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Article (Published Version)

Barrett, John W, Deckelnick, Klaus and Styles, Vanessa (2017) Numerical analysis for a system coupling curve evolution to reaction-diffusion on the curve. SIAM Journal on Numerical Analysis (SINUM), 55 (2). pp. 1080-1100. ISSN 0036-1429

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NUMERICAL ANALYSIS FOR A SYSTEM COUPLING CURVE EVOLUTION TO REACTION DIFFUSION ON THE CURVE∗

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Abstract. We consider a finite element approximation for a system consisting of the evolution of a closed planar curve by forced curve shortening flow coupled to a reaction-diffusion equation on the evolving curve. The scheme for the curve evolution is based on a parametric description allowing for tangential motion, whereas the discretization for the PDE on the curve uses an idea from [G. Dziuk and C. M. Elliott, IMA J. Numer. Anal., 27 (2007), pp. 262–292]. We prove optimal error bounds for the resulting fully discrete approximation and present numerical experiments. These confirm our estimates and also illustrate the advantage of the tangential motion of the mesh points in practice.

Key words. surface PDE, forced curve shortening flow, diffusion induced grain boundary motion, parametric finite elements, tangential motion, error analysis

AMS subject classifications. 65M60, 65M15, 35K55, 53C44, 74N20

DOI. 10.1137/16M1083682

1. Introduction. The aim of this paper is to analyze a fully discrete numerical scheme for approximating a solution of the following system: find a family of planar, closed curves \( \Gamma(t) \) for \( t \in [0, T] \) and a function \( w : \bigcup_{t \in [0, T]} (\Gamma(t) \times \{t\}) \to \mathbb{R} \) such that

\[
\begin{align*}
\frac{v}{s} &= \kappa + f(w) & &\text{on } \Gamma(t), \quad t \in (0, T], \\
\partial_s^* w &= d w_{ss} + \kappa s v w + g(v, w) & &\text{on } \Gamma(t), \quad t \in (0, T],
\end{align*}
\]

subject to the initial conditions

\[
\Gamma(0) = \Gamma^0, \quad w(\cdot, 0) = w^0 \text{ on } \Gamma^0.
\]

Here, \( v \) and \( \kappa \) are the normal velocity and the curvature of \( \Gamma(t) \) corresponding to the choice \( \vec{\nu} \) of a unit normal, while \( s \) is the arc length parameter on \( \Gamma(t) \). Furthermore, \( \partial_s^* w \) denotes the material derivative of \( w \), i.e., \( \partial_s^* w = w_t + v \frac{w_s}{s} \). Finally, \( d \in \mathbb{R}_{>0}, \ f : \mathbb{R} \to \mathbb{R}, \ g : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \), the closed curve \( \Gamma^0 \), and \( w^0 : \Gamma^0 \to \mathbb{R} \) are all given. The system (1.1a), (1.1b) couples the evolution of the curves \( (\Gamma(t))_{t \in [0, T]} \) by forced curve shortening flow to a parabolic PDE on the moving curves. It occurs, for example, as a sharp-interface model for diffusion induced grain boundary motion: in this setting \( \Gamma(t) \) represents a grain boundary separating the crystals of a thin polycrystalline film of metal that is placed in a vapor containing another metal; atoms from the vapor diffuse into the film along the grain boundaries causing them to move. A
thorough description of the physical setup can be found in [11], while an existence
and uniqueness result has been obtained in [12].

In what follows we shall describe the evolving curves $\Gamma(t)$ with the help of a
parametrization $\vec{x}(\cdot, t) : \mathbb{I} \rightarrow \mathbb{R}^2$, where $\mathbb{I} := \mathbb{R} \setminus \mathbb{Z}$ is the periodic unit interval. Then
\begin{equation}
(1.3) \quad \vec{\tau} = \vec{x}_s = \frac{\vec{x}_\rho}{|\vec{x}_\rho|}, \quad \vec{\nu} = \vec{\tau}^\perp
\end{equation}
are a unit tangent and unit normal to $\Gamma(t)$, respectively, where $(\cdot)^\perp$ denotes counter-
clockwise rotation by $\frac{\pi}{2}$. The normal and tangential velocities of $\Gamma(t)$ are then
\begin{equation}
(1.4) \quad v = \vec{x}_t \cdot \vec{\nu}, \quad \psi = \vec{x}_t \cdot \vec{\tau}.
\end{equation}
Note that (1.1a) only prescribes $v$ and with it the shape of $\Gamma(t)$, so that there is a
certain freedom in choosing the tangential velocity. Since $\vec{x}_{ss} = \kappa \vec{\nu}$ one may consider
\begin{equation}
(1.5) \quad \vec{x}_t = \vec{x}_{ss} + f(\vec{w}) \vec{\nu} = \frac{1}{|\vec{x}_\rho|} \left( \frac{\vec{x}_\rho}{|\vec{x}_\rho|} \right)_\rho + f(\vec{w}) \vec{\nu},
\end{equation}
where $\vec{w}(\rho, t) := w(\vec{x}(\rho, t), t), (\rho, t) \in \mathbb{I} \times [0, T]$. Clearly, (1.1a) holds with the additional
property that the velocity vector $\vec{x}_t$ points in the normal direction. Coupling
(1.5) to the PDE satisfied by $\vec{w}$, Pozzi and Stinner have derived and analyzed in [13]
a finite element scheme for (1.1a), (1.1b) (with $g \equiv 0$) based on continuous piecewise
linears. They are able to prove the following error bounds in the spatially discrete
case:
\begin{equation}
\sup_{[0,T]} \int_{\mathbb{I}} \left( |\vec{x}_\rho - \vec{x}_h^\rho|^2 + |\vec{w} - \vec{w}_h|^2 \right) d\rho + \int_0^T \int_{\mathbb{I}} \left( |\vec{x}_t - \vec{x}_h^t|^2 + |\vec{w}_\rho - \vec{w}_h^\rho|^2 \right) d\rho \, dt \leq C h^2.
\end{equation}
A major difficulty in the analysis arises from the fact that (1.5) is only weakly
parabolic. Following [5], this problem is solved in [13] by deriving additional equations for the continuous and discrete length elements and by splitting the error $\vec{x}_\rho - \vec{x}_h^\rho$
into a tangent part and a length element part. Apart from these analytical difficul-
ties, the motion in purely the normal direction also may lead to the accumulation of
mesh points in numerical simulations. A natural way to handle the abovementioned
difficulties is to introduce a tangential part in the velocity, which can be seen as a
reparametrization, an approach that has recently been explored in a systematic way
by Elliott and Fritz in [7]. The underlying idea uses the DeTurck trick in coupling
the motion of the curve to the harmonic map heat flow, which results in the following
equation replacing (1.5) (cf. [7, (3.1)]):
\begin{equation}
(1.6) \quad \alpha \vec{x}_t + (1 - \alpha)(\vec{x}_t \cdot \vec{\nu}) \vec{\nu} = \frac{\vec{x}_{\rho \rho}}{|\vec{x}_\rho|^2} + f(\vec{w}) \vec{\nu}.
\end{equation}
In the above, $\alpha \in (0, 1]$ is a parameter so that $\frac{1}{\alpha}$ corresponds to the diffusion coefficient
in the harmonic map heat flow. Note that we obtain (1.1a) by taking the scalar
product of (1.6) with $\vec{\nu}$. It turns out that $\vec{x}(\cdot, t)$ gets closer to a parametrization
proportional to arc length as $\alpha$ gets small. At the numerical level this means that
mesh points along the curve become more and more equidistributed. Setting formally
$\alpha = 0$, one recovers an approach introduced by Barrett, Garcke, and Nürnberg; see
[1], [2]. A nice feature of (1.6) is that the problem now is strictly parabolic allowing
for a more straightforward error analysis; for the spatially discrete case with \( f = 0 \), see [3], [4] for \( \alpha = 1 \), and [7] for \( \alpha \in (0,1] \).

It remains to derive the equation for \( \tilde{w} \), which in contrast to [13] will also involve the tangential velocity \( (\vec{x}_t \cdot \vec{r}) \). It is easily seen that

\[
\tilde{w}_t(\rho,t) = \frac{d}{dt}[w(\vec{x}(\rho,t),t)] = \partial_t \hat{w}(\vec{x}(\rho,t),t) + (\vec{x}_t(\rho,t) \cdot \vec{r}(\vec{x}(\rho,t))) \left( \nabla_{\Gamma}w(\vec{x}(\rho,t),t) \cdot \vec{r}(\vec{x}(\rho,t)) \right),
\]

where \( \nabla_{\Gamma}w \) is the tangential gradient of \( w \). Inserting this relation into (1.1b) and recalling (1.4), we obtain

\[
(1.7) \quad \tilde{w}_t - (\vec{x}_t \cdot \vec{r}) \frac{1}{|\vec{x}_\rho|} \tilde{w}_\rho - \frac{d}{|\vec{x}_\rho|} \left( \frac{\tilde{w}_\rho}{|\vec{x}_\rho|} \right)_\rho - \kappa v \tilde{w} = g(v,\tilde{w}).
\]

The initial conditions for (1.6), (1.7) now are \( \vec{x}(\rho,0) = \vec{x}^0(\rho) \), \( \tilde{w}(\rho,0) = \tilde{w}^0(\rho) := w^0(\vec{x}^0(\rho)), \rho \in I \), where \( \vec{x}^0 \) is a parametrization of \( \Gamma^0 \). We shall derive in section 2 a weak formulation for (1.6), (1.7) which forms the basis for a discretization by continuous piecewise linear finite elements in space and a backward Euler scheme in time. As the main result of our paper we shall present an error analysis for the resulting fully discrete scheme. The precise result will be formulated at the end of section 2, while the proof is carried out in detail in section 3. In section 4 we present a series of test calculations that confirm our theoretical results and demonstrate the abovementioned improvement of the mesh quality for small values of \( \alpha \).

Finally, we end this section with a few comments about notation. We adopt the standard notation for Sobolev spaces, denoting the norm of \( W^{\ell,p}(G) \) (\( \ell \in \mathbb{N} \), \( p \in [1, \infty] \), and \( G \) a bounded interval in \( \mathbb{R} \)) by \( \| \cdot \|_{\ell,p,G} \) and the seminorm by \( | \cdot |_{\ell,p,G} \). For \( p = 2 \), \( W^{\ell,2}(G) \) will be denoted by \( H^\ell(G) \) with the associated norm and seminorm written as, respectively, \( \| \cdot \|_{\ell,G} \) and \( | \cdot |_{\ell,G} \). For ease of notation, in the common case when \( G \equiv I \) the subscript “I” will be dropped on the above norms and seminorms. The above are naturally extended to vector functions, and we will write \( [W^{\ell,p}(G)]^2 \) for a vector function with two components. In addition, we adopt the standard notation \( W^{\ell,p}(a,b;X) \) (\( \ell \in \mathbb{N} \), \( p \in [1, \infty] \), \( (a,b) \) an interval in \( \mathbb{R} \), \( X \) a Banach space) for time dependent spaces with norm \( \| \cdot \|_{\ell,p,a,b;X} \). Once again, we write \( H^\ell(a,b;X) \) if \( p = 2 \). Furthermore, \( C \) denotes a generic constant independent of the mesh parameter \( h \) and the time step \( \Delta t \); see below.

### 2. Weak formulation and finite element approximation

We shall assume that the data and the solution of (1.6), (1.7) (writing again \( w \) instead of \( \tilde{w} \) for ease of notation) satisfy

\[
(2.1a) \quad f, g \in C(\mathbb{R},\mathbb{R}) \text{ such that } \forall L > 0, \exists C_L \geq 0 : \\
|f(w_1) - f(w_2)| \leq C_L |w_1 - w_2| \quad \forall |w_1|, |w_2| \leq L, \\
|g(w_1, w) - g(v, w)| \leq C_L |v - w| \quad \forall |w| \leq L, \forall v, w, \\
|g(v, w_1) - g(v, w_2)| \leq C_L |w_1 - w_2| \quad \forall |w_1|, |w_2|, |v| \leq L;
\]

\[
(2.1b) \quad \vec{x} \in W^{1,\infty}(0,T; [H^2(\Omega)^2] \cap H^2(0,T; [H^1(\Omega)^2]) \cap W^{2,\infty}(0,T; [L^2(\Omega)^2]));
\]

\[
(2.1c) \quad w \in C([0,T]; H^2(\Omega)) \cap W^{1,\infty}(0,T; H^1(\Omega)) \cap H^2(0,T; L^2(\Omega));
\]

\[
(2.1d) \quad 0 < m \leq |\vec{x}_\rho| \leq M \quad \text{on } I \times [0,T], \text{ for some } m, M \in \mathbb{R}_{>0}.
\]

Since our error analysis will also include the discretization in time our regularity assumptions are slightly stronger than those made in [13]; see Assumption 2.2 there.
Let us fix $\alpha \in (0, 1]$. For a test function $\xi \in [H^1(I)]^2$, we obtain on multiplying (1.6) by $|\vec{x}_\rho|^2 \xi$ that

$$
(2.2) \quad \int_{I_\rho} |\vec{x}_\rho|^2 [\alpha \vec{x}_t + (1 - \alpha) (\vec{x}_t \cdot \vec{v}) \vec{v}] \cdot \xi \, d\rho + \int_{I_\rho} \vec{x}_\rho \cdot \vec{\xi}_\rho \, d\rho = \int_{I_\rho} |\vec{x}_\rho|^2 f(w) \vec{v} \cdot \xi \, d\rho.
$$

In order to derive a weak formulation for (1.7) we employ an idea from [6], [9], and calculate for $\eta \in H^1(I)$, with the help of integration by parts and (1.3),

$$
(2.7) \quad \frac{d}{dt} \int_I |\vec{x}_\rho| w \eta \, d\rho = \int_I |\vec{x}_\rho| w_t \eta \, d\rho + \int_I \vec{x}_\rho \cdot \vec{x}_{\rho,t} \eta \, d\rho - \int_I (\vec{x}_t \cdot \vec{v}) w \eta \, d\rho - \int_I \kappa (\vec{x}_t \cdot \vec{v}) w \eta |\vec{x}_\rho| \, d\rho
$$

where we have also used (1.4), (1.7), and the fact that $\vec{v} = \kappa |\vec{x}_\rho|$. We now use (2.2), (2.3) in order to discretize our system and begin by introducing the decomposition $I = \bigcup_{j=1}^J \sigma_j$, where $\sigma_j = (\rho_{j-1}, \rho_j)$. We set $h := \max_{j=1,\ldots,J} h_j$, where $h_j := \rho_j - \rho_{j-1}$ and assume that

$$
(2.4) \quad h \leq C h_j, \quad j = 1, \ldots, J.
$$

Let

$$
(2.5) \quad V_1^h := \{ \chi \in C(I) : \chi|_{\sigma_j} \text{ is affine, } j = 1, \ldots, J \} \subset H^1(I)
$$

and $I^h : C(I) \to V_1^h$ be the standard Lagrange interpolation operator such that $(I^h \eta)(\rho_j) = \eta(\rho_j)$, $j = 1, \ldots, J$. We require also the local interpolation operator $I_{\sigma_j}^h \equiv I_{\sigma_j}^h$, $j = 1, \ldots, J$, and recall for $p \in (1, \infty)$, $k \in \{0, 1\}$, $\ell \in \{1, 2\}$, and $j = 1, \ldots, J$ that

$$
(2.6a) \quad h_{j}^k |\eta|_{0, \infty, \sigma_j} + h_{j}^k |\eta|_{1, p, \sigma_j} \leq C \| \eta_{h} \|_{0, p, \sigma_j} \quad \forall \eta \in V_1^h,
$$

$$
(2.6b) \quad |(I - I_{\sigma_j}^h)\eta|_{k, p, \sigma_j} \leq C h_{j}^{k-1} \| \eta \|_{\ell, p, \sigma_j} \quad \forall \eta \in W^{\ell, p}(\sigma_j),
$$

$$
(2.6c) \quad |(I - I_{\sigma_j}^h)\eta|_{\ell-1, \infty, \sigma_j} \leq C h_{j}^\ell \| \eta \|_{\ell, \sigma_j} \quad \forall \eta \in H^\ell(\sigma_j).
$$

As well as the standard $L^2(I)$ inner product $(\cdot, \cdot)$, we introduce the discrete inner product $(\cdot, \cdot)^h$ defined by

$$
(2.7) \quad (\eta_1, \eta_2)^h := \sum_{j=1}^J \int_{\sigma_j} I_{\sigma_j}^h (\eta_1, \eta_2),
$$

where $\eta_j$ are piecewise continuous functions on the partition $\bigcup_{j=1}^J \sigma_j$ of $I$. We note for $j = 1, \ldots, J$ and for all $\eta_{h}, \chi_{h} \in V_1^h$ that

$$
(2.8a) \quad \int_{\sigma_j} |\eta_{h}|^2 \, d\rho \leq \int_{\sigma_j} I_{\sigma_j}^h [|\eta_{h}|^2] \, d\rho \leq 3 \int_{\sigma_j} |\eta_{h}|^2 \, d\rho,
$$

$$
(2.8b) \quad \left| \int_{\sigma_j} (I - I_{\sigma_j}^h)(\eta_{h} \chi_{h}) \, d\rho \right| \leq C h_{j}^2 |\eta_{h}|_{1, \sigma_j} |\chi_{h}|_{1, \sigma_j} \leq C h_{j} |\eta_{h}|_{1, \sigma_j} |\chi_{h}|_{0, \sigma_j}.
$$
The result (2.8b) follows immediately from (2.6a), (2.6b). The inner products $(\cdot, \cdot)$ and $(\cdot, \cdot)_h$ are naturally extended to vector functions. In addition to the above spatial discretization, let $0 \equiv t_0 < t_1 < \cdots < t_{N-1} < t_N \equiv T$ be a partitioning of $[0, T]$ with time steps $\Delta t_n := t_n - t_{n-1}$, $n = 1, \ldots, N$, and $\Delta t := \max_{n=1,\ldots,N} \Delta t_n$. Before we define our scheme we assign to an element $\tilde{X}^n \in [V_1^h]^2$ (the upper index referring to the time level $n$) a piecewise constant discrete unit tangent and normal by

\begin{equation}
\tilde{T}^n = \frac{\tilde{X}^n}{|\tilde{X}^n|} \quad \tilde{N}^n = (\tilde{T}^n)^\perp \quad \text{on } \sigma_j, \ j = 1, \ldots, J.
\end{equation}

(2.9)

Our discretization of (2.2) now reads: given $\tilde{X}^{n-1} \in [V_1^h]^2$ and $W^{n-1} \in V_1^h$, find $\tilde{X}^n \in [V_1^h]^2$ such that

\begin{equation}
\left(\tilde{X}^{n-1}_\rho\right)^2 \left[ \alpha D_t \tilde{X}^n + (1 - \alpha) \left( D_t \tilde{X}^n, \tilde{N}^{n-1} \right), \tilde{N}^{n-1} \right] + \left( \tilde{X}^n, \tilde{N}^{n-1}_\rho \right) = \left( \tilde{X}^{n-1}_\rho \right)^2 f(W^{n-1}, \tilde{N}^{n-1}, \tilde{N}^{n-1}_\rho) \quad \forall \tilde{N}^{n-1}_\rho \in [V_1^h]^2.
\end{equation}

Here, and in what follows, we abbreviate $D_t a^n := \frac{a^n - a^{n-1}}{\Delta t_n}$. We next use the solution $\tilde{X}^n$ of (2.10) in order to discretize (2.3). To do so, we define approximations $V^n, \Psi^n$ of the normal and tangential velocities by

\begin{equation}
V^n = D_t \tilde{X}^n \quad \text{and} \quad \Psi^n = D_t \tilde{X}^n \tilde{T}^n \quad \text{on } \sigma_j, \ j = 1, \ldots, J.
\end{equation}

(2.11)

Then find $W^n \in V_1^h$ such that

\begin{equation}
D_t \left[ \left( |\tilde{X}^{n-1}_\rho| W^n, \eta^h \right)^h \right] + d \left( \frac{W^n}{|\tilde{X}^{n-1}_\rho|} \eta^h \right) + (\Psi^n W^n, \eta^h)^h = \left( |\tilde{X}^{n-1}_\rho| g(V^n, W^{n-1}), \eta^h \right)^h \quad \forall \eta^h \in V_1^h.
\end{equation}

(2.12)

Let us formulate the main result of this paper, which will be proved in section 3.

**Theorem 2.1.** Let $\tilde{X}^0 = I^h \tilde{x}^0 \in [V_1^h]^2$ and $W^0 = I^h w^0 \in V_1^h$. There exists $h^* > 0$ such that for all $h \in (0, h^*)$ and $\Delta t \leq C h$ the discrete problem (2.10), (2.12) has a unique solution $(\tilde{X}^n, W^n) \in [V_1^h]^2 \times V_1^h, n = 1, \ldots, N$, and the following error bounds hold:

\begin{equation}
\sup_{n=0,\ldots,N} \left[ |\tilde{x}^n - \tilde{X}^n|_1^2 + |w^n - W^n|_0^2 \right] + \sum_{n=1}^N \Delta t_n \left[ |\tilde{x}^n_1 - D_t \tilde{X}^n|_0^2 + |w^n - W^n|_1^2 \right] \leq C h^2,
\end{equation}

(2.13)

where $\tilde{x}^n := \tilde{x}(\cdot, t_n), w^n := w(\cdot, t_n), \tilde{x}^n_1 := \tilde{x}_t(\cdot, t_n), n = 0, \ldots, N$.

**3. Error analysis.** To begin, it follows from (2.1b)–(2.1d) and (1.4) that for $n = 0, \ldots, N$,

\begin{equation}
\|\tilde{x}^n\|_2 + \|\tilde{x}^n_1\|_1 + \|\tilde{v}^n\|_1 + \|\tilde{v}^n_1\|_1 + \|\tilde{w}^n\|_2 \leq C,
\end{equation}

(3.1)

where $\tilde{x}^n := \tilde{r}(\cdot, t_n), v^n := v(\cdot, t_n), w^n := \psi(\cdot, t_n)$. We abbreviate for $n = 0, \ldots, N$

\begin{equation}
\tilde{E}^n := I^h \tilde{x}^n - \tilde{X}^n \in [V_1^h]^2 \quad \text{and} \quad Z^n := I^h w^n - W^n \in V_1^h.
\end{equation}

(3.2)
In the same way as (3.6), we obtain from (2.8a), (3.7), (3.3), and (3.5) that

$$ |E^{n-1}|^2_1 + \beta^2 \left( |X^{n-1} \cdot Z^{n-1}| \right)^h \leq h^2 e^{\gamma E^{n-1} \zeta(t)} dt \quad \text{for } h \in (0, h^*], $$

where the function $\zeta(t)$ is defined by

$$ \zeta(t) := 1 + \|\vec{x}_t(t)\|^2_1 + |w_t(t)|^2_0 $$

and $h^* > 0$ is chosen so small that

$$ (h^*)^2 e^{\gamma K} \leq \beta \quad \text{with } K := \int_{0}^{t} \zeta(t) \, dt. $$

Here, $\beta \in (0, 1]$ and $\gamma > 0$ are independent of $h$ and $\Delta t$, and will be chosen a posteriori. Clearly, (3.3) holds for $n = 1$ in view of our choice of initial data for $\vec{X}^0$ and $\vec{W}^0$.

It follows from (2.4), (2.6c), (3.3), (3.1), and (3.5) that

$$ |\vec{x}^{n-1} - \vec{X}^{n-1}|_{1, \infty} \leq |E^{n-1}|_{1, \infty} + |(I - I h) \vec{x}^{n-1}|_{1, \infty} \leq C h^{-\frac{1}{2}} |E^{n-1}|_1 + C h^\frac{1}{2} |\vec{x}^{n-1}|_2 $$

$$ \leq Ch^\frac{1}{2} \left( e^{\frac{2\gamma K}{T}} + 1 \right) \leq C (h^*)^\frac{1}{2} e^{\gamma K} \leq C \beta \leq \min \left\{ \frac{m}{2}, M \right\}, $$

provided that $\beta$ is chosen small enough. Combining this inequality with (2.1d) we infer that

$$ 0 < \frac{m}{2} \leq |\vec{x}^{n-1}|_1 \leq 2 M \quad \text{on } I \text{ for } h \in (0, h^*]. $$

If we use (3.7) and argue similarly to (3.6) we further obtain for any $h \in (0, h^*]$

$$ |\vec{x}^{n-1} - \vec{T}^{n-1}|_0 + |\vec{v}^{n-1} - \vec{V}^{n-1}|_0 \leq C |\vec{x}^{n-1} - \vec{X}^{n-1}|_1 \leq C \left[ |E^{n-1}|_1 + h \right], $$

$$ |\vec{x}^{n-1} - \vec{T}^{n-1}|_{0, \infty} + |\vec{v}^{n-1} - \vec{V}^{n-1}|_{0, \infty} \leq C |\vec{x}^{n-1} - \vec{X}^{n-1}|_{1, \infty} $$

$$ \leq C h^{-\frac{1}{2}} \left[ |E^{n-1}|_1 + h \right]. $$

In the same way as (3.6), we obtain from (2.8a), (3.7), (3.3), and (3.5) that

$$ |Z^{n-1}|_0^2 \leq C \left( |X^{n-1} \cdot Z^{n-1}| \right)^h \leq \frac{C}{\beta^2} h^2 e^{\gamma K} \leq \frac{C}{\beta^2} h^* e^{2\gamma K} \leq C h, $$

which combined with (2.6a), (2.6c), (2.4), and (3.1) yields for $h \in (0, h^*]$

$$ |W^{n-1}|_{0, \infty} \leq |Z^{n-1}|_{0, \infty} + |I h w^{n-1}|_{0, \infty} \leq C h^{-\frac{1}{2}} |Z^{n-1}|_0 + C \leq C. $$

### 3.1. The curve equation.

We assume throughout that $h \in (0, h^*]$. Since (2.10) forms a linear problem it is easily seen that (3.7) implies the existence of a unique solution $\vec{X}^n \in [V^n]^2$ to (2.10). We deduce from (2.10) and (2.2) with $\vec{\xi} = \vec{\xi}^h =$...
\[ \Delta t_n D_t \tilde{E}^n \in [V^h_1]^2 \]

\[
LHS := \Delta t_n \left( \left[ |X_{\rho}^{n-1}|^2 \left[ \alpha D_t \tilde{E}^n + (1 - \alpha) \left( D_t \tilde{x}^n \cdot \tilde{v}^{n-1} \right) \tilde{v}^{n-1} \right], D_t \tilde{E}^n \right]_h \right.
+ \Delta t_n \left( \tilde{E}^n_\rho, (D_t \tilde{E}^n)_\rho \right) \\
= \Delta t_n \left( \left[ |X_{\rho}^{n-1}|^2 \left[ \alpha D_t \tilde{x}^n + (1 - \alpha) \left( D_t \tilde{x}^n \cdot \tilde{v}^{n-1} \right) \tilde{v}^{n-1} \right], D_t \tilde{E}^n \right]_h \right.
+ \Delta t_n \left( \tilde{x}^n_\rho, (D_t \tilde{E}^n)_\rho \right) - \Delta t_n \left( |X_{\rho}^{n-1}|^2 f(W^{n-1}) \tilde{v}^{n-1}, D_t \tilde{E}^n \right)_h \\
= \Delta t_n \left( \left[ |X_{\rho}^{n-1}|^2 \left[ \alpha D_t \tilde{x}^n + (1 - \alpha) \left( D_t \tilde{x}^n \cdot \tilde{v}^{n-1} \right) \tilde{v}^{n-1} \right], D_t \tilde{E}^n \right]_h \right.
- \left( |X_{\rho}^{n}|^2 \left[ \alpha \tilde{x}_t(t_n) + (1 - \alpha) (\tilde{x}_t(t_n) \cdot \tilde{v}^n) \tilde{v}^n \right], D_t \tilde{E}^n \right)_h \\
+ \Delta t_n \left[ \left( |X_{\rho}^{n}|^2 f(W^n) \tilde{v}^n, D_t \tilde{E}^n \right) - \left( |X_{\rho}^{n-1}|^2 f(W^{n-1}) \tilde{v}^{n-1}, D_t \tilde{E}^n \right)_h \right] \\
(3.11) \quad =: A_1 + A_2.
\]

Using (3.7) and (2.8a) we find, with the help of an elementary calculation, that

\[
(3.12) \quad LHS \geq \Delta t_n \alpha \frac{n^2}{4} \left| D_t \tilde{E}^n \right|_0^2 + \frac{1}{2} \left| \tilde{E}^n \right|_0^2 + \left| \tilde{E}^n - \tilde{E}^{n-1} \right|_1^2 - \left| \tilde{E}^{n-1} \right|_1^2.
\]

Let us analyze the \( A_1 \) term defined in (3.11) and note that

\[
A_1 = \Delta t_n \left[ \left( |X_{\rho}^{n-1}|^2 \left[ \alpha D_t (I^h \tilde{x}^n) + (1 - \alpha) \left( D_t (I^h \tilde{x}^n) \cdot \tilde{v}^{n-1} \right) \tilde{v}^{n-1} \right], D_t \tilde{E}^n \right]_h \right.
- \left( |X_{\rho}^{n-1}|^2 \left[ \alpha D_t (I^h \tilde{x}^n) + (1 - \alpha) \left( D_t (I^h \tilde{x}^n) \cdot \tilde{v}^{n-1} \right) \tilde{v}^{n-1} \right], D_t \tilde{E}^n \right)_h \\
+ \Delta t_n \left[ \left( |X_{\rho}^{n-1}|^2 \left[ D_t (I^h \tilde{x}^n) - \tilde{x}_t^n \right], D_t \tilde{E}^n \right) \\
+ \left( |X_{\rho}^{n-1}|^2 \left[ D_t (I^h \tilde{x}^n) - \tilde{x}_t^n \right] \tilde{v}^{n-1}, D_t \tilde{E}^n \right)_h \\
+ \Delta t_n \left( |X_{\rho}^{n-1}|^2 \left[ (\tilde{x}_t^n \cdot \tilde{v}^{n-1}) \tilde{v}^{n-1} - (\tilde{x}_t^n \cdot \tilde{v}^n) \tilde{v}^n \right], D_t \tilde{E}^n \right)_h \\
+ \left( |X_{\rho}^{n-1}|^2 \left[ |X_{\rho}^{n-1}|^2 \right] \left[ \alpha \tilde{x}_t^n + (1 - \alpha) (\tilde{x}_t^n \cdot \tilde{v}^n) \tilde{v}^n \right], D_t \tilde{E}^n \right)_h \\
= : \sum_{i=1}^{3} A_{1,i}.
\]

(3.13)

We now bound the terms \( A_{1,i} \) defined in (3.13) on recalling (2.1b), (2.1d), (3.7), (1.3), and (2.9). It follows from (2.8b) and (2.6b) that

\[
|A_{1,1}| \leq C h \Delta t_n \left| D_t (I^h \tilde{x}^n) \right|_1 \left| D_t \tilde{E}^n \right|_0 \leq C h \Delta t_n \left| D_t \tilde{E}^n \right|_0
\]

(3.14)

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If we combine (3.13)–(3.20) with Young's inequality and the definition of $\zeta$, we infer, as $\Delta t_n \leq C h$, that

\begin{align*}
|A_1 + A_2| &\leq \delta \Delta t_n |D_t \overline{E}_n|^2 + C(\delta) \left[ \Delta t_n \left( |\overline{E}^{n-1}_n|^2 + |Z^{n-1}_n|^2 \right) + h^2 \int_{t_{n-1}}^{t_n} \zeta(t) \, dt \right].
\end{align*}
Inserting (3.12) and (3.21) into (3.11) and choosing \( \delta \) sufficiently small we obtain with the help of (3.7) and (2.8a)

\[
\Delta t_n \frac{m^2}{4} |D_t \vec{E}^n|^2 + |\vec{E}^n|^2 + |\vec{E}^n - \vec{E}^{n-1}|^2
\]

(3.22) \leq |\vec{E}^{n-1}|^2 + C \Delta t_n \left[ |\vec{E}^{n-1}|^2 + (|X_p^{n-1}| Z^{n-1}, Z^{n-1}) + C h^2 \int_{t_{n-1}}^{t_n} \zeta(t) \, dt \right].

The induction hypothesis (3.3) together with the fact that \( \zeta \geq 1 \) then yields

\[
\Delta t_n \frac{m^2}{4} |D_t \vec{E}^n|^2 + |\vec{E}^n|^2 + |\vec{E}^n - \vec{E}^{n-1}|^2
\]

(3.23) \leq h^2 e^{\gamma L^{n-1} \zeta(t)} dt \left[ 1 + C \left( 1 + \frac{1}{\beta^2} \right) \int_{t_{n-1}}^{t_n} \zeta(t) \, dt \right]

so that

(3.24) \quad |\vec{E}^n|^2 \leq h^2 e^{\gamma L^{n-1} \zeta(t)} dt \quad \text{provided that } \gamma \geq C \left( 1 + \frac{1}{\beta^2} \right).

In particular, (3.23), this choice of \( \gamma \) and (3.5) imply that

(3.25) \Delta t_n \frac{m^2}{4} |D_t \vec{E}^n|^2 + |\vec{E}^n|^2 + |\vec{E}^n - \vec{E}^{n-1}|^2 \leq h^2 e^{\gamma K} \leq h^3 (h^*)^2 e^{\gamma K} \leq \beta h^2.

In the same way as in (3.6)–(3.8a), (3.8b) we infer from (3.24) for \( h \in (0, h^*) \)

(3.26a) \quad 0 < \frac{m}{2} \leq |\vec{X}^n_2| \leq 2 M \quad \text{on } \Omega,

(3.26b) \quad |\vec{\nu} - \vec{\nu}|_0 + |\vec{\nu} - \vec{\nu}|_0 + h^\frac{1}{2} \left( |\vec{\nu} - \vec{\nu}|_{0, \infty} + |\vec{\nu} - \vec{\nu}|_{0, \infty} \right) \leq C \left[ |\vec{E}^n|_1 + h \right].

In addition, we deduce from (1.4), (2.11), (3.1), (3.26b), (2.1b), and as \( \Delta t_n \leq C h \), that

\[
|\psi - \psi|_0 + |\psi - \psi|_0 \\
\leq |t^\alpha_t \cdot (t^\alpha - \vec{T}^\alpha)|_0 + |t^\alpha_t \cdot (t^\alpha - \vec{T}^\alpha)|_0 + 2 |t^\alpha_t - D_t(I^h \vec{\nu})|_0 + 2 \left| D_t \vec{E}^n \right|_0 \\
\leq C \left[ |\vec{E}^n|_1 + h \right] + C \Delta t_n \left| t^\alpha_t \right|_{L^\infty(t_{n-1}, t_n; L^2(\Omega))} + 2 \left| D_t \vec{E}^n \right|_0 \\
(3.27) \quad \leq C \left[ |\vec{E}^n|_1 + h + \left| D_t \vec{E}^n \right|_0 \right],
\]

where the term \( |t^\alpha_t - D_t(I^h \vec{\nu})|_0 \) is bounded as in (3.15). Furthermore, we conclude from (2.6a,b), (2.4), (3.27), (3.1), (3.25), and as \( \Delta t_n \leq C h \), that

\[
\Delta t_n \left| \psi \right|_{0, \infty}^2 \leq 2 \Delta t_n \left[ I^h \psi|_{0, \infty}^2 + |I^h \psi - \psi|_{0, \infty}^2 \right] \\
\leq 2 \Delta t_n \left[ \psi|_{0, \infty}^2 + C h^{-1} |I^h \psi - \psi|_{0}^2 \right] \\
\leq 2 \Delta t_n \left| \psi|_{0, \infty}^2 + C \Delta t_n h^{-1} \left[ \psi|_{0}^2 + |(I - I^h) \psi|_0^2 \right] \right] \\
\leq C \left( \| \psi \|_{1}^2 + C \Delta t_n |\vec{E}^n|_1^2 + h^2 \right) + C h^{-1} \Delta t_n \left| D_t \vec{E}^n \right|_0^2 \\
(3.28) \quad \leq C h^{\frac{3}{2}} < 4 d
for $h \in (0, h^*]$, provided that $h^*$ is small enough. Finally, on noting (2.6a), (2.6b), (2.4), (2.1b), and that $|D_t(I_hx^n)|_1 \leq C$ as in (3.14), we have

\begin{equation}
|\bar{X}^n - \bar{X}^{n-1}|_1 \leq C \Delta t_n \left[ |D_t \bar{E}^n|_1 + |D_t(I_hx^n)|_1 \right] \leq C \Delta t_n \left[ h^{-1} |D_t \bar{E}^n|_0 + 1 \right].
\end{equation}

### 3.2. The scalar equation

We assume throughout that $h \in (0, h^*)$. Let us first establish the existence and uniqueness of $W^n \in V^h_1$. Since the system (2.12) is linear, existence follows from uniqueness which in turn is a consequence of the estimate

$$ \left| (\Psi^h \eta^h, \eta^h) \right| \leq d\left( \frac{\eta^h}{|X|^h}, \eta^h \right) + \frac{1}{4d} \left( |\bar{X}^h| |(\Psi^h)^2\eta^h, \eta^h | \right) \leq d\left( \frac{\eta^h}{|X|^h}, \eta^h \right) + \frac{1}{4d} \left( |\bar{X}^h| |\eta^h, \eta^h | \right) $$

for all nontrivial $\eta^h \in V^h_1$, which follows from (3.28). We deduce from (2.12) with $\eta^h = Z^n$ that

$$ D_t \left[ \frac{|\bar{X}^h|}{|X|^h} Z^n, Z^n \right] = D_t \left[ \frac{|\bar{X}^h|}{|X|^h} I^h w^n, Z^n \right] + (\Psi^h W^n, Z^n)^h - \left[ |\bar{X}^h| g(V^n, W^n-1), Z^n \right]^h. $$

Observing that for all $a \in R$

$$ D_t [a^b b^n]^h = \frac{1}{2} D_t [a^b (b^n)^2] + \frac{1}{2} \Delta t_n (b^n - b^{n-1})^2 + \frac{1}{2} \Delta t_n (a^n - a^{n-1}) (b^n)^2 $$

and multiplying (3.30) by $\Delta t_n$ we obtain with the help of (2.3) and (3.26a)

$$\begin{align*}
\frac{1}{2} \left( |\bar{X}^h| Z^n, Z^n \right)^h &+ \frac{d}{2M} \Delta t_n |Z^n|^2 + \frac{1}{2} \left( |\bar{X}^h| - |\bar{X}^{n-1}| \right) (Z^n - Z^{n-1}) \left( Z^n - Z^{n-1} \right)^h \\
&\leq \frac{1}{2} \left( |\bar{X}^h| Z^n, Z^n \right)^h + \frac{1}{2} \left( \left[ |\bar{X}^h| - |\bar{X}^{n-1}| \right] Z^n, Z^n \right)^h \\
&+ \Delta t_n \left( D_t |\bar{X}^h| I^h w^n, Z^n \right)^h - \Delta t_n \left( (|\bar{X}^h| \cdot w^n)(t^n), Z^n \right) \\
&+ \Delta t_n d\left( \frac{(I^h w^n)^h}{|X|^h} - \frac{w^n}{|X|^h} Z^n \right) + \Delta t_n \left[ (\Psi^h (I^h w^n), Z^n)^h - (w^n Z^n) \right] \\
&- \Delta t_n (\Psi^h Z^n, Z^n)^h + \Delta t_n \left[ (|\bar{x}^n|^2 g(v^n, w^n), Z^n) - \left( |\bar{X}^h| g(V^n, W^n-1), Z^n \right)^h \right] \\
&= \frac{1}{2} \left( |\bar{X}^h| Z^n, Z^n \right)^h + \sum_{i=1}^6 B_i.
\end{align*}\n
We now estimate the $B_i$ terms defined in (3.31). In order to treat $B_i$ we first observe that

$$ |\bar{X}^h| - |\bar{X}^{n-1}| = \bar{t}^{n-1} \cdot \left( \bar{X}^h - \bar{X}^{n-1} \right) + \frac{1}{2} |\bar{X}^h| |\bar{t}^n - \bar{t}^{n-1}|^2, $$

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so that (2.8b), (2.6a), (2.4), and integration by parts imply

\[
2B_1 \leq - \left( \tilde{f}^{n-1} : (\tilde{X}^n \prec X^n) Z^n, Z^n \right) + C \int_{\Omega} \left| \tilde{X}^n \prec X^n \right| \left| Z^n \right| \left| Z^n \right| \nabla Z^n.
\]

Using (2.6a), (2.4), (3.8a), (3.29), (3.1), the Gagliardo–Nirenberg interpolation inequality \( |\eta|_{0, \infty} \leq C |\eta|_0 |\eta|_1, \) (3.14), and the fact that \( |\tilde{E}^{n-1}| \leq h^\frac{1}{2} \) (cf. (3.3) and (3.5)) we infer that

\[
B_1 \leq C \left[ \left| \tilde{f}^{n-1} \right|_{0} \left| \tilde{X}^n \prec X^n \right|_{1} \left| Z^n \right|_{0, \infty} \right] + C \Delta t_n \left[ h^{-1} |D_t \tilde{E}^n|_0 + 1 \right] |Z^n|_0 |Z^n|_1
+ C \Delta t_n \left| D_t (I^n \tilde{E}^n) \right| |Z^n|_{1, \infty} + C \Delta t_n |D_t \tilde{E}^n|_0 h^{-\frac{1}{2}} |Z^n|_0 |Z^n|_1
\]

(3.32) \( \leq C \Delta t_n \left[ h^{-\frac{1}{2}} |D_t \tilde{E}^n|_0 + h^\frac{1}{2} \right] |Z^n|_0 |Z^n|_1 + C \Delta t_n |Z^n|_{1, \infty} |Z^n|_1.
\]

Let us postpone the rather complicated analysis of \( B_2 \) and first deal with \( B_3, \ldots, B_6. \)

To bound \( B_3, \) we first note that

\[
\left( \frac{(w^n - I^n w^n)}{|X^n|^\frac{3}{2}}, Z^n \right) = 0.
\]

It then follows from (2.1b), (2.1d), (3.26a), (2.6b), and Sobolev embedding that

\[
B_3 = \Delta t_n d \left( \frac{|\tilde{x}^n| - |X^n|}{|X^n|} w^n, Z^n \right) \leq C \Delta t_n |\tilde{x}^n - X^n|_1 |w^n|_{1, \infty} |Z^n|_1
\]

(3.33) \( \leq C \Delta t_n \left[ |\tilde{E}^n|_1 + \left| (I^n \tilde{E}^n) \right|_1 \right] |Z^n|_1 \leq C \Delta t_n \left[ |\tilde{E}^n|_1 + h \right] |Z^n|_1.
\]

Next we note from (2.8a), (2.8b), (2.6a), (2.6b), (2.4), Sobolev embedding, (3.1), and (3.27) that

\[
B_4 = \Delta t_n \left( \left( \Psi^n - I^n \psi^n \right) I^n w^n, Z^n \right)^h + \Delta t_n \left( \left( (I^n \psi^n) I^n w^n, Z^n \right)^h - \left( (I^n \psi^n) I^n w^n, Z^n \right) \right)
+ \Delta t_n \left( (I^n w^n) (I^n - I) \psi^n + \psi^n (I^n - I) w^n, Z^n \right)
\leq C \Delta t_n \left[ h^n - \Psi^n \right]_0 |I^n w^n|_{0, \infty} \left| Z^n \right|_1 + h^2 |I^n \psi^n|_1 |I^n w^n|_{1, \infty} \left| Z^n \right|_1
+ C \Delta t_n h \left( |w^n|_{0, \infty} |\psi^n|_1 + |\psi^n|_{0, \infty} |w^n|_1 \right) |Z^n|_1
\]

(3.34) \( \leq C \Delta t_n \left[ D_t \tilde{E}^n \right]_0 + \left[ \tilde{E}^n \right]_1 + h \left| Z^n \right|_1.
\]

On noting (2.7), (2.8a), (2.6a), (2.6b), (2.4), Sobolev embedding, (3.1), and (3.27),
we have that
\[ B_5 = -\Delta t_n \left( I^h \psi^n Z^n, Z^n_h \right) - \Delta t_n \left( (\Psi^n - I^h \psi^n) Z^n, Z^n_h \right) \]
\[ \leq C \Delta t_n \left( |\psi^n|, |Z^n| \right) + C \Delta t_n \left( |\Psi^n - I^h \psi^n|, |Z^n| \right) \]
\[ \leq C \Delta t_n \left( |Z^n| + C \Delta t_n \left( |E^n_1| + |D_t E^n_1| + \tilde{h}^{\frac{1}{2}} \right) \right) \]
\[ \leq C \Delta t_n \left( |Z^n| + \left( 1 + h^{\frac{1}{2}} \right) \left( |E^n_1| + |D_t E^n_1| \right) \right). \]

Finally, we deduce from (2.8a), (2.8b) the Cauchy–Schwarz inequality for \((\cdot, \cdot)^n\), (2.1a), (3.1), and (2.6b) that
\[ B_6 = \Delta t_n \left[ \left( \left| \frac{\partial}{\partial t} \mathcal{X}_n \right| \cdot g(v^n, w^n), Z^n \right) + \left( \left| \frac{\partial}{\partial t} \mathcal{X}_n \right| \cdot (I - I^h) g(v^n, w^n), Z^n \right) \right] \]
\[ + \Delta t_n \left[ \left( \left| \frac{\partial}{\partial t} \mathcal{X}_n \right| \cdot I^h \left[ g(v^n, w^n) \right], Z^n \right) - \left( \left| \frac{\partial}{\partial t} \mathcal{X}_n \right| \cdot I^h \left[ g(v^n, w^n) \right], Z^n_h \right) \right] \]
\[ + \Delta t_n \left[ \left( \left| \frac{\partial}{\partial t} \mathcal{X}_n \right| \cdot g(I^h v^n, I^h w^n - g(V^n, W^{n-1})), Z^n \right) \right] \]
\[ \leq C \Delta t_n \left[ \left| \mathcal{X}_n \right| (v^n, w^n)|_{t=0} + \left| (I - I^h) g(v^n, w^n) \right|_{t=0} \right] |Z^n| \]
\[ + C \Delta t_n \left[ h \left| I^h [g(v^n, w^n)] \right| + \sqrt{(r^n, r^n)^h} \right] |Z^n| \]
\[ \leq C \Delta t_n \left[ \left| \mathcal{X}_n \right| (v^n, w^n)|_{t=0} + h |v^n + w^n|_1 \right] |Z^n| \]
\[ + C \Delta t_n \left[ h \left| I^h v^n + I^h w^n \right| + \sqrt{(r^n, r^n)^h} \right] |Z^n| \]
\[ \leq C \Delta t_n \left[ \left| \mathcal{X}_n \right| (v^n, w^n)|_{t=0} + h |v^n + w^n|_1 \right] |Z^n| \]
\[ + C \Delta t_n \left[ h \left| I^h v^n + I^h w^n \right| + \sqrt{(r^n, r^n)^h} \right] |Z^n| \]
\[ \leq C \Delta t_n \left( |E^n_1| + h + \sqrt{(r^n, r^n)^h} \right) |Z^n|_1. \]

where we have set \( r^n = g(I^h v^n, I^h w^n) - g(V^n, W^{n-1}) \). Using (2.1a) together with
the fact that \( W^{n-1} \leq C \), recall (3.10), we infer that
\[ |r^n| \leq g \left( I^h v^n, I^h w^n \right) - g \left( I^h v^n, W^{n-1} \right) \leq \left[ \left| I^h v^n - W^{n-1} \right| + \left| I^h v^n - V^n \right| \right] \]
\[ \leq C \left[ \left| I^h v^n - W^{n-1} \right| + \left| I^h v^n - V^n \right| \right] \]
\[ \leq C \left[ \left| I^h (w^n - w^{n-1}) \right| + \left| Z^n - V^n \right| + \left| I^h v^n - V^n \right| \right] \]
from which we deduce that
\[ \sqrt{(r^n, r^n)^h} \leq C \left[ \left| I^h (w^n - w^{n-1}) \right|_0 + \left| Z^n - V^n \right|_0 + \left| I^h v^n - V^n \right|_0 \right] \]
on noting (2.7) and (2.8a). Inserting this bound into (3.36) and recalling (3.1), (2.1c), (2.6b), (2.27), as well as \( \Delta t_n \leq Ch \), we have
\[ B_6 \leq C \Delta t_n \left( |E^n_1| + |Z^n - V^n|_0 + \left( D_t E^n_1 \right) \right) |Z^n|_0. \]

It remains to analyze \( B_2 \). We claim that
\[ |B_2| \leq C \Delta t_n \left[ \left| D_t E^n_1 \right| + \left| E^n_1 \right| + \left| E^n - V^n \right|_1 + h \right] \]
\[ + C \Delta t_n \left[ D_t E^n_1 \right]^2 \left( h^{-\frac{1}{2}} \right) |Z^n| + C \left| Z^n \right|_{t=0} \left( E^n - E^n_1 \right)^2 \]
\[ + C h \left( \Delta t_n \int_{t_{n-1}}^{t_n} \zeta(t) dt \right)^{\frac{3}{2}} |Z^n|_1. \]

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In order to prove (3.38), we first write

\[
B_2 = \left[ (\frac{\nabla}{\rho} - \nabla_{\rho} \nabla w^n - \nabla_{\rho} w^{n-1}, Z^n) - h - \left( (\frac{\nabla}{\rho} - \nabla_{\rho} w^n - \nabla_{\rho} w^{n-1}, Z^n) \right) \nabla \right] + \left( (\nabla^2 - I) w^n - \nabla_{\rho} w^{n-1}, Z^n \right)
\]

Similarly, we deduce from (3.7), Sobolev embedding, (2.6b), (2.1c), and (3.29) that

\[
(3.39) + \left( (\nabla - \nabla_{\rho}) w^n - \left( (\nabla_{\rho} - \nabla_{\rho} w^{n-1}, Z^n \right) \right) = \sum_{i=1}^{4} B_{2,i}.
\]

We now bound the $B_{2,i}$ terms defined in (3.39). It follows from (2.8b), (3.7), (2.6b), (3.1), (3.29), and as $w_t \in L^\infty(0, T; H^1(\Omega))$, that

\[
|B_{2,1}| \leq C \int |\nabla_{\rho} w^n - \nabla_{\rho} w^{n-1} - I| Z^n| \leq C \int |\nabla - \nabla_{\rho} w^n - \nabla_{\rho} w^{n-1} - I| Z^n| \leq C |\nabla - \nabla_{\rho} w^n - \nabla_{\rho} w^{n-1} - I| Z^n|
\]

Similarly, we deduce from (3.7), Sobolev embedding, (2.6b), (2.1c), and (3.29) that

\[
|B_{2,2}| \leq C \left[ \nabla - \nabla_{\rho} w^n - \nabla_{\rho} w^{n-1} - I \right] \leq C \int |\nabla - \nabla_{\rho} w^n - \nabla_{\rho} w^{n-1} - I| Z^n| \leq C |\nabla - \nabla_{\rho} w^n - \nabla_{\rho} w^{n-1} - I| Z^n|
\]

In view of (2.1b), (2.1c), we have that $\bar{\xi}_t \in L^\infty(0, T; [W^{-1,\infty}(\Omega)]^2)$, $w_t \in L^\infty(0, T; L^\infty(\Omega))$, and hence, as $\Delta t \leq C h$,

\[
|B_{2,3}| \leq C \int_{t_n}^{t_{n-1}} \left[ |\partial \bar{\xi}_t| w_t + |\partial \bar{\xi}_t| w_t^2 \right] \leq C \int_{t_n}^{t_{n-1}} \left[ |w_t| + |\partial \bar{\xi}_t| w_t^2 \right] \leq C \int_{t_n}^{t_{n-1}} \left[ |w_t| + |\partial \bar{\xi}_t| w_t^2 \right]
\]

It remains to analyze $B_{2,4}$, which requires some more intricate arguments. To begin,

\[
B_{2,4} = \left( (\nabla_{\rho} - \nabla_{\rho} w^n - \nabla_{\rho} w^{n-1}, Z^n) \right) = \left( (\nabla_{\rho} - \nabla_{\rho} w^n - \nabla_{\rho} w^{n-1}, Z^n) \right) + \left( (\nabla_{\rho} - \nabla_{\rho} w^n - \nabla_{\rho} w^{n-1}, Z^n) \right)
\]

(3.43) $= B_{2,4,1} + B_{2,4,2}$.
On recalling (2.9), we rewrite
\begin{align*}
B_{2,4,1} &= \frac{1}{2} \left( \| \hat{X}_n - \hat{\bar{T}}_{n-1} \|^2 w_n - \| \hat{X}_n - \hat{\bar{T}}_{n-1} \|^2 w_n - \| \hat{X}_n - \hat{\bar{T}}_{n-1} \|^2 w_n, Z^n \right) \\
&= \frac{1}{2} \left( \| \hat{X}_n - \hat{\bar{T}}_{n-1} \|^2 w_n, Z^n \right) \\
&\quad + \frac{1}{2} \left( \| \hat{X}_n - \hat{\bar{T}}_{n-1} \|^2 w_n - \| \hat{X}_n - \hat{\bar{T}}_{n-1} \|^2 w_n, Z^n \right) \\
&\quad + \frac{1}{2} \left( \| \hat{X}_n - \hat{\bar{T}}_{n-1} \|^2 w_n, Z^n \right)
\end{align*}

(3.44)

\begin{align*}
+ \frac{1}{2} \left( \| \hat{X}_n - \hat{\bar{T}}_{n-1} \|^2 w_n - \| \hat{X}_n - \hat{\bar{T}}_{n-1} \|^2 w_n, Z^n \right) =: I + II + III.
\end{align*}

Noting (3.29), (3.26b), Sobolev embedding, and (3.25), we obtain that
\begin{align*}
|I| &\leq C \left\| \hat{X}_n - \hat{\bar{T}}_{n-1} \right\|_{\infty} \left\| \rho \right\|_{\infty} \left\| \hat{T} - \hat{\bar{T}}_{n-1} \right\|_{\infty} \left\| Z^n \right\|_{\infty} \\
&\leq C \Delta t_n \left( h^{-1} \| D_1 \hat{E}_n^0 \|_0 + 1 \right) h^{-1} \left\| \hat{E}_n^1 \|_1 + h \right\| \| Z^n \|_1 \\
&\leq C \Delta t_n \left[ D_1 \hat{E}_n^0 \|_0 + h \right] \| Z^n \|_1.
\end{align*}

(3.45)

Since \( w_t \in L^\infty(0, T; L^\infty) \) and \( \| \hat{\bar{T}}_{n-1} \|_{0, \infty} \leq 2 \), we deduce from (3.7) and (3.8a) that
\begin{align*}
|II| &\leq C \Delta t_n \left\| \hat{X}_n - \hat{\bar{T}}_{n-1} \right\|_{0} \left\| Z^n \right\|_{0} \leq C \Delta t_n \left[ \| \hat{E}_n^1 \|_1 + h \right] \| Z^n \|_0.
\end{align*}

In order to treat term \( III \), we first observe that
\begin{align*}
\| \hat{X}_n - \hat{\bar{T}}_{n-1} \|^2 - \| \hat{X}_n - \hat{\bar{T}}_{n-1} \|^2 &= -(\hat{X}_n - \hat{\bar{T}}_{n-1}) \cdot (\hat{\bar{T}}_{n-1} + \hat{\bar{T}}_{n-1}) \\
&\quad - (\hat{\bar{T}}_{n-1} - \hat{\bar{T}}_{n-1}) \cdot (\hat{\bar{T}}_{n-1} + \hat{\bar{T}}_{n-1}).
\end{align*}

(3.47)

Furthermore, a straightforward calculation shows that
\begin{align*}
\hat{\bar{T}}_{n-1} - \hat{\bar{T}}_{n-1} &= \frac{1}{\| \hat{X}_n - \hat{\bar{T}}_{n-1} \|} \left[ I - \hat{\bar{T}}_{n-1} \otimes \hat{\bar{T}}_{n-1} \right] \left( \hat{X}_n - \hat{\bar{T}}_{n-1} \right) \\
&= \frac{1}{\| \hat{X}_n - \hat{\bar{T}}_{n-1} \|} (P^n + R^n) \left( \hat{X}_n - \hat{\bar{T}}_{n-1} \right),
\end{align*}

(3.48a)

where
\begin{align*}
P^n &= I - \hat{\bar{T}}_{n-1} \otimes \hat{\bar{T}}_{n-1} \quad \text{and} \quad |R^n| \leq C \| \hat{X}_n - \hat{\bar{T}}_{n-1} \|.
\end{align*}

(3.48b)

Inserting (3.47) and (3.48a), (3.48b) into \( III \) and observing that \( (\hat{\bar{T}}_{n-1} - \hat{\bar{T}}_{n-1}) \cdot (\hat{\bar{T}}_{n-1} + \hat{\bar{T}}_{n-1}) = 0 \), we derive
\begin{align*}
2III &= -(\hat{X}_n - \hat{\bar{T}}_{n-1}) \cdot (\hat{\bar{T}}_{n-1} + \hat{\bar{T}}_{n-1}) w_n, Z^n \\
&\quad - (P^n + R^n) \left( \hat{X}_n - \hat{\bar{T}}_{n-1} \right) \cdot (\hat{\bar{T}}_{n-1} + \hat{\bar{T}}_{n-1}) w_n, Z^n \\
&\quad - (\hat{X}_n - \hat{\bar{T}}_{n-1}) \cdot (\hat{\bar{T}}_{n-1} + \hat{\bar{T}}_{n-1} - \hat{\bar{T}}_{n-1}) w_n, Z^n \\
&\quad - \left( \left( r \hat{\bar{X}}_n \right)_{n} - \left( r \hat{\bar{X}}_{n-1} \right)_{n} \right) \cdot (P^n + R^n) \left( \hat{\bar{T}}_{n-1} + \hat{\bar{T}}_{n-1} \right) w_n, Z^n \\
&\quad + (P^n - E_n + E_{n-1}^1) \cdot (\hat{\bar{T}}_{n-1} + \hat{\bar{T}}_{n-1}) w_n, Z^n \\
&\quad + (R^n - E_n + E_{n-1}^1) \cdot (\hat{\bar{T}}_{n-1} + \hat{\bar{T}}_{n-1}) w_n, Z^n \\
&= III_1 + III_2 + III_3 + III_4.
\end{align*}

(3.49)
It follows from (3.7), (3.8a), (3.26b), Sobolev embedding, and (2.1b), (2.1c) that
\[
|III_1| \leq |\bar{\tau}^m - \bar{\tau}^{m-1}|_{0,\infty} \left| \tau^m - \bar{\tau}^m \right|_0 + |\bar{\tau}^{m-1} - \bar{\tau}^{m-2}|_0 \| w^n \|_{0,\infty} \| Z^n \|_0 \\
\leq C \Delta t_n \left[ |\bar{E}^n|_1 + |\bar{E}^{n-1}|_1 + h \right] \| Z^n \|_0,
\]
(3.50)
as \bar{x}_t \in L^\infty(t_{n-1}, t_n; [W^{1,\infty}(1)]^2). Recalling the definition of $P^n$ we may write
\[
P^n (\bar{\tau}^m + \bar{\tau}^{m-1}) = (\bar{\tau}^m - \bar{\tau}^m) + (\bar{\tau}^{m-1} - \bar{\tau}^m) + \frac{1}{2} \left[ |\bar{\tau}^m - \bar{\tau}^m|^2 + |\bar{\tau}^{m-1} - \bar{\tau}^m|^2 \right] \bar{\tau}^n,
\]
so, similarly to (3.50), we have from (3.48b), (3.8a), (3.26b), Sobolev embedding, and (2.1b), (2.1c) that
\[
|III_2| \leq C \Delta t_n \left[ |\bar{\tau}^m - \bar{\tau}^m|_0 + |\bar{\tau}^{m-1} - \bar{\tau}^{m-1}|_0 + |\bar{X}^n - \bar{X}^{n-1}|_1 \right] \| w^n \|_{0,\infty} \| Z^n \|_0 \\
\leq C \Delta t_n \left[ |\bar{E}^n|_1 + |\bar{E}^{n-1}|_1 + h \right] \| Z^n \|_0,
\]
(3.51)since $\Delta t_n \leq C h$. Next, performing integration by parts on the subintervals $\sigma_j$ we derive that
\[
III_3 = - \left( P^n (\bar{E}^n - \bar{E}^{n-1}), [(\bar{\tau}^m + \bar{\tau}^{m-1}) w^n]_\rho \right) Z^n + (\bar{\tau}^{m-1} + \bar{\tau}^m) w^n Z^n \\
- \sum_{j=1}^J \left( P^n |_{\sigma_{j+1}} - P^n |_{\sigma_j} \right) \left( \bar{E}^n - \bar{E}^{n-1} \right) (\rho_j) \cdot (\bar{\tau}^m + \bar{\tau}^{m-1}) w^n (\rho_j) Z^n (\rho_j),
\]
(3.52)where $\sigma_{j+1} \equiv \sigma_j$. Introducing the nodal basis functions $\chi_j \in V^h_i$ such that $\chi_j(\rho_i) = \delta_{i,j}$, $i, j = 1, \ldots, J$, and then choosing $\tilde{\xi}^h = \chi_j \tilde{\xi}$, for any $\tilde{\xi} \in \mathbb{R}^2$, in (2.10) we obtain
\[
\left[ \bar{X}_\rho |_{\sigma_{j+1}} - \bar{X}_\rho |_{\sigma_j} \right] \cdot \tilde{\xi} \\
= \left[ (\bar{X}_\rho)^{n-1} \right]^T f (W^{n-1}) \tilde{v}^{n-1} (\rho_j) \chi_j \tilde{\xi}^h \\
+ \left[ (\bar{X}_\rho)^{n-1} \right]^T \left[ \alpha D_t \bar{X}^n + (1 - \alpha) \left( D_t \bar{X}^n \cdot \tilde{v}^{n-1} \right) \tilde{v}^{n-1} \right] \chi_j \tilde{\xi}^h.
\]
We deduce from this and (3.48b), with the help of (3.7), (3.10), Sobolev embedding, and (2.1b), that for $j = 1, \ldots, J$
\[
\left| P^n |_{\sigma_{j+1}} - P^n |_{\sigma_j} \right| \leq C \left[ \bar{X}_\rho |_{\sigma_{j+1}} - \bar{X}_\rho |_{\sigma_j} \right] \\
\leq C h \left[ f (W^{n-1}) |_{0,\infty} + |(D_t \bar{X}^n) (\rho_j)| \right] \\
\leq C h \left[ 1 + |(D_t \bar{E}^n) |_{0,\infty} + |(D_t \bar{E}^n) (\rho_j)| \right] \\
\leq C h \left[ 1 + |(D_t \bar{E}^n) (\rho_j)| \right].
\]
Inserting this estimate into (3.52) and recalling (3.48b), (2.1d), (2.4), (2.8a), Sobolev embedding, (3.1), and (2.6a), we deduce that
\[
|III_3| \leq C \Delta t_n |D_t \bar{E}^n|_0 \| Z^n \|_1 + C \Delta t_n \| D_t \bar{E}^n |_0^2 \| Z^n \|_{0,\infty} \\
\leq C \Delta t_n |D_t \bar{E}^n|_0 \| Z^n \|_1 + C \Delta t_n \| D_t \bar{E}^n |_0^2 h^{-\frac{1}{2}} \| Z^n \|_0.
\]
(3.53)
Finally, we have, on noting (3.48b), (3.14), Sobolev embedding, and (3.1), that
\[ |III_{4}| \leq C |\tilde{E}^{n} - \tilde{E}^{n-1}|_1 \left[ |\tilde{E}^{n} - \tilde{E}^{n-1}|_1 + |I^h (\tilde{x}^{n} - \tilde{x}^{n-1})|_1 \right] |w^n|_{0,\infty} |Z^n|_{0,\infty} \]
\[ \leq C |Z^n|_{0,\infty} |\tilde{E}^{n} - \tilde{E}^{n-1}|^2_1 + C \Delta t_n |\tilde{E}^{n} - \tilde{E}^{n-1}|_1 |Z^n|_1. \]

Combining (3.44)–(3.46), (3.49)–(3.51), (3.53), and (3.54), we conclude that
\[ |B_{2,4,1}| \leq C \Delta t_n \left[ |D_t \tilde{E}^{n}|_0 + |\tilde{E}^{n}|_1 + |\tilde{E}^{n-1}|_1 + h \right] |Z^n|_1 \]
\[ + C \Delta t_n |D_t \tilde{E}^{n}|_0^2 h^{-\frac{1}{2}} |Z^n|_0 + C |Z^n|_{0,\infty} |\tilde{E}^{n} - \tilde{E}^{n-1}|^2_1. \]

For the second term in (3.43) we obtain with the help of integration by parts that
\[ B_{2,4,2} \]
\[ = \Delta t_n \left( D_t [(\tilde{X}^{n} - \tilde{x}^{n})_\rho \cdot \tilde{x}^{n} \cdot w^n], Z^n \right) \]
\[ = \Delta t_n \left( D_t [(\tilde{X}^{n} - \tilde{x}^{n})_\rho \cdot \tilde{x}^{n} \cdot w^n], Z^n \right) + \Delta t_n \left( (\tilde{X}^{n-1} - \tilde{x}^{n-1})_\rho \cdot D_t [\tilde{x}^{n} \cdot w^n], Z^n \right) \]
\[ = - \Delta t_n \left( D_t (\tilde{X}^{n} - \tilde{x}^{n}), (w^n \cdot Z^n \cdot \tilde{x}^{n})_\rho \right) + \Delta t_n \left( (\tilde{X}^{n-1} - \tilde{x}^{n-1})_\rho \cdot D_t [\tilde{x}^{n} \cdot w^n], Z^n \right). \]

Similarly to (3.50), it follows from (2.6b), (2.1b), (2.1d), (3.1), and Sobolev embedding that
\[ |B_{2,4,2}| \leq C \Delta t_n \left[ |(I - I^h) D_t \tilde{x}^{n}|_0 + \Delta t_n |D_t \tilde{E}^{n}|_0 \right] |Z^n|_1 \]
\[ + C \Delta t_n \left[ |(I - I^h) \tilde{x}^{n-1}|_1 + |\tilde{E}^{n-1}|_1 \right] |Z^n|_0 \]
\[ \leq C \Delta t_n \left[ |D_t \tilde{E}^{n}|_0 + |\tilde{E}^{n-1}|_1 + h \right] |Z^n|_1. \]

Thus, (3.43), (3.55), and (3.56) yield the bound on $B_{2,4}$, which together with (3.39)–(3.42), implies (3.38) on recalling (3.4).

Hence we have bounded all the terms on the right-hand side of (3.31), so we obtain from (3.32)–(3.35), (3.37), and (3.38) that
\[ \frac{1}{2} \left( |\tilde{X}^n|_Z \cdot Z^n \right)^h + \frac{d}{2M} \Delta t_n |Z^n|^2_1 + \frac{1}{2} \left( |\tilde{X}^{n-1}|_Z \cdot (Z^n - Z^{n-1}) \right)^h \]
\[ \leq \frac{1}{2} \left( |\tilde{X}^{n-1}|_Z \cdot Z^{n-1} \right)^h + C \Delta t_n \delta^{-\frac{1}{2}} \left[ |\tilde{E}^{n}|_1 + |D_t \tilde{E}^{n}|_0 \right] |Z^n|_1 |Z^n|_1 \]
\[ + C \Delta t_n \left[ |\tilde{E}^{n}|_1 + |\tilde{E}^{n-1}|_1 + |Z^{n-1}|_0 + |D_t \tilde{E}^{n}|_0 + h \right] |Z^n|_1 \]
\[ + C \Delta t_n |D_t \tilde{E}^{n}|_0^2 h^{-\frac{1}{2}} |Z^n|_0 + C |Z^n|_{0,\infty} |\tilde{E}^{n} - \tilde{E}^{n-1}|^2_1 \]
\[ + C h \left( \Delta t_n \int_{t_{n-1}}^{t_n} \xi(t) dt \right) \delta^{-\frac{1}{2}} |Z^n|_1 + C \Delta t_n |Z^n|^2_0 |Z^n|^2_1 \]
\[ \leq \frac{1}{2} \left( |\tilde{X}^{n-1}|_Z \cdot Z^{n-1} \right)^h + \delta \Delta t_n |Z^n|^2_1 + C(\delta) \Delta t_n \left[ |Z^n|^2_0 + |Z^{n-1}|^2_0 \right] \]
\[ + C(\delta) h^2 \int_{t_{n-1}}^{t_n} \xi(t) dt + C(\delta) \Delta t_n \left[ |\tilde{E}^{n}|^2_1 + |\tilde{E}^{n-1}|^2_1 + |D_t \tilde{E}^{n}|^2_0 \right] \]
\[ + C(\delta) \Delta t_n h^{-1} |Z^n|^2_0 \left[ |\tilde{E}^{n}|^2_1 + |D_t \tilde{E}^{n}|^2_0 \right] + C |Z^n|_{0,\infty} |\tilde{E}^{n} - \tilde{E}^{n-1}|^2_1. \]
Let us examine the last two terms appearing on the right-hand side of (3.57). We deduce from (3.9), (3.25), (2.6a), and (2.4) that

\[
\Delta t_n h^{-1} |Z^n_{10}|^2 \left[ |\tilde{E}^n_{10}|^2 + |D_t \tilde{E}^n_{10}|^2 \right] \\
\leq C h^{-1} \left[ |Z^n_{10}|^2 + |Z^n - Z^{n-1}_{10}| \Delta t_n \left[ |\tilde{E}^n_{10}|^2 + |D_t \tilde{E}^n_{10}|^2 \right] \right] \\
\leq C \Delta t_n \left[ |\tilde{E}^n_{10}|^2 + |D_t \tilde{E}^n_{10}|^2 \right] + C h^\frac{1}{2} |Z^n - Z^{n-1}|_{10}^2
\]

and

\[
|Z^n_{0,\infty} |\tilde{E}^n - \tilde{E}^{n-1}|_{10}^2 \leq C h^{-\frac{1}{2}} \left[ |Z^n_{10} - Z^{n-1}_{10}| + |Z^n - Z^{n-1}| \right] |\tilde{E}^n - \tilde{E}^{n-1}|_{10}^2 \\
\leq C |\tilde{E}^n - \tilde{E}^{n-1}|_{10}^2 + C h^\frac{1}{2} |Z^n - Z^{n-1}| \tilde{E}^n - \tilde{E}^{n-1}|_{10} \\
\leq C |\tilde{E}^n - \tilde{E}^{n-1}|_{10}^2 + C h^\frac{1}{2} |Z^n - Z^{n-1}|_{10}^2.
\]

Combining (3.57) and (3.58a), (3.58b) and choosing \( \delta \) sufficiently small yields, on noting (3.7), (2.8a), and \( \Delta t_n \leq C h \),

\[
\left( |\tilde{X}^n_{\rho}| Z^n, Z^n \right) + \frac{d}{2M} \Delta t_n |Z^n|_{10}^2 \\
+ \left( 1 - C h^\frac{1}{2} \right) \left( |\tilde{X}^{n-1}_{\rho}| Z^n - Z^{n-1}, Z^n - Z^{n-1} \right) \\
\leq \left( |\tilde{X}^{n-1}_{\rho}| Z^n - Z^{n-1}, Z^n - Z^{n-1} \right) + C \Delta t_n \left[ \left| |\tilde{X}^{n-1}_{\rho}| Z^n - Z^{n-1} \right| + |\tilde{E}^{n-1}|_{10}^2 \right] \\
+ C \Delta t_n |D_t \tilde{E}^{n-1}|_{10}^2 + C |\tilde{E}^n - \tilde{E}^{n-1}|_{10}^2 + C h^2 \int_{t_{n-1}}^{t_n} \zeta(t) \, dt.
\]

We now proceed by choosing \( h^* > 0 \) so small that \( C (h^*)^{\frac{1}{2}} \leq 1 \) in (3.59). Multiplying (3.59) by \( \beta^2 \) and adding to (3.22) yields that

\[
\Delta t_n \frac{\alpha n^2}{4} |D_t \tilde{E}^{n-1}|_{10}^2 + |\tilde{E}^{n-1}|_{10}^2 + |\tilde{E}^n|_{10}^2 + \beta^2 \left( |\tilde{X}^n_{\rho}| Z^n, Z^n \right) \\
\leq |\tilde{E}^{n-1}|_{10}^2 + \beta^2 \left( |\tilde{X}^{n-1}_{\rho}| Z^n - Z^{n-1}, Z^n - Z^{n-1} \right) \\
+ C \Delta t_n \left[ |\tilde{E}^{n-1}|_{10}^2 + \left( |\tilde{X}^{n-1}_{\rho}| Z^n - Z^{n-1}, Z^n - Z^{n-1} \right) \right] \\
+ C h^2 \int_{t_{n-1}}^{t_n} \zeta(t) \, dt + C \beta^2 \Delta t_n |D_t \tilde{E}^{n-1}|_{10}^2 + C \beta^2 \tilde{E}^n - \tilde{E}^{n-1}|_{10}^2.
\]

Choosing \( \beta \in (0, 1] \) in such a way that \( C \beta^2 \leq \min\{ \alpha n^2, 1 \} \) we find, with the help of the induction hypothesis (3.3), that for any \( h \in (0, h^*) \)

\[
|\tilde{E}^{n}|_{10}^2 + \beta^2 \left( |\tilde{X}^n_{\rho}| Z^n, Z^n \right) \\
\leq \left( 1 + \frac{C \Delta t_n}{\beta^2} \right) \left( |\tilde{E}^{n-1}|_{10}^2 + \beta^2 \left( |\tilde{X}^{n-1}_{\rho}| Z^n - Z^{n-1}, Z^n - Z^{n-1} \right) \right) \\
+ C \beta^2 \int_{t_{n-1}}^{t_n} \zeta(t) \, dt
\]

\[
\leq h^2 e^{\gamma J_{0}^{n-1}} \zeta(t) \, dt \left( 1 + C \left( 1 + \frac{1}{\beta^2} \right) \right) \int_{t_{n-1}}^{t_n} \zeta(t) \, dt \leq h^2 e^{\gamma J_{0}^{n-1}} \zeta(t) \, dt,
\]

provided that \( \gamma \geq C(1 + \frac{1}{\beta^2}) \). Hence, on assuming (3.3) for some \( n \in \{1, \ldots, N\} \) we have shown that it holds for \( n + 1 \), (3.61), provided that \( \beta \in (0, 1] \) is chosen as...
required in deriving (3.6) and (3.61), γ as required by (3.24) and in deriving (3.61), and finally \( h^* \) so that (3.5), (3.28), and the condition for deriving (3.60) are satisfied.

Therefore, under the above constraints on \( \beta, \gamma, \) and \( h^* \), we have that (3.61) holds for \( n = 0, \ldots, N \) and (3.7) holds for \( n = 1, \ldots, N + 1 \). We deduce from these, on noting (2.8a), that

\[
\left| \bar{E}^n_{1} \right|^2 + \left| Z^n_{10} \right|^2 \leq C h^2. \tag{3.62}
\]

Moreover, we have that (3.22) and (3.59) hold for \( n = 1, \ldots, N \). Summing these from \( n = 1 \) to \( N \) yields, on noting (3.62), (3.7), and (2.8a), that

\[
\sum_{n=1}^{N} \Delta t_n \left[ \left| D_t \bar{E}^n_{10} \right|^2 + \left| Z^n_{10} \right|^2 \right] \leq C h^2. \tag{3.63}
\]

Finally, (3.62), (3.63), (3.2), (2.6b), (2.1b), (2.1c), (3.15), and \( \Delta t_n \leq C h \) yield the desired error bounds (2.13) of Theorem 2.1.

4. Numerical results. Throughout this section we set \( d = 1 \). Let us begin by investigating the experimental order of convergence (eoc). We use the following example from [13, section 6.4] and consider for \( \rho \in \mathbb{I}, t \in [0, 1] \)

\[
\bar{x}(\rho, t) = \left( \frac{1}{2} \sin(2\pi t) \cos(2\pi \rho), \frac{1}{2} \sin(2\pi t) \sin(2\pi \rho) \right), \quad w(\rho, t) = t \cos(8\pi \rho) + (1 - t) \sin(6\pi \rho),
\]

as well as \( f(w) = 2w, g(v, w) = 0 \). With these choices, (1.6), (1.7) are satisfied with additional terms \( \bar{S}, S_w \) on the right-hand side, respectively, so that we replace \( f(W^{n-1}) \bar{Y}^{n-1} \) by \( f(W^{n-1}) \bar{Y}^{n-1} + \bar{S} \) in (2.10) and \( g(V^n, W^{n-1}) \) by \( g(V^n, W^{n-1}) + S_w \) in (2.12). In what follows we choose a uniform element length \( h = 1/J \) and, unless stated otherwise, a uniform time step \( \Delta t = h^2 \) throughout the computations in this section. We monitor the following errors:

\[
\mathcal{E}_1 := \sup_{n=0,\ldots,N} \left| Z^n_{10} \right|^2, \quad \mathcal{E}_2 := \sum_{n=1}^{N} \Delta t_n \left| Z^n_{10} \right|^2, \quad \mathcal{E}_3 := \sup_{n=0,\ldots,N} \left| \bar{E}^n_{1} \right|^2, \quad \mathcal{E}_4 := \sum_{n=1}^{N} \Delta t_n \left| D_t \bar{E}^n_{10} \right|^2.
\]

In Tables 1 and 2 we display the values of \( \mathcal{E}_i, i = 1, \ldots, 4 \), for \( \alpha = 1 \) with \( \Delta t = h^2 \) and \( \Delta t = 0.5h \) respectively. For \( \Delta t = 0.5h \) we see eocs close to two, while for \( \Delta t = h^2 \) we see eocs close to four. When \( \Delta t = h^2 \) the eocs for \( \mathcal{E}_2 \) and \( \mathcal{E}_3 \) are better than the spatial approximation error, thus demonstrating superconvergence. Tables 3 and 4 show the corresponding results for \( \alpha = 0.1 \): again we see eocs close to two for \( \Delta t = 0.5h \) and eocs close to four for \( \Delta t = h^2 \). We saw similar results when we computed \( \mathcal{E}_i, i = 1, \ldots, 4 \), for the scheme in [13]. The above results confirm the bounds obtained in Theorem 2.1. It will be a subject of future research to rigorously prove the better rates in the case \( \Delta t = h^2 \).

In our second example we present numerical results that highlight the advantage of using \( \alpha \ll 1 \) for practical choices of \( J \). We set \( f(w) = 0.5w^2, g(v, w) = 0, \)

\[
\bar{x}(\rho, 0) = \left( \cos(2\pi \rho), 0.9 \cos^2(2\pi \rho) + 0.1 \sin(2\pi \rho) \right), \quad w(\rho, 0) = \sin(6\pi \rho), \quad \rho \in \mathbb{I}.
\]

For \( J = 1200 \) the results obtained by plotting \( \bar{X}(\rho, t) \) for \( t \in [0, 0.15] \), with \( \alpha = 1 \) and \( \alpha = 0.1 \) are visually indistinguishable. However when \( J = 60 \) at \( t = 0.15 \) there is a marked difference between the two solutions. This can be seen in Figure 1 where we display \( \bar{X}(\rho, t) \) (red lines) at times \( t = 0.05, 0.1, \) and \( 0.15 \) for \( \alpha = 1 \) (upper plots).
and $\alpha = 0.1$ (lower plots). In these results we include $\bar{X}(\rho, t)$ (blue lines), obtained by setting $J = 1200$ and $\alpha = 1$, to act as a comparison to the “true solution.” We see that the additional tangential motion of the nodes obtained by setting $\alpha = 0.1$ results in a better approximation at $t = 0.15$ to the “true solution” than the approximation obtained by setting $\alpha = 1$. The importance of the tangential motion of the nodes can be further seen in Figure 2 where we display $\bar{X}(\rho, t)$ (red lines) at times $t = 0.05, 0.1, \text{ and } 0.15$ for the scheme in [13] with $J = 60$, in which there is no tangential motion. Again we include the “true solution” (blue lines) as in Figure 1. The absence of tangential motion leads to a severe accumulation of the nodes at $t = 0.15$; however, as the value of $J$ increases this accumulation effect becomes less pronounced.

We conclude our numerical results with a simulation of diffusion induced grain boundary motion. With $\bar{v}$ being the inward normal, physically meaningful choices for $f(w)$ and $g(v, w)$ in (1.1a), (1.1b) are $f(w) = w^2$ and $g(v, w) = (C - w) - |v| w$, where $C$ is the concentration of solute in the vapor. Note that these choices satisfy (2.1a).

We consider the experimental setup in which the parametrization of the initial grain boundary and the initial concentration of solute are given by

$$
\bar{\xi}(\rho, 0) = \left(\frac{\cos(2\pi \rho)}{2\sin(2\pi \rho) + \sin(\cos(2\pi \rho)) + \sin(2\pi \rho)[\frac{1}{2} + \sin(2\pi \rho)\sin^2(6\pi \rho)]}, \rho \in \mathbb{I},
\right)
$$

Table 1

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<thead>
<tr>
<th>$J$</th>
<th>$\xi_1 \times 10^4$</th>
<th>$\cos_1$</th>
<th>$\xi_2$</th>
<th>$\cos_2$</th>
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<th>$\cos_3$</th>
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<th>$\cos_2$</th>
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<th>$\cos_3$</th>
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Table 4

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<th>$\cos_2$</th>
<th>$\xi_3$</th>
<th>$\cos_3$</th>
<th>$\xi_4 \times 10^4$</th>
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</thead>
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</table>
and \( w(\rho, 0) = 0, \rho \in \mathbb{I} \), respectively, while \( C = 1 \). In the simulations presented we set \( \alpha = 0.1 \) and \( J = 120 \). Figure 3 displays the grain boundary, \( \vec{X}(\rho, t) \), (blue lines) at times \( t = 0.105, t = 0.21 \), and \( t = 0.315 \) together with its initial position (red lines).
In this simulation we see bidirectional motion of the grain boundary; see [8], [10]. In Figure 4 we plot the concentration $w(\rho, t)$ at $t = 0.105, t = 0.21, \text{ and } t = 0.315$.

REFERENCES


