rho_1 \mathbf{sA} \ dt \right\} \mathbf{P} \right]_{qj} \omega_j, \]

where in concluding the last line in both equalities we have used the fact that the only components of \( \omega \) depending explicitly on the \( t \) variable are \( \omega_p = \rho_1 \) and \( \omega_q = \rho_2 \) where in each case one is excluded from the summation sign and the other has a zero coefficient in view of the skew-symmetry of the matrix preceding it.

However in view of the specific manner in which \( \mathbf{s} \) depends on \( t \) [see (3.22)] it follows that both integrals vanish and so as a result \( II = III = 0. \)  \[^4\] Hence returning to (3.21) and utilising the skew-symmetry on \( \mathbf{F} \) and (3.16) we can write

\[ I = \int_0^{2\pi} (\mathbf{F}_{pq} \rho_2^2 - \mathbf{F}_{qp} \rho_1^2) \ dt \]
\[ = \int_0^{2\pi} (\rho_1^2 + \rho_2^2) \mathbf{F}_{pq} \ dt \]
\[ = \int_0^{2\pi} r^{n+1} (\rho_1^2 + \rho_2^2) \mathbf{s} \left[ \mathbf{P}' \hat{\mathbf{A}} \mathbf{P} \right]_{pq} \ dt \]
\[ = r^{n+1} (\rho_1^2 + \rho_2^2) \left\{ \int_0^{2\pi} \mathbf{s} \ dt \right\} \left[ \mathbf{P}' \hat{\mathbf{A}} \mathbf{P} \right]_{pq} = 0. \]

Thus as \( s > 0 \) for \( \rho_1^2 + \rho_2^2 \neq 0 \) it follows that \( \left[ \mathbf{P}' \hat{\mathbf{A}} \mathbf{P} \right]_{pq} = 0. \) Since for the latter range of \( p, q \) we have that \( \mathbf{D}_{pq} = 0 \) referring to (3.16) it immediately follows that \( \mathbf{F}_{pq} = 0. \)

\[^4\]It can be easily shown that as a result of periodicity the following identities hold:
\[ \int_0^{2\pi} s (\sin^2 t, \cos^2 t) \sin t \ dt = 0, \]
\[ \int_0^{2\pi} s (\sin^2 t, \cos^2 t) \cos t \ dt = 0. \]
Hence summarising we have shown that in both cases \([1]\) and \([2]\) for fixed \(r \in [a, b]\) we have \(F_{pq}(r, \cdot) = 0\) outside a copy of \(S^{n-3}\). By continuity of \(F_{pq}(r, \cdot)\) on \(S^{n-1}\) this gives (3.17) and as a result (3.11). The proof of step 1 is therefore complete.

Step 2. \([A^2 = -\sigma I_n]\) Here we establish the remaining part of \([2]\) namely that \(A^2 = -\sigma I_n\) for some \(\sigma \in C^1[a, b]\) with \(\sigma \geq 0\). To this end, we first observe that by utilising (3.11) the vector field \(v\) can be considerably simplified and re-written in the form [as in (3.12)]

\[
v = r s A^2 \theta.
\]

Now for \(v = (v_1, v_2, \ldots, v_n)\) to be a gradient it is necessary that the differential 1-form \(\omega = v_1 dx_1 + \cdots + v_n dx_n\) be closed. In other words \(d\omega = 0\) which in turn amounts to

\[
\frac{\partial v_q}{\partial x_p} - \frac{\partial v_p}{\partial x_q} = 0,
\]

for all \(1 \leq p, q \leq n\). Setting \(F = A^2\) we have that

\[
\frac{\partial v_q}{\partial x_p} = r \frac{\partial s}{\partial x_p} [F \theta]_q + rs [\dot{F} \theta]_q \theta_p + s F_{qp},
\]

and in a similar way

\[
\frac{\partial v_p}{\partial x_q} = r \frac{\partial s}{\partial x_q} [F \theta]_p + rs [\dot{F} \theta]_p \theta_q + s F_{pq}.
\]

Thus in view of the symmetry of \(F\) for the latter range of \(p, q\) we have that

\[
0 = \frac{\partial v_q}{\partial x_p} - \frac{\partial v_p}{\partial x_q} = r \frac{\partial s}{\partial x_p} [F \theta]_q - r \frac{\partial s}{\partial x_q} [F \theta]_p + rs \left( [\dot{F} \theta \otimes \theta]_{pq} - [F \theta \otimes \theta]_{qp} \right).
\]

Alternatively using tensor notation this can be simplified in the form

\[
0 = \nabla s \otimes F \theta - F \theta \otimes \nabla s +
\]

\[
s(\theta \otimes \dot{F} \theta - \dot{F} \theta \otimes \theta) = \frac{1}{2} \beta r^2 \langle \dot{F} \theta, \theta \rangle (F \theta \otimes \theta - \theta \otimes F \theta) +
\]

\[
s(\theta \otimes \dot{F} \theta - \dot{F} \theta \otimes \theta).
\]
where in concluding the second identity we have used

\[
\nabla s = \nabla (n + r^2 |A\theta|^2)^{\frac{p-2}{2}}
\]

\[
= \nabla (n - r^2(\mathbf{F}\theta, \theta))^{\frac{p-2}{2}}
\]

\[
= -\beta \left[ \frac{1}{2} r^2 (\mathbf{F}\theta, \theta) |A\theta|^2 \right]^\frac{p-2}{2}
\]

with \( \beta = \beta(r, \theta, p) := (p - 2)(n - r^2(\mathbf{F}\theta, \theta))^{\frac{p-2}{2}} \). Next a straight-forward calculation using (3.11) gives

\[
\dot{\mathbf{F}} = -2 \left( \frac{n + 1}{r} + \frac{s_r}{s} \right) \mathbf{F}.
\] (3.24)

Therefore substituting this into (3.23) results in

\[
0 = \frac{1}{2} \beta r^2 (\dot{\mathbf{F}}\theta, \theta)(\mathbf{F}\theta \otimes \theta - \theta \otimes \mathbf{F}\theta) -
\]

\[
s(\dot{\mathbf{F}}\theta \otimes \theta - \theta \otimes \dot{\mathbf{F}}\theta)
\]

\[
= \left( 2 \left( \frac{n + 1}{r} + \frac{s_r}{s} \right) \left( s - \frac{1}{2} r^2(\mathbf{F}\theta, \theta) \right) \right) (\mathbf{F}\theta \otimes \theta - \theta \otimes \mathbf{F}\theta)
\]

\[
= \gamma \times (\mathbf{F}\theta \otimes \theta - \theta \otimes \mathbf{F}\theta),
\] (3.25)

where for the sake of convenience we have introduced

\[
\gamma = \gamma(r, \theta, p)
\]

\[
= 2 \left( \frac{n + 1}{r} + \frac{s_r}{s} \right) \left( s - \frac{1}{2} r^2(\mathbf{F}\theta, \theta) \right).
\] (3.26)

**Claim 2.** Let \( p \in [1, \infty[ \). Then \( \gamma = \gamma(r, \theta, p) > 0 \) for all \( r \in ]a, b[ \) and \( \theta \in S^{n-1} \).

The proof of this claim follows by direct verification. Indeed here a straight-forward differentiation gives

\[
s_r = \frac{\partial s}{\partial r} = \frac{\partial}{\partial r} \left( n + r^2 |A\theta|^2 \right)^{\frac{p-2}{2}}
\]

\[
= \frac{\partial}{\partial r} \left( n - r^2(\mathbf{F}\theta, \theta) \right)^{\frac{p-2}{2}}
\]

\[
= -\beta \left[ r(\mathbf{F}\theta, \theta) + \frac{1}{2} r^2(\dot{\mathbf{F}}\theta, \theta) \right].
\]
Now eliminating the term $\langle \dot{F}_\theta, \theta \rangle$ in the above expression with the aid of (3.24) results in

$$s_r = \frac{nr_0s(F_\theta, \theta)}{s - r^2\beta(F_\theta, \theta)}.$$  

(See below for a justification that $s - r^2\beta(F_\theta, \theta) \neq 0$.) Hence referring to (3.26) we can write

$$\gamma = 2 \left( \frac{n+1}{r} + \frac{s_r}{s} \right) \left( s - \frac{1}{2}r^2\beta(F_\theta, \theta) \right)$$

$$= \frac{(n+1)s - r^2\beta(F_\theta, \theta)}{r(s - r^2\beta(F_\theta, \theta))} \left( 2s - r^2\beta(F_\theta, \theta) \right)$$

$$= \frac{I}{II} \times III.$$  

We now proceed by evaluating each term separately. Indeed with regards to the first term we have that

$$I = (n+1)s - r^2\beta(F_\theta, \theta)$$

$$= (n - r^2\beta(F_\theta, \theta)) \frac{n+1}{r} \left[ n(n+1) - (n+p-1)r^2\beta(F_\theta, \theta) \right],$$

and in a similar way

$$II = r(s - r^2\beta(F_\theta, \theta))$$

$$= r(n - r^2\beta(F_\theta, \theta)) \frac{n}{r} \left[ n - (p-1)r^2\beta(F_\theta, \theta) \right],$$

and

$$III = (2s - r^2\beta(F_\theta, \theta))$$

$$= (n - r^2\beta(F_\theta, \theta)) \frac{n}{r} \left[ 2n - pr^2\beta(F_\theta, \theta) \right].$$

Now in view of $-\langle F_\theta, \theta \rangle = \langle A^T A \theta, \theta \rangle = |A\theta|^2 \geq 0$ for all $r \in [a, b]$ and $\theta \in S^{n-1}$ along with $p \in [1, \infty]$, it follows that all the terms $I$, $II$, and $III$ are strictly positive. As a result

$$\gamma > 0,$$  

(3.27)

and so the claim is justified.

Now returning to the identity (3.25) it follows as a result of (3.27) that necessarily

$$F_\theta \otimes \theta - \theta \otimes F_\theta = 0,$$  

(3.28)
for all \( r \in ]a, b[ \) and \( \theta \in \mathbb{S}^{n-1} \). The conclusion in step 2 is now an immediate result of the following statement.

**Claim 3.** Let \( F \in \mathbb{M}_{n \times n} \). Then (3.28) holds for all \( \theta \in \mathbb{S}^{n-1} \) if and only if there exists \( -\sigma \in \mathbb{R} \) such that \( F = -\sigma I_n \).

For a proof of claim 3 we refer the interested reader to Proposition B.0.6 in Appendix B. Finally \( \sigma \in C^1]a, b[ \) and \( \sigma \geq 0 \) are consequences of the representation above and the hypothesis of the theorem. With this the proof of Theorem 3.4.2 is complete.

**Theorem 3.4.3.** Let \( \Omega = \{ x \in \mathbb{R}^n : a < |x| < b \} \) and consider the vector field \( v \) as defined in Theorem 3.4.2. Then the following are equivalent.

1. \( v \) is a gradient,
2. \( A = \mu J \) for some \( \mu \in C^1]a,b[ \) with \( \mu \geq 0 \), \( J \in \mathbb{M}_{n \times n} \) skew-symmetric with \( J^2 = -I_n \) and \( ddr (r^{n+1}s\mu) = 0 \), (3.29) in \( ]a,b[ \). Here \( s = (n + r^2\mu^2)^{\frac{n-2}{2}} \).

**Proof.** [2] \( \implies [1] \) The argument here is similar to that in Theorem 3.4.2 and so will be abbreviated.

[1] \( \implies [2] \) Let \( v \) be a gradient. Then according to [2] in Theorem 3.4.2, \( A^2 = -\sigma I_n \) for some \( \sigma \in C^1]a,b[ \) with \( \sigma \geq 0 \) and so \( A = \sqrt{\sigma} J \) where \( J = J(r) \) and \( J^2 = -I_n \). The aim is to show that \( J \) is independent of \( r \). Note that in general there is no uniqueness or even finiteness associated with the choice of a square root of a matrix! Thus an argument purely based on continuity would not yield the aforementioned claim and it is crucial to additionally utilise (3.11). To this end we proceed as follows. Indeed according to [2] in Theorem 3.4.2,

\[
\frac{d}{dr} (r^{n+1}sA) = 0.
\]

Integrating the above equation gives \( r^{n+1}sA = \xi \) for some constant \( \xi \in \mathbb{M}_{n \times n} \). Moreover,

\[
-s(r^{n+1}s)^2\sigma I_n = (r^{n+1}sA)^2 = \xi^2,
\]

(3.30)
giving \( (r^{n+1}s)^2 \sigma \equiv c \) for some non-negative constant \( c \). Thus either \( \sigma \equiv 0 \) in which case \( A \equiv 0 \) on \( ]a,b[ \) and so the choice \( \mu \equiv 0 \) gives the conclusion or else \( \sigma > 0 \) on \( ]a,b[ \) and so setting

\[
J := \frac{1}{\sqrt{c}} \xi,
\]
we have as a result of (3.30) that \( J^2 = -I_n \). Furthermore setting

\[
\mu := \frac{1}{\sqrt{c}} r^{n+1} \sigma,
\]

it follows that \( \mu \in C^1[a, b] \), \( \mu^2 = \sigma \) and by substitution \( A = \mu J \). As a result \( \mu \) also satisfies (3.29). The proof of the theorem is thus complete. \( \square \)

**Remark 3.4.4.** Referring to the above proof it follows from \( r^{n+1} \sigma = c \) on \( [a, b] \) that when \( p > 1 \) the function \( \mu \) remains bounded on \( [a, b] \).

**Theorem 3.4.5.** Let \( \Omega = \{ x \in \mathbb{R}^n : a < |x| < b \} \) and \( u \in \mathcal{A}_p \) with \( p \in [1, \infty) \) be a generalised twist whose corresponding twist loop \( Q \in C^2([a, b], \text{SO}(n)) \). Then the following are equivalent.

1. \( u \) is a classical solution to the Euler-Lagrange equations associated with \( F_p \) over \( \mathcal{A}_p \),
2. depending on whether \( n \) is even or odd we have that

   a. \( n = 2k \) there exist \( g = g(r) \in C[a, b] \cap C^2[a, b] \) with \( g(a), g(b) \in 2\pi \mathbb{Z} \) and \( P \in O(n) \) such that

   \[
   Q = P \text{diag}(\mathcal{R}(g), \mathcal{R}(g), \ldots, \mathcal{R}(g))P^t,
   \]

   whilst \( g \) is a solution on \( [a, b] \) to

   \[
   \frac{d}{dr} \left\{ r^{n+1} (n + r^2 g'^2) \frac{r^2}{2} g' \right\} = 0,
   \]

   or

   b. \( n = 2k + 1 \) necessarily \( u = x \) on \( \bar{\Omega} \).

**Proof.** [1] \( \implies \) [2] Let \( u = Q(r)x \) be a classical solution to the stated Euler-Lagrange equations. Then setting \( A = Q^t \dot{Q} \) an application of Proposition 3.1.4 in conjunction with Theorem 3.4.3 gives

\[
\frac{d}{dr} Q = \mu QJ, \tag{3.32}
\]

where \( \mu \in C^1[a, b] \) satisfies (3.29) and \( J^2 = -I_n \). Moreover either \( \mu \equiv 0 \) or else \( \mu > 0 \) and bounded on \( [a, b] \). (See Remark 3.4.4.) We now consider the cases [2a] and [2b] separately.

[2a] \( n = 2k \) Let \( g \in C[a, b] \cap C^2[a, b] \) be a primitive of \( \mu \) satisfying \( g(a) \in 2\pi \mathbb{Z} \). (The continuity of \( g \) on \( [a, b] \) follows from \( g \) being monotone and \( g' = \mu \) being bounded on \( [a, b] \).) Next, a straight-forward calculation gives

\[
s = (n + r^2 |A\theta|^2)^{\frac{n-2}{2}}
= (n + r^2 g'^2 |J\theta|^2)^{\frac{n-2}{2}}.
\]
\[(n + r^2 g^2)^{\frac{2-2}{2}}.\]

Thus in view of (3.29) \(g\) satisfies (3.31) on \([a, b]\). An application of Tonelli and Hilbert-Weierstrass differentiability theorems (see, e.g., [18] pp. 57-61) now gives \(g \in C^2[a, b]\) and so in particular \(\mu \in C^1[a, b]\). As will be seen in the next section (3.31) is the Euler-Lagrange equation corresponding to the energy functional \(G_p\) over the space \(G_p^m\) [see (3.33), (3.35)]. In particular it follows that \(g \in C^\infty[a, b]\).

With this introduction now put \(C = gJ\). Then \(A = g'J = \mu J\). In particular \(A\) and \(C\) commute and so we have that

\[
\frac{d}{dr} e^C = e^CA = g'e^CJ = \mu e^CJ.
\]

Thus \(e^C\) is a solution to (3.32). Moreover by bringing \(C\) into a block diagonal form we can write

\[
C = gPJ_nP^t
\]

where \(P \in O(n)\) and \(J_n = diag(J_2, J_2, \ldots, J_2)\). As a result

\[
e^C = e^{gPJ_nP^t} = P e^{gJ_n}P^t = P diag(\mathcal{R}(g), \mathcal{R}(g), \ldots, \mathcal{R}(g))P^t.
\]

Since \(g(a) \in 2\pi\mathbb{Z}\) the above shows that \(e^C|_{r=a} = Q(a) = I_n\) and so by uniqueness of solutions to initial value problems \(Q = e^C\) on \([a, b]\). Since \(Q(b) = I_n\) it follows in a similar way that \(g(b) \in 2\pi\mathbb{Z}\).

[2b] \((n = 2k + 1)\) Here in view of the skew-symmetry of \(Q^t\dot{Q}\), pre-multiplying (3.32) by \(Q^t\) and then taking determinants from both sides, \(\mu \equiv 0\) and so \(\dot{Q} \equiv 0\) on \([a, b]\). As \(Q(a) = Q(b) = I_n\) this gives \(Q \equiv I_n\) on \([a, b]\) and so \(u = x\) on \(\bar{\Omega}\).

[2] \(\Rightarrow\) [1] For the case [2b] this is trivial and for [2a] it is enough to note that for such \(u\), (3.31) is equivalent to (3.11).

\[\square\]

### 3.5 A characterisation of all twist solutions

In section 3.3 we proved the existence of multiple \(p\)-stationary loops by directly minimizing the energy functional \(E_p\) over the homotopy classes \(\mathcal{C}_*\mathcal{E}_p\) of the loop space \(\mathcal{E}_p\). By contrast in this section we focus on the Euler-Lagrange equation itself and present a class of \(p\)-stationary loops that in turn will prove fruitful in discussing the existence of multiple solutions to the Euler-Lagrange equations associated with the energy functional \(F_p\) over the space \(\mathcal{A}_p\).
To this end we consider the case of even dimensions \((n = 2k)\) and for \(p \in [1, \infty[\) and \(m \in \mathbb{N}\) set

\[
G_p^m = G_p^m(a, b) := \left\{ g = g(r) \in W^{1,p}(a, b) : g(a) = 0, g(b) = 2\pi m \right\}.
\] (3.33)

Now for \(g \in G_p^m\) and \(P \in O(n)\) set

\[
Q = P \operatorname{diag}(R(g), R(g), \ldots, R(g)) P^t.
\] (3.34)

It is then evident that the path \(Q\) so defined forms an \textit{admissible} loop, i.e., lies in \(E_p\). It is thus natural to set

\[
G_p[g] := E_p[Q] = \int_a^b \int_{\mathbb{S}^{n-1}} (n + r^2 |\dot{Q}\theta|^2)^{\frac{p}{2}} r^{n-1} d\mathcal{H}^{n-1}(\theta) dr
\]

\[
= n\omega_n \int_a^b (n + r^2 g'^2)^{\frac{p}{2}} r^{n-1} dr.
\] (3.35)

An application of the direct methods of the calculus of variations and standard \textit{regularity} theory (see, e.g., [18] pp. 57-61) leads us to the following statement.

**Theorem 3.5.1.** Let \(p \in ]1, \infty[\) and consider the energy functional \(G_p\) over the space \(G_p^m\). Then for each \(m \in \mathbb{N}\) there exists a unique \(g = g(r; m, a, b) \in G_p^m\) such that

\[
G_p[g] = \inf_{G_p^m} G_p.
\]

Moreover \(g(r; m, a, b)\) satisfies the corresponding Euler-Lagrange equation

\[
\frac{d}{dr} \left\{ r^{n+1} (n + r^2 g'^2)^{\frac{p-2}{2}} g' \right\} = 0,
\] (3.36)

on \([a, b]\). Additionally \(g \in C^\infty[a, b]\).

**Remark 3.5.2.** The Euler-Lagrange equation (3.36) for \(g\) is \textit{equivalent} to equation (3.9) for the twist loop \(Q\) defined through (3.34) and imply the Euler-Lagrange equation (3.9) or alternatively that given in Proposition 3.2.3 for \(A = Q' \dot{Q}\). Hence for every \(P \in O(n)\) and every \(m \in \mathbb{Z}\) the corresponding \(Q\) given by (3.34) with \(g = g(r; m, a, b)\) is a \(p\)-stationary loop.

**Theorem 3.5.3.** Let \(\Omega = \{ x \in \mathbb{R}^n : a < |x| < b \}\). Consider the energy functional \(F_p\) with \(p \in ]1, \infty[\) over the space \(A_p\). Then the set \(\mathcal{G}\) of all generalised twist solutions to the corresponding Euler-Lagrange equations can be characterised as follows.
(1) \((n = 2k)\) \(S\) is infinite and any generalised twist \(u \in S\) can be described as

\[
\begin{align*}
u &= rQ(r; a, b, m)\theta \\
&= rP\text{diag}(H(g), H(g), \ldots, H(g))(r)P^t \theta,
\end{align*}
\]

where \(P \in O(n)\) and \(g \in C^\infty[a, b]\) satisfies

\[
\frac{d}{dr} \left\{ r^{n+1}(n + r^2g'^2) \frac{g'}{g} \right\} = 0,
\]

with \(g(a), g(b) \in 2\pi\mathbb{Z}\),

(2) \((n = 2k + 1)\) \(S\) consists of the single map \(u = x\).

Proof. This is an immediate consequence of Theorem 3.4.5 and Theorem 3.5.1. 

Remark 3.5.4. Is it possible to consider generalised twists \(u\) whose twist loop lies in other spaces [than \(SO(n)\) already considered] with the hope of finding new classes of classical solutions to the Euler-Lagrange equations associated with the energy functional \(F_p\) over \(A_p\)?

Motivated by the requirement \(\det \nabla u = 1\) on such maps the choice of loops in \(SL(n) \supset SO(n)\) seems a natural one. However it turns out that the choice \(SO(n)\) is no less general than \(SL(n)\)!

Claim. Let \(\Omega = \{x \in \mathbb{R}^n : a < |x| < b\}\). For \(p \in [1, \infty]\) consider the map \(u \in C(\Omega, \Omega)\) defined via

\[
u = F(r)x,
\]

where \(r = |x|\) and \(F \in W^{1,p}([a, b], SL(n))\). Then

\[
u \in A_p(\Omega) \implies F \in W^{1,p}([a, b], SO(n)).
\]

Proof. A straight-forward calculation as in the proof of Proposition 3.1.2 gives

\[
\nabla \nu = F + r\dot{F}\theta \otimes \theta = F(I_n + rF^{-1}\dot{F}\theta \otimes \theta).
\]

\footnote{Recall that for every non-negative integer \(n\) we have that

\[
SL(n) = SL(\mathbb{R}, n) := \left\{ F \in M_{n \times n}(\mathbb{R}) : \det F = 1 \right\}.
\]}
Hence in view of $\det F = 1$ we can write

\[
\det \nabla u = \det(F + r \hat{F}\theta \otimes \theta) \\
= \det(I_n + rF^{-1}\hat{F}\theta \otimes \theta) \\
= 1 + r(F^{-1}\hat{F}\theta, \theta).
\]

Evidently $u \in A_p(\Omega)$ provided that

(i) $u = x$ on $\partial \Omega$,

(ii) $\det \nabla u = 1$ in $\Omega$, and,

(iii) $\|u\|_{W^{1,p}(\Omega)} < \infty$.

Now again referring to the proof of Proposition 3.1.2 we have that

(i) $\iff F(a) = F(b) = I_n$,

whilst

(ii) $\iff (F^{-1}\hat{F}\theta, \theta) = 0$ for all $\theta \in S^{n-1} \iff F^{-1}\hat{F} + \hat{F}F^{-t} = 0$.

However, anticipating on the latter, we can write

\[
F^{-1}\hat{F} + \hat{F}F^{-t} = 0 \iff \hat{F} + FF^{-1}\hat{F} = 0, \\
\iff \hat{F}F + FF\hat{F} = 0 \\
\iff \frac{d}{dr}(FF^t) = 0.
\]

This together with (i) and the continuity of $F$ on $[a, b]$ gives $FF^t = I_n$ and so the conclusion follows.

### 3.6 Limiting behaviour of the generalised twists as the inner hole shrinks to a point

In this section we consider the case where $b = 1$ and $a = \varepsilon > 0$ with the aim of discussing the limiting properties of the generalised twists from Theorem 3.5.3 as $\varepsilon \downarrow 0$. This is particularly interesting since in the limit (the punctured ball), by Remark 2.1.4, all components of the function space collapse to a single one and so it is important to have a clear understanding as to how the twist solutions and their energies [for each fixed integer $m$] behave.
To this end, let $\Omega := \{ x \in \mathbb{R}^n : \varepsilon < |x| < 1 \}$ where $n = 2k$ and for each $m \in \mathbb{Z}$ let $u_\varepsilon \in A_p$ denote the generalised twist from [1] in Theorem 3.5.3, that is, with the notation $x = r\theta$,

$$u_\varepsilon = rQ(r; \varepsilon, 1, m)\theta$$

$$= rP_\varepsilon \text{diag}(\mathcal{R}(g_\varepsilon), \mathcal{R}(g_\varepsilon), \ldots, \mathcal{R}(g_\varepsilon))P_\varepsilon^t \theta,$$

where $P_\varepsilon \in O(n)$ and $g_\varepsilon(r) = g(r; \varepsilon, 1, m)$.

In order to make the study of the limiting properties of $u_\varepsilon$ more tractable, we fix the domain to be the unit ball and extend each map by identity off $\Omega_\varepsilon$. [In what follows, unless otherwise stated, we speak of $u_\varepsilon$ in this extended sense.] Thus, here, we have that

$$u_\varepsilon : (r, \theta) \mapsto (r, G_\varepsilon(r)\theta),$$

(3.37)

where

$$G_\varepsilon(r) = P_\varepsilon \text{diag}(\mathcal{R}(g_\varepsilon), \mathcal{R}(g_\varepsilon), \ldots, \mathcal{R}(g_\varepsilon))P_\varepsilon^t,$$

and

$$g_\varepsilon(r) = \begin{cases} 
0 & r \leq \varepsilon, \\
g(r; \varepsilon, 1, m) & \varepsilon < r \leq 1.
\end{cases}$$

In discussing the limiting properties of $u_\varepsilon$ it is convenient to introduce a so-called comparison map. Indeed, fix $m \in \mathbb{Z}$ and consider the generalised twist

$$v_\varepsilon : (r, \theta) \mapsto (r, H_\varepsilon(r)\theta).$$

(3.38)

where

$$H_\varepsilon(r) = P_\varepsilon \text{diag}(\mathcal{R}(h_\varepsilon), \mathcal{R}(h_\varepsilon), \ldots, \mathcal{R}(h_\varepsilon))P_\varepsilon^t,$$

and

$$h_\varepsilon(r) := \begin{cases} 
0 & r \in (0, \varepsilon), \\
2m\pi(\frac{r}{\varepsilon} - 1) & r \in (\varepsilon, 2\varepsilon), \\
2m\pi & r \in (2\varepsilon, 1).
\end{cases}$$

Proposition 3.6.1. Let $p \in ]1, \infty[$. The family of generalised twists $(v_\varepsilon)$ enjoys the following properties.

[1] $v_\varepsilon \to x$ in $W^{1,p}(\mathbb{B}, \mathbb{R}^n)$,

[2] $v_\varepsilon \to x$ uniformly on $\mathbb{B}$.
Proof. [1] Using (3.38) and a straight-forward calculation we have that,

\[ \|v_\varepsilon - x\|^p_{\mathcal{W}^1,p} = \int_{\mathcal{B}} |\nabla v_\varepsilon - I_n|^p \]

\[ = \int_{\mathcal{B} \setminus \mathcal{B}_\varepsilon} |\nabla v_\varepsilon - I_n|^p \leq 2^{p-1} \int_{\mathcal{B} \setminus \mathcal{B}_\varepsilon} |\nabla v_\varepsilon|^p + |I_n|^p. \]

Furthermore, referring to Proposition 3.1.2 [see (3.5)] we can write

\[ \int_{\mathcal{B} \setminus \mathcal{B}_\varepsilon} |\nabla v_\varepsilon|^p = \int_{\varepsilon}^{2\varepsilon} \int_{\mathbb{S}^{n-1}} (n + r^2 |H_\varepsilon \theta|^2) \frac{1}{2} r^{n-1} d\mathcal{H}^{n-1}(\theta) dr \]

\[ = n\omega_n \int_{\varepsilon}^{2\varepsilon} (n + r^2 |h_\varepsilon|^2) \frac{1}{2} r^{n-1} dr \]

\[ \leq \omega_n (2^n - 1) \varepsilon^n |n + 4(2m\pi)| \frac{1}{2}. \] (3.39)

The above estimates when combined give [1] as a result of Poincaré inequality.

[2] By direct verification we have that

\[ |v_\varepsilon - x|^2 = |rH_\varepsilon(r)\theta - r\theta|^2 \]

\[ = r^2 \left| \mathbf{P}_\varepsilon \text{diag}(\mathfrak{R}(h_\varepsilon), \ldots, \mathfrak{R}(h_\varepsilon)) \mathbf{P}_\varepsilon^t \theta - \theta \right|^2 \]

\[ = r^2 \left| \mathbf{P}_\varepsilon \left[ \text{diag}(\mathfrak{R}(h_\varepsilon), \ldots, \mathfrak{R}(h_\varepsilon)) - I_n \right] \mathbf{P}_\varepsilon^t \theta \right|^2 \]

\[ = r^2 \left| \text{diag}(\mathfrak{R}(h_\varepsilon), \ldots, \mathfrak{R}(h_\varepsilon)) - I_n \right| \omega_\varepsilon^2 \quad (\omega_\varepsilon := \mathbf{P}_\varepsilon^t \theta) \]

\[ = \frac{1}{2} r^2 |\mathfrak{R}(h_\varepsilon) - I_2|^2. \] (3.40)

However a straight-forward calculation gives

\[ |\mathfrak{R}(h_\varepsilon) - I_2|^2 = 4(1 - \cos h_\varepsilon) = 8 \sin^2 \frac{h_\varepsilon}{2}. \]
Thus combining the above and referring to the definition of $h_\varepsilon$ we arrive at the bound

$$\sup_B |v_\varepsilon - x| = \sup_{[\varepsilon, 2\varepsilon]} 2r \left| \sin \frac{h_\varepsilon}{2} \right| \leq 4\varepsilon,$$

which gives the required conclusion. \hfill \Box

Let $p \in ]1, \infty]$ and fix $m \in \mathbb{Z}$. Then $g_\varepsilon, h_\varepsilon \in G^m_p(\varepsilon, 1)$ [see (3.33)] and so according to the minimizing property of $g_\varepsilon$ we have that

$$F_p[u_\varepsilon, B] = \frac{1}{p} E_p[G_\varepsilon] = \frac{1}{p} G_p[g_\varepsilon] \leq \frac{1}{p} G_p[h_\varepsilon] = \frac{1}{p} E_p[H_\varepsilon] = F_p[v_\varepsilon, B].$$

This in conjunction with [1] in Proposition 3.6.1 implies the boundedness of $(u_\varepsilon)$ in $W^{1,p}(\mathbb{B}, \mathbb{R}^n)$ and so as a result $(u_\varepsilon)$ admits a weakly convergent subsequence. Indeed more is true!

**Theorem 3.6.2.** Let $\Omega_\varepsilon := \{ x \in \mathbb{R}^n : \varepsilon < |x| < 1 \}$. For $p \in ]1, \infty]$ and $m \in \mathbb{Z}$ let $(u_\varepsilon)_{\varepsilon > 0}$ denote the family of generalised twists as in (3.37). Then,

1. $u_\varepsilon \to x$ in $W^{1,p}(\mathbb{B}, \mathbb{R}^n)$,
2. $u_\varepsilon \to x$ uniformly in $\overline{\mathbb{B}}$.

Before proof we note that here both convergences are in reference to the entire sequence and not merely a subsequence as was implied in discussing the weak convergence prior to the proposition. The argument is standard and will be abbreviated.

**Proof.** [1] Fix $m \in \mathbb{Z}$ and let $v_\varepsilon$ be as in (3.38). Then referring to (3.41) it follows that by passing to a subsequence (not re-labeled) $u_\varepsilon \to u$ in $W^{1,p}(\mathbb{B}, \mathbb{R}^n)$. Appealing to the sequential weak lower semicontinuity of $F_p$ and [1] in Proposition 3.6.1 we can write

$$F_p[x, B] \leq F_p[u, B] \leq \liminf_{\varepsilon \downarrow 0} F_p[u_\varepsilon, B] \leq \limsup_{\varepsilon \downarrow 0} F_p[u_\varepsilon, B] \leq \lim_{\varepsilon \downarrow 0} F_p[v_\varepsilon, B] = F_p[x, B].$$

This in view of the strict convexity of $F_p$ (on $W^{1,p}$) gives $u = x$. As a result of the uniform convexity of the $p$-norm ($p > 1$) the aforementioned weak convergence can now be improved to strong convergence and this gives [1].

[2] By [1] we can assume without loss of generality that $u_\varepsilon \to x \mathcal{L}^n$-a.e. in $\mathbb{B}$. To justify the uniform convergence in [2] let $g_\varepsilon$ be as that described in (3.37) and fix $\sigma \in (0, 1)$. Then we claim that $g_\varepsilon \to 2m\pi$ uniformly on $[\sigma, 1]$. Indeed, $(u_\varepsilon)$ bounded in $W^{1,p}(\mathbb{B}, \mathbb{R}^n)$ gives $(u_\varepsilon)$ bounded in
Chapter 3. Measure-preserving maps and generalised twists

$W^{1,p}(B \setminus \tilde{B}_\sigma, \mathbb{R}^n)$ and so referring to (3.5) and using a calculation similar to that in (3.39) we have $(g_\varepsilon)$ bounded in $W^{1,p}(\sigma, 1)$. Hence, there exists $f = f_\sigma \in W^{1,p}(\sigma, 1)$ so that passing to a subsequence (not relabeled)

$$
\begin{align*}
g_\varepsilon &\to f \quad \text{in } W^{1,p}(\sigma, 1), \\
g_\varepsilon &\to f \quad \text{in } L^\infty[\sigma, 1], \\
f(1) &= 2m\pi.
\end{align*}
$$

In addition referring again to (3.37) we can assume in view of $O(n)$ being compact, that by passing to a further subsequence (again, not relabeled) $P_\varepsilon \to P$ for some $P \in O(n)$. Hence for $L^n$-a.e. $x \in B \setminus \tilde{B}_\sigma$ we can write

$$
x = r\theta = \lim_{\varepsilon \downarrow 0} u_\varepsilon(x) \\
= \lim_{\varepsilon \downarrow 0} rG_\varepsilon(r)\theta \\
= \lim_{\varepsilon \downarrow 0} rP_\varepsilon\text{diag}(\mathcal{R}(g_\varepsilon), \ldots, \mathcal{R}(g_\varepsilon))P^t_\varepsilon\theta \\
= rP\text{diag}(\mathcal{R}(f), \ldots, \mathcal{R}(f))P^t\theta,
$$

giving $\mathcal{R}(f) = I_2$ and in turn that $f = 2\pi n(r)$ for some $n(r) \in \mathbb{Z}$. The continuity of $f$ along with $f(1) = 2m\pi$ now gives $f = 2m\pi$ on $[\sigma, 1]$ justifying the assertion. Next, arguing as in (3.40) we can write

$$
|u_\varepsilon - x|^2 = |rG_\varepsilon(r)\theta - r\theta|^2 \\
= 2r^2(1 - \cos g_\varepsilon) \\
= 4r^2 \sin^2 \frac{g_\varepsilon}{2}.
$$

Thus, to conclude [2] fix $\delta > 0$ and first take $\varepsilon \in (0, 2^{-1}\delta]$ and then $\varepsilon_0$ such that $|\sin(2^{-1}g_\varepsilon)| \leq 2^{-1}\delta$ on $[\sigma, 1]$ for $\varepsilon < \varepsilon_0$. Then $\sup_B |u_\varepsilon - x| \leq \max(2\sigma, \delta) = \delta$. \hfill \square

The uniform convergence in [2] above looks at first counter-intuitive, as, how can $u_\varepsilon$ and $x$ be uniformly close when $u_\varepsilon$ twists $m$ times while the limit $x$ none? Indeed a careful consideration reveals that the latter twists occur at a distance $\varepsilon$ from the origin and within a layer of thickness $O(\varepsilon)$ and this is in no conflict with the stated uniform convergence!
Chapter 4

Quasiconvexity and uniqueness of stationary points

Throughout this chapter we assume $\Omega \subset \mathbb{R}^n$ to be a bounded starshaped domain and consider the energy functional

$$ F[u, \Omega] := \int_{\Omega} F(\nabla u(x)) \, dx, $$

over the space of measure-preserving maps

$$ A_p(\Omega) := \left\{ u \in \xi x + W_0^{1,p}(\Omega, \mathbb{R}^n) : \det \nabla u = 1 \text{ a.e. in } \Omega \right\}, $$

with $p \in [1, \infty]$, $\xi \in M_{n \times n}$ satisfying $\det \xi = 1$. The hypotheses $[\text{H1}]-[\text{H3}]$ on the integrand $F$ here refer to those outline in Chapter 1. We address the question of uniqueness for solutions of the corresponding system of Euler-Lagrange equations. In particular we give a short and new proof of the celebrated result of Knops & Sturat [44] using the method based on comparison with the homogeneous degree-one extension maps. The material in this chapter is taken from Shahrokhi-Dehkordi & Taheri [58].

4.1 Quasiconvexity and uniqueness in starshaped domain

Let $\Omega \subset \mathbb{R}^n$ be a $C^1$ bounded starshaped domain (with respect to the origin). Without loss of generality we assume in the sequel that there exists a strictly positive function $d : S^{n-1} \to \mathbb{R}$ of class

1Recall that a set $\mathcal{S} \subset \mathbb{R}^n$ is said to be starshaped with respect to the point $x_0 \in \mathcal{S}$ if and only if whenever the point $x$ belongs to $\mathcal{S}$, the straight line segment joining $x_0$ to $x$ also lies in $\mathcal{S}$.}
\[
\partial \Omega = \left\{ \omega \neq 0 : |\omega| = d(\frac{\omega}{|\omega|}) \right\}.
\]

It is then clear that \( \Omega = \{0\} \cup \{x \neq 0 : |x| < d(x/|x|)\} \). Moreover the unit outward normal to the boundary at a point \( \omega \in \partial \Omega \) is given by

\[
\nu = \frac{1}{\alpha(\theta)} \left[ \theta - (I_n - \theta \otimes \theta) \nabla d(\theta) d(\theta) \right],
\]

where \( \alpha(\theta) = d(\theta)^{-1} \sqrt{d(\theta)^2 + |\nabla d(\theta)|^2 - (\theta, \nabla d(\theta))^2} \) and \( \theta = \omega/|\omega| \).

**Definition 4.1.1.** (Classical solution)

A pair \((u, p)\) is said to be a classical solution to the Euler-Lagrange equations associated with \( F \) over \( A_p(\Omega) \) if and only if the following hold.

1. \( u \in C^2(\Omega, \mathbb{R}^n) \cap C^1(\bar{\Omega}, \mathbb{R}^n) \),
2. \( p \in C^1(\Omega) \cap C(\bar{\Omega}) \),
3. \((u, p)\) satisfy the system of equations

\[
\begin{align*}
\text{div} \left\{ F_\xi(\nabla u) - p \text{cof} \nabla u \right\} &= 0 \quad \text{in } \Omega, \\
det \nabla u &= 1 \quad \text{in } \Omega, \\
u &= v \quad \text{on } \partial \Omega.
\end{align*}
\]

Although we are primarily concerned with the case \( v = \bar{\xi}x \), for reasons that will become clear later, we allow \( v \in C^1(\bar{\Omega}, \mathbb{R}^n) \) to be arbitrary. Now suppose that \((u, p)\) is a classical solution as described in Definition 4.1.1. We set

\[
G(x, z, \xi) = G(x, z, \xi; p) := F(\xi) - p(x)[\det \xi - 1],
\]

for all \( x \in \Omega, z \in \mathbb{R}^n \) and \( \xi \in M_{n \times n} \). Next with the aid of \( G \) we introduce the Hamilton [or the energy-momentum] tensor

\[
T^i_n(x, z, \xi) := \xi^i_n G_{\xi_j}(x, z, \xi) - \delta^i_n G(x, z, \xi).
\]

**Theorem 4.1.2.** Let \((u, p)\) be a classical solution to the Euler-Lagrange equations associated with \( F \) over \( A_p(\Omega) \). Let \( F \) be of class \( C^2 \). Then with \( G \) and \( T \) as in (4.1) and (4.2) we have that

\[
\text{div}\{T(x, u, \nabla u)\} + G_x(x, u, \nabla u) = 0,
\]

(4.3)
in $\Omega$.

Proof. (By direct verification) Indeed expanding the above identity componentwise we have that

$$L_\alpha := \left[ \text{div}\{T(x, u, \nabla u)\} + G_x(x, u, \nabla u) \right]_\alpha$$

$$= \frac{\partial T_\alpha}{\partial x_\beta}(x, u, \nabla u) + G_{x_\alpha}(x, u, \nabla u)$$

$$= \frac{\partial}{\partial x_\alpha} \left( u^i_\alpha \left( F_{\xi^i_\beta} - p(x)[\text{cof} \nabla u]_{i\beta} \right) \right) -$$

$$\frac{\partial}{\partial x_\alpha} \left( F - p(x)[\det \nabla u - 1] \right) - \frac{\partial p}{\partial x_\alpha}(x)[\det \nabla u - 1].$$

Therefore taking advantage of $\det \nabla u = 1$ and by direct differentiation we can write

$$L_\alpha = u^i_\alpha \beta \left( F_{\xi^i_\beta} - p(x)[\text{cof} \nabla u]_{i\beta} \right) +$$

$$u^i_\alpha \frac{\partial}{\partial x_\beta} \left( F_{\xi^i_\beta} - p(x)[\text{cof} \nabla u]_{i\beta} \right) - F_{\xi^i_\beta} u^i_\alpha \beta$$

$$= - p(x) \frac{\partial}{\partial x_\alpha} \det \nabla u + u^i_\alpha \frac{\partial}{\partial x_\beta} \left( F_{\xi^i_\beta} - p(x)[\text{cof} \nabla u]_{i\beta} \right)$$

$$= - u^i_\alpha \frac{\partial}{\partial x_\beta} \left( F_{\xi^i_\beta} - p(x)[\text{cof} \nabla u]_{i\beta} \right) = 0,$$

which is the required conclusion. □

We note that the equation (4.3) is the so-called stationarity condition in its strong form as opposed to its weak form given by (4.4) below. For the sake of future reference we next introduce the unconstrained energy functional

$$G[u, p; \Omega] := \int_\Omega G(x, u, \nabla u) \, dx$$

$$= \int_\Omega \left( F(\nabla u) - p(x)[\det \nabla u - 1] \right) \, dx.$$

Then setting $u_\varepsilon(x) := u(x + \varepsilon \varphi)$ with $\varphi \in C^\infty_c(\Omega, \mathbb{R}^n)$ an application of Theorem 4.1.2 and the divergence theorem along with a straight-forward calculation gives

$$\frac{d}{d\varepsilon} G[u_\varepsilon, p; \Omega] \bigg|_{\varepsilon=0} = \int_\Omega \left( T^\alpha_{\alpha \varphi^{(\alpha)}} - G_{x_\alpha} \varphi^{(\alpha)} \right) \, dx$$

$$= \int_\Omega \left( u^i_\alpha G_{\xi^i_\beta \varphi^{(\beta)}} - \delta^i_\beta G_{\varphi^{(\alpha)}} - G_{x_\alpha} \varphi^{(\alpha)} \right) \, dx = 0. \quad (4.4)$$

In the course of the proof of next theorem we make repeated use of the following integration formula.
**Proposition 4.1.3.** For every \( f \in L^1(\Omega) \) we have that

\[
\int_{\Omega} f(x) \, dx = \int_0^1 \int_{\partial \Omega} \rho^{n-1} \frac{d(\theta)}{\alpha(\theta)} f(\rho \omega) \, dH^{n-1}(\omega) d\rho.
\]

**Proof.** As \( d \) and \( \alpha \) are bounded away from zero a straight-forward proof of this assertion follows from co-area formula (see e.g., [28], Theorem 3.2.12, pp. 249) with the particular choice of \( f(x) = |x|/d(x/||x||) \) there. \( \square \)

**Theorem 4.1.4.** Let \((u, p)\) be a classical solution to the Euler-Lagrange equations associated with \( F \) over \( A_p(\Omega) \). Assume that

1. \( F \) is of class \( C^2 \),
2. \( F \) satisfies \( [H4]_\xi \) for all \( \xi \in \{\nabla u(\omega) : \omega \in \partial \Omega\} \).

Then with \( G \) and \( T \) as in (4.1) and (4.2) we have that

\[
G[u, p; \Omega] \leq G[\bar{u}, \bar{p}; \Omega],
\]

where \( \bar{u}, \bar{p} \) denote the homogeneous degree-one and degree-zero extensions of \( u, p \) to \( \Omega \) respectively, that is,

\[
\bar{u}(x) := r \frac{d(\theta)}{d(\theta)} u(\theta d(\theta)),
\]

and

\[
\bar{p}(x) := p(\theta d(\theta)),
\]

for \( x \in \bar{\Omega} \) where \( r = |x| \) and \( \theta = x/|x| \).

**Proof.** For the sake of clarity and convenience we present this in the following two steps.

**Step 1.** \((G[u, p; \Omega] \text{ as a boundary integral})\) For \( t \in [0, 1] \) and \( \varepsilon > 0 \) put

\[
s_\varepsilon(t) = \begin{cases} 
1 & \text{for } 0 \leq t \leq 1 - \varepsilon, \\
1 - \frac{t - (1-\varepsilon)}{\varepsilon} & \text{for } 1 - \varepsilon \leq t \leq 1,
\end{cases}
\]

and set

\[
\varphi(x) = s_\varepsilon\left(\frac{|x|}{d(\theta)}\right)x.
\]

Then one can easily verify that

\[
\nabla \varphi(x) = s_\varepsilon\left(\frac{|x|}{d(\theta)}\right)I_n + |x| \frac{1}{d(\theta)} \alpha(\theta) \frac{|x|}{d(\theta)} \theta \otimes \left(\theta - (I_n - \theta \otimes \theta) \nabla d(\theta)\right)
\]

\[
= s_\varepsilon\left(\frac{|x|}{d(\theta)}\right)I_n + |x| \alpha(\theta) \frac{|x|}{d(\theta)} \theta \otimes \nu,
\]
where $\theta = x/|x|$ and $\nu = \nu(\theta d(\theta))$ is the unit outward normal to $\partial \Omega$. Moreover it is evident that

$$I_\Omega = \lim_{\varepsilon \downarrow 0} s_\varepsilon \left(\frac{|x|}{d(\theta)}\right),$$  \hspace{1cm} (4.7)

where the limit is being understood both as $L^\infty$-a.e. in $\Omega$ and strongly in $L^1(\Omega)$. Now upon substituting $\varphi$ as given by (4.6) into (4.4) and re-arranging terms it follows after taking into account (4.7) that

\begin{equation*}
\begin{split}
nG[u, p; \Omega] &= \lim_{\varepsilon \downarrow 0} \int_\Omega n s_\varepsilon \left(\frac{|x|}{d(\theta)}\right) G(x, u, \nabla u) \, dx \\
&= \lim_{\varepsilon \downarrow 0} \int_\Omega \left\{ - \frac{\alpha(\theta)}{d(\theta)} |x| (\theta \cdot \nu) s'_\varepsilon \left(\frac{|x|}{d(\theta)}\right) G(x, u, \nabla u) + \\
&\quad s_\varepsilon \left(\frac{|x|}{d(\theta)}\right) (G_\xi(x, u, \nabla u), \nabla u) + \\
&\quad \frac{\alpha(\theta)}{d(\theta)} |x| s'_\varepsilon \left(\frac{|x|}{d(\theta)}\right) (G_\xi(x, u, \nabla u), \nabla u \theta \otimes \nu) \right\} \, dx \\
&= \lim_{\varepsilon \downarrow 0} \left\{ I + II + III \right\}. \hspace{1cm} (4.8)
\end{split}
\end{equation*}

We now proceed by considering each term separately. Indeed, with regards to the first term we have that

\begin{equation*}
\begin{split}
I &= I(\varepsilon) = \int_\Omega - \frac{\alpha(\theta)}{d(\theta)} |x| (\theta \cdot \nu) s'_\varepsilon \left(\frac{|x|}{d(\theta)}\right) G(x, u, \nabla u) \, dx \\
&= \int_\Omega - \frac{1}{d(\theta)} |x| s'_\varepsilon \left(\frac{|x|}{d(\theta)}\right) F(\nabla u(x)) \, dx \\
&= \int_{1-\varepsilon}^1 \int_{\partial \Omega} \frac{1}{\alpha(\theta)} \rho^n d(\theta) F(\nabla u(\rho \omega)) \, dH^{n-1}(\omega) d\rho.
\end{split}
\end{equation*}

Thus by passing to the limit $\varepsilon \downarrow 0$ we have that

$$\lim_{\varepsilon \downarrow 0} I = \lim_{\varepsilon \downarrow 0} \int_{1-\varepsilon}^1 \int_{\partial \Omega} \frac{1}{\alpha(\theta)} \rho^n d(\theta) F(\nabla u(\rho \omega)) \, dH^{n-1}(\omega) d\rho = \int_{\partial \Omega} \frac{d(\theta)}{\alpha(\theta)} F(\nabla u(\omega)) \, dH^{n-1}(\omega).$$

In a similar way with regards to the second term we have that

\begin{equation*}
\begin{split}
II &= II(\varepsilon) = \int_\Omega s_\varepsilon \left(\frac{|x|}{d(\theta)}\right) (G_\xi(x, u, \nabla u), \nabla u) \, dx \\
&= \int_\Omega s_\varepsilon \left(\frac{|x|}{d(\theta)}\right) (F_\xi(\nabla u) - p(x) [\text{cof} \nabla u], \nabla u) \, dx.
\end{split}
\end{equation*}
Chapter 4. Quasiconvexity and uniqueness of stationary points

Utilising (4.7) and Lebesgue’s theorem on dominated convergence, passing to the limit \( \varepsilon \downarrow 0 \) yields

\[
\lim_{\varepsilon \downarrow 0} \mathbf{II} = \lim_{\varepsilon \downarrow 0} \int_{\Omega} s_\varepsilon(\frac{|x|}{d(\theta)}) (\mathbf{F}_\varepsilon(\nabla u) - p(x)[\text{cof } \nabla u], \nabla u) \, dx
\]
\[
= \int_{\Omega} (\mathbf{F}_\varepsilon(\nabla u) - p(x)[\text{cof } \nabla u], \nabla u) \, dx
\]
\[
= \int_{\partial \Omega} (\mathbf{F}_\varepsilon(\nabla u(\omega)) - p(\omega)[\text{cof } \nabla u(\omega)], u(\omega) \otimes \nu) \, d\mathcal{H}^{n-1}(\omega),
\]

where in the second identity we have appealed to the divergence theorem along with the fact that \((u, p)\) is a solution to the Euler-Lagrange equations associated with \(F\) over \(A_p\).

Finally with regards to the third term we can write

\[
\mathbf{III} = \mathbf{III}(\varepsilon) = \int_{\Omega} \frac{\alpha(\theta)}{d(\theta)} |x| s_\varepsilon^\prime(\frac{|x|}{d(\theta)}) (\mathbf{G}_\varepsilon(x, u, \nabla u), \nabla u \theta \otimes \nu) \, dx
\]
\[
= \int_{\Omega} \frac{\alpha(\theta)}{d(\theta)} |x| s_\varepsilon^\prime(\frac{|x|}{d(\theta)}) (\mathbf{F}_\varepsilon(\nabla u) - p(x)[\text{cof } \nabla u], \nabla u \theta \otimes \nu) \, dx
\]
\[
= \int_{1-\varepsilon}^{1} \int_{\partial \Omega} -\frac{1}{\varepsilon} \rho^\alpha d(\theta) \times
\]
\[
\left\{ (\mathbf{F}_\varepsilon(\nabla u(\rho \omega)) - p(\rho \omega)[\text{cof } \nabla u(\rho \omega)], \nabla u(\rho \omega) \theta \otimes \nu) \right\} \, d\mathcal{H}^{n-1}(\omega) d\rho.
\]

Thus by passing to the limit \( \varepsilon \downarrow 0 \) we have that

\[
\lim_{\varepsilon \downarrow 0} \mathbf{III} = \lim_{\varepsilon \downarrow 0} \int_{1-\varepsilon}^{1} \int_{\partial \Omega} -\frac{1}{\varepsilon} \rho^\alpha d(\theta) \times
\]
\[
\left\{ (\mathbf{F}_\varepsilon(\nabla u(\rho \omega)) - p(\rho \omega)[\text{cof } \nabla u(\rho \omega)], \nabla u(\rho \omega) \theta \otimes \nu) \right\} \, d\mathcal{H}^{n-1}(\omega) d\rho
\]
\[
= \int_{\partial \Omega} -d(\theta)(\mathbf{F}_\varepsilon(\nabla u(\omega)) - p(\omega)[\text{cof } \nabla u(\omega)], \nabla u(\omega) \theta \otimes \nu) \, d\mathcal{H}^{n-1}(\omega).
\]

Hence referring to (4.8) and summarising the above conclusions we have that

\[
n \mathcal{G}[u, p; \Omega] = \int_{\Omega} n \mathbf{G}(x, u, \nabla u) \, dx
\]
\[
= \int_{\partial \Omega} \frac{d(\theta)}{\alpha(\theta)} \mathbf{F}(\nabla u(\omega)) \, d\mathcal{H}^{n-1}(\omega) +
\]
\[
\int_{\partial \Omega} (\mathbf{F}_\varepsilon(\nabla u(\omega)) - p(\omega)[\text{cof } \nabla u(\omega)], u(\omega) \otimes \nu) \, d\mathcal{H}^{n-1}(\omega) -
\]
\[
\int_{\partial \Omega} d(\theta)(\mathbf{F}_\varepsilon(\nabla u(\omega)) - p(\omega)[\text{cof } \nabla u(\omega)], \nabla u(\omega) \theta \otimes \nu) \, d\mathcal{H}^{n-1}(\omega).
\]

(4.9)

**Step 2.** (A lower bound on \( \mathcal{G}[\bar{u}, \bar{p}; \Omega] \)) Recall that the homogeneous degree-one extension of \(u\) to \(\Omega\) is given by

\[
\bar{u}(x) = \frac{|x|}{d(\theta)}u(\theta d(\theta)),
\]
for $x \in \bar{\Omega}$ with $\theta = x/|x|$. It can therefore be easily checked that

$$
\nabla u(x) = \nabla u(\theta d(\theta)) + \left\{ \frac{u(\theta d(\theta))}{d(\theta)} - \nabla u(\theta d(\theta)) \theta \right\} \otimes \left( \theta - (I_n - \theta \otimes \theta) \frac{\nabla d(\theta)}{d(\theta)} \right)
$$

$$
= \nabla u(\omega) + \frac{\alpha(\theta)}{d(\theta)} \left\{ [u(\omega) - d(\theta) \nabla u(\omega) \theta] \otimes \nu \right\},
$$

(4.10)

for $x \in \bar{\Omega}$ where $\omega = \theta d(\theta) \in \partial \Omega$. In particular we have that

$$
det \nabla \bar{u}(x) = det \nabla u(\omega) + \frac{\alpha(\theta)}{d(\theta)} \langle [\nabla u(\omega)]^{-1} [u(\omega) - d(\theta) \nabla u(\omega) \theta], \nu \rangle
$$

$$
= 1 + \frac{\alpha(\theta)}{d(\theta)} \langle \operatorname{cof} \nabla u(\omega)[\nabla u(\omega)]^{-1} [u(\omega) - d(\theta) \nabla u(\omega) \theta], \nu \rangle.
$$

(4.11)

Thus we can write

$$
nG[\bar{u}, \bar{p}; \Omega] = \int_{\Omega} \frac{n}{\alpha(\theta)} \| \nabla (\nabla \bar{u}) - \bar{p}(\omega) \| [\nabla (\nabla \bar{u}) - 1] \, dx
$$

$$
= \int_{\Omega} n \frac{n}{\alpha(\theta)} \| \nabla \bar{u}(\omega) - \bar{p}(\omega) \| [\nabla \bar{u}(\omega) - 1] \, dx
$$

$$
= \int_0^1 \int_{\partial \Omega^2} \rho^{n-1} \frac{d(\theta)}{\alpha(\theta)} \times
$$

$$
\left\{ \int \frac{d(\theta)}{\alpha(\theta)} \left\{ \frac{F(\nabla \bar{u}(\rho \omega)) - \bar{p}(\rho \omega) [\nabla \bar{u}(\rho \omega) - 1]}{\nabla \bar{u}(\rho \omega) - 1} \right\} dH^{n-1}(\omega) \right\}
$$

(4.12)

where in concluding the last line we have used the identities $\nabla u(\rho \omega) = \nabla \bar{u}(\omega)$ and $\bar{p}(\rho \omega) = \bar{p}(\omega)$ for $\rho \in [0, 1]$ and $\omega \in \partial \Omega$ as a consequence of homogeneity.

Now anticipating on the integral on the right in (4.12) we first note that in view of the rank-one convexity of $F$ at the points $\nabla u(\omega)$ using (4.10) [with $x = \omega$] we have that

$$
F(\nabla \bar{u}(\omega)) = F \left( \nabla u(\omega) + \frac{\alpha(\theta)}{d(\theta)} [u(\omega) - d(\theta) \nabla u(\omega) \theta] \otimes \nu \right)
$$

$$
\geq F \left( \nabla u(\omega) + \frac{\alpha(\theta)}{d(\theta)} (F_{\xi}(\nabla u(\omega)), [u(\omega) - d(\theta) \nabla u(\omega) \theta] \otimes \nu) \right).
$$

(4.13)

Hence substituting from (4.11) and (4.13) into (4.12) and making note of the inequality $d/\alpha > 0$ we can write

$$
nG[\bar{u}, \bar{p}; \Omega] = \int_{\partial \Omega} \frac{d(\theta)}{\alpha(\theta)} \left\{ F(\nabla \bar{u}(\omega)) - \bar{p}(\omega) [\nabla \bar{u}(\omega) - 1] \right\} dH^{n-1}(\omega)
$$

$$
\geq \int_{\partial \Omega} \frac{d(\theta)}{\alpha(\theta)} F(\nabla u(\omega)) dH^{n-1}(\omega) +
$$
\[ \int_{\partial \Omega} (F_\xi(\nabla u(\omega)), [u(\omega) - d(\theta)\nabla u(\omega)\theta] \otimes \nu) \, dH^{n-1}(\omega) - \int_{\partial \Omega} p(\omega) \langle [\text{cof } \nabla u(\omega)]^t [u(\omega) - d(\theta)\nabla u(\omega)\theta], \nu \rangle \, dH^{n-1}(\omega). \]

Finally, re-arranging terms and comparing the expression on the right in the above with (4.9) immediately yields

\[ nG[\bar{u}, \bar{p}; \Omega] \geq \int_{\partial \Omega} d(\theta) F(\nabla u(\omega)) \, dH^{n-1}(\omega) + \int_{\partial \Omega} (F_\xi(\nabla u(\omega)) - p(\omega)[\text{cof } \nabla u(\omega)], u(\omega) \otimes \nu) \, dH^{n-1}(\omega) - \int_{\partial \Omega} d(\theta)(F_\xi(\nabla u(\omega)) - p(\omega)[\text{cof } \nabla u(\omega)], \nabla u(\omega) \theta \otimes \nu) \, dH^{n-1}(\omega) \geq nG[u, p; \Omega], \]

which is the required conclusion. \( \square \)

### 4.2 Uniqueness theorems on starshaped domain

**Theorem 4.2.1.** (Uniqueness I) Let \( \Omega \subset \mathbb{R}^n \) be a \( C^1 \) bounded starshaped domain and consider the energy functional \( F \) over \( A_{p}(\Omega) \). Assume that

1. \( F \) is of class \( C^2 \),
2. \( F \) satisfies \( [H1] \) and \( [H3]_\xi \),
3. \( (u, p) \) is a classical solution (see Definition 4.1.1),
4. \( F \) satisfies \( [H4]_\xi \) for all \( \xi \in \{ \nabla u(\omega) : \omega \in \partial \Omega \} \).

Then,

\[ F[u, \Omega] = F[\xi x, \Omega] = \inf_{A_p(\Omega)} F[\cdot, \Omega]. \]

If, additionally, \( F \) is strictly quasiconvex at \( \xi \) then \( u = \xi x \) on \( \bar{\Omega} \).

**Proof.** Evidently \( \bar{u} = \xi x \) and therefore \( \det \nabla \bar{u} = 1 \) in \( \Omega \). It should note that in general \( \det \nabla \bar{u} = 1 \) is **false!** [See (4.11)] However, interestingly, subject to \( u = \xi x \) on \( \partial \Omega \) as described the latter identity holds throughout \( \Omega \). Hence referring to the estimate (4.5) in Theorem 4.1.4 and the quasiconvexity of \( F \) at \( \xi \) we can write

\[ F[\bar{u}, \Omega] \leq F[u, \Omega] = G[u, p; \Omega] \leq G[\bar{u}, \bar{p}; \Omega] = F[\bar{u}, \Omega]. \]
The remaining assertion is now a trivial consequence of the latter and the strict quasiconvexity of $F$ at $\xi$. □

**Remark 4.2.2.** The proof of Theorem 4.1.4 and Theorem 4.2.1 remain unchanged if $F$ is of class $C^1$ and in Definition (4.1.1), [1] is replaced by $u \in C^1(\bar{\Omega}, \mathbb{R}^n)$, [2] by $p \in C(\bar{\Omega})$ and [3] by $(u, p)$ being a weak solution to the corresponding system of Euler-Lagrange equation provided that additionally (4.4) holds.

**Theorem 4.2.3.** (Uniqueness II) Let $\Omega \subset \mathbb{R}^n$ be a bounded starshaped domain and consider the energy functional $F$ over $A_p(\Omega)$. Assume that

1. $F$ is of class $C$,
2. $F$ satisfies $[H1]$ and $[H3]_\xi$,
3. $u \in A_p(\Omega)$ is a strong local minimizer of $F$, i.e., that there exists $\rho = \rho(u) > 0$ such that $F[u, \Omega] \leq F[w, \Omega]$ for all $w \in A_p(\Omega)$ with $\|u - w\|_{L^1} \leq \rho$.

Then,

$$F[u, \Omega] = F[\xi x, \Omega] = \inf_{A_p(\Omega)} F[\cdot, \Omega].$$

(4.14)

If, additionally, $F$ is strictly quasiconvex at $\xi$ then $u = \xi x$ on $\Omega$.

**Proof.** The second identity in (4.14) is a result of [1], [2] and a straight-forward approximation and so it suffices to justify only the first equality. Indeed for the sake of a contradiction assume $F[u, \Omega] > F[\xi x, \Omega]$ and for $\delta \in (0, 1]$ and $x \in \Omega$ set

$$u_\delta(x) := \begin{cases} 
\delta u(\xi) & x \in \Omega_\delta, \\
\xi x & x \in \Omega \setminus \Omega_\delta,
\end{cases}$$

where $\Omega_\delta = \delta \Omega$. Then $\det \nabla u_\delta = 1$ $\mathcal{L}^n$-a.e. in $\Omega$ and so $u_\delta \in A_p(\Omega)$. Moreover, a straight-forward calculation gives

$$F[u_\delta, \Omega] = F[u, \Omega] + (1 - \delta^n) \left\{ F[\xi x, \Omega] - F[u, \Omega] \right\}$$

$$< F[u, \Omega],$$

whilst $u_\delta \rightarrow u$ in $W^{1,p}$ as $\delta \uparrow 1$. This contradicts [3] and so the assertion is justified. The final part is now a trivial consequence of the latter and the strict quasiconvexity of $F$ at $\xi$. □
Chapter 5

Polyconvexity and generalised twists

In this chapter we consider the energy functional $F$ as given by (1.6) over the space of orientation preserving maps $A(\Omega)$ as defined by (1.7) in the first chapter and discuss the question of existence of multiple strong local minimizers for $F$. Motivated by their signification in topology and the study of mapping class groups, we consider a class of maps, referred to as generalised twists as defined in Chapter 3, and examine them in connection with the corresponding Euler-Lagrange equation and we show that in even dimensions the latter system of equations admits infinitely many smooth solutions, modulo isometries, amongst such maps. In odd dimensions this number reduces to one. The material in this chapter is taken from Shahrokhi-Dehkordi & Taheri [57].

5.1 Generalised twists and the space of orientation preserving maps

We start this section by recalling the definition of a generalised twist. Let $\Omega = \{x \in \mathbb{R}^n : a < |x| < b\}$. A map $u \in C(\bar{\Omega}, \bar{\Omega})$ is referred to as a generalised twist if and only if it can be expressed as

$$u(x) = G(r)\theta,$$

with

$$G(r) = f(r)Q(r),$$

where $r = |x|$, $\theta = x/|x|$, $Q \in C([a, b], \text{SO}(n))$ and $f \in C[a, b]$. 

Chapter 5. Polyconvexity and generalised twists

Notice that as a result of the basic requirement \( u \in C(\bar{\Omega}, \Omega) \) built into the definition of a generalised twist it follows in particular that \( a \leq |f| \leq b \) on \([a, b]\) see (5.5). The continuous function \( G \) in the above definition will be referred to as the twist path. When additionally \( G(a) = G(b) \) we refer to \( G \) as the twist loop.

**Proposition 5.1.1.** Let \( \Omega = \{ x \in \mathbb{R}^n : a < |x| < b \} \). A generalised twist \( u \) lies in \( A(\Omega) \) provided that the following set of conditions hold.

1. Conditions on \( Q \):
   1a) \( Q \in W^{1,2}([a, b], SO(n)) \),
   1b) \( Q(a) = Q(b) = I_n \),

2. Conditions on \( f \):
   2a) \( f \in W^{1,2}(a, b) \),
   2b) \( \dot{f} > 0 \) \( L^1 \)-a.e. on \((a, b)\),
   2c) \( f(a) = a \) and \( f(b) = b \).

**Proof.** Let \( u \) be a generalised twist as in Definition 3.1.1. Then \( u \) lies in \( A(\Omega) \) if and only if the following conditions hold.

(i) \( u = x \) on \( \partial \Omega \),

(ii) \( \det \nabla u > 0 \) \( L^n \)-a.e. in \( \Omega \), and,

(iii) \( \| u \|_{W^{1,2}(\Omega)} < \infty \).

Evidently [1b] and [2c] together give (i). In addition a straight-forward differentiation reveals that

\[
\nabla u = \frac{\dot{f}}{r}Q + (\dot{f} - \frac{\dot{f}}{r})Q\theta \otimes \theta + f\dot{Q}\theta \otimes \theta. \quad (5.1)
\]

Here we have denoted \( \dot{f} := \frac{d}{dr} f \) and in a similar way \( \dot{Q} := \frac{d}{dr} Q \). Therefore using the latter we can write

\[
\det \nabla u = \det \left[ \frac{\dot{f}}{r}Q + (\dot{f} - \frac{\dot{f}}{r})Q\theta \otimes \theta + f\dot{Q}\theta \otimes \theta \right]
= \det \left[ \frac{\dot{f}}{r}Q \right] \det \left[ I_n + \left( \frac{\dot{f}}{f} - 1 \right)Q\theta \otimes Q\theta + r(\dot{Q}\theta \otimes Q\theta) \right]
= \left( \frac{\dot{f}}{f} \right)^n \left[ 1 + \left( \frac{\dot{f}}{f} - 1 \right)(Q\theta, Q\theta) + r(Q\theta, Q\theta) \right]
= \dot{f} \left( \frac{\dot{f}}{f} \right)^{n-1}, \quad (5.2)
\]
where in concluding the last identity we have used the fact that \( \langle Q\theta, Q\theta \rangle = |\theta|^2 = 1 \) for all \( \theta \in S^{n-1} \) and so as a result

\[
\langle Q\theta, \dot{Q}\theta \rangle = \langle \dot{Q}\theta, Q\theta \rangle = \frac{1}{2} \frac{d}{dr} \langle Q\theta, Q\theta \rangle = \frac{1}{2} \left[ \langle Q\theta, \dot{Q}\theta \rangle + \langle Q\theta, Q\theta \rangle \right] = 0.
\]  

(5.3)

Since as a result of [2a], [2b] and [2c] we have that \( f \in C[a, b] \) and \( f(r) \in [a, b] \) for all \( r \in [a, b] \) this immediately gives \( (ii) \). Now to justify \( (iii) \) we begin by first noting that

\[
|\nabla u|^2 = tr\left\{ [\nabla u][\nabla u]^t \right\}
= tr\left\{ \left[ \frac{f}{r} Q + (\frac{j}{r} - \frac{f}{r})Q\theta \otimes \theta + f\dot{Q}\theta \otimes \theta \right] \times \right.
\left[ \frac{f}{r} Q + (\frac{j}{r} - \frac{f}{r})Q\theta \otimes \theta + f\dot{Q}\theta \otimes \theta \right] \right\}
= tr\left\{ \left[ \frac{f}{r} I_n + (\frac{j}{r} - \frac{f}{r})Q\theta \otimes Q\theta + fQ\theta \otimes \dot{Q}\theta \right] + 
(\frac{j}{r} - \frac{f}{r}) \left[ \frac{f}{r} Q\theta \otimes Q\theta + (\frac{j}{r} - \frac{f}{r})Q\theta \otimes Q\theta + fQ\theta \otimes \dot{Q}\theta \right] + 
\left[ f \left( \frac{f}{r} Q\theta \otimes Q\theta + (\frac{j}{r} - \frac{f}{r})\dot{Q}\theta \otimes Q\theta + f\dot{Q}\theta \otimes \dot{Q}\theta \right) \right] \right\}
= n\left( \frac{f}{r} \right)^2 + (\frac{j}{r} - \frac{f}{r})^2 \langle Q\theta, Q\theta \rangle + 2f \frac{j}{r} \langle Q\theta, \dot{Q}\theta \rangle + f^2 \langle \dot{Q}\theta, \dot{Q}\theta \rangle
= (n - 1)\left( \frac{f}{r} \right)^2 + \frac{j^2}{r^2} + f^2 \langle \dot{Q}\theta \rangle^2.
\]  

(5.4)

Next it is evident that

\[
|u|^2 = \langle G(r)\theta, G(r)\theta \rangle = \langle f(r)\theta, f(r)\theta \rangle = |f|^2.
\]  

(5.5)

Hence by combining the latter we can write \(^1\)

\[
\left\| u \right\|^2_{W^{1,2}(\Omega)} = \int_a^b \int_{S^{n-1}} \left\{ f^2 + (n-1) \left( \frac{f}{r} \right)^2 + f^2 +
\int_{\theta} f^2 \langle \dot{Q}\theta \rangle^2 \right\} r^{n-1} dH^{n-1}(\theta) dr
= \omega_n \int_a^b \left\{ f^2 \left[ \frac{n + n(n-1)}{r^2} + \langle \dot{Q}\theta \rangle^2 \right] + nf^2 \right\} r^{n-1} dr,
\]  

(5.6)

\(^1\)Here we are taking advantage of the identity

\[
\int_{S^{n-1}} \langle F, \theta \rangle \ dH^{n-1}(\theta) = \omega_n tr F,
\]

that holds for any given \( F \in M_{n \times n} \). A straight-forward proof of this assertion is in Proposition B.0.8, Appendix B.
and so referring again to $[1a]$ and $[2a]$ the conclusion follows. □

**Proposition 5.1.2.** Suppose that $u$ is a generalised twist as in Definition 3.1.1. Then subject to $Q \in C^2([a,b], \text{SO}(n))$ and $f \in C^2([a,b])$ we have that

$$
\Delta u = [\alpha Q + \beta \dot{Q} + f \ddot{Q}] \theta, \tag{5.7}
$$

where

$$
\alpha := \ddot{f} + \frac{n-1}{r} (\dot{f} - \frac{f}{r}),
$$

and

$$
\beta := 2\dot{f} + \frac{n-1}{r} f.
$$

**Proof.** Referring to Definition 3.1.1 and using the notation $u = (u_1, u_2, \ldots, u_n)$ we can write with the aid of (5.1) in Proposition 5.1.1 that

$$
\Delta u_i = \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left\{ \frac{f}{r} \dot{Q}_{ij} + (\dot{f} - \frac{f}{r}) \sum_{k=1}^{n} Q_{ik} \theta_k \theta_j + f \sum_{k=1}^{n} \dot{Q}_{ik} \theta_k \theta_j \right\}
$$

$$
= \sum_{j=1}^{n} \left\{ \frac{r \dot{f} - f}{r^2} Q_{ij} \theta_j + \frac{f}{r} \dot{Q}_{ij} \theta_j + (\dot{f} - \frac{r \dot{f} - f}{r^2}) \sum_{k=1}^{n} Q_{ik} \theta_k \theta_j^2 + \right. 
$$

$$
\left. (\dot{f} - \frac{f}{r}) \left[ \sum_{k=1}^{n} Q_{ik} \theta_k \theta_j^2 + \frac{1}{r} \sum_{k=1}^{n} Q_{ik} (\delta_{kj} - \theta_k \theta_j) \theta_j + \frac{1}{r} \sum_{k=1}^{n} Q_{ik} \theta_k \theta_j^2 \right] + \dot{f} \sum_{k=1}^{n} \dot{Q}_{ik} \theta_k \theta_j^2 + f \sum_{k=1}^{n} \dot{Q}_{ik} \theta_k \theta_j^2 + \right.
$$

$$
\left. \frac{f}{r} \sum_{k=1}^{n} \dot{Q}_{ik} (\delta_{kj} - \theta_k \theta_j) \theta_j + \frac{f}{r} \sum_{k=1}^{n} \dot{Q}_{ik} \theta_k (1 - \theta_j^2) \right\}.
$$

Hence we have that

$$
\Delta u_i = \dot{f} \sum_{k=1}^{n} Q_{ik} \theta_k + \frac{(n-1)}{r} (\dot{f} - \frac{f}{r}) \sum_{k=1}^{n} Q_{ik} \theta_k + 
$$

$$
2\dot{f} \sum_{k=1}^{n} \dot{Q}_{ik} \theta_k + f \sum_{k=1}^{n} \dot{Q}_{ik} \theta_k + \frac{(n-1)}{r} f \sum_{k=1}^{n} \dot{Q}_{ik} \theta_k + 
$$

$$
\left[ \dot{f} + \frac{(n-1)}{r} (\dot{f} - \frac{f}{r}) \right] \sum_{k=1}^{n} Q_{ik} \theta_k + 
$$

$$
\left[ 2\dot{f} + \frac{(n-1)}{r} f \right] \sum_{k=1}^{n} \dot{Q}_{ik} \theta_k + f \sum_{k=1}^{n} \dot{Q}_{ik} \theta_k.
$$
As this is true for every $1 \leq i \leq n$ using vector notation we can write
\[
\Delta u = \left\{ \left[ \tilde{f} + \frac{(n-1)}{r}(f - \frac{f}{r}) \right] Q + \left[ 2\tilde{f} + \frac{(n-1)}{r}f \right] \dot{Q} + f \ddot{Q} \right\} \theta,
\]
which is the required identity. \hfill \Box

### 5.2 The energy restricted to the space of twists

For a generalised twist $u$ as in Definition 3.1.1 using (5.1) and (5.2) in Proposition 5.1.1 we can write
\[
F[u, \Omega] = \int_{\Omega} \left[ \frac{1}{2}|\nabla u|^2 + \phi(\det(\nabla u)) \right] \, dx
= \frac{1}{2} \int_{a}^{b} \int_{\mathbb{S}^{n-1}} \left\{ (n-1)\left( \frac{f'}{r} \right)^2 + \dot{f}^2 + f^2 |\dot{Q}|^2 + 2\phi(f'\left( \frac{f}{r} \right)^{n-1}) \right\} r^{n-1} \, d\mathcal{H}^{n-1}(\theta) \, dr
= \frac{\omega_n}{2} \int_{a}^{b} \left\{ f^2 \left[ n(n-1)\frac{1}{r^2} + |\dot{Q}|^2 \right] + n\dot{f}^2 + 2n\phi(f'\left( \frac{f}{r} \right)^{n-1}) \right\} r^{n-1} \, dr.
\]
Motivated by the above representation in what follows we introduce the energy functional
\[
E[Q, f] := \int_{a}^{b} \left\{ f^2 \left[ n(n-1)\frac{1}{r^2} + |\dot{Q}|^2 \right] + n\dot{f}^2 + 2n\phi(f'\left( \frac{f}{r} \right)^{n-1}) \right\} r^{n-1} \, dr,
\]
over the space of admissible maps
\[
\mathcal{E} := \left\{ (Q, f) : \begin{array}{l}
Q \in W^{1,2}([a, b], SO(n)), \\
Q(a) = Q(b) = I_n,
\end{array}
\begin{array}{l}
f \in W^{1,2}[a, b], \\
f' > 0 \mathcal{L}^{1}\text{-a.e. on } (a, b), \\
f(a) = a, f(b) = b.
\end{array}\right\}
\]
Our primary objective here is to obtain the Euler-Lagrange equations associated with the energy functional $E$ over the space $\mathcal{E}$. Before that, we recall the Proposition 3.2.1 which in effect gives a
characterisation of the tangent space to the orthogonal group at an arbitrary matrix $Q \in \text{SO}(n)$. This will turn useful in computing the aforementioned Euler-Lagrange equations.

**Proposition 5.2.1.** Let $(Q, f) \in \mathcal{E}$ with $Q \in C^2([a, b], \text{SO}(n))$, $f \in C^2([a, b])$ and $\dot{f} > 0$ on $[a, b]$. Then assuming $\mathbb{E}[Q, f] < \infty$ the Euler-Lagrange equations associated with $\mathbb{E}$ over $\mathcal{E}$ at $(Q, f)$ take the form

$$\mathbb{E}[Q, f] = 0,$$

that is,

$$\begin{cases}
(i) & \frac{d}{dr} \left[ r^{n-1} f^2 Q^t \frac{d}{dr} Q \right] = 0, \\
(ii) & \frac{d}{dr} \left[ r^{n-1} \dot{f} + f^{-1} \phi' \right] = (n-1) [r^{n-3} \dot{f} + \dot{f} f^{-2} \phi'] + \frac{1}{n} r^{n-1} f |\dot{Q}|^2,
\end{cases}$$

on $[a, b]$ where $\phi' = \phi'(\dot{f})^{n-1}$.

**Proof.** First fix $Q$ as described and for $\varepsilon \in \mathbb{R}$ put $Q, f \varepsilon = Q + \varepsilon Q (F - F^t)$ where $F \in C^\infty_0([a, b], \mathbb{M}_{n \times n})$. Then by utilising Proposition 3.2.1 we can write

$$0 = \frac{d}{d\varepsilon} \mathbb{E}[Q, f] \bigg|_{\varepsilon=0}$$

$$0 = \frac{d}{d\varepsilon} \left[ \int_a^b \left\{ f^2 \left[ n(n-1) \frac{1}{r^2} + |\dot{Q}|^2 \right] + n \dot{f}^2 + 2n \phi(f^{(\frac{1}{r})^{n-1}}) \right\} r^{n-1} dr \right] \bigg|_{\varepsilon=0}$$

$$= \int_a^b r^{n-1} f^2 \dot{Q}^t (Q^t \dot{Q}, (F - F^t)) dr$$

$$= - \int_a^b \left( \frac{d}{dr} [r^{n-1} f^2 Q^t \dot{Q}] (F - F^t) \right) dr.$$

Note that in concluding the last line we have used integration by parts together with the boundary conditions $F(a) = F(b) = 0$. Now in view of $Q^t \dot{Q}$ being skew-symmetric it follows that

$$\frac{d}{dr} \left[ r^{n-1} f^2 Q^t \frac{d}{dr} Q \right] = 0,$$

which is the first equation in the system.

Next fix $f$ as described and for $\varepsilon \in \mathbb{R}$ put $f, f \varepsilon = f + \varepsilon g$ where $g \in C^\infty_0([a, b])$. As $\dot{f} \in C([a, b])$ and $K := \text{supp } g \subset [a, b]$ is compact it follows that $\dot{f} \geq c > 0$ on $K$. Thus for $|\varepsilon|$ sufficiently small ($|\varepsilon| \times \text{sup}_{[a, b]} |\dot{g}| < c$) we have $\dot{f} > 0$ on $[a, b]$ and so $(Q, f, f \varepsilon) \in \mathcal{E}$. 


In addition by choosing $\varepsilon$ smaller we have $E[\mathbf{Q}, f_\varepsilon] < \infty$. The latter follows from the observation that $\dot{f}_\varepsilon = \dot{f}$ on $[a, b]\setminus \mathbf{K}$, the assumption $E[\mathbf{Q}, f] < \infty$ and the lower and upper bounds

$$\frac{c}{2} \leq \dot{f}_\varepsilon = \dot{f} + \varepsilon \dot{g} \leq \sup_{\mathbf{K}} \dot{f} + |\varepsilon| \sup_{[a, b]} |\dot{g}|,$$

on $\mathbf{K}$ provided that $c \geq 2|\varepsilon| \sup_{[a, b]} |\dot{g}|$.

We can now proceed by considering the variations of $E$ along the path $(\mathbf{Q}, f_\varepsilon)$ and as a result we can write

$$0 = \frac{d}{d\varepsilon} E[\mathbf{Q}, f_\varepsilon] \bigg|_{\varepsilon = 0}$$

$$= \frac{d}{d\varepsilon} \left[ \int_a^b \left\{ f_\varepsilon^2 \left[ n(n-1)\frac{1}{r^2} + |\dot{\mathbf{Q}}|^2 \right] + n \dot{f}_\varepsilon^2 + 2n\phi(\dot{f}_\varepsilon r^{n-1}) \right\} r^{n-1} dr \right] \bigg|_{\varepsilon = 0}$$

$$= \int_a^b \left\{ 2fg \left[ n(n-1)\frac{1}{r^2} + |\dot{\mathbf{Q}}|^2 \right] + 2n\dot{f}\dot{g} + 2n \left( \left( \frac{f}{r} \right)^{n-1}\dot{g} + (n-1)\frac{1}{r^{n-1}} \dot{f} f^{n-2}g \right) \phi' \right\} r^{n-1} dr$$

$$= \int_a^b \left( n-1 \right) \left[ r^{n-3}f + \dot{f} f^{n-2}\phi' + \frac{1}{n} r^{n-1} f|\dot{\mathbf{Q}}|^2 \right] g dr + \int_a^b \left[ r^{n-1}\dot{f} + f^{n-1}\phi' \right] \dot{g} dr.$$ 

Now using integration by parts on the second term on the right together with the fact that $g(a) = g(b) = 0$ we obtain

$$0 = \int_a^b \left( n-1 \right) \left[ r^{n-3}f + \dot{f} f^{n-2}\phi' + \frac{1}{n} r^{n-1} f|\dot{\mathbf{Q}}|^2 \right] g dr - \int_a^b \frac{d}{dr} \left[ r^{n-1}\dot{f} + f^{n-1}\phi' \right] g dr.$$

As the latter is true for all $g \in C_0^\infty([a, b])$ it follows that

$$\frac{d}{dr} \left[ r^{n-1}\dot{f} + f^{n-1}\phi' \right] = (n-1)\left[ r^{n-3}f + \dot{f} f^{n-2}\phi' + \frac{1}{n} r^{n-1} f|\dot{\mathbf{Q}}|^2 \right],$$

on $[a, b]$ which is the second equation in the system. This completes the proof. \hfill \Box

Any twist loop $\mathbf{G} = f\mathbf{Q}$ forming a solution to the Euler-Lagrange equations associated with $E$ over $\mathcal{E}$ [i.e., whose corresponding $f$, $\mathbf{Q}$ satisfy (i), (ii) above] will be referred to as a stationary loop.
5.3 Energy minimizing loops in homotopy classes

Consider as in the previous section the energy functional

\[ E(Q, f) := \int_a^b \left\{ f^2 \left[ (n-1) \frac{1}{r^2} + |\dot{Q}|^2 \right] + n \dot{f}^2 + 2n \phi \left( f \right)^n \right\} r^{n-1} dr, \]

over the space of admissible maps

\[
\mathcal{E} := \left\{ (Q, f) : \begin{array}{l}
Q \in W^{1,2}([a, b], \text{SO}(n)), \\
Q(a) = Q(b) = I_n, \\
f \in W^{1,2}[a, b], \\
\dot{f} > 0 \text{ } \mathcal{L}^1\text{-a.e. on } (a, b), \\
f(a) = a, f(b) = b.
\end{array} \right\}
\]

According to an elementary version of the Sobolev embedding theorem any pair \((Q, f) \in \mathcal{E}\) has a continuous representative [again denoted \((Q, f)\)]. In particular each such \(Q\) represents an element of the fundamental group \(\pi_1[\text{SO}(n)]\) denoted \([Q]\). As is well-known

\[ \pi_1[\text{SO}(n)] \cong \begin{cases} 
\mathbb{Z} & \text{when } n = 2, \\
\mathbb{Z}_2 & \text{when } n \geq 3,
\end{cases} \]

and these facts together enables one to introduce the following partitioning of the Sobolev space \(\mathcal{E}\).

[1] \((n = 2)\) for each \(m \in \mathbb{Z}\) put

\[ c_m[\mathcal{E}] := \left\{ (Q, f) \in \mathcal{E} : |Q| = m \right\}. \]

As a result the latter are pairwise disjoint and that

\[ \mathcal{E} = \bigcup_{m \in \mathbb{Z}} c_m[\mathcal{E}]. \]

[2] \((n \geq 3)\) for each \(\alpha \in \mathbb{Z}_2 = \{0, 1\}\) put

\[ c_{\alpha}[\mathcal{E}] := \left\{ (Q, f) \in \mathcal{E} : |Q| = \alpha \right\}. \]
As a result, again, the latter are pairwise disjoint and that
\[ \mathcal{E} = \bigcup_{\alpha \in \mathbb{Z}_2} \mathcal{c}_\alpha[\mathcal{E}] \].

An application of the direct methods of the calculus of variations to the energy functional \( \mathcal{E} \) together with the observation that the homotopy classes \( \mathcal{c}_*[\mathcal{E}] \subset \mathcal{E} \) are sequentially weakly closed results in the existence of [multiple] minimizing stationary loops (See Theorem 5.3.2.). We note that the sequential weak closedness of the homotopy classes \( \mathcal{c}_*[\mathcal{E}] \) is a result of \( \text{SO}(n) \) having a tubular neighbourhood that projects back onto it and this in turn follows from \( \text{SO}(n) \) being a smooth compact manifold.

We begin by first establishing the following straight-forward lower bound on \( \mathcal{E} \) amounting to it being coercive on \( \mathcal{E} \).

**Proposition 5.3.1.** (Coercivity). There exists \( d = d(n,a,b) > 0 \) such that
\[
\mathcal{E}[Q,f] \geq d\left(\|Q\|_{W^{1,2}}^2 + \|f\|_{W^{1,2}}^2\right),
\]
for all \( (Q,f) \in \mathcal{E} \).

**Proof.** Since for all \( (Q,f) \in \mathcal{E} \) we have that \( a \leq f \leq b \) on \([a,b]\) by taking into account that \( \phi \geq 0 \) we can write
\[
\mathcal{E}[Q,f] = \int_a^b \left\{ f^2 \left[ n(n - 1)\frac{1}{r^2} + |\dot{Q}|^2 \right] + n\dot{f}^2 + 2n\phi(\frac{f}{r})^{n-1} \right\} r^{n-1} \, dr 
\geq \int_a^b \left\{ n \left[ (n - 1)\frac{1}{r^2} f^2 + \dot{f}^2 \right] + f^2|\dot{Q}|^2 \right\} r^{n-1} \, dr.
\]
The conclusion now follows by utilising the Poincaré inequality. \( \square \)

**Theorem 5.3.2.** (Existence of energy minimizing loops).
Consider the energy functional \( \mathcal{E} \) over the space of admissible maps \( \mathcal{E} \). Then,

1. (\( n = 2 \)) for each \( m \in \mathbb{Z} \) there exists \( (Q_m,f_m) \in \mathcal{c}_m[\mathcal{E}] \) such that
\[
\mathcal{E}[Q_m,f_m] = \inf_{\mathcal{c}_m[\mathcal{E}]} \mathcal{E}.
\]
2. (\( n \geq 3 \)) for each \( \alpha \in \mathbb{Z}_2 \) there exists \( (Q_\alpha,f_\alpha) \in \mathcal{c}_\alpha[\mathcal{E}] \) such that
\[
\mathcal{E}[Q_\alpha,f_\alpha] = \inf_{\mathcal{c}_\alpha[\mathcal{E}]} \mathcal{E}.
\]
Proof. First of all we note that each homotopy class \( c_\ast[E] \subset \mathcal{E} \) admits a pair \((Q, f)\) for which \( E(Q, f) < \infty \) follows by taking, e.g., \( f = r \) and \( Q \) a smooth loop representing the corresponding element of \( \pi_1(\text{SO}(n)) \). Let \((Q_j, f_j) \subset c_\ast[E]\) denote an infimizing sequence for \( E \) over \( c_\ast[E] \). Then appealing to Proposition 5.3.1 it follows that by passing to a subsequence (not re-labeled) we have that

\[
\begin{cases}
Q_j \to Q & \text{in } C([a, b], \text{SO}(n)), \\
Q_j & \text{in } W^{1,2}([a, b], \text{SO}(n)), \\
f_j & \text{in } C[a, b], \\
f_j & \text{in } W^{1,2}(a, b).
\end{cases}
\]

As a result we conclude in particular that \( a \leq f \leq b \) on \([a, b]\) and additionally

\[
f_j \dot{Q}_j \rightharpoonup f \dot{Q}
\]

\[
f_j \left( \frac{f_j}{r} \right)^{n-1} \rightharpoonup f \left( \frac{f}{r} \right)^{n-1},
\]

where both convergences are interpreted as weakly in \( L^2 \). Therefore by standard lower semicontinuity results (see, e.g., [18]) we have that

\[
\inf_{c_\ast[E]} E \leq E(Q, f) \\
\leq \liminf_{j \to \infty} E(Q_j, f_j) \\
\leq \inf_{c_\ast[E]} E < \infty.
\]

(5.8)

The above firstly implies that \( \dot{f} > 0 \) \( L^1 \)-a.e. on \([a, b]\) and as a result \((Q, f) \in \mathcal{E}\). This in view of the closedness of the homotopy classes \( c_\ast[E] \subset \mathcal{E} \) with respect to the topology of uniform convergence gives

\[
|Q| = |Q_j|,
\]

and therefore \((Q, f) \in c_\ast[E]\). A second appeal to (5.8) now implies \((Q, f)\) to be the required minimizer on \( c_\ast[E]\). \[\square\]

Remark 5.3.3. It can be shown that in [1] and [2] above the resulting minimizers satisfy the corresponding Euler-Lagrange equations described in Proposition 5.2.1. The argument here will follow closely that given in detail in the proof of Theorem 5.4.3 and hence will be abbreviated.
5.4 Alternative construction of multiple stationary loops

In section 5.3 we proved the existence of multiple stationary loops by directly minimizing the energy functional $E$ over the homotopy classes $c_*[\mathcal{E}]$ of the loop space $\mathcal{E}$. By contrast in this section we focus on the Euler-Lagrange equation itself and present a class of stationary loops that in turn will prove fruitful in discussing the existence of multiple solutions to the Euler-Lagrange equations associated with the energy functional $F$ over the space $A(\Omega)$.

Indeed here we establish the existence of multiple (infinitely many) stationary loops $G = fQ$ where the pair $(Q, f) \in \mathcal{E}$, depending on whether the dimension $n$ is even or odd, has one of the following specific forms.

[1] $(n = 2k)$

$$Q = Q[g] := P \text{diag}(\mathcal{R}(g_1), \mathcal{R}(g_2), \ldots, \mathcal{R}(g_k))P^t,$$

[2] $(n = 2k + 1)$

$$Q = Q[g] := P \text{diag}(\mathcal{R}(g_1), \mathcal{R}(g_2), \ldots, \mathcal{R}(g_k), 1)P^t,$$

where $P \in O(n)$ is fixed, $g = (g_1, g_2, \ldots, g_k) \in J_m$ (see below) and the matrix $\mathcal{R} \in M_{2 \times 2}$ is given by

$$\mathcal{R}(s) := \begin{bmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{bmatrix}.$$ 

Indeed for $m = (m_1, m_2, \ldots, m_k) \in \mathbb{Z}^k$ we put

$$J_m = J_m(a, b) := \left\{ (g, f) : \begin{array}{l} g \in W^{1,2}(a, b)^k, \\
g(a) = 0, g(b) = 2\pi m, \\
f \in W^{1,2}(a, b), \\
f' > 0 \text{ } \mathcal{L}^1\text{-a.e. on } (a, b), \\
f(a) = a, f(b) = b \end{array} \right\}$$

It is thus evident that for each such $m$ and fixed $P \in O(n)$ and with $Q = Q[g]$ we have that

$$(g, f) \in J_m \iff (Q, f) \in \mathcal{E}.$$
Next for \((g, f) \in J_m\) as described above and fixed \(P \in O(n)\) denoting again \(Q = Q[g]\) we introduce

\[
J[g, f] := \mathbb{E}[Q, f]
\]

\[
= \int_a^b \left\{ f^2 \left[ \frac{1}{r^2} n(n-1) + 2 |\dot{Q}|^2 \right] + n \dot{f}^2 + 2n \phi(\frac{f}{r})^{n-1} \right\} r^{n-1} dr
\]

\[
= \int_a^b \left\{ f^2 \left[ \frac{1}{r^2} n(n-1) + 2 |\dot{g}|^2 \right] + n \dot{f}^2 + 2n \phi(\frac{f}{r})^{n-1} \right\} r^{n-1} dr
\]

\[= : \int_a^b J(r, \dot{g}, f, \dot{f}) dr, \quad (5.9)\]

where we have set \(|\dot{g}|^2 = \sum_{j=1}^k \dot{g}_j^2\) whilst

\[
J(r, s, z, p) := n \left\{ z^2 \left[ \frac{1}{r^2} (n-1) + \frac{2}{n} |g|^2 \right] + p^2 + 2 \phi(p(z)^{n-1}) \right\} r^{n-1}.
\]

**Proposition 5.4.1.** There exists \(L > 0\) and \(\sigma > 0\) so that for all \(\alpha > 0\) satisfying \(|\alpha - 1| < \sigma\) we have that

\[
\left| zJ_z(r, s, \alpha z, p) \right| \leq L \left( J(r, s, z, p) + 1 \right),
\]

for all \(r \in [a, b], s \in \mathbb{R}^k, z \in [0, \infty[\text{ and } p \in [0, \infty[.\)

**Proof.** This follows by direct verification and use of \([h5]\). \(\square\)

**Proposition 5.4.2.** For fixed \(r \in [a, b], s \in \mathbb{R}^k\) and \(z \in [0, \infty[\) the function \(J_p\) has the following limiting behaviours

\[
\lim_{p \downarrow 0} J_p(r, s, z, p) = -\infty,
\]

\[
\lim_{p \uparrow \infty} J_p(r, s, z, p) = \infty.
\]

**Proof.** This is an immediate consequence of \([h2]\)-\([h4]\). \(\square\)

We are now in a position to state the main result of this section on existence of infinitely many stationary loops \((Q, f) \in \mathcal{E}\) for which \(Q = Q[g]\).

**Theorem 5.4.3.** (Existence and regularity of multiple stationary loops). Consider the energy functional \(J\) over the space \(J_m\). Then for each \(m \in \mathbb{Z}^k\) there exists \((g, f) \in J_m\) such that

\[
J[g, f] = \inf_{J_m} J[\cdot].
\]
In addition the pair \((g, f)\) satisfies the corresponding Euler-Lagrange equations

\[
\text{EL}[g, f] = 0,
\]

that is,

\[
\begin{align*}
\frac{d}{dt} \left[ (n-1)f^2 \dot{g} \right] &= 0, \\
\frac{d}{dt} \left[ (n-1)\dot{f} + f^{n-1}\phi' \right] &= (n-1)[r^{n-3}f + \dot{f}f^{n-2}\phi'] + \frac{2}{n}r^{n-1}f|\dot{g}|^2,
\end{align*}
\]

on \([a, b]\) where \(\phi' = \phi'(f_r^{(n-1)})\) whilst \((g, f)\) ∈ \(\mathbb{C}_2[a, b] \times \cdots \times \mathbb{C}_2[a, b]\) and \(\dot{f} > 0\) on \([a, b]\).

Note that the above Euler-Lagrange equations will be shown to be satisfied by the pair \((g, f)\) as a result of its minimizing property. One can then verify that the latter equations result from those in Proposition 5.2.1 upon making the substitution \((Q, f) = (Q[g], f)\). Thus any such \((g, f)\) gives rise to an associated stationary loop!

**Proof.** (Existence) Let \((g_j, f_j) \subset J_m\) denote an infimizing sequence for \(J\) over \(J_m\). An application of Proposition 5.3.1 (with \(Q_j := Q[g_j]\)) gives

\[
\infty > J[g_j, f_j] = E[Q_j, f_j] \geq d(\|Q_j\|_{W^{1,2}}^2 + \|f_j\|_{W^{1,2}}^2) \\
\geq d \left[ n(b - a) + 2\|g_j\|_{L^2}^2 + \|f_j\|_{W^{1,2}}^2 \right],
\]

and so as a result \((g_j, f_j) \subset J_m\) is bounded. It thus follows that by passing to a subsequence (not re-labeled) we have that

\[
\begin{align*}
g_j &\rightarrow g \quad \text{in } \mathbb{C}[a, b], \\
g_j &\rightarrow g \quad \text{in } W^{1,2}(a, b), \\
f_j &\rightarrow f \quad \text{in } \mathbb{C}[a, b], \\
f_j &\rightarrow f \quad \text{in } W^{1,2}(a, b).
\end{align*}
\]

Hence in particular \(a \leq f \leq b\) on \([a, b]\) and that

\[
\begin{align*}
f_j \dot{g}_j &\rightarrow f\dot{g} \\
\left( f_r \right)^{n-1} &\rightarrow \left( f_r \right)^{n-1},
\end{align*}
\]
where both convergences are interpreted as \textit{weakly} in $L^2$. Therefore, again, by standard lower semicontinuity results we have that

\begin{align*}
\inf_{J_m} J & \leq J[g, f] \\
& \leq \liminf_{j \to \infty} J[g_j, f_j] \\
& \leq \inf_{J_m} E < \infty. \tag{5.10}
\end{align*}

The above \textit{firstly} implies that $\dot{f} > 0$ $L^1$-a.e. on $[a, b]$ which gives $(g, f) \in J_m$ and \textit{secondly} that $(g, f)$ is the required minimizer.

\textit{(Regularity)} Let $(g, f) \in J_m$ be the minimizer from the previous part. For the sake of clarity and convenience we present the proof of this in the following \textit{three} steps.

\textbf{Step 1} $(g \in C^1[a, b])$ Evidently $f \in C[a, b]$ and $a \leq f \leq b$ on $[a, b]$. Hence the assertion follows immediately in view of $g$ minimizing the integral

$$h \mapsto \int_a^b |\dot{h}|^2 f^2 r^{n-1} dr,$$

among all $h$ with $(h, f) \in J_m$.

\textbf{Step 2} $(f \in C^1[a, b])$ The argument here is based upon \textit{suitably} modifying a well-known technique from [4], Theorem 7.3. To this end for $j \in \mathbb{N}$ put

$$E_j := \left\{ r \in [a, b] : j^{-1} \leq \dot{f}(r) \leq j \right\}.$$

Then $(E_j)$ is \textit{monotone} increasing and $L^1([a, b] \cup \cup_{j=1}^{\infty} E_j) = 0$. Now fix $j$ and pick $w \in L^\infty(a, b)$ such that

$$\int_{E_j} w = \int_a^b w 1_{E_j} = 0. \tag{5.11}$$

For $\varepsilon \in \mathbb{R}$ put

$$f_\varepsilon(r) := f(r) + \varepsilon \int_a^r w 1_{E_j}.$$

Then we have that

1. $f_\varepsilon(a) = f(a) = a$,
2. $f_\varepsilon(b) = f(b) = b$,
3. $\dot{f_\varepsilon}(r) = \dot{f}(r)$ for $L^1$-a.e. $r \notin E_j$,
4. $\dot{f_\varepsilon}(r) > 0$ $L^1$-a.e. on $[a, b]$, provided that $|\varepsilon| \times ||w||_{L^\infty(a, b)} < j^{-1}.$
Chapter 5. Polyconvexity and generalised twists

The aim is now to derive the Euler-Lagrange equation associated with \( f \) as a result of differentiating the energy functional \( J \) along \( f_\varepsilon \) at \( \varepsilon = 0 \). To this end consider first the difference quotient

\[
\left| \frac{J(r, \dot{g}, f_\varepsilon, \dot{f}_\varepsilon) - J(r, \dot{g}, f, \dot{f})}{\varepsilon} \right| \leq \left| \frac{J(r, \dot{g}, f_\varepsilon) - J(r, \dot{g}, f)}{\varepsilon} \right| + \left| \frac{J(r, \dot{g}, \dot{f}_\varepsilon) - J(r, \dot{g}, \dot{f})}{\varepsilon} \right| = I + II.
\]

Then an application of the mean value theorem gives

\[
I = \left| \frac{J(r, \dot{g}, f_\varepsilon) - J(r, \dot{g}, f)}{\varepsilon} \right| \leq c,
\]

where \( c = c(j) > 0 \) is independent of \( \varepsilon \). [Notice that indeed \( I = 0 \) for \( \mathcal{L}^1 \)-a.e. \( r \notin E_j \).] In a similar way we have that

\[
II = \left| \frac{J(r, \dot{g}, f_\varepsilon, \dot{f}) - J(r, \dot{g}, f, \dot{f})}{\varepsilon} \right| = \left| J_z(r, \dot{g}, f + \theta[f_\varepsilon - f], \dot{f}) \right| \left| \frac{f_\varepsilon - f}{\varepsilon} \right|,
\]

where \( \theta = \theta(\varepsilon, r) \in [0, 1] \). However, since

\[
f + \theta[f_\varepsilon - f] = f \left[ 1 + \theta \frac{f_\varepsilon - f}{f} \right] = f \left[ 1 + \varepsilon \theta \frac{1}{f} \int_a^r w 1_{E_j} \right],
\]

it follows from Proposition 5.4.1 that upon choosing \( \varepsilon \) sufficiently small we can write

\[
II \leq L \left[ J(r, \dot{g}, f, \dot{f}) + 1 \right] \int_a^r w 1_{E_j} =: F(r),
\]

where \( F \in L^1(a, b) \) [note that \( J[g, f] < \infty \)]. Hence an application of Lebesgue’s theorem on dominated convergence gives

\[
0 = \frac{d}{d\varepsilon} \left. J[g, f_\varepsilon] \right|_{\varepsilon = 0} = \lim_{\varepsilon \to 0} \int_a^b \frac{J(r, \dot{g}, f_\varepsilon, \dot{f}_\varepsilon) - J(r, \dot{g}, f, \dot{f})}{\varepsilon} dr = \int_a^b \left[ J_p(r, \dot{g}, f, \dot{f}) w 1_{E_j} + J_z(r, \dot{g}, f, \dot{f}) \int_a^r w 1_{E_j} \right] dr.
\]
\[ = \int_{a}^{b} w_{1_{E_{j}}} \left[ \mathbf{J}_{p}(r, \dot{g}, f, \dot{f}) - \int_{a}^{r} \mathbf{J}_{z}(\rho, \dot{g}, f, \dot{f}) \, d\rho \right] \, dr, \]

where in concluding the last line we have used a convenient form of integration by parts. Therefore recalling (5.11) it follows from the above (note that arguing as in estimating \( \| \| \) above [see Proposition 5.4.1] and taking into account \( \mathbf{J}[g, f] < \infty \) and \( a \leq f \leq b \) it follows that \( \mathbf{J}_{z}(r, \dot{g}, f, \dot{f}) \) is summable on \( ]a, b[ \) for \( \mathcal{L}^{1}\)-a.e. \( r \in E_{j} \). [Here \( c_{j} \) is an arbitrary constant.] Now in view of \( \bigcup_{j=1}^{\infty} E_{j} \) having full measure in \( ]a, b[ \) it follows that \( \mathbf{J}_{p}(r, \dot{g}, f, \dot{f}) = c + \int_{a}^{r} \mathbf{J}_{z}(\rho, \dot{g}, f, \dot{f}) \, d\rho \), (5.12) for \( \mathcal{L}^{1}\)-a.e. \( r \in ]a, b[ \). As the term on the right is absolutely continuous on \([a, b],[/\) using [h2] and Proposition 5.4.2, it follows that by modifying \( \dot{f} \) on an \( \mathcal{L}^{1}\)-null set, we have \( \dot{f} > 0 \) and equality in (5.12) holds everywhere on \([a, b],[/\) (hence \( \mathbf{J}_{p}(r, \dot{g}, f, \dot{f}) \) is continuous on \([a, b],[/\). Standard arguments (see, e.g., [4] pp. 584 or [18] pp. 57-61) now give the continuity of \( \dot{f} \) on \([a, b],[/\). A close inspection of the proof of Theorem 2.6(ii) in [18] reveals that having (5.12) the same conclusion holds if the assumption \( \mathbf{J} \) being of class \( \mathbf{C}^{1}\) is replaced by \( \mathbf{J}_{\dot{f}} \) being of class \( \mathbf{C}^{1}\). In a similar way the conclusion of Theorem 2.6(iii) holds if the assumption \( \mathbf{J} \) being of class \( \mathbf{C}^{2}\) is replaced by \( \mathbf{J}_{\dot{f}} \) being of class \( \mathbf{C}^{1}\).

**5.5 The restricted versus the full Euler-Lagrange equations**

In this section we discuss in detail the implications that the Euler-Lagrange equations associated with the energy functional \( \mathcal{F} \) will exert upon the twist loop \( \mathbf{G} = fQ \) of a generalised twist \( u \in \mathcal{A}(\Omega) \).

**Theorem 5.5.1.** Let \( \Omega = \{ x \in \mathbb{R}^{n} : a < |x| < b \} \) and let \( u \) be a generalised twist as in Definition 3.1.1. Assume in addition that the twist loop \( \mathbf{G} = fQ \) satisfies the following assumptions.

[1] \( Q \in \mathcal{C}^{2}([a, b], \text{SO}(n)) \),

[2] \( f \in \mathcal{C}^{2}([a, b]) \),

[3] \( \dot{f} > 0 \) on \( ]a, b[ \),

[4] \( (Q, f) \in \mathcal{E} \).
Then the following implication holds.

\[ \text{EL}[(Q, f)] = 0 \implies \mathbb{E}[u] := [\nabla u]^t \Delta u + (\det \nabla u) \nabla \left[ \phi'(\det \nabla u) \right] = f \left[ \frac{1}{n} \dot{f}|Q|^2 I_n - \frac{f}{r} Q^t \dot{Q} - (\dot{f} - \frac{f}{r}) |\dot{Q}|^2 I_n \right] \theta. \]

**Proof.** We proceed by evaluating each of the expressions in \( \mathbb{E}[u] \) separately. Indeed with regards to the first term using (5.1) in Proposition 5.1.1 in conjunction with (5.7) in Proposition 5.1.2 we can write

\[
[\nabla u]^t \Delta u = \left\{ \frac{f}{r} Q^t + (\dot{f} - \frac{f}{r}) \theta \otimes Q \theta + f \theta \otimes \dot{Q} \right\} \left[ \alpha Q + \beta \dot{Q} + f \ddot{Q} \right] \theta
\]

\[
= \left\{ \alpha \frac{f}{r} I_n + \beta \frac{f}{r} Q^t \dot{Q} + \frac{f^2}{r} Q^t \ddot{Q} + \frac{f}{r} \alpha (Q \theta, Q \theta) + \beta (Q \theta, \dot{Q} \theta) + f(Q \theta, \ddot{Q} \theta) \right\} \left( \dot{f} - \frac{f}{r} \right) I_n + \alpha \frac{f}{r} I_n + \beta (Q \theta, \dot{Q} \theta) I_n + \alpha \dot{f} I_n + \beta (Q \theta, \ddot{Q} \theta) I_n + \frac{f}{r} (Q \theta, \ddot{Q} \theta) I_n + \alpha \dot{f} I_n + \beta (Q \theta, \ddot{Q} \theta) I_n + \frac{f}{r} (Q \theta, \ddot{Q} \theta) I_n \right\} \theta
\]

\[
= \left\{ \frac{f}{r} \left[ \beta Q^t \dot{Q} + f Q^t \ddot{Q} \right] + f(\dot{f} - \frac{f}{r})(Q \theta, \dot{Q} \theta) I_n + \alpha \dot{f} I_n + \beta (Q \theta, \ddot{Q} \theta) I_n + \frac{f}{r} (Q \theta, \ddot{Q} \theta) I_n \right\} \theta,
\]

where in concluding the last equation we have made repeated use of the identity (5.3).

Now referring to the Euler-Lagrange equations in Proposition 5.2.1 it follows upon expansion of (i) that

\[
\frac{1}{r^{n-1}} \frac{d}{dr} \left[ r^{n-1} f^2 Q^t \ddot{Q} \right] = (n-1) \frac{f^2}{r} Q^t \dot{Q} + 2f \dot{f} Q^t \dot{Q} + f^2 Q^t \ddot{Q} + f^2 Q^t \ddot{Q}
\]

\[
= f \left\{ \left( (n-1) \frac{f}{r} + 2 \dot{f} \right) Q^t \dot{Q} + f Q^t \dot{Q} + f Q^t \ddot{Q} \right\}
\]

\[
= f \left[ \beta Q^t \dot{Q} + f Q^t \ddot{Q} + f Q^t \ddot{Q} \right] = 0.
\]

(5.14)

By pre-multiplying (5.14) with \( \dot{Q} \theta \) and ignoring the non-zero factor \( f \) we can thus conclude that

\[
0 = \dot{Q} \theta \left[ \beta Q^t \dot{Q} + f Q^t \dot{Q} + f Q^t \ddot{Q} \right]
\]

\[
= \beta \dot{Q} \theta + f Q \theta Q^t \dot{Q} + f \dot{Q} \theta \ddot{Q}.
\]
However in view of the second term in the last line being skew-symmetric the above equation in particular implies that for all $\theta \in S^{n-1}$ we have

$$
\langle [\beta \dot{Q}' \dot{Q} + f \dot{Q}' \dot{Q}' \dot{Q} + f \dot{Q}' \ddot{Q}] \theta, \theta \rangle = \beta |\dot{Q} \theta|^2 + f \langle \dot{Q} \theta, \ddot{Q} \theta \rangle = 0.
$$

(5.15)

In a similar way referring to (5.3) we have that

$$
\frac{d}{dr} \langle Q \theta, \dot{Q} \theta \rangle = \langle \dot{Q} \theta, \dot{Q} \theta \rangle + \langle Q \theta, \ddot{Q} \theta \rangle = |\dot{Q} \theta|^2 + \langle Q \theta, \ddot{Q} \theta \rangle = 0.
$$

(5.16)

Therefore by substituting (5.14), (5.15) and (5.16) into (5.13) respectively we arrive at the identity

$$
[\nabla u]^t \Delta u = \left\{ \frac{f}{r} \left[ \beta Q' \dot{Q} + f Q' \ddot{Q} \right] + f (\dot{f} - \frac{f}{r^2}) \langle Q \theta, \ddot{Q} \theta \rangle I_n + \alpha f I_n + \left[ \beta |\dot{Q} \theta|^2 + f \langle \dot{Q} \theta, \ddot{Q} \theta \rangle \right] I_n \right\} \theta,
$$

(5.17)

Next referring again to the Euler-Lagrange equation in Proposition 5.2.1 it follows upon expansion of (ii) that

$$
(\det \nabla u) \nabla \left[ \phi' (\det \nabla u) \right] = \frac{f}{r} (\frac{f}{r})^{n-1} \frac{d}{dr} \left[ \phi' (\frac{f}{r})^{n-1} \right] \theta
$$

$$
= - \left[ \frac{(n-1)}{r} \dot{f} \ddot{f} + f \ddot{f} - \frac{(n-1)}{r^2} \dot{f} \dddot{f} - \frac{1}{n} f \dot{f} |\dot{Q}'|^2 \right] \theta
$$

$$
= - \left[ \alpha \dot{f} - \frac{1}{n} f \dot{f} |\dot{Q}'|^2 \right] \theta.
$$

(5.18)

Therefore, by combining (5.17) and (5.18), we arrive at

$$
\mathbb{E} \nabla u = [\nabla u]^t \Delta u + (\det \nabla u) \nabla \left[ \phi' (\det \nabla u) \right]
$$

$$
= [\nabla u]^t \Delta u + (\det \nabla u) \phi'' (\det \nabla u) \nabla \left[ \det \nabla u \right]
$$

$$
= \left[ - \frac{f^2}{r} \dot{Q}' \ddot{Q} - f (\dot{f} - \frac{f}{r}) |\dot{Q} \theta|^2 I_n + \alpha f I_n + \right.
$$
Chapter 5. Polyconvexity and generalised twists

\[
\frac{1}{n} f \dot{j} [\dot{Q}]^2 I_n - \alpha f I_n \theta \\
= \left[ \frac{1}{n} f \dot{j} [\dot{Q}]^2 I_n - \frac{f^2}{r} \dot{Q}' \dot{Q} - f (\dot{j} - \frac{f}{r}) [\dot{Q} \theta]^2 I_n \right] \theta ,
\]

which is the required conclusion. □

**Theorem 5.5.2.** Let \( \Omega = \{ x \in \mathbb{R}^n : a < |x| < b \} \) and let \( u \) be a generalised twist as in Definition 3.1.1. Assume \( F[u, \Omega] < \infty \) and that the twist loop \( G = fQ \) satisfies the following assumptions.

1. \( Q \in C^2([a, b], SO(n)) \),
2. \( f \in C^2([a, b]) \),
3. \( \dot{f} > 0 \) on \( [a, b] \),
4. \( (Q, f) \in \mathcal{E} \).

Then the following equivalence between the full and the restricted Euler-Lagrange equations holds. ²

\[
\therefore \text{EL}[u] = 0 \iff \begin{cases} 
\text{(i)} \, \text{EL}[(Q, f)] = 0, \\
\text{(ii)} \, \dot{Q}(r) \in \mathbb{R} SO(n) \text{ for all } r \in [a, b].
\end{cases}
\]

**Proof.** Let \( u \) be a generalised twist and let \( G = fQ \) denote its twist loop. Then in view of [1]-[4] above an application of Theorem 5.5.1 gives

\[
\text{EL}[u] = 0 \iff [\nabla u]^t \Delta u + (\det \nabla u) \nabla \left[ \phi'(\det \nabla u) \right] = 0 \\
\iff \left[ \frac{1}{n} f \dot{j} [F]^2 I_n - (\dot{j} - \frac{f}{r}) [F \theta]^2 I_n - \frac{f}{r} F' F \right] \theta = 0, \tag{5.19}
\]

with \( F = \dot{Q}(r) \). Moreover, we have that

\[
\therefore \text{EL}[u] = 0 \implies \text{EL}[(Q, f)] = 0. \tag{5.20}
\]

(This follows, e.g., by arguing as in Proposition 5.2.1 and noting that the equation on the left results from taking a larger class of variations in \( F \) than that on the right.)

With the aid of the equivalence and the implications in (5.19) and (5.20) we now proceed by establishing the two implications in the conclusion of the theorem separately.

(\textbf{Sufficiency } "\iff") Fix \( r \in [a, b] \) and assume that \( F := \dot{Q}(r) \in \mathbb{R} SO(n) \). Then by definition

²Recall that for every non-negative integer \( n \) we have that \( \mathbb{R} SO(n) := \left\{ F : F = \rho Q \text{ where } \rho \in \mathbb{R}, Q \in SO(n) \right\} \).
there exists $\rho = \rho(r) \in \mathbb{R}$ and $R = R(r) \in \text{SO}(n)$ such that

$$F = \rho R.$$ 

A straight-forward calculation now gives

$$0 = \left[ f^{-1} \langle f, \dot{f} \rangle \rho^2 - \frac{f}{r} \rho^2 \right] I_n = \frac{1}{n} \frac{f}{|F|^2} I_n - \left( \frac{\dot{f}}{r} \right) I_n - \frac{f}{r} F^t F$$

$$= \frac{1}{n} \frac{f}{|Q|^2} I_n - \left( \frac{\dot{f}}{r} \right) \dot{Q}^t \dot{Q} - \frac{f}{r} \dot{Q}^t \dot{Q}.$$ 

Therefore if $\mathbb{E}[Q, f] = 0$ an application of Theorem 5.5.1 immediately gives $\mathbb{E}[u] = 0$.

(*Necessity "$\implies$"*) Assume that $\mathbb{E}[u] = 0$. Fix $r \in [a, b]$ and put $Q := Q(r)$ and $F := \dot{Q}(r)$. Then referring to (5.19) for every $\theta \in \mathbb{S}^{n-1}$ we have that

$$0 = \left( \frac{1}{n} \frac{f}{|F|^2} I_n - \left( \frac{\dot{f}}{r} \right) I_n - \frac{f}{r} F^t F \right) \theta, \theta$$

$$= \frac{1}{n} \frac{f}{|F|^2} - \left( \frac{\dot{f}}{r} \right) I_n - \frac{f}{r} |F|^2$$

$$= \frac{1}{n} |F|^2 - |F|^2.$$ 

In view of the latter being true for all $\theta \in \mathbb{S}^{n-1}$ [and that $f(r) \neq 0$] it follows that $F \in \mathbb{R}O(n)$. Indeed fix $F \in M_{n \times n}$ and put $A := F^t F$. Then it is evident that

$$\frac{1}{n} |F|^2 = \langle F \theta, F \theta \rangle \iff \frac{1}{n} tr A = \langle A \theta, \theta \rangle,$$

[for all $\theta \in \mathbb{S}^{n-1}$]. Since $A$ is symmetric and non-negative its eigen-values are real and satisfy $0 \leq \lambda_1[A] \leq \cdots \leq \lambda_n[A]$. Testing the above identity in turn with corresponding eigen-vectors gives at once $\lambda_1[A] = \cdots = \lambda_n[A] := \lambda$ and so $A = \lambda I_n$. This can now easily be seen to give $F \in \mathbb{R}O(n)$.

However as $QF'$ is skew-symmetric it follows from Proposition A.0.5 that $QF' \in \mathbb{R}SO(n)$ and so $F \in \mathbb{R}SO(n)$. This together with (5.20) completes the proof. $\square$

**Theorem 5.5.3.** Let $\Omega = \{ x \in \mathbb{R}^n : a < |x| < b \}$ and let $u$ be a generalised twist as in Definition 3.1.1. Assume $F[u, \Omega] < \infty$ and that the twist loop $G = fQ$ satisfies the following assumptions.

1. $Q \in C^2([a, b], \text{SO}(n))$,
2. $f \in C^2([a, b])$,
3. $\dot{f} > 0$ on $[a, b]$,
Then the following equivalence between the full and the restricted Euler-Lagrange equations hold.

\[ \text{EL}[u] = 0 \iff \begin{cases} [a] \ (n = 2k) \text{ there exist } g = g(r) \in C^2[a, b] \text{ with } \\
g(a), g(b) \in 2\pi Z \text{ and } P \in O(n) \text{ so that} \\
\quad (i) \ Q = P \text{diag} \{R(g), \ldots, R(g)\}P^t, \\
\quad (ii) \ \text{EL}[g, f] = 0. \\
\end{cases} \]

Moreover we have that

\[ \begin{cases} (iii) \ f \in C^2[a, b]. \\
\end{cases} \]

\[ [b] \ (n = 2k + 1) \]

\[ \begin{cases} (i) \ Q = I_n, \\
(ii) \ \text{EL}[0, f] = 0, \\
\end{cases} \]

Moreover we have that

\[ \begin{cases} (iii) \ f \in C^2[a, b]. \\
\end{cases} \]

Note that in [a](ii) and [b](ii) above we have denoted

\[ \text{EL}[g, f] = 0, \]

as an abbreviation for the second order system

\[ \begin{cases} \frac{d}{dr} \left[ r^{n-1} f^2 \dot{g} \right] = 0, \\
\frac{d}{dr} \left[ r^{n-1} \dot{f} + f^{n-1} \phi' \right] = (n-1)[r^{n-3} f + \dot{f} f^{n-2} \phi'] + r^{n-1} f \dot{g}^2, \\
\end{cases} \]

where \( \phi' = \phi'(\dot{f}(\frac{f}{r})^{n-1}) \) on \( [a, b] \).

Proof. We establish each of the two implications in the conclusion of the theorem separately.

(Sufficiency "\( \Rightarrow \)"") We restrict to the case [a] only as for [b] the conclusion is trivially true. Indeed let \( g, P \) and \( Q \) be as described. Then a straight-forward differentiation gives

\[ \dot{Q}^t \dot{Q} = [P \text{diag} \{R(g), \ldots, R(g)\}P^t]^t \times \\
\quad [P \text{diag} \{R(g), \ldots, R(g)\}P^t] \\
\quad = \ddot{g}^2 PI_n P^t = g^2 I_n, \]
while \( \text{diag}({\mathfrak{h}}(g), \ldots, {\mathfrak{h}}(g)) \in \mathbb{R}SO(n) \). Hence \( \dot{Q}(r) \in \mathbb{R}SO(n) \) for all \( r \in ]a, b[ \). Next, using the same expression for \( Q \) we can verify that

\[
Q' \dot{Q} = [P \text{diag}({\mathfrak{h}}(g), \ldots, {\mathfrak{h}}(g))P']' \times
\]

\[
= gP \text{diag}(J_2, \ldots, J_2)P',
\]

and in a similar way that

\[
|\dot{Q}|^2 = tr[\dot{Q}' \dot{Q}] = tr[\dot{g}^2 I] = ng^2.
\]

Therefore referring to Proposition 5.2.1 it follows that

\[
\text{EL}[(Q, f)] = \text{EL}[g, f] = 0,
\]

where in concluding the second equality we have appealed to [a](ii) above. The assertion is now easily seen to follow from Theorem 5.5.2.

**Necessity "\( \Rightarrow \)"** Assume that \( \text{EL}[u] = 0 \). Then according to Theorem 5.5.2 we have that

\[
\begin{cases}
(i) & \text{EL}[(Q, f)] = 0, \\
(ii) & \dot{Q}(r) \in \mathbb{R}SO(n) \text{ for all } r \in ]a, b[.
\end{cases}
\]

Now referring to (i) above by integrating the first equation in the corresponding system (see Proposition 5.2.1) we can write

\[
r^{n-1} f^2 Q' \frac{d}{dr} Q = A,
\]

where \( A \in \mathbb{M}_{n \times n} \) is skew-symmetric and by (ii) above \( A \in \mathbb{R}SO(n) \). We now consider the cases [a] and [b] separately.

[a] \((n = 2k)\) By utilising Proposition A.0.5 there exist \( \alpha \in \mathbb{R}, P \in O(n) \) such that we can re-write the above equation in the more convenient form

\[
\frac{d}{dr} Q = \alpha \frac{1}{r^{n-1} f^2} QP \text{diag}(J_2, J_2, \ldots, J_2)P' =: \mu QJ.
\]
Let $g \in C^1[a, b] \cap C^2[a, b]$ be a primitive of $\mu$ satisfying $g(a) \in 2\pi \mathbb{Z}$ and then fix $\alpha$ so that $g(b) \in 2\pi \mathbb{Z}$. Then a straightforward differentiation gives

$$\frac{d}{dr} e^{gJ} = \dot{g} e^{gJ} J = \mu e^{gJ} J,$$

whilst

$$e^{gJ} = e^{P \text{diag}(J_2, \ldots, J_2) P^t}
= P e^{\text{diag}(J_2, \ldots, J_2)} P^t
= P \text{diag}(\mathcal{N}(g), \ldots, \mathcal{N}(g)) P^t.$$

Hence by the uniqueness of solutions to initial value problems [applied to (5.22)] it follows that $Q = e^{gJ}$ on $[a, b]$. This gives $[a](i)$. Using the latter conclusion $[a](ii)$ follows as in the proof of the sufficiency part using $(i)$ above. Finally that $g, f \in C^2[a, b]$ follows by using an adaptation of the argument in Theorem 5.4.3 along with the well-known Hilbert-Weierstrass differentiability theorem (See [4] pp. 584 and [18] pp. 57-61). As will be seen in the next section $\text{EL}[g, f] = 0$ is a genuine Euler-Lagrange equation [in fact corresponding to the energy functional $\mathcal{J}$ over the space $(2\pi m, 0) + J_{m_n - m_n}$ (see Section 5.6)].

$[b] \ (n = 2k + 1)$ An application of Proposition A.0.5 gives $A = 0$. Hence referring to (5.21) together with the boundary conditions $Q(a) = Q(b) = I_n$ it follows that $Q = I_n$ on $[a, b]$. This gives $[b](i)$. Finally according to $(i)$ above we have that

$$\text{EL}[0, f] = \text{EL}[I_n, f] = 0$$

which gives $[b](ii)$. The proof is thus complete.

\[ \square \]

5.6 A characterisation of all twist solutions

In the previous section we discussed the implications that the Euler-Lagrange equations associated with the energy functional $F$ exerted upon the twist loop $G = fQ$ corresponding to a generalised twist $u \in \mathcal{A}(\Omega)$ in order for the latter to furnish a solution to these equations. In this section we show how this analysis enables one to give a complete characterisation of all such twist solutions. (See Definition 5.6.2.)
We begin by considering the case of even dimensions \((n = 2k)\). Here for each fixed \(m \in \mathbb{Z}\) we set

\[
J_m := J_m(a, b) := \begin{cases} 
  g \in W^{1,2}(a, b), \\
  g(a) = 0, g(b) = 2\pi m, \\
  f \in W^{1,2}(a, b), \\
  \dot{f} > 0 \ \mathcal{L}^1\text{-a.e. on } (a, b), \\
  f(a) = a, f(b) = b,
\end{cases}
\]

and

\[
\mathcal{J}[g, f] := J[g, f] = \int_a^b \left\{ f^2 \left[ \frac{n(n-1)}{r^2} + n\dot{g}^2 \right] + nf^2 + 2n\phi(\frac{f}{r})^{n-1} \right\} r^{n-1} dr,
\]

where \(g = (g, g, \ldots, g)\). With the aid of this notations we have the following statement.

**Theorem 5.6.1.** (Existence and regularity of special stationary loops).

Consider the energy functional \(\mathcal{J}\) over the space \(J_m\). Then for each \(m \in \mathbb{Z}\) there exist \(g = g(r; a, b, m)\) and \(f = f(r; a, b, m)\) with \((g, f) \in J_m\) such that

\[
\mathcal{J}[g, f] = \inf_{\mathcal{J}_m} \mathcal{J}[\cdot].
\]

Moreover the pair \((g, f)\) satisfies the corresponding Euler-Lagrange equations

\[
\mathcal{EL}[g, f] = 0,
\]

that is,

\[
\begin{align*}
\frac{d}{dr} \left[ r^{n-1} \dot{f}^2 \dot{g} \right] &= 0, \\
\frac{d}{dr} \left[ r^{n-1} \dot{f} + f^{n-1} \phi' \right] &= (n-1)[r^{n-3} \dot{f} + \dot{f} f^{n-2} \phi'] + r^{n-1} f \dot{g}^2,
\end{align*}
\]

on \([a, b]\) where \(\phi' = \phi'(\frac{f}{r})^{n-1}\). Additionally \((g, f) \in C^2[a, b] \times C^2[a, b]\) and \(\dot{f} > 0\) on \([a, b]\).

**Proof.**

The argument here is similar to that used in Theorem 5.4.3 and hence will be abbreviated.

We now return to the energy functional \(\mathcal{F}\) defined over the space of admissible maps \(A(\Omega)\). For the sake of clarity and future reference we proceed with the following definition.
Definition 5.6.2. (Classical solution)
An admissible map $u \in A(\Omega)$ is referred to as a classical solution to the Euler-Lagrange equations associated with the energy functional (1.6) over the space (1.7) if and only if the following hold:

1. $\mathcal{F}[u, \Omega] < \infty$,
2. $u \in C^2(\Omega, \mathbb{R}^n) \cap C(\bar{\Omega}, \mathbb{R}^n)$,
3. $u$ satisfies the system of equations

$$
\begin{align*}
\nabla u^t \Delta u + \det \nabla u \nabla \phi'(\det \nabla u) &= 0 \quad \text{in } \Omega, \\
\det \nabla u &> 0 \quad \text{in } \Omega, \\
u &= x \quad \text{on } \partial \Omega.
\end{align*}
$$

Note that when speaking of a classical solution in the form of a generalised twist [i.e., $u(x) = f(r)Q(r)\theta$] in connection with [2] above we implicitly assume the pair $(Q, f)$ to be of class $C^2$, i.e., that $f \in C^2[a, b]$ and $Q \in C^2([a, b], \text{SO}(n))$. Moreover, in connection with [3] we have $\det \nabla u > 0$ in $\Omega \iff \dot{f} > 0 \text{ in } ]a, b[$. [See (5.2).]

We are now in a position to present the main result of this chapter which is a complete characterisation of all twist solutions to the Euler-Lagrange equations associated with the energy functional $\mathcal{F}$.

Theorem 5.6.3. (Characterisation of all twist solutions).
Let $\Omega = \{x \in \mathbb{R}^n : a < |x| < b\}$ and consider the energy functional $\mathcal{F}$ over the space $A(\Omega)$. Then the set $\mathcal{S}$ of all classical solutions in the form of generalised twists to the corresponding Euler-Lagrange equations can be characterised as follows.

1. $(n = 2k)$ $\mathcal{S}$ is infinite and any generalised twist $u \in \mathcal{S}$ can be described as

$$
u = G(r)\theta = f(r)P \text{diag}(R(g), \ldots, R(g))P^t \theta,$$

where $P \in O(n)$ and $f, g \in C^2[a, b]$ satisfy the second order system [notation as in Theorem 5.6.1]

$$\mathcal{E}L[g, f] = 0.$$

2. $(n = 2k + 1)$ $\mathcal{S}$ consist of the single map $u = x$.

Proof. [1] That $\mathcal{S}$ is infinite follows from Theorem 5.6.1. The remaining assertions follow from [a] in Theorem 5.5.3.
[2] Assume \( u \in \mathcal{S} \). Then referring to [b] in Theorem 5.5.3 it follows that \( Q = I_n \) while \( f \in C^2[a, b] \) and \( \text{EL}[0, f] = 0 \). Evidently \( f = r \) is a solution to the latter. An application of the phase-plane argument in [55] (see pp. 111-117) shows that the latter is indeed the only solution. \( \square \)

5.7 The limiting behaviour of twists when the inner boundary converges to a point

In this section we consider the case where \( b = 1 \) and \( a = \varepsilon > 0 \) with the aim of discussing the limiting properties of the generalised twists from Theorem 5.6.3 as \( \varepsilon \downarrow 0 \). This is particularly interesting since in the limit (the punctured ball) all components of the function space collapse to a single one and so it is important to have a clear understanding as to how the twist solutions and their energies [for each fixed integer \( m \)] behave.

To this end, let \( \Omega_\varepsilon := \{ x \in \mathbb{R}^n : \varepsilon < |x| < 1 \} \) where \( n = 2k \) and for each \( m \in \mathbb{Z} \) let \( u_\varepsilon \in A(\Omega) \) denote the generalised twist from [1] in Theorem 5.6.3, that is, with the notation \( x = r\theta \), set

\[
\begin{align*}
  u_\varepsilon &= G(r; \varepsilon, 1, m)\theta \\
  &= f_\varepsilon(r)P_\varepsilon \text{diag}(\Re(g_\varepsilon), \ldots, \Re(g_\varepsilon))P_\varepsilon^t \theta,
\end{align*}
\]

where \( P_\varepsilon \in O(n) \), \( f_\varepsilon(r) = f(r; \varepsilon, 1, m) \) and \( g_\varepsilon(r) = g(r; \varepsilon, 1, m) \).

In order to make the study of the limiting properties of \( u_\varepsilon \) more tractable, we fix the domain to be the unit ball and extend each map by identity off \( \Omega_\varepsilon \). [In what follows, unless otherwise stated, we speak of \( u_\varepsilon \) in this extended sense.] Thus, here, we have that

\[
  u_\varepsilon : (r, \theta) \mapsto (f_\varepsilon(r), Q_\varepsilon(r)\theta), \tag{5.23}
\]

where

\[
  Q_\varepsilon(r) = P_\varepsilon \text{diag}(\Re(g_\varepsilon), \ldots, \Re(g_\varepsilon))P_\varepsilon^t,
\]

and

\[
  g_\varepsilon(r) = \begin{cases} 
  0 & r \leq \varepsilon, \\
  g(r; \varepsilon, 1, m) & \varepsilon \leq r \leq 1,
\end{cases}
\]

while

\[
  f_\varepsilon(r) = \begin{cases} 
  r & r \leq \varepsilon, \\
  f(r; \varepsilon, 1, m) & \varepsilon \leq r \leq 1.
\end{cases}
\]
In discussing the limiting properties of \( u_\varepsilon \) it is convenient to introduce a so-called comparison map. Indeed, fix \( m \in \mathbb{Z} \) and consider the generalised twist

\[
v_\varepsilon = H_\varepsilon(r)\theta = rP_\varepsilon \text{diag}(\mathcal{R}(h_\varepsilon), \ldots, \mathcal{R}(h_\varepsilon))P_\varepsilon^t \theta,
\]

where \( P_\varepsilon \in O(n) \) is as above and

\[
h_\varepsilon(r) := \begin{cases} 
0 & r \in (0, \varepsilon), \\
2m\pi \left( \frac{r}{\varepsilon} - 1 \right) & r \in (\varepsilon, 2\varepsilon), \\
2m\pi & r \in (2\varepsilon, 1).
\end{cases}
\]

Thus in particular we can write

\[
v_\varepsilon : (r, \theta) \mapsto (r, R_\varepsilon(r)\theta),
\]

where

\[
R_\varepsilon(r) = P_\varepsilon \text{diag}(\mathcal{R}(h_\varepsilon), \ldots, \mathcal{R}(h_\varepsilon))P_\varepsilon^t.
\]

The following proposition describes some of the basic properties of the family of comparison maps \((v_\varepsilon)\).

**Proposition 5.7.1.** The family of comparison maps \((v_\varepsilon)\) enjoys the following properties.

1. \( \det \nabla v_\varepsilon = 1 \) in \( B \),
2. \( v_\varepsilon \to x \) in \( W^{1,2}(\bar{B}, \mathbb{R}^n) \),
3. \( v_\varepsilon \to x \) uniformly on \( \bar{B} \).

**Proof.**

1. Evidently \( v_\varepsilon \) is a generalised twist with the corresponding twist loop

\[
H_\varepsilon(r) := rR_\varepsilon(r) = rP_\varepsilon \text{diag}(\mathcal{R}(h_\varepsilon), \ldots, \mathcal{R}(h_\varepsilon))P_\varepsilon^t.
\]

An application of (5.2) in Proposition 5.1.1 [with the choice \( f = r \)] immediately gives [1].

2. Indeed referring to the definition of \( v_\varepsilon \) we can write

\[
\|v_\varepsilon - x\|_{W^{1,2}}^2 = \int_{\bar{B}} |\nabla v_\varepsilon - I_n|^2 \, dx \leq 2 \int_{\mathbb{R}^n} \left( |\nabla v_\varepsilon|^2 + |I_n|^2 \right) \, dx.
\]
However in view of (5.4) in Proposition 5.1.1 [again with the choice \( f = r \)] we have that

\[
|\nabla v_\varepsilon|^2 = n + r^2|\dot{\mathbf{R}}_\varepsilon \theta|^2.
\]

Therefore we can write

\[
\int_{B_{2\varepsilon} \setminus B_\varepsilon} |\nabla v_\varepsilon|^2 \, dx = \int_\varepsilon^{2\varepsilon} \int_{\mathbb{S}^{n-1}} (n + r^2|\dot{\mathbf{R}}_\varepsilon \theta|^2) r^{n-1} \, d\mathcal{H}^{n-1}(\theta) \, dr
\]

\[
= n\omega_n \int_\varepsilon^{2\varepsilon} (n + r^2 h_\varepsilon^2) r^{n-1} \, dr
\]

\[
\leq \omega_n (2^n - 1) |n + 4(2m\pi)^2| \varepsilon^n.
\]

The above estimates when combined give [2] as a result of Poincaré inequality.

[3] Again by direct verification we have that

\[
|v_\varepsilon - x|^2 = |\mathbf{H}_\varepsilon(r) \theta - r\theta|^2
\]

\[
= r^2 |\mathbf{P}_\varepsilon \text{diag}(\mathbb{R}(h_\varepsilon), \ldots, \mathbb{R}(h_\varepsilon)) \mathbf{P}_\varepsilon^t \theta - \theta|^2
\]

\[
= r^2 |\mathbf{P}_\varepsilon \left[ \text{diag}(\mathbb{R}(h_\varepsilon), \ldots, \mathbb{R}(h_\varepsilon)) - \mathbf{I}_n \right] \mathbf{P}_\varepsilon^t \theta|^2
\]

\[
= r^2 \left| \text{diag}(\mathbb{R}(h_\varepsilon), \ldots, \mathbb{R}(h_\varepsilon)) - \mathbf{I}_n \right| \omega_\varepsilon \]

\[
= \frac{1}{2} r^2 |\mathbb{R}(h_\varepsilon) - \mathbf{I}_2|^2.
\]

However a straight-forward calculation gives

\[
|\mathbb{R}(h_\varepsilon) - \mathbf{I}_2|^2 = 8 \sin^2 \frac{h_\varepsilon}{2}.
\]

Thus by substitution and referring to the definition of \( h_\varepsilon \) we immediately arrive at the bound

\[
\sup_{x} |v_\varepsilon - x| = \sup_{\varepsilon, 2\varepsilon} 2r |\sin \frac{h_\varepsilon}{2}| \leq 4\varepsilon,
\]

which is the required conclusion. This complete the proof. \( \Box \)

Fix \( m \in \mathbb{Z} \) and let \( g_\varepsilon := (g_\varepsilon, \ldots, g_\varepsilon) \), \( h_\varepsilon := (h_\varepsilon, \ldots, h_\varepsilon) \). It is evident that the pairs \((g_\varepsilon, f_\varepsilon), (h_\varepsilon, r) \in \mathcal{J}_m(\varepsilon, 1) \) and so according to the minimizing property of \((g_\varepsilon, f_\varepsilon) \) we have that

\[
\frac{2}{\omega_n} \mathbb{E}[u_\varepsilon, B] = \mathbb{E}[Q_\varepsilon, f_\varepsilon] = \mathcal{J}[g_\varepsilon, f_\varepsilon]
\]

\[
\leq \mathcal{J}[h_\varepsilon, r] = \mathbb{E}[\mathbf{R}_\varepsilon, r] = \frac{2}{\omega_n} \mathbb{E}[v_\varepsilon, B].
\]
This in conjunction with [1], [2] in Proposition 5.7.1 implies the boundedness of \((u_\varepsilon)\) in \(W^{1,2}(B, \mathbb{R}^n)\) and as a result \((u_\varepsilon)\) admits a weakly convergent subsequence. Indeed more is true!

**Theorem 5.7.2.** (Limiting behaviour of twists).

Let \(\Omega_\varepsilon := \{x \in \mathbb{R}^n : \varepsilon < |x| < 1\}\). For fixed \(m \in \mathbb{Z}\) let \((u_\varepsilon)\) denote the family of generalised twists as in (5.23). Then we have the following convergences.

1. \(u_\varepsilon \to x\) in \(W^{1,2}(B, \mathbb{R}^n)\),
2. \(u_\varepsilon \to x\) uniformly on \(\overline{B}\).

**Proof.** [1] Fix \(m \in \mathbb{Z}\) and let \(v_\varepsilon\) be as in (5.24). Then referring to (5.26) it follows that by passing to a subsequence (not re-labeled) we have that

\[
\begin{align*}
&u_\varepsilon \rightharpoonup u \quad \text{in} \quad W^{1,2}(B, \mathbb{R}^n), \\
u_\varepsilon \to u \quad \text{\(L^n\)-a.e. in} \quad B.
\end{align*}
\]

In addition we can write

\[
F[x, B] \leq \lim_{\varepsilon \downarrow 0} \inf F[u_\varepsilon, B] \\
\leq \lim_{\varepsilon \downarrow 0} \sup F[u_\varepsilon, B] \\
\leq \lim_{\varepsilon \downarrow 0} F[v_\varepsilon, B] = F[x, B].
\tag{5.27}
\]

Now fix \(\sigma \in (0,1)\) and recall the pair \((g_\varepsilon, f_\varepsilon)\) used in expressing \((u_\varepsilon)\). Then \((u_\varepsilon)\) bounded in \(W^{1,2}(B, \mathbb{R}^n)\) gives \((u_\varepsilon)\) bounded in \(W^{1,2}(B\setminus \overline{B}_\sigma, \mathbb{R}^n)\) and so as a result \((g_\varepsilon, f_\varepsilon)\) is bounded in \([W^{1,2}(\sigma, 1)]^2\).

In particular there exist \((g, f) \in [W^{1,2}(\sigma, 1)]^2\) such that passing to a subsequence (not re-labeled) we have that

\[
\begin{align*}
g_\varepsilon &\to g \quad \text{in} \quad W^{1,2}(\sigma, 1), \\
g_\varepsilon &\to g \quad \text{in} \quad C[\sigma, 1], \\
f_\varepsilon &\to f \quad \text{in} \quad W^{1,2}(\sigma, 1), \\
f_\varepsilon &\to f \quad \text{in} \quad C[\sigma, 1], \\
g(1) &= 2m\pi, \\
f(1) &= 1.
\end{align*}
\]

As a consequence we have in particular that

\[
\begin{align*}
f_\varepsilon g_\varepsilon &\to fg, \\
f_\varepsilon \left(\frac{f_\varepsilon}{r}\right)^{n-1} &\to f\left(\frac{f}{r}\right)^{n-1},
\end{align*}
\]
where both convergences are interpreted as \textit{weakly} in $L^2(\sigma, 1)$. Therefore [using the \textit{same} notation as in (5.9)] standard lower semicontinuity results (see, e.g., [18]) give

$$\int_{\sigma}^{1} J(r, \dot{g}, f, \dot{f}) \, dr \leq \liminf_{\varepsilon \downarrow 0} \int_{\sigma}^{1} J(r, \dot{g}_\varepsilon, f_\varepsilon, \dot{f}_\varepsilon) \, dr. \quad (5.28)$$

Now referring to (5.23) we can assume that $P_\varepsilon \rightarrow P$ as a result of $O(n)$ being a compact manifold. Hence with the aid of the above we can write

$$\lim_{\varepsilon \downarrow 0} u_\varepsilon(x) = \lim_{\varepsilon \downarrow 0} G_\varepsilon(r) \theta = \lim_{\varepsilon \downarrow 0} f_\varepsilon P_\varepsilon \text{diag}(\mathcal{R}(g_\varepsilon), \ldots, \mathcal{R}(g_\varepsilon)) P_\varepsilon^t \theta := w, \quad (5.29)$$

where the convergence is interpreted as \textit{uniformly} on $B \setminus \overline{B}_\sigma$. Recalling the pointwise convergence of $(u_\varepsilon)$ we thus conclude that $u = w$ $\mathcal{L}^n$-a.e. in $B \setminus \overline{B}_\sigma$. Hence by combining (5.27) and (5.28) we have that

$$F[u, B \setminus \overline{B}_\sigma] = F[w, B \setminus \overline{B}_\sigma] \leq \liminf_{\varepsilon \downarrow 0} F[u_\varepsilon, B \setminus \overline{B}_\sigma] \leq \limsup_{\varepsilon \downarrow 0} F[u_\varepsilon, B] \leq F[x, B].$$

Note that the energy functional $F[\cdot, \Omega]$ is \textit{not} sequentially \textit{weakly} lower semicontinuous on $\mathcal{A}(\Omega)$. However (5.28) demonstrates that the same is true if one \textit{restricts to generalised} twists! An application of Lebesgue’s theorem on \textit{monotone} convergence now gives

$$F[x, B] \leq F[u, B] = \lim_{\varepsilon \downarrow 0} F[u_\varepsilon, B \setminus \overline{B}_\sigma] \leq F[x, B].$$

Hence $F[x, B] = F[u, B]$ and this in turn together with the \textit{strict} quasiconvexity of $F$ gives $u = x$ in $\mathcal{B}$. Finally referring again to (5.27) we have that

$$\begin{cases} u_\varepsilon \rightharpoonup u, \\
 u = x, \\
 F[u_\varepsilon, B] \rightarrow F[x, B], \end{cases} \quad \Rightarrow \quad u_\varepsilon \rightarrow x,$$

which is the required conclusion in [1].
In view of the assertion in [1] and the characterisation of the pointwise limit of the family \((u_\varepsilon)\) in (5.29) we have that

\[
w = f(r) P \text{diag}(\mathcal{R}(g), \ldots, \mathcal{R}(g)) P^t = r \theta = x,
\]

\(L^n\)-a.e. in \(\mathcal{B}\). Therefore according to \(f, g\) both being continuous on the interval \([0, 1] = \cup_{\sigma \in [0,1]} [\sigma, 1]\) it follows that

\[f \mathcal{R}(g) = r I_2\]

on \([0, 1]\). This gives \(f = r\) (e.g., by taking the norm of both sides and noting that \(f\) is non-negative) and \(\mathcal{R}(g) = I_2\) which in turn gives \(g(r) = 2\pi n(r)\) for some \(n(r) \in \mathbb{Z}\). Referring to \(g(1) = 2\pi m\) it follows again by appealing to the continuity of \(g\) that \(g(r) = 2\pi m\) on \([0, 1]\). Next, arguing as in (5.25) we can write

\[
|u_\varepsilon - x|^2 = |G_\varepsilon(r) \theta - r \theta|^2 \\
= 2r^2 (1 - \cos g_\varepsilon) \\
= 4r^2 \sin^2 \frac{g_\varepsilon}{2}.
\]

Thus, to conclude \([2]\) fix \(\delta > 0\) and first take \(\sigma \in (0, 2^{-1} \delta]\) and then \(\varepsilon_0\) such that \(|\sin(2^{-1} g_\varepsilon)| \leq 2^{-1} \delta\) on \([\sigma, 1]\) for \(\varepsilon < \varepsilon_0\). Then \(\sup_\mathcal{B} |u_\varepsilon - x| \leq \max(2\sigma, \delta) = \delta\). \(\square\)
Appendix A

Skew-symmetric matrices and the orthogonal group

Recall from linear algebra that all eigen-values of a real skew-symmetric matrix have zero real parts. Hence they either appear as purely imaginary conjugate pairs or zero. In particular when \( n \) is odd there is necessarily a zero eigen-value. Thus distinguishing between the cases when \( n \) is even and odd respectively we can bring every skew-symmetric matrix to a block diagonal form. In what follows we set

\[
J_2 := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.
\]

Proposition A.0.3. Let \( A \in \mathbb{M}_{n \times n} \) be skew-symmetric. Then there exist \( (\lambda_j)_{j=1}^k \subset \mathbb{R} \) and \( P \in \text{SO}(n) \) such that

1. \( n = 2k \)

\[
A = P \text{diag}(\lambda_1 J_2, \lambda_2 J_2, \ldots, \lambda_k J_2) P^t.
\]

2. \( n = 2k + 1 \)

\[
A = P \text{diag}(\lambda_1 J_2, \lambda_2 J_2, \ldots, \lambda_k J_2, 0) P^t.
\]

Proof. Indeed, here, \( A \) is normal [i.e., it commutes with its transpose \( A^t = -A \)] and so the conclusion follows from the well-known spectral theorem.

We note that by allowing \( P \in \text{O}(n) \) we can additionally arrange for the sequence \( (\lambda_j)_{j=1}^k \) to be non-negative. On the other hand the choices of \( P \) and \( (\lambda_j)_{j=1}^k \) are in general non-unique. Indeed it is a trivial matter to see that by suitably adjusting \( P \) one can replace any \( \lambda_j \) with \(-\lambda_j\).
Appendix A. Skew-symmetric matrices and orthogonal group

In what follows we set $\mathcal{R}(s) := \begin{bmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{bmatrix}$.

**Proposition A.0.4.** Let $A \in M_{n \times n}$ be skew-symmetric. Then using the notation in Proposition A.0.3 we have that

1. $(n = 2k)$
   $$e^{sA} = P \text{diag}(\mathcal{R}(s\lambda_1), \mathcal{R}(s\lambda_2), \ldots, \mathcal{R}(s\lambda_k)) P^t,$$

2. $(n = 2k + 1)$
   $$e^{sA} = P \text{diag}(\mathcal{R}(s\lambda_1), \mathcal{R}(s\lambda_2), \ldots, \mathcal{R}(s\lambda_k), 1) P^t.$$

**Proof.** A straight-forward calculation gives

$$e^{sJ} = \sum_{n=0}^{\infty} \frac{1}{n!} s^n J_2^n = \mathcal{R}(s).$$

The conclusion now follows by noting that $e^A = e^{PD^t} = P e^D P^t$. 

**Proposition A.0.5.** Let $A \in M_{n \times n}$ be skew-symmetric. Assume in addition that $A \in \mathbb{R}O(n)$. Then the following hold.

1. $(n = 2k)$ there exists $\alpha \in \mathbb{R}$ and $P \in \mathbb{O}(n)$ such that
   $$A = \alpha P \text{diag}(J_2, J_2, \ldots, J_2) P^t,$$

2. $(n = 2k + 1)$ necessarily $A = 0$.

Therefore it follows that indeed $A \in \mathbb{R}SO(n)$.

**Proof.** In view of $A \in \mathbb{R}O(n)$ there exists $\alpha \in \mathbb{R}$ such that $A^t A = \alpha I_n$. In what follows we proceed by considering each of the cases $n = 2k$ and $n = 2k + 1$ separately.

1. Since $A$ is skew-symmetric it follows from [1] in Proposition A.0.3 that there exist $(\lambda_j)_{j=1}^k$ and $R \in \mathbb{O}(n)$ such that $A = R \text{diag}(\lambda_1 J_2, \lambda_2 J_2, \ldots, \lambda_k J_2) R^t$. Hence
   $$A^t A = [R \text{diag}(\lambda_1 J_2, \lambda_2 J_2, \ldots, \lambda_k J_2) R^t]^t \times [R \text{diag}(\lambda_1 J_2, \lambda_2 J_2, \ldots, \lambda_k J_2) R^t] = R \text{diag}(\lambda_1^2 I_2, \lambda_2^2 I_2, \ldots, \lambda_k^2 I_2) R^t = \alpha^2 I_n$$
and so $\lambda_1^2 = \lambda_2^2 = \ldots = \lambda_k^2 = \alpha^2$. In particular there exists $(\beta_j)_{j=1}^k \subset \{\pm 1\}$ such that $A = \alpha R \text{diag}(\beta_1 J_2, \beta_2 J_2, \ldots, \beta_k J_2) R^t$. The conclusion now follows by post-multiplying $R$ with suitable orthogonal matrices through an application of the following trivial identity relating $-J_2$ to $J_2$,

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

[2] This is an immediate consequence of $\det A = 0$. 

For more details and basics properties related to the matrix exponential as a mapping between the spaces of skew-symmetric matrices and the special orthogonal groups, we refer the interested reader to the books [21], [39] and [54].
Appendix B

Symmetric matrices and vector fields

**Proposition B.0.6.** Let $F \in M_{n \times n}$ be fixed and consider for $\theta \in S^{n-1}$ the identity

$$F \theta \otimes \theta - \theta \otimes F \theta = 0.$$  

Then (B.1) holds for all $\theta \in S^{n-1}$ if and only if there exists $\sigma \in \mathbb{R}$ such that $F = \sigma I_n$.

Proof. (Sufficiency) If $F = \sigma I_n$ for some $\sigma \in \mathbb{R}$ then (B.1) is trivially true for all $\theta \in S^{n-1}$.

(Necessity) Assume that (B.1) holds for all $\theta \in S^{n-1}$. To justify the assertion it suffices to consider the following steps.

[1] By substituting the choices $\theta \in \{e_1, e_2, \ldots, e_n\}$ (the standard basis) it follows that $F$ must be diagonal.

[2] Assume now that $F = \text{diag}(d_1, d_2, \ldots, d_n)$. Then (B.1) is equivalent to the set of equations

$$\theta_i \theta_j (d_i - d_j) = 0,$$

for $1 \leq i, j \leq n$ where $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$. It thus follows that $d_1 = d_2 = \cdots = d_n$ and so denoting the common value as $\sigma$ gives the conclusion. $\square$

**Proposition B.0.7.** Let $\Omega = \{x \in \mathbb{R}^n : a < |x| < b\}$ and consider the vector field $v = A(r) x$ in $\Omega$ where $A \in C^1([a, b], M_{n \times n})$ is symmetric. Then the following are equivalent.

[1] $v$ is a gradient,

[2] $A = sI_n + K$ for some $s \in C^1[a, b]$ and constant symmetric matrix $K \in M_{n \times n}$. 

92
Appendix B. Symmetric matrices and vector fields

Proof. ([1] $\implies$ [2])

If $v = (v_1, v_2, \ldots, v_n)$ is a gradient field in $\Omega$ then it is necessary that for all $1 \leq p, q \leq n$,

$$\frac{\partial v_q}{\partial x_p} - \frac{\partial v_p}{\partial x_q} = 0.$$

Substituting for $v$ and denoting $r = |x|$ this means that

$$0 = \frac{\partial}{\partial x_p} \sum_{j=1}^{n} A_{qj} x_j - \frac{\partial}{\partial x_q} \sum_{j=1}^{n} A_{pj} x_j$$

$$= \left\{ r \sum_{j=1}^{n} \dot{A}_{qj} \theta_j \theta_p + \sum_{j=1}^{n} A_{qj} \delta_{jp} \right\} - \left\{ r \sum_{j=1}^{n} \dot{A}_{pj} \theta_j \theta_q + \sum_{j=1}^{n} A_{pj} \delta_{jq} \right\},$$

or in view of $A$ being symmetric that

$$0 = \sum_{j=1}^{n} \left\{ \dot{A}_{qj}(r) \theta_j \theta_p - \dot{A}_{pj}(r) \theta_j \theta_q \right\}$$

$$= \left[ \dot{A}(r) \theta \otimes \theta - \theta \otimes \dot{A}(r) \theta \right]_{qp},$$

for $r \in (a, b)$ and $\theta \in S^{n-1}$. An application of Proposition B.0.6 [with $F = \dot{A}(r)$] now gives $\dot{A}(r) = \sigma(r) I_n$ where $\sigma \in C[a, b]$. Consequently by integration we arrive at

$$A = s I_n + K,$$

on $]a, b]$ where $s \in C^1[a, b]$ is a suitable primitive for $\sigma$ and $K \in M_{n \times n}$ is constant and symmetric. This gives [2].

([2] $\implies$ [1])

Assume now $A(r) = s(r) I_n + K$ then clearly $v = s(r)x + Kx$ in $\Omega$. To show that $v$ is a gradient it suffices to consider $f(x) := \rho(r) + \frac{1}{2} \langle Kx, x \rangle$ for some $\rho \in C^2[a, b]$ to be determined. Then as $K$ being a symmetric matrix we have

$$\nabla f(x) = \dot{\rho}(r) \frac{x}{r} + Kx,$$

which in turn gives $\dot{\rho}(r) = rs(r)$. An integration now leads to $\rho$ and so here $v = \nabla f$. \qed

Proposition B.0.8. Let $F \in M_{n \times n}$. Then

$$\int_{S^{n-1}} \langle F \theta, \theta \rangle d\mathcal{H}^{n-1}(\theta) = \omega_n tr F,$$
where $\omega_n = \mathcal{L}^n(B)$.

**Proof.** Consider the vector field $v := Fx$ for $x \in \bar{B}$. Then an application of the *divergence* theorem gives

$$
\int_{\partial B} (F\theta, \theta) \, d\mathcal{H}^{n-1}(\theta) = \int_{\partial B} (v(\theta), \theta) \, d\mathcal{H}^{n-1}(\theta) = \int_{B} \text{div} v(x) \, dx \bigg| = \int_{\bar{B}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} (F_{ij} x_j) = \omega_n \text{tr} F.
$$

□
Bibliography


Bibliography


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